Tropical curves and metric graphs

by

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Abstract

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In just ten years, tropical geometry has established itself as an important new field bridging algebraic geometry and combinatorics whose techniques have been used to attack problems in both fields. It also has important connections to areas as diverse as geometric group theory, mirror symmetry, and phylogenetics. Our particular interest here is the tropical geometry associated to algebraic curves over a field with nonarchimedean valuation. This dissertation examines tropical curves from several angles.

An *abstract tropical curve* is a vertex-weighted metric graph satisfying certain conditions (see Definition 2.2.1), while an *embedded tropical curve* takes the form of a 1-dimensional balanced polyhedral complex in $\mathbb{R}^n$. Both combinatorial objects inform the study of algebraic curves over nonarchimedean fields. The connection between the two perspectives is also very rich and is developed e.g. in [Pay09] and [BPR11]; we give a brief overview in Chapter 1 as well as a contribution in Chapter 4.

Chapters 2 and 3 are contributions to the study of abstract tropical curves. We begin in Chapter 2 by studying the moduli space of abstract tropical curves of genus g, the moduli space of principally polarized tropical abelian varieties, and the tropical Torelli map, as initiated in [BMV11]. We provide a detailed combinatorial and computational study of these objects and give a new definition of the category of stacky fans, of which the aforementioned moduli spaces are objects and the Torelli map is a morphism.

In Chapter 3, we study the locus of tropical hyperelliptic curves inside the moduli space of tropical curves of genus g. Our work ties together two strands in the tropical geometry literature, namely the study of the *tropical moduli space of curves* and *tropical Brill-Noether theory*. Our methods are graph-theoretic and extend much of the work of Baker and Norine [BN09] on harmonic morphisms of graphs to the case of metric graphs. We also provide new computations of tropical hyperelliptic loci in the form of theorems describing their specific combinatorial structure.

Chapter 4 presents joint work with Bernd Sturmfels and is a contribution to the study of tropical curves as balanced embedded 1-dimensional polyhedral complexes. We say that a plane cubic curve, defined over a field with valuation, is in *honeycomb form* if its tropicalization exhibits the standard hexagonal cycle shown in Figure 4.1. We explicitly compute such representations from a given j-invariant with negative valuation, we give

an analytic characterization of elliptic curves in honeycomb form, and we offer a detailed analysis of the tropical group law on such a curve.

Chapter 5 is joint work with Anders Jensen and Elena Rubei and is a departure from the subject of tropical curves. In this chapter, we study tropical determinantal varieties and prevarieties. After recalling the definitions of tropical prevarieties, varieties, and bases, we present a short proof that the 4×4 minors of a $5 \times n$ matrix of indeterminates form a tropical basis. The methods are combinatorial and involve a study of arrangements of tropical hyperplanes. Our result together with the results in [DSS05], [Shi10], [Shi11] answer completely the fundamental question of when the $r \times r$ minors of a $d \times n$ matrix form a tropical basis; see Table 5.1.

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> "And remember, also," added the Princess of Sweet Rhyme, "that many places you would like to see are just off the map and many things you want to know are just out of sight or a little beyond your reach. But someday you'll reach them all, for what you learn today, for no reason at all, will help you discover all the wonderful secrets of tomorrow."
>
> Norton Juster, *The Phantom Tollbooth*

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Chapter 1

Introduction

In just ten years, tropical geometry has established itself as an important new field bridging algebraic geometry and combinatorics whose techniques have been used successfully to attack problems in both fields. Tropical geometry also has important connections to areas as diverse as geometric group theory, mirror symmetry, and phylogenetics.

There are several different ways to describe tropical geometry. On the one hand, it is a "combinatorial shadow" of algebraic geometry [MS10]. Let us start with the following definition. Let K be a field, which we will assume for now to be algebraically closed and complete with respect to a nonarchimedean valuation $\mathrm{val} : K^* \to \mathbb{R}$ on it. Suppose X is an algebraic subvariety of the torus $(K^*)^n$, that is, the solution set to a system of Laurent polynomials over K. Then the *tropicalization* of X is the set

$$\mathrm{Trop}(X) = \{(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) \in \mathbb{R}^n : (x_1, \ldots, x_n) \in X\}.$$

(more precisely, it is the closure in $\mathbb{R}^n$, under its usual topology, of this set.) By the theory of initial degenerations (see [Stu96], [MS10, Theorem 3.2.4]), these tropical varieties are made of polyhedral pieces and have many nice combinatorial properties. Furthermore, they remember information about classical varieties, for example, their dimensions [BG84]. So, if X was a plane curve, then $\mathrm{Trop}(X)$ would be made of 1-dimensional polyhedra, i.e. line segments and rays, in $\mathbb{R}^2$. A tropical plane curve of degree 3 is shown in Figure 1.1; this particular cubic will be revisited in Chapter 4. Note that different embeddings of a variety X can yield very different tropicalizations.

There is a complementary perspective from which tropical geometry is a tool for taking finite snapshots of Berkovich analytifications. This perspective has been made explicit in the paper [BPR11] in the case of curves. Suppose X is a smooth curve over the field K. Then X has a space X^{an} intrinsically associated to it called its *Berkovich analytification* [Ber90]. Furthermore, there is a metric on X^{an}, or more precisely on $X^{an} \setminus X$, and one views the original points of the curve X as infinitely far away. The space X^{an} is a very useful object to study because it has some key desirable properties that X lacks: it admits a good notion of an analytic function on it, along the lines of Tate's pioneering work on rigid geometry; but in addition, it has a more desirable topology than X, which is a totally disconnected space since the nonarchimedean field K itself is. Furthermore,

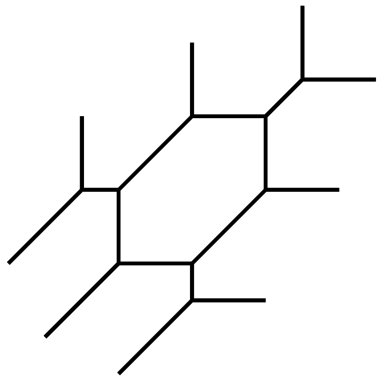


Figure 1.1: A tropical plane cubic curve.

X^{an} has a canonical deformation retract down to a finite metric graph Γ sitting inside it, called its *Berkovich skeleton.* There is a canonical choice of such a Γ for curves of genus at least 2, or for genus 1 curves with a marked point, i.e. elliptic curves. This combinatorial core Γ, decorated with some nonnegative integer weights, is visible in sufficiently nice tropicalizations (see Theorem 6.20 of [BPR11] for the precise theorem, which is actually stronger and involves an extended notion of tropicalization inside toric varieties in the sense of [Pay09]). Thus, we call this decorated metric graph Γ an (abstract) *tropical curve.* More precisely:

Definition. *An* abstract tropical curve *is a triple* (G, w, ℓ)*, where* G *is a connected multigraph,* $\ell : E(G) \to \mathbb{R}_{>0}$ *is a length function on the edges of* G*, and* $w : V(G) \to \mathbb{Z}_{\geq 0}$ *is a weight function on the vertices of* G *such that if* $w(v) = 0$*, then* v *has valence at least* 3*. The* genus *of the curve is* $\dim H_1(G, \mathbb{R}) + \sum w(v)$*.*

Thus, as suggested above, we have a natural map

$$\mathrm{trop} : \mathcal{M}_{g,n}(K) \longrightarrow M_{g,n}^{tr}$$

sending a genus g curve over K with n marked points to its skeleton, canonically defined when $g \geq 2$ or $g = n = 1$ [BPR11, Remark 5.52].

Abstract tropical curves play a central role in Chapters 2 and 3 of this dissertation. In Chapter 2, we study the moduli space of abstract tropical curves of genus g, the moduli space of principally polarized tropical abelian varieties, and the tropical Torelli map, a study initiated in [BMV11]. In Chapter 3, we study the locus of tropical hyperelliptic curves inside the moduli space of tropical curves of genus g. We view both chapters as contributions to understanding the combinatorial side of the study of tropical curves, moduli spaces, and Brill-Noether theory. We hope that this work will serve as the combinatorial underpinning of future developments tightening the relationships between relating algebraic and tropical curves and their moduli, for example by studying the fibers of the map of moduli spaces above.

In Chapter 4, we study elliptic curves in *honeycomb form*, that is, plane cubic curves whose tropicalizations are in the combinatorially desirable form shown in Figure 1.1. An elliptic E curve can be put into such a form if and only if the valuation of its j-invariant is negative; equivalently, the abstract tropical curve associated to it consists of a single vertex plus a loop edge based at that vertex of length $-\operatorname{val}(j(E))$. Put differently, the tropicalization of an elliptic curve in honeycomb form faithfully represents the cycle in E^{an}. Our work can thus be viewed as making more explicit, in the case of elliptic curves with bad reduction, the abstract tropicalization map trop defined above. It can also be viewed as a computational algebra supplement to [BPR11, §7.1], in which faithful tropicalizations elliptic curves are considered. In particular, we use explicit computations with both classical (nonarchimedean) and tropical theta functions to gather specific combinatorial information about elliptic curves and the tropical group law on them, as suggested by Matt Baker. Chapter 4 is joint work with Bernd Sturmfels.

In Chapter 5, we turn our attention to the study of tropical determinantal varieties and prevarieties, and we present a short proof that the 4×4 minors of a $5 \times n$ matrix of indeterminates form a tropical basis. This chapter is joint work with Anders Jensen and Elena Rubei.

Chapter 2

Tropical curves, abelian varieties, and the Torelli map

This chapter presents the paper "Combinatorics of the tropical Torelli map" [Cha11a], which will appear in *Algebra and Number Theory*, with only minor changes.

2.1 Introduction

In this chapter, we undertake a combinatorial and computational study of the tropical moduli spaces M_g^{tr} and A_g^{tr} and the tropical Torelli map.

There is, of course, a vast literature on the subjects of algebraic curves and moduli spaces in algebraic geometry. For example, two well-studied objects are the moduli space $\mathcal{M}_g$ of smooth projective complex curves of genus g and the moduli space $\mathcal{A}_g$ of g-dimensional principally polarized complex abelian varieties. The Torelli map

$$t_g : \mathcal{M}_g \to \mathcal{A}_g$$

then sends a genus g algebraic curve to its Jacobian, which is a certain g-dimensional complex torus. The image of t_g is called the Torelli locus or the Schottky locus. The problem of how to characterize the Schottky locus inside $\mathcal{A}_g$ is already very deep. See, for example, the survey of Grushevsky [Gru09].

The perspective we take here is the perspective of tropical geometry [MS10]. From this viewpoint, one replaces algebraic varieties with piecewise-linear or polyhedral objects. These latter objects are amenable to combinatorial techniques, but they still carry information about the former ones. Roughly speaking, the information they carry has to do with what is happening "at the boundary" or "at the missing points" of the algebraic object.

For example, the tropical analogue of $\mathcal{M}_g$, denoted M_g^{tr}, parametrizes certain weighted metric graphs, and it has a poset of cells corresponding to the boundary strata of the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of $\mathcal{M}_g$. Under this correspondence, a stable curve C in $\overline{\mathcal{M}_g}$ is sent to its so-called dual graph. The irreducible components of C, weighted by their geometric genus, are the vertices of this graph, and each node in the intersection of two components is recorded with an edge. The correspondence in genus 2 is shown in Figure 2.4. A rigorous proof of this correspondence was given by Caporaso in [Cap10, Section 5.3].

We remark that the correspondence above yields dual graphs that are just graphs, not metric graphs. One can refine the correspondence using Berkovich analytifications, whereby an algebraic curve over a complete nonarchimedean valued field is associated to its Berkovich skeleton, which is intrinsically a metric graph. In this way, one obtains a map between classical and tropical moduli spaces. This very interesting perspective, developed in [BPR11, Section 5], was already mentioned in Chapter 1, and additionally plays a crucial role in Chapter 4, a study of tropical elliptic curves in honeycomb form.

The starting point of this chapter is the recent paper by Brannetti, Melo, and Viviani [BMV11]. In that paper, the authors rigorously define a plausible category for tropical moduli spaces called stacky fans. (The term "stacky fan" is due to the authors of [BMV11], and is unrelated to the construction of Borisov, Chen, and Smith in [BCS05]). They further define the tropical versions M_g^{tr} and A_g^{tr} of $\mathcal{M}_g$ and $\mathcal{A}_g$ and a tropical Torelli map between them, and prove many results about these objects, some of which we will review here.

Preceding that paper is the foundational work of Mikhalkin in [Mik06b] and of Mikhalkin and Zharkov [MZ07], in which tropical curves and Jacobians were first introduced and studied in detail. The notion of tropical curves in [BMV11] is slightly different from the original definition, in that curves now come equipped with vertex weights. We should also mention the work of Caporaso [Cap10], who proves geometric results on M_g^{tr} considered just as a topological space, and Caporaso and Viviani [CV10], who prove a tropical Torelli theorem stating that the tropical Torelli map is "mostly" injective, as originally conjectured in [MZ07].

In laying the groundwork for the results we will present here, we ran into some inconsistencies in [BMV11]. It seems that the definition of a stacky fan there is inadvertently restrictive. In fact, it excludes M_g^{tr} and A_g^{tr} themselves from being stacky fans. Also, there is a topological subtlety in defining A_g^{tr}, which we will address in §4.4. Thus, we find ourself doing some foundational work here too.

We begin in Section 2 by recalling the definition in [BMV11] of the tropical moduli space M_g^{tr} and presenting computations, summarized in Theorem 2.2.12, for $g \leq 5$. With M_g^{tr} as a motivating example, we attempt a better definition of stacky fans in Section 3. In Section 4, we define the space A_g^{tr}, recalling the beautiful combinatorics of Voronoi decompositions along the way, and prove that it is Hausdorff. Note that our definition of this space, Definition 2.4.10, is different from the one in [BMV11, Section 4.2], and it corrects a minor error there. In Section 5, we study the combinatorics of the zonotopal subfan. We review the tropical Torelli map in Section 6; Theorem 2.6.4 presents computations on the tropical Schottky locus for $g \leq 5$. Tables 1 and 2 compare the number of cells in the stacky fans M_g^{tr}, the Schottky locus, and A_g^{tr} for $g \leq 5$. In Section 7, we partially answer a question suggested by Diane Maclagan: we give finite-index covers of A_2^{tr} and A_3^{tr} that satisfy a tropical-type balancing condition.

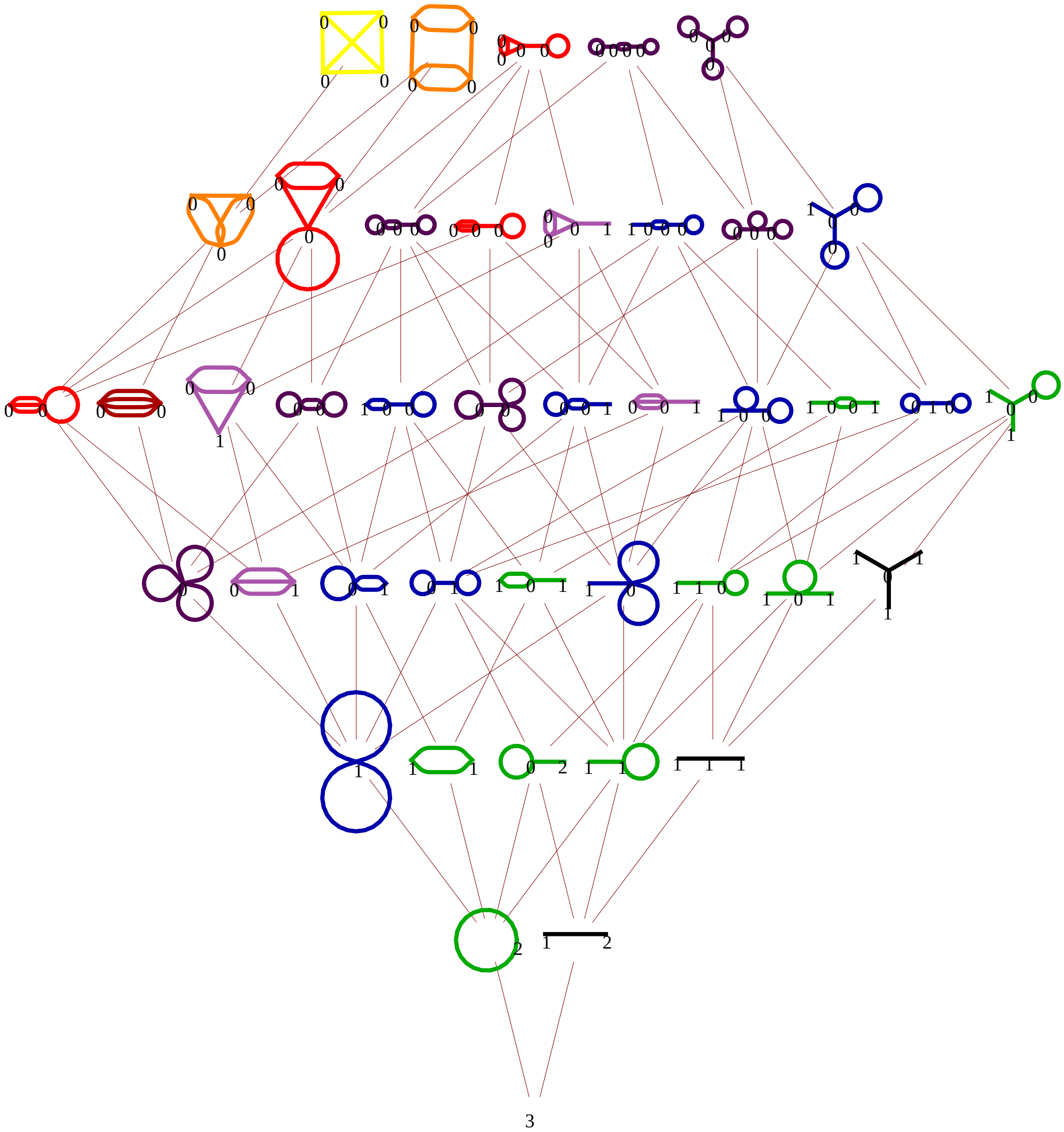


Figure 2.1: Poset of cells of M_3^{tr}, color-coded according to their images in A_3^{tr} via the tropical Torelli map.

2.2 The moduli space of tropical curves

In this section, we review the construction in [BMV11] of the moduli space of tropical curves of a fixed genus g (see also [Mik06b]). This space is denoted M_g^{tr}. Then, we present explicit computations of these spaces in genus up to 5.

We will see that the moduli space M_g^{tr} is not itself a tropical variety, in that it does not have the structure of a balanced polyhedral fan ([MS10, Definition 3.3.1]). That would be too much to expect, as it has automorphisms built into its structure that precisely give rise to "stackiness." Contrast this with the situation of moduli space $M_{0,n}$ of tropical rational curves with n marked points, constructed and studied in [GKM09], [Mik06a], and [SS04]. As expected by analogy with the classical situation, this latter space is well known to have the structure of a tropical variety that comes from the tropical Grassmannian $Gr(2,n)$.

§2.2.1 Definition of tropical curves.

Before constructing the moduli space of tropical curves, let us review the definition of a tropical curve.

First, recall that a **metric graph** is a pair (G, l), where G is a finite connected graph, loops and parallel edges allowed, and l is a function

$$l : E(G) \to \mathbb{R}_{>0}$$

on the edges of G. We view l as recording lengths of the edges of G. The **genus** of a graph G is the rank of its first homology group:

$$g(G) = |E| - |V| + 1.$$

Definition 2.2.1. A **tropical curve** C is a triple (G, l, w), where (G, l) is a metric graph (so G is connected), and w is a weight function

$$w : V(G) \to \mathbb{Z}_{\geq 0}$$

on the vertices of G, with the property that every weight zero vertex has degree at least 3.

Definition 2.2.2. Two tropical curves (G, l, w) and (G', l', w') are isomorphic if there is an isomorphism of graphs $G \xrightarrow{\cong} G'$ that preserves edge lengths and preserves vertex weights.

We are interested in tropical curves only up to isomorphism. When we speak of a tropical curve, we will really mean its isomorphism class.

Definition 2.2.3. Given a tropical curve $C = (G, l, w)$, write

$$|w| := \sum_{v \in V(G)} w(v).$$

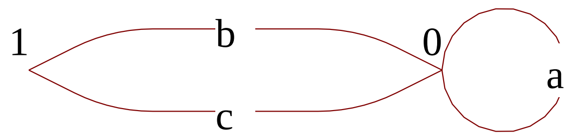


Figure 2.2: A tropical curve of genus 3. Here, a, b, c are fixed positive real numbers.

Then the **genus** of C is defined to be

$$g(C) = g(G) + |w|.$$

In this chapter, we will restrict our attention to tropical curves of genus at least 2.

The **combinatorial type** of C is the pair (G, w), in other words, all of the data of C except for the edge lengths.

Remark 2.2.4. Informally, we view a weight of k at a vertex v as k loops, based at v, of infinitesimally small length. Each infinitesimal loop contributes once to the genus of C. Furthermore, the property that only vertices with positive weight may have degree 1 or 2 amounts to requiring that, were the infinitesimal loops really to exist, every vertex would have degree at least 3.

Permitting vertex weights will ensure that the moduli space M_g^{tr}, once it is constructed, is complete. That is, a sequence of genus g tropical curves obtained by sending the length of a loop to zero will still converge to a genus g curve. Furthermore, permitting vertex weights allows the combinatorial types of genus g tropical curves to correspond precisely to dual graphs of stable curves in $\overline{\mathcal{M}_g}$, as discussed in the introduction and in [Cap10, Section 5.3]. See Figure 2.4.

Figure 2.2 shows an example of a tropical curve C of genus 3. Note that if we allow the edge lengths l to vary over all positive real numbers, we obtain all tropical curves of the same combinatorial type as C. This motivates our construction of the moduli space of tropical curves below. We will first group together curves of the same combinatorial type, obtaining one cell for each combinatorial type. Then, we will glue our cells together to obtain the moduli space.

§2.2.2 Definition of the moduli space of tropical curves

Fix $g \geq 2$. Our goal now is to construct a moduli space for genus g tropical curves, that is, a space whose points correspond to tropical curves of genus g and whose geometry reflects the geometry of the tropical curves in a sensible way. The following construction is due to the authors of [BMV11].

First, fix a combinatorial type (G, w) of genus g. What is a parameter space for all tropical curves of this type? Our first guess might be a positive orthant $\mathbb{R}_{>0}^{|E(G)|}$, that is, a choice of positive length for each edge of G. But we have overcounted by symmetries of the combinatorial type (G, w). For example, in Figure 2.2, $(a, b, c) = (1, 2, 3)$ and $(a, b, c) = (1, 3, 2)$ give the same tropical curve.

Furthermore, with foresight, we will allow lengths of zero on our edges as well, with the understanding that a curve with some zero-length edges will soon be identified with the curve obtained by contracting those edges. This suggests the following definition.

Definition 2.2.5. Given a combinatorial type (G, w), let the **automorphism group** $\mathrm{Aut}(G, w)$ be the set of all permutations $\varphi : E(G) \to E(G)$ that arise from weight-preserving automorphisms of G. That is, $\mathrm{Aut}(G, w)$ is the set of permutations $\varphi : E(G) \to E(G)$ that admit a permutation $\pi : V(G) \to V(G)$ which preserves the weight function w, and such that if an edge $e \in E(G)$ has endpoints v and w, then $\varphi(e)$ has endpoints $\pi(v)$ and $\pi(w)$.

Now, the group $\mathrm{Aut}(G, w)$ acts naturally on the set $E(G)$, and hence on the orthant $\mathbb{R}_{\geq 0}^{E(G)}$, with the latter action given by permuting coordinates. We define $\overline{C(G, w)}$ to be the topological quotient space

$$\overline{C(G, w)} = \frac{\mathbb{R}_{\geq 0}^{E(G)}}{\mathrm{Aut}(G, w)}.$$

Next, we define an equivalence relation on the points in the union

$$\coprod \overline{C(G, w)},$$

as (G, w) ranges over all combinatorial types of genus g. Regard a point $x \in \overline{C(G, w)}$ as an assignment of lengths to the edges of G. Now, given two points $x \in \overline{C(G, w)}$ and $x' \in \overline{C(G', w')}$, let $x \sim x'$ if the two tropical curves obtained from them by contracting all edges of length zero are isomorphic. Note that contracting a loop, say at vertex v, means deleting that loop and adding 1 to the weight of v. Contracting a nonloop edge, say with endpoints v_1 and v_2, means deleting that edge and identifying v_1 and v_2 to obtain a new vertex whose weight is $w(v_1) + w(v_2)$.

Now we glue the cells $\overline{C(G, w)}$ along $\sim$ to obtain our moduli space:

Definition 2.2.6. The **moduli space** M_g^{tr} is the topological space

$$M_g^{\mathrm{tr}} \; := \; \coprod \overline{C(G, w)} / \sim,$$

where the disjoint union ranges over all combinatorial types of genus g, and $\sim$ is the equivalence relation defined above.

In fact, the space M_g^{tr} carries additional structure: it is an example of a stacky fan. We will define the category of stacky fans in Section 3.

Example 2.2.7. Figure 2.3 is a picture of M_2^{tr}. Its cells are quotients of polyhedral cones; the dotted lines represent symmetries, and faces labeled by the same combinatorial type are in fact identified. The poset of cells, which we will investigate next for higher g, is shown in Figure 2.4. It has two vertices, two edges and two 2-cells. $\diamond$

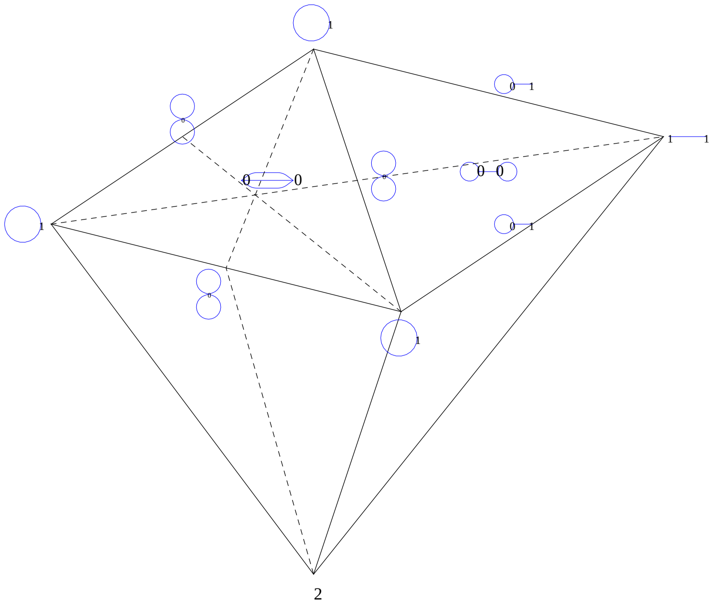


Figure 2.3: The stacky fan M_2^{tr}.

Remark 2.2.8. One can also construct the moduli space of genus g tropical curves with n marked points using the same methods, as done for example in the recent survey of Caporaso [Cap11a].

§2.2.3 Explicit computations of M_g^{tr}

Our next goal will be to compute the space M_g^{tr} for g at most 5. The computations were done in Mathematica.

What we compute, to be precise, is the partially ordered set P_g on the cells of M_g^{tr}. This poset is defined in Lemma 2.2.10 below. Our results, summarized in Theorem 2.2.12 below, provide independent verification of the first six terms of the sequence A174224 in [OEIS], which counts the number of tropical curves of genus g:

$$0, 0, 7, 42, 379, 4555, 69808, 1281678, \ldots$$

This sequence, along with much more data along these lines, was first obtained by Maggiolo and Pagani by an algorithm described in [MP].

Definition 2.2.9. Given two combinatorial types (G, w) and (G', w') of genus g, we say that (G', w') is a **specialization**, or **contraction**, of (G, w), if it can be obtained from (G, w) by a sequence of edge contractions. Here, contracting a loop means deleting it and adding 1 to the weight of its base vertex; contracting a nonloop edge, say with endpoints v_1 and v_2, means deleting the edge and identifying v_1 and v_2 to obtain a new vertex whose weight we set to $w(v_1) + w(v_2)$.

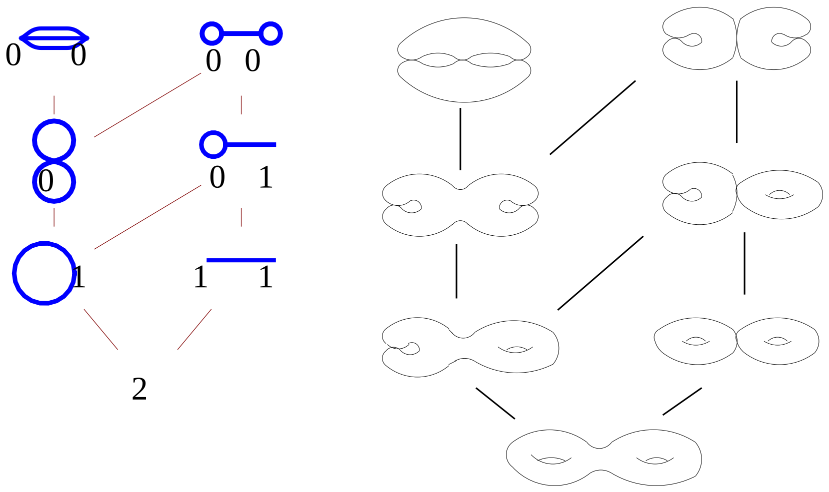


Figure 2.4: Posets of cells of M_2^{tr} (left) and of $\overline{\mathcal{M}_2}$ (right).

Lemma 2.2.10. *The relation of specialization on genus g combinatorial types yields a graded partially ordered set P_g on the cells of M_g^{tr}. The rank of a combinatorial type (G, w) is $|E(G)|$.*

Proof. It is clear that we obtain a poset; furthermore, (G', w') is covered by (G, w) precisely if (G', w') is obtained from (G, w) by contracting a single edge. The formula for the rank then follows. □

For example, P_2 is shown in Figure 2.4; it also appeared in [BMV11, Figure 1]. The poset P_3 is shown in Figure 2.1. It is color-coded according to the Torelli map, as explained in Section 6.

Our goal is to compute P_g. We do so by first listing its maximal elements, and then computing all possible specializations of those combinatorial types. For the first step, we use Proposition 3.2.4(i) in [BMV11], which characterizes the maximal cells of M_g^{tr}: they correspond precisely to combinatorial types $(G, \overline{0})$, where G is a connected 3-regular graph of genus g, and $\overline{0}$ is the zero weight function on $V(G)$. Connected, 3-regular graphs of genus g are equivalently characterized as connected, 3-regular graphs on $2g-2$ vertices. These have been enumerated:

Proposition 2.2.11. *The number of maximal cells of M_g^{tr} is equal to the $(g-1)^{st}$ term in the sequence*

$$2, 5, 17, 71, 388, 2592, 21096, 204638, 2317172, 30024276, 437469859, \ldots$$

Proof. This is sequence A005967 in [OEIS], whose g^{th} term is the number of connected 3-regular graphs on $2g$ vertices. □

In fact, the connected, 3-regular graphs of genus g have been conveniently written down for g at most 6. This work was done in the 1970s by Balaban, a chemist whose interests along these lines were in molecular applications of graph theory. The graphs for $g \leq 5$ appear in his article [Bal76], and the 388 genus 6 graphs appear in [Bal70].

Given the maximal cells of M_g^{tr}, we can compute the rest of them:

Input: Maximal cells of M_g^{tr}
Output: Poset of all cells of M_g^{tr}

1. Initialize P_g to be the set of all maximal cells of M_g^{tr}, with no relations. Let L be a list of elements of P_g.
2. While L is nonempty:

   Let (G, w) be the first element of L. Remove (G, w) from L. Compute all 1-edge contractions of (G, w).
   For each such contraction (G', w'):

   If (G', w') is isomorphic to an element (G'', w'') already in the poset P_g, add a cover relation $(G'', w'') \leq (G, w)$.
   Else, add (G', w') to P_g and add a cover relation $(G', w') \leq (G, w)$. Add (G', w') to the list L.
3. Return P_g.

We implemented this algorithm in MATHEMATICA. The most costly step is computing graph isomorphisms in Step 2. Our results are summarized in the following theorem. By an f-vector of a poset, we mean the vector whose i-th entry is the number of elements of rank $i-1$. (The term "f-vector" originates from counting faces of polytopes).

Theorem 2.2.12. *We obtained the following computational results:*

(i) The moduli space M_3^{tr} has 42 cells and f-vector

$$(1, 2, 5, 9, 12, 8, 5).$$

Its poset of cells P_3 is shown in Figure 2.1.

(ii) The moduli space M_4^{tr} has 379 cells and f-vector

$$(1, 3, 7, 21, 43, 75, 89, 81, 42, 17).$$

(iii) The moduli space M_5^{tr} has 4555 cells and f-vector

$$(1, 3, 11, 34, 100, 239, 492, 784, 1002, 926, 632, 260, 71).$$

Remark 2.2.13. The data of P_3, illustrated in Figure 2.1, is related, but not identical, to the data obtained by T. Brady in [Bra93, Appendix A]. In that paper, the author enumerates the cells of a certain deformation retract, called K_3, of Culler-Vogtmann Outer Space [CV84], modulo the action of the group $\mathrm{Out}(F_3)$. In that setting, one only needs to consider bridgeless graphs with all vertices of weight 0, thus throwing out all but 8 cells of the poset P_3. In turn, the cells of $K_3/\mathrm{Out}(F_n)$ correspond to chains in the poset on those eight cells. It is these chains that are listed in Appendix A of [Bra93].

Note that the pure part of $M_g^{tr'}$, that is, those tropical curves in $M_g^{tr'}$ with all vertex weights zero, is a quotient of rank g Outer Space by the action of the outer automorphism group $Out(F_g)$. We believe that further exploration of the connection between Outer Space and M_g^{tr} would be interesting to researchers in both tropical geometry and geometric group theory.

Remark 2.2.14. What is the topology of M_g^{tr}? Of course, M_g^{tr} is always contractible: there is a deformation retract onto the unique 0-dimensional cell. So to make this question interesting, we restrict our attention to the subspace $M_g^{tr'}$ of M_g^{tr} consisting of graphs with total edge length 1, say. For example, by looking at Figure 2.3, we can see that $M_2^{tr'}$ is still contractible. We would like to know if the space $M_g^{tr'}$ is also contractible for larger g.

2.3 Stacky fans

In Section 2, we defined the space M_g^{tr}. In Sections 4 and 6, we will define the space A_g^{tr} and the Torelli map $t_g^{\mathrm{tr}} : M_g^{\mathrm{tr}} \to A_g^{\mathrm{tr}}$. For now, however, let us pause and define the category of stacky fans, of which M_g^{tr} and A_g^{tr} are objects and t_g^{tr} is a morphism. The reader is invited to keep M_g^{tr} in mind as a running example of a stacky fan.

The purpose of this section is to offer a new definition of stacky fan, Definition 2.3.2, which we hope fixes an inconsistency in the definition by Brannetti, Melo, and Viviani, Definition 2.1.1 of [BMV11]. We believe that their condition for integral-linear gluing maps is too restrictive and fails for M_g^{tr} and A_g^{tr}. See Remark 2.3.6. However, we do think that their definition of a stacky fan morphism is correct, so we repeat it in Definition 2.3.5. We also prove that M_g^{tr} is a stacky fan according to our new definition. The proof for A_g^{tr} is deferred to §4.3.

Definition 2.3.1. A **rational open polyhedral cone** in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form $\{a_1x_1+\cdots+a_tx_t : a_i \in \mathbb{R}_{>0}\}$, for some fixed vectors $x_1,\ldots,x_t \in \mathbb{Z}^n$. By convention, we also allow the trivial cone $\{0\}$.

Definition 2.3.2. Let $X_1 \subseteq \mathbb{R}^{m_1},\ldots,X_k \subseteq \mathbb{R}^{m_k}$ be full-dimensional rational open polyhedral cones. For each $i = 1,\ldots,k$, let G_i be a subgroup of $GL_{m_i}(\mathbb{Z})$ which fixes the cone X_i setwise, and let X_i/G_i denote the topological quotient thus obtained. The action of G_i on X_i extends naturally to an action of G_i on the Euclidean closure $\overline{X_i}$, and we let $\overline{X_i}/G_i$ denote the quotient.

Suppose that we have a topological space X and, for each $i = 1, \ldots, k$, a continuous map

$$\alpha_i : \frac{\overline{X_i}}{G_i} \to X.$$

Write $C_i = \alpha_i\left(\frac{X_i}{G_i}\right)$ and $\overline{C_i} = \alpha_i\left(\frac{\overline{X_i}}{G_i}\right)$ for each i. Given $Y \subseteq X_i$, we will abuse notation by writing $\alpha_i(Y)$ for α_i applied to the image of Y under the map $\overline{X_i} \twoheadrightarrow \overline{X_i}/G_i$.

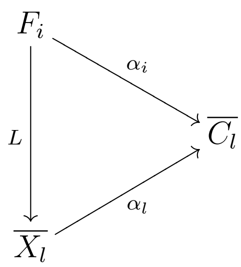
Suppose that the following properties hold for each index i:

(i) The restriction of α_i to $\frac{X_i}{G_i}$ is a homeomorphism onto C_i,

(ii) We have an equality of sets $X = \coprod C_i$,

(iii) For each cone $\overline{X_i}$ and for each face F_i of $\overline{X_i}$, $\alpha_i(F_i) = \overline{C_l}$ for some l. Furthermore, $\dim F_i = \dim \overline{X_l} = m_l$, and there is an $\mathbb{R}$-invertible linear map $L : \operatorname{span}\langle F_i \rangle \cong \mathbb{R}^{m_l} \to \mathbb{R}^{m_l}$ such that

- $L(F_i) = \overline{X_l}$,
- $L(\mathbb{Z}^{m_i} \cap \operatorname{span}(F_i)) = \mathbb{Z}^{m_l}$, and
- the following diagram commutes:



We say that $\overline{C_l}$ is a **stacky face** of $\overline{C_i}$ in this situation.

(iv) For each pair i, j,

$$\overline{C_i} \cap \overline{C_j} = C_{l_1} \cup \cdots \cup C_{l_t},$$

where $C_{l_1}, \ldots, C_{l_t}$ are the common stacky faces of $\overline{C_i}$ and $\overline{C_j}$.

Then we say that X is a **stacky fan**, with cells $\{X_i/G_i\}$.

Remark 2.3.3. Condition (iii) in the definition above essentially says that $\overline{X_i}$ has a face F_i that looks "exactly like" $\overline{X_l}$, even taking into account where the lattice points are. It plays the role of the usual condition on polyhedral fans that the set of cones is closed under taking faces. Condition (iv) replaces the usual condition on polyhedral fans that the intersection of two cones is a face of each. Here, we instead allow unions of common faces.

Theorem 2.3.4. *The moduli space* M_g^{tr} *is a stacky fan with cells*

$$C(G,w) = \frac{\mathbb{R}_{>0}^{E(G)}}{\mathrm{Aut}(G,w)}$$

as (G,w) *ranges over genus* g *combinatorial types. Its points are in bijection with tropical curves of genus* g.

Proof. Recall that

$$M_g^{\mathrm{tr}} = \frac{\coprod \overline{C(G,w)}}{\sim},$$

where $\sim$ is the relation generated by contracting zero-length edges. Thus, each equivalence class has a unique representative (G_0, w, l) corresponding to an honest metric graph: one with all edge lengths positive. This gives the desired bijection.

