



Research article

Monotone Dynamical Systems with Polyhedral Order Cones and Dense Periodic Points

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Abstract: Let $X \subset \mathbb{R}^n$ be a set whose interior is connected and dense in X , ordered by a closed convex cone $K \subset \mathbb{R}^n$ having nonempty interior. Let $T: X \approx X$ be an order-preserving homeomorphism. The following result is proved: Assume the set of periodic points of T is dense in X , and K is a polyhedron. Then T is periodic.

Keywords: Dynamical systems; ordered spaces; convex cones; periodic orbits

1. Introduction

The following postulates and notation are used throughout:

- $K \subset \mathbb{R}^n$ (Euclidean n -space) is a *solid order cone*: a closed convex cone that has nonempty interior $\text{Int}(K)$ and contains no affine line.
- \mathbb{R}^n has the (partial) order \geq determined by K :

$$y \geq x \iff y - x \in K,$$

referred to as the K -order.

- $X \subset \mathbb{R}^n$ is a nonempty set whose $\text{Int}(X)$ is connected and dense in X .
- $T: X \approx X$ is homeomorphism that is *monotone* for the K -order:

$$x \geq y \implies Tx \geq Ty.$$

A point $x \in X$ has *period* k provided k is a positive integer and $T^k x = x$. The set of such points is $\mathcal{P}_k = \mathcal{P}_k(T)$, and the set of periodic points is $\mathcal{P} = \mathcal{P}(T) = \bigcup_k \mathcal{P}_k$. T is *periodic* if $X = \mathcal{P}_k$, and *pointwise periodic* if $X = \mathcal{P}$.

Our main concern is the following speculation:

Conjecture. *If \mathcal{P} is dense in X , then T is periodic.*

The assumptions on X show that T is periodic iff $T| \text{Int}(X)$ is periodic. Therefore we assume henceforth:

- X is connected and open \mathbb{R}^n .

We prove the conjecture under the additional assumption that K is a *polyhedron*, the intersection of finitely many closed affine halfspaces of \mathbb{R}^n :

Theorem 1 (MAIN). *Assume K is a polyhedron, $T : X \approx X$ is monotone for the K -order, and \mathcal{P} is dense in X . Then T is periodic.*

For analytic maps there is an interesting contrapositive:

Theorem 2. *Assume K is a polyhedron and $T : X \approx X$ is monotone for the K -order. If T is analytic but not periodic, \mathcal{P} is nowhere dense.*

Proof. As X is open and connected but not contained in any of the closed sets \mathcal{P}_k , analyticity implies each \mathcal{P}_k is nowhere dense. Since $\mathcal{P} = \bigcup_{k=1}^{\infty} \mathcal{P}_k$, a well known theorem of Baire [1] implies \mathcal{P} is nowhere dense. ■

The following result of D. MONTGOMERY [4]* is crucial for the proof of the Main Theorem:

Theorem 3 (MONTGOMERY). *Every pointwise periodic homeomorphism of a connected manifold is periodic.*

Notation

i, j, k, l denote positive integers. Points of \mathbb{R}^n are denoted by $a, b, p, q, u, v, w, x, y, z$.

$x \leq y$ is a synonym for $y \geq x$. If $x \leq y$ and $x \neq y$ we write $x <$ or $y > x$.

The relations $x \ll y$ and $y \gg x$ mean $y - x \in \text{Int}(K)$.

A set S is *totally ordered* if $x, y \in S \implies x \leq y$ or $x \geq y$.

If $x \leq y$, the *order interval* $[x, y]$ is $\{z : x \leq z \leq y\} = K_x \cap -K_y$.

The translation of K by $x \in \mathbb{R}^n$ is $K_x := \{w + x, w \in K\}$

The image of a set or point ξ under a map H is denoted by $H\xi$ or $H(\xi)$. A set S is *positively invariant* under H if $HS \subset S$, *invariant* if $H\xi = \xi$, and *periodically invariant* if $H^k\xi = \xi$.

2. Proof of the Main Theorem

The following four topological consequences of the standing assumptions are valid even if K is not polyhedral.

Proposition 4. *Assume $p, q \in \mathcal{P}_k$ are such that*

$$p \ll q, \quad p, q \in \mathcal{P}_k, \quad [p, q] \subset X.$$

Then $T^k([p, q]) = [p, q]$.

*See also S. KAUL [3].

