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SAN DIEGO STATE UNIVERSITY

The Hidden Strand of Mathematical Proficiency: Defining and Assessing for Productive Disposition in Elementary School Teachers' Mathematical Content Knowledge

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Mathematics and Science Education

by

John (Zig) Michael Siegfried

Committee in charge:

University of California, San Diego

Professor Christopher Halter
Professor Gail Heyman

San Diego State University

Professor Randolph Philipp, Chair
Professor Jessica Bishop
Professor Victoria Jacobs

2012

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The Dissertation of John (Zig) Michael Siegfried is approved, and it is acceptable in quality and for publication on microfilm and electronically:

Chair

University of California, San Diego

San Diego State University

2012

DEDICATION

This is dedicated to Jack Perry, Janice Ulloa-Brown, and Erica Hendrix.

EPIGRAPH

Mathematical discoveries, small or great, are never born of spontaneous generation. They always presuppose a soil seeded with preliminary knowledge and well prepared by labour, both conscious and subconscious.

Jules Henri Poincaré

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Principal Investigators	Randy Philipp, PI Vicki Jacobs, co-PI
Faculty Associates	Lisa Lamb Jessica Pierson Bishop
Research Associate	Bonnie Schappelle
Project Coordinator	Candace Cabral
Student Assistants	Amber Curtis, Jennifer Cumiskey, Darcy Fishbaugh, Kelly Humphrey, Chris Macias-Papierniak, Courtney White

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Lewis, J., Philipp, R. A., Siegfried, J. M., Williams, A., & Hix, S. L. (2011, January). *Using the "Strands of Mathematical Proficiency" to Improve Mathematics Instruction in Multiple Contexts.* Presentation, Association of Mathematics Teacher Educators Annual Conference, Irvine, CA.

Philipp, R. A., Siegfried, J. M., Jacobs, V. R., & Lamb, L. C. (2010, April). *Productive Disposition: The Missing Component of Mathematical Proficiency*. Presentation, Annual Meeting of the National Council of Teachers of Mathematics Research Presession, San Diego, CA.

Philipp, R. A., Jacobs, V. R., Lamb, L. C., & Siegfried, J. M. (2010, April). *Using Video and Student Work Focused on Children's Thinking to Help Professional Developers Support K-3 Teachers in Transforming Their Teaching*. Presentation, National Council of Supervisors of Mathematics, San Diego, CA.

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Philipp, R. A., Siegfried, J. M., Williams, A., & Cline, L. (2010, January). *Studying and Developing Productive Disposition of Elementary School Teachers and Their Students*. Presentation, Association of Mathematics Teacher Educators Annual Conference, Irvine, CA.

Philipp, R. A., Siegfried, J. M., Schappelle, B. P., Jacobs, V. R., & Lamb, L. C. (2009, April). *Assessing Grades K-3 Teachers' Mathematical Content Knowledge: Practice and Theory*. Working Group Presentation, Annual Meeting of the National Council of Teachers of Mathematics Research Presession, Washington, DC.

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Philipp, R. A., Jacobs, V., Lamb, L. L. C., Schappelle, B. P., & Siegfried, J. M. (2008, November). *Studying Teachers' Evolving Perspectives.* Presentation to a Group of Elementary School Teacher Leaders, Encinitas, CA.

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Philipp, R. A., & Siegfried, J. M. (2008, January). *Engaging Prospective Elementary School Teachers in Mathematics Through Children's Thinking.* Presentation at the Joint Meeting of the American Mathematical Society and the Mathematical Association of America, San Diego, California.

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ABSTRACT OF THE DISSERTATION

The Hidden Strand of Mathematical Proficiency: Defining and Assessing for Productive Disposition in Elementary School Teachers' Mathematical Content Knowledge

by

John (Zig) Michael Siegfried

Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2012
San Diego State University, 2012

Professor Randolph Philipp, Chair

Teachers' mathematical content knowledge is one of the most important constructs considered by researchers studying elementary mathematics education (Fennema & Franke, 1992). One component of mathematical content knowledge that is complicated, ill-defined, and oft-ignored is *productive disposition,* defined as the "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (NRC, 2001, p. 116). In this

dissertation I undertook two separate but related studies about the productive dispositions of K–3 elementary school teachers to better understand the construct.

In Study 1, my overarching research question was *What differences in evidence for productive disposition can be found through the analysis of teachers' engagement in a mathematical task?* To answer this question, I assessed for differences and highlighted what might be taken as evidence for teachers' holding strong productive dispositions among136 preservice and in-service K–3 teachers who worked on a challenging mathematical task in focus groups. Building from the mathematics education literature and the focus-group data, I created a list of potential productive-disposition indicators.

In Study 2, my overarching question was *What evidence for productive disposition is self-reported by teachers who have been identified as having strong productive dispositions?* I observed 10 of the in-service elementary school teachers who had participated in Study 1 while they engaged in several mathematical tasks. These teachers were also asked to complete a mathematical autobiography and were interviewed individually. My purpose was to verify with the participants my interpretation of the indicators they exhibited when working on the mathematical tasks and to identify others not recognized in the analysis. Seven productive-disposition traits resonated with all 10 of the teachers, and 3 new traits arose from our discussions. I conclude by discussing why mathematics educators and mathematics teachers should care about productive dispositions.

CHAPTER 1

INTRODUCTION

Section 1.1—Background

Before beginning my graduate program at San Diego State University and the University of California, San Diego, I was a teaching assistant in the mathematics department of the University of California, Santa Barbara. At the time, I thought that my destiny was to become a mathematics professor, but I quickly became disenchanted with teaching at the college level. To me, my students seemed to lack the prerequisite knowledge for college mathematics. They often sought help with homework problems after having spent only five minutes thinking about the problem themselves. From my point of view, nothing seemed to make sense to them, and they would often protest, “Why do we need to learn this?!”

One day, when feeling particularly frustrated, I complained at length to my adviser about my students. I did not understand how they had come to be college students without being more mathematically proficient. After I had vented, instead of explaining that these freshmen were not unusual, my adviser suggested, “Well, why don’t you try working in the schools for a while and see why they are not ready for college?”

Fortunately, I had worked with local teachers in summer workshops to help them understand and use different forms of representations, such as ratio tables and area models, in their classrooms. At one session, I offered to visit the teachers’ classrooms to help them find ways to integrate workshop content into their lessons. In the process, I hoped that they could show me what mathematics was taught and how the students learned in their classrooms.

I quickly learned that the teachers were doing the best they could to present the mathematical content knowledge to the students. Moreover, the teachers generally seemed to have many of the same complaints about their own students that I had about my students: The students often made little effort; they often failed to make mathematical sense; and they did not see the usefulness in the mathematics they were learning.

I began to believe that lack of content knowledge was not necessarily the biggest problem. Although I still believed that understanding mathematical content was important and that many of the students were deficient in that area, I found many students who, I concluded, had the mathematical understandings they needed to solve their problems yet were unable to complete the tasks or even make progress on them. What is wrong with a system that does not actively support more students in developing the kinds of reasoning mathematics educators value? What could I and other teachers do to help these students see mathematics as more useful to their lives and worthwhile studying? It was these nagging questions that piqued my interest in mathematics education and brought me to this study.

Section 1.2—So What Is This *Thing*?

Why do some students actively engage with mathematical tasks, whereas others do not, even though the students may have similar levels of mathematical content understanding? What is this *thing* that some students seem to have that supports them when they engage in mathematical reasoning? What might we teachers do to support more students in developing this quality? Moreover, how can we ensure that *all* students can become mathematically proficient, including having these positive orientations toward learning and engaging with mathematics?

Through a review of the literature in mathematics education, I found a construct that seems to closely describe these orientations toward engaging in and with mathematics. In 2001, the National Research Council [NRC] released its consensus report titled *Adding It Up*. In it, they proposed five strands that constitute one's mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. (I consider these five strands further in Chapter 2.) Although four of the strands seem to refer to mathematical content knowledge, the last term, *productive disposition*, refers to something different. The authors defined productive disposition as "the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (NRC, 2001, p. 131). This strand is related not directly to mathematical knowledge but instead to the attitudes and beliefs people have toward mathematics as well as how closely they identify with being a learner and doer of mathematics. The authors described people with productive dispositions as being confident in their mathematical knowledge and abilities and as able to continue to increase their knowledge and abilities by steady effort and the gaining of new mathematical experiences.

My concerns about my students and those of other teachers about theirs seem to be caused by the students’ not being supported and encouraged to develop productive dispositions. Students lacking productive dispositions may not see themselves as learners or doers of mathematics. They might not see value in steady effort in mathematics. They might not even think that mathematics is supposed to make sense. Productive disposition seems to be precisely what I and other teachers felt these students needed support to develop.

However, in this study, I did not want to assume that productive disposition, as presented in *Adding It Up*, is a complete definition of this thing that supports people engaged in mathematical reasoning. I was open to the idea that it might have other parts as well, and my goal in this study was to identify them. Consider mathematical proficiency as a supporting wall. To strengthen the wall, one needs to fill any holes in the wall. One of these holes can be filled with the orientations people need to actively engage in mathematics. Productive disposition fills this hole quite satisfactorily. On the one hand, other holes may exist that can be filled with things like mathematical integrity or self-efficacy, which, in conjunction with productive disposition, may better support the wall. On the other hand, perhaps productive disposition fills the only hole, and nothing more (at least as identified to date) is needed.

I am open to the idea that students might need support in more areas than are included in the current definition of productive disposition. One of the goals of this study is to look for these additional areas, which might be categorized as mathematical orientations that support people in the engagement in mathematics. As I began the study, I thought that perhaps only one or two additional constructs, along with productive disposition, would fully comprise these mathematical orientations. If so, I would propose to simply add these constructs to the definition of productive disposition so that all orientations that support one in engaging with mathematics would be just one's productive disposition. (I prefer not to invent a new term if slightly extending a currently used term is adequate.) However, I was also open to the possibility that productive disposition is only a small part of these desired mathematical orientations. If so, I felt it may be best to create a new term to describe these mathematical orientations that support people in the engagement with mathematics. This distinction, though, may simply be an argument over semantics. To simplify the issue, I will only be using the term *productive disposition* through this paper.