Now we prove that M_g^{tr} is a stacky fan. For each (G,w), let

$$\alpha_{G,w} : \overline{C(G,w)} \to \frac{\coprod \overline{C(G',w')}}{\sim}$$

be the natural map. Now we check each of the requirements to be a stacky fan, in the order (ii), (iii), (iv), (i).

For (ii), the fact that

$$M_g^{\mathrm{tr}} = \coprod C(G,w)$$

follows immediately from the observation above.

Let us prove (iii). Given a combinatorial type (G,w), the corresponding closed cone is $\mathbb{R}_{\geq 0}^{E(G)}$. A face F of $\mathbb{R}_{\geq 0}^{E(G)}$ corresponds to setting edge lengths of some subset S of the edges to zero. Let (G',w') be the resulting combinatorial type, and let $\pi : E(G) \setminus S \to E(G')$ be the natural bijection (it is well-defined up to (G',w')-automorphisms, but this is enough). Then π induces an invertible linear map

$$L_\pi : \mathbb{R}^{E(G)\setminus S} \longrightarrow \mathbb{R}^{E(G')}$$

with the desired properties. Note also that the stacky faces of $\overline{C(G,w)}$ are thus all possible specializations $\overline{C(G',w')}$.

For (iv), given two combinatorial types (G,w) and (G',w'), then

$$\overline{C(G,w)} \cap \overline{C(G',w')}$$

consists of the union of all cells corresponding to common specializations of (G,w) and (G',w'). As noted above, these are precisely the common stacky faces of $\overline{C(G,w)}$ and $\overline{C(G',w')}$.

For (i), we show that $\alpha_{G,w}$ restricted to $C(G,w) = \mathbb{R}_{>0}^{E(G)} / \mathrm{Aut}(G,w)$ is a homeomorphism onto its image. It is continuous by definition of $\alpha_{G,w}$ and injective by definition of $\sim$. Let V be closed in $C(G,w)$, say $V = W \cap C(G,w)$ where W is closed in $\overline{C(G,w)}$. To show that $\alpha_{G,w}(V)$ is closed in $\alpha_{G,w}(C(G,w))$, it suffices to show that $\alpha_{G,w}(W)$ is closed in M_g^{tr}. Indeed, the fact that the cells $C(G,w)$ are pairwise disjoint in M_g^{tr} implies that

$$\alpha_{G,w}(V) = \alpha_{G,w}(W) \cap \alpha_{G,w}(C(G,w)).$$

Now, note that M_g^{tr} can equivalently be given as the quotient of the space

$$\coprod_{(G,w)} \mathbb{R}_{\geq 0}^{E(G)}$$

by all possible linear maps L_π arising as in the proof of (iii). All of the maps L_π identify faces of cones with other cones. Now let $\tilde{W}$ denote the lift of W to $\mathbb{R}_{\geq 0}^{E(G)}$; then for any other type (G', w'), we see that the set of points in $\mathbb{R}_{\geq 0}^{E(G')}$ that are identified with some point in $\tilde{W}$ is both closed and $\mathrm{Aut}(G', w')$-invariant, and passing to the quotient $\mathbb{R}_{\geq 0}^{E(G')} / \mathrm{Aut}(G', w')$ gives the claim. □

We close this section with the definition of a morphism of stacky fans. The tropical Torelli map, which we will define in Section 6, will be an example.

Definition 2.3.5. [BMV11, Definition 2.1.2] Let

$$X_1 \subseteq \mathbb{R}^{m_1}, \ldots, X_k \subseteq \mathbb{R}^{m_k},\ Y_1 \subseteq \mathbb{R}^{n_1}, \ldots, Y_l \subseteq \mathbb{R}^{n_l}$$

be full-dimensional rational open polyhedral cones. Let $G_1 \subseteq GL_{m_1}(\mathbb{Z}), \ldots, G_k \subseteq GL_{m_k}(\mathbb{Z}), H_1 \subseteq GL_{n_1}(\mathbb{Z}), \ldots, H_l \subseteq GL_{n_l}(\mathbb{Z})$ be groups stabilizing $X_1, \ldots, X_k, Y_1, \ldots, Y_l$ respectively. Let X and Y be stacky fans with cells

$$\left\{\frac{X_i}{G_i}\right\}_{i=1}^{k}, \left\{\frac{Y_j}{H_j}\right\}_{j=1}^{l},$$

Denote by α_i and β_j the maps $\frac{\overline{X_i}}{G_i} \to X$ and $\frac{\overline{Y_j}}{H_j} \to Y$ that are part of the stacky fan data of X and Y.

Then a **morphism of stacky fans** from X to Y is a continuous map $\pi : X \to Y$ such that for each cell X_i/G_i there exists a cell Y_j/H_j such that

(i) $\pi\left(\alpha_i\left(\frac{X_i}{G_i}\right)\right) \subseteq \beta_j\left(\frac{Y_j}{H_j}\right)$, and

(ii) there exists an integral-linear map

$$L : \mathbb{R}^{m_i} \to \mathbb{R}^{n_j},$$

that is, a linear map defined by a matrix with integer entries, restricting to a map

$$L : X_i \to Y_j,$$

such that the diagram below commutes:

$$\begin{array}{ccc} X_i & \longrightarrow & \alpha_i(X_i/G_i) \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y_j & \longrightarrow & \beta_j(Y_j/H_j) \end{array}$$

Remark 2.3.6. Here is why we believe the original definition of a stacky fan, Definition 2.1.1 of [BMV11], is too restrictive. The original definition requires that for every pair of cones $\overline{X_i}$ and $\overline{X_j}$, there exists a linear map $L : \overline{X_i} \to \overline{X_j}$ that induces the inclusion

$$\alpha_i\left(\frac{\overline{X_i}}{G_i}\right) \cap \alpha_j\left(\frac{\overline{X_j}}{G_j}\right) \hookrightarrow \alpha_j\left(\frac{\overline{X_j}}{G_j}\right).$$

We claim that such a map does not always exist in the cases of M_g^{tr} and A_g^{tr}. For example, let $\overline{X_i}$ be the maximal cone of M_2^{tr} drawn on the left in Figure 2.3, and let $\overline{X_j}$ be the maximal cone drawn on the right. There is no map from $\overline{X_i}$ to $\overline{X_j}$ that takes each of the three facets of $\overline{X_i}$ isomorphically to a single facet of $\overline{X_j}$, as would be required. There is a similar problem for A_g^{tr}.

2.4 Principally polarized tropical abelian varieties

The purpose of this section is to construct the moduli space of principally polarized tropical abelian varieties, denoted A_g^{tr}. Our construction is different from the one in [BMV11], though it is still very much inspired by the ideas in that paper. The reason for presenting a new construction here is that a topological subtlety in the construction there prevents their space from being a stacky fan as claimed in [BMV11, Thm. 4.2.4].

We begin in §4.1 by recalling the definition of a principally polarized tropical abelian variety. In §4.2, we review the theory of Delone subdivisions and the main theorem of Voronoi reduction theory. We construct A_g^{tr} in §4.3 and prove that it is a stacky fan and that it is Hausdorff. We remark on the difference between our construction and the one in [BMV11] in §4.4.

§2.4.1 Definition of principally polarized tropical abelian variety

Fix $g \geq 1$. Following [BMV11] and [MZ07], we define a **principally polarized tropical abelian variety**, or **pptav** for short, to be a pair

$$(\mathbb{R}^g/\Lambda, Q),$$

where Λ is a lattice of rank g in $\mathbb{R}^g$ (i.e. a discrete subgroup of $\mathbb{R}^g$ that is isomorphic to $\mathbb{Z}^g$), and Q is a positive semidefinite quadratic form on $\mathbb{R}^g$ whose nullspace is rational with respect to Λ. We say that the nullspace of Q is **rational** with respect to Λ if the subspace $\ker(Q) \subseteq \mathbb{R}^g$ has a vector space basis whose elements are each of the form

$$a_1\lambda_1 + \cdots + a_k\lambda_k, \quad a_i \in \mathbb{Q}, \lambda_i \in \Lambda.$$

We say that Q has **rational nullspace** if its nullspace is rational with respect to $\mathbb{Z}^g$.

We say that two pptavs $(\mathbb{R}^g/\Lambda, Q)$ and $(\mathbb{R}^g/\Lambda', Q')$ are isomorphic if there exists a matrix $X \in GL_g(\mathbb{R})$ such that

- left multiplication by X^{-1} sends Λ isomorphically to Λ', that is, the map X^{-1} : $\mathbb{R}^g \to \mathbb{R}^g$ sending a column vector v to $X^{-1}v$ restricts to an isomorphism of lattices Λ and Λ'; and
- $Q' = X^TQX$.

Note that any pptav $(\mathbb{R}^g/\Lambda, Q)$ is isomorphic to one of the form $(\mathbb{R}^g/\mathbb{Z}^g, Q')$, namely by taking X to be any matrix sending $\mathbb{Z}^g$ to Λ and setting $Q' = X^TQX$. Furthermore, $(\mathbb{R}^g/\mathbb{Z}^g, Q)$ and $(\mathbb{R}^g/\mathbb{Z}^g, Q')$ are isomorphic if and only if there exists $X \in GL_g(\mathbb{Z})$ with $X^TQX = Q'$.

Remark 2.4.1. Since we are interested in pptavs only up to isomorphism, we might be tempted to define the moduli space of pptavs to be the quotient of the topological space $\widetilde{S}^g_{\geq 0}$, the space of positive semidefinite matrices with rational nullspace, by the action of $GL_g(\mathbb{Z})$. That is what is done in [BMV11]. That quotient space is the correct moduli space of pptavs set-theoretically. But it has an undesirable topology: as we will see in Section 4.4, it is not even Hausdorff!

We will fix this problem by putting a different topology on the set of pptavs. We will first group matrices together into cells according to their Delone subdivisions, and then glue the cells together to obtain the full moduli space. We review the theory of Delone subdivisions next.

§2.4.2 Voronoi reduction theory

Recall that a matrix has rational nullspace if its kernel has a basis consisting of vectors with entries in $\mathbb{Q}$.

Definition 2.4.2. Let $\widetilde{S}^g_{\geq 0}$ denote the set of $g \times g$ positive semidefinite matrices with rational nullspace. By regarding a $g \times g$ symmetric real matrix as a vector in $\mathbb{R}^{\binom{g+1}{2}}$, with one coordinate for each diagonal and above-diagonal entry of the matrix, we view $\widetilde{S}^g_{\geq 0}$ as a subset of $\mathbb{R}^{\binom{g+1}{2}}$.

The group $GL_g(\mathbb{Z})$ acts on $\widetilde{S}^g_{\geq 0}$ on the right by changing basis:

$$Q \cdot X = X^TQX, \text{ for all } X \in GL_g(\mathbb{Z}), Q \in \widetilde{S}^g_{\geq 0}.$$

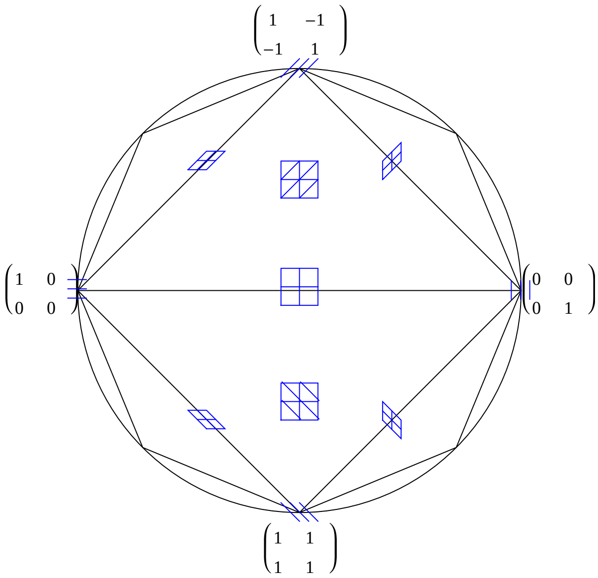


Figure 2.5: Infinite decomposition of $\widetilde{S}^2_{\geq 0}$ into secondary cones.

Definition 2.4.3. Given $Q \in \widetilde{S}^g_{\geq 0}$, define $\mathrm{Del}(Q)$ as follows. Consider the map $l : \mathbb{Z}^g \to \mathbb{Z}^g \times \mathbb{R}$ sending $x \in \mathbb{Z}^g$ to $(x, x^T Q x)$. View the image of l as an infinite set of points in $\mathbb{R}^{g+1}$, one above each point in $\mathbb{Z}^g$, and consider the convex hull of these points. The lower faces of the convex hull (the faces that are visible from $(0, -\infty)$) can now be projected to $\mathbb{R}^g$ by the map $\pi : \mathbb{R}^{g+1} \to \mathbb{R}^g$ that forgets the last coordinate. This produces an infinite periodic polyhedral subdivision of $\mathbb{R}^g$, called the **Delone subdivision** of Q and denoted $\mathrm{Del}(Q)$.

Now, we group together matrices in $\widetilde{S}^g_{\geq 0}$ according to the Delone subdivisions to which they correspond.

Definition 2.4.4. Given a Delone subdivision D, let

$$\sigma_D = \{Q \in \widetilde{S}^g_{\geq 0} : \mathrm{Del}(Q) = D\}.$$

Proposition 2.4.5. *[Vor09]The set σ_D is an open rational polyhedral cone in $\widetilde{S}^g_{\geq 0}$.*

Let $\overline{\sigma_D}$ denote the Euclidean closure of σ_D in $\mathbb{R}^{\binom{g+1}{2}}$, so $\overline{\sigma_D}$ is a closed rational polyhedral cone. We call it the **secondary cone** of D.

Example 2.4.6. Figure 2.5 shows the decomposition of $\widetilde{S}^2_{\geq 0}$ into secondary cones. Here is how to interpret the picture. First, points in $\widetilde{S}^2_{\geq 0}$ are 2×2 real symmetric matrices, so let us regard them as points in $\mathbb{R}^3$. Then $\widetilde{S}^2_{\geq 0}$ is a cone in $\mathbb{R}^3$. Instead of drawing the cone in $\mathbb{R}^3$, however, we only draw a hyperplane slice of it. Since it was a cone, our drawing does not lose information. For example, what looks like a point in the picture, labeled by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, really is the ray in $\mathbb{R}^3$ passing through the point $(1, 0, 0)$. $\diamond$

Now, the action of the group $GL_g(\mathbb{Z})$ on $\widetilde{S}^g_{\geq 0}$ extends naturally to an action (say, on the right) on subsets of $\widetilde{S}^g_{\geq 0}$. In fact, given $X \in GL_g(\mathbb{Z})$ and D a Delone subdivision,

$$\sigma_D \cdot X = \sigma_{X^{-1}D} \quad \text{and} \quad \overline{\sigma_D} \cdot X = \overline{\sigma_{X^{-1}D}}.$$

So $GL_g(\mathbb{Z})$ acts on the set

$$\{\overline{\sigma_D} : D \text{ is a Delone subdivision of } \mathbb{R}^g\}.$$

Furthermore, $GL_g(\mathbb{Z})$ acts on the set of Delone subdivisions, with action induced by the action of $GL_g(\mathbb{Z})$ on $\mathbb{R}^g$. Two cones σ_D and $\sigma_{D'}$ are $GL_g(\mathbb{Z})$-equivalent iff D and D' are.

Theorem 2.4.7 (Main theorem of Voronoi reduction theory [Vor09])**.** *The set of secondary cones*

$$\{\overline{\sigma_D} : D \textit{ is a Delone subdivision of } \mathbb{R}^g\}$$

yields an infinite polyhedral fan whose support is $\widetilde{S}^g_{\geq 0}$, known as the **second Voronoi decomposition**. *There are only finitely many $GL_g(\mathbb{Z})$-orbits of this set.*

§2.4.3 Construction of A_g^{tr}

Equipped with Theorem 2.4.7, we will now construct our tropical moduli space A_g^{tr}. We will show that its points are in bijection with the points of $\widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$, and that it is a stacky fan whose cells correspond to $GL_g(\mathbb{Z})$-equivalence classes of Delone subdivisions of $\mathbb{R}^g$.

Definition 2.4.8. Given a Delone subdivision D of $\mathbb{R}^g$, let

$$\mathrm{Stab}(\sigma_D) = \{X \in GL_g(\mathbb{Z}) : \sigma_D \cdot X = \sigma_D\}$$

be the setwise stabilizer of σ_D.

Now, the subgroup $\mathrm{Stab}(\sigma_D) \subseteq GL_g(\mathbb{Z})$ acts on the open cone σ_D, and we may extend this action to an action on its closure $\overline{\sigma_D}$.

Definition 2.4.9. Given a Delone subdivision D of $\mathbb{R}^g$, let

$$C(D) \;=\; \overline{\sigma_D}/\,\mathrm{Stab}(\sigma_D).$$

Thus, $C(D)$ is the topological space obtained as a quotient of the rational polyhedral cone $\overline{\sigma_D}$ by a group action.

Now, by Theorem 2.4.7, there are only finitely many $GL_g(\mathbb{Z})$-orbits of secondary cones $\overline{\sigma_D}$. Thus, we may choose $D_1, \ldots, D_k$ Delone subdivisions of $\mathbb{R}^g$ such that $\overline{\sigma_{D_1}}, \ldots, \overline{\sigma_{D_k}}$ are representatives for $GL_g(\mathbb{Z})$-equivalence classes of secondary cones. (Note that we do not need anything like the Axiom of Choice to select these representatives. Rather, we can use Algorithm 1 in [Val03]. We start with a particular Delone triangulation and then walk across codimension 1 faces to all of the other ones; then we compute the faces of these maximal cones to obtain the nonmaximal ones. The key idea that allows the algorithm to terminate is that all maximal cones are related to each other by finite sequences of "bistellar flips" as described in Section 2.4 of [Val03]).

Definition 2.4.10. Let $D_1, \ldots, D_k$ be Delone subdivisions such that $\overline{\sigma_{D_1}}, \ldots, \overline{\sigma_{D_k}}$ are representatives for $GL_g(\mathbb{Z})$-equivalence classes of secondary cones in $\mathbb{R}^g$. Consider the disjoint union

$$C(D_1) \coprod \cdots \coprod C(D_k),$$

and define an equivalence relation $\sim$ on it as follows. Given $Q_i \in \overline{\sigma(D_i)}$ and $Q_j \in \overline{\sigma(D_j)}$, let $[Q_i]$ and $[Q_j]$ be the corresponding elements in $C(D_i)$ and $C(D_j)$, respectively. Now let

$$[Q_i] \sim [Q_j]$$

if and only if Q_i and Q_j are $GL_g(\mathbb{Z})$-equivalent matrices in $\widetilde{S}^g_{\geq 0}$. Since $\mathrm{Stab}(\sigma_{D_i})$, $\mathrm{Stab}(\sigma_{D_j})$ are subgroups of $GL_g(\mathbb{Z})$, the relation $\sim$ is defined independently of the choice of representatives Q_i and Q_j, and is clearly an equivalence relation.

We now define the **moduli space of principally polarized tropical abelian varieties**, denoted A_g^{tr}, to be the topological space

$$A_g^{\mathrm{tr}} \quad = \quad \coprod_{i=1}^{k} C(D_k)/\sim .$$

Example 2.4.11. Let us compute A_2^{tr}. Combining the taxonomies in Sections 4.1 and 4.2 of [Val03], we may choose four representatives D_1, D_2, D_3, D_4 for orbits of secondary cones as in Figure 2.6.

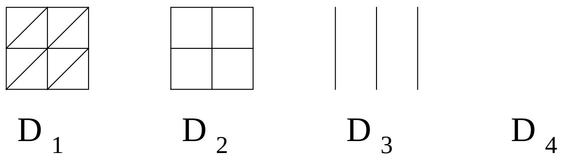


Figure 2.6: Cells of A_2^{tr}. Note that D_4 is the trivial subdivision of $\mathbb{R}^2$, consisting of $\mathbb{R}^2$ itself.

We can describe the corresponding secondary cones as follows: let $R_{12} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $R_{13} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $R_{23} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned}
\overline{\sigma_{D_1}} &= \mathbb{R}_{\geq 0}\langle R_{12}, R_{13}, R_{23}\rangle, \\
\overline{\sigma_{D_2}} &= \mathbb{R}_{\geq 0}\langle R_{13}, R_{23}\rangle, \\
\overline{\sigma_{D_3}} &= \mathbb{R}_{\geq 0}\langle R_{13}\rangle, \text{ and} \\
\overline{\sigma_{D_4}} &= \{0\}.
\end{aligned}$$

Note that each closed cone $\overline{\sigma_{D_2}}, \overline{\sigma_{D_3}}, \overline{\sigma_{D_4}}$ is just a face of $\overline{\sigma_{D_1}}$. One may check – and we will, in Section 5 – that for each $j = 2, 3, 4$, two matrices Q, Q' in $\overline{\sigma_{D_j}}$ are $\mathrm{Stab}(\sigma_{D_j})$-equivalent if and only if they are $\mathrm{Stab}(\sigma_{D_1})$-equivalent. Thus, gluing the cones $C(D_2)$,

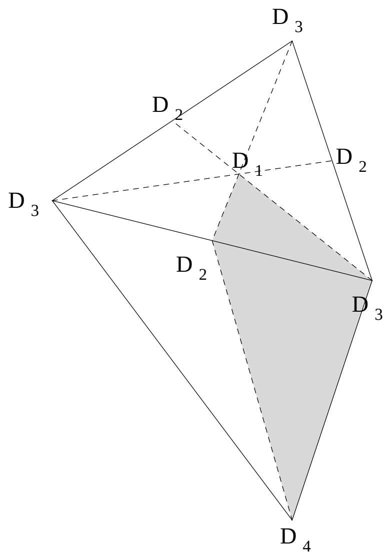


Figure 2.7: The stacky fan A_2^{tr}. The shaded area represents a choice of fundamental domain.

$C(D_3)$, and $C(D_4)$ to $C(D_1)$ does not change $C(D_1)$. We will see in Theorem 2.5.10 that the action of $\mathrm{Stab}(\sigma_{D_1})$ on $\overline{\sigma_{D_1}}$ is an S_3-action that permutes the three rays of $\overline{\sigma_{D_1}}$. So we may pick a fundamental domain, say the closed cone

$$C = \mathbb{R}_{\geq 0} \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \right\rangle,$$

and conclude that $C(D_1)$, and hence A_2^{tr}, is homeomorphic to C. See Figure 2.7 for a picture of A_2^{tr}. Of course, A_2^{tr} has further structure, as the next theorem shows. $\diamond$

Theorem 2.4.12. *The space A_g^{tr} constructed in Definition 2.4.10 is a stacky fan with cells $\sigma_{D_i}/\operatorname{Stab}(\sigma_{D_i})$ for $i = 1, \ldots, k$.*

Proof. For each $i = 1, \ldots, k$, let α_i be the composition

$$\frac{\overline{\sigma_{D_i}}}{\mathrm{Stab}(\sigma_{D_i})} \overset{\gamma_i}{\hookrightarrow} \coprod_{j=1}^{k} C(D_j) \xrightarrow{q} \left(\coprod_{j=1}^{k} C(D_j) \right) / \sim,$$

where γ_i is the inclusion of $C(D_i) = \frac{\overline{\sigma_{D_i}}}{\mathrm{Stab}(\sigma_{D_i})}$ into $\coprod_{j=1}^{k} C(D_j)$ and q is the quotient map. Now we check the four conditions listed in Definition 2.3.2 for A_g^{tr} to be a stacky fan.

First, we prove that the restriction of α_i to $\frac{\sigma_{D_i}}{\mathrm{Stab}(\sigma_{D_i})}$ is a homeomorphism onto its image. Now, α_i is continuous since both γ_i and q are. To show that $\alpha_i|_{\frac{\sigma_{D_i}}{\mathrm{Stab}(\sigma_{D_i})}}$ is one-to-one onto its image, let $Q, Q' \in \sigma_{D_i}$ such that $\alpha_i([Q]) = \alpha_i([Q'])$. Then $[Q] \sim [Q']$, so there exists $A \in GL_g(\mathbb{Z})$ such that $Q' = A^T Q A$. Hence $Q' \in A^T \sigma_{D_i} A = \sigma_{A^{-1}D_i}$. Thus $\sigma_{A^{-1}D_i}$ and σ_{D_i} intersect, hence $\sigma_{A^{-1}D_i} = \sigma_{D_i}$ and $A \in \mathrm{Stab}(\sigma_{D_i})$. So $[Q] = [Q']$.

Thus, $\alpha_i|_{\frac{\sigma_{D_i}}{\mathrm{Stab}(\sigma_{D_i})}}$ has a well-defined inverse map, and we wish to show that this inverse map is continuous. Let $X \subseteq \frac{\sigma_{D_i}}{\mathrm{Stab}\,\sigma_{D_i}}$ be closed; we wish to show that $\alpha_i(X)$ is closed in $\alpha_i\left(\frac{\sigma_{D_i}}{\mathrm{Stab}\,\sigma_{D_i}}\right)$. Write $X = Y \cap \frac{\sigma_{D_i}}{\mathrm{Stab}\,\sigma_{D_i}}$ where $Y \subseteq \frac{\overline{\sigma_{D_i}}}{\mathrm{Stab}\,\sigma_{D_i}}$ is closed. Then

$$\alpha_i(X) = \alpha_i(Y) \cap \alpha_i\left(\frac{\sigma_{D_i}}{\mathrm{Stab}\,\sigma_{D_i}}\right);$$

this follows from the fact that $GL_g(\mathbb{Z})$-equivalence never identifies a point on the boundary of a closed cone with a point in the relative interior. So we need only show that $\alpha_i(Y)$ is closed in A_g^{tr}. To be clear: we want to show that given any closed $Y \subseteq \frac{\overline{\sigma_{D_i}}}{\mathrm{Stab}\,\sigma_{D_i}}$, the image $\alpha_i(Y) \subseteq A_g^{\mathrm{tr}}$ is closed.

Let $\tilde{Y} \subseteq \overline{\sigma_{D_i}}$ be the preimage of Y under the quotient map

$$\overline{\sigma_{D_i}} \twoheadrightarrow \frac{\overline{\sigma_{D_i}}}{\mathrm{Stab}\,\sigma_{D_i}}.$$

Then, for each $j = 1, \ldots, k$, let

$$\tilde{Y}_j = \{Q \in \overline{\sigma_{D_j}} : Q \equiv_{GL_g(\mathbb{Z})} Q' \text{ for some } Q' \in \tilde{Y}\} \subseteq \overline{\sigma_{D_j}}.$$

We claim each $\tilde{Y}_j$ is closed in $\overline{\sigma_{D_j}}$. First, notice that for any $A \in GL_g(\mathbb{Z})$, the cone $A^T \overline{\sigma_{D_i}} A$ intersects $\overline{\sigma_{D_j}}$ in a (closed) face of $\overline{\sigma_{D_j}}$ (after all, the cones form a polyhedral subdivision). In other words, A defines an integral-linear isomorphism $L_A : F_{A,i} \to F_{A,j}$ sending $X \mapsto A^T X A$, where $F_{A,i}$ is a face of $\overline{\sigma_{D_i}}$ and $F_{A,j}$ is a face of $\overline{\sigma_{D_j}}$. Moreover, the map L_A is entirely determined by three choices: the choice of $F_{A,i}$, the choice of $F_{A,j}$, and the choice of a bijection between the rays of $F_{A,i}$ and $F_{A,j}$. Thus there exist only finitely many distinct such maps. Therefore

$$\tilde{Y}_j = \bigcup_{A \in GL_g(\mathbb{Z})} L_A(\tilde{Y} \cap F_{A,i}) = \bigcup_{k=1}^{s} L_{A_k}(\tilde{Y} \cap F_{A_k,i})$$

for some choice of finitely many matrices $A_1, \ldots, A_s \in GL_g(\mathbb{Z})$. Now, each L_A is a homeomorphism, so each $L_A(\tilde{Y} \cap F_{A,i})$ is closed in $F_{A,j}$ and hence in $\overline{\sigma_{D_j}}$. So $\tilde{Y}_j$ is closed.

Finally, let Y_j be the image of $\tilde{Y}_j \subseteq \overline{\sigma_{D_j}}$ under the quotient map

$$\overline{\sigma_{D_j}} \xrightarrow{\pi_i} \frac{\overline{\sigma_{D_j}}}{\mathrm{Stab}\,\sigma_{D_j}}.$$

Since $\pi_j^{-1}(Y_j) = \tilde{Y}_j$, we have that Y_j is closed. Then the inverse image of $\alpha_i(Y)$ under the quotient map

$$\coprod_{j=1}^{k} C(D_j) \longrightarrow \left(\coprod_{j=1}^{k} C(D_j)\right)/\!\sim$$

is precisely $Y_1 \coprod \cdots \coprod Y_k$, which is closed. Hence $\alpha_i(Y)$ is closed. This finishes the proof that $\alpha_i|_{\frac{\sigma_{D_i}}{\mathrm{Stab}(\sigma_{D_i})}}$ is a homeomorphism onto its image.

Property (ii) of being a stacky fan follows from the fact that any matrix $Q \in \widetilde{S}^g_{\geq 0}$ is $GL_g(\mathbb{Z})$-equivalent only to some matrices in a single chosen cone, say σ_{D_i}, and no others. Here, $\mathrm{Del}(Q)$ and D_i are $GL_g(\mathbb{Z})$-equivalent. Thus, given a point in A_g^{tr} represented by $Q \in \widetilde{S}^g_{\geq 0}$, Q lies in $\alpha_i\left(\frac{\sigma_{D_i}}{\mathrm{Stab}\,\sigma_{D_i}}\right)$ and no other $\alpha_j\left(\frac{\sigma_{D_j}}{\mathrm{Stab}\,\sigma_{D_j}}\right)$, and is the image of a single point in $\frac{\sigma_{D_i}}{\mathrm{Stab}\,\sigma_{D_i}}$ since α_i was shown to be bijective on $\frac{\sigma_{D_i}}{\mathrm{Stab}\,\sigma_{D_i}}$. This shows that $A_g^{\mathrm{tr}} = \coprod_{i=1}^k \alpha_i\left(\frac{\sigma_{D_i}}{\mathrm{Stab}\,\sigma_{D_i}}\right)$ as a set.

Third, a face F of some cone $\overline{\sigma_{D_i}}$ is $\overline{\sigma_{D(F)}}$, where $D(F)$ is a Delone subdivision that is a coarsening of D_i [Val03, Proposition 2.6.1]. Then there exists D_j and $A \in GL_g(\mathbb{Z})$ with $\overline{\sigma_{D(F)}} \cdot A = \overline{\sigma_{D_j}}$ (recall that A acts on a point $p \in \widetilde{S}^g_{\geq 0}$ by $p \mapsto A^T p A$). Restricting A to the linear span of $\overline{\sigma_{D(F)}}$ gives a linear map

$$L_A : \mathrm{span}(\overline{\sigma_{D(F)}}) \longrightarrow \mathrm{span}(\overline{\sigma_{D_j}})$$

with the desired properties. Note, therefore, that $\overline{\sigma_{D_k}}$ is a stacky face of $\overline{\sigma_{D_i}}$ precisely if D_k is $GL_g(\mathbb{Z})$-equivalent to a coarsening of D_i.

The fourth property then follows: the intersection

$$\alpha_i(\overline{\sigma_{D_i}}) \cap \alpha_j(\overline{\sigma_{D_j}}) = \bigcup \alpha_k(\sigma_{D_k})$$

where σ_{D_k} ranges over all common stacky faces. □

Proposition 2.4.13. *The construction of A_g^{tr} in Definition 2.4.10 does not depend on our choice of $D_1, \ldots, D_k$. More precisely, suppose $D'_1, \ldots, D'_k$ are another choice of representatives such that D'_i and D_i are $GL_g(\mathbb{Z})$-equivalent for each i. Let $A_g^{tr'}$ be the corresponding stacky fan. Then there is an isomorphism of stacky fans between A_g^{tr} and $A_g^{tr'}$.*

Proof. For each i, choose $A_i \in GL_g(\mathbb{Z})$ with

$$\sigma_{D_i} \cdot A_i = \sigma_{D'_i}.$$

Then we obtain a map

$$C(D_1) \coprod \cdots \coprod C(D_k) \xrightarrow{(A_1, \ldots, A_k)} C(D'_1) \coprod \cdots \coprod C(D'_k)$$

descending to a map

$$A_g^{\mathrm{tr}} \longrightarrow A_g^{\mathrm{tr}'},$$

and this map is an isomorphism of stacky fans, as evidenced by the inverse map $A_g^{\mathrm{tr}'} \to A_g^{\mathrm{tr}}$ constructed from the matrices $A_1^{-1}, \ldots, A_k^{-1}$. □

Theorem 2.4.14. *The moduli space A_g^{tr} is Hausdorff.*

Remark 2.4.15. Theorem 2.4.14 complements the theorem of Caporaso that M_g^{tr} is Hausdorff [Cap10, Theorem 5.2].

Proof. Let $\overline{\sigma_{D_1}}, \ldots, \overline{\sigma_{D_k}}$ be representatives for $GL_g(\mathbb{Z})$-classes of secondary cones. Let us regard A_g^{tr} as a quotient of the cones themselves, rather than the cones modulo their stabilizers, thus

$$A_g^{\mathrm{tr}} = \left(\coprod_{i=1}^{k} \overline{\sigma_{D_k}} \right) / \sim$$

where $\sim$ denotes $GL_g(\mathbb{Z})$-equivalence as usual. Denote by β_i the natural maps

$$\beta_i : \overline{\sigma_{D_i}} \longrightarrow A_g^{\mathrm{tr}}.$$

Now suppose $p \neq q \in A_g^{\mathrm{tr}}$. For each $i = 1, \ldots, k$, pick disjoint open sets U_i and V_i in $\overline{\sigma_{D_i}}$ such that $\beta_i^{-1}(p) \subseteq U_i$ and $\beta_i^{-1}(q) \subseteq V_i$. Let

$$\begin{aligned} U &:= \{x \in A_g^{\mathrm{tr}} \ : \ \beta_i^{-1}(x) \subseteq U_i \text{ for all } i\}, \\ V &:= \{x \in A_g^{\mathrm{tr}} \ : \ \beta_i^{-1}(x) \subseteq V_i \text{ for all } i\}. \end{aligned}$$

By construction, we have $p \in U$ and $q \in V$. We claim that U and V are disjoint open sets in A_g^{tr}.

Suppose $x \in U \cap V$. Now $\beta_i^{-1}(x)$ is nonempty for some i, hence $U_i \cap V_i$ is nonempty, contradiction. Hence U and V are disjoint. So we just need to prove that U is open (similarly, V is open). It suffices to show that for each $j = 1, \ldots, k$, the set $\beta_j^{-1}(U)$ is open. Now,

$$\begin{aligned} \beta_j^{-1}(U) &= \{y \in \overline{\sigma_{D_j}} \ : \ \beta_i^{-1}\beta_j(y) \subseteq U_i \text{ for all } i\}, \\ &= \bigcap_i \{y \in \overline{\sigma_{D_j}} \ : \ \beta_i^{-1}\beta_j(y) \subseteq U_i\}. \end{aligned}$$

Write U_{ij} for the sets in the intersection above, so that $\beta_j^{-1}(U) = \bigcap_i U_{ij}$, and let $Z_i = \overline{\sigma_{D_i}} \backslash U_i$. Note that U_{ij} consists of those points in $\overline{\sigma_{D_j}}$ that are not $GL_g(\mathbb{Z})$-equivalent to any point in Z_i. Then, just as in the proof of Theorem 2.4.12, there exist finitely many matrices $A_1, \ldots, A_s \in GL_g(\mathbb{Z})$ such that

$$\begin{aligned} \overline{\sigma_{D_j}} \setminus U_{ij} &= \{y \in \overline{\sigma_{D_j}} \ : \ y \sim z \text{ for some } z \in Z_i\} \\ &= \bigcup_{l=1}^{s} \left(A_l^T Z_i A_l \cap \overline{\sigma_{D_j}} \right), \end{aligned}$$

which shows that $\overline{\sigma_{D_j}} \setminus U_{ij}$ is closed. Thus the U_{ij}'s are open and so $\beta_j^{-1}(U)$ is open for each j. Hence U is open, and similarly, V is open. □

Remark 2.4.16. Actually, we could have done a much more general construction of A_g^{tr}. We made a choice of decomposition of $\widetilde{S}^g_{\geq 0}$: we chose the second Voronoi decomposition, whose cones are secondary cones of Delone subdivisions. This decomposition has the advantage that it interacts nicely with the Torelli map, as we will see. But, as rightly pointed out in [BMV11], we could use any decomposition of $\widetilde{S}^g_{\geq 0}$ that is "$GL_g(\mathbb{Z})$-admissible." This means that it is an infinite polyhedral subdivision of $\widetilde{S}^g_{\geq 0}$ such that $GL_g(\mathbb{Z})$ permutes its open cones in a finite number of orbits. See [AMRT75, Section II] for the formal definition. Every result in this section can be restated for a general $GL_g(\mathbb{Z})$-admissible decomposition: each such decomposition produces a moduli space which is a stacky fan, which is independent of any choice of representatives, and which is Hausdorff. The proofs are all the same. Here, though, we chose to fix a specific decomposition purely for the sake of concreteness and readability, invoking only what we needed to build up to the definition of the Torelli map.

§2.4.4 The quotient space $\widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$

We briefly remark on the construction of A_g^{tr} originally proposed in [BMV11]. There, the strategy is to try to equip the quotient space $\widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$ directly with a stacky fan structure. To do this, one maps a set of representative cones σ_D, modulo their stabilizers $\operatorname{Stab}(\sigma_D)$, into the space $\widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$, via the map

$$i_D : \sigma_D/\operatorname{Stab}(\sigma_D) \to \widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$$

induced by the inclusion $\sigma_D \hookrightarrow \widetilde{S}^g_{\geq 0}$.

The problem is that the map i_D above may not be a homeomorphism onto its image. In fact, the image of $\sigma_D/\operatorname{Stab}(\sigma_D)$ in $\widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$ may not even be Hausdorff, even though $\sigma_D/\operatorname{Stab}(\sigma_D)$ certainly is. The following example shows that the cone σ_{D_3}, using the notation of Example 2.4.11, exhibits such behavior. Note that $\operatorname{Stab}(\sigma_{D_3})$ happens to be trivial in this case.

Example 2.4.17. Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be the sequences of matrices

$$X_n = \begin{pmatrix} 1 & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n^2} \end{pmatrix}, \ Y_n = \begin{pmatrix} \frac{1}{n^2} & 0 \\ 0 & 0 \end{pmatrix}$$

in $\widetilde{S}^2_{\geq 0}$. Then we have

$$\{X_n\} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \{Y_n\} \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

On the other hand, for each n, $X_n \equiv_{GL_2(\mathbb{Z})} Y_n$ even while $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \not\equiv_{GL_2(\mathbb{Z})} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. This example then descends to non-Hausdorffness in the topological quotient. It can easily be generalized to $g > 2$. $\diamond$

Thus, we disagree with the claim in the proof of Theorem 4.2.4 of [BMV11] that the open cones σ_D, modulo their stabilizers, map homeomorphically onto their image in $\widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$. However, we emphasize that our construction in Section 4.3 is just a minor modification of the ideas already present in [BMV11].