Proof. It suffices to take $k = 1$. Evidently $T\mathcal{P} = \mathcal{P}$, and $T[p, q] \subset [p, q]$ because T is monotone, whence $\text{Int}([p, q]) \cap \mathcal{P}$ is positively invariant under T . The conclusion follows because $\text{Int}([p, q]) \cap \mathcal{P}$ is dense in $[p, q]$ and T is continuous. ■

Proposition 5. *Assume $a, b \in \mathcal{P}_k, a \ll b$, and $[a, b] \subset X$. There is a compact arc $J \subset \mathcal{P}_k \cap [a, b]$ that joins a to b , and is totally ordered by \ll .[†]*

Proof. An application of Zorn's Lemma yields a maximal set $J \subset [a, b] \cap \mathcal{P}$ such that: J is totally ordered by \ll , $a = \max J$, $b = \min J$. Maximality implies J is compact and connected and $a, b \in J$, so J is an arc (WILDER [7], Theorem I.11.23). ■

Proposition 6. *Let $M \subset X$ be a homeomorphically embedded topological manifold of dimension $n - 1$, with empty boundary.*

(i) \mathcal{P} is dense in M .

(ii) *If M is periodically invariant, it has a neighborhood base \mathcal{B} of periodically invariant open sets.*

Proof. (i) M locally separates X , by Lefschetz duality [5] (or dimension theory [6]). Therefore we can choose a family \mathcal{V} of nonempty open sets in X that the family of sets $\mathcal{V}_M := \{V \cap M : V \in \mathcal{V}\}$ satisfies:

- \mathcal{V}_M is a neighborhood basis of M ,
- each set $V \cap M$ separates V .

By Proposition 5, for each $V \in \mathcal{V}$ there is a compact arc $J_V \cap \mathcal{P} \cap V$ whose endpoints a_V, b_V lie in different components of $V \setminus M$. Since J_V is connected, it contains a point in $V \cap M \cap \mathcal{P}$. This proves (i).

(ii) With notation as above, let $B_V := [a_V, b_V] \setminus \partial[a_V, b_V]$. The desired neighborhood basis is $\mathcal{B} := \{B_V : V \in \mathcal{V}\}$. ■

From Propositions 4 and 6 we infer:

Proposition 7. *Suppose $p, q \in \mathcal{P}, p \ll q$ and $[p, q] \subset X$. Then \mathcal{P} is dense in $\partial[p, q]$.* ■

Let $\mathcal{T}(m)$ stand for the statement of Theorem 1 for the case $n = m$. Then $\mathcal{T}(0)$ is trivial, and we use the following inductive hypothesis:

Hypothesis (INDUCTION). $n \geq 1$ and $\mathcal{T}(n - 1)$ holds.

Let $Q \subset \mathbb{R}^n$ be a compact n -dimensional polyhedron. Its boundary ∂Q is the union of finitely many convex compact $(n - 1)$ -cells, the *faces* of Q . Each face F is the intersection of $\partial[p, q]$ with a unique affine hyperplane E^{n-1} . The corresponding *open face* $F^\circ := F \setminus \partial F$ is an open $(n - 1)$ -cell in E^{n-1} . Distinct open faces are disjoint, and their union is dense and open in ∂Q .

Proposition 8. *Assume $p, q \in \mathcal{P}_k, p \ll q, [p, q] \subset X$. Then $T|\partial[p, q]$ is periodic.*

[†]This result is adapted from HIRSCH & SMITH [2], Theorems 5.11 & 5.15.

Proof. $[p, q]$ is a compact, convex n -dimensional polyhedron, invariant under T^k (Proposition 4). By Proposition 6 applied to $M := \partial[p, q]$, there is a neighborhood base \mathcal{B} for $\partial[p, q]$ composed of periodically invariant open sets. Therefore if $F^\circ \subset \partial[p, q]$ is an open face of $[p, q]$, the family of sets

$$\mathcal{B}_{F^\circ} := \{W \in \mathcal{B} : W \subset F^\circ\}$$

is a neighborhood base for F° , and each $W \in \mathcal{B}_{F^\circ}$ is a periodically invariant open set in which \mathcal{P} is dense.

For every face F of $[p, q]$ the Induction Hypothesis shows that $F^\circ \subset \mathcal{P}$. Therefore Montgomery's Theorem implies $T|F^\circ$ is periodic, so $T|F$ is periodic by continuity. Since $\partial[p, q]$ is the union of the finitely many faces, it follows that $T|\partial[p, q]$ is periodic. ■

To complete the inductive proof of the Main Theorem, it suffices by Montgomery's theorem to prove that an arbitrary $x \in X$ is periodic. As X is open in \mathbb{R}^n and \mathcal{P} is dense in X , there is an order interval $[a, b] \subset X$ such that

$$a \ll x \ll b, \quad a, b \in \mathcal{P}_k.$$

By Proposition 5, a and b are the endpoints of a compact arc $J \subset \mathcal{P}_k \cap [a, b]$, totally ordered by \ll . Define $p, q \in J$:

$$p := \sup \{y \in J : y \leq x\}, \quad q := \inf \{y \in J : y \geq x\}.$$

If $p = q = x$ then $x \in \mathcal{P}_k$. Otherwise $p \ll q$, implying $x \in \partial[p, q]$, whence $x \in \mathcal{P}$ by Proposition 8. ■

Conflict of Interest

The author declares no conflicts of interest in this paper.

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