Section 1.3—Contributions to the Field of Mathematics Education

I believe for several reasons that this study will contribute to the field of mathematics education. First, this study is focused on the least researched strand of mathematical proficiency, productive disposition. Second, for many, productive disposition is not considered to be part of mathematically proficiency. For example, my adviser’s students, when asked to describe the qualities in a person they consider to be mathematically proficient, often mention knowing concepts, knowing procedures, and being able to solve problems, and they sometimes mention reasoning and justifying. They rarely if ever mention having a positive outlook toward mathematics. In other words, using their own terminology, they often include four of the strands in their own definitions of mathematical proficiency but overlook productive disposition. Third, I believe that this strand is worthy of further research. My hope is that other studies of this construct will follow this study.

Fourth, note that productive disposition is a relatively new construct to the field of mathematical education research. New constructs arise after being identified and validated in the real world, but they may then need to be modified and refined. For this reason, I approach the study by looking for the mathematical orientations that support engagement in mathematics, not limited by the existing definition of productive disposition in my effort to add to and refine the construct of productive disposition. More generally, this study is designed to answer the question *What is productive disposition?* To do so, I take a step back from the construct to study it in comparison with ideas that seem related and then to refine the definition.

Fifth, productive disposition is worthy of study if it is useful in the field and adds to educators' and researchers' understanding of mathematics education. We as teachers would like to see more of our students become mathematically proficient. This proficiency includes believing that hard mathematical work is beneficial, mathematics should make sense, and anyone can become a learner and doer of mathematics. Therefore, this research is needed so that we can better understand how to help our students build stronger productive dispositions and thus help them become more mathematically proficient.

However, the participants in this study are not K–12 students, but rather teachers. One might question this choice, but it was made for two reasons. First, teachers were chosen because the dispositions they have greatly affect the dispositions of their students:

> The teacher of mathematics plays a critical role in encouraging students to maintain positive attitudes toward mathematics. How a teacher views mathematics and its learning affects the teacher's teaching practice, which ultimately affects not only what the students learn but how they view themselves as mathematics learners. (NRC, 2001, p. 132)

The study was designed to learn more about the productive dispositions of teachers, including ways to assess them and, potentially, support their development. In doing so, one may also learn about the dispositions of their students. Because mathematics educators care about the productive dispositions of both students *and* teachers and because teachers' productive dispositions influence their students' productive dispositions, studying the teachers seemed important.

The second reason for studying teachers was a practical one. Our research team for the project Studying Teachers' Evolving Perspectives (STEP) had collected from four teacher groups videotaped focus-group data potentially rich for assessing mathematical orientations because of differences in the groups. Further, because adults are likely to be more articulate about their own thinking than are young students, indications of productive dispositions and mathematical orientations that support engaging in mathematical tasks might be more evident in adults' conversations than in students' discussions.

Section 1.4—Research Goal and Research Questions

My overarching goal for this study was to better understand productive disposition, an integral part of mathematical proficiency. However, many students do not seem to hold strong productive dispositions. Although educators assume that these dispositions can be developed and supported in learners, currently they are not a focus of mathematics education at any level of schooling. Although studying how learners might develop strong productive dispositions is important, it was not my purpose in this study. Instead, I examined the foundations of this construct to determine what it is and how one might assess for it. In the study I plan to answer the following research questions:

- What differences in evidence for one's productive disposition can be found through the analysis of teachers' actively engaging in a mathematical task?
- What evidence for productive disposition is self-reported by teachers who have been identified as having strong productive dispositions?

Research Question 1. What differences in evidence for one's productive disposition can be found through the analysis of teachers' actively engaging in a mathematical task?

In Study 1, I used a preliminary rubric to analyze video data from focus groups of teachers working on a problematic mathematical task: the Savannah Task. These teachers were previously found to vary significantly on several measures, including their content knowledge, their beliefs about mathematics and mathematics teaching and learning, and their professional noticing of children's mathematical thinking. Thus, teachers with these differences will likely have a range of mathematical orientations as well. The rubric was be used to assess for the differences in the individuals' indicators of productive disposition. In the process, the list of indicators was refined when new pieces of evidence about one's productive disposition were uncovered in the data.

Research Question 2. What evidence for productive disposition is self-reported by teachers who have been identified as having strong productive dispositions?

The data from the STEP study provided a starting point to determine whether the rubric can be used to find differences in the evidence for people's productive dispositions. It did not, however, fully answer the questions of what is a productive disposition and how to identify it when it is being enacted. In Study 2, I investigated whether specific characteristics always arise in people with strong productive dispositions or if there are different ways in which strong productive dispositions can be enacted. Study 2 provided an opportunity to determine whether certain characteristics of strong productive dispositions were hidden in the video data.

I worked with a group of ten K–3 teachers who in the previous study were found to hold strong productive dispositions. These teachers completed four mathematical tasks, reflections about each task, and a mathematical autobiography. They were then interviewed by me to verify whether inferences about the evidence they showed were being interpreted correct. More importantly, I wanted to see if they talked about other traits that people with strong productive dispositions hold that did not appear in any of the data. The goal was to get a fuller sense of the qualities and characteristics of people with a strong productive dispositions.

Summary

At the conclusion of the study, answers to these research questions will be useful for educators' better understanding productive disposition. The studies are designed to show (a) what constitutes productive disposition; (b) ways to assess for productive disposition; and (c) the variety of ways one's productive disposition is expressed in learning and engaging with mathematics. Thus, by examining orientations of a variety of teachers and by studying in-depth the evidence provided by teachers who seem to hold strong productive dispositions, I intend to better understand what exactly productive disposition is and what evidence is shown when it is being used to help solve mathematical tasks.

CHAPTER 2

REVIEW OF THE LITERATURE

Section 2.1—Introduction

My purpose in this chapter is to unpack the definition of productive disposition, repack it using other work from the literature in mathematics education, and then create a initial rubric to assess for the productive dispositions of teachers actively engaging with mathematical tasks. To begin this chapter, I describe the five strands of mathematical proficiency, including productive disposition, as defined in *Adding It Up* (National Research Council [NRC], 2001). I then give an example from my work with the STEP Project showing how four of the strands can be assessed with a paper-and-pencil task, but that there are problems in assessing the fifth strand, productive disposition. By using constructs related to productive disposition, I then try to unpack the four parts of the definition of productive disposition and connect each part to other research in the mathematics education field. Finally, to repackage the construct of productive disposition, a rubric was created designed as a way to assess evidence of an individual's productive disposition. The types of evidence listed in the rubric are drawn from the definition of productive disposition as well as from the constructs associated with the definition. My goal in this chapter was to create a rubric to be operationalized in the two studies designed to assess for individuals' productive dispositions. Through these two studies, indicators of evidence were added to the rubric and other indicators were reorganized into more appropriate categories.

In doing so, I hope to have developed a more refined rubric that can be used to assess teachers' productive dispositions.

Section 2.2—Mathematical Proficiency

The mathematics education of United States students is of great concern because, as has been shown by a number of studies, many lack mathematical proficiency. For example, the 1996 National Assessment of Educational Progress (NAEP) showed that only 21% of 4th-grade students, 24% of 8th-grade students, and 16% of 12th-grade students were proficient in mathematics. Further, for those grades, fewer than 70% of the students held the knowledge and skills fundamental for proficient grade-level work (National Assessment of Educational Progress, 1999). International studies have shown similar results. In the Third International Mathematics and Science Study (1996), 8th-grade U.S. students were shown to have scored below the international average in mathematics and 12th-grade students scored the worst of any country in the study in mathematics general knowledge.

This poor mathematical performance is troubling news. Teachers, parents, school administrators, and educational researchers all want students to be more mathematically proficient, but what constitutes proficiency in mathematics? Various conceptions of being "good in mathematics" permeate our culture: having good number sense, being proficient in using algorithms, or being skilled in logic and reasoning.

To help clarify this issue, in 2001 the National Research Council (NRC) released its consensus report, *Adding It Up,* to give "a more rounded portrayal of the mathematics children need to learn, how they learn it, and how it might be taught to them more effectively" (p. xiv). The report was the 16-member committee's synthesis of research in mathematics education in grades K–8. To reach their goal, the NRC adopted a definition of mathematical proficiency that represented a comprehensive view of what they thought constituted successful mathematics learning (NRC, 2001). They stated that *mathematical proficiency* consists of five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. They noted that "the five strands are interwoven and interdependent in the development of mathematical proficiency" (p. 116). To understand what *mathematical proficiency* entails, one must first consider the strands individually and then consider the interdependent nature of the strands (see Figure 2.1).

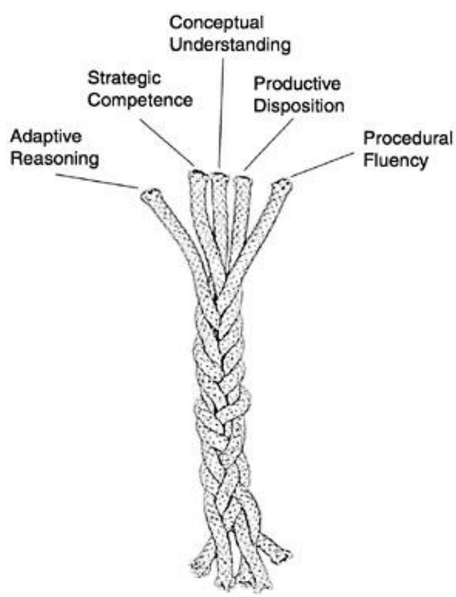

Figure 2.1. The strands of mathematical proficiency. Source: NRC, p. 117 (2001).

Conceptual Understanding

Conceptual understanding is defined as the "comprehension of mathematical concepts, operations, and procedures" (NRC, 2001, p. 116). People with conceptual understanding have their mathematical knowledge organized in such a way that they can easily use it in appropriate contexts. Moreover, their knowledge is not in separate, disconnected pieces, but rather their knowledge builds from their old conceptions to newer ones. Hiebert and Lefevre (1986) emphasized the importance of the relationships in conceptual understanding:

> It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual faces and propositions so that all pieces of information are linked to some network. (pp. 3–4)

These connections are important, for if a person forgets a fact or procedure, she can recreate it by building on her previous understandings, making mathematical knowledge easier to use and easier to remember and providing a basis from which to build new understandings (NRC, 2001).

According to the National Research Council (2001), "A significant indicator of conceptual knowledge is being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes" (p. 119). The more connections to different representations a person has, the richer her conceptual understanding is. For example, suppose that the decimal .45 was given to a student with strong conceptual understandings of place value and number sense. She may know that .45 is 4 tenths and 5 hundreds, which is the same as 45 hundredths (or 450 thousandths). She might be able to represent the number using base-ten blocks: If she represents 1 with a flat, then .45 could be represented by 4 longs and 5 cubes or 45 cubes or 2 longs and 25 cubes. She might be able to place it on a number line slightly to the left of .5 or ½. She might be able to connect it to her knowledge of fractions and see .45 as the same as 45/100 or 9/20. It is all these connections among differing representations that constitute conceptual understanding.