2.5 Regular matroids and the zonotopal subfan

In the previous section, we defined the moduli space A_g^{tr} of principally polarized tropical abelian varieties. In this section, we describe a particular stacky subfan of A_g^{tr} whose cells are in correspondence with simple regular matroids of rank at most g. This subfan is called the zonotopal subfan and denoted A_g^{zon} because its cells correspond to those classes of Delone triangulations which are dual to zonotopes; see [BMV11, Section 4.4]. The zonotopal subfan A_g^{zon} is important because, as we shall see in Section 6, it contains the image of the Torelli map. For $g \geq 4$, this containment is proper. Our main contribution in this section is to characterize the stabilizing subgroups of all zonotopal cells.

We begin by recalling some basic facts about matroids. A good reference is [Oxl92]. The connection between matroids and the Torelli map seems to have been first observed by Gerritzen [Ger82], and our approach here can be seen as an continuation of his work in the late 1970s.

Definition 2.5.1. A matroid is said to be **simple** if it has no loops and no parallel elements.

Definition 2.5.2. A matroid M is **regular** if it is representable over every field; equivalently, M is regular if it is representable over $\mathbb{R}$ by a totally unimodular matrix. (A totally unimodular matrix is a matrix such that every square submatrix has determinant in $\{0, 1, -1\}$.)

Next, we review the correspondence between simple regular matroids and zonotopal cells.

Construction 2.5.3. *Let M be a simple regular matroid of rank at most g, and let A be a $g \times n$ totally unimodular matrix that represents M. Let $v_1, \ldots, v_n$ be the columns of A. Then let $\sigma_A \subseteq \mathbb{R}^{\binom{g+1}{2}}$ be the rational open polyhedral cone*

$$\mathbb{R}_{>0}\langle v_1 v_1^T, \ldots, v_n v_n^T \rangle.$$

Example 2.5.4. Here is an example of Construction 2.5.3 at work. Let M be the uniform matroid $U_{2,3}$; equivalently M is the graphic matroid $M(K_3)$. Then the 2×3 totally unimodular matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

represents M, and σ_A is the open cone generated by matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It is the cone σ_{D_1} in Example 2.4.11 and is shown in Figure 2.7. ◇

Proposition 2.5.5. *[BMV11, Lemma 4.4.3, Theorem 4.4.4] Let M be a simple regular matroid of rank at most g, and let A be a $g \times n$ totally unimodular matrix that represents M. Then the cone σ_A, defined in Construction 2.5.3, is a secondary cone in $\widetilde{S}^g_{\geq 0}$. Choosing a different totally unimodular matrix A' to represent M produces a cone $\sigma_{A'}$ that is $GL_g(\mathbb{Z})$-equivalent to σ_A. Thus, we may associate to M a unique cell of A_g^{tr}, denoted $C(M)$.*

Definition 2.5.6. The **zonotopal subfan** A_g^{zon} is the union of cells in A_g^{tr}

$$A_g^{\text{zon}} = \bigcup_{\substack{M \text{ a simple regular} \\ \text{matroid of rank } \leq g}} C(M).$$

We briefly recall the definition of the Voronoi polytope of a quadratic form in $\widetilde{S}^g_{\geq 0}$, just in order to explain the relationship with zonotopes.

Definition 2.5.7. Let $Q \in \widetilde{S}^g_{\geq 0}$, and let $H = (\ker Q)^\perp \subseteq \mathbb{R}^g$. Then

$$\text{Vor}(Q) = \{x \in H : x^T Q x \leq (x - \lambda)^T Q (x - \lambda)\ \forall \lambda \in \mathbb{Z}^g\}$$

is a polytope in $H \subseteq \mathbb{R}^g$, called the **Voronoi polytope** of Q.

Theorem 2.5.8. *[BMV11, Theorem 4.4.4, Definition 4.4.5] The zonotopal subfan A_g^{zon} is a stacky subfan of A_g^{tr}. It consists of those points of the tropical moduli space A_g^{tr} whose Voronoi polytope is a zonotope.*

Remark 2.5.9. Suppose σ is an open rational polyhedral cone in $\mathbb{R}^n$. Then any $A \in GL_n(\mathbb{Z})$ such that $A\sigma = \sigma$ must permute the rays of $\overline{\sigma}$, since the action of A on $\overline{\sigma}$ is linear. Furthermore, it sends a first lattice point on a ray to another first lattice point; that is, it preserves lattice lengths. Thus, the subgroup $\text{Stab}(\sigma) \subseteq GL_n(\mathbb{Z})$ realizes some subgroup of the permutation group on the rays of $\overline{\sigma}$ (although if σ is not full-dimensional then the action of $\text{Stab}(\sigma)$ on its rays may not be faithful).

Now, given a simple regular matroid M of rank $\leq g$, we have almost computed the cell of A_g^{tr} to which it corresponds. Specifically, we have computed the cone $\overline{\sigma_A}$ for A a matrix representing M, in Construction 2.5.3. The remaining task is to compute the action of the stabilizer $\text{Stab}(\sigma_A)$.

Note that $\overline{\sigma_A}$ has rays corresponding to the columns of A: a column vector v_i corresponds to the ray generated by the symmetric rank 1 matrix $v_i v_i^T$. In light of Remark 2.5.9, we might conjecture that the permutations of rays of $\overline{\sigma_A}$ coming from the stabilizer are the ones that respect the matroid M, i.e. come from matroid automorphisms. That is precisely the case and provides valuable local information about A_g^{tr}.

Theorem 2.5.10. *Let A be a $g \times n$ totally unimodular matrix representing the simple regular matroid M. Let H denote the group of permutations of the rays of σ_A which are realized by the action of $Stab(\sigma_A)$. Then*

$$H \cong \text{Aut}(M).$$

Remark 2.5.11. This statement seems to have been known to Gerritzen in [Ger82], but we present a new proof here, one which might be easier to read. Our main tool is the combinatorics of unimodular matrices.

Here is a nice fact about totally unimodular matrices: they are essentially determined by the placement of their zeroes.

Lemma 2.5.12. *[Tru92, Lemma 9.2.6] Suppose A and B are $g \times n$ totally unimodular matrices with the same support, i.e. $a_{ij} \neq 0$ if and only if $b_{ij} \neq 0$ for all i, j. Then A can be transformed into B by negating rows and negating columns.*

Lemma 2.5.13. *Let A and B be $g \times n$ totally unimodular matrices, with column vectors $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ respectively. Suppose that the map $v_i \mapsto w_i$ induces an isomorphism of matroids $M[A] \xrightarrow{\cong} M[B]$, i.e. takes independent sets to independent sets and dependent sets to dependent sets. Then there exists $X \in GL_g(\mathbb{Z})$ such that*

$$Xv_i = \pm w_i, \text{ for each } i = 1, \ldots, n.$$

Proof. First, let $r = \operatorname{rank}(A) = \operatorname{rank}(B)$, noting that the ranks are equal since the matroids are isomorphic. Since the statement of Lemma 2.5.13 does not depend on the ordering of the columns, we may simultaneously reorder the columns of A and the columns of B and so assume that the first r rows of A (respectively B) form a basis of $M[A]$ (respectively $M[B]$). Furthermore, we may replace A by ΣA and B by $\Sigma' B$, where $\Sigma, \Sigma' \in GL_g(\mathbb{Z})$ are appropriate permutation matrices, and assume that the upper-left-most $r \times r$ submatrix of both A and B have nonzero determinant, in fact determinant ± 1. Then, we can act further on A and B by elements of $GL_g(\mathbb{Z})$ so that, without loss of generality, both A and B have the form

$$\left[\begin{array}{c|c} \mathrm{Id}_{r\times r} & * \\ \hline 0 & 0 \end{array}\right]$$

Note that after these operations, A and B are still totally unimodular; this follows from the fact that totally unimodular matrices are closed under multiplication and taking inverses. But then A and B are totally unimodular matrices with the same support. Indeed, the support of a column v_i of A, for each $i = r+1, \ldots, n$, is determined by the fundamental circuit of v_i with respect to the basis $\{v_1, \ldots, v_r\}$ in $M[A]$, and since $M[A] \cong M[B]$, each v_i and w_i have the same support.

Thus, by Lemma 2.5.12, there exists a diagonal matrix $X \in GL_g(\mathbb{Z})$, whose diagonal entries are ± 1, such that XA can be transformed into B by a sequence of column negations. This is what we claimed. □

Proof of Theorem 2.5.10. Let $v_1, \ldots, v_n$ be the columns of A. Let $X \in \operatorname{Stab} \sigma_A$. Then X acts on the rays of $\overline{\sigma_A}$ via

$$(v_i v_i^T) \cdot X = X^T v_i v_i^T X = v_j v_j^T \text{ for some column } v_j.$$

So $v_j = \pm X^T v_i$. But X^T is invertible, so a set of vectors $\{v_{i_1}, \ldots, v_{i_k}\}$ is linearly independent if and only if $\{X^T v_{i_1}, \ldots, X^T v_{i_k}\}$ is, so X induces a permutation that is in $\mathrm{Aut}(M)$.

Conversely, suppose we are given $\pi \in \mathrm{Aut}(M)$. Let B be the matrix

$$B = \begin{bmatrix} | & & | \\ v_{\pi(1)} & \cdots & v_{\pi(n)} \\ | & & | \end{bmatrix}.$$

Then $M[A] = M[B]$, so by Lemma 2.5.13, there exists $X \in GL_g(\mathbb{Z})$ such that $X^T \cdot v_i = \pm v_{\pi(i)}$ for each i. Then

$$X^T v_i v_i^T X = (\pm v_{\pi(i)})(\pm v_{\pi(i)}{}^T) = v_{\pi(i)} v_{\pi(i)}{}^T$$

so X realizes π as a permutation of the rays of $\overline{\sigma_A}$. □

2.6 The tropical Torelli map

The classical Torelli map $t_g : \mathcal{M}_g \to \mathcal{A}_g$ sends a curve to its Jacobian. Jacobians were developed thoroughly in the tropical setting in [MZ07] and [Zha07]. Here, we define the tropical Torelli map following [BMV11], and recall the characterization of its image, the so-called Schottky locus, in terms of cographic matroids. We then present a comparison of the number of cells in M_g^{tr}, in the Schottky locus, and in A_g^{tr}, for small g.

Definition 2.6.1. The tropical Torelli map

$$t_g^{\mathrm{tr}} : M_g^{\mathrm{tr}} \to A_g^{\mathrm{tr}}$$

is defined as follows. Consider the first homology group $H_1(G, \mathbb{R})$ of the graph G, whose elements are formal sums of edges with coefficients in $\mathbb{R}$ lying in the kernel of the boundary map. Given a genus g tropical curve $C = (G, l, w)$, we define a positive semidefinite form Q_C on $H_1(G, \mathbb{R}) \oplus \mathbb{R}^{|w|}$, where $|w| := \sum w(v)$. The form is 0 whenever the second summand $\mathbb{R}^{|w|}$ is involved, and on $H_1(G, \mathbb{R})$ it is

$$Q_C(\sum_{e \in E(G)} \alpha_e \cdot e) \;=\; \sum_{e \in E(G)} \alpha_e^2 \cdot l(e).$$

Here, the edges of G are oriented for reference, and the α_e are real numbers such that $\sum \alpha_e \cdot e \in H_1(G, \mathbb{R})$.

Now, pick a basis of $H_1(G, \mathbb{Z})$; this identifies $H_1(G, \mathbb{Z}) \oplus \mathbb{Z}^{|w|}$ with the lattice $\mathbb{Z}^g$, and hence $H_1(G, \mathbb{R}) \oplus \mathbb{R}^{|w|}$ with $\mathbb{R}^g = \mathbb{Z}^g \otimes_{\mathbb{Z}} \mathbb{R}$. Thus Q_C is identified with an element of $\widetilde{S}^g_{\geq 0}$. Choosing a different basis gives another element of $\widetilde{S}^g_{\geq 0}$ only up to a $GL_g(\mathbb{Z})$-action, so we have produced a well-defined element of A_g^{tr}, called the **tropical Jacobian** of C.

Theorem 2.6.2. *[BMV11, Theorem 5.1.5] The map*

$$t_g^{tr} : M_g^{tr} \to A_g^{tr}$$

is a morphism of stacky fans.

Note that the proof by Brannetti, Melo, and Viviani of Theorem 2.6.2 is correct under the new definitions. In particular, the definition of a morphism of stacky fans has not changed.

The following theorem tells us how the tropical Torelli map behaves, at least on the level of stacky cells. Given a graph G, its cographic matroid is denoted $M^*(G)$, and $\widetilde{M^*(G)}$ is then the matroid obtained by removing loops and replacing each parallel class with a single element. See [BMV11, Definition 2.3.8].

Theorem 2.6.3. *[BMV11, Theorem 5.1.5] The map t_g^{tr} sends the cell $C(G,w)$ of M_g^{tr} surjectively to the cell $C(\widetilde{M^*(G)})$.*

We denote by A_g^{cogr} the stacky subfan of A_g^{tr} consisting of those cells

$$\{C(M) : M \text{ a simple cographic matroid of rank } \leq g\}.$$

The cell $C(M)$ was defined in Construction 2.5.3. Note that A_g^{cogr} sits inside the zonotopal subfan of Section 5:

$$A_g^{\text{cogr}} \subseteq A_g^{\text{zon}} \subseteq A_g^{\text{tr}}.$$

Also, $A_g^{\text{cogr}} = A_g^{\text{tr}}$ when $g \leq 3$, but not when $g \geq 4$ ([BMV11, Remark 5.2.5]). The previous theorem says that the image of t_g^{tr} is precisely $A_g^{\text{cogr}} \subseteq A_g^{\text{tr}}$. So, in analogy with the classical situation, we call A_g^{cogr} the **tropical Schottky locus**.

Figures 2.1 and 2.8 illustrate the tropical Torelli map in genus 3. The cells of M_3^{tr} in Figure 2.1 are color-coded according to the color of the cells of A_3^{tr} in Figure 2.8 to which they are sent. These figures serve to illustrate the correspondence in Theorem 2.6.3.

Our contribution in this section is to compute the poset of cells of A_g^{cogr}, for $g \leq 5$, using Mathematica. First, we computed the cographic matroid of each graph of genus $\leq g$, and discarded the ones that were not simple. Then we checked whether any two matroids obtained in this way were in fact isomorphic. Part of this computation was done by hand in the genus 5 case, because it became intractable to check whether two 12-element matroids were isomorphic. Instead, we used some heuristic tests and then checked by hand that, for the few pairs of matroids passing the tests, the original pair of graphs were related by a sequence of vertex-cleavings and Whitney flips. This condition ensures that they have the same cographic matroid; see [Oxl92].

Theorem 2.6.4. *We obtained the following computational results:*

(i) The tropical Schottky locus A_3^{cogr} has nine cells and f-vector

$$(1, 1, 1, 2, 2, 1, 1).$$

Its poset of cells is shown in Figure 2.8.

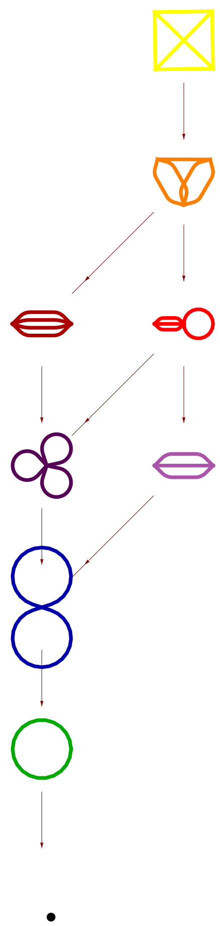


Figure 2.8: Poset of cells of $A_3^{\text{tr}} = A_3^{\text{cogr}}$. Each cell corresponds to a cographic matroid, and for convenience, we draw a graph G in order to represent its cographic matroid $M^*(G)$.

(ii) The tropical Schottky locus A_4^{cogr} has 25 cells and f-vector

$$(1,1,1,2,3,4,5,4,2,2).$$

(iii) The tropical Schottky locus A_5^{cogr} has 92 cells and f-vector

$$(1,1,1,2,3,5,9,12,15,17,15,7,4).$$

Remark 2.6.5. Actually, since $A_3^{\mathrm{cogr}} = A_3^{\mathrm{tr}}$, the results of part (i) of Theorem 2.6.4 were already known, say in [Val03].

Tables 1 and 2 show a comparison of the number of maximal cells and the number of total cells, respectively, of M_g^{tr}, A_g^{cogr}, and A_g^{tr}. The numbers in the first column of Table 2 were obtained in [MP] and in Theorem 2.2.12. The first column of Table 1 is the work of Balaban [Bal76]. The results in the second column are our contribution in Theorem 2.6.4. The third columns are due to [Eng02a] and [Eng02b]; computations for $g > 5$ were done by Vallentin [Val03].

g	M_g^{tr}	A_g^{cogr}	A_g^{tr}
2	2	1	1
3	5	1	1
4	17	2	3
5	71	4	222

Table 2.1: Number of maximal cells in the stacky fans M_g^{tr}, A_g^{cogr}, and A_g^{tr}.

g	M_g^{tr}	A_g^{cogr}	A_g^{tr}
2	7	4	4
3	42	9	9
4	379	25	61
5	4555	92	179433

Table 2.2: Total number of cells in the stacky fans M_g^{tr}, A_g^{cogr}, and A_g^{tr}.

It would be desirable to extend our computations of A_g^{cogr} to $g \geq 6$, but this would require some new ideas on effectively testing matroid isomorphisms.

2.7 Tropical covers via level structure

All tropical varieties are stacky fans: at least in the "constant coefficient" case (see [MS10]), tropical varieties are polyhedral fans, and all polyhedral fans are stacky fans in which every cone has only trivial symmetries. On the other hand, stacky fans are not always tropical varieties. Indeed, one problem with the spaces M_g^{tr} and A_g^{tr} is that although they are tropical moduli spaces, they do not "look" very tropical: they do not satisfy a tropical balancing condition (see [MS10]).

But what if we allow ourselves to consider finite-index covers of our spaces – can we then produce a more tropical object? In what follows, we answer this question for the spaces A_2^{tr} and A_3^{tr}. The uniform matroid U_4^2 and the Fano matroid F_7 play a role. We are grateful to Diane Maclagan for suggesting this question and the approach presented here.

Given $n \geq 1$, let $\mathbb{FP}^n$ denote the complete polyhedral fan in $\mathbb{R}^n$ associated to projective space $\mathbb{P}^n$, regarded as a toric variety. Concretely, we fix the rays of $\mathbb{FP}^n$ to be generated by

$$e_1, \ldots, e_n, \quad e_{n+1} := -e_1 - \cdots - e_n,$$

and each subset of at most n rays spans a cone in $\mathbb{FP}^n$. So $\mathbb{FP}^n$ has $n+1$ top-dimensional cones. Given $S \subseteq \{1, \ldots, n+1\}$, let $\mathrm{cone}(S)$ denote the open cone $\mathbb{R}_{>0}\{e_i : i \in S\}$ in $\mathbb{FP}^n$, let $\mathrm{cone}(\hat{\imath}) := \mathrm{cone}(\{1, \ldots, \hat{\imath}, \ldots, n+1\})$, and let $\overline{\mathrm{cone}(S)}$ be the closed cone corresponding to S. Note that the polyhedral fan $\mathbb{FP}^n$ is also a stacky fan: each open cone can be equipped with trivial symmetries. Its support is the tropical variety corresponding to all of $\mathbb{T}^n$.

By a **generic point** of A_g^{tr}, we mean a point x lying in a cell of A_g^{tr} of maximal dimension such that any positive semidefinite matrix X representing x is fixed only by the identity element in $GL_g(\mathbb{Z})$.

§2.7.1 A tropical cover for A_3^{tr}

By the classification in Sections 4.1–4.3 of [Val03], we note that

$$A_3^{\mathrm{tr}} = \left(\coprod_{M \subseteq MK_4} C(M) \right) / \sim .$$

In the disjoint union above, the symbol MK_4 denotes the graphic (equivalently, in this case, cographic) matroid of the graph K_4, and $M \subseteq M'$ means that M is a submatroid of M', i.e. obtained by deleting elements. The cell $C(M)$ of a regular matroid M was defined in Construction 2.5.3. There is a single maximal cell $C(MK_4)$ in A_3^{tr}, and the other cells are stacky faces of it. The cells are also listed in Figure 2.8.

Now define a continuous map

$$\pi : \mathbb{FP}^6 \to A_3^{\mathrm{tr}}$$

as follows. Let A be a 3×6 unimodular matrix representing MK_4, for example

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix},$$

and let $\overline{\sigma_A}$ be the cone in $\widetilde{S}^3_{\geq 0}$ with rays $\{v_i v_i^T\}$, where the v_i's are the columns of A, as in Construction 2.5.3. Fix, once and for all, a Fano matroid structure on the set $\{1, \ldots, 7\}$. For example, we could take F_7 to have circuits $\{124, 235, 346, 457, 156, 267, 137\}$.

Now, for each $i = 1, \ldots, 7$, the deletion $F_7 \setminus \{i\}$ is isomorphic to MK_4, so let

$$\pi_{\hat{\imath}} : [7] \setminus \{i\} \to E(MK_4)$$

be any bijection inducing such an isomorphism. Now define

$$\alpha_{\hat{\imath}} : \overline{\text{cone}(\hat{\imath})} \to A_3^{\text{tr}}$$

as the composition

$$\overline{\text{cone}(\hat{\imath})} \xrightarrow{L_{\hat{\imath}}} \overline{\sigma_A} \twoheadrightarrow \frac{\overline{\sigma_A}}{\text{Stab}\,\sigma_A} = C(MK_4) \longrightarrow A_3^{\text{tr}}$$

where $L_{\hat{\imath}}$ is the integral-linear map arising from $\pi_{\hat{\imath}}$.

Now, each $\alpha_{\hat{\imath}}$ is clearly continuous, and to paste them together into a map on all of $\mathbb{FP}^6$, we need to show that they agree on intersections. Thus, fix $i \neq j$ and let $S \subseteq \{1, \ldots, 7\} \setminus \{i, j\}$. We want to show that

$$\alpha_{\hat{\imath}} = \alpha_{\hat{\jmath}} \text{ on } \overline{\text{cone}(S)}.$$

Indeed, the map $L_{\hat{\imath}}$ sends $\overline{\text{cone}(S)}$ isomorphically to $\overline{\sigma_{A|_{\pi_{\hat{\imath}}(S)}}}$, where $A|_{\pi_{\hat{\imath}}(S)}$ denotes the submatrix of A gotten by taking the columns indexed by $\pi_{\hat{\imath}}(S)$. Furthermore, the bijection on the rays of the cones agrees with the isomorphism of matroids

$$F_7|_S \xrightarrow{\cong} MK_4|_{\pi_{\hat{\imath}}(S)}.$$

Similarly, $L_{\hat{\jmath}}$ sends $\overline{\text{cone}(S)}$ isomorphically to $\overline{\sigma_{A|_{\pi_{\hat{\jmath}}(S)}}}$, and the map on rays agrees with the matroid isomorphism

$$F_7|_S \xrightarrow{\cong} MK_4|_{\pi_{\hat{\jmath}}(S)}.$$

Hence $MK_4|_{\pi_{\hat{\imath}}(S)} \cong MK_4|_{\pi_{\hat{\jmath}}(S)}$ and by Theorem 2.5.10, there exists $X \in GL_3(\mathbb{Z})$ such that the diagram commutes:

$$\begin{array}{ccc} & & \overline{\sigma_{A|_{\pi_{\hat{\imath}}(S)}}} \\ & \nearrow^{L_{\hat{\imath}}} & \\ \overline{\text{cone}(S)} & & \big\downarrow X \\ & \searrow_{L_{\hat{\jmath}}} & \\ & & \overline{\sigma_{A|_{\pi_{\hat{\jmath}}(S)}}} \end{array}$$

We conclude that $\alpha_{\hat{\imath}}$ and $\alpha_{\hat{\jmath}}$ agree on $\overline{\text{cone}(S)}$, since $L_{\hat{\imath}}$ and $L_{\hat{\jmath}}$ differ only by a $GL_3(\mathbb{Z})$-action.

Therefore, we can glue the seven maps $\alpha_{\hat{\imath}}$ together to obtain a continuous map $\alpha : \mathbb{FP}^6 \to A_3^{\text{tr}}$.

Theorem 2.7.1. *The map $\alpha : \mathbb{FP}^6 \to A_3^{tr}$ is a surjective morphism of stacky fans. Each of the seven maximal cells of $\mathbb{FP}^6$ is mapped surjectively onto the maximal cell of A_3^{tr}. Furthermore, the map α has finite fibers, and if $x \in A_3^{tr}$ is a generic point, then $|\alpha^{-1}(x)| = 168$.*

Proof. By construction, α sends each cell cone(S) of $\mathbb{FP}^6$ surjectively onto the cell of A_3^{tr} corresponding to the matroid $F_7|_S$, and each of these maps is induced by some integral-linear map $L_{\hat{\imath}}$. That α is surjective then follows from the fact that every submatroid of MK_4 is a proper submatroid of F_7. Also, by construction, α maps each maximal cell cone$(\hat{\imath})$ of $\mathbb{FP}^6$ surjectively to the cell $C(MK_4)$ of A_3^{tr}.

By definition of the map α_i, each $x \in A_3^{\mathrm{tr}}$ has only finitely many preimages $\alpha_i^{-1}(x)$ in $\overline{\mathrm{cone}(\hat{\imath})}$, so α has finite fibers. If $x \in A_3^{\mathrm{tr}}$ is a generic point, then x has $24 = |\operatorname{Aut}(MK_4)|$ preimages in each of the seven maximal open cones cone$(\hat{\imath})$, so $|\alpha^{-1}(x)| = 168$. □

§2.7.2 A tropical cover for A_2^{tr}

Our strategy in Theorem 2.7.1 for constructing a covering map $\mathbb{FP}^6 \to A_3^{\mathrm{tr}}$ was to use the combinatorics of the Fano matroid to paste together seven copies of MK_4 in a coherent way. In fact, an analogous, and easier, argument yields a covering map $\mathbb{FP}^3 \to A_2^{\mathrm{tr}}$. We will use U_4^2 to paste together four copies of U_3^2. Here, U_n^d denotes the uniform rank d matroid on n elements.

The space A_2^{tr} can be given by

$$A_2^{\mathrm{tr}} = \left(\coprod_{M \subseteq U_3^2} C(M) \right) / \sim .$$

It has a single maximal cell $C(U_3^2)$, and the three other cells are stacky faces of it of dimensions 0, 1, and 2. See Figure 2.7.

Analogously to Section 7.1, let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

say, and for each $i = 1, \ldots, 4$, define

$$\beta_{\hat{\imath}} : \overline{\mathrm{cone}(\hat{\imath})} \to A_2^{\mathrm{tr}}$$

by sending $\overline{\mathrm{cone}(\hat{\imath})}$ to $\overline{\sigma_A}$ by a bijective linear map preserving lattice points. Here, any of the 3! possible maps will do, because the matroid U_3^2 has full automorphisms.

Just as in Section 7.1, we may check that the four maps $\alpha_{\hat{\imath}}$ agree on their overlaps, so we obtain a continuous map

$$\beta : \mathbb{FP}^3 \to A_2^{\mathrm{tr}}.$$

Proposition 2.7.2. *The map $\beta : \mathbb{FP}^3 \to A_2^{tr}$ is a surjective morphism of stacky fans. Each of the four maximal cells of $\mathbb{FP}^3$ maps surjectively onto the maximal cell of A_2^{tr}. Furthermore, the map β has finite fibers, and if $x \in A_2^{tr}$ is a generic point, then we have $|\beta^{-1}(x)| = 24$.*

Proof. The proof is exactly analogous to the proof of Theorem 2.7.1. Instead of noting that every one-element deletion of F_7 is isomorphic to MK_4, we make the easy observation that every one-element deletion of U_4^2 is isomorphic to U_3^2. If $x \in A_2^{\mathrm{tr}}$ is a generic point, then x has $6 = |\operatorname{Aut}(U_3^2)|$ preimages in each of the four maximal open cones of $\mathbb{FP}^3$. □

Remark 2.7.3. We do not know a more general construction for $g \geq 4$. We seem to be relying on the fact that all cells of A_g^{tr} are cographic when $g = 2, 3$, but this is not true when $g \geq 4$: the Schottky locus is proper.

Remark 2.7.4. Although our constructions look purely matroidal, they come from level structures on A_2^{tr} and A_3^{tr} with respect to the primes $p = 3$ and $p = 2$, respectively. More precisely, in the genus 2 case, consider the decomposition of $\widetilde{S}^2_{\geq 0}$ into secondary cones as in Theorem 2.4.7, and identify rays vv^T and ww^T if $v \equiv \pm w \pmod 3$. Then we obtain $\mathbb{FP}^3$. The analogous statement holds, replacing the prime 3 with 2, in genus 3.

Chapter 3

Tropical hyperelliptic curves

This chapter covers the material in [Cha11b], which will appear in the Journal of Algebraic Combinatorics, with only minor changes in wording.

3.1 Introduction

Our aim in this chapter is to study the locus of hyperelliptic curves inside the moduli space of tropical curves of a fixed genus g. Our work ties together two strands in the recent tropical geometry literature: tropical Brill-Noether theory on the one hand [Bak08], [Cap11b], [CDPR10], [LPP11]; and tropical moduli spaces of curves on the other [BMV11], [Cap10], [Cap11a], [Cha11a], [MZ07].

The work of Baker and Norine in [BN07] and Baker in [Bak08] has opened up a world of fascinating connections between algebraic and tropical curves. The Specialization Lemma of Baker [Bak08], recently extended by Caporaso [Cap11b], allows for precise translations between statements about divisors on algebraic curves and divisors on tropical curves. One of its most notable applications is to *tropical Brill-Noether theory.* The classical Brill-Noether theorem in algebraic geometry, proved by Griffiths and Harris, [GH80], is the following.

Theorem 3.1.1. *Suppose $g, r,$ and d are positive numbers and let*

$$\rho(g, r, d) = g - (r + 1)(g - d + r).$$

(i) If $\rho \geq 0$, then every smooth projective curve X of genus g has a divisor of degree d and rank at least r. In fact, the scheme $W^r_d(X)$ parametrizing linear equivalence classes of such divisors has dimension $\min\{\rho, g\}$.

(ii) If $\rho < 0$, then the general smooth projective curve of genus g has no divisors of degree d and rank at least r.

A tropical proof of the Brill-Noether theorem by way of the Specialization Lemma was conjectured in [Bak08] and obtained by Cools, Draisma, Payne, and Robeva in [CDPR10]. See [Cap11b] and [LPP11] for other advances in tropical Brill-Noether theory.

Another strand in the literature concerns the $(3g-3)$-dimensional moduli space M_g^{tr} of tropical curves of genus g. This space was considered by Mikhalkin and Zharkov in [Mik06b] and [MZ07], in a more limited setting (i.e. without vertex weights). It was constructed and studied as a topological space by Caporaso, who proved that M_g^{tr} is Hausdorff and connected through codimension one [Cap10], [Cap11a]. In [BMV11], Brannetti, Melo, and Viviani constructed it explicitly in the category of stacky polyhedral fans (see Definition 2.3.2). In [Cha11a], we gained a detailed understanding of the combinatorics of M_g^{tr}, which deeply informs the present study.

Fix g, r, and d such that $\rho(g,r,d) < 0$. Then the *Brill-Noether locus* $\mathcal{M}_{g,d}^r \subset \mathcal{M}_g$ consists of those genus g curves which are exceptional in Theorem 3.1.1(ii) in the sense that they do admit a divisor of degree d and rank at least r. The tropical Brill-Noether locus $M_{g,d}^{r,\mathrm{tr}} \subset M_g^{\mathrm{tr}}$ is defined in exactly the same way.

In light of the recent advances in both tropical Brill-Noether theory and tropical moduli theory, it is natural to pose the following

Problem 3.1.2. *Characterize the tropical Brill-Noether loci $M_{g,d}^{r,tr}$ inside M_g^{tr}.*

The case $r = 1$ and $d = 2$ is, of course, the case of hyperelliptic curves, and the combinatorics is already very rich. Here, we are able to characterize the hyperelliptic loci in each genus. The main results of this chapter are the follwing three theorems, proved in Sections 3, 4, and 5, respectively.

Theorem 3.1.3. *Let Γ be a metric graph with no points of valence 1, and let (G, l) denote its canonical loopless model. Then the following are equivalent:*

(i) Γ is hyperelliptic.

(ii) There exists an involution $i : G \to G$ such that G/i is a tree.

(iii) There exists a nondegenerate harmonic morphism of degree 2 from G to a tree, or $|V(G)| = 2$. (See Figure 3.1).

Theorem 3.1.4. *Let $g \geq 3$. The locus of 2-edge-connected genus g tropical hyperelliptic curves is a $(2g-1)$-dimensional stacky polyhedral fan whose maximal cells are in bijection with trees on $g-1$ vertices with maximum valence 3. (See Figure 3.3).*

Theorem 3.1.5. *Let $X \subseteq \mathbb{T}^2$ be a hyperelliptic curve of genus $g \geq 3$ over a complete, algebraically closed nonarchimedean valuated field K, and suppose X is defined by a polynomial of the form $P = y^2 + f(x)y + h(x)$. Let $\widehat{X}$ be its smooth completion. Suppose the Newton complex of P is a unimodular subdivision of the triangle with vertices $(0,0), (2g+2, 0)$, and $(0,2)$, and suppose that the core of* $\operatorname{Trop} X$ *is bridgeless. Then the skeleton Σ of the Berkovich analytification $\widehat{X}^{\mathrm{an}}$ is a standard ladder of genus g whose opposite sides have equal length. (See Figure 3.9).*

We begin in Section 2 by giving new definitions of harmonic morphisms and quotients of metric graphs, and recalling the basics of divisors on tropical curves. In Section 3, we

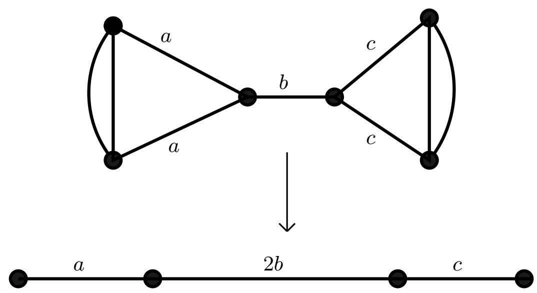


Figure 3.1: A harmonic morphism of degree two. Here, a, b, and c are positive real numbers.

prove the characterization of hyperellipticity stated in Theorem 3.1.3. This generalizes a central result of [BN09] to metric graphs. See also [HMY09, Proposition 45] for a proof of one of the three parts. In Section 4, we build the space of hyperelliptic tropical curves and the space of 2-edge-connected hyperelliptic tropical curves. We then explicitly compute the hyperelliptic loci in M_g^{tr} for $g = 3$ and $g = 4$. See Figures 3.2 and 3.3. Note that all genus 2 tropical curves are hyperelliptic, as in the classical case. Finally, in Section 5, we establish a connection to embedded tropical curves in the plane, of the sort shown in Figure 3.9, and prove Theorem 3.1.5.

Our work in Section 5 represents a first step in studying the behavior of hyperelliptic curves under the map

$$\text{trop} : M_g(K) \to M_g^{\text{tr}}$$

in [BPR11, Remark 5.51], which sends a curve X over a algebraically closed, complete nonarchimedean field K to its Berkovich skeleton, or equivalently to the dual graph, appropriately metrized, of the special fiber of a semistable model for X. Note that algebraic curves that are not hyperelliptic can tropicalize to hyperelliptic curves [Bak08, Example 3.6], whereas tropicalizations of hyperelliptic algebraic curves are necessarily hyperelliptic [Bak08, Corollary 3.5]. It would be very interesting to study further the behavior of hyperelliptic loci, and higher Brill-Noether loci, under the tropicalization map above.

3.2 Definitions and notation

We start by defining harmonic morphisms, quotients, divisors, and rational functions on metric graphs. These concepts will be used in Theorem 3.3.12 to characterize hyperellipticity.

§3.2.1 Metric graphs and harmonic morphisms

Throughout, all of our graphs are connected, with loops and multiple edges allowed. A **metric graph** is a metric space Γ such that there exists a graph G and a length function

$$l : E(G) \to \mathbb{R}_{>0}$$

so that Γ is obtained from (G, l) by gluing intervals $[0, l(e)]$ for $e \in E(G)$ at their endpoints, as prescribed by the combinatorial data of G. The distance $d(x, y)$ between two points x and y in Γ is given by the length of the shortest path between them. We say that (G, l) is a **model** for Γ in this case. We say that (G, l) is a loopless model if G has no loops. Note that a given metric graph Γ admits many possible models (G, l). For example, a line segment of length a can be subdivided into many edges whose lengths sum to a. The reason for working with both metric graphs and models is that metric graphs do not have a preferred vertex set, but choosing some vertex set is convenient for making combinatorial statements.

Suppose Γ is a metric graph, and let us exclude, once and for all, the case that Γ is homeomorphic to the circle S^1. We define the **valence** $\mathrm{val}(x)$ of a point $x \in \Gamma$ to be the number of connected components in $U_x \setminus \{x\}$ for any sufficiently small neighborhood U_x of x. Hence almost all points in Γ have valence 2. By a **segment** of Γ we mean a subset s of Γ isometric to a real closed interval, such that any point in the interior of s has valence 2.

If $V \subseteq \Gamma$ is a finite set which includes all points of Γ of valence different from 2, then define a model (G_V, l) as follows. The vertices of the graph G_V are the points in V, and the edges of G_V correspond to the connected components of $\Gamma \setminus V$. These components are necessarily isometric to open intervals, the length of each of which determines the function $l : E(G_V) \to \mathbb{R}_{>0}$. Then (G_V, l) is a model for Γ.

The **canonical model** (G_0, l) for a metric graph Γ is the model obtained by taking

$$V = \{x \in \Gamma \ : \ \mathrm{val}(x) \neq 2\}.$$

The **canonical loopless model** (G_-, l) for the metric graph Γ is the model obtained from the canonical model by placing an additional vertex at the midpoint of each loop edge. Thus, if a vertex $v \in V(G_-)$ has degree 2, then the two edges incident to v necessarily have the same endpoints and the same length.

Suppose (G, l) and (G', l') are loopless models for metric graphs Γ and Γ', respectively. A **morphism of loopless models** $\phi : (G, l) \to (G', l')$ is a map of sets

$$V(G) \cup E(G) \xrightarrow{\phi} V(G') \cup E(G')$$

such that

(i) $\phi(V(G)) \subseteq V(G')$,

(ii) if $e = xy$ is an edge of G and $\phi(e) \in V(G')$ then $\phi(x) = \phi(e) = \phi(y)$,

(iii) if $e = xy$ is an edge of G and $\phi(e) \in E(G')$ then $\phi(e)$ is an edge between $\phi(x)$ and $\phi(y)$, and

(iv) if $\phi(e) = e'$ then $l'(e')/l(e)$ is an integer.

For simplicity, we will sometimes drop the length function from the notation and just write $\phi : G \to G'$. An edge $e \in E(G)$ is called **horizontal** if $\phi(e) \in E(G')$ and **vertical** if $\phi(e) \in V(G')$.

Now, the map ϕ induces a a map $\tilde{\phi} : \Gamma \to \Gamma'$ of topological spaces in the natural way. In particular, if $e \in E(G)$ is sent to $e' \in E(G')$, then we declare $\tilde{\phi}$ to be linear along e. Let

$$\mu_\phi(e) = l'(e')/l(e) \in \mathbb{Z}$$

denote the slope of this linear map.