Procedural Fluency

Procedural fluency is defined as "the knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (NRC, 2001, p. 121). The procedural knowledge referred to is any and all methods one might use to solve a mathematical problem, including (but not limited to) written procedures, mental procedures, computer or calculator use, and modeling with manipulatives (NRC, 2001).

Note that procedural fluency is not in opposition to conceptual understanding; indeed, the two work together to help build mathematical proficiency. Procedural knowledge without conceptual understanding leads to learning algorithms by rote, without understanding the underlying mathematics and the constraints on their appropriate use. However, procedural knowledge can also lead to new conceptual understandings.

For example, in one of several mathematical tasks posed to a group of in-service teachers, they were asked to explain an elementary school student's nonstandard and somewhat confusing method for solving a division story problem. In addition to explaining the child's steps, they were asked to use reasoning similar to his to solve another division problem. This task was difficult for most of the teachers. One teacher explained that although she was unable initially to explain all the child's steps, she was able to use the child's reasoning to solve the similar division problem. Using the child's strategy enabled her to make sense of and recognize the validity of each step. In this case, the teacher used her procedural fluency to build her conceptual understanding.

Strategic Competence

Strategic competence is defined as "the ability to formulate mathematical problems, represent them, and solve them" (NRC, 2001, p. 124). Having strategic competence enables one to formulate a problem mathematically and then use his or her knowledge to solve it. Strategic competence is invoked in deciding which strategies might be useful in solving the problem and in finding connections to previous mathematical experiences in which similar problems were solved. This skill is important not only for the mathematics classroom but also for problematic situations in real life. Unlike the classroom, the real world lacks neatly set-up problems with well-defined procedures for solving them, so one must be able to construct a model of the situation, find the relevant mathematical terms, and think flexibly about which approach to use in a solution.

Numerous factors affect one's use of strategic competence. In trying to develop strategic competence in a group of fifth-grade students, Townsend, Lannin, and Barker (2009) found that students who were given opportunities to generalize algebraic tasks used recursive and proportional reasoning and were explicit in their choice and use of strategies. These students differed from students taught to focus on *key words* and to use those words to determine the operations needed (Mayer & Hegarty, 1996) instead of trying to make sense of and formulate a mathematical problem relevant in the given situation.

Adaptive Reasoning

Adaptive reasoning is defined as “the capacity to think logically about the relationship among concepts and situations” (NRC, 2001, p. 129). Ability in adaptive reasoning enables one to consider alternative approaches, to follow the mathematical logic of a proposed proof, to note logical inconsistencies or contradictions, and to justify any conclusions. The justifications need not be formal proofs, but rather, as the NRC (2001) noted, would “provide sufficient reasons” (p. 130)—as do deductive reasoning and two-column geometry proofs but also a kindergartener’s explanation that there are two 5s in 10 because he can split the 10 in half and count the halves and arrive at the same amount. Most important, people with adaptive reasoning know when their solutions are correct, not because of the particular procedures they used but because they could follow the steps they used to solve them in a logical manner and justify their solutions.

Productive Disposition

Productive disposition is defined as “the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (NRC, 2001, p. 131). Students who have productive dispositions see mathematics not as a set of arbitrary rules that one must memorize but as a system of connected conceptions that, with diligent effort, can be understood. This strand is very different from the other strands; it encompasses issues such as a person's affect, beliefs, and identity,[1] whereas the other four strands focus mainly on cognitive processes. However, productive disposition is needed to build the other four strands (NRC, 2001). Moreover, the strengthening of the other four strands helps to build one's productive disposition. This symbiotic relationship might be clarified with an example. Suppose that a student given a mathematical task new to her does not see herself as having ability to think mathematically or does not believe that mathematics makes sense; she might make no attempt to solve the problem. If she does not envision success even if she works diligently on the problem or is unwilling to struggle if the problem seems difficult, she may give up after only a short time. However, if she has a positive productive disposition, she will likely make significant progress on the problem, if not solve it altogether, using the other strands of proficiency as tools to help her. By making progress on this problem (or by solving it altogether), she will have built on and strengthened one or more of the other four strands of mathematical proficiency. Simultaneously, she will confirm for herself the benefit of applying effort to a mathematical problem and recognize her ability to think about and engage with difficult mathematical tasks.

[1] More about these constructs in section 2.4.

The previous example indicates that productive disposition should be seen not as a fixed characteristic but as a quality that can be changed and *does* change through interaction with novel mathematical tasks. Resnick's (1987) conclusion that "the term *disposition* should not be taken to imply a biological or inherited trait," but that a disposition "is more akin to a *habit* of thought, one that can be learned and, therefore, taught" (p. 41) has the important implication that humans are not born predisposed to liking or disliking mathematics. Productive dispositions in mathematics can be developed in all learners, and teachers can play active roles in the construction.

Summary

Mathematical proficiency, as defined in *Adding It Up* (NRC, 2001), consists of five interwoven and interdependent strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. The authors noted that proficiency is not an all or nothing prospect; learners build proficiency over time through understanding and connecting mathematical concepts. Further, students cannot be thought of as being proficient in mathematics if they are deficient in one or more of the strands.

In the next section, I explain that in many assessments one strand of proficiency, productive disposition, is ignored. I show a paper-and-pencil task useful for assessing the other four strands but note its shortcoming for assessing productive disposition. This difficulty in assessing productive disposition leads me to consider research on constructs related to productive disposition that might be more assessable.

Section 2.3—Productive Disposition: The Hidden Strand

I work on a research project, Studying Teachers' Evolving Perspectives (STEP), in which the research team is analyzing 131 preservice and in-service teachers' beliefs about mathematics and mathematics teaching and learning, noticing of children's mathematical thinking, responsiveness to children's thinking in interviewing children, and content knowledge (I discuss another aspect of the STEP Project in Chapter 3). We are analyzing participants' mathematical content knowledge using the lens of mathematical proficiency from *Adding It Up* (NRC, 2001), as described in the previous section. Participants completed seven paper-and-pencil tasks and one computer-based task, all relevant to teachers teaching K–3-level mathematics with a focus on children's mathematical thinking.

For one task, the Andrew Task (see Figure 2.2), participants were asked, first, why Andrew's strategy makes mathematical sense, and second, to use Andrew's reasoning to solve 432 – 162.

In March, Andrew, a second grader, solved 63 – 25 = ☐ as shown below:

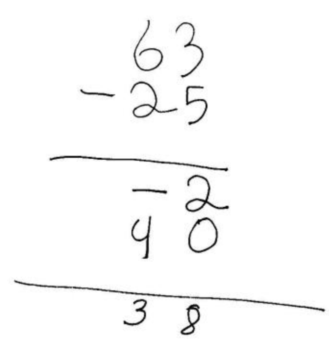

Figure 2.2. Andrew's strategy for 63 – 25.

In analyzing the solutions to this task, we found evidence to assess participants' holding of four of the five strands of mathematical proficiency. Conceptual understanding was needed to make sense of Andrew's –2, which could be seen conceptually in at least two ways: as a negative 2 that results from Andrew's subtracting 5 from 3 in the one's column or as 2 more to be taken away after 3 of the 5 had been subtracted from the 3. Procedural fluency is needed both to understand the standard subtraction algorithm *and* to make sense of Andrew's modified version of it. Adaptive reasoning is used to explain why Andrew's reasoning makes mathematical sense, that is, in justifying his reasoning. Participants showed strategic competence in applying Andrew's strategy to a similar problem, 432 – 162.

We found, however, that we were unable to assess the fifth strand of mathematical proficiency, productive disposition, through the written work of the participants on this task or any other. We claim not that the participants used no productive disposition in working out their solutions but only that their written work gives us no indication of their effort or mathematical engagement. For example, consider two participants. Participant A has a strong understanding of place value and integers and immediately recognized that Andrew could have thought the following: 3 – 5 = –2, 60 – 20 = 40, and –2 + 40 = 38. Participant A was then able to explain why this thinking made mathematical sense and to use the strategy to solve 432 – 162. Participant B, meanwhile, struggled to make sense of Andrew's strategy. She believed that mathematics makes sense and thought that she could, with effort, find an explanation for the strategy. After trying several possible solutions, Participant B figured out and explained that the –2 represents two more Andrew has to take away, and then uses this strategy to solve for 432 – 162.

Although this difference may have occurred in two participants' minds, we would be unable to discern the difference in their written responses, which could look nearly identical. On the basis of the written work alone, we could not discern that Participant B used her strong productive disposition to solve this problem. We would not claim that Participant A is lacking in productive disposition. In this case, however, because her other strands of mathematical proficiency relating to place value and negative numbers were so strong, she would not necessarily rely on her productive disposition to solve the problem. Participant B, however, would previously have used her productive disposition to strengthen the other strands and build her mathematical knowledge.

The example above illustrates why I call productive disposition the hidden strand of mathematical proficiency: People use their productive dispositions to help them solve problems and build new mathematical knowledge, but their dispositions generally are not evident in their written work. Consider the exams in your life as a mathematics student: I doubt that any assessed productive disposition; thus, they did not fully assess your mathematical proficiency. Recall the National Research Council's (2001) conclusion: "Students should not be thought of as having proficiency when one or more strands are undeveloped" (p. 135). Thus, mathematics educators need to understand more about productive disposition and how to identify it in students. In the next section, I discuss three constructs that partially overlap productive disposition. Although these three constructs are quite broad, I attend particularly to those aspects related to the definition of productive disposition or to assessing for one's mathematical orientations that lead to engagement in mathematics.

Section 2.4—Constructs Connected to Productive Disposition

My overarching goal in this study is to determine what productive disposition is and how to identify it when it is being enacted. As was illustrated above, fulfilling these goals can be difficult. Little research has been conducted in the area of productive disposition, but several areas closely related to productive disposition have been studied extensively and may prove useful in researching the construct. In the following section, I begin by describing the four elements of productive disposition from the definition in *Adding It Up*: "the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (NRC, 2001, p. 131). I then connect each of these to other related areas of research in mathematics education. My goal here is not to create a new definition of productive disposition but rather to start to identify components that comprise it. In doing so, I hope to use the research from these related constructs to strengthen the definition of productive disposition as well as give rise to observable evidence for a person's productive disposition.