A morphism of loopless models $\phi : (G, l) \to (G', l')$ is said to be **harmonic** if for every $x \in V(G)$, the nonnegative integer

$$m_\phi(x) = \sum_{\substack{e \in E(G) \\ x \in e,\ \phi(e) = e'}} \mu_\phi(e)$$

is the same over all choices of $e' \in E(G')$ that are incident to the vertex $\phi(x)$. The number $m_\phi(x)$ is called the **horizontal multiplicity** of ϕ at x. We say that ϕ is **nondegenerate** if $m_\phi(x) > 0$ for all $x \in V(G)$. The **degree** of ϕ is defined to be

$$\deg \phi = \sum_{\substack{e \in E(G) \\ \phi(e) = e'}} \mu_\phi(e)$$

for any $e' \in E(G')$. One can check that the number $\deg \phi$ does not depend on the choice of e' ([BN09, Lemma 2.4], [Ura00, Lemma 2.12]). If G' has no edges, then we set $\deg \phi = 0$.

We define a **morphism of metric graphs** to be a continuous map $\tilde{\phi} : \Gamma \to \Gamma'$ which is induced from a morphism $\phi : (G, l) \to (G', l')$ of loopless models, for some choice of models (G, l) and (G', l'). It is **harmonic** if ϕ is harmonic as a morphism of loopless models, and we define its degree $\deg(\tilde{\phi}) = \deg(\phi)$. For any point $x \in \Gamma$, we define the **horizontal multiplicity** of $\tilde{\phi}$ at x as

$$m_{\tilde{\phi}}(x) = \begin{cases} m_\phi(x) & \text{if } x \in V(G) \\ 0 & \text{if } x \in e^\circ \text{ and } \phi(e) \in V(G') \\ \mu_\phi(e) & \text{if } x \in e^\circ \text{ and } \phi(e) \in E(G'). \end{cases}$$

Here, e° denotes the interior of e. One can check that the harmonicity of $\tilde{\phi}$ is independent of choice of ϕ, as are the computations of $\deg(\tilde{\phi})$ and $m_\phi(x)$.

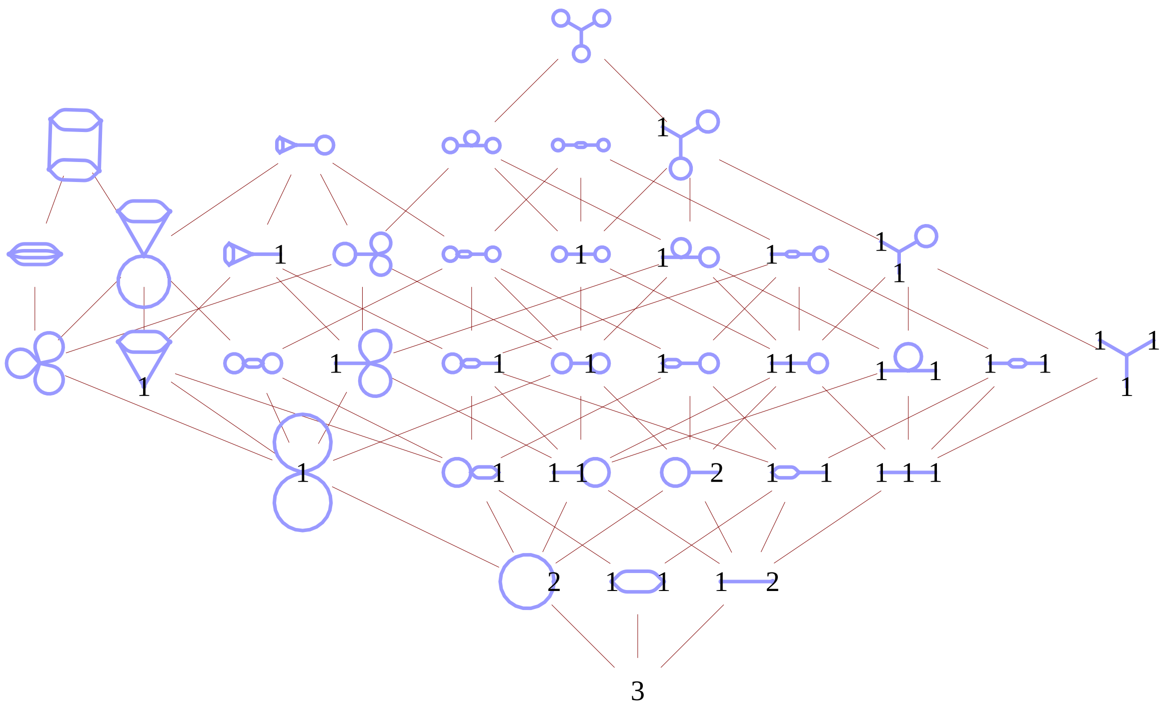


Figure 3.2: The tropical hyperelliptic curves of genus 3. Here, unmarked vertices have weight 0, and edges that form a 2-edge-cut are required to have the same length. The space of such curves sits inside the moduli space M_3^{tr} shown in Figure 2.1.

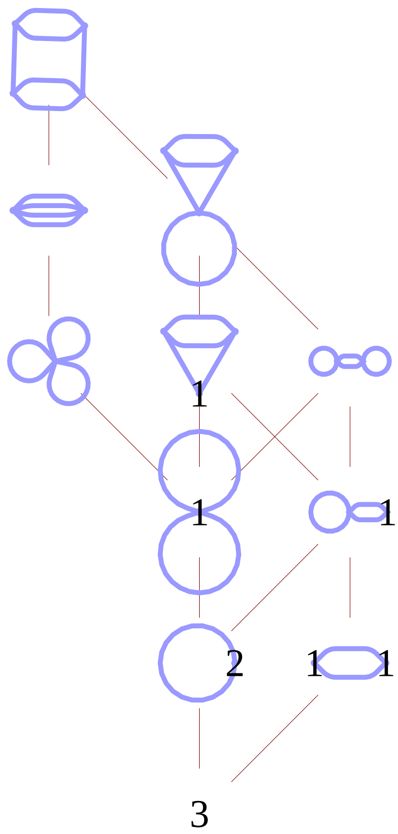


Figure 3.3: The 2-edge-connected tropical hyperelliptic curves of genus 3. Here, unmarked vertices have weight 0, and edges that form a 2-edge-cut are required to have the same length. The space of such curves sits inside the moduli space M_3^{tr} shown in Figure 2.1.

§3.2.2 Automorphisms and quotients

Let (G,l) be a loopless model for a metric graph Γ. An **automorphism** of (G,l) is a harmonic morphism $\phi : (G,l) \to (G,l)$ of degree 1 such that the map $V(G) \cup E(G) \to V(G) \cup E(G)$ is a bijection. Equivalently, an automorphism of (G,l) is an automorphism of G that also happens to preserve the length function l. The group of all such automorphisms is denoted $\mathrm{Aut}(G,l)$.

An automorphism of Γ is an isometry $\Gamma \to \Gamma$. We assume throughout that Γ is not a circle, so the automorphisms $\mathrm{Aut}(\Gamma)$ of Γ form a finite group, since they permute the finitely many points of Γ of valence $\neq 2$. Then automorphisms of Γ correspond precisely to automorphisms of the canonical loopless model (G_-,l) for Γ. An automorphism i is an **involution** if $i^2 = \mathrm{id}$.

Let (G,l) be a loopless model, and let K be a subgroup of $\mathrm{Aut}(G,l)$. We will now define the **quotient loopless model** $(G/K,l')$. The vertices of G/K are the K-orbits of $V(G)$, and the edges of G/K are the K-orbits of those edges xy of G such that x and y lie in distinct K-orbits. If $e \in E(G)$ is such an edge, let $[e] \in E(G/K)$ denote the edge corresponding to its K-orbit. The length function

$$l' : E(G/K) \to \mathbb{R}_{>0}$$

is given by

$$l'([e]) = l(e) \cdot |\,\mathrm{Stab}(e)|$$

for every edge $e \in E(G)$ with K-inequivalent ends. Notice that l' is well-defined on $E(G/K)$. Note also that $(G/K,l')$ is a loopless model.

Remark 3.2.1. If K is generated by an involution i on (G,l), then computing the length function l' on the quotient G/K is very easy. By definition, edges of G in orbits of size 2 have their length preserved, edges of G flipped by i are collapsed to a vertex, and edges of G fixed by i are stretched by a factor of 2. See Figure 3.1.

The definition of the quotient metric graph follows the definition in [BN09] in the case of nonmetric graphs, but has the seemingly strange property that edges can be stretched by some integer factor in the quotient. The reason for the stretching is that it allows the natural quotient morphism to be harmonic. Indeed, let (G,l) be a loopless model, let K be a subgroup of $\mathrm{Aut}(G,l)$, and let $(G/K,l')$ be the quotient. Define a morphism of loopless models

$$\pi_K : (G,l) \to (G/K,l')$$

as follows. If $v \in V(G)$, let $\pi_K(v) = [v]$. If $e \in E(G)$ has K-equivalent ends x and y, then let $\pi_K(e) = [x] = [y]$. If $e \in E(G)$ has K-inequivalent ends, then let $\pi_K(e) = [e]$.

Lemma 3.2.2. *Let (G,l) be a loopless model, let K be a subgroup of $\mathrm{Aut}(G,l)$, and let $(G/K,l')$ be the quotient. Then the quotient morphism $\pi_K : (G,l) \to (G/K,l')$ constructed above is a harmonic morphism. If the graph G/K does not consist of a single vertex, then π_K is nondegenerate of degree $|K|$.*

Proof. Let $x \in V(G)$. Then for all $e' \in E(G/K)$ incident to $\pi_K(x)$, we have

$$\sum_{\substack{e \in E(G) \\ x \in e,\ \phi(e)=e'}} \mu_\phi(e) = \sum_{\substack{e \in E(G) \\ x \in e,\ \phi(e)=e'}} |Stab(e)| = |Stab(x)|$$

is indeed independent of choice of e'. The last equality above follows from applying the Orbit-Stabilizer formula to the transitive action of $Stab(x)$ on the set $\{e \in E(G) : x \in e,\ \phi(e) = e'\}$. Furthermore, for any $e' \in E(G/K)$, we have

$$\deg(\pi_K) = \sum_{\substack{e \in E(G) \\ \phi(e)=e'}} \frac{l'(e')}{l(e)} = \sum_{\substack{e \in E(G) \\ \phi(e)=e'}} |Stab(e)| = |K|,$$

where the last equality again follows from the Orbit-Stabilizer formula.

Finally, suppose the graph G/K is not a single vertex. Then every $x' \in V(G/K)$ is incident to some edge e'. Therefore, any $x \in V(G)$ with $\pi_K(x) = x'$ must be incident to some $e \in E(G)$ with $\pi_K(e) = e'$, which shows that $m_{\pi_K}(x) \neq 0$. Hence $\pi_K(x)$ is nondegenerate. □

Figure 3.1 shows an example of a quotient morphism which is harmonic by Lemma 3.2.2. Here, the group K is generated by the involution which flips the loopless model G across its horizontal axis. The quotient G/K is then a path. We will see in Theorem 3.3.12 that G is therefore hyperelliptic.

§3.2.3 Divisors on metric graphs and tropical curves

The basic theory of divisors on tropical curves that follows is due to [BN07] in the non-metric case and to [GK08], [MZ07] in the metric case. Let Γ be a metric graph. Then the **divisor group** $\mathrm{Div}(\Gamma)$ is the group of formal $\mathbb{Z}$-sums of points of Γ, with all but finitely many points appearing with coefficient 0. If

$$D = \sum_{x \in \Gamma} D(x) \cdot x,\ D(x) \in \mathbb{Z}$$

is an element of $\mathrm{Div}(\Gamma)$, then define the **degree** of D to be

$$\deg D = \sum_{x \in \Gamma} D(x) \in \mathbb{Z}.$$

Denote by $\mathrm{Div}^0(\Gamma)$ the subgroup of divisors of degree 0.

A **rational function** on Γ is a function $f : \Gamma \to \mathbb{R}$ that is continuous and piecewise-linear, with integer slopes along its domains of linearity. We further require that there are only finitely many points in the interiors of edges at which f is not differentiable. If f is a rational function on Γ, then define the divisor $\mathrm{div}\, f$ associated to it as follows: at any given point $x \in \Gamma$, let $(\operatorname{div} f)(x)$ be the sum of all slopes that f takes on along edges emanating from x.

The group of **principal divisors** is defined to be

$$\operatorname{Prin}(\Gamma) = \{\operatorname{div} f \ : \ f \text{ a rational function on } \Gamma\}.$$

One may check that $\operatorname{Prin}(\Gamma) \subseteq \operatorname{Div}^0(\Gamma)$. Then the **Jacobian** $\operatorname{Jac}(\Gamma)$ is defined to be

$$\operatorname{Jac}(\Gamma) = \frac{\operatorname{Div}^0(\Gamma)}{\operatorname{Prin}(\Gamma)}.$$

Note that $\operatorname{Jac}(\Gamma) = 1$ if and only if Γ is a tree; this follows from [BF11, Corollary 3.3].

A divisor $D \in \operatorname{Div}(\Gamma)$ is **effective**, and we write $D \geq 0$, if $D = \sum_{x \in \Gamma} D(x) \cdot x$ with $D(x) \in \mathbb{Z}_{\geq 0}$ for all $x \in X$. We say that two divisors D and D' are **linearly equivalent**, and we write $D \sim D'$, if $D - D' \in \operatorname{Prin}(\Gamma)$. Then the **rank** of a divisor D is defined to be

$$\max\{k \in \mathbb{Z} \ : \text{ for all } E \geq 0 \text{ of degree } k, \text{ there exists } E' \geq 0 \text{ such that } D - E \sim E'\}.$$

Note that a divisor not equivalent to any effective divisor thus has rank -1.

Definition 3.2.3. A metric graph is **hyperelliptic** if it has a divisor of degree 2 and rank 1.

Let us extend the above definition to abstract tropical curves. The definition we will give is due to [ABC11]. First, we recall:

Definition 3.2.4. An **abstract tropical curve** is a triple (G, w, l), where (G, l) is a model for a metric graph Γ, and

$$w : V(G) \to \mathbb{Z}_{\geq 0}$$

is a weight function satisfying a stability condition as follows: every vertex v with $w(v) = 0$ has valence at least 3.

Now given a tropical curve (G, w, l) with underlying metric graph Γ, let Γ^w be the metric graph obtained from Γ by adding at each vertex $v \in V(G)$ a total of $w(v)$ loops of any lengths.

Definition 3.2.5. An abstract tropical curve (G, w, l) with underlying metric graph Γ is **hyperelliptic** if Γ^w is hyperelliptic, that is, if Γ^w admits a divisor of degree 2 and rank 1.

Remark 3.2.6. We will see in Theorem 3.3.12 that the definition of hyperellipticity does not depend on the lengths of the loops that we added to Γ.

§3.2.4 Pullbacks of divisors and rational functions

Let $\phi : \Gamma \to \Gamma'$ be a harmonic morphism, and let $g : \Gamma' \to \mathbb{R}$ be a rational function. The **pullback** of g is the function $\phi^* g : \Gamma \to \mathbb{R}$ defined by

$$\phi^* g = g \circ \phi.$$

One may check that $\phi^* g$ is again a rational function. The **pullback** map on divisors

$$\phi^* : \operatorname{Div} \Gamma' \to \operatorname{Div} \Gamma$$

is defined as follows: given $D' \in \operatorname{Div} \Gamma'$, let

$$\left(\phi^* \left(D'\right)\right)(x) = m_\phi \left(x\right) \cdot D' \left(\phi(x)\right)$$

for all $x \in \Gamma$. The following extends [BN09, Proposition 4.2(ii)] to metric graphs. It will be used in the proof of Theorem 3.3.2.

Proposition 3.2.7. *Let $\phi : \Gamma \to \Gamma'$ be a harmonic morphism of metric graphs, and let $g : \Gamma' \to \mathbb{R}$ be a rational function. Then*

$$\phi^* \operatorname{div} g = \operatorname{div} \phi^* g.$$

Remark 3.2.8. The two occurrences of ϕ^* are really different operations, one on divisors and the other on rational functions. Also, note that there is an analogous statement for pushforwards along ϕ, as well as many other analogues of standard properties of holomorphic maps, but we do not need them here.

Proof. It is straightforward (and we omit the details) to show that we may break Γ and Γ' into sets S and S' of segments along which $\phi^* g$ and g, respectively, are linear, and furthermore, such that each segment $s \in S$ is mapped linearly to some $s' \in S'$ or collapsed to a point. Then at any point $x' \in \Gamma$, we have

$$(\operatorname{div} g)(x') = \sum_{s'=x'y' \in S'} \frac{g(y') - g(x')}{l(s')}.$$

So for any $x \in \Gamma$,

$$\begin{aligned}
(\phi^* \operatorname{div} g)(x) &= m_\phi(x) \sum_{s'=\phi(x)y' \in S'} \frac{g(y') - g(\phi(x))}{l(s')} \\
&= \sum_{s'=\phi(x)y' \in S'} \sum_{s=xy \in \phi^{-1}(s')} \frac{l(s')}{l(s)} \cdot \frac{g(y') - g(\phi(x))}{l(s')} \\
&= \sum_{s'=\phi(x)y' \in S'} \sum_{s=xy \in \phi^{-1}(s')} \frac{g(\phi(y)) - g(\phi(x))}{l(s)} \\
&= (\operatorname{div} \phi^* g)(x).
\end{aligned}$$

□

3.3 When is a metric graph hyperelliptic?

Our goal in this section is to prove Theorem 3.3.12, which characterizes when a metric graph hyperelliptic. It is the metric analogue of one of the main theorems of [BN09]. We will first prove the theorem for 2-edge-connected metric graphs in Theorem 3.3.2 and then prove the general case.

Given f a rational function on a metric graph Γ, let $M(f)$ (respectively $m(f)$) denote the set of points of Γ at which f is maximized (respectively minimized). Recall that a graph is said to be **k-edge-connected** if removing any set of at most $k-1$ edges from it yields a connected graph. A metric graph Γ is said to be k-edge-connected if the underlying graph G_0 of its canonical model (G_0, l) is k-edge-connected. Thus a metric graph is 2-edge-connected if and only if any model for it is 2-edge-connected, although the analogous statement is false for $k > 2$.

The next lemma studies the structure of tropical rational functions with precisely two zeroes and two poles. It is important to our study of hyperelliptic curves because if Γ is a graph with a degree two harmonic morphism ϕ to a tree T and $D = \tilde{y} - \tilde{x} \in \operatorname{Prin} T$, then the pullback $\phi^*(D)$ is a principal divisor on Γ with two zeroes and two poles by Proposition 3.2.7.

Lemma 3.3.1. *Let Γ be a 2-edge connected metric graph; let f be a rational function on Γ such that*

$$\operatorname{div} f = y + y' - x - x'$$

for $x, x', y, y' \in \Gamma$ such that the sets $\{x, x'\}$ and $\{y, y'\}$ are disjoint. Then

(i) $\partial M(f) = \{x, x'\}$ and $\partial m(f) = \{y, y'\}$, where ∂ denotes the boundary. In fact, there are precisely two edges leaving $M(f)$, one at x and one at x', and precisely two edges leaving $m(f)$, one at y and one at y'. Furthermore, there is an x–x' path in $M(f)$, and a y–y' path in $m(f)$.

(ii) f never takes on slope greater than 1. Furthermore, for any $w \in \Gamma$, let $s^+(w)$, (resp. $s^-(w)$) denote the total positive (resp. negative) outgoing slope from w. Then

$$s^+(w) \leq 2 \text{ and } s^-(w) \geq -2.$$

(iii) Given $w \in \Gamma$ with $(\operatorname{div} f)(w) = 0$, the multiset of outgoing nonzero slopes at w is $\emptyset$, $\{1, -1\}$, or $\{1, 1, -1, -1\}$.

Proof. Let $D = \operatorname{div} f$. Note that $D < 0$ at any point in $\partial M(f)$, so $\partial M(f) \subset \{x, x'\}$. Now, $\partial M(f)$ must be nonempty since $\operatorname{div} f \neq 0$. If $|\partial M(f)| = 1$, then since $M(f)$ has at least two edges leaving it (since Γ is 2-edge-connected), and f is decreasing along them, then we must have $x = x'$ and $\partial M(f) = \{x\}$ as desired. Otherwise, $|\partial M(f)| = 2$ and again $\partial M(f) = \{x, x'\}$. Now, f must decrease along any segment leaving $M(f)$, and there are at least two such segments by 2-edge-connectivity of Γ. By inspecting D, we conclude that there exactly two such segments, one at x and one at x'. Furthermore, suppose $x \neq x'$. Then deleting the segment leaving $M(f)$ at x cannot separate x from $m(f)$, again by 2-edge-connectivity, so there is an x–x' path in $M(f)$. The analogous results hold for $m(f)$. This proves (i).

Let us prove (ii). Pick any $w \in \Gamma$. We will show that $s^+(w) \leq 2$, and moreover, that if $s^+(w) = 2$ then there are precisely two directions at w along which f has outgoing slope $+1$. An analogous argument holds for $s^-(w)$, so (ii) will follow.

Let U be the union of all paths in Γ that start at w and along which f is nonincreasing. Let $w_1, \ldots, w_l$ be the set of points in U that are either vertices of Γ or are points at which f is not differentiable. Let $W = \{w, w_1, \ldots, w_l\}$. Then $U \backslash W$ consists of finitely many open segments. Let $S = \{s_1, \ldots, s_k\}$ be the set of closures of these segments. So the closed segments $s_1, \ldots, s_k$ cover U and intersect at points in W, and f is linear along each of them. Orient each segment in S for reference, and for each $y \in W$, let

$$\delta^+(y) = \{j \;:\; s_j \text{ is outgoing at } y\},$$
$$\delta^-(y) = \{j \;:\; s_j \text{ is incoming at } y\}.$$

Finally, for each $i = 1, \ldots, k$, let $m_i \in \mathbb{Z}$ be the slope that f takes on along the (oriented) segment s_i.

The key observation regarding U is that the slope of f along any edge e leaving U must be positive, because otherwise e would lie in U. Therefore we have

$$D(w_i) \geq \sum_{j \in \delta^+(w_i)} m_j - \sum_{j \in \delta^-(w_i)} m_j \qquad \text{for } i = 1, \ldots, l, \tag{3.1}$$
$$D(w) \geq \sum_{j \in \delta^+(w)} m_j - \sum_{j \in \delta^-(w)} m_j - s^+(w). \tag{3.2}$$

Summing (3.1) and (3.2), we have

$$D(w) + D(w_1) + \cdots + D(w_l) \geq s^+(w). \tag{3.3}$$

Since $D = y + y' - x - x'$, we must have $s^+(w) \leq 2$.

Suppose $s^+(w) = 2$. Then the inequality (3.3) must be an equality, and so (3.1) and (3.2) must also be equalities. In particular, no segment leaves U except at w. Since Γ is 2-edge-connected and $\partial U = \{w\}$, it follows that there are precisely two directions at w along which f has outgoing slope $+1$. This proves (ii).

Finally, part (iii) follows directly from (ii). □

We now come to the metric version of [BN09, Theorem 5.12], which gives equivalent characterizations of hyperellipticity for 2-edge-connected metric graphs.

Theorem 3.3.2. *Let Γ be a 2-edge-connected metric graph, and let (G, l) denote its canonical loopless model. Then the following are equivalent:*

(i) Γ is hyperelliptic.

(ii) There exists an involution $i : G \to G$ such that G/i is a tree.

(iii) There exists a nondegenerate harmonic morphism of degree 2 *from* G *to a tree, or* $|V(G)| = 2$.

Proof. First, if $|V(G)| = 2$, then G must be the unique graph on 2 vertices and n edges, and so all three conditions hold. So suppose $|V(G)| > 2$. Let us prove (i) $\Rightarrow$ (ii).

Let $D \in \operatorname{Div}\Gamma$ be a divisor of degree 2 and rank 1. We will now define an involution $i : \Gamma \to \Gamma$ and then show that it induces an involution $i : G \to G$ on its model. By slight abuse of notation, we will call both of these involutions i. We will then prove that G/i is a tree.

We define $i : \Gamma \to \Gamma$ as follows. Given $x \in \Gamma$, since D has rank 1, there exists $x' \in \Gamma$ such that $D \sim x + x'$. Since Γ is 2-edge-connected, this x' is unique. Then let $i(x) = x'$. Clearly i is an involution on sets.

Claim 3.3.3. *The map* $i : \Gamma \to \Gamma$ *is an isometry.*

Proof of Claim 3.3.3. Let $x, y \in \Gamma$, and let $x' = i(x)$, $y' = i(y)$. We will show that

$$d(x, y) = d(x', y')$$

with respect to the shortest path metric on Γ. We may assume $x \neq y$ and $x \neq y'$, for otherwise the statement is clear.

By construction, we have $x + x' \sim y + y'$, so let $f : \Gamma \to \mathbb{R}$ be a rational function on Γ such that

$$\operatorname{div} f = y + y' - x - x'.$$

By Lemma 3.3.1(i), we have $\partial M(f) = \{x, x'\}$ and $\partial m(f) = \{y, y'\}$. Furthermore, $M(f)$ has precisely two edges leaving it, one at x and one at x'. Similarly, $m(f)$ has precisely two edges leaving it, one at y and one at y'.

By a **path down from** x we mean a path in Γ that starts at x and along which f is decreasing, and which is as long as possible given these conditions. By Lemma 3.3.1(ii), f must have constant slope -1 along such a path. Furthermore, by Lemma 3.3.1(iii), any path down from x must end at y or y'. After all, if it ended at some point w with $\operatorname{div} f(w) = 0$, then it would not be longest possible. Similarly, every path down from x' must end at y or y'.

So let P be a path down from x, and let l be its length in Γ. Suppose that P ends at y. Note that $d(x, y) = l$. Indeed if there were a shorter path from x to y, then f would have an average slope of less than -1 along it, contradicting Lemma 3.3.1(ii). Also, by Lemma 3.3.1(iii), there must exist a path P' down from x' to $m(f)$ that is edge-disjoint from P, so P' must end at y'. Now, P and P' have the same length since f decreases from its maximum to its minimum value at constant slope -1 along each of them. Thus $d(x', y') = l$, and $d(x, y) = d(x', y')$ as desired.

So we may assume instead that every path P down from x ends at y'. Then every path down from x' must end at y, and no pair of x–y' and x–y paths has a common vertex, for otherwise we could find a path down from x that ends at y. Thus $x \neq x'$ and $y \neq y'$ and, by Lemma 3.3.1, there is in fact a unique path $P_{xy'}$ down from x ending at y' and a unique path $P_{x'y}$ down from x' ending at y. Then $P_{xy'}$ and $P_{x'y}$ have the same length, say l, by the argument above. Let δ be the length of a shortest path S in Γ between $P_{xy'}$ and $P_{x'y}$, and note that f must be constant on S. Then

$$d(x,y) = \delta + l = d(x',y'),$$

proving the claim. □

Thus the involution $i : \Gamma \to \Gamma$ induces an automorphism $i : G \to G$ of canonical loopless models. Then by Lemma 3.2.2, it induces a quotient morphism

$$\pi : G \to G/i$$

which is a nondegenerate harmonic morphism of degree two. In particular, we note that G/i does not consist of a single vertex, since G had more than two vertices by assumption.

The final task is to show that G/i is a tree. In [BN09], this is done by by showing that $\operatorname{Jac}(G/i) = 1$ using the pullback of Jacobians along harmonic morphisms. Here, we will instead prove directly, in several steps, that the removal of any edge $\tilde{e}$ from G/i disconnects it. If so, then G/i is necessarily a tree.

Claim 3.3.4. *Let $\tilde{e} \in E(G/i)$. Then $\pi^{-1}(\tilde{e})$ consists of two edges.*

Proof of Claim 3.3.4. Suppose instead that $\pi^{-1}(\tilde{e})$ consists of a single edge $e = xy \in E(G)$. Then i fixes e, and $i(x) = x$ and $i(y) = y$. Regarding e as a segment of Γ, choose points x_0, y_0 in the interior of e such that the subsegment x_0y_0 of e has length $< d(x,y)$. Then $i(x_0) = x_0$ and $i(y_0) = y_0$, so $D \sim 2x_0 \sim 2y_0$. So there exists $f : G \to \mathbb{R}$ with $\operatorname{div} f = 2x_0 - 2y_0$. Then there are two paths down from x_0 to y_0, necessarily of the same length in G; but one of them is the segment x_0y_0 and the other one passes through x and y, contradiction. □

Claim 3.3.5. *Let $\tilde{e} \in E(G/i)$, and let $\pi^{-1}(\tilde{e}) = \{e, e'\}$. Let e have vertices x, y and e' have vertices x', y', labeled so that $\pi(x) = \pi(x')$ and $\pi(y) = \pi(y')$. Then in $G \setminus \{e, e'\}$, there is no path from $\{x, x'\}$ to $\{y, y'\}$.*

Proof of Claim 3.3.5. By definition of the quotient map π, we have $i(x) = x'$ and $i(y) = y'$. Since $x + x' \sim y + y'$, we may pick a rational function f on Γ such that

$$\operatorname{div} f = y + y' - x - x'.$$

Then f is linear along both e and e'. By Lemma 3.3.1(i), we have $\partial M(f) = \{x, x'\}$ and $\partial m(f) = \{y, y'\}$, and by Lemma 3.3.1(ii), f must have constant slope 1 along each of e and e', decreasing from $\{x, x'\}$ to $\{y, y'\}$. Then again by Lemma 3.3.1(i), e and e' separate $\{x, x'\}$ from $\{y, y'\}$. □

Claim 3.3.6. *Any edge $\tilde{e} \in E(G/i)$ is a cut edge, that is, its removal disconnects G/i.*

Proof of Claim 3.3.6. By Claim 3.3.4, we may let $\pi^{-1}(\tilde{e}) = \{e, e'\}$. Let e have vertices x, y and e' have vertices x', y', and let $\tilde{x} = \pi(x) = \pi(x')$ and $\tilde{y} = \pi(y) = \pi(y')$ be the vertices of $\tilde{e}$.

Suppose for a contradiction that there is a path in G/i from $\tilde{x}$ to $\tilde{y}$ not using $\tilde{e}$. By nondegeneracy of π, we may lift that path to a path in G from x to y or y' that does not use e or e'. But this contradicts Claim 3.3.5. □

We conclude that G/i is a tree. We have shown (i) implies (ii).

(ii) ⇒ (iii). This is precisely Lemma 3.2.2.

(iii) ⇒(i). This argument follows [BN09]. Let $\phi : (G, l) \to (T, l')$ be a nondegenerate degree 2 harmonic morphism of loopless models, where T is a tree. Let $\phi : \Gamma \to T$ be the induced map on metric graphs, also denoted ϕ by abuse of notation. Pick any $y_0 \in T$ and, regarding y_0 as a divisor of degree 1, let $D = \phi^*(y_0)$. So D is an effective divisor on Γ of degree 2, say $D = y + y'$. Now we have $r(D) \leq 1$, for otherwise, for any $z \in \Gamma$, we would have $(y + y') - (z + y') \sim 0$, so $y \sim z$, so Γ itself would be a tree. It remains to show $r(D) \geq 1$. Let $x \in \Gamma$. Then $\phi(x) \sim y_0$, since $\operatorname{Jac} T = 1$. Then by Proposition 3.2.7, we have

$$\begin{aligned} D &= \phi^*(y_0) \sim \phi^*(\phi(x)) \\ &= m_\phi(x) \cdot x + E \end{aligned}$$

for E an effective divisor on Γ. Since ϕ is nondegenerate, we have $m_\phi(x) \geq 1$, so $D \sim x + E'$ for some effective E', as desired. This completes the proof of Theorem 3.3.2. □

In the next section, we will use the fact that the hyperelliptic involution in Theorem 3.3.2(ii) is unique. To prove uniqueness, we need the following fact.

Proposition 3.3.7. *[BN09, Proposition 5.5] If D and D' are degree 2 divisors on a metric graph Γ with $r(D) = r(D') = 1$, then $D \sim D'$.*

Proof. The proof of [BN09, Proposition 5.5] extends immediately to metric graphs. □

Corollary 3.3.8. *[BN09, Corollary 5.14] Let Γ be a 2-edge-connected hyperelliptic metric graph. Then there is a* unique *involution $i : \Gamma \to \Gamma$ such that Γ/i is a tree.*

Proof. Suppose i and i' are two such involutions. From the proof of Theorem 3.3.2, we see that for any $x \in \Gamma$, $x + i(x)$ and $x + i'(x)$ are both divisors of rank 1. Then by Proposition 3.3.7, $x + i(x) \sim x + i'(x)$, so $i(x) \sim i'(x)$. Since Γ is 2-edge-connected, it follows that $i(x) = i'(x)$. □

Our next goal is to prove Theorem 3.3.12, which extends Theorem 3.3.2 to graphs with bridges. Theorem 3.3.12 has no analogue in [BN09] because it takes advantage of the integer stretching factors present in morphisms of metric graphs which do not occur in combinatorial graphs.

Let us say that a point s in a metric graph Γ *separates* $y, z \in \Gamma$ if every y–z path in Γ passes through s, or equivalently, if y and z lie in different connected components of $\Gamma \setminus s$. A *cut vertex* is a point $x \in \Gamma$ such that $\Gamma \setminus x$ has more than one connected component.

Lemma 3.3.9. *Let Γ be a 2-edge-connected hyperelliptic metric graph with hyperelliptic involution i, and let x be a cut vertex of Γ. Then $i(x) = x$.*

Proof. Suppose $i(x) \neq x$. Since x is a cut vertex, we may pick some $y \in \Gamma$ such that x separates y and $i(x)$, so $i(x)$ does not separate x and y. Furthermore, since i is an automorphism, $i(x)$ separates $i(y)$ and x. Therefore $i(x)$ separates y and $i(y)$. But Lemma 3.3.1(i), applied to a rational function f with

$$\operatorname{div} f = y + i(y) - x - i(x),$$

yields the existence of a y—$i(y)$ path in $m(f)$ not passing through $i(x)$, contradiction. □

Lemma 3.3.10. *Let Γ be a metric graph, and let Γ' be the metric graph obtained from Γ by contracting all bridges. Let $\varphi : \Gamma \to \Gamma'$ be the natural contraction morphism. Given $D \in \operatorname{Div} \Gamma$, let $D' \in \operatorname{Div} \Gamma'$ be given by*

$$D' = \sum_{x \in \Gamma} D(x) \cdot \varphi(x).$$

Then $D \in \operatorname{Prin} \Gamma$ if and only if $D' \in \operatorname{Prin} \Gamma'$.

Proof. By induction, we may assume that Γ' was obtained by contracting a single bridge e in Γ, and that D is not supported on the interior of e. Let e have endpoints x_1 and x_2, and for $i = 1, 2$, let Γ_i denote the connected component of x_i in $\Gamma \setminus e$. Let $\Gamma'_i = \varphi(\Gamma_i)$, and let $x' = \varphi(x_1) = \varphi(x_2) = \varphi(e)$.

Now suppose $D \in \operatorname{Prin} \Gamma$, so let $D = \operatorname{div} f$ for f a rational function on Γ. Define f' on Γ as follows: let $f'(\varphi(y)) = f(y)$ for $y \in \Gamma_1$, and $f'(\varphi(y)) = f(y) + f(x_1) - f(x_2)$ for $y \in \Gamma_2$. This uniquely defines a rational function f' on Γ', and one can check that $\operatorname{div} f' = D'$.

Conversely, suppose $D' = \operatorname{div} f' \in \operatorname{Prin} \Gamma'$. Let

$$m = D(x_1) - \sum \text{outgoing slopes from } x' \text{ into } \Gamma'_1.$$

Now define f on Γ as follows: let $f(x) = f'(\varphi(x))$ if $x \in \Gamma_1$, let $f(x) = f'(\varphi(x)) + m \cdot l(e)$ if $x \in \Gamma_2$, and let f be linear with slope m along e. Then one can check that f is a rational function on Γ with $\operatorname{div} f = D$. □

It follows from Lemma 3.3.10 that the rank of divisors is preserved under contracting bridges. The argument can be found in [BN09, Corollaries 5.10, 5.11], and we will not repeat it here. In particular, we obtain

Corollary 3.3.11. *Let Γ be a metric graph, and let Γ' be the metric graph obtained from Γ by contracting all bridges. Then Γ is hyperelliptic if and only if Γ' is hyperelliptic.*

We can finally prove the main theorem of the section, which generalizes Theorem 3.3.2.

Theorem 3.3.12. *Let Γ be a metric graph with no points of valence 1, and let (G, l) denote its canonical loopless model. Then the following are equivalent:*

(i) Γ *is hyperelliptic.*

(ii) There exists an involution $i : G \to G$ such that G/i is a tree.

(iii) There exists a nondegenerate harmonic morphism of degree 2 from G to a tree, or $|V(G)| = 2$.

Proof. As in the proof of Theorem 3.3.2, we may assume that $|V(G)| > 2$. In fact, the proofs of (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) from Theorem 3.3.2 still hold here, since they do not rely on 2-edge-connectivity. We need only show (i) $\Rightarrow$ (ii).

Let Γ be a hyperelliptic metric graph with no points of valence 1, and let Γ' be obtained by contracting all bridges of Γ. Then Γ' is hyperelliptic by Corollary 3.3.11. Since Γ had no points of valence 1, the image of any bridge in Γ is a cut vertex $x \in \Gamma'$. By Lemma 3.3.9, all such vertices are fixed by the hyperelliptic involution i' on Γ'. Thus, we can extend i' uniquely to an involution i on Γ by fixing, pointwise, each bridge of Γ. Then i is also an involution on the canonical loopless model (G, l), and G/i is a tree whose contraction by the images of the bridges of G is the tree Γ'/i'. □

Remark 3.3.13. The requirement that Γ has no points of valence 1 is not important, because Corollary 3.3.11 allows us to contract such points away. Also, because of Definition 3.2.5 and the stability condition on tropical curves, we will actually never encounter points of valence 1 in the tropical context.

3.4 The hyperelliptic locus in tropical M_g

Which tropical curves of genus g are hyperelliptic? In this section, we will use the main combinatorial tool we have developed, Theorem 3.3.12, to construct the hyperelliptic locus H_g^{tr} in the moduli space M_g^{tr} of tropical curves. The space M_g^{tr} was defined in [BMV11] and computed explicitly for $g \leq 5$ in [Cha11a]. It is $(3g-3)$-dimensional and has the structure of a stacky fan, as defined in Definition 2.3.2.

It is a well-known fact that the classical hyperelliptic locus $\mathcal{H}_g \subset \mathcal{M}_g$ has dimension $2g-1$. Therefore, it is surprising that H_g^{tr} is actually $(3g-3)$-dimensional, as observed in [LPP11], especially given that tropicalization is a dimension-preserving operation in many important cases [BG84]. However, if one considers only 2-edge-connected tropical curves, then the resulting locus, denoted $H_g^{(2),\text{tr}}$, is in fact $(2g-1)$-dimensional. The combinatorics of $H_g^{(2),\text{tr}}$ is nice, too: in Theorem 3.4.9 we prove that the $(2g-1)$-dimensional cells are graphs that we call ladders of genus g. See Definition 3.4.7 and Figure 3.6. We then explicitly compute the spaces H_3^{tr}, $H_3^{(2),\text{tr}}$ and $H_4^{(2),\text{tr}}$.

§3.4.1 Construction of H_g^{tr} and $H_g^{(2),\mathrm{tr}}$

We will start by giving a general framework for constructing parameter spaces of tropical curves in which edges may be required to have the same length. We will see that the loci H_g^{tr} and $H_g^{(2),\mathrm{tr}}$ of hyperelliptic and 2-edge-connected hyperelliptic tropical curves, respectively, fit into this framework.

Recall that a **combinatorial type** of a tropical curve is a pair (G, w), where G is a connected, non-metric multigraph, possibly with loops, and $w : V(G) \to \mathbb{Z}_{\geq 0}$ satisfies the stability condition that if $w(v) = 0$ then v has valence at least 3. The **genus** of (G, w) is

$$|E(G)| - |V(G)| + 1 + \sum_{v \in V(G)} w(v).$$

Now, a **constrained type** is a triple (G, w, r), where (G, w) is a combinatorial type and r is an equivalence relation on the edges of G. We regard r as imposing the constraint that edges in the same equivalence class must have the same length. Given a constrained type (G, w, r) and a union of equivalence classes $S = \{e_1, \ldots, e_k\}$ of r, define the contraction along S as the constrained type (G', w', r'). Here, (G', w') is the combinatorial type obtained by contracting all edges in S. Contracting a loop, say at vertex v, means deleting it and adding 1 to $w(v)$. Contracting a nonloop edge, say with endpoints v_1 and v_2, means deleting that edge and identifying v_1 and v_2 to obtain a new vertex whose weight is $w(v_1) + w(v_2)$. Finally, r' is the restriction of r to $E(G) \setminus S$.

An automorphism of (G, w, r) is an automorphism φ of the graph G which is compatible with both w and r. Thus, for every vertex $v \in V(G)$, we have $w(\varphi(v)) = w(v)$, and for every pair of edges $e_1, e_2 \in E(G)$, we have that $e_1 \sim_r e_2$ if and only if $\varphi(e_1) \sim_r \varphi(e_2)$. Let N_r denote the set of equivalence classes of r. Note that the group of automorphisms $\mathrm{Aut}(G, w, r)$ acts naturally on the set N_r, and hence on the orthant $\mathbb{R}_{\geq 0}^{N_r}$, with the latter action given by permuting coordinates. We define $\overline{C(G, w, r)}$ to be the topological quotient space

$$\overline{C(G, w, r)} = \frac{\mathbb{R}_{\geq 0}^{N_r}}{\mathrm{Aut}(G, w, r)}.$$

Now, suppose $\mathcal{C}$ is a collection of constrained types that is closed under contraction. Define an equivalence relation $\sim$ on the points in the union

$$\coprod_{(G,w,r) \in \mathcal{C}} \overline{C(G, w, r)}$$

as follows. Regard a point $x \in \overline{C(G, w, r)}$ as an assignment of lengths to the edges of G such that r-equivalent edges have the same length. Given two points $x \in \overline{C(G, w, r)}$ and $x' \in \overline{C(G', w', r')}$, let $x \sim x'$ if the two tropical curves obtained by contracting all edges of length zero are isomorphic.

Now define the topological space $M_{\mathcal{C}}$ as

$$M_{\mathcal{C}} = \coprod \overline{C(G, w, r)} / \sim,$$

where the disjoint union ranges over all types $(G, w, r) \in \mathcal{C}$. Since $\mathcal{C}$ is closed under contraction, it follows that the points of $M_{\mathcal{C}}$ are in bijection with r-compatible assignments of positive lengths to $E(G)$ for some $(G, w, r) \in \mathcal{C}$.

Theorem 3.4.1. *Let $\mathcal{C}$ be a collection of constrained types, as defined above, that is closed under contraction. Then the space $M_{\mathcal{C}}$ is a stacky fan, with cells corresponding to types (G, w, r) in $\mathcal{C}$.*

Proof. For the definition of a stacky fan, see Definition 2.3.2. The proof of Theorem 3.4.1 is entirely analogous to the proof that the moduli space of genus g tropical curves is a stacky fan, so we refer the reader to Theorem 2.3.4. □

Remark 3.4.2. Note that the stacky fan $M_{\mathcal{C}}$ is not in general a stacky subfan of M_g^{tr}. Instead, it may include only parts of the cells of M_g^{tr}, since edges may be required to have equal length.

Our next goal is to define the collections $\mathcal{C}_g$ and $\mathcal{C}_g^2$ of hyperelliptic and 2-edge-connected hyperelliptic types of genus g. If (G, w) is a combinatorial type, let G^w denote the graph obtained from G by adding $w(v)$ loops at each vertex v. Let G^w_- be the loopless graph obtained by adding a vertex to the interior of each loop in G^w. Now let G and G' be loopless graphs. Then a morphism $\phi : G \to G'$ is a map of sets $V(G) \cup E(G) \to V(G') \cup E(G')$ such that

(i) $\phi(V(G)) \subseteq V(G')$,

(ii) if $e = xy$ is an edge of G and $\phi(e) \in V(G')$ then $\phi(x) = \phi(e) = \phi(y)$, and

(iii) if $e = xy$ is an edge of G and $\phi(e) \in E(G')$ then $\phi(e)$ is an edge between $\phi(x)$ and $\phi(y)$.

The morphism ϕ is harmonic if the map $\bar{\phi} : (G, \mathbf{1}) \to (G', \mathbf{1})$ of loopless models is a harmonic morphism, where $\mathbf{1}$ denotes the function assigning length 1 to every edge. The definitions above follow [BN09].

Definition 3.4.3. A constrained type (G, w, r) is **2-edge-connected hyperelliptic** if

(i) G is 2-edge-connected,

(ii) the loopless graph G^w_- has a nondegenerate harmonic morphism ϕ of degree 2 to a tree, or $|V(G)| = 2$, and

(iii) the relation r is induced by the fibers of ϕ on nonloop edges of G, and is trivial on the loops of G.

The type (G, w, r) is said to be **hyperelliptic** if r is the trivial relation on bridges and the type (G', w', r') obtained by contracting all bridges is 2-edge-connected hyperelliptic in the sense we have just defined.

Let $\mathcal{C}_g$ denote the collection of hyperelliptic types of genus g, and $\mathcal{C}_g^{(2)}$ the collection of 2-hyperelliptic types of genus g.

Proposition 3.4.4. *The collections $\mathcal{C}_g$ and $\mathcal{C}_g^2$ of hyperelliptic and 2-edge-connected hyperelliptic types defined above are closed under contraction.*

Proof. We will check that $\mathcal{C}_g^{(2)}$ is closed under contraction. If so, then $\mathcal{C}_g$ is too, since it was defined precisely according to contractions to types in $\mathcal{C}_g^{(2)}$. Note also that by definition, contraction preserves the genus g, whether the contracted edge was a loop or a nonloop edge.

Let (G, w, r) be a 2-hyperelliptic type, and let

$$\varphi : G_-^w \to T$$

be the unique harmonic morphism to a tree T that is either nondegenerate of degree 2, or T has 1 vertex and G_-^w has 2 vertices. Let S be a class of the relation r. Then there are three cases: either $S = \{e\}$ and e is a loop; or $S = \{e\}$, e is a nonloop edge, and $\varphi(e) \in V(T)$; or $S = \{e_1, e_2\}$ and $\varphi(e_1) = \varphi(e_2)$ is an edge of T.

Suppose $S = \{e\}$ and e is a loop. Then the contraction of (G, w, r) by e is $(G/e, w', r|_{E(G/e)})$, where w' is obtained from w by adding 1 at v. Then

$$(G/e)_-^{w'} = G_-^w$$

so the same morphism φ shows that $(G/e, w', r|_{E(G/e)})$ is a 2-hyperelliptic type.

Suppose $S = \{e\}$, e is a nonloop edge, and, regarding e as an edge of G_-^w, we have $\varphi(e) = v \in V(T)$. Let k be the number of edges in G that are parallel to e, and let T' be the tree obtained from T by adding k leaves at v. Then we may construct a morphism

$$\varphi' : (G/e)_-^{w'} \to T'$$

which in particular sends the k edges parallel to e to the k new leaf edges of T', and which has the required properties. See Figure 3.4.

Suppose $S = \{e_1, e_2\}$ where e_1 and e_2 are nonloop edges of G and $\varphi(e_1) = \varphi(e_2) = e \in E(T)$. Now, contracting $\{e_1, e_2\}$ in G_-^w may create new loops, say k of them, as illustrated in Figure 3.5. Let T' be the tree obtained from T by adding k leaves at either end of the edge e and then contracting e. Then we may construct a harmonic morphism

$$\varphi' : (G/\{e_1, e_2\})_-^w \to T'$$

which sends the k new loops to the k new leaves of T', and which has the required properties. $\square$

Definition 3.4.5. The space H_g^{tr} of tropical hyperelliptic curves of genus g is defined to be the space $M_{\mathcal{C}_g}$, where $\mathcal{C}_g$ is the collection of hyperelliptic combinatorial types of genus g defined above.

The space $H_g^{(2),\mathrm{tr}}$ of 2-edge-connected tropical hyperelliptic curves of genus g is defined to be the space $M_{\mathcal{C}_g^{(2)}}$, where $\mathcal{C}_g^{(2)}$ is the collection of 2-edge-connected hyperelliptic combinatorial types of genus g defined above.

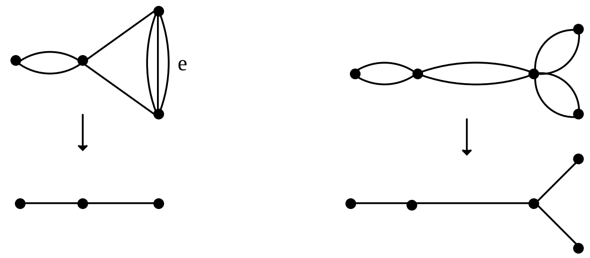


Figure 3.4: Contracting a vertical edge.

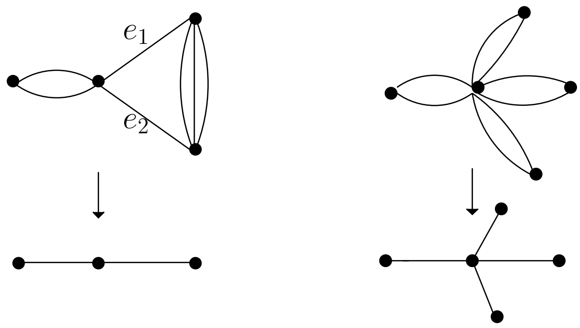


Figure 3.5: Contracting two horizontal edges.

Proposition 3.4.6.

(i) The points of $H_g^{(2),\mathrm{tr}}$ are in bijection with the 2-edge-connected hyperelliptic tropical curves of genus g.

(ii) The points of H_g^{tr} are in bijection with the hyperelliptic tropical curves of genus g.

Proof. Regard a point in $H_g^{(2),\mathrm{tr}}$ as an assignment $l' : E(G) \to \mathbb{R}$ of positive lengths to the edges of G, where (G, w, r) is some 2-edge-connected hyperelliptic type. Then there is a degree 2 harmonic morphism of loopless graphs $G^w_- \to T$ inducing the relation r on $E(G)$, or $|V(G)| = 2$ in which case (G, l') is clearly hyperelliptic. Then there is a a degree 2 harmonic morphism of loopless models $\phi : (G^w_-, l) \to (T, l'')$. Here, l agrees with l' on nonloop edges of G, is uniformly 1, say, on the $2w(v)$ added half-loops of G^w_- at each vertex, and is $l'(e)/2$ on the half-loops of G^w_- corresponding to each loop $e \in E(G)$. Furthermore, l'' is defined in the natural way, so that the harmonic morphism ϕ uses only the stretching factor 1.

Then by Theorem 3.3.2, the loopless model we constructed is 2-edge-connected hyperelliptic. Reversing the construction above shows that the map from $H_g^{(2),\mathrm{tr}}$ to the set of 2-edge-connected tropical hyperelliptic curves is surjective. On the other hand, it is injective by Corollary 3.3.8. Finally, part (ii) follows from part (i), using Corollary 3.3.11 and the fact that the hyperelliptic types are constructed by adding bridges to 2-edge-connected hyperelliptic types. $\square$

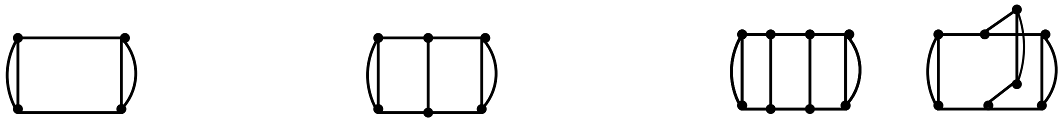


Figure 3.6: The ladders of genus 3, 4, and 5.

§3.4.2 Maximal cells of $H_g^{(2),\mathrm{tr}}$

Now we will prove that $H_g^{(2),\mathrm{tr}}$ is pure of dimension $2g-1$, and we will characterize its maximal cells. Recall that the dimension of a cell of the form $\mathbb{R}_{\geq 0}^N/G$ is equal to N, and the dimension of a stacky fan is the largest dimension of one of its cells. It is **pure** if all of its maximal cells have the same dimension.

First, we define the graphs which, as it turns out, correspond to the maximal cells of $H_g^{(2),\mathrm{tr}}$.

Definition 3.4.7. Let T be any nontrivial tree with maximum valence ≤ 3. Construct a graph $L(T)$ as follows. Take two disjoint copies of T, say with vertex sets $\{v_1,\ldots,v_n\}$ and $\{v'_1,\ldots,v'_n\}$, ordered such that v_i and v'_i correspond. Now, for each $i=1,\ldots,n$, consider the degree $d=\deg v_i=\deg v'_i$. If $d=1$, add two edges between v_i and v'_i. If $d=2$, add one edge between v_i and v'_i. The resulting graph $L(T)$ is a loopless connected graph, and by construction, every vertex has valence 3. We call a graph of the form $L(T)$ for some tree T a **ladder**. See Figure 3.6.

Lemma 3.4.8. *Let T be a tree on n vertices with maximum degree at most 3. Then the genus of the graph $L(T)$ is $n+1$.*

Proof. For $i=1,2,3$, let n_i denote the number of vertices in T of degree i. Then $L(T)$ has $2n$ vertices and $2(n-1)+2n_1+n_2$ edges; hence its genus is $g=2n_1+n_2-1$. Also, double-counting vertex-edge incidences in T gives $2(n-1)=n_1+2n_2+3n_3$. Adding, we have

$$g+2(n-1)=3n_1+3n_2+3n_3-1=3n-1,$$

so $g=n+1$. □

Theorem 3.4.9. *Fix $g\geq 3$. The space $H_g^{(2),\mathrm{tr}}$ of 2-edge-connected hyperelliptic tropical curves of genus g is a stacky fan which is pure of dimension $2g-1$. The maximal cells correspond to ladders of genus g.*

Remark 3.4.10. Note that the stacky fan $H_g^{(2),\mathrm{tr}}$ is naturally a closed subset of M_g^{tr}, but it is not a stacky subfan because sometimes edges are required to have equal lengths. So its stacky fan structure is more refined than that of M_g^{tr}. Compare Figure 3.3 and Figure 2.1.

Proof. Let $(G, w, r) \in \mathcal{C}_g^{(2)}$ be the type of a maximal cell in $H_g^{(2),\mathrm{tr}}$. Our goal is to show that $w \equiv 0$, that $G = L(T)$ for some tree T, and that r is the relation on $E(G)$ induced by the natural harmonic morphism

$$\varphi_T : L(T) \to T$$

of degree 2.

First, we observe that the dimension of the cell $\overline{C(G, w, r)}$ is, by construction, the number of equivalence classes of r. Now, we have, by definition of $\mathcal{C}_g^{(2)}$, a morphism

$$\varphi : G_-^w \to T$$

where either G_-^w has 2 vertices and T is trivial, or T is nontrivial and φ is a nondegenerate harmonic morphism of degree 2.

We immediately see that w is uniformly zero, for otherwise the cell $\overline{C(G^w, \underline{0}, r')}$ would contain $\overline{C(G, w, r)}$, contradicting maximality of the latter. Here, r' denotes the relation on $E(G^w)$ that is r on $E(G)$ and trivial on the added loops of G^w.

Let us also dispense with the special case that G_- has 2 vertices: if so, then G is the unique graph consisting of $g + 1$ parallel edges. But then our cell $\overline{C(G, \underline{0}, r)}$ is far from maximal; in fact, it is contained in each cell corresponding to a ladder.

Therefore we may assume that T is nontrivial and $\varphi : G_- \to T$ is a nondegenerate degree 2 harmonic morphism.

Next, we claim that every vertex $v \in V(G_-)$ has horizontal multiplicity $m_\varphi(v) = 1$. This claim shows in particular that G has no loops. In fact, the intuition behind the claim is simple: if $m_\varphi(v) = 2$, then split v into two adjacent vertices v' and v'' with $\varphi(v') = \varphi(v'')$ to make a larger cell. However, because G might a priori contain loops and thus G_- might have vertices of degree 2, we will need to prove the claim carefully. The proof is illustrated in Figure 3.7.

To prove the claim, suppose $v \in V(G_-)$ is such that $m_\varphi(v) = 2$. Let us assume that $\deg(v) > 2$, i.e. that v is not the midpoint of some loop l of G, for if it is, then pick the basepoint of l instead. Let $w = \varphi(v)$. Let $e_1, \ldots, e_k \in E(T)$ be the edges of T incident to w, and let $w_1, \ldots, w_k$ denote their respective endpoints that are different from w. Now, for each $i = 1, \ldots, k$, the set $\varphi^{-1}(e_i)$ consists of two edges of G_-; call them a_i and b_i. By renumbering them, we may assume that the first j of the pairs form loops in G. That is, if $1 \leq i \leq j$, then the edges a_i and b_i have common endpoints v_i and v, with $\deg(v_i) = 2$.

Let us construct a new graph G' and relation r' such that $(G, \underline{0}, r)$ is a contraction of $(G', \underline{0}, r')$. Replace the vertex v with two vertices, v_a and v_b, where v_a is incident to edges $a_1, \ldots, a_k$ and v_b is incident to edges $b_1, \ldots, b_k$. Now suppress the degree 2 vertices $v_1, \ldots, v_j$, so that for $1 \leq i \leq j$, the pair of edges $\{a_i, b_i\}$ becomes a single edge e_i. Finally, add an edge e between v_a and v_b. Call the resulting graph G'. See Figure 3.7.

By construction, the graph G' is a stable, 2-edge-connected graph with $G'/e = G$ and with the same genus as G. Let T' be the tree obtained from T by deleting $e_1, \ldots, e_j$. Then there is a natural degree 2 harmonic morphism

$$\varphi' : G'_- \to T'$$

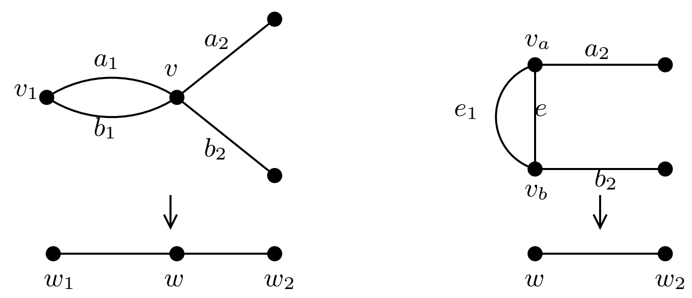


Figure 3.7: Splitting a vertex v of horizontal multiplicity 2 in the proof of Theorem 3.4.9.

as shown in Figure 3.7, inducing a relation r' on $E(G')$. Finally, note that $\overline{C(G',\underline{0},r')} \supseteq \overline{C(G,\underline{0},r)}$ is a larger cell in $H_g^{(2),\mathrm{tr}}$, contradiction. We have proved the claim that every $v \in V(G_-)$ satisfies $m_\varphi(v) = 1$. As a consequence, every vertex of T has precisely two preimages under φ. Hence $G = G_-$ has no loops.

Next, we claim that G consists of two disjoint copies of T, say T_1 and T_2, that are sent by φ isomorphically to T, plus some vertical edges. The intuition is clear: the harmonicity of φ implies that the horizontal edges of G form a twofold cover of T, which must be two copies of T since T is contractible. More formally, pick any $v \in V(T)$ and let $\varphi^{-1}(v) = \{v_1, v_2\}$. For $i = 1, 2$, let T_i be the union of all paths from v_i using only horizontal edges of G. Since no horizontal edges leave T_i, we have $\varphi(T_i) = T$, and since $m_\varphi(x) = 1$ for all $x \in V(G)$, it follows that T_1 and T_2 are disjoint and that

$$\varphi|_{T_i} : T \to T_i$$

is harmonic of degree 1 and nondegenerate, hence an isomorphism. Finally, since each edge $e \in E(T)$ has only two preimages, T_1 and T_2 account for all of the horizontal edges in G, and hence only vertical edges are left.

Where can the vertical edges of G be? Let $v \in V(T)$ be any vertex. If v has degree 1, then G must have at least two vertical edges above v, and if v has degree 2, then G must have at least one vertical edge above v, for otherwise G would not be stable. In fact, we claim that G cannot have any vertical edges other than the aforementioned ones. Otherwise, G would have vertices of degree at least 4, and splitting these vertices horizontally produces a larger graph G', the cell of which contains $\overline{C(G,w,r)}$. See Figure 3.8.

For the same reason, T cannot have vertices of degree ≥ 4. We have shown that every maximal cell of $H_g^{(2),\mathrm{tr}}$ is a ladder of genus g. If $L(T)$ is a ladder of genus g with tree T, then T has $g-1$ nodes by Lemma 3.4.8, and the dimension of the cell of $L(T)$ is $(g-2)+2n_1+n_2 = (g-2)+(g+1) = 2g-1$. So all genus g ladders yield equidimensional cells and none contains another. Therefore the genus g ladders are precisely the maximal cells of $H_g^{(2),\mathrm{tr}}$, and each cell has dimension $2g-1$. □

Corollary 3.4.11. *The maximal cells of $H_g^{(2),\mathrm{tr}}$ are in bijection with the trees on $g-1$ vertices with maximum degree 3.*

Proof. Each ladder $L(T)$ of genus g corresponds to the tree T. □

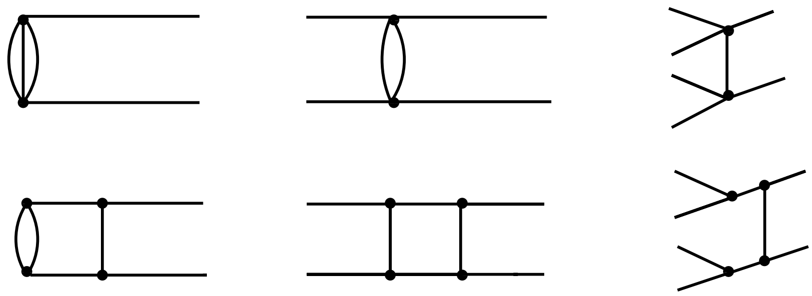


Figure 3.8: Horizontal splits in G above vertices in T of degrees 1, 2, and 3, respectively.

Corollary 3.4.12. *Let $g \geq 3$. The number of maximal cells of $H_g^{(2),\mathrm{tr}}$ is equal to the $(g-2)^{nd}$ term of the sequence*

$$1, 1, 2, 2, 4, 6, 11, 18, 37, 66, 135, 265, 552, 1132, 2410, 5098, \ldots$$

Proof. This is sequence A000672 in [OEIS], which counts the number of trees of maximum degree 3 on a given number of vertices. Cayley obtained the first twelve terms of this sequence in 1875 [Cay75]; see [RS99] for a generating function. See Figure 3.6 for the ladders of genus 3,4, and 5, corresponding to the first three terms 1, 1, and 2 of the sequence. □

Remark 3.4.13. As pointed out in [LPP11], the stacky fan H_g^{tr} is full-dimensional for each g. Indeed, take any 3-valent tree with g leaves, and attach a loop at each leaf. The resulting genus g graph indexes a cell of dimension $3g-3$. Thus it is more natural to consider the sublocus $H_g^{(2),\mathrm{tr}}$, at least from the point of view of dimension.

One can use Theorem 3.4.9 to compute $H_g^{(2),\mathrm{tr}}$ and H_g^{tr}, the latter by adding all possible bridges to the cells of $H_g^{(2),\mathrm{tr}}$. We will explicitly compute $H_3^{(2),\mathrm{tr}}$, H_3^{tr}, and $H_4^{(2),\mathrm{tr}}$ next. The space H_4^{tr} is too large to compute by hand.

§3.4.3 Computations

We now apply Theorem 3.4.9 to explicitly compute the spaces $H_3^{(2),\mathrm{tr}}$, H_3^{tr}, and $H_4^{(2),\mathrm{tr}}$.

Theorem 3.4.14. *The moduli space $H_3^{(2),\mathrm{tr}}$ of 2-edge-connected tropical hyperelliptic curves has 11 cells and f-vector*

$$(1, 2, 2, 3, 2, 1).$$

Its poset of cells is shown in Figure 3.3.

Proof. The space M_3^{tr} was computed explicitly in Theorem 2.2.12. A poset of genus 3 combinatorial types is shown in Figure 2.1. Now use Theorem 3.4.9. □

Theorem 3.4.15. *The moduli space H_3^{tr} of tropical typerelliptic curves has 36 cells and f-vector*

$$(1, 3, 6, 11, 9, 5, 1).$$

Its poset of cells is shown in Figure 3.2.

Proof. Apply Theorem 3.4.14 and Lemma 3.3.10. □

Theorem 3.4.16. *The moduli space $H_4^{(2),\mathrm{tr}}$ of 2-edge-connected tropical hyperelliptic curves of genus 4 has 31 cells and f-vector*

$$(1, 2, 5, 6, 7, 6, 3, 1).$$

Proof. The space M_4^{tr} was computed explicitly in Theorem 2.2.12, so we can apply Theorem 3.4.9 directly to it. □

3.5 Berkovich skeletons and tropical plane curves

In this section, we are interested in hyperelliptic curves in the plane over a complete, nonarchimedean valuated field K. Every such curve X is given by a polynomial of the form

$$P = y^2 + f(x)y + h(x)$$

for $f, h \in K[x]$. We suppose that the Newton polygon of P is the lattice triangle in $\mathbb{R}^2$ with vertices $(0, 0), (2g + 2, 0)$, and $(0, 2)$. We write $\Delta_{2g+2,2}$ for this triangle. Since $\Delta_{2g+2,2}$ has g interior lattice points, the curve X has genus g. The main theorem in this section, Theorem 3.5.3, says that under certain combinatorial conditions, the Berkovich skeleton of X is a ladder over the path P_{g-1} on $g-1$ vertices. The main tool is [BPR11, Corollary 6.27], which states that under nice conditions, an embedded tropical plane curve, equipped with a lattice length metric, faithfully represents the Berkovich skeleton of X.

Recall that an embedded tropical curve $T \subseteq \mathbb{R}^2$ can be regarded as a metric space with respect to **lattice length**. Indeed, if e is a 1-dimensional segment in T, then it has a rational slope. Then a ray from the origin with the same slope meets some first lattice point $(p, q) \in \mathbb{Z}^2$. Then the length of e in the metric space T is its Euclidean length divided by $\sqrt{p^2 + q^2}$. By a **standard ladder** of genus g, we mean the graph $L(T)$ defined above for T a path on $g - 1$ vertices. We will denote this graph L_g. Figure 3.6 shows the standard ladders of genus 3, 4, and 5, as well as a nonstandard ladder of genus 5.

In Theorem 3.5.3, we will consider unimodular triangulations of $\Delta_{2g+2,2}$ with bridgeless dual graph. Such triangulations contain a maximally triangulated trapezoid of height 1 in their bottom half (see Figure 3.9), so we study height 1 trapezoids in the next lemma.

Lemma 3.5.1. *Fix $a, b, c, d \in \mathbb{Z}$ with $a < b$ and $c < d$, and let Q be the lattice polytope with vertices $(a, 0), (b, 0), (c, 1)$, and $(d, 1)$. Then any unimodular triangulation of Q has*

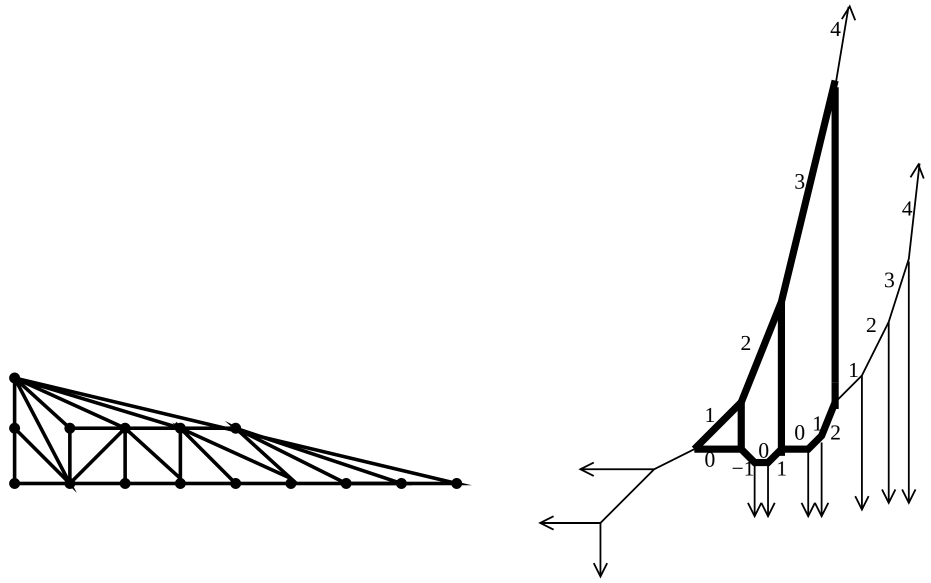


Figure 3.9: On the left, a unimodular triangulation N of $\Delta_{2g+2,2}$ that happens to include the edge $\{(0,2),(1,0)\}$ but not the edge $\{(0,2),(2g+1,0)\}$. On the right, the tropical plane curve dual to N has a core (shown in thick edges) that is a standard genus 3 ladder. The numbers on the edges indicate their slopes. We have chosen $g = 3$ in this illustration.

$N = b - a + d - c + 1$ nonhorizontal edges $w_1, \ldots, w_N$ satisfying $w_1 = \{(a,0),(c,1)\}$, $w_N = \{(b,0),(d,1)\}$, and for each $i = 1, \ldots, N-1$, if w_i has the form $\{(x,0),(y,1)\}$ for some $x, y \in \mathbb{Z}$, then

$$w_{i+1} = \{(x+1,0),(y,1)\} \text{ or } w_{i+1} = \{(x,0),(y+1,1)\}.$$

Thus there are $\binom{b-a+d-c}{b-a}$ such triangulations.

Proof. The edge $\{(a,0),(c,1)\}$ is contained in some unimodular triangle Δ, so either $\{(a+1,0),(c,1)\}$ or $\{(a,0),(c+1,1)\}$ must be present. Now induct, replacing Q by $Q - \Delta$. $\square$

Definition 3.5.2. Let $T \subseteq \mathbb{R}^n$ be an embedded tropical curve with $\dim H_1(T,\mathbb{R}) > 0$. We define the **core** of T to be the smallest subspace $Y \subseteq T$ such that there exists a deformation retract from T to Y. See Figure 3.9.

Theorem 3.5.3. *Let $X \subseteq \mathbb{T}^2$ be a plane hyperelliptic curve of genus $g \geq 3$ over a complete nonarchimedean valuated field K, defined by a polynomial of the form $P = y^2 + f(x)y + h(x)$. Let $\widehat{X}$ be its smooth completion. Suppose the Newton complex of P is a unimodular triangulation of the lattice triangle $\Delta_{2g+2,2}$, and suppose that the core of* $\operatorname{Trop} X$ *is bridgeless. Then the skeleton Σ of the Berkovich analytification $\widehat{X}^{\mathrm{an}}$ is a standard ladder of genus g whose opposite sides have equal length.*

Proof. Let Σ' denote the core of $T = \operatorname{Trop} X$, and let N denote the Newton complex of f, to which T is dual. Note that the lattice triangle $\Delta_{2g+2,2}$ has precisely g interior lattice points $(1,1),\ldots,(g,1)$. Now, we claim that N cannot have any edge from $(0,2)$ to any point on the x-axis such that this edge partitions the interior lattice points nontrivially. (For example, the edge $\{(0,2),(3,0)\}$ cannot be in N). Indeed, if such an edge were present in N, then the edge in T dual to it would be a bridge in Σ', contradiction. Then since all the triangles incident to the interior lattice points are unimodular, it follows that each edge

$$s_1 = \{(1,1),(2,1)\},\ldots,s_{g-1} = \{(g-1,1),(g,1)\}$$

is in N, and for the same reason, that each edge

$$\{(0,2),(1,1)\},\ldots,\{(0,2),(g,1)\}$$

is in N. For $i = 1,\ldots,g-1$, the 2-face of N above (respectively, below) s_i corresponds to a vertex of T; call this vertex V_i (respectively W_i).

Now, either the edge $\{(0,2),(1,0)\}$ is in N, or it is not. If it is, then unimodularity implies that the edges $\{(0,1),(1,0)\}$ and $\{(1,1),(1,0)\}$ are also in N. If not, then unimodularity again implies that $\{(0,1),(1,1)\}$ is in N. Similarly, if $\{(0,2),(2g+1,0)\}$ is in N, then the edges $\{(g,1),(2g+1,0)\}$ and $\{(g-1,1),(2g+1,0)\}$ are in N. In each of these cases, N contains a unimodular triangulation of the trapezoidal region with vertices

$$(a,0),(b,0),(1,1),(g,1)$$

where we take $a = 0$ or $a = 1$ according to whether the edge $\{(0,2),(1,0)\}$ is in N, and similarly we take $b = 2g+1$ or $b = 2g+2$ according to whether the edge $\{(0,2),(2g+1,0)\}$ is in N. For example, in Figure 3.9, we take $a = 1$ and $b = 2g+2$.

In each case, we see that the dual graph to N is a standard ladder of genus g, with vertices $V_1,\ldots,V_{g-1},W_1,\ldots,W_{g-1}$. In particular, there are two paths in T from V_1 to W_1 not passing through any other V_i or W_i, and similarly, there are two paths from V_{g-1} to W_{g-1}.

Moreover, we claim that for each $i = 1,\ldots,g-2$, the lattice-lengths of the paths in Σ' between V_i and V_{i+1} and between W_i and W_{i+1} are equal. Indeed, for each i, write $V_i = (V_{i,1},V_{i,2})$ and $W_i = (W_{i,1},W_{i,2})$. Notice that $V_{i,1} = W_{i,1}$ for each i, because each segment $s_i \in N$ is horizontal so its dual in T is vertical. Furthermore, the path from W_i to W_{i+1} in Σ' consists of a union of segments of integer slopes, so their lattice-length distance is the horizontal displacement $W_{i+1,1} - W_{i,1}$. Similarly, the distance in Σ' from V_i to V_{i+1} is $V_{i+1,1} - V_{i,1} = W_{i+1,1} - W_{i,1}$, as desired. So Σ' is a standard ladder of genus g, with opposite sides of equal length. Finally, we apply [BPR11, Corollary 6.27] to conclude that the tropicalization map on the Berkovich analytification $\widehat{X}^{\mathrm{an}}$ of $\widehat{X}$ induces an isometry of the Berkovich skeleton Σ of $\widehat{X}^{\mathrm{an}}$ onto Σ'. This completes the proof of Theorem 3.5.3. □

In the proof above, we gained a refined combinatorial understanding of the unimodular triangulations of $\Delta_{2g+2,2}$, which we make explicit in the following corollary.

Corollary 3.5.4. *Consider unimodular triangulations of* $\Delta_{2g+2,2}$ *whose dual complex contains a bridgeless connected subgraph of genus* g*. There are* $\binom{3g+3}{g+1}$ *such triangulations that use neither the edge* $e_1 = \{(0,2),(1,0)\}$ *nor the edge* $e_2 = \{(0,2),(2g+1,0)\}$*, there are* $2 \cdot \binom{3g+1}{g}$ *triangulations using one of* e_1 *and* e_2*, and there are* $\binom{3g-1}{g-1}$ *triangulations using both* e_1 *and* e_2*.*

Proof. Use the proof of Theorem 3.5.3, which characterizes precisely which triangulations can occur, and Lemma 3.5.1. □

Chapter 4

Elliptic curves in honeycomb form

The content of this chapter is the paper [CS12] joint with Bernd Sturmfels and has been submitted to the Proceedings of the CIEM Workshop on tropical geometry. The version here incorporates only minor changes.

4.1 Introduction

Suppose K is a field with a nonarchimedean valuation $\mathrm{val} : K^* \to \mathbb{R}$, such as the rational numbers $\mathbb{Q}$ with their p-adic valuation for some prime $p \geq 5$ or the rational functions $\mathbb{Q}(t)$ with the t-adic valuation. Throughout, we shall assume that the residue field of K has characteristic different from 2 and 3.

We consider a ternary cubic polynomial whose coefficients c_{ijk} lie in K:

$$\begin{aligned} f(x,y,z) \;=\; & c_{300}x^3 + c_{210}x^2y + c_{120}xy^2 + c_{030}y^3 + c_{021}y^2z \\ & + c_{012}yz^2 + c_{003}z^3 + c_{102}xz^2 + c_{201}x^2z + c_{111}xyz. \end{aligned} \tag{4.1}$$

Provided the discriminant of $f(x, y, z)$ is non-zero, this cubic represents an elliptic curve E in the projective plane $\mathbb{P}^2_K$. The group $\mathrm{GL}(3, K)$ acts on the projective space $\mathbb{P}^9_K$ of all cubics. The field of rational invariants under this action is generated by the familiar *j-invariant*, which we can write explicitly (with coefficients in $\mathbb{Z}$) as

$$j(f) \;=\; \frac{\text{a polynomial of degree 12 in the } c_{ijk} \text{ having 1607 terms}}{\text{a polynomial of degree 12 in the } c_{ijk} \text{ having 2040 terms}}. \tag{4.2}$$

The Weierstrass normal form of an elliptic curve can be obtained from $f(x, y, z)$ by applying a matrix in $\mathrm{GL}(3, \overline{K})$. From the perspective of tropical geometry, however, the Weierstrass form is too limiting: its tropicalization never has a cycle. One would rather have a model for plane cubics whose tropicalization looks like the graphs in Figures 4.1, 4.3 and 4.5. If this holds then we say that f is in *honeycomb form*. Cubic curves in honeycomb form are the central object of interest in this chapter.

Honeycomb curves of arbitrary degree were studied in [Spe05a, §5]; they are dual to the standard triangulation of the Newton polygon of f. For cubics in honeycomb form,

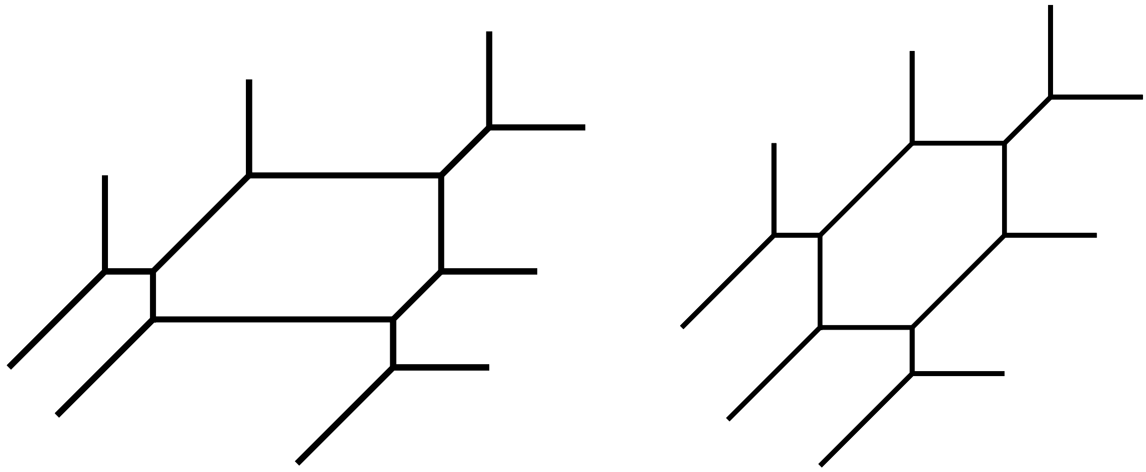


Figure 4.1: Tropicalizations of plane cubic curves in honeycomb form. The curve on the right is in symmetric honeycomb form; the one on the left is not symmetric.

by [KMM08], the lattice length of the hexagon equals $-\mathrm{val}(j(f))$. Moreover, by [BPR11], a honeycomb cubic faithfully represents a subgraph of the Berkovich curve E^{an}.