For ease of reference, I include below brief definitions of the eight constructs I consider to be related to productive disposition. These definitions are not from any one source but rather are my condensed definitions built from many researchers' work.

In the following sections, I provide more detail about each construct and discuss work from which these shortened definitions were built.

- *Affect*—a person's feelings and attitudes that shape the way one looks at the world
- *Beliefs*—psychological understandings about how one perceives the world to be
- *Goals*—the states that human beings desire to obtain
- *Identity*—qualities people recognize in themselves or that are recognized by others (a person's type or kind)
- *Mathematical Integrity*—knowing what one knows, knowing what was does not know, and being honest about these assessments
- *Motivation*—the inclination people have to do certain things and avoid doing others
- *Risk Taking*—willingness to ask questions or share ideas that may expose one's misconceptions or weaknesses
- *Self-efficacy*—a person's own belief in his or her ability to take action on a particular problematic situation

Figure 2.3. Eight constructs related to productive disposition.

Tendency to See Sense in Mathematics

The first part of the definition of productive disposition is the tendency to see sense in mathematics. To develop the other four strands of mathematical proficiency, people "must believe that mathematics is understandable, not arbitrary" (NRC, 2001, p. 131). The tendency to see sense in mathematics, essentially, is a belief about the nature of mathematics. In the 1996 National Assessment of Educational Progress (NAEP) study, researchers found that 54% of fourth-grade students and 40% of eighth-grade students believed that mathematics is comprised mostly of rules and that learning mathematics consists of simply memorizing these rules (NRC, 2001). To change the beliefs of their students, teachers must also believe that mathematics makes sense. More teachers need to see that the conceptions of their students may contain deep mathematical complexity, even in elementary mathematics. In fact, teachers can learn mathematical concepts and strategies from these interactions, leading to their being more comfortable with mathematics and their viewing mathematics as being a connected set of concepts that make sense.

However, I also think that *mathematical integrity*, which is having a sense for what one does and does not understand mathematically, is part of the tendency to see sense in mathematics. I next explore the constructs of both beliefs about mathematics and mathematical integrity and connect them to this part of the definition of productive disposition.

Beliefs About Mathematics

As noted above, the tendency to see sense in mathematics in essence is a belief about the nature of mathematics. In this section, I provide background on beliefs and show their connection to productive disposition. Specifically, I focus on beliefs about mathematics, including the belief that mathematics should make sense.

Beliefs are ways in which one views the world, psychological understandings about "how things really are." Perhaps the easiest way to define beliefs is to contrast beliefs with knowledge. Beliefs differ from knowledge in several ways. First, beliefs can be held at various levels of conviction; knowledge cannot. Second, beliefs are not agreed upon by all, whereas knowledge generally is (Thompson, 1992). Third, beliefs are not certain or immutable; they can grow or change drastically over time. Last, knowledge needs to be tested and proven before it becomes accepted; beliefs, however, go through no vigorous testing.

McLeod (1992) identified four categories of students' mathematical beliefs: beliefs about mathematics, beliefs about self, beliefs about teaching, and beliefs about social contexts. *Beliefs about mathematics* include whether the student believes that mathematics is (a) based purely on rules, (b) a difficult or a manageable subject, and (c) important and useful in the real world. (I discuss the other categories later.) McLeod's category of beliefs about mathematics seems to be focused mainly on the perception of the nature of mathematics. In particular, many people believe that mathematics is based purely on arbitrary rules and see it as an incoherent subject, and thus they are unlikely to see sense in mathematics. Adding to this problem is the fact that most Americans believe that people are either born with mathematical ability or they are not. This belief in mathematical skill as an inborn trait seems to have cultural origins. Stevenson, Lee, and Stigler (1986) found that American children held this belief much more strongly than their Chinese and Japanese counterparts.

Ideally all people would come to believe that mathematics can and does make logical sense. Frank (1988), after working with junior-high students on a 2-week summer mathematics program, found that even gifted students believed that (a) mathematics is solely about computations, (b) students are supposed to receive mathematical knowledge transmitted by the teacher, (c) mathematics problems should be solvable in a few steps, and (d) the goal of mathematics is to find the correct answer. Garofalo (1989), in addition to finding views similar to those found by Frank, found that his secondary-school students believed mathematics to be a disjoint set of rules and procedures, not a highly related web of conceptual ideas. Garofalo's students believed that only very creative and prodigious people could create new mathematics and that the source for learning mathematics for everyone else is someone who is much more knowledgeable in mathematics. Finally, he found that students believed that they should accept mathematical knowledge from their teachers without question, not try to make sense of the concepts for themselves. For people to build more productive dispositions, these beliefs about the nature of mathematics need to be changed.

Like productive disposition as a whole, these beliefs about the nature of mathematics are not immutable; teachers directly affect the beliefs their students hold and can actively change their beliefs. Stodolsky (1985) found that the effect of teachers' often having their students work alone in their seats from the textbook was that the students believed that the only way one learns mathematics is through an expert (be it the teacher or the text). Moreover, they came to believe that either they did or did not have mathematical abilities. They did not believe that they could learn mathematics from working with others or by making sense of it on their own. This finding shows that beliefs are affected by one's mathematical experiences, and thus a way to assess for one's beliefs may be to find out about the person's previous mathematical experiences, both in and out of the classroom.

Assessing for beliefs has been a difficult task, but a few methods have been developed by researchers. Several had participants complete questionnaires (Brown, Carpenter, Koube, Lindquist, Silver, & Swafford, 1988; Fleener, 1996; Kouba & McDonald, 1986; Spangler, 1992). These questionnaires took different forms, including having participants agree or disagree with statements, complete Likert scales, or provide short responses. Carey and Franke (1993) interviewed first-grade students to get a sense of their mathematical beliefs. They asked questions like "What would you tell other students about first grade mathematics?" and categorized the responses by what the students believed constituted doing first-grade mathematics. Several researchers used observations of people working on mathematical problems along with semistructured interviews to discern beliefs (Diaz-Obando, Plasencia-Cruz, & Solano-Alvarado, 2003; Frank, 1988; Garofalo, 1989). In these interviews, participants were not asked directly about their beliefs about mathematics, but their beliefs were inferred from the ways in which the participants talked about their experiences in mathematics (either their experiences recently in solving particular problems or their more general mathematics experiences overall).

Although not all mathematical beliefs are connected to this aspect of productive disposition, beliefs about the nature of mathematics help determine whether a person has a tendency to see sense in mathematics. For example, if one believes that mathematics is just a set of rules one must memorize and apply, one might think that these mathematical rules are completely arbitrary with no connection to sense making whatsoever. However, I think that a person with a strong productive disposition not only believes that mathematics should make sense but also knows when he or she is actually making sense in mathematics. “Students who have developed a productive disposition are confident in their knowledge and ability” (NRC, 2001, p. 133), and part of this confidence comes from know what one does or does not understand mathematically. This understanding of when one is or is not making mathematical sense is a construct known as *mathematical integrity*.

Mathematical Integrity

As defined by DeBellis and Goldin (1999), *mathematical integrity* is “an individual’s affect[ive] psychological posture in relation to when mathematics is ‘right,’ when a problem is solved satisfactorily, when the learner’s understanding is sufficient, or when mathematical achievement is deserving of respect or commendation” (p. 253). Mathematical integrity can be seen, then, as knowing what one knows, knowing what one does not know, and being honest about these assessments. As noted before, part of having a productive disposition is having the tendency to see sense in mathematics, and, to me, part of that tendency includes knowing what one does or does not know; otherwise, one cannot know what parts of the mathematics make sense to the individual and what parts do not.

DeBellis and Goldin listed the affective components of mathematical integrity as being able to recognize when one has insufficient knowledge, being able to decide whether further action is needed, and what that action might be (DeBellis & Goldin, 2006). They noted that these self-judgments may pose value, moral, or ethical dilemmas (DeBellis and Goldin considered values, morals, and ethics to be part of affect). Mathematical integrity also seems to be a connection between one's knowledge and one's affect. It is making an honest assessment as to what knowledge the learner does or does not have (Carlson & Bloom, 2005) and then using that assessment to decide what actions to take next.

Part of doing mathematics is knowing when a problem has been solved. Learners need to know when a problem is solved to know whether to continue their effort. A student with mathematical integrity, however, will have the confidence to know, truly *know*, when he or she has completed a problem. Moreover, the learner will also be able to tell when a problem has not been satisfactorily completed. One pleasing, yet intimidating, aspect about mathematics is that the solving of one problem generally leads to several more questions' arising. However, for learners with mathematical integrity, this idea of endless open mathematical questions seems exciting and full of opportunities instead of frightening and, in turn, gives rise to the belief that they are learners and doers of mathematics.

Most that has been written on assessing for one's mathematical integrity is from a larger study focused on affect (more about affect is in a later section). Carlson and Bloom (2005) used mathematical integrity as part of their taxonomy for problem solving. They looked for integrity and the other constructs in their taxonomy in expert mathematicians by observing the participants working on four or five challenging problems and then interviewing them. The researchers then inferred mathematical integrity from their observations of the participants at work and from their response in the interviews. DeBellis and Goldin (2006) conducted a similar study with children aged 9 to 10. Like Carlson and Bloom, DeBellis and Goldin gave the participants challenging problems to solve and followed up with interviews, but in a longitudinal study they examined whether the participants changed over time.

Although little has been written about mathematical integrity, I consider it important to include in this section because for a learner to have the tendency to see sense in mathematics, that learner must truthfully assess what he or she does and does not know. Mathematical integrity also helps people see themselves as able to solve mathematical problems and understand mathematical concepts. However, mathematical integrity alone is not enough; people also need to believe that they can take action on mathematical problems and to build strong mathematical identities for themselves. All of these factors play a role in seeing oneself as a learner and doer of mathematics, the next part of the definition of productive disposition.

Seeing Oneself As an Effective Learner and Doer of Mathematics

A second part of the definition of productive disposition is seeing oneself as an effective learner and doer of mathematics. To actively engage with the mathematics, people need to see themselves as having the capacity to learn mathematics and to make progress on mathematical tasks. Doing so not only enables people to make progress on a given mathematical task but also helps develop the other strands of mathematical proficiency. When people see themselves "as capable of learning mathematics and using it to solve problems, they become able to develop further their procedural fluency or their adaptive reasoning abilities" (NRC, 2001, p. 131).