A standard Newton subdivision argument [MS10] shows that a cubic f is in honeycomb form if and only if the following nine scalars in K have positive valuation:

$$\frac{c_{021}c_{102}}{c_{111}c_{012}}, \frac{c_{012}c_{120}}{c_{111}c_{021}}, \frac{c_{201}c_{012}}{c_{111}c_{102}}, \frac{c_{210}c_{021}}{c_{111}c_{120}}, \frac{c_{102}c_{210}}{c_{111}c_{201}}, \frac{c_{120}c_{201}}{c_{111}c_{210}}, \tag{4.3}$$

$$\frac{c_{111}c_{003}}{c_{012}c_{102}}, \frac{c_{111}c_{030}}{c_{021}c_{120}}, \frac{c_{111}c_{300}}{c_{201}c_{210}}. \tag{4.4}$$

If the six ratios in (4.3) have the same positive valuation, and also the three ratios in (4.4) have the same positive valuation, then we say that f is in *symmetric honeycomb form*. So f is in symmetric honeycomb form if and only if the lattice lengths of the six sides of the hexagon are equal, and the lattice lengths of the three bounded segments coming off the hexagon are also equal, as in Figure 4.1 on the right.

Our contributions here are as follows. In Section 2 we focus on symmetric honeycomb cubics. We present a symbolic algorithm whose input is an arbitrary cubic f with $\mathrm{val}(j(f)) < 0$ and whose output is a 3×3-matrix M such that $f \circ M$ is in symmetric honeycomb form. This answers a question raised by Buchholz and Markwig (cf. [Buc10, §6]). We pay close attention to the arithmetic of the entries of M. Our key tool is the relationship between honeycombs and the *Hesse pencil* [AD09, Nob11]. Results similar to those in Section 2 were obtained independently by Helminck [Hel11].

Section 3 discusses the *Tate parametrization* [Sil94] of elliptic curves using theta functions. Our approach is similar to that used by Speyer in [Spe07] for lifting tropical curves. We present an analytic characterization of honeycomb cubics with prescribed j-invariant, and we give a numerical algorithm for computing such cubics.

Section 4 explains a combinatorial rule for the *tropical group law* on a honeycomb cubic C. Our object of study is the tropicalization of the surface $\{u, v, w \in C^3 \mid u \star v \star w = \mathrm{id}\} \subset (\mathbb{P}^2)^3$. Here $\star$ denotes multiplication on C. We explain how to compute this tropical surface in $(\mathbb{R}^2)^3$. See Corollary 4.4.4 for a concrete instance. Our results complete the partially defined group law found by Vigeland [Vig09].

Practitioners of computational algebraic geometry are well aware of the challenges involved in working with algebraic varieties over a valued field K. One aim of this article is to demonstrate how these challenges can be overcome in practice, at least for the basic case of elliptic curves. In that sense, this chapter is a computational algebra supplement to the work of Baker, Payne and Rabinoff [BPR11].

Many of our methods have been implemented in MATHEMATICA. In our test implementations, the input data are assumed to lie in the field $K = \mathbb{Q}(t)$, and scalars in $\overline{K}$ are represented as truncated Laurent series with coefficients in $\overline{\mathbb{Q}}$. This is analogous to the representation of scalars in $\mathbb{R}$ by floating point numbers.

4.2 Symmetric cubics

We begin by establishing the existence of symmetric honeycomb forms for elliptic curves whose j-invariant has negative valuation. Consider a symmetric cubic

$$g \;=\; a \cdot (x^3 + y^3 + z^3) \,+\, b \cdot (x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2) \,+ xyz. \tag{4.5}$$

The conditions in (4.3)-(4.4) imply that g is in symmetric honeycomb form if and only if

$$\mathrm{val}(a) \;>\; 2 \cdot \mathrm{val}(b) \;>\; 0. \tag{4.6}$$

Our aim in this section is to transform arbitrary cubics (4.1) to symmetric cubics in honeycomb form. In other words, we seek to achieve both (4.5) and (4.6). Note that $a = 0$ is allowed by the valuation inequalities (4.6), but b must be non-zero in (4.5). The classical *Hesse normal form* of [AD09, Lemma 1], whose tropicalization was examined recently by Nobe [Nob11], is therefore ruled out by the honeycomb condition.

Proposition 4.2.1. *Given any two scalars ι and a in K with $\mathrm{val}(\iota) < 0$ and $\mathrm{val}(a) + \mathrm{val}(\iota) > 0$, there exist precisely six elements b in the algebraic closure $\overline{K}$, defined by an equation of degree 12 over K, such that the cubic g above has j-invariant $j(g) = \iota$ and is in symmetric honeycomb form.*

Proof. First, consider the case $a = 0$, so that $\mathrm{val}(a) = \infty$. By specializing (4.2), we deduce that the j-invariant of $g = b(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2) + xyz$ is

$$j(g) \;=\; \frac{(48b^3 - 24b^2 + 1)^3}{b^6(2b-1)^3(3b-1)^2(6b+1)}. \tag{4.7}$$

Our task is to find $b \in \overline{K}$ such that $j(g) = \iota$. The expansion of this equation equals

$$\begin{aligned} &432\iota b^{12} - 864\iota b^{11} + 648\iota b^{10} - (208\iota + 110592)b^9 + (15\iota + 165888)b^8 \\ &+ (6\iota - 82944)b^7 - (\iota - 6912)b^6 + 6912b^5 - 1728b^4 - 144b^3 + 72b^2 - 1 \;=\; 0. \end{aligned} \tag{4.8}$$

We examine the Newton polygon of this equation. It is independent of K because the characteristic of the residue field of K is not 2 or 3. Since ι has negative valuation, we see that (4.8) has six solutions $b \in \overline{K}$ with $\mathrm{val}(b) = 0$ and six solutions b with $\mathrm{val}(b) = -\mathrm{val}(\iota)/6$. The latter six solutions are indexed by the choice of a sixth root of ι^{-1}. They share the following expansion as a Laurent series in $\iota^{-1/6}$:

$$\begin{aligned} b \;=\; & \iota^{-1/6} + \iota^{-1/3} - 5\iota^{-1/2} - 7\iota^{-2/3} + 30\iota^{-5/6} + 43\iota^{-1} - 60\iota^{-7/6} - 15\iota^{-4/3} \\ & -731\iota^{-3/2} - 1858\iota^{-5/3} + 11676\iota^{-11/6} + 22091\iota^{-2} - 30612\iota^{-13/6} + \cdots \end{aligned}$$

These six values of b establish the assertion in Proposition 4.2.1 when $a = 0$.

Now suppose $a \neq 0$. Then our equation $j(g) = \iota$ has the more complicated form

$$\frac{(6a-1)^3(72ab^2 - 48b^3 - 36a^2 + 24b^2 - 6a - 1)^3}{(3a+6b+1)(3a-3b+1)^2(9a^3 - 3ab^2 + 2b^3 - 3a^2 - b^2 + a)^3} \;=\; \iota. \tag{4.9}$$

Our hypotheses on K and a ensure that $\mathrm{val}(a)$ is large enough so as to not interfere with the lowest order terms when solving this equation for b. In particular, the degree 12 equation in the unknown b with coefficients in K resulting from (4.9) has the same Newton polygon as equation (4.8). As before, this equation has 12 solutions $b = b(\iota, a)$ that are scalars in $\overline{K}$, and six of the solutions satisfy $\mathrm{val}(b) = 0$ while the other six satisfy $\mathrm{val}(b) = -\mathrm{val}(\iota)/6$. The latter six establish our assertion. □

We have proved the existence of a symmetric honeycomb form for any nonsingular cubic whose j-invariant has negative valuation. Our main goal in what follows is to describe an algorithm for computing a 3×3 matrix that transforms a given cubic into that form. Our method is to compute the nine inflection points of each cubic and find a suitable projective transformation that takes one set of points to the other. Computing the inflection points is a relatively easy task in the special case of symmetric cubics. The result of that computation is the following lemma.

Lemma 4.2.2. *Let C be a nonsingular cubic curve defined over K by a symmetric polynomial g as in (4.5), fix a primitive third root of unity ξ in $\overline{K}$, and set*

$$\omega \;=\; \frac{3a+6b+1}{-3a+3b-1}. \tag{4.10}$$

Then the nine inflection points of C in $\mathbb{P}^2$ are given by the rows of the matrix

$$A_\omega \;=\; \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1+\omega^{1/3} & 1+\xi\omega^{1/3} & 1+\xi^2\omega^{1/3} \\ 1+\xi\omega^{1/3} & 1+\xi^2\omega^{1/3} & 1+\omega^{1/3} \\ 1+\xi^2\omega^{1/3} & 1+\omega^{1/3} & 1+\xi\omega^{1/3} \\ 1+\xi\omega^{1/3} & 1+\omega^{1/3} & 1+\xi^2\omega^{1/3} \\ 1+\omega^{1/3} & 1+\xi^2\omega^{1/3} & 1+\xi\omega^{1/3} \\ 1+\xi^2\omega^{1/3} & 1+\xi\omega^{1/3} & 1+\omega^{1/3} \end{bmatrix}. \tag{4.11}$$

The matrix A_ω has precisely the following vanishing 3×3-minors:

$$123,\ 147,\ 159,\ 168,\ 249,\ 258,\ 267,\ 348,\ 357,\ 369,\ 456,\ 789. \tag{4.12}$$

This list of triples is the classical *Hesse configuration* of 9 points and 12 lines.

Next, for an arbitrary nonsingular cubic f as in (4.1), the nine inflection points can be expressed in radicals in the ten coefficients $c_{300}, c_{210}, \ldots, c_{111}$, since their Galois group is solvable [AD09, §4]. How can we compute these inflection points? Consider the *Hesse pencil* $\mathrm{HP}(f) = \{s \cdot f + s' \cdot H_f : s, s' \in K\}$ of plane cubics spanned by f and its Hessian H_f. Each cubic in $\mathrm{HP}(f)$ passes through the nine inflection points of f since both f and H_f do, and in fact every such cubic is in $\mathrm{HP}(f)$. In particular, the four systems of three lines through the nine points are precisely the four reducible members of $\mathrm{HP}(f)$. Indeed, if $g = l \cdot h \in \mathrm{HP}(f)$ where l is a line passing through three inflection points, then h passes through the remaining six and thus must itself be two lines by Bézout's Theorem. So we may compute any two of the four such systems of three lines, and take pairwise intersections of their lines to obtain the nine desired inflection points. This algorithm was extracted from Salmon's book [Sal79], and it runs in exact arithmetic. We now make it more precise.

We introduce four unknowns u, v, w, s, and we consider the condition that a cubic $s \cdot f + H_f$ is divisible by the linear form $ux + vy + wz$. That condition translates into a system of polynomials that are cubic in the unknowns u, v, w and linear in s. We derive this system by specializing the following universal solution, found by a `Macaulay2` computation.

Lemma 4.2.3. *The condition that a linear form $ux + vy + wz$ divides a cubic (4.1) is given by a prime ideal in the polynomial ring $K[u, v, w, c_{300}, c_{210}, \ldots, c_{003}]$ in 13 unknowns. This prime ideal is of codimension 4 and degree 28. It has 96 minimal generators, namely 25 quartics, 15 quintics, 21 sextics and 35 octics.*

Consider the polynomials in u, v, w, s that are obtained by specializing the c_{ijk} in the 96 ideal generators above to the coefficients of $s \cdot f + H_f$. After permuting coordinates if necessary, we may set $w = 1$ and work with the resulting polynomials in u, v, s. The lexicographic Gröbner basis of their ideal has the special form

$$\left\{\, \underline{s^4} + \alpha_2 s^2 + \alpha_1 s + \alpha_0\,,\ \underline{v^3} + \beta_2(s)v^2 + \beta_1(s)v + \beta_0(s)\,,\ \underline{u} + \gamma(s, v) \,\right\},$$

where the α_i are constants in K, the β_j are univariate polynomials, and γ is a bivariate polynomial. These equations have 12 solutions $(s_i, u_{ij}, v_{ij}) \in (\overline{K})^3$ where $i = 1, 2, 3, 4$ and $j = 1, 2, 3$. The leading terms in the Gröbner basis reveal that the coordinates of these solutions can be expressed in radicals over K, since we need only solve a quartic in s, a cubic in v, and a degree 1 equation in u, in that order.

For each of the nine choices of $j, k \in \{1, 2, 3\}$, the two linear equations

$$u_{1j}x + v_{1j}y + z \;=\; u_{2k}x + v_{2k}y + z \;=\; 0$$

have a unique solution $(b_1^{jk} : b_2^{jk} : b_3^{jk})$ in the projective plane $\mathbb{P}^2$ over $\overline{K}$. We can write its coordinates b_l^{jk} in radicals over K. Let B denote the 9×3-matrix whose rows are the vectors $(b_1^{jk}, b_2^{jk}, b_3^{jk})$ for $j, k \in \{1, 2, 3\}$. While the entries of B have been written in radicals over K, they can also be represented as formal series in the completion of $\overline{K}$, which we can approximate by a suitable truncation.

To summarize our discussion up to this point: we have shown how to compute the inflection points of a plane cubic, and we have written them as the rows of a 9×3-matrix B whose entries are expressed in radicals over K. For the special case of symmetric cubics, the specific 9×3-matrix A_ω in (4.11) gives the inflection points.

Now, we return to our main goal. Suppose we are given a nonsingular ternary cubic f whose j-invariant $\iota = j(f)$ has negative valuation. We then choose $a, b \in \overline{K}$ as prescribed in Proposition 4.2.1, and we define ω by the ratio in (4.10). The scalars a and b define a symmetric honeycomb cubic g as in (4.5). Let $\mathcal{A}_\omega$ and $\mathcal{B}$ denote the sets of inflection points of the cubic curves $V(g)$ and $V(f)$ respectively. Thus $\mathcal{A}_\omega$ and $\mathcal{B}$ are unordered 9-element subsets of $\mathbb{P}^2$, represented by the rows of our matrices A_ω and B. There exists an automorphism ϕ of $\mathbb{P}^2$ taking $V(f)$ to $V(g)$, since their j-invariants agree. Clearly, any such automorphism ϕ takes $\mathcal{B}$ to $\mathcal{A}_\omega$.

We write $\mathcal{B} = \{b_1, b_2, \ldots, b_9\}$, where the labeling is such that b_i, b_j, b_k are collinear in $\mathbb{P}^2$ if and only if ijk appears on the list (4.12). The automorphism group of the Hesse configuration (4.12) has order $9 \cdot 8 \cdot 6 = 432$. Hence precisely 432 of the 9! possible bijections $\mathcal{B} \to \mathcal{A}_\omega$ respect the collinearities of the inflection points. For each such bijection $\pi_i : \mathcal{B} \to \mathcal{A}_\omega$, $i = 1, 2, \ldots, 432$, we associate a unique projective transformation $\sigma_i : \mathbb{P}^2 \to \mathbb{P}^2$ by requiring that $\sigma_i(b_1) = \pi_i(b_1)$, $\sigma_i(b_2) = \pi_i(b_2)$, $\sigma_i(b_4) = \pi_i(b_4)$ and $\sigma_i(b_5) = \pi_i(b_5)$. We emphasize that σ_i may or may not induce a bijection $\mathcal{B} \to \mathcal{A}_\omega$ on all nine points. We write M_i for the unique (up to scaling) 3×3-matrix with entries in $\overline{K}$ that represents the projective transformation σ_i.

The simplest version of our algorithm constructs all matrices $M_1, M_2, \ldots, M_{432}$. One of these matrices, say M_j, represents the automorphism ϕ of $\mathbb{P}^2$ in the second-to-last paragraph. The ternary cubics $f \circ M_j$ and g are equal up to a scalar. To find such an index j, we simply check, for each $j \in \{1, 2, \ldots, 432\}$, whether $f \circ M_j$ is in symmetric honeycomb form. The answer will be affirmative for at least one index j, and we set $M = M_j$. This resolves the question raised by Markwig and Buchholz [Buc10, §6]. The following theorem summarizes the problem and our solution.

Theorem 4.2.4. *Let f be a nonsingular cubic with* $\mathrm{val}(j(f)) < 0$*. If M is the* 3×3*-matrix over $\overline{K}$ constructed above then $f \circ M$ is a symmetric honeycomb cubic.*

Next, we discuss a refinement of the algorithm above that reduces the number of matrices to check from 432 to 12. It takes advantage of the detailed description of the Hessian group in [AD09]. Given a plane cubic f, the *Hessian group* G_{216} consists of those linear automorphisms of $\mathbb{P}^2$ that preserve the pencil $\mathrm{HP}(f)$. This group was first described by C. Jordan in [Jor77]. The elements of G_{216} naturally act on the subset $\mathcal{A}_\omega$ of $\mathbb{P}^2$ given by the rows $a_1, a_2, \ldots, a_9$ of A_ω. Of the 432 automorphisms of (4.12), precisely half are realized by the action of G_{216}. The group G_{216} is isomorphic to the semidirect product $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathrm{SL}(2,3)$. The first factor sends f to itself and permutes $\mathcal{A}_\omega$ transitively. The second factor, of order 24, sends f to each of the 12 cubics in $\mathrm{HP}(f)$ isomorphic to it. The quotient of $\mathrm{SL}(2,3)$ by the 2-element stabilizer of f is isomorphic to $\mathrm{PSL}(2,3)$, with 12 elements. Identifying G_{216} with the subgroup of S_9 permuting $\mathcal{A}_\omega$, a set of coset representatives for $\mathrm{PSL}(2,3)$ inside G_{216} consists of the following 12 permutations (in cycle notation):

$$\begin{gathered}\mathrm{id},\ (456)(987),\ (654)(789),\ (2437)(5698),\ (246378)(59),\ (254397)(68)\\ (249)(375), (258)(963), (2539)(4876), (852)(369), (287364)(59), (2836)(4975)\end{gathered} \tag{4.13}$$

With this notation, an example of an automorphism of the Hesse configuration (4.12) that is not realized by the Hessian group G_{216} is the permutation $\tau = (47)(58)(69)$.

Here is now our refined algorithm for the last step towards Theorem 4.2.4. Let $f, g, \mathcal{B}, \mathcal{A}_\omega$ be given as above. For any automorphism ρ of (4.12), we denote by $\phi_\rho : \mathbb{P}^2 \to \mathbb{P}^2$ the projective transformation $(b_1, b_2, b_4, b_5) \mapsto (a_{\rho(1)}, a_{\rho(2)}, a_{\rho(4)}, a_{\rho(5)})$. To find a transformation from f to g, we proceed as follows. First, we check to see whether ϕ_{id} maps $\mathcal{B}$ to $\mathcal{A}_\omega$. (It certainly maps four elements of $\mathcal{B}$ to $\mathcal{A}_\omega$ but maybe not all nine). If $\phi_{\mathrm{id}}(\mathcal{B}) = \mathcal{A}_\omega$ then $\phi_{\mathrm{id}}(f)$ is in the Hesse pencil $\mathrm{HP}(g)$. This implies that one of the 12 maps ϕ_σ, where σ runs over (4.13), takes f to g. If $\phi_{\mathrm{id}}(\mathcal{B}) \neq \mathcal{A}_\omega$ then ϕ_τ must map $\mathcal{B}$ to $\mathcal{A}_\omega$, since G_{216} has index 2 in the automorphism group of the Hesse configuration and τ represents the nonidentity coset. Then one of the 12 maps $\phi_{\tau\sigma}$, where σ runs over (4.13), takes f to g. In either case, after computing ϕ_{id}, we only have to check 12 maps, and one of them will work.

We close with two remarks. First, the set of matrices $M \in \mathrm{GL}(3, \overline{K})$ that send a given cubic f into honeycomb form is a rigid analytic variety, since the conditions on the entries of M are inequalities in valuations of polynomial expressions therein. It would be interesting to study this space further.

The second remark concerns the arithmetic nature of the output of our algorithm. The entries of the matrix M were constructed to be expressible in radicals over $K(\omega)$, with ω as in (4.10). However, as it stands, we do not know whether they can be expressed in radicals over the ground field K. The problem lies in the application of Proposition 4.2.1. Our first step was to chose a scalar $a \in K$ whose valuation is large enough. Thereafter, we computed b by solving a univariate equation of degree 12. This equation is generally irreducible with non-solvable Galois group. Perhaps it is possible to choose a and b simultaneously, in radicals over K, so that (a, b) lies on the curve (4.9), but at present, we do not know how to make this choice.

4.3 Parametrization and implicitization

A standard task of computer algebra is to go back and forth between parametric and implicit representations of algebraic varieties. Of course, these transformations are most transparent when the variety is rational. If the variety is not unirational then parametric representations typically involve transcendental functions. In this section, we use nonarchimedean theta functions to parametrize planar cubics, we demonstrate how to implicitize this parametrization, and we derive an intrinsic characterization of honeycomb cubics in terms of their nonarchimedean geometry.

In this section we assume that K is an algebraically closed field which is complete with respect to a nonarchimedean valuation. Fix a scalar $\iota \in K$ with $\mathrm{val}(\iota) < 0$. According to Tate's classical theory [Sil94], the unique elliptic curve E over K with $j(E) = \iota$ is analytically isomorphic to $K^*/q^{\mathbb{Z}}$, where $q \in K^*$ is a particular scalar with $\mathrm{val}(q) > 0$, called the *Tate parameter* of E. The symbol $q^{\mathbb{Z}}$ denotes the multiplicative group generated by q. The Tate parameter of E is determined from the j-invariant by inverting the power series relation

$$j \;=\; \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots \qquad (4.14)$$

This relation can be derived and computed to arbitrary precision from the identity

$$j = \frac{(1-48a_4(q))^3}{\Delta(q)},$$

where the invariant a_4 and the discriminant Δ are given by

$$a_4(q) \;=\; -5\sum_{n\geq 1}\frac{n^3q^n}{1-q^n} \qquad \text{and} \qquad \Delta(q) \;=\; q\prod_{n\geq 1}(1-q^n)^{24}.$$

We refer to Silverman's text book [Sil94] for an introduction to the relevant theory of elliptic curves, and specifically to [Sil94, Theorems V.1.1, V.3.1] for the above results.

Our aim in this section is to work directly with the analytic representation

$$E \;=\; K^*/q^{\mathbb{Z}},$$

and to construct its honeycomb embeddings into the plane $\mathbb{P}^2_K$. In our explicit computations, scalars in K are presented as truncated power series in a uniformizing parameter. The arithmetic is numerical rather than symbolic. Thus, this section connects the emerging fields of tropical geometry and numerical algebraic geometry.

By a *holomorphic function* on K^* we mean a formal Laurent series $\sum a_n x^n$ which converges for every $x \in K^*$. A *meromorphic function* is a ratio of two holomorphic functions; they have a well-defined notion of zeroes and poles as usual. A *theta function* on K^*, relative to the subgroup $q^{\mathbb{Z}}$, is a meromorphic function on K^* whose divisor is periodic with respect to $q^{\mathbb{Z}}$. Hence theta functions on K^* represent divisors on E. The *fundamental theta function* $\Theta : K^* \to K$ is defined by

$$\Theta(x) \;=\; \prod_{n>0}(1-q^n x)\prod_{n\geq 0}(1-\frac{q^n}{x}).$$

Note that Θ has a simple zero at the identity of E and no other zeroes or poles. Furthermore, given any $a \in K^*$, we define the shifted theta function

$$\Theta_a(x) \;=\; \Theta(x/a).$$

The function Θ_a represents the divisor $[a] \in \mathrm{Div}E$, where $[a]$ denotes the point of the elliptic curve E represented by a. One can also check that $\Theta_a(x/q) = -\frac{x}{a}\Theta_a(x)$.

Now suppose $D = n_1p_1 + \cdots + n_sp_s$ is a divisor on E that satisfies $\deg(D) = 0$ and $p_1^{n_1} \cdots p_s^{n_s} = 1$, as an equation in the multiplicative group $K^*/q^{\mathbb{Z}}$. We can use theta functions to exhibit D as a principal divisor, as follows. Pick lifts $\tilde{p}_1, \ldots, \tilde{p}_s \in K^*$ of $p_1, \ldots, p_s$, respectively, such that $\tilde{p}_1^{n_1} \cdots \tilde{p}_s^{n_s} = 1$ as an equation in K^*. Let

$$f(x) \;=\; \Theta_{\tilde{p}_1}(x)^{n_1} \cdots \Theta_{\tilde{p}_s}(x)^{n_s}.$$

This defines a function $f : K^* \to K$ that is q-periodic because

$$f(x/q) \;=\; \frac{(-x)^{n_1+\cdots+n_s}}{\tilde{p}_1^{n_1} \cdots \tilde{p}_s^{n_s}} f(x) \;=\; (-x)^{n_1+\cdots+n_s} f(x) \;=\; f(x).$$

The last equation holds because we assumed that $\deg(D) = n_1 + \cdots + n_s$ is zero. We conclude that f descends to a meromorphic function on $K^*/q^{\mathbb{Z}}$ with divisor D.

We now present a parametric representation of plane cubic curves that will work well for honeycombs. In what follows, we write $(z_0 : z_1 : z_2)$ for the coordinates on $\mathbb{P}^2$. Fix scalars $a, b, c, p_1, \ldots, p_9$ in K^* that satisfy the conditions

$$p_1p_2p_3 \;=\; p_4p_5p_6 \;=\; p_7p_8p_9 \quad \text{and} \quad p_i/p_j \notin q^{\mathbb{Z}} \text{ for } i \neq j. \tag{4.15}$$

The following defines a map from $E = K^*/q^{\mathbb{Z}}$ into the projective plane $\mathbb{P}^2$ as follows:

$$x \;\mapsto\; \big(a \cdot \Theta_{p_1}\Theta_{p_2}\Theta_{p_3}(x) : b \cdot \Theta_{p_4}\Theta_{p_5}\Theta_{p_6}(x) : c \cdot \Theta_{p_7}\Theta_{p_8}\Theta_{p_9}(x)\big). \tag{4.16}$$

This map embeds the elliptic curve $E = K^*/q^{\mathbb{Z}}$ analytically as a plane cubic:

Lemma 4.3.1. *If the image of the map (4.16) has three distinct intersection points with each of the three coordinate lines $\{z_i = 0\}$, then it is a cubic curve in $\mathbb{P}^2$. Every nonsingular cubic with this property and having Tate parameter q arises this way.*

Proof. By construction, the following two functions $K^* \to K$ are q-periodic:

$$f(x) \;=\; \frac{a \cdot \Theta_{p_1}\Theta_{p_2}\Theta_{p_3}(x)}{c \cdot \Theta_{p_7}\Theta_{p_8}\Theta_{p_9}(x)} \quad \text{and} \quad g(x) \;=\; \frac{b \cdot \Theta_{p_4}\Theta_{p_5}\Theta_{p_6}(x)}{c \cdot \Theta_{p_7}\Theta_{p_8}\Theta_{p_9}(x)}. \tag{4.17}$$

Hence f and g descend to meromorphic functions on the elliptic curve $E = K^*/q^{\mathbb{Z}}$. The map (4.16) can be written as $x \mapsto (f(x) : g(x) : 1)$ and this defines a map from E into $\mathbb{P}^2$. The divisor $D = p_7+p_8+p_9$ on E has degree 3. By the Riemann-Roch argument in [Har77, Example 3.3.3], its space of sections $L(D)$ is 3-dimensional. Moreover, the assumption about having three distinct intersection points implies that the meromorphic functions f, g and 1 form a basis of the vector space $L(D)$. The image of E in $\mathbb{P}(L(D)) \simeq \mathbb{P}^2$ is a cubic curve because $L(3D)$ is 9-dimensional.

For the second statement, we take C to be any nonsingular cubic curve with Tate parameter q that has distinct intersection points with the three coordinate lines in $\mathbb{P}^2$.

There exists a morphism ϕ from the abstract elliptic curve $E = K^*/q^{\mathbb{Z}}$ into $\mathbb{P}^2$ whose image equals C. Let $\{p_1, p_2, p_3\}$, $\{p_4, p_5, p_6\}$ and $\{p_7, p_8, p_9\}$ be the preimages under ϕ of the triples $C \cap \{z_0 = 0\}$, $C \cap \{z_1 = 0\}$ and $C \cap \{z_2 = 0\}$ respectively. The divisors $D_1 = p_1 + p_2 + p_3 - p_7 - p_8 - p_9$ and $D_2 = p_4 + p_5 + p_6 - p_7 - p_8 - p_9$ are principal, and hence $\frac{p_1p_2p_3}{p_7p_8p_9} = \frac{p_4p_5p_6}{p_7p_8p_9} = 1$ in the multiplicative group $E = K^*/q^{\mathbb{Z}}$. We choose preimages $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_9$ in K^* such that (4.15) holds for these scalars.

Our map ϕ can be written in the form $x \mapsto (f(x) : g(x) : 1)$ where $\mathrm{div}(f) = D_1$ and $\mathrm{div}(g) = D_2$. By the Abel-Jacobi Theorem (cf. [Roq70, Proposition 1]), the function f is uniquely determined, up to a multiplicative scalar, by the property $\mathrm{div}(f) = D_1$, and similarly for g and D_2. Then there exist $\gamma_1, \gamma_2 \in K^*$ such that

$$f(x) \;=\; \gamma_1 \frac{\Theta_{p_1}\Theta_{p_2}\Theta_{p_3}(x)}{\Theta_{p_7}\Theta_{p_8}\Theta_{p_9}(x)} \quad \text{and} \quad g(x) \;=\; \gamma_2 \frac{\Theta_{p_4}\Theta_{p_5}\Theta_{p_6}(x)}{\Theta_{p_7}\Theta_{p_8}\Theta_{p_9}(x)}.$$

We conclude that ϕ has the form (4.16). □

It is a natural ask to what extent the parameters in the representation (4.16) of a plane cubic are unique. The following result answers this question.

Proposition 4.3.2. *Two vectors $(a, b, c, p_1, \ldots, p_9)$ and $(a', b', c', p_1', \ldots, p_9')$ in $(K^*)^{12}$, both satisfying (4.15), define the same plane cubic if and only if the latter vector can be obtained from the former by combining the following operations:*

(a) Permute the sets $\{p_1, p_2, p_3\}$, $\{p_4, p_5, p_6\}$ and $\{p_7, p_8, p_9\}$.

(b) Scale each of a, b and c by the same multiplier $\lambda \in K^*$.

(c) Scale each p_i by the same multiplier $\lambda \in K^*$.

(d) Replace each p_i by its multiplicative inverse $1/p_i$.

(e) Multiply each p_i by q^{n_i} for some $n_i \in \mathbb{Z}$, where $n_1+n_2+n_3 = n_4+n_5+n_6 = n_7+n_8+n_9$, and set $a' = p_1^{n_1}p_2^{n_2}p_3^{n_3}a$, $b' = p_4^{n_4}p_5^{n_5}p_6^{n_6}b$, $c' = p_7^{n_7}p_8^{n_8}p_9^{n_9}c$.

Proof. Clearly the relabeling in (a) and the scaling in (b) preserve the curve $C \subset \mathbb{P}^2$. For (c), we note that scaling each p_i by the same constant $\lambda \in K^*$ produces a reparametrization of the same curve; only the location of the identity point changes. For part (e), note that $\Theta_{aq^i}(x) = \Theta_a(q^{-i}x) = (-x/a)^i\Theta_a(x)$. Suppose $(a, b, c, p_1, \ldots, p_9)$ and $(a', b', c', p_1', \ldots, p_9')$ satisfy the conditions in (e). Then

$$\begin{aligned}
&(a' \cdot \Theta_{p_1'}\Theta_{p_2'}\Theta_{p_3'} : b' \cdot \Theta_{p_4'}\Theta_{p_5'}\Theta_{p_6'} : c' \cdot \Theta_{p_7'}\Theta_{p_8'}\Theta_{p_9'}) \\
= \;& \left(\frac{(-x)^n a'\Theta_{p_1}\Theta_{p_2}\Theta_{p_3}}{p_1^{n_1}p_2^{n_2}p_3^{n_3}} : \frac{(-x)^n b'\Theta_{p_4}\Theta_{p_5}\Theta_{p_6}}{p_4^{n_4}p_5^{n_5}p_6^{n_6}} : \frac{(-x)^n c'\Theta_{p_7}\Theta_{p_8}\Theta_{p_9}}{p_7^{n_7}p_8^{n_8}p_9^{n_9}} \right) \\
= \;& (a \cdot \Theta_{p_1}\Theta_{p_2}\Theta_{p_3} : b \cdot \Theta_{p_4}\Theta_{p_5}\Theta_{p_6} : c \cdot \Theta_{p_7}\Theta_{p_8}\Theta_{p_9}),
\end{aligned}$$

where $n = n_1 + n_2 + n_3 = n_4 + n_5 + n_6 = n_7 + n_8 + n_9$.

Finally, for (d), one may check the identity $x\Theta(x^{-1}) = \Theta(x)$ directly from the definition of the fundamental theta function. In light of (4.15), this implies

$$\begin{array}{rc} & (a \cdot \Theta_{p_1}\Theta_{p_2}\Theta_{p_3} : b \cdot \Theta_{p_4}\Theta_{p_5}\Theta_{p_6} : c \cdot \Theta_{p_7}\Theta_{p_8}\Theta_{p_9}) \\ = & (\frac{-x^3 a}{p_1p_2p_3}\Theta(\frac{p_1}{x})\Theta(\frac{p_2}{x})\Theta(\frac{p_3}{x}) : \frac{-x^3 b}{p_4p_5p_6}\Theta(\frac{p_4}{x})\Theta(\frac{p_5}{x})\Theta(\frac{p_6}{x}) : \frac{-x^3 c}{p_7p_8p_9}\Theta(\frac{p_7}{x})\Theta(\frac{p_8}{x})\Theta(\frac{p_9}{x})) \\ = & (a\,\Theta_{\frac{1}{p_1}}(\frac{1}{x})\Theta_{\frac{1}{p_2}}(\frac{1}{x})\Theta_{\frac{1}{p_3}}(\frac{1}{x}) : b\,\Theta_{\frac{1}{p_4}}(\frac{1}{x})\Theta_{\frac{1}{p_5}}(\frac{1}{x})\Theta_{\frac{1}{p_6}}(\frac{1}{x}) : c\,\Theta_{\frac{1}{p_7}}(\frac{1}{x})\Theta_{\frac{1}{p_8}}(\frac{1}{x})\Theta_{\frac{1}{p_0}}(\frac{1}{x})). \end{array}$$

This is the reparametrization of the elliptic curve E under the involution $x \mapsto 1/x$ in the group law. We have thus proved the if direction of Proposition 4.3.2.

For the only-if direction, we write ψ and ψ' for the maps $E \to C \subset \mathbb{P}^2$ defined by $(a, b, \ldots, p_9)$ and $(a', b', \ldots, p_9')$ respectively. Then $(\psi')^{-1} \circ \psi$ is an automorphism of the elliptic curve E. The j-invariant of E is neither of the special values 0 or 1728. By [Sil86, Theorem 10.1], the only automorphisms of E are the involution $x \mapsto 1/x$ and multiplication by some fixed element in the group law. These are precisely the operations we discussed above, and they can be realized by the transformations from $(a, b, \ldots, p_9)$ to $(a', b', \ldots, p_9')$ that are described in (c) and (d).

Finally, if $(\psi')^{-1} \circ \psi$ is the identity on E, then plugging in $x = p_i$ for $i = 1, 2, 3$ shows that p_i is a zero of $a\Theta_{p_1}\Theta_{p_2}\Theta_{p_3}$ and hence of $a'\Theta_{p_1'}\Theta_{p_2'}\Theta_{p_3'}$. The same holds for $i = 4, 5, 6$ and $i = 7, 8, 9$. This accounts for the operations (a), (b) and (e). □

Our main result in this section is the following characterization of honeycomb curves, in terms of the analytic representation of plane cubics in (4.16). Writing $\mathbb{S}^1$ for the circle, let $\mathcal{V} : K^* \to \mathbb{S}^1$ denote the composition $K^* \stackrel{\text{val}}{\to} \mathbb{R} \to \mathbb{R}/\text{val}(q) \simeq \mathbb{S}^1$.

Theorem 4.3.3. *Let $a, b, c, p_1, \ldots, p_9 \in K^*$ as in Lemma 4.3.1. Suppose the values $\mathcal{V}(p_1), \ldots, \mathcal{V}(p_9)$ occur in cyclic order on $\mathbb{S}^1$, with $\mathcal{V}(p_3) = \mathcal{V}(p_4), \mathcal{V}(p_6) = \mathcal{V}(p_7)$, $\mathcal{V}(p_9) = \mathcal{V}(p_1)$, and all other values $\mathcal{V}(p_i)$ are distinct. Then the image of the map (4.16) is an elliptic curve in honeycomb form. Conversely, any elliptic curve in honeycomb form arises in this manner, after a suitable permutation of the indices.*

We shall present two alternative proofs of Theorem 4.3.3. These will highlight different features of honeycomb curves and how they relate to the literature. The first proof is computational and relates our study to the *tropical theta functions* studied by Mikhalkin and Zharkov [MZ07]. The second proof is more conceptual. It is based on the nonarchimedean Poincaré-Lelong formula for *Berkovich curves* [BPR11, Theorem 5.68]. Both approaches were suggested to us by Matt Baker.

First proof of Theorem 4.3.3. We shall examine the naive tropicalization of the elliptic curve $E = K^*/q^{\mathbb{Z}}$ under its embedding (4.16) into $\mathbb{P}^2$. Set $Q = \text{val}(q) \in \mathbb{R}_{>0}$. If $a \in K^*$ with $A = \text{val}(a) \in \mathbb{R}$ then the *tropicalization* of the theta function $\Theta_a : K^* \to K$ is obtained by replacing the infinite product of binomials in the definition of $\Theta(x)$ by an infinite sum of pairwise minima. The result is the function

$$\text{trop}(\Theta_a) : \mathbb{R} \to \mathbb{R}, \;\; X \mapsto \sum_{n>0} \min(0, nQ+X-A) + \sum_{n\geq 0} \min(0, nQ+A-X). \qquad (4.18)$$

For any particular real number X, only finitely many summands are non-zero, and hence $\mathrm{trop}(\Theta_a)(X)$ is a well-defined real number. A direct calculation shows that

$$\mathrm{trop}(\Theta_a)(X) \;=\; \min\left\{ \frac{m^2-m}{2}\cdot Q + m\cdot(A-X) \;:\; m\in\mathbb{Z} \right\}. \tag{4.19}$$

Indeed, the distributive law transforms the tropical product of binomials on the right hand side of (4.18) into the tropical sum in (4.19). The representation (4.19) is essentially the same as the *tropical theta function* of Mikhalkin and Zharkov [MZ07].

The tropical theta function is a piecewise linear function on $\mathbb{R}$, and we can translate (4.19) into an explicit description of the linear pieces of its graph. We find

$$\mathrm{trop}(\Theta_a)(X) \;=\; \frac{m^2-m}{2}\cdot Q + m\cdot(A-X) \tag{4.20}$$

where m is the unique integer satisfying $mQ \leq A - X < (m+1)Q$. In particular, for arguments X in this interval, the function is linear with slope $-m$.

The tropical theta function approximates the valuation of the theta function. These two functions agree unless there is some cancellation because the two terms in some binomial factor of Θ_a have the same order.

The gap between the tropical theta function and the valuation of the theta function is crucial in understanding the tropical geometry of the map $\mathcal{V}: K^* \to \mathbb{S}^1$. Our next definition makes this precise. If $x, y \in K^*$ with $\mathcal{V}(x) = \mathcal{V}(y)$ then we set

$$\delta(x,y) \;:=\; \mathrm{val}\big(1 - \frac{x}{y}q^i\big) \tag{4.21}$$

where $i \in \mathbb{Z}$ is the unique integer satisfying $\mathrm{val}(x) + \mathrm{val}(q^i) = \mathrm{val}(y)$. It is easy to check that the quantity defined in (4.21) is symmetric, i.e. $\delta(x,y) = \delta(y,x)$. With this notation, the following formula characterizes the gap between the tropical theta function and the valuation of the theta function. For any $a, x \in K^*$, we have

$$\mathrm{val}\big(\Theta_a(x)\big) - \mathrm{trop}(\Theta_a)(\mathrm{val}(x)) \;=\; \begin{cases} \delta(x,a) & \text{if } \mathcal{V}(a) = \mathcal{V}(x), \\ 0 & \text{otherwise.} \end{cases} \tag{4.22}$$

Consider now any three scalars $x, y, z \in K^*$ that lie in the same fiber of the map from K^* onto the unit circle $\mathbb{S}^1$. In symbols, $\mathcal{V}(x) = \mathcal{V}(y) = \mathcal{V}(z)$. Then

$$\text{the minimum of } \delta(x,y),\ \delta(x,y) \text{ and } \delta(y,z) \text{ is attained twice.} \tag{4.23}$$

This follows from the identity

$$(xq^i - yq^j) + (yq^j - z) + (z - xq^i) \;=\; 0$$

where $i, j \in \mathbb{Z}$ are defined by $\mathrm{val}(x) + \mathrm{val}(q^i) = \mathrm{val}(y) + \mathrm{val}(q^j) = \mathrm{val}(z)$.

We are now prepared to prove Theorem 4.3.3. Set $Q = \mathrm{val}(q)$. For $i = 1, 2, \ldots, 9$, let $P_i = \mathrm{val}(p_i)$ and write $P_i = n_iQ + r_i$ where $n_i \in \mathbb{Z}$ and $r_i \in [0, Q)$. Rescaling the p_i's by a common factor and inverting them all does not change the cubic curve, by Proposition 4.3.2. After performing such operations if needed, we can assume

$$0 = r_9 = r_1 < r_2 < r_3 = r_4 < r_5 < r_6 = r_7 < r_8 < Q. \tag{4.24}$$

The hypothesis (4.15) implies

$$r_1 + r_2 + r_3 \equiv r_4 + r_5 + r_6 \equiv r_7 + r_8 + r_9 \pmod{Q}.$$

Together with the chain of inequalities in (4.24), this implies

$$\begin{array}{rcccl} r_1 + r_2 + r_3 & = & r_4 + r_5 + r_6 - Q & = & r_7 + r_8 + r_9 - Q, \\ n_1 + n_2 + n_3 & = & n_4 + n_5 + n_6 + 1 & = & n_7 + n_8 + n_9 + 1. \end{array} \tag{4.25}$$

We now examine the naive tropicalization $\mathbb{R} \to \mathbb{TP}^2$ of our map (4.16). It equals

$$X \mapsto \begin{array}{c} \big(A + \mathrm{trop}(\Theta_{p_1})(X) + \mathrm{trop}(\Theta_{p_2})(X) + \mathrm{trop}(\Theta_{p_3})(X) : \\ B + \mathrm{trop}(\Theta_{p_4})(X) + \mathrm{trop}(\Theta_{p_5})(X) + \mathrm{trop}(\Theta_{p_6})(X) : \\ C + \mathrm{trop}(\Theta_{p_7})(X) + \mathrm{trop}(\Theta_{p_8})(X) + \mathrm{trop}(\Theta_{p_9})(X) \big). \end{array} \tag{4.26}$$

Here $A = \mathrm{val}(a), B = \mathrm{val}(b), C = \mathrm{val}(c)$. The image of this piecewise-linear map coincides with the tropicalization of the image of (4.16) for generic values of $X = \mathrm{val}(x)$. Indeed, if X lies in the open interval $(0, r_2)$ then, by (4.20), the map (4.26) is linear and its image in the tropical projective plane $\mathbb{TP}^2$ is a segment with slope

$$(-n_1 - n_2 - n_3 - 1 : -n_4 - n_5 - n_6 : n_7 - n_8 - n_9 - 1) = (1 : 1 : 0).$$

Here we fix tropical affine coordinates with last coordinate 0. Similarly,

- the slope is $(0 : 1 : 0)$ for $X \in (r_2, r_3)$,
- the slope is $(-1 : 0 : 0)$ for $X \in (r_4, r_5)$,
- the slope is $(-1 : -1 : 0)$ for $X \in (r_5, r_6)$,
- the slope is $(0 : -1 : 0)$ for $X \in (r_7, r_8)$,
- the slope is $(1 : 0 : 0)$ for $X \in (r_8, Q)$.

These six line segments form a hexagon in $\mathbb{TP}^2$. The vertices of that hexagon are the images of the six distinct real numbers r_i in (4.24) under the map (4.26).

Finally, we examine the special values $x \in K^*$ for which the naive tropicalization (4.26) does not compute the correct image in $\mathbb{TP}^2$. This happens precisely when some of the nine theta functions in (4.16) have a valuation gap when passing to (4.26).

Suppose, for instance, that $\mathcal{V}(x) = \mathcal{V}(p_1) = \mathcal{V}(p_9)$. From (4.22) we have

$$\mathrm{val}\big(\Theta_{p_i}(x)\big) - \mathrm{trop}(\Theta_{p_i})(\mathrm{val}(x)) = \begin{cases} \delta(x, p_i) & \text{if } i = 1 \text{ or } i = 9, \\ 0 & \text{if } i \neq 1, 9. \end{cases} \tag{4.27}$$

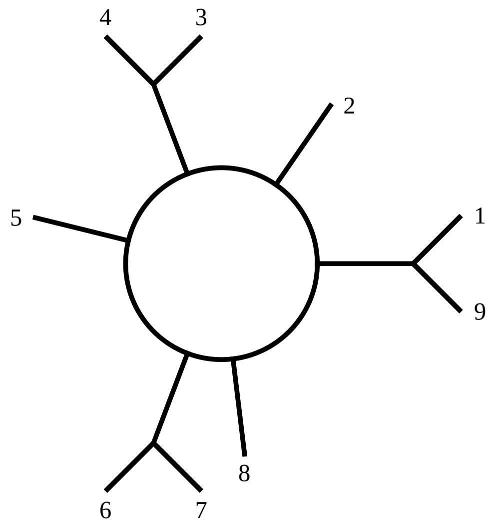


Figure 4.2: The Berkovich skeleton Σ of an elliptic curve with honeycomb punctures

We know from (4.23) that the minimum of $\delta(p_1, p_9), \delta(x, p_1), \delta(x, p_9)$ is achieved twice. Moreover, by varying the choice of the scalar x with $\mathcal{V}(x) = r_1$, the latter two quantities can attain any non-negative value that is compatible with this constraint. This shows that the image of the set of such x under the tropicalization of the map (4.16) consists of one bounded segment and two rays in $\mathbb{TP}^2$. The segment meets the hexagon described above at the vertex corresponding to $r_1 = r_9$, and consists of the images of the points x such that $\delta(p_1, p_9) \geq \delta(p_1, x) = \delta(p_9, x)$. Since Θ_{p_1} and Θ_{p_9} occur in the first and third coordinates, respectively, of (4.16), the slope of the segment is $(1 : 0 : 1)$, and its length is $\delta(p_1, p_9)$. Similarly, the image of the points x such that $\delta(p_1, x) \geq \delta(p_1, p_9) = \delta(p_9, x)$ is a ray of slope $(1 : 0 : 0)$, and the image of the points x such that $\delta(p_9, x) \geq \delta(p_1, p_9) = \delta(p_1, x)$ is a ray of slope $(0 : 0 : 1)$. Note that these three slopes obey the balancing condition for tropical curves.

A similar analysis determines all six connected components of the complement of the hexagon in the tropical curve, and we see that the tropical curve is a honeycomb when the asserted conditions on $a, b, c, p_1, p_2, \ldots, p_9$ are satisfied.

The derivation of the converse direction, that any honeycomb cubic has the desired parametrization, will be deferred to the second proof. It seems challenging to prove this without [BPR11]. See also the problem stated at the end of this section. $\square$

Second proof of Theorem 4.3.3. We work in the setting of Berkovich curves introduced by Baker, Payne and Rabinoff in [BPR11]. Let E^{an} denote the *analytification* of the elliptic curve E, and let Σ denote the minimal skeleton of E^{an}, as defined in [BPR11, §5.14], with respect to the given set $D = \{p_1, p_2, \ldots, p_9\}$ of nine punctures. Our standing assumption $\mathrm{val}(j) < 0$ ensures that the Berkovich curve E^{an} contains a unique cycle $\mathbb{S}^1$, and Σ is obtained from that cycle by attaching trees with nine leaves in total. In close analogy to [BPR11, §7.1], we consider the retraction map onto $\mathbb{S}^1$:

$$E(K)\backslash D \hookrightarrow E^{an}\backslash D \to \Sigma \to \mathbb{R}/\mathrm{val}(q)\mathbb{Z} \simeq \mathbb{S}^1.$$

The condition in Theorem 4.3.3 states that, under this map, the points of $E(K)$ given by $p_1, p_2, \ldots, p_9$ retract to six distinct points on $\mathbb{S}^1$, in cyclic order with fibers

$$\{p_9, p_1\}, \{p_2\}, \{p_3, p_4\}, \{p_5\}, \{p_6, p_7\}, \{p_8\}.$$

This means that Σ looks precisely like the graph in Figure 4.2. This picture is the Berkovich model of a honeycomb cubic. To see this, we shall apply the nonarchimedean Poincaré-Lelong Formula [BPR11, Theorem 5.69] in conjunction with a combinatorial argument about balanced graphs in $\mathbb{R}^2$.

The rational functions f and g in (4.17) are well-defined on $E^{an}\backslash D$, and we may consider the negated logarithms of their norms: $F = -\log|f|$ and $G = -\log|g|$. Our tropical curve can be identified with its image in $\mathbb{R}^2$ under the map (F, G). According to part (1) of [BPR11, Theorem 5.69], this map factors through the retraction of $E^{an}\backslash D$ onto Σ. By part (2), the function (F, G) is linear on each edge of Σ.

We shall argue that the graph Σ in Figure 4.2 is mapped isometrically onto a tropical honeycomb curve in $\mathbb{R}^2$. Using part (5) of [BPR11, Theorem 5.69], we can determine the slopes of the nine unbounded edges. Namely, F has slope 1 on the rays in Σ towards p_1, p_2 and p_3, and slope -1 on the rays towards p_7, p_8 and p_9. Similarly, G has slope 1 on the rays in Σ towards p_4, p_5 and p_6, and slope -1 on the rays towards p_7, p_8 and p_9. By part (4), the functions F and G are harmonic, which means that the image in $\mathbb{R}^2$ satisfies the balancing condition of tropical geometry. This requirement uniquely determines the slopes of the nine bounded edges in the image of Σ. For the three edges not on the cycle this is immediate, and for the six edges on the cycle, this follows by solving a linear system of equations. The unique solution to these constraints is a balanced planar graph that must be a honeycomb cubic. Conversely, every tropical honeycomb cubic in $\mathbb{R}^2$ is trivalent with all multiplicities one. By [BPR11, Corollary 6.27(1)], the skeleton Σ must look like Figure 4.2, and the corresponding map (F, G) is an isometry onto the cubic. □

In the rest of Section 3, we discuss computational aspects of the representation of plane cubics given in Lemma 4.3.1 and Theorem 4.3.3. We begin with the *implicitization problem*: Given $a, b, c, p_1, \ldots, p_9 \in K^*$, how can we compute the implicit equation (4.1)? Write $\big(f(x) : g(x) : h(x)\big)$ for the analytic parametrization in (4.16). Then we seek to compute the unique (up to scaling) coefficients c_{ijk} in a K-linear relation

$$\begin{aligned} &c_{300}f(x)^3 + c_{210}f(x)^2g(x) + c_{120}f(x)g(x)^2 \\ &+ c_{030}g(x)^3 + c_{021}g(x)^2h(x) + c_{012}g(x)h(x)^2 + c_{003}h(x)^3 \\ &+ c_{102}f(x)h(x)^2 + c_{201}f(x)^2h(x) + c_{111}f(x)g(x)h(x) \qquad = \quad 0. \end{aligned} \tag{4.28}$$

Evaluating this relation at $x = p_i$, and noting that $\Theta_{p_i}(p_i) = 0$, we get nine linear equations for the nine c_{ijk}'s other than c_{111}. These equations are

$$\begin{aligned} c_{300}f(p_i)^3 + c_{210}f(p_i)^2g(p_i) + c_{120}f(p_i)g(p_i)^2 + c_{030}g(p_i)^3 &= 0 \quad \text{for } i = 7, 8, 9, \\ c_{003}h(p_i)^3 + c_{102}f(p_i)h(p_i)^2 + c_{201}f(p_i)^2h(p_i) + c_{300}f(p_i)^3 &= 0 \quad \text{for } i = 4, 5, 6, \\ c_{030}g(p_i)^3 + c_{021}g(p_i)^2h(p_i) + c_{012}g(p_i)h(p_i)^2 + c_{003}h(p_i)^3 &= 0 \quad \text{for } i = 1, 2, 3. \end{aligned}$$

The first group of equations has a solution $(c_{300}, c_{210}, c_{120}, c_{030})$ that is unique up to scaling. Namely, the ratios $c_{300}/c_{030}, c_{210}/c_{030}, c_{120}/c_{030}$ are the elementary symmetry functions in the three quantities

$$\frac{b \cdot \Theta(\frac{p_7}{p_4})\Theta(\frac{p_7}{p_5})\Theta(\frac{p_7}{p_6})}{a \cdot \Theta(\frac{p_7}{p_1})\Theta(\frac{p_7}{p_2})\Theta(\frac{p_7}{p_3})}, \quad \frac{b \cdot \Theta(\frac{p_8}{p_4})\Theta(\frac{p_8}{p_5})\Theta(\frac{p_8}{p_6})}{a \cdot \Theta(\frac{p_8}{p_1})\Theta(\frac{p_8}{p_2})\Theta(\frac{p_8}{p_3})} \quad \text{and} \quad \frac{b \cdot \Theta(\frac{p_9}{p_4})\Theta(\frac{p_9}{p_5})\Theta(\frac{p_9}{p_6})}{a \cdot \Theta(\frac{p_9}{p_1})\Theta(\frac{p_9}{p_2})\Theta(\frac{p_9}{p_3})}.$$

The analogous statements hold for the second and third group of equations.

We are thus left with computing the middle coefficient c_{111} in the relation (4.28). We do this by picking any $v \in K$ with $f(v)g(v)h(v) \neq 0$. Then (4.28) gives

$$c_{111} \;=\; -\frac{1}{f(v)g(v)h(v)}\big(c_{300}f(v)^3 + c_{210}f(v)^2g(v) + \cdots + c_{201}f(v)^2h(v)\big).$$

We have implemented this implicitization method in Mathematica, for input data in the field $K = \mathbb{Q}(t)$ of rational functions with rational coefficients.

The *parameterization problem* is harder. Here we are given the 10 coefficients c_{ijk} of a honeycomb cubic that has three distinct intersection points with each coordinate line $z_0 = 0, z_1 = 0, z_2 = 0$. The task is to compute 12 scalars $a, b, c, p_1, \ldots, p_9 \in K^*$ that represent the cubic as in Lemma 4.3.1. The 12 output scalars are not unique, but the degree of non-uniqueness is characterized exactly by Proposition 4.3.2. This task amounts to solving an analytic system of equations. We shall leave it to a future project to design an algorithm for doing this in practice.

4.4 The tropical group law

In this section, we present a combinatorial description of the group law on a honeycomb elliptic curve based on the parametric representation in Section 3. We start by studying the inflection points of such a curve. We continue to assume that K is algebraically closed and complete with respect to a nonarchimedean valuation.

Let $E \xrightarrow{\psi} C \subset \mathbb{P}^2$ be a honeycomb embedding of the abstract elliptic curve $E = K^*/q^{\mathbb{Z}}$. Let v_1, v_2, v_3, v_4, v_5 and v_6 denote the vertices of the hexagon in the tropical cubic $\mathrm{trop}(C)$, labeled as in Figure 4.3. Let e_i denote the edge between v_i and v_{i+1}, with the convention $v_7 = v_1$, and let ℓ_i denote the lattice length of e_i. By examining the width and height of the hexagon, we see that the six lattice lengths $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6$ satisfy two linearly independent relations:

$$\ell_1 + \ell_2 \;= \ell_4 + \ell_5 \quad \text{and} \quad \ell_2 + \ell_3 \;= \ell_5 + \ell_6.$$

We first prove the following basic fact about the inflection points on the cubic C.

Lemma 4.4.1. *The tropicalizations of the nine inflection points of the cubic curve $C \subset \mathbb{P}^2$ retract to the hexagonal cycle of* $\mathrm{trop}(C) \subset \mathbb{R}^2$ *in three groups of three.*

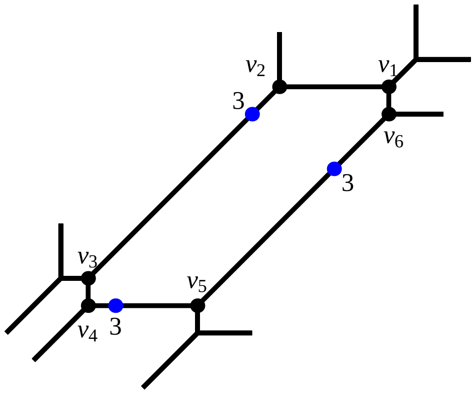


Figure 4.3: A honeycomb cubic and its nine inflection points in groups of three

Proof. This lemma is best understood from the perspective of Berkovich theory. The analytification E^{an} retracts onto its skeleton, namely the unique cycle, which is isometrically embedded into $\text{trop}(C)$ as the hexagon. Thus every point of E retracts onto a unique point in the hexagon. In fact, this retraction is given by

$$K^*/q^{\mathbb{Z}} \cong E(K) \hookrightarrow E^{\text{an}} \twoheadrightarrow \mathbb{S}^1 \cong \mathbb{R}/\text{val}(q)\mathbb{Z} \tag{4.29}$$

and is the natural map induced from the valuation homomorphism $(K^*, \cdot) \to (\mathbb{R}, +)$. We refer the reader unfamiliar with the map (4.29) to [BPR11, Theorem 7.2].

Now, after a multiplicative translation, we may assume that ψ sends the identity of E to an inflection point. Then the inflection points of C are the images of the 3-torsion points of $E = K^*/q^{\mathbb{Z}}$. These can be written as $\omega^i \cdot q^{j/3}$ for $\omega^3 = 1$, for $q^{1/3}$ a cube root of q, and for $i, j = 0, 1, 2$. Note that $\text{val}(\omega) = 0$ whereas $\text{val}(q^{1/3}) = \text{val}(q)/3 > 0$. Hence the valuations of the scalars $\omega^i \cdot q^{j/3}$ are 0, $\text{val}(q)/3$, and $2\text{val}(q)/3$, and each group contains three of these nine scalars. □

Our next result refines Lemma 4.4.1. It is a very special case of a theorem due to Brugallé and de Medrano [BM11] which covers honeycomb curves of arbitrary degree.

Lemma 4.4.2. *Let $P \in \text{trop}(C)$ be the tropicalization of an inflection point on the cubic curve $C \subset \mathbb{P}^2$. Then there are three possibilities, as indicated in Figure 4.3:*

- *The point P lies on the longer of e_1 or e_2, at distance $|\ell_2 - \ell_1|/3$ from v_2.*
- *The point P lies on the longer of e_3 or e_4, at distance $|\ell_4 - \ell_3|/3$ from v_4.*
- *The point P lies on the longer of e_5 or e_6, at distance $|\ell_6 - \ell_5|/3$ from v_6.*

The nine inflection points fall into three groups of three in this way.

In the special case that $\ell_1 = \ell_2$ (and similarly $\ell_3 = \ell_4$ or $\ell_5 = \ell_6$), the lemma should be understood as saying that P lies somewhere on the ray emanating from v_2.

Proof. Consider the tropical line whose node lies at v_2, and let L be any classical line in $\mathbb{P}^2_K$ that is generic among lifts of that tropical line. Then the three points in $L \cap C$ tropicalize to the stable intersection points of $\text{trop}(L)$ with $\text{trop}(C)$, namely the vertices v_1, v_2 and v_3. Let x denote the counterclockwise distance along the hexagon from v_2 to P. By applying a multiplicative translation in E, we fix the identity to be an inflection point on C that tropicalizes to P. Then $v_1 + v_2 + v_3 = 0$ in the group $\mathbb{S}^1 \simeq \mathbb{R}/\text{val}(q)\mathbb{Z}$. This observation implies the congruence relation

$$(\ell_1 + x) + x + (x - \ell_2) \;\equiv\; 0 \mod \text{val}(q),$$

and hence $3x \equiv \ell_2 - \ell_1$. One solution to this congruence is $x = (\ell_2 - \ell_1)/3$. This means that the point P lies on the longer edge, e_1 or e_2, at distance $|\ell_1 - \ell_2|/3$ from v_2. The analysis is identical for the vertex v_4 and for the vertex v_6, and it identifies two other locations on $\mathbb{S}^1$ for retractions of inflection points. □

Next, we wish to review some basic facts about the group structures on a plane cubic curve C. The abstract elliptic curve $E = K^*/q^{\mathbb{Z}}$ is an abelian group, in the obvious sense, as a quotient of the abelian group $(K^*, \cdot)$. Knowledge of that group structure is equivalent to knowing the surface $\{(r,s,t) \in E^3 : r \cdot s \cdot t = q^{\mathbb{Z}}\}$.

The group structure on E induces a group structure $(C, \star)$ on the plane cubic curve $C = \psi(E)$. While the isomorphism ψ is analytic, the group operation on C is actually algebraic. Equivalently, the set $\{(u,v,w) \in C^3 : u \star v \star w = \text{id}\}$ is an algebraic surface in $(\mathbb{P}^2)^3$. However, this surface depends heavily on the choice of parametrization ψ. Different isomorphisms from the abstract elliptic curve E onto the plane cubic C will result in different group structures in $\mathbb{P}^2$.

The most convenient choices of ψ are those that send the identity element $q^{\mathbb{Z}}$ of E to one of the nine inflection points of C. Such maps ψ are characterized by the condition that $u \star v \star w = \text{id}$ if and only if u, v and w are collinear in $\mathbb{P}^2$. If so, the data of the group law can be recorded in the surface

$$\big\{(u,v,w) \in C^3 \,:\, u,\, v \text{ and } w \text{ lie on a line in } \mathbb{P}^2 \big\} \quad \subset \quad (\mathbb{P}^2)^3. \tag{4.30}$$

We emphasize, however, that we have no reason to assume a priori that the identity element $\text{id} = \psi(q^{\mathbb{Z}})$ on the plane cubic C is an inflection point: *every point on C is the identity in some group law.* Indeed, we can simply replace ψ by its composition with a translation $x \mapsto r \cdot x$ of the group E. Our analysis below covers all cases: our combinatorial description of the group law on $\text{trop}(C)$ is always valid, regardless of whether the identity is an inflection point or not.

A partial description of the tropical group law was given by Vigeland in [Vig09]. Specifically, his choice of the fixed point $\mathcal{O}$ corresponds to the point $\text{trop}(\psi(q^{\mathbb{Z}}))$. Vigeland's group law is a combinatorial extension of the polyhedral surface

$$\big\{(U,V,W) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \,:\, U + V + W = \mathcal{O}\big\} \tag{4.31}$$

This torus comes with a distinguished subdivision into polygons since $\mathbb{S}^1$ has been identified with the hexagon. The polyhedral torus (4.31) is illustrated in Figure 4.4.

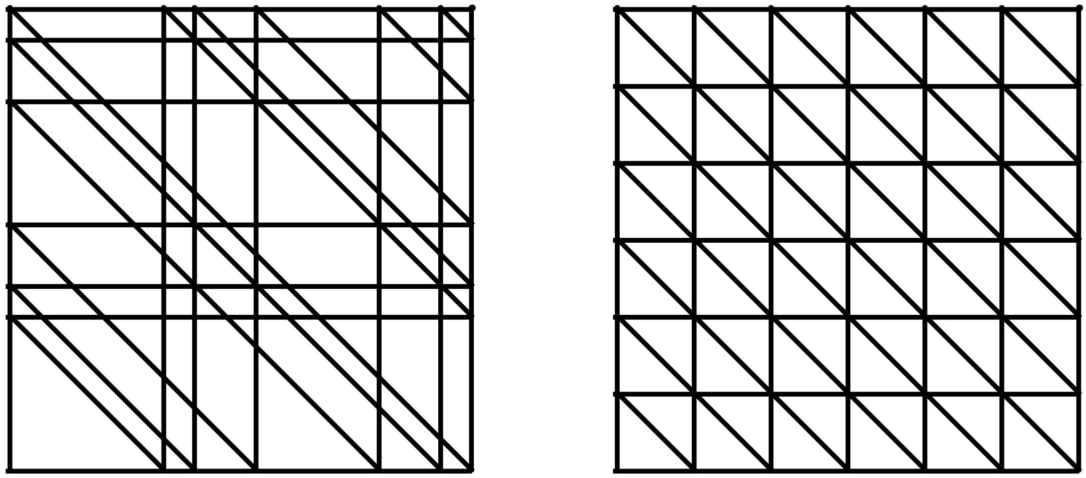


Figure 4.4: The torus in the tropical group law surface (4.31). The two pictures represent the honeycomb cubic and the symmetric honeycomb cubic shown in Figure 4.1.

Our goal is to characterize the tropical group law as follows. For a given honeycomb embedding $\psi : E \to \mathbb{P}^2$, we define the *tropical group law surface* to be

$$\mathrm{TGL}(\psi) = \big\{(\mathrm{trop}(\psi(x)), \mathrm{trop}(\psi(y)), \mathrm{trop}(\psi(z))) : x, y, z \in E, x \cdot y \cdot z = \mathrm{id}\big\} \subset (\mathbb{R}^2)^3.$$

If ψ sends the identity to an inflection point, then this is the tropicalization of (4.30):

$$\mathrm{TGL}(\psi) \;=\; \{(\mathrm{trop}(u), \mathrm{trop}(v), \mathrm{trop}(w)) : u, v, w \in C \text{ lie on a line in } \mathbb{P}^2\} \subset (\mathbb{R}^2)^3.$$

The tropical group law surface is a tropical algebraic variety of dimension 2, and can in principle be computed, for $K = \mathbb{Q}(t)$, using the software `gfan` [Jen]. This surface contains all the information of the *tropical group law*. We shall explain how $\mathrm{TGL}(\psi)$ can be computed combinatorially, even if $\psi(\mathrm{id})$ is not an inflection point.

Our approach follows directly from Section 3. Let $\psi : E \to \mathbb{P}^2$ be a honeycomb embedding of an elliptic curve E, and let $V : K^* \to \mathbb{S}^1$ denote the composition $K^* \stackrel{\mathrm{val}}{\to} \mathbb{R} \to \mathbb{R}/\mathrm{val}(q) \simeq \mathbb{S}^1$. By Theorem 4.3.3, there exist $a, b, c, p_1, \ldots, p_9 \in K^*$ with $\mathcal{V}(p_1), \ldots, \mathcal{V}(p_9)$ occurring in cyclic order on $\mathbb{S}^1$, with $\mathcal{V}(p_3) = \mathcal{V}(p_4), \mathcal{V}(p_6) = \mathcal{V}(p_7), \mathcal{V}(p_9) = \mathcal{V}(p_1)$, and all other values $\mathcal{V}(p_i)$ distinct, such that ψ is given by

$$x \mapsto \big(a \cdot \Theta_{p_1}\Theta_{p_2}\Theta_{p_3}(x) : b \cdot \Theta_{p_4}\Theta_{p_5}\Theta_{p_6}(x) : c \cdot \Theta_{p_7}\Theta_{p_8}\Theta_{p_9}(x)\big).$$

Now, again as in Section 3, given $x, y \in K^*$ with $\mathcal{V}(x) = \mathcal{V}(y)$, we set

$$\delta(x, y) \;=\; \mathrm{val}\big(1 - \frac{x}{y} q^i\big),$$

where $i \in \mathbb{Z}$ is specified by $\mathrm{val}(x) + \mathrm{val}(q^i) = \mathrm{val}(y)$. We have seen that if $x \in K^*$ satisfies $\mathcal{V}(x) = \mathcal{V}(p_i)$, then $\mathrm{trop}(x)$ lies at distance $\delta(x, p_i)$ from the hexagon along the tentacle associated to p_i. By the *tentacle* of p_i we mean the union of the ray associated to p_i and the bounded segment to which that ray is attached.

The next proposition is our main result in Section 4. We shall construct the tropical group law surface by way of its projection to the first two coordinates

$$\pi : \mathrm{TGL}(\psi) \to \mathrm{trop}(C) \times \mathrm{trop}(C) \subset (\mathbb{R}^2)^2.$$

As before, we identify the hexagon of $\mathrm{trop}(C)$ with the circle $\mathbb{S}^1 \simeq \mathbb{R}/\mathrm{val}(q)$. Given $U, V \in \mathrm{trop}(C)$, let $U \circ V \in \mathbb{S}^1$ denote the sum of the retractions of U and V to the hexagon. The location of $U \circ V$ depends on the choice of ψ and in particular the location of $\mathcal{O} = \mathrm{trop}(\psi(\mathrm{id}))$ on the hexagon $\mathbb{S}^1$. Let $-(U \circ V)$ denote the inverse of $U \circ V$, again under addition on $\mathbb{S}^1$. By the *distance* of a point $U \in \mathrm{trop}(C)$ to the hexagon $\mathbb{S}^1$ we mean the lattice length of the unique path in $\mathrm{trop}(C)$ from U to $\mathbb{S}^1$. Finally, we say that $u \in K^*$ is a lift of $U \in \mathrm{trop}(C)$ if $\mathrm{trop}(\psi(u \cdot q^{\mathbb{Z}})) = U$. The following proposition characterizes the fiber of the map π over a given pair (U, V).

Proposition 4.4.3. *Let $\psi : E \to C \subset \mathbb{P}^2$ be a honeycomb embedding, with the operation $\circ : \mathrm{trop}(C) \times \mathrm{trop}(C) \to \mathbb{S}^1$ defined as above, and "is a vertex" refers to the hexagon $\mathbb{S}^1$. For any U and $V \in \mathrm{trop}(C)$, exactly one of the following occurs:*

(i) If $-(U \circ V)$ is not a vertex, then $\pi^{-1}(U, V)$ is the singleton $\{-(U \circ V)\}$.

(ii) If $-(U \circ V)$ is a vertex adjacent to a single unbounded ray R_i towards the point p_i, then $\pi^{-1}(U, V)$ is the set of points on R_i whose distance to $\mathbb{S}^1$ equals $\delta(u^{-1}v^{-1}, p_i)$ for some lifts $u, v \in K^*$ of U and V.

(iii) If $-(U \circ V)$ is a vertex adjacent to a bounded segment B, along with two rays R_j and R_k, toward the points p_j and p_k, then $\pi^{-1}(U, V)$ consists of

- *the points on B whose distance to $\mathbb{S}^1$ is equal to $\delta(u^{-1}v^{-1}, p_j) = \delta(u^{-1}v^{-1}, p_k)$ for some lifts $u, v \in K^*$ of U and V,*
- *the points on R_j whose distance to $\mathbb{S}^1$ is equal to $\delta(u^{-1}v^{-1}, p_j) > \delta(u^{-1}v^{-1}, p_k)$ for some lifts $u, v \in K^*$ of U and V, and*
- *the points on R_k whose distance to $\mathbb{S}^1$ is equal to $\delta(u^{-1}v^{-1}, p_k) > \delta(u^{-1}v^{-1}, p_j)$ for some lifts $u, v \in K^*$ of U and V.*

Proof. For any lifts $u, v \in K^*$ of U, V, we have $\mathcal{V}(u^{-1}v^{-1}) + \mathcal{V}(u) + \mathcal{V}(v) = 0$ in $\mathbb{S}^1 = \mathbb{R}/\mathrm{val}(q)\mathbb{Z}$. This equation determines the retraction $-(U \circ V)$ of the point $\mathrm{trop}(\psi(u^{-1}v^{-1}))$ to the hexagon. If $-(U \circ V)$ is not a vertex, then $\mathrm{trop}(\psi(u^{-1}v^{-1}))$ must be precisely the point $-(U \circ V)$, as no other points retract to it.

If instead $-(U \circ V)$ is a vertex of the hexagon, then we either have $\mathcal{V}(u^{-1}v^{-1}) = \mathcal{V}(p_i)$ for exactly one p_i, or $\mathcal{V}(u^{-1}v^{-1}) = \mathcal{V}(p_j) = \mathcal{V}(p_k)$ for exactly two points p_j and p_k, depending on whether one or two rays emanate from $-(U \circ V)$. We have seen (in our first proof of Theorem 4.3.3) that the distance of $\mathrm{trop}(\psi(u^{-1}v^{-1}))$ to the hexagon, measured along the tentacle associated to p_i, is $\delta(u^{-1}v^{-1}, p_i)$. In either case, these distances uniquely determine the location of $\mathrm{trop}(\psi(u^{-1}v^{-1}))$. ∎

We demonstrate this method in a special example which was found in discussion with Spencer Backman. Pick $r, s \in K^*$ with $r^6 = q$ and $\mathrm{val}(1-s) =: \beta > 0$. Let

$$\begin{array}{ccc} p_1 = r^{-1}s^{-1}, & p_2 = 1, & p_3 = rs, \\ p_4 = rs^{-1}, & p_5 = r^2, & p_6 = r^{-3}s, \\ p_7 = r^3 s^{-1}, & p_8 = r^{-2}, & p_9 = r^{-1}s. \end{array} \tag{4.32}$$

We also set $a = b = c = 1$ in (4.16). This choice produces a symmetric honeycomb embedding $\psi : E \to C \subset \mathbb{P}^2$. The parameter β is the common length of the three bounded segments adjacent to $\mathbb{S}^1$. Note that $p_1p_2p_3 = p_4p_5p_6 = p_7p_8p_9 = 1$ in K^*. This condition implies that the identity of E is mapped to an inflection point in C.

Corollary 4.4.4. *For the elliptic curve in honeycomb form defined by (4.32), the tropical group law* $\mathrm{TGL}(\psi)$ *is a polyhedral surface consisting of 117 vertices, 279 bounded edges, 315 rays, 54 squares, 108 triangles, 279 flaps, and 171 quadrants.*

Here, a "flap" is a product of a bounded edge and a ray, and a "quadrant" is a product of two rays. Note that the Euler characteristic is $117 - 279 + 54 + 108 = 0$. This is consistent with the fact that the Berkovich skeleton of $(E \times E)^{an}$ is a torus.

Proof. Let $X = \mathrm{trop}(\psi(E))$ and $\pi : \mathrm{TGL}(\psi) \to X \times X$ as in Proposition 4.4.3. We modify the tropical surface $X \times X$ by attaching the fiber $\pi^{-1}(U, V)$ to each point $(U, V) \in X \times X$. These modifications change the polyhedral structure of $X \times X$. For example, the torus $\mathbb{S}^1 \times \mathbb{S}^1$ in $X \times X$ consists of 36 squares, but the modifications subdivide each square into two triangles as in Figure 4.4 on the right.

We give three examples of explicit computations of fibers $\pi^{-1}(U, V)$ but omit the full analysis. For convenience, say that a point $U \in X$ *prefers* p_i if it lies on the infinite ray associated to p_i. For our first example, suppose U prefers p_3 and V prefers p_6, and suppose $u, v \in K^*$ are any lifts of U and V. We set $\rho = u/p_3$ and $\sigma = v/p_6$. These two scalars in K^* satisfy $u^{-1}v^{-1} = p_5 s^{-2} \rho^{-1} \sigma^{-1}$ and

$$\mathrm{val}(1-\rho) = \delta(p_3, u) > \beta \quad \text{and} \quad \mathrm{val}(1-\sigma) = \delta(p_6, v) > \beta. \tag{4.33}$$

It is a general fact that $\mathrm{val}(1 - xy) = \min\{\mathrm{val}(1-x), \mathrm{val}(1-y)\}$ if $x, y \in K^*$ have valuation 0 and $\mathrm{val}(1-x) \neq \mathrm{val}(1-y)$. Combining this fact with (4.33), we find

$$\delta(u^{-1}v^{-1}, p_5) \;=\; \mathrm{val}(1 - s^2\rho\sigma) \;=\; \mathrm{val}(1-s^2) \;=\; \mathrm{val}(1-s) \;=\; \beta.$$

We conclude that $\mathrm{trop}(\psi(u^{-1}v^{-1}))$ prefers p_5 and is at lattice distance β from the hexagon. Thus we do not modify $X \times X$ above (U, V) since the fiber is a single point.

As a second example, suppose U prefers p_1 and V prefers p_3. If $u, v \in K^*$ are lifts of U and V then $\mathrm{trop}(\psi(u^{-1}v^{-1}))$ prefers p_2. Direct computation shows that the minimum of $\delta(u, p_1), \delta(v, p_3), \delta(u^{-1}v^{-1}, p_2)$ is achieved twice, and this condition characterizes the possibilities for $\mathrm{trop}(\psi(u^{-1}v^{-1}))$. For example, if $\delta(u, p_1) = \delta(v, p_3) = d$, then $\mathrm{trop}(\psi(u^{-1}v^{-1}))$ can be any point that prefers p_2 and is distance at least d from the hexagon. Thus, we

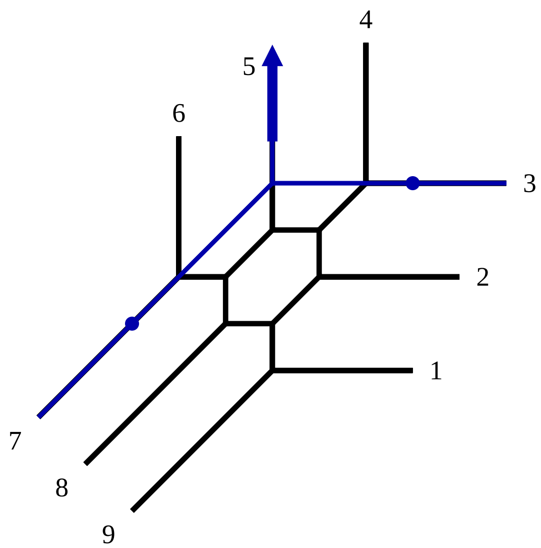


Figure 4.5: Constraints on the intersection points of a cubic and a line.

modify $X \times X$ at (U, V) by attaching a ray representing the points on the ray of p_2 at distance $\geq d$ from the hexagon.