The importance of seeing oneself as a learner and doer of mathematics might be best illustrated in a story told to me by Ms. Williams, a colleague who teaches first grade. Ms. Williams, in a study of her students' productive dispositions, and looked for indicators of it. She found one of her students to be particularly interesting: The student believed that mathematics could make sense, that doing mathematics was worthwhile, and that steady effort would lead to progress in mathematics. Although she held these indicators of productive disposition, she held these beliefs about *other* students, not *herself*. Because of this attitude, Ms. William's student felt not need to exert effort because she had no confidence that she would succeed.

The story about Ms. William's student is illustrative, but unfortunately it is not unique. Many people hold stereotypes about what sorts of people can learn and do mathematics. Some of these stereotypes are gender based, ethnicity based, or based on the notion that mathematics is a trait one inherits. However, other people, like the student described, may not hold these stereotypes broadly but may simply believe that they themselves cannot learn and do mathematics. To help such people develop stronger productive dispositions, we must change the mathematical identities people hold for themselves and for others.

Mathematical Identity

Part of being able to see oneself as being a learner and doer of mathematics is holding a strong *mathematical identity* for oneself. One's identity comes from interacting with others in a context. Others can recognize a person as a *type* or *kind*, such as student, activist, or food connoisseur. *Identity*, then, consists of the qualities people recognize in themselves or that are recognized by others (Gee, 2001). Identity can shift from moment to moment, and a person can hold more than one identity at a time. These identities can be seen as created solely by humans and are continually recreated through interaction with others (Holland & Lave, 2001). For example, right now I consider myself a graduate student and a researcher, but last night while watching a game, I considered myself a hockey fan. I am not any one of these identities, but rather I take on differing identities, depending on a context.

Sfard and Prusak's (2005) view of identity differs from Gee's. Although they considered Gee's (2000) definition a good starting place, Sfard and Prusak stated that it failed to clarify how one can determine who or what kind of person an individual is and was based on some *reality* of person—who the person is—completely independent of the person's actions. Sfard and Prusak (2005) defined *identity* as the collection of personal stories that are *reifying, endorsable,* and *significant*. To be *reifying,* the stories should include verbs such as *be, have*, or *can,* with qualifying adverbs such as *usually* or *never*. To be *endorsable,* the story must be considered true by the person telling the story in respect to the way the storyteller views the world. To be *significant,* the story must be stable so that any change in the story would drastically change the storyteller's views about the person in the story.

Both Gee's (2001) and Sfard and Prusak's (2005) research is about identities in general. Of more direct importance to my research are people's mathematical identities. Building from Gee's (2001) categories, Anderson (2007) found four avenues through which identity is constructed in a mathematics learner: engagement, imagination, alignment, and nature. First, mathematical identity is built through engagement by direct experiences with mathematics and interacting with others. People who develop their own strategies and build mathematical meaning with others begin to view themselves as members of the mathematical community. Second, using imagination, people can build on their mathematical identities by envisioning how mathematics will play a broad role in their lives, such as its use in current or future employment. Third, related to alignment, mathematics may become part of a person's identity because of the need for certain mathematics classes as prerequisites for college or for a change in occupation. Last, nature comes into play for those who believe that they are (or are not) inherently "math people." Each of these constructs, however, can positively or negatively influence people's mathematical identities, depending on their experiences and reinforcements. Perhaps more important for this study, each of these four constructs could lead to potential questions to probe for one's mathematical identity during interviews.

Using interviews appears to be the most common way to assess for one's identity. In several studies, researchers used analysis of interviews in conjunction with other data sources to find evidence for participants' identities. Anderson (2007) used interviews along with a survey and a questionnaire to investigate participants' identities, whereas Sugrue (1997) used interview transcripts to look for emerging themes. Mawhinney and Xu (1997) and Drake, Spillane, and Hufferd-Ackles (2001) looked for emerging themes in interviews but also in the observational data of their studies. Thus, using interviews, I might gain insight into participants' identities and, in turn, into how they view themselves in relation to learning and doing mathematics.

People's mathematical identities are how people see themselves (and how others see them) in relation to mathematics. In particular, identity is important in relation to whether people view themselves as "learners and doers of mathematics" (NRC, 2001, p. 131). Boaler (2002) recognized this connection between identity and productive disposition in noting that in "building up" their mathematical identities, students not only increased their mathematical content knowledge but also became engaged in the practice of doing mathematics and developed productive relations with the mathematics. In this way, identity seems directly connected to both having students see themselves as learners of mathematics and having them view mathematics as a subject that is worthwhile and useful. However, I think that having a strong mathematical identity alone is not enough; people must also believe that they can *take action* on a particular mathematical task. In other words, to believe that one can learn and do mathematics, one must also have a high degree of self-efficacy.

Self-Efficacy

In a previous section I discussed beliefs about the nature of mathematics, but in this section I explore one specific belief, *self-efficacy,* which McLeod (1992) might describe as a belief about self. The concept of self-efficacy comes from a social-cognition perspective (Bandura, 1986). *Self-efficacy* is a person's own confidence in his or her ability to take action on a particular problematic situation (Bandura, 1977). Learners with high self-efficacies motivate themselves, set challenging goals for themselves, and use strategies appropriate for obtaining their goals (Bandura, 1977; Schunk, 1990; Zimmerman, Bandura, & Martinez-Pons, 1992). Self-regulated learning is not only a set of cognitive skills but also includes "the self-regulation of motivation, the learning environment, and social supports for self-directedness" (Zimmerman et al., 1992, p. 664). Although complex in its description, self-efficacy can be seen, in the simplest terms, as one's confidence in one's self to make progress or succeed at a given academic task at a given time (Ferla, Valcke, & Cai, 2009).

Note that self-efficacy is not the same as outcome expectancy. *Outcome expectancy* is the effectiveness of a given behavior in producing a desired outcome, whereas self-efficacy is "the conviction that one can successfully execute the behavior required to produce the outcomes" (Bandura, 1977, p. 193). Like Ms. William's student, individuals might think that given the right behavior, one could obtain a certain outcome, yet not believe that they themselves are able to produce such a behavior (Bandura, 1977). However, self-efficacy *is* similar (if not identical) to other expectancy constructs, such as expectancy for success, perceived control, perceived ability, and many others. As Pajares (1996) commented, these constructs are typically defined almost identically to one another and used somewhat interchangeably. [For more about this abundance of similar terminology, see Bong, 1996.]

Self-efficacy closely corresponds with "to see oneself as an effective learner and doer of mathematics" in the definition of productive disposition (NRC, 2001, p. 131) in referring to people's beliefs about themselves as learners, how they put those beliefs into action, and their willingness to persevere to successfully complete tasks. As Schunk (1984) noted,

> Self-efficacy refers to personal judgments of how well one can perform actions in specific situations that may contain ambiguous, unpredictable, and stressful features. Self-efficacy is hypothesized to influence one's choice of activities, effort expended, perseverance when difficulties are encountered, and skill performance. (p. 29)

To succeed on any task, a person must not only have the prerequisite knowledge and skills for the task but also view himself or herself as having the ability to accomplish (or, at least, make progress on) the task. Depending on fluctuations in one's self-efficacy, people with the same knowledge and skills might perform poorly, moderately well, or extremely well (Bandura, 1993).

Thus, one might find it unsurprising that self-efficacy closely correlates with academic performance, and these correlations have been shown to be higher in studies in mathematics than in other fields, such as reading or writing (Pajares, 1996). Zimmerman et al. (1992) found that students' goals and their perceived self-efficacies accounted for 31% of the variance in the students' final grades in their course. In fact, Multon, Brown, and Lent (1991), in a meta-analysis of 36 studies, found that about 14% of the variation in academic performances could be attributed to a person's self-efficacy. Similarly, studies in college science courses have shown that high self-efficacy correlates with high academic achievement and the persistence needed to maintain that achievement (Lent, Brown, & Larkin, 1984). In other words, people with high self-efficacies in mathematics will rightfully see themselves as being able to learn mathematics and perform well on mathematical tasks.

Bandura (1977) found four influences on self-efficacy: *personal accomplishments*, how well or poorly the individual has done on similar tasks in the past; *vicarious experience*, seeing others perform on similar tasks; *verbal persuasion*, suggestions by others on how the individual might perform; and *emotional arousal*, how anxious, vulnerable, or stressful the individual feels. These four influences have been studied to assess self-efficacy through questioning individuals or analyzing their work with others on a mathematical task. The other seemingly most common way researchers have assessed self-efficacy is via questionnaires. Participants generally responded on a Likert scale to statements about their own perceived self-efficacies (Zimmerman, Bandura, Martinez-Pons, 1992) or rated their own self-efficacies in relation to specific problems on a 10–100 scale (Bandura & Schunk, 1981; Schunk, 1983, 1984). Unlike with assessment of other beliefs, participants in these studies answered direct questions about their own perceived self-efficacies.

A person's self-efficacy has a significant effect on the amount of effort one is willing to put forward on a mathematical task. According to Zimmerman et al. (1992), "Perceived self-efficacy influences the level of goal challenge people set for themselves, the amount of effort they mobilize, and their persistence in the face of difficulties" (p. 664). People with low self-efficacies may see certain types of problems as being too difficult for them to solve so that any effort on such problems would be in vain, which can in turn lead to stress and feelings that mathematical problems are far more difficult than they really are (Bandura, 1993; Pajares, 1996). People with high self-efficacies may feel more calm and collected when tackling a difficult problem and, therefore, may exert more effort, even on a challenging task (Bandura, 1977, 1993; Pajares, 1996). However, self-efficacy is just one part of the belief that one can make progress on a mathematical task if one puts effort forth. This belief in effort leading to progress is another part of the definition of productive disposition (NRC, 2001). As will be discussed in the next section, I feel that learning goals and the willingness to take academic risks also play a role in determining how much effort one is willing to put forth on a mathematical task.

The Belief That Steady Effort in Learning Mathematics Pays Off

A third part of the definition of productive disposition is the belief that steady effort in learning mathematics is worthwhile. This belief is an integral part of having a productive disposition, but unfortunately many people give up quickly and put forth little effort when facing a problematic mathematical task. Silver (1985) famously showed that students often believe that mathematical problems should be solvable in 5 minutes or less. This belief restricts students from putting forth effort for more than a short period of time, which in turn interferes with their seeing that exerting effort on mathematical tasks can be effective. This result might be categorized as a teaching or learning belief (McLeod, 1992). Effects of this belief become increasingly detrimental when older students face more complicated and nonroutine problems, so that students eventually avoid higher level mathematics courses altogether.