Our third example is similar, but we display it visually. Let U, V be the points shown in blue in Figure 4.5, each at distance 2β from the hexagon. Then $\pi^{-1}(U, V)$ is the thick blue subray that starts at distance β from the node of the tropical line.

In this way, we modify the surface $X \times X$ at each point (U, V) as prescribed by Proposition 4.4.3. A detailed case analysis yields the f-vector in Corollary 4.4.4. □

We note that the combinatorics of the tropical group law surface TGL(ψ) depends very much on ψ. For example, if ψ is a non-symmetric honeycomb embedding, then the torus in TGL(ψ) can contain quadrilaterals and pentagons, as shown in Figure 4.4. For this reason, it seems that there is no "generic" combinatorial description of the surface TGL(ψ) as ψ ranges over all honeycomb embeddings.

Chapter 5

Tropical bases and determinantal varieties

This chapter is based on the paper "The 4×4 minors of a $5 \times n$ matrix are a tropical basis" [CJR09] with Anders N. Jensen and Elena Rubei.

5.1 Introduction

The tropical semi-ring $(\mathbb{R}, \oplus, \odot)$, consisting of the real numbers equipped with tropical addition and multiplication

$$x \oplus y := \min(x, y) \qquad \text{and} \qquad x \odot y := x + y \qquad \text{for all} \qquad x, y \in \mathbb{R},$$

gives rise to three distinct notions of rank of a tropical matrix $A \in \mathbb{R}^{d \times n}$. These, tropical rank, Kapranov rank, and Barvinok rank, were studied in [DSS05]. They arise as the tropicalizations of three equivalent characterizations of matrix rank in the usual sense.

Indeed, classically, a $d \times n$ matrix with entries in a field K has rank at most r if and only if all of its $(r+1) \times (r+1)$ submatrices are singular. Equivalently, the set of $d \times n$ matrices of rank at most r is the determinantal variety defined by the ideal $J_r^{dn} \subseteq K[x_{11}, \ldots, x_{dn}]$ generated by the $(r+1) \times (r+1)$ minors of a $d \times n$-matrix of variables. Finally, this algebraic variety is the image of the matrix product map $\phi : K^{d \times r} \times K^{r \times n} \to K^{d \times n}$.

Accordingly, the set of matrices of tropical rank $\leq r$ is defined to be the intersection of the tropical hypersurfaces defined by the $(r+1) \times (r+1)$ minors in $K[x_{11}, \ldots, x_{dn}]$. The set of matrices of Kapranov rank $\leq r$ is defined to be the tropical variety $T(J_r^{dn})$, while the set of matrices of Barvinok rank $\leq r$ is the image of the tropicalization of ϕ. We will revisit these definitions in Section 5.2. We note that $T(J_r^{dn})$ can be regarded as the space of n labeled points in $\mathbb{TP}^{d-1}$ (defined in Section 2) for which there exists a tropicalized $r-1$ plane containing them; here, we consider matrices up to the equivalence relation of tropically scaling columns.

Since the intersection of the tropical hypersurfaces defined by a set of polynomials does not always equal the tropical variety of the ideal they generate, we do not expect

Kapranov rank and tropical rank to be the same. Similarly, the tropicalization of the image of a polynomial function is not always equal to the image of its tropicalization; therefore, we do not expect Barvinok rank and Kapranov rank to be the same. However, in both of these cases one containment is true, implying

$$\text{Tropical rank}(A) \leq \text{Kapranov rank}(A) \leq \text{Barvinok rank}(A). \tag{5.1}$$

as shown in [DSS05, Theorem 1.4].

We are interested in studying Kapranov rank and tropical rank. The question of whether these coincide is really a question about tropical bases. Recall that a *tropical basis* for an ideal $I \subseteq K[x_1, \ldots, x_n]$, where K is the field of generalized Laurent series to be defined in Section 2, is a finite generating set with hypersurface intersection equal to $T(I)$. The authors of [BJSST07] prove that any ideal I generated by polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ has a tropical basis. It is of fundamental interest to understand the geometry of intersections of tropical hypersurfaces and varieties, and to develop methods to recognize tropical bases. We note the tropical varieties are most naturally defined for ideals I of the Laurent polynomial ring $K[x_1^{\pm 1}, \ldots, x_n^{\pm}1]$, and that it is still possible to define tropical bases in this context; see [MS10, Section 2.5]. However, we cannot define a Gröbner complex for these ideals and therefore cannot give $T(I)$ a reasonable polyhedral structure, so we will stick to ideals $I \subseteq K[x_1, \ldots, x_n]$ here.

Using the language of tropical bases, it is natural to ask:

Question 5.1.1. *For which numbers d, n, and r do the $(r+1) \times (r+1)$-minors of a $d \times n$ matrix form a tropical basis? Equivalently, for which d, n, r does every $d \times n$ matrix of tropical rank at most r have Kapranov rank at most r?*

Theorems 5.5 and 6.5 in [DSS05] show that if the tropical rank or the Kapranov rank of a matrix is 1, 2, or $\min(d, n)$, then these two ranks are equal. As a corollary, if A is a $d \times n$ real matrix with d or $n \leq 4$, then tropical rank of A equals its Kapranov rank. In this chapter, we prove the following result.

Theorem 5.1.2. *For $n \geq 4$, the 4×4 minors of a $5 \times n$ matrix form a tropical basis. Thus, if A is a $d \times n$ matrix over $\mathbb{R}$ with d or $n \leq 5$, then the tropical rank of A equals the Kapranov rank of A.*

Our theorem specifically answers the open question [DSS05, Section 8,(6)].

The organization of this chapter is as follows. In the next section, we review the basic definitions of tropical prevarieties and varieties, with a focus on the determinantal case. Then, in Section 5.3, we give a proof of Theorem 5.1.2 using the technique of stable intersections and an analysis of types similar to those in [AD09]. We refer the reader to [CJR09] for a computational proof, which makes use of the software `gfan` [Jen], that shows that the 4×4 minors of a 5×5 matrix form a tropical basis, and [Rub07] for an alternative, longer proof of Theorem 5.1.2.

After the publication of [CJR09], Y. Shitov subsequently solved the remaining cases in [Shi10] and [Shi11], settling Question 5.1.1 completely. The results are summarized in Table 5.1.

r, d	3	4	5	6	7	8
3	yes [DSS05]	yes [DSS05]	yes [DSS05]	yes [DSS05]	yes [DSS05]	yes [DSS05]
4		yes [DSS05]	yes [CJR09]	yes [Shi11]	no [DSS05]	no [DSS05]
5			yes [DSS05]	no [Shi10]	no [Shi10]	no [DSS05, Shi10]
6				yes [DSS05]	no [Shi10]	no [Shi10]
7					yes [DSS05]	no [Shi10]
8						yes [DSS05]

Table 5.1: Do the $r \times r$ minors of a $d \times n$ matrix of indeterminates, with $d \leq n$, form a tropical basis for the ideal they generate?

We close this section by noting that the authors of [DSS05] were far from the only ones to consider notions of rank in the min-plus setting. Rather, there is a substantial body of literature along these lines. See, for instance, the work of Akian, Gaubert, and Guterman on linear independence in the setting of a symmetrized max-plus semiring in [AGG09a] and [AGG09b] and the work of Izhakian and Rowen [IR09] on matrix ranks over the extended tropical semiring. We also refer to the paper of Kim and Roush [KR06] on the computational complexity of computing Kapranov rank. Finally, see Figure 1 in [AGG09a] for a partially ordered set comparing ten different notions of min-plus rank. These include Barvinok rank, referred to as factor rank in that paper, and tropical rank. Of the three different ranks studied in [DSS05], Kapranov rank is in some sense the newest and the least well-studied: it is intrinsically a tropical algebro-geometric notion and has no elementary description in terms of min-plus operations. This chapter can be viewed as a contribution to its study.

5.2 Tropical determinantal varieties

We remind the reader of the basic definitions in tropical geometry. Let K be the field whose elements are generalized Laurent series in t with complex coefficients and real exponents, such that the set of exponents involved in a series is a well-ordered subset of $\mathbb{R}$. The valuation map val : $K^* \to \mathbb{R}$ takes a series to the exponent of its lowest order term. Denote by val : $(K^*)^N \to \mathbb{R}^N$ the N-fold Cartesian product of val. The **tropicalization** of a subvariety $V(I)$ of the torus $(K^*)^N$ defined by an ideal $I \subseteq K[x_1, \ldots, x_N]$ is $\mathrm{val}(V(I)) \subseteq \mathbb{R}^N$. With small modifications to our definitions, the results in this chapter should hold for any algebraically closed field K with a nonarchimedean valuation whose image is dense in the reals; these assumptions are necessary to accomodate our use of the Fundamental Theorem of Tropical Geometry, the characterization of Kapranov rank in terms of tropicalized linear spaces in [DSS05, Theorem 3.3], and the results on stable intersections [Spe05b, Proposition 4.4.1, Theorem 4.4.6, Proposition 4.5.3].

For $\omega \in \mathbb{R}^N$, the **ω-degree** of a monomial $cx^a = cx_1^{a_1} \cdots x_N^{a_N}$ is $\mathrm{val}(c) + \langle \omega, a \rangle$. The **initial form** $\mathrm{in}_\omega(f) \in \mathbb{C}[x_1, \ldots, x_N]$ of a polynomial $f \in K[x_1, \ldots,$ $x_N]$ with respect to ω is the sum of terms of the form $\gamma t^b x^a$ ($\gamma \in \mathbb{C}$) in f with minimal ω-degree, but with 1 substituted for t. Define the **initial ideal**

$$\mathrm{in}_\omega(I) := \langle \mathrm{in}_\omega(f) : f \in I \rangle \subseteq \mathbb{C}[x_1, \ldots, x_N].$$

The Fundamental Theorem of Tropical Geometry, variously attributed to Draisma, Kapranov, Speyer-Sturmfels (see [Dra08],[SS04]), says that $\mathrm{val}(V(I))$ equals the **tropical variety** $T(I)$, with

$$T(I) := \{\omega \in \mathbb{R}^N : \mathrm{in}_\omega(I) \text{ does not contain a monomial}\}.$$

The **Gröbner complex** $\Sigma(I)$ of a homogeneous ideal I, see [MS10], is the polyhedral complex consisting of all polyhedra

$$C_\omega(I) := \overline{\{\omega' \in \mathbb{R}^N : \mathrm{in}_\omega(I) = \mathrm{in}_{\omega'}(I)\}},$$

where ω runs through $\mathbb{R}^N$, and the closure is taken in the usual Euclidean topology of $\mathbb{R}^N$. It is clear that the tropical variety $T(I)$ is the support of a subcomplex of $\Sigma(I)$, and we shall not distinguish between $T(I)$ and this subcomplex.

By the **linear span** of a polyhedron $P \subseteq \mathbb{R}^N$ we mean the $\mathbb{R}$-span of $P - P := \{p - p' : p, p' \in P\}$. The intersection of the linear spans of all the polyhedra in a complex is called the **lineality space** of the complex. A complex is invariant under translation by elements of its lineality space. Since I is homogeneous, the lineality space of $\Sigma(I)$ contains the $(1, \ldots, 1)$ vector and it makes sense to consider $T(I)$ in the **tropical torus** $\mathbb{TP}^{N-1} := \mathbb{R}^N / \sim$, where we quotient out by coordinate-wise tropical multiplication by a constant.

If I is a principal ideal $\langle f \rangle$, where $f = \sum_i c_i x^{a_i}$ with $c_i \in K^*$, the tropical variety is called a **hypersurface**. It consists of all $\omega \in \mathbb{R}^N$ such that the minimum

$$\bigoplus_i \mathrm{val}(c_i) \odot \langle \omega, a_i \rangle \tag{5.2}$$

is attained at least twice.

In the special case where I is defined by polynomials with coefficients in $\mathbb{C}$, the complex $\Sigma(I)$ is a fan. In this chapter, we study two kinds of tropical varieties: those defined by linear ideals in $K[x_1, \ldots, x_N]$, which yield polyhedral complexes; and those which are sets of matrices of Kapranov rank at most r, and are therefore polyhedral fans (since their ideals are defined over $\mathbb{C}$). In the latter case, we use the terms **Gröbner fan** and **Gröbner cones** for $\Sigma(I)$ and its cones.

Definition 5.2.1. Let A, B be polyhedral fans. The **common refinement** of A and B is the fan

$$A \wedge B := \{a \cap b : a \text{ is a cone of } A \text{ and } b \text{ is a cone of } B\}.$$

The *support* of a fan A is the set of points $p \in \mathbb{R}^n$ such that p lies in some cone of A. Note that the support of $A \wedge B$ is the intersection of the support of A and the support of B.

Definition 5.2.2. Given a set $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq K[x_1, \ldots, x_N]$, its **tropical prevariety** is the intersection

$$\bigcap_i T(\langle f_i \rangle).$$

The set $\mathcal{F}$ is a **tropical basis** if its prevariety equals $T(\langle f_1, \ldots, f_m \rangle)$. If each $T(\langle f_i \rangle)$ is a fan, then the prevariety can be regarded as their common refinement and hence is a fan.

We can now give precise definitions of rank.

Definition 5.2.3. Let $\mathcal{F}_r^{dn} \subseteq K[x_1, \ldots, x_{dn}]$ be the set of $(r+1) \times (r+1)$ minors of the $d \times n$ matrix $\{x_{ij}\}$. Let $J_r^{dn} = \langle f : f \in \mathcal{F}_r^{dn} \rangle$, and $A \in \mathbb{R}^{d \times n}$.

- A has **tropical rank** at most r if $A \in \bigcap_{f \in \mathcal{F}_r^{dn}} T(\langle f \rangle)$.
- A has **Kapranov rank** at most r if $A \in T(J_r^{dn})$.

Equivalently, by the Fundamental Theorem, a matrix A has Kapranov rank at most r if it has a **lift** $\tilde{A}$ over K of rank at most r. By a lift we mean a matrix $\tilde{A}$ such that $\mathrm{val}(\tilde{A}) = A$.

Example 5.2.4. Let $f \in K[x_{11}, \ldots, x_{33}]$ be the 3×3 determinant,

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \tilde{A} = \begin{pmatrix} 1 & t & t^2 \\ 2t & 3t & 5t \\ 1+2t & 4t & 5t+t^2 \end{pmatrix}.$$

The tropical hypersurface $T(\langle f \rangle)$ contains A since (5.2) attains its minimum three times. Hence, A has tropical rank at most 2. Equivalently, $\mathrm{in}_A(f) = x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} + x_{12}x_{23}x_{31}$ is not a monomial. The tropical rank is not less than 2, since $\mathrm{in}_A(x_{11}x_{22} - x_{12}x_{21}) = x_{11}x_{22}$. Thus the tropical rank is 2. To argue about the Kapranov rank, we consider the lift $\tilde{A} \in K^{3\times 3}$ above. The classical rank of $\tilde{A}$ is 2. By the fundamental theorem, or since $\mathrm{in}_A(J_2^{33})$ is monomial-free, A has Kapranov rank at most 2. Since its Kapranov rank is at least its tropical rank, it is equal to 2. $\diamond$

5.3 The 4×4 minors of a $5 \times n$ matrix are a tropical basis

The goal of this section is to prove our main theorem, Theorem 5.1.2. For an alternative proof, see [Rub07]. For a computational proof of the 5×5 case, see [CJR09].

The idea of the proof presented here is as follows.. Given a $5 \times n$ matrix of tropical rank at most 3, we will produce five tropical hyperplanes, each containing the n column vectors of A. Then, we will argue that some pair of these hyperplanes must contain these n points in their stable intersection and conclude that the Kapranov rank is at most 3. The central argument is an analysis of the possible combinatorial types of the point-hyperplane incidences. We begin with some definitions. In the following the ambient space will be $\mathbb{TP}^{d-1}$ with d general, but we will later specialize to the case where d is the number of rows of the matrix, in our case $d = 5$.

Definition 5.3.1. Let H be the tropical hyperplane in $\mathbb{TP}^{d-1}$ corresponding to the tropical polynomial $h_1 \odot x_1 \oplus \cdots \oplus h_d \odot x_d$; that is, H is the set of points $(x_1, \ldots, x_d)$ at which $h_1 \odot x_1 \oplus \cdots \oplus h_d \odot x_d$ attains its minimum at least twice, and we write

$$H = T(h_1 \odot x_1 \oplus \cdots \oplus h_d \odot x_d).$$

Let $w = (w_1, \ldots, w_d)$ be a point in $\mathbb{TP}^{d-1}$.

Then the **type** of w with respect to H, denoted $\operatorname{type}_H w$, is the subset of $[d]$ of those indices at which the minimum of $h_1 \odot w_1, \ldots, h_d \odot w_d$ is attained. Thus, $|\operatorname{type}_H w| \geq 2$ if and only if $w \in H$.

Note that our definition of type is similar to the definition by Ardila and Develin in [AD09] which gives rise to tropical oriented matroids. The only difference is that our types are taken with respect to a single hyperplane instead of a hyperplane arrangement.

Types have a natural geometric interpretation, as follows. A tropical hyperplane H divides $\mathbb{TP}^{d-1}$ into d **sectors**: the ith closed sector consists of those points w for which the minimum when the tropical function corresponding to H is evaluated at w is attained at coordinate i. The type of a point records precisely in which closed sectors it lies. Figure 5.1 illustrates the case of a tropical line in $\mathbb{TP}^2$.

Recall that a **tropicalized linear space** is the tropicalization of a classical linear variety in $K[x_1, \ldots, x_d]$. If the linear space is a classical hyperplane, then we just call its tropicalization a **tropical hyperplane** as in Definition 5.3.1. The **stable intersection** of two tropical linear spaces L and L' is

$$\lim_{v \to 0} L \cap (L' + v)$$

where v approaches 0 along a ray whose direction lies in an open dense subset of S^{n-1} (see [Spe08, p. 1545] for a discussion on the topology on the set of the polyhedral complexes). It is itself a tropicalized linear space, see [Spe05b, Proposition 4.4.1, Theorem 4.4.6]. To clarify, a point w lies in the stable intersection of L and L' if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ and an open dense subset $A \subseteq S^{n-1}$ such that for each $v \in \mathbb{R}^n$ with

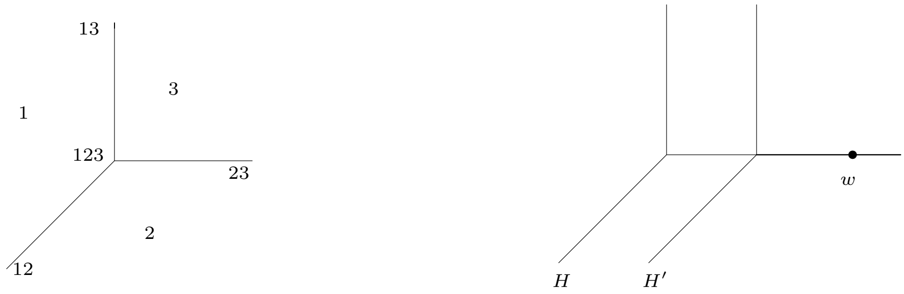


Figure 5.1: A hyperplane in $\mathbb{TP}^2$ partitions the points of the plane according to their types. On the right, a point $w \in \mathbb{TP}^2$ can lie in the intersection of two hyperplanes H and H' but not in their stable intersection.

$\|v\|_\infty < \delta$ and direction in A, there exists $\tilde{w} \in L \cap (L' + v)$ with $\|\tilde{w} - w\|_\infty < \varepsilon$. (We use the L^∞ norm in our definition for ease of exposition; it is equivalent to using the L^2 norm by a standard argument in analysis.)

Figure 5.1 shows a point that is contained in two hyperplanes but not in their stable intersection. The following proposition characterizes this situation in terms of types.

Proposition 5.3.2. *Let H, H' be hyperplanes in $\mathbb{TP}^{d-1}$, and let $w \in \mathbb{TP}^{d-1}$ be a point lying on both H and H'. Then w does not lie in their stable intersection precisely when* $\operatorname{type}_H w = \operatorname{type}_{H'} w$ *and they are a set of size two.*

Proof. Given w, H, and H', let Δ be the difference between the minimum and the second smallest number when the tropical function corresponding to H is evaluated at w, or ∞ if only one value occurs. Define Δ' with respect to H' similarly. Also, write

$$H = T(a_1 \odot x_1 \oplus \cdots \oplus a_d \odot x_d), H' = T(b_1 \odot x_1 \oplus \cdots \oplus b_d \odot x_d),$$

for real numbers a_i, b_i.

Suppose $\operatorname{type}_H w \neq \operatorname{type}_{H'} w$ or $|\operatorname{type}_H w| \geq 3$ or $|\operatorname{type}_{H'} w| \geq 3$, that is $|\operatorname{type}_H w \cup \operatorname{type}_{H'} w| \geq 3$. Given $\varepsilon > 0$, let $\delta = \frac{1}{2}\min\{\varepsilon, \Delta, \Delta'\}$. Let $v \in \mathbb{R}^n$ satisfy $\|v\|_\infty < \delta$. We wish to find a point $\tilde{w} \in H \cap (H' + v)$ such that $\|\tilde{w} - w\|_\infty < \varepsilon$.

If $w \in H' + v$ then we may choose $\tilde{w} = w$ and we are done, so assume instead that the minimum when $H' + v$ is evaluated at w is achieved uniquely, say at coordinate i. Furthermore, since $\|v\|_\infty < \frac{1}{2}\Delta'$, the fact that $i \in \operatorname{type}_{H'+v} w$ implies $i \in \operatorname{type}_{H'} w$ (in fact suppose $i \notin \operatorname{type}_{H'} w$ and let $k \in \operatorname{type}_{H'} w$, then, for the definition of Δ', we have $b_i + w_i - b_k + w_k \geq \Delta' > v_i - v_k$, therefore $b_i + w_i - v_i > b_k + w_k - v_k$, which contradicts $i \in \operatorname{type}_{H'+v} w$).

We have two cases:

1) $\operatorname{type}_H w$ does not have 2 elements different from i; thus $\operatorname{type}_H w$ has exactly two elements, say $1, 2$, and one of them is i. Pick some $k \in \operatorname{type}_{H'} w \setminus \operatorname{type}_H w$ that is $k \in \operatorname{type}_{H'} w \setminus \{1, 2\}$. This is possible since $|\operatorname{type}_{H'} w| \geq 2$ and $|\operatorname{type}_H w \cup \operatorname{type}_{H'} w| \geq 3$.

2) $\operatorname{type}_H w$ has at least two elements different from i. Then pick some $k \in \operatorname{type}_{H'} w \setminus \{i\}$.

Let

$$t = \begin{cases} \min_{3 \le j \le d}(b_j - v_j + w_j) - (b_i - v_i + w_i) & \text{in case 1,} \\ \min_{j \in [d] \setminus \{i\}}(b_j - v_j + w_j) - (b_i - v_i + w_i) & \text{in case 2,} \end{cases}$$

and let

$$\tilde{w} = \begin{cases} (w_1 + t, w_2 + t, w_3, \ldots, w_d) & \text{in case 1,} \\ (w_1, \ldots, w_i + t, \ldots, w_d) & \text{in case 2.} \end{cases}$$

Now, we claim that, in both cases, $t < 2\delta$:

$$\begin{aligned} t &\le b_k - v_k + w_k - (b_i - v_i + w_i) \\ &\le |b_k - v_k + w_k - (b_k + w_k)| + |(b_k + w_k) - (b_i + w_i)| \\ &\quad + |(b_i + w_i) - (b_i - v_i + w_i)| \\ &\le |v_k| + |v_i| < 2\delta, \end{aligned}$$

where the fact that $b_k + w_k = b_i + w_i$ follows from the fact that $\{i, k\} \subseteq \operatorname{type}_{H'} w$. Thus, $\|\tilde{w} - w\|_\infty = t < 2\delta \le \varepsilon$.

In Case 1, $\tilde{w}$ lies on H since $\operatorname{type}_H w = \{1, 2\}$ and $t < 2\delta \le \Delta$, and $\tilde{w}$ lies on $H' + v$ by construction. Also, in Case 2, $\tilde{w}$ lies on H since $\operatorname{type}_H w$ has at least two elements different from i, and $\tilde{w}$ lies on $H' + v$ by construction, as desired.

For the converse, suppose that $\operatorname{type}_H w = \operatorname{type}_{H'} w$ is a set of size two; we may assume it is $\{1, 2\}$. Let P be the affine linear span of the face in H containing w. This equals the affine linear span of the face in H' containing w since $\operatorname{type}_H w = \operatorname{type}_{H'} w$. Since the faces of H (and H') are closed, and $|\operatorname{type}_H w| = 2$ (and $|\operatorname{type}_{H'} w| = 2$) implies that w is contained in just one face of H (and H'), there exists $\varepsilon > 0$ such that

$$H \cap B(w, 2\varepsilon) = P \cap B(w, 2\varepsilon) = H' \cap B(w, 2\varepsilon),$$

where $B(w, \varepsilon)$ is the ε-ball centered at w. For any $\delta > 0$, pick v with first coordinate different from the second and norm less or equal than $\min(\delta/2, \varepsilon)$. Now

$$\begin{aligned} B(w, \varepsilon) \cap (H' + v) &= ((B(w, \varepsilon) - v) \cap H') + v \\ &\subseteq (B(w, 2\varepsilon) \cap H') + v \\ &= (B(w, 2\varepsilon) \cap P) + v \subseteq P + v \end{aligned}$$

which shows that $B(w, \varepsilon) \cap (H' + v) \cap H \subseteq (P + v) \cap P = \emptyset$.

□

Proposition 5.3.3. *Let H, H' be tropical hyperplanes in $\mathbb{TP}^{d-1}$. Then there exists a codimension 2 linear space L over K whose tropicalization is the stable intersection of H and H'.*

Proof. By [Spe05b, Proposition 4.5.3], we may lift H and H' generically to classical hyperplanes $\mathcal{H}$ and $\mathcal{H}'$ over K such that the tropicalization of $\mathcal{H} \cap \mathcal{H}'$ is the stable intersection of H and H'.

□

Now let W be a set of points in $\mathbb{TP}^{d-1}$, and let $i \in \{1, \ldots, d\}$. We say that a hyperplane H that contains each point in W is an **i-coordinate hyperplane** for W if $\operatorname{type}_H w$ does not contain i for any $w \in W$. That is, no point in W lies in the ith closed sector of H.

Next, suppose w is a point contained in two hyperplanes H and H' but not in their stable intersection. Then, by Proposition 5.3.3, w has type $\{a, b\}$ with respect to both H and H', for some a and b. Then we say that w is a **witness of type ab** to the nonstable intersection of H and H'. In the case that H and H' are k- and l-coordinate hyperplanes, respectively, for a set of points W containg w, we note that the sets $\{a, b\}$ and $\{k, l\}$ must be disjoint.

We are now ready to state a proposition which will serve as the combinatorial heart of our proof of Theorem 5.1.2.

Proposition 5.3.4. *Let $W = \{w_1, \ldots, w_n\}$ be a subset of points in $\mathbb{TP}^4$, and for each i with $1 \le i \le 5$, let H_i be a hyperplane containing each point in W such that H_i is an i-coordinate hyperplane for W. Suppose further that for every pair of hyperplanes H_i and H_j, the intersection $H_i \cap H_j$ is not the stable intersection of H_i and H_j and some $w_s \in W$ witnesses this nonstable intersection.*

Let i, j, k, l, m be distinct elements in $\{1, 2, 3, 4, 5\}$. Suppose H_i and H_j have a witness in W of type kl. Then any witness in W for H_i and H_k has type lm.

Proposition 5.3.4 follows from the following two lemmas, whose proofs we postpone to the end of the section.

Lemma 5.3.5. *Let $W = \{w_1, \ldots, w_n\}$, $H_1, \ldots, H_5$ be as in the first paragraph of Proposition 5.3.4. Let i, j, k, l be distinct elements in $\{1, \ldots, 5\}$, and assume without loss of generality that H_i and H_j have a witness in W of type kl. Then any witness in W for H_i and H_k must be of type containing l.*

Lemma 5.3.6. *Let $W = \{w_1, \ldots, w_n\}$, $H_1, \ldots, H_5$ be as in the first paragraph of Proposition 5.3.4. Let i, j, k, l be distinct elements in $\{1, \ldots, 5\}$. Then it is not possible that H_i and H_k have a witness in W of type jl and that H_i and H_j have a witness in W of type kl.*

Proof of Proposition 5.3.4. H_i and H_k have some witness to their nonstable intersection; it must be of type jl, jm, or lm. But it is not of type jm by Lemma 5.3.5, and it is not of type jl by Lemma 5.3.6.

□

Before proving the lemmas, we first prove that Proposition 5.3.4 implies the main result.

Proof of Theorem 5.1.2. Fix $n \ge 4$; let A be a $5 \times n$ real matrix, and let $W = \{w_1, \ldots, w_n\}$ be the set of its column vectors. Suppose that the tropical rank of A is ≤ 3. We wish to show that the Kapranov rank is ≤ 3, or equivalently, that there exists a 3-dimensional subspace in K^5 whose tropicalization contains each point $w_1, \ldots, w_n$.

Let A' be the $4 \times n$ matrix obtained by deleting the first row of A. Then the tropical rank of A' is ≤ 3, so by [DSS05, Theorem 5.5, 6.5], the Kapranov rank of A' is ≤ 3, so the columns of A' lie on some tropical hyperplane, say

$$T(h_{12} \odot x_2 \oplus h_{13} \odot x_3 \oplus h_{14} \odot x_4 \oplus h_{15} \odot x_5).$$

Then, for N sufficiently large,

$$H_1 := T(N \odot x_1 \oplus h_{12} \odot x_2 \oplus h_{13} \odot x_3 \oplus h_{14} \odot x_4 \oplus h_{15} \odot x_5)$$

is a hyperplane containing the columns of A, indeed a 1-coordinate hyperplane, where none of $w_1, \ldots, w_n$ has type containing 1.

Similarly, we may choose H_2, H_3, H_4, H_5 to be 2, 3, 4, 5-coordinate hyperplanes, respectively, for the points $w_1, \ldots, w_n$.

We claim that for some i, j with $1 \leq i < j \leq 5$, H_i and H_j contain each $w_1, \ldots, w_n$ in their stable intersection. If so, we are done by Proposition 5.3.3.

Suppose, then, that the claim is not true, so that for every i, j with $1 \leq i < j \leq 5$, some point in W witnesses the nonstable intersection of H_i and H_j. We derive a contradiction as follows.

By symmetry, we may assume that H_1 and H_2 have a witness of type 34. We now apply Proposition 5.3.4 four times to get a contradiction. First, H_1 and H_3 have a witness in W to their nonstable intersection by assumption; it is of type 45 by Proposition 5.3.4. Similarly, H_1 and H_4 have a witness in W of type 35. Applying Proposition 5.3.4 to these two facts, we get that any witness in W for H_1 and H_5 must have type 24, and similarly, that any witness in W for H_1 and H_5 must have type 23. Since H_1 and H_5 do have a witness in W, by assumption, this is a contradiction. □

Proof of Lemma 5.3.5. By symmetry, assume $i = 1, j = 2, k = 4, l = 5$. Suppose H_1 and H_2 have a witness of type 45, and that H_1 and H_4 have a witness of type not containing 5 – that is, we assume that H_1 and H_4 have a witness of type 23. We wish to derive a contradiction.

For each s with $1 \leq s \leq 5$, write

$$H_s = T(h_{s1} \odot x_1 \oplus \cdots \oplus h_{s5} \odot x_5), \text{with } h_{sr} \in \mathbb{R}.$$

By translating each hyperplane and each point, we may assume that

$$H_1 = T(0 \odot x_1 \oplus \cdots \oplus 0 \odot x_5),$$

and for each s with $2 \leq s \leq 5$, we may assume, by tropically scaling the coefficients of H_s, that $h_{s1} = 0$. Furthermore, since H_1 and H_2 have a witness of type 45, and $h_{14} = h_{15}$, it follows that $h_{24} = h_{25}$. Similarly, $h_{42} = h_{43}$. Summarizing, we have

$$\begin{aligned} H_1 &= T(0x_1 \oplus 0x_2 \oplus 0x_3 \oplus 0x_4 \oplus 0x_5), \\ H_2 &= T(0x_1 \oplus ex_2 \oplus bx_3 \oplus ax_4 \oplus ax_5), \\ H_4 &= T(0x_1 \oplus cx_2 \oplus cx_3 \oplus fx_4 \oplus dx_5), \end{aligned}$$

with $a, b, c, d, e, f \in \mathbb{R}$. By symmetry (i.e. switching 2 with 4 and 3 with 5), we may assume $a \leq c$.

Now, we claim $b > a$. Indeed, let $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$ be a witness of type 23 for H_1 and H_4. Since $\text{type}_{H_1}(\gamma) = 23$, we have that the minimum of $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ is attained twice, in fact, precisely at γ_2 and γ_3. Tropically rescaling, we may assume that $\gamma_2 = \gamma_3 = 0$ and that $\gamma_1, \gamma_4, \gamma_5 > 0$. Since $\text{type}_{H_4}(\gamma) = 23$, we have that $\min(\gamma_1, c+\gamma_2, c+\gamma_3, f+\gamma_4, d+\gamma_5)$ is attained precisely at $c+\gamma_2 = c+\gamma_3 = c$, so $\gamma_1 > c$. Finally, since $\gamma \in H_2$, and H_2 is a 2-coordinate hyperplane, we have that $\min(\gamma_1, b+\gamma_3, a+\gamma_4, a+\gamma_5)$ is achieved twice. Since $\gamma_1 > c \geq a$, $\gamma_3 = 0$, $\gamma_4, \gamma_5 > 0$, this is only possible if $b > a$.

Next, we claim $d > a$. The proof is similar. Let $\chi = (\chi_1, \ldots, \chi_5) \in \mathbb{TP}^4$ be a witness of type 45 for H_1 and H_2. Using that $\text{type}_{H_1} \chi = \text{type}_{H_2} \chi = \{4, 5\}$ and tropically rescaling, we have $\chi_1 > a$, $\chi_2 > 0$, $\chi_3 > 0$, $\chi_4 = \chi_5 = 0$. Together with $a \leq c$ this implies $c + \chi_2, c + \chi_3 > a$. But $\chi \in H_4$ and H_4 is a 4-coordinate hyperplane, so $\min(\chi_1, c+\chi_2, c+\chi_3, d+\chi_5)$ is attained twice, and since $\chi_1, c+\chi_2, c+\chi_3$ are all $> a$ and $\chi_5 = 0$, we have $d > a$.

Now, H_2 and H_4 have some witness of nonstable intersection, say $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \in \mathbb{TP}^4$, where ψ is a witness of type 13, 15, or 35. Since $h_{21} = h_{41} = 0$, but $h_{25} = a \neq d = h_{45}$, it is not type 15, so it is of type 13 or 35.

Suppose ψ is of type 35, so $\text{type}_{H_2} \psi = \text{type}_{H_4} \psi = \{3, 5\}$. Then, rescaling, we may assume $\psi_3 = a$, $\psi_5 = b$, $\psi_2 > a$, and $\psi_4 > b$. But we showed $b > a$, so $\min(\psi_2, \psi_3, \psi_4, \psi_5)$ is attained uniquely, contradicting that $\psi \in H_1$ and H_1 is a 1-coordinate hyperplane.

So ψ must be a witness of type 13 for H_2 and H_4. Since $h_{21} = h_{41} = 0$, we have $h_{23} = h_{43}$, that is, $b = c$. Furthermore, using $\text{type}_{H_2} \psi = \text{type}_{H_4} \psi = \{1, 3\}$, and rescaling ψ, we may assume that $\psi_1 = b$, $\psi_2 > 0$, $\psi_3 = 0$, $\psi_4 > b - a$, $\psi_5 > b - a$. But since $b - a > 0$ and H_1 is a 1-coordinate hyperplane, $\psi \notin H_1$, contradiction. This proves Lemma 5.3.5. □

Proof of Lemma 5.3.6. By symmetry, we may assume $i = 1, j = 4, k = 3, l = 2$, and we suppose for a contradiction that H_1 and H_3 have a witness, γ, of type 24, and H_1 and H_4 have a witness, χ, of type 23. We may assume, by translating and rescaling, that

$$\begin{aligned} H_1 &= T(0x_1 \oplus 0x_2 \oplus 0x_3 \oplus 0x_4 \oplus 0x_5), \\ H_3 &= T(0x_1 \oplus ax_2 \oplus ex_3 \oplus ax_4 \oplus bx_5), \\ H_4 &= T(0x_1 \oplus cx_2 \oplus cx_3 \oplus fx_4 \oplus dx_5), \end{aligned}$$

for some $a, b, c, d, e, f \in \mathbb{R}$. Then, rescaling, we may assume that $\gamma = (\gamma_1, 0, \gamma_3, 0, \gamma_5)$ where $\gamma_1 > a$, $\gamma_3 > 0$, and $\gamma_5 > a - b$. Similarly, we may assume that $\chi = (\chi_1, 0, 0, \chi_4, \chi_5)$, where $\chi_1 > c$, $\chi_4 > 0$, and $\chi_5 > c - d$.

Now, by hypothesis, H_3 and H_4 have some witness ψ to their nonstable intersection; its type must be 12, 15, or 25, since types containing 3 or 4 may not occur.

Suppose it is type 12. Then $a = c$. That γ lies on H_4 implies that $\min(\gamma_1, c, c+\gamma_3, d+\gamma_5)$ is attained twice; since $\gamma_1 > a = c$ and $c+\gamma_3 > c$, we have $c = d+\gamma_5$. Since $\gamma_5 > a-b$, we have $c > d + a - b$, so $b > d$. Symmetrically, $\chi \in H_3$ implies $\min(\chi_1, a, a+\chi_4, b+\chi_5)$

is achieved twice; since $\chi_1 > c = a$, $a + \chi_4 > a$ and $b + \chi_5 > b + c - d = a + b - d$, it follows that $d > b$, contradiction.

Next, suppose ψ is a witness for H_3 and H_4 of type 15. Then $b = d$. Then $\gamma \in H_4$ implies that $\min(\gamma_1, c, c + \gamma_3, d + \gamma_5)$ is achieved twice; since $\gamma_1 > a$, $\gamma_3 > 0$ and $d + \gamma_5 > d + a - b = a$, it follows that $c > a$ (otherwise the minimum is achieved uniquely at c). Symmetrically, $\chi \in H_3$ implies $\min(\chi_1, a, a + \chi_4, b + \chi_5)$ is achieved twice; since $\chi_1 > c$, $a + \chi_4 > a$ and $b + \chi_5 > b + c - d = c$, it follows that $a > c$, contradiction.

Finally, suppose ψ is a witness of type 25. Then $a + d = b + c$, and $\gamma \in H_4$ implies that $\min(\gamma_1, c, c+\gamma_3, d+\gamma_5)$ is attained twice; since $\gamma_1 > a$, $c+\gamma_3 > c$, and $d+\gamma_5 > d+a-b = c$, we have $c > a$. Symmetrically, $\chi \in H_3$ implies $\min(\chi_1, a, a + \chi_4, b + \chi_5)$ is achieved twice; since $\chi_1 > c$, $a + \chi_4 > a$ and $b + \chi_5 > b + c - d = a$, it follows that $a > c$. This is a contradiction and proves Lemma 5.3.6. $\square$

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