However, I feel that this part of the definition of productive disposition is more than just the belief one holds about whether one can make progress with effort on mathematical tasks. To put forth effort, one must first have a goal beyond avoiding failure, a goal of making sense of mathematical problems. (Goals will be discussed later in this section.) Also, part of putting effort forth in mathematics is trying new procedures and strategies, asking questions, or sharing possible solution strategies. To engage in these ways, one must be willing to take academic risks.

Academic Risk Taking

As noted above, to put forth effort on a problem, one must be willing to take *academic risks*. The concept of academic risk taking developed from theories about decision making in humans. Decision-making theories were based on the idea that humans generally assess the likelihood of outcomes before attempting to act (Clifford, 1991). When adopted by educational researchers, *academic risk taking* was defined as occurring when a person freely attempts a task in which the person has no more than .50 probability of success (Clifford, 1991), but the construct has been redefined in less probabilistic terms. Beghetto (2009) defined *academic* (or *intellectual*) *risk taking* as "engaging in adaptive behavior (sharing tentative ideas, asking questions, attempting to do and learn new things) that place the learner at risk of making mistakes or appearing less competent than others" (p. 210).

Academic risk takers generally exhibit three traits: They prefer difficult but doable tasks, they have a relatively high tolerance for failure, and they tend to use various strategies flexibly to overcome obstacles (Clifford, 1988; Meyer, Turner, & Spencer, 1997). Although less risky than jumping from airplanes, sharing ideas and offering possible solutions with others can be seen as risky by people who may be proven wrong or ridiculed by their peers (Beghetto, 2009). In truth, any learning involves risk, because of the uncertainty involved in new knowledge. However, without taking academic risks, people may come to believe that even with significant effort they will be unable to make mathematical progress, potentially leading them to lose opportunities to form knowledge and belief in themselves as mathematical learners and doers (Streitmatter, 1997).

Studies have shown that higher levels of students' academic risk taking are positively correlated with schools' self-reported tolerance of errors and mistakes (Clifford, Lan, Chou, & Qi, 1989). Meyer et al. (1997) found that risk-takers not only tolerated errors more than did other people but also formed positive associations with the tasks and mathematics as a whole. In fact, even moderate academic risk taking was linked to increases in attention, concentration, and persistence (Csikszentmihalyi, 1978). These findings indicate that academic risk takers are more likely than others to put effort forth and persist, even on difficult mathematical tasks.

Thus, one way to assess for academic risk taking is to watch participants while they work on difficult problems and see how long they persist at trying to solve them. These observations could be done in either an artificial test-taking environment (Csikszentmihalyi, 1978) or in the natural classroom setting (Streitmatter, 1997). Researchers have also assessed for risk taking by analyzing the level of problem difficulty participants choose when given a choice of problems at various known levels (Harter, 1978; Miller & Byrnes, 1997). These studies generally involved engaging in a few practice tasks with the participants to give a sense of how the problems were to be solved and the differing levels of problem difficulty. Then, the participants' risk taking was assessed by noting the difficulty level of problems they choose to solve.

Generally, risk takers choose more challenging tasks, because these tasks offer more opportunities to refine and build their knowledge (Streitmatter, 1997). Risk takers usually are more concerned with understanding concepts than with grades or praise, enabling them to be more tolerant of errors and confusion because they keep the goal of understanding in mind (Meyer et al., 1997). In other words, most academic risk takers have task-focused goals. Like academic risk taking, these goals also correlate with the belief that effort in mathematics is worthwhile.

Goals and Goal Setting

Goals are the states that, through affective, cognitive, and biochemical body regulations, human beings desire to obtain (Ford, 1992). Primarily two types of goals have been researched in mathematics education: *task-focused goals* and *ability-focused goals*. Learners with task-focused goals master tasks simply for the intrinsic value of learning, whereas learners with ability-focused goals see their goals as demonstrating their abilities or to do better than others (Anderman & Maehr, 1994). These two types of goals have been given a plethora of names: Dweck and Elliot (1983) referred to them as *learning goals* and *performance goals*; Ames (1992) called them *mastery goals* and *performance goals*; Urdan and Maehr (1995) used the terms *task goals* and *ability goals*; and Nicholls (1984) used the terms *task involvement* and *ego involvement*. Although the terminology may differ among researchers, their definitions have only slight differences and generally describe the same ideas (Ames, 1992; Eccles & Wigfield, 2002; Middleton & Midgley, 2002).

Anderman and Maehr (1994) expounded on differences between learners with task-focused and ability-focused goals. Task-focused learners place value on attempting difficult tasks and putting forth effort, gaining satisfaction from making progress and mastering a skill. They view errors as useful for personal growth and beneficial in the learning process. Ability-focused learners place value on avoiding failure; their satisfaction comes from getting high grades and being deemed "the best." Errors are seen as exposing weakness and lack of ability. Urdan and Maehr (1995) noted that different goals "do not necessarily affect the *amount* of motivation a student has to perform in a given situation. Instead, they affect the *quality* of the motivation, which affects behavioral, cognitive, and affective outcomes" (p. 215). According to Anderman and Maehr (1994), these two type of goals "are orthogonal and not simply opposite ends of a continuum" (p. 295). In fact, a learner can hold these two goals simultaneously.

The goal one holds when working on a task has a great effect on how that individual chooses to engage with the task and for how long (Anderman & Maehr, 1994). Learners who adopt mastery-oriented goals enjoy being challenged and will engage with difficult tasks longer than learners with ability-focused goals. Dweck and Leggett (1988) observed that all individuals need to know when a task should be abandoned, but that learners who have mastery-oriented goals can work through most periods of difficulty, maximizing their learning in the long run. In other words, learners with mastery-oriented or learning goals are more likely to persist on difficult mathematical tasks and to see that effort in mathematics is beneficial.

> Those with learning goals were more likely to view effort as a means or strategy for activating or manifesting their ability for mastery. Here effort and ability are seen as *positively related*: Greater effort activates and makes manifest more ability. (Dweck & Leggett, 1988, p. 261)

People with learning goals generally seek and persist with challenges that foster learning, whether their confidence in their abilities is high *or* low. Similarly, people with performance goals *and* high confidence in their abilities generally seek and persist with challenges that foster learning. The case is different for people with performance goals and low confidence in their abilities; these people often avoid challenges or fail to persist in putting effort forth when difficulties arise (Dweck, 1986). Gabriele and Montecinos (2001) found that low-achieving students with learning goals learned more than low-achieving students without learning goals. However, they found no discernable difference in the ways the two groups of students collaborated and participated with a higher achieving student. This finding indicates that goals are more indicative of individual cognitive performances than collaborative ones.

Assessing for one's goals has been a somewhat difficult task. In many studies, the participants completed survey-type instruments in which they denoted how closely they agreed with given goal-related statements on a Likert scale (Ames & Archer, 1988; Middleton & Midgley, 1997, 2002; Wentzel, 1989). More recently, researchers have used observational data along with surveys to help verify their inferences about the participants' goals (Morrone, Harkness, D'Ambrosio, & Caulfield, 2004; Patrick, Anderman, Ryan, Edelin, & Midgley, 2001). I think that one of the more promising ways of assessing for goals comes from the work of Gabriele and Montecinos (2001). They videotaped participants working together in pairs on mathematical tasks and used those data along with data from surveys and interviews to code for the likely goals of each participant.

People with learning goals see an ideal task as one that is challenging enough to advance their understandings. These people would disengage with a task not because it is difficult but because is too easy or intellectually boring (Dweck & Leggett, 1988). They find more challenging mathematical tasks to be interesting and worth pursuing. Compared with people who have performance goals, people who have mastery goals are more likely to consider how pieces of knowledge relate to one another and are capable of making more complex connections (Anderman & Maehr, 1994). This understanding of how different pieces of knowledge relate helps them to see mathematics as useful and worthwhile, but I believe that other factors play a role as well. For example, a person's affect and motivation also determine whether or not a person sees the subject of mathematics as useful and worthwhile.

Perceiving Mathematics As Both Useful and Worthwhile

The last element of the definition of productive disposition is the perception of mathematics as both useful and worthwhile. I have had personal experience with people who see sense in mathematics, that effort in mathematics is worthwhile, and that they themselves can engage with mathematics, but they did not view mathematics as useful and worthwhile, and thus saw no reason to pursue mathematical endeavors.

Young students generally hold the view that mathematics is worthwhile. The 1996 NAEP study found that 69% of fourth graders and 70% of eighth graders believed that mathematics is useful in solving everyday problems (NRC, 2001).

Unfortunately, Wilkins and Ma (2003) found that students' beliefs about the social importance of mathematics steadily declined in grades 7 through 12 and that this decline was influenced by the students' teachers and peers. Interestingly, they also found that students who spent significant time on their mathematics homework (averaging 3–10 hours a week) had a slower decline in their mathematical beliefs than had students who spent less than 3 hours on average and that students whose parents believed mathematics to be useful and worthwhile often held the same beliefs.

I think that this belief that mathematics is useful and worthwhile shows up not only in the goals one has when working on a mathematical problem, as was noted in the previous section, but also in the affect a person has toward learning and doing mathematics as well as in the motivation one has toward mathematics tasks. In the next sections, I explore these constructs in more detail.

Affect Toward Mathematics

McLeod (1992) defined *affect* as "a wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition" (p. 575). Affect is generally considered to be composed of three constituent parts: *emotions, attitudes*, and *beliefs* (McLeod, 1992; Philipp, 2007). *Emotions* are generally considered to be a person's feelings. These feelings can change quickly. People feel emotions more strongly and intensely than attitudes or beliefs, and emotions have less of a cognitive aspect than do attitudes or beliefs (Philipp, 2007). Emotions in humans are often unconscious or subconscious and may be somewhat unpredictable. They can also be hard for people to verbalize, analyze, or decipher in themselves (DeBellis & Goldin, 2006).

Attitudes are the ways a person thinks or feels that have the effect of showing the person's opinions. Attitudes change more slowly, are felt less strongly, and are more cognitive than emotions (Philipp, 2007). Attitudes include liking or disliking a certain topic or finding certain areas of study interesting or boring.

Beliefs are ways in which one views the world. They are more cognitive than attitudes and are very slow to change. Note that whereas some who study affect consider beliefs to be part of affect, some who study beliefs consider beliefs separately from affect. For the purposes of this paper, I consider affect to consist of emotions and attitudes and view beliefs as a separate category (discussed previously). Further, DeBellis and Goldin (2006) considered a fourth category, *values, ethics, and morals*, as part of affect. However, this addition to the definition of affect does not seem to predominate in the literature at this time.

Not all affect influences one's productive disposition. Of interest here is the affect one has toward engaging with mathematics and toward mathematics as a subject, which for many is quite strong. If positive, the affect can help stimulate people to continue working on and thinking about difficult problems, as well as see their endeavor as useful and worthwhile; if negative, it can lead people to quickly give up attempting to solve problems or to feel that there is no point in doing mathematics (DeBellis & Goldin, 2006).

DeBellis and Goldin (2006) described how affect can affect one's productive disposition. In the researchers' ideal positive pathway, students start a mathematics problem with curiosity and puzzlement, leading the students to start to solve the problem. Feelings of confusion or frustration can lead the students to realize that they may have reached a dead end, so they might try employing new strategies. After trying new methods, the students may become successful, leading to feelings of pleasure or accomplishment. These positive emotions help the students to believe that, with effort, they can succeed on mathematical tasks. DeBellis and Goldin also described a negative pathway, in which frustration leads to anxiety, which in turn leads to the students' avoiding mathematics and feeling as if they are not capable mathematicians.

Attitudes toward mathematics are thought to be formed in one of two ways. First, they can be brought about by one's repeatedly experiencing similar feelings toward mathematics (McLeod, 1992). For example, if a person generally has good emotional experiences while solving geometrical tasks, that person is likely to view geometry as useful or to find it worthwhile to engage in similar geometrical activities. Second, attitudes can be formed by people's making connections from an area in which they have already formed an opinion to a new related area (McLeod, 1992). For example, someone may already have the attitude of disliking algebra and may come to extend that dislike to any mathematics task in which a variable or unknown is used. This attitude may lead to the person's viewing all algebra or anything related to algebra as useless or unhelpful. Thus, asking participants about their previous experiences in mathematics and other areas the participants feel are connected to mathematics is one possible way to assess for their attitudes. These data could also be used in conjunction with noting the emotions people express while working on mathematical tasks to gain a more full sense of their affects toward mathematics.

Affect has a direct effect on the learning and doing of mathematics. Tsamir and Tirosh (2002) found that affect directly influences one's thinking processes and motivation while he or she is engaged with mathematics, showing the power that emotions and attitudes can have on learners. Kloosterman (2002) also found that affect influences an individual's interest in and motivation for learning mathematics. I believe that, like affect, motivation plays a significant role in whether people view mathematics as useful and worthwhile.

Motivation

In the simplest of terms, *motivation* can be seen as "the inclination to do certain things and avoid doing some others" (Hannula, 2006, p. 165). One's motivation is the reason a person behaves in a certain way given a certain situation. Motivation is directly related to one's beliefs and goals; it determines whether or not a person will view a particular pursuit as useful or worthwhile (Ames, 1992; Middleton & Spanias, 1999).

An important distinction made in the fields of psychology and education research is between *intrinsic* and *extrinsic* motivation:

> Intrinsic motivated behavior was defined as behavior undertaken for its own sake, for the enjoyment it provides, the learning it permits, or the feelings of accomplishment it evokes. Extrinsic motivated behavior, by contrast, involved actions undertaken *in order* to obtain some reward or avoid some punishment external to the activity itself. (Lepper, 1988, p. 292)

For example, a person might be motivated to complete work on a mathematical task either for intrinsic reasons, such as a sense of accomplishment or gaining new knowledge, or for extrinsic reasons, such as getting a prize or praise. Intrinsic and extrinsic motivations are not mutually exclusive; people can have both intrinsic and extrinsic motivations to pursue some activity.

In particular, mathematics education researchers are interested in *academic intrinsic motivation*, which is the drive or desire to engage in learning for its own sake. People with such motivation seek out learning activities for the sheer joy of learning; they enjoy thinking about the tasks (Middleton & Spanias, 1999). Note that some researchers use the terms *intrinsic motivation* and *fun* interchangeably, mainly because those outside of the mathematics education community may not know the former term and the latter term has connotations of positive affect (Middleton, 1995).

Stevens, Olivarez, Lan, and Tallent-Runnels (2004) looked across ethnicities to see how motivation affected mathematics performance. They found that, for all groups, the students who reported greater intrinsic motivation toward mathematics were more likely to view taking additional mathematics course as worthwhile and considered strong mathematical skills useful for their future career choices.

Remember that mathematics itself is not inherently motivating, inasmuch as students of equal ability levels may or may not be intrinsically motivated by mathematics (Middleton, 1995). People find the subject motivating because of their own backgrounds and beliefs. The building of intrinsic motivation in mathematics, though, may lead to more students' undertaking (and enjoying) more mathematics classes.

Many traits of learners with productive dispositions are found in people who have intrinsic motivations. Intrinsically motivated people tend to exhibit desirable learning traits, such as persisting, selecting tasks of appropriate levels (problems that are challenging but not impossible to solve), monitoring their own comprehension, risk taking, attempting varied strategies, and choosing activities that yield no extrinsic reward (Lepper, 1988; Middleton & Spanias, 1999). Intrinsic motivation has been found to have a stronger correlation with these variables in mathematics than in other subjects (Gottfried, 1985; Middleton & Spanias, 1999). These are traits for which one assessing motivations could look in studying participants working on mathematical problems.

Conversely, many of the traits of learners with productive dispositions were not found in people whose motivations for completing mathematical tasks were solely extrinsic. An extrinsically motivated person might perform well on a highly algorithmic task by using a rote method but would be unlikely to gain any conceptual learning from the task (Middleton & Spanias, 1999). Given a more complex task that requires more conceptual understanding, an extrinsically motivated person is likely to lack the insight, creativity, or effort needed to make progress (Lepper, 1988). Thus, people's motivations directly affect their productive dispositions.

Summary

Productive disposition consists of seeing sense in mathematics, seeing oneself as a learner and doer of mathematics, believing that steady effort is fruitful, and believing that mathematics is useful and worthwhile. To more fully elaborate the definition of *productive disposition,* I considered connections of these four elements of the definition to eight constructs from the research literature in mathematics education: beliefs, mathematical integrity, identity, self-efficacy, risk taking, goals, affect, and motivation. These constructs were used in this study in the differing ways. The constructs of affect, beliefs, and identity have features in common with the definition of productive disposition, but because of their breadth, they may not be useful in totality. However, specific aspects of these constructs were helpful to consider in looking for evidence of one's productive disposition. Evidence of strong productive dispositions will be examined as related to the five narrower constructs.

I do not claim that this list is exhaustive of the constructs in the educational research literature connected to productive disposition and as such I considered other orientations that arose from the analysis of the data. However, by building from the research on these constructs, I considered ways one might assess for the productive dispositions of teachers actively engaged in mathematical tasks. To conclude this chapter, I summarize the ways one can assess for the eight constructs and use them to build a preliminary rubric designed to assess for productive disposition. This rubric was used as a starting point to assess the productive dispositions of teachers working on mathematical tasks.

Section 2.5—Assessing for Productive Disposition

As noted in the previous section, my overarching goal in this study was to determine what productive disposition is and how to identify it being enacted in others. To accomplish this goal, one needs to be able to assess for productive disposition. However, because little research has been done on this relatively new construct, assessing for productive disposition may be a somewhat daunting task. To undertake this task, I used a combination of methods that researchers have used (as described in the previous sections) to assess the eight constructs connected to productive disposition, with the three most common being observing people at work on mathematical tasks, having the participants reflect about themselves or mathematical problems they have solved (either by asking the participants open-ended questions or by using Likert-scale questions), and conducting interviews with the participants.

In this initial attempt to assess productive disposition, I used all these general methods in my two studies. I found that some methods are more useful than others and that the different methods glean different information about the participants' productive dispositions, as will be seen in Chapter 4. Such observations will be beneficial for future research on productive disposition.

Indicators

Using these general methods to assess for productive disposition, I refined the assessment by looking for specific indicators in participants, whether they are actively engaging in a mathematical task or being interviewed about their previous mathematical experiences, that reveal the strengths of their productive dispositions. In this section, I describe indicators researchers have used to assess for affect, beliefs, identity, mathematical integrity, risk taking, goals, motivation, and self-efficacy. These indicators helped shape a rubric, a description of evidence one would see in a person holding a strong productive disposition.

Affect includes the attitudes and moods of people, and one of the most direct ways to assess for affect is to directly ask people how they are feeling. Researchers have asked the participants in their studies to select their emotions relating to a mathematical task from a set of choices (Gomez-Chacon, 2000) or asked them to describe when they enjoy learning mathematics, how they feel when engaging in mathematical activity, what they like least about mathematics, and other similar questions (Haladyna, Shaughnessy, & Shaughnessy, 1983; Ruffel, Mason, & Allen, 1998). Although several researchers (e.g., Ma, 2006) had participants complete Likert-scale evaluations of statements such as "I enjoy math," "Math is useful in everyday problems," and "Math helps me think logically," I looked for these and similar statements that reveal a strong productive disposition in the conversations the participants have and in the interviews. I also looked for expressions of one's emotions, such as interest in the task or frustration (Gomez-Chacon, 2000). Last, building on the work of Hannula (2002), I observed the emotions experienced during mathematics-related activities and asked about the emotions the participants associate with the concept of *mathematics.*

Beliefs-assessing methods have been similar to those for assessing affect. Many researchers ask the participants direct questions about their beliefs, such as "What do you think math is about," "What do you think is the best way for students to learn math," and "Are there students who cannot learn math, or can everyone learn if they try" (Kloosterman, Raymond, & Emenaker, 1996; Muis, 2004; Raymond, 1997). On the basis of this work, I first watched the focus-group videos and, in Study 2, read participants' mathematical autobiographies to collect beliefs statements the participants offer freely. Indicators that the participants have positive beliefs about mathematics, and thus positive productive dispositions, include statements that everyone can do mathematics if they try, that mathematics is a series of interconnected concepts, that mathematical problems can be solved in a variety of ways, and that one's goal in working on a mathematics problem is more than simply to get the correct answer as quickly as possible (Muis, 2004, NRC, 2001). Some of these beliefs were hard to infer from the aforementioned data, so in Study 2 I also asked participants about their beliefs more directly in the interviews, both to ensure that any inferences made about their beliefs were correct and to gather additional data about their stances toward mathematics.

Assessing identity, like assessing affect and beliefs, has previously been accomplished by asking participants directly about their previous experiences with mathematics, including asking them to identify high points or turning points in their mathematics experiences and to explain whether their mathematical experiences were steady or more like a roller coaster (Drake, Spillane, & Hufferd-Ackles, 2001). Such questions served as prompts for participants' mathematical autobiographies. To understand more about their mathematical identities, and in turn their productive dispositions, I observed how they engaged with mathematics and ask how they envisioned mathematics fitting into their broader lives, what "type of person" they saw themselves as being in relation to mathematics, and how much of their mathematical identities they attributed to nature and how much they saw as an inborn trait (Anderson, 2007; Gee, 2001). In particular, I attended to statements of the form "I am..." or "I have..." and to their use of future-tense verbs, such as should, ought, must, or can. Such statements often reveal the mathematical identities the participants hold (Sfard & Prusak, 2005).

Unlike for affect, beliefs, and identity, little research has been done in assessing for mathematical integrity. However, DeBellis and Goldin (2006) suggested considering whether the participants know when a solution is correct, when a problem has been solved satisfactorily, when one's understanding suffices, and when mathematical achievement deserves commendation. They suggested further looking for when participants recognize that they have insufficient mathematics understanding or when they decide to take further action on a problem. Participants with mathematical integrity may look for a deeper structure, solve a related problem, or admit when something does not make sense. In essence, these authors suggested looking for evidence that the participants are honest and open about their mathematical abilities.

To externally display mathematical integrity, though, one must be willing to take academic risks. Assessing for academic risk taking was done in several parts of the studies. First, in reviewing the focus-group interactions, I looked for evidence of participants' sharing tentative ideas, asking questions about others' or their own solutions, and exhibiting willingness to make mistakes or appear less than completely competent (Beghetto, 2009; Clifford, 1991). Second, in the task reflections, the participants were asked how successful they felt, how certain they were about their solutions, and whether there were any mathematical ideas they were unwilling to share previously (Meyer, Turner, & Spencer, 1997). Last, in the interviews, the participants were asked their feelings about failure, about their willingness to make errors or offer partial solutions, and their preferred level of mathematical-task difficulty (Clifford, 1991). These data was used as evidence for the participants' willingness to take academic risks.

Participants' preferred level of mathematics-task difficulty is also evidence of the goals the participants hold. Participants with task-focused goals choose tasks that lead to higher mathematical abilities, not tasks that can easily be performed with one's current knowledge (Nicholls, 1984). Other evidence of holding task-focused goals includes toleration of failure, a focus on effort and learning when engaging with tasks, and active engagement and interest in mathematical tasks (Ames, 1992). Conversely, people with ability-focused goals want to be the only one with a solution, to look smart, and to not embarrass themselves in front of others (Middleton & Midgley, 2002). Because people with productive dispositions are more likely to have task-focused goals than ability-focus goals, I looked for evidence of participants' holding such task-focused goals as a belief in efficacy of effort, attentiveness and focus on the task, and enjoyment in a challenge (Dweck & Leggett, 1988).

Many task-focused-goals indicators are similar to intrinsic-motivation indicators. Lepper (1988) noted that intrinsically motivated people spend more time on an activity, are more attentive and focused on the task, and tolerate failure more readily than the extrinsically motivated. The data for such evidence of both goals and intrinsic motivation came from the focus groups. However, the motivation-assessment literature also included interview questions ranging from what participants think accounts for their motivations on tasks (Middleton, 1995) to why they proceeded on a task and what made them eventually stop (Matsumoto & Sanders, 1988) to how much of their effort they feel was due simply to natural ability (MacIver, Stipek, & Davis, 1991). These data gave me evidence for the participants' motivations and, in turn, for their productive dispositions.

The last piece of the productive-disposition puzzle is self-efficacy. Assessing for self-efficacy is, in essence, looking for confidence (Ferla, Valcke, & Cai, 2009; Midgley, Feldlaufer, & Eccles, 1989; Pajares, 1996), which has two components: confidence in one's own skills and abilities to solve a problem and confidence in one's own knowledge (Pajares, 1996). I found it difficult to assess for self-efficacy through evidence of confidence in the video data. However, the reflections and interview questions about the participants' confidence levels during the tasks in Study 2 were more enlightening.

From these eight constructs, I have many possible indicators of a participant's holding a strong productive disposition. However, I needed to reorganize and refine this list, especially because many of the indicators appear multiple times. In the next section, I arrange these indicators into an initial rubric so that they can effectively be operationalized.

The Rubric

In the previous section, I described a plethora of sources of evidence for indicators of one's holding a strong productive disposition. To operationalize these indicators of these sometimes overlapping constructs, the Savannah focus-group team created a rubric that is based on both the definition of productive disposition and the research in areas directly connected to that definition. The rubric was designed to apply to discussions by teachers actively engaging with mathematical tasks that they can, with effort, complete. The rubric includes categories of evidence one might use for understanding a participant's productive disposition. The data-analysis approach for this study falls midway along the inductive to *a priori* continuum of approaches for coding data (Miles & Huberman, 1994) and is similar to typological analysis as described by Hatch (2002).

Several potential indicators (listed in Table 2.1) were derived from the research literature and preliminary analysis of focus-group videos. Additional indicators were later encountered in the study data and the table was adapted. One of my goals for this study is to identify such additional indicators and refine the categories by watching teachers actively engaging on mathematical tasks. (For more about the changes to this table, see the results of Study 1 in Chapter 4.)

Table 2.1

Potential Indicators of Strong Productive Dispositions (Initial Version)

Potential categories	Evidence
1. View mathematics as a sense-making endeavor	a. Tries to make sense of the task b. Considers alternate approaches c. Asks if answer seems logical
2. View mathematics as beautiful or useful and worthwhile	a. Interest in the task, evidenced by one's engagement or one's comments about the task
3. Sense one can, with appropriate effort, learn mathematics	a. Sense one can make progress on the task on which one is working b. Defining progress as learning through grappling, not just getting an answer
4. Approaching mathematics with particular habits of mind	a. Asks questions about the mathematics, or about one's or another's approach b. Persistence c. Seeks and provides clarifications
5. Mathematical integrity	a. Having a sense for when one has completed a task (whether or not one continues) b. Appreciating one's solution
6. Does not avoid frustrations	a. Sense of wonder b. Shows pleasure or excitement about a particular way of reasoning c. Recognizes worthy and unworthy confusions

The first four categories of indicators were based on the four parts of the definition of productive disposition: seeing sense in mathematics, believing mathematics to be useful and worthwhile, believing that steady effort is beneficial, and seeing oneself as a learner and doer of mathematics. The fourth of these categories was broadened to include the habits of mind mathematical learners and doers might see themselves as having. In other words, part of being a mathematician is questioning others' (and your own) approaches to solving problems and seeking needed clarifications. Thus, people who view themselves as learners and doers of mathematics should also hold and display these habits of mind, especially when engaging with a mathematical task.

Aspects of the constructs of affect, beliefs, and identity were included in these first four categories. Indicators for affect include showing interest for the task, both in the comments about the task and engagement in the task. Believing that one can make progress on the task or asking if potential solutions seem logical are two possible indicators for one's beliefs both about mathematics as a subject and about the value of effort. Indicators for one's mathematical identity, more difficult to decipher, may be revealed in the evidence one shows for mathematical habits of mind.

The fifth category of indicators was derived from the constructs of mathematical integrity and academic risk taking. Mathematical integrity entails a person's being able to self-assess what he or she does and does not know and being honest about that assessment. These self-assessments might be solely internal, and, as such, no evidence for these assessments might be shown. However, if one is a risk taker and is willing to offer tentative ideas or ask questions about a particular strategy, what the person asking does and does not know is exposed. Thus, when one takes an academic risk, his or her mathematical integrity is evident. For one unwilling to take these risks, we will likely be unable to ascertain evidence for the person's mathematical integrity.

The sixth indicator category was drawn from the constructs of goals and motivation, paired because of the symbiotic nature of the two. As noted earlier, people who are extrinsically motivated by rewards such as grades and approval generally tend to have performance goals, whereas people who are intrinsically motivated and are moved by understanding and connecting concepts generally tend to have learning goals (Ames, 1992; Dweck, 1986; Middleton & Spanias, 1999). Thus, separating the evidence for these two constructs is difficult. For example, on the one hand, people might willingly reengage with a difficult mathematical task because they have the goal of understanding the mathematics in the task. On the other hand, they might find the activity intellectually interesting and relevant to their learning interests, and thus be motivated to reengage with the task. A person who is willing to reengage with a difficult mathematical task likely has both the goal to understand the mathematics and the motivation to do so. As such, evidence one might show for holding these two constructs was included in one category. (This category was later renamed *positive goals and motivation*.)

Missing here is the construct of self-efficacy. Self-efficacy is a person's own belief in his or her ability to take action in a particular problematic situation (Bandura, 1977). Although this belief seems important for the productive dispositions people need to engage with mathematical tasks, I initially felt that one's self-efficacy may not be fully evident in the focus-group data. Part of self-efficacy is confidence, which may be evident in the data. However, beliefs about the participants' own perceived abilities, especially as related to the task given, may not. Moreover, one may not look confident in his or her mathematical understandings and skills yet may persevere on mathematical tasks or believe that the tasks are doable and should make sense. However, as was noted previously, this rubric was only a list of potential indicators for one's productive disposition. The list changed significantly after looking at the focus-group data. Part of those changes included adding a seventh category based on self-efficacy. (Again, for more about the changes to this table, see the results of Study 1 in Chapter 4.)

Section 2.6 – Chapter Summary

The rubric posited above is based on a review of the literature and was an initial attempt to assess for productive disposition indicators, but needed to be operationalized to see if it determined differences in teachers' productive dispositions. In Study 1, I used this rubric to help answer this research question: ***What differences in evidence for one's productive disposition can be found through the analysis of teachers' active engagement in a mathematical task?***

In Study 1, I used the preliminary rubric to analyze existing video data from the STEP Project's Savannah Task. These data consist of videotapes of 3–6 teachers engaged in a focus-group discussion of the mathematics in the work of a student, Savannah (see Appendix A for the task). Because 136 teachers participated in the STEP Project, the data are from 30 focus groups. Using the rubric, I sought evidence to assess for the teachers' productive dispositions, but I also modified the rubric during my data analysis as warranted by evidence not previously considered.

STEP participants were grouped on the basis of their years of sustained professional development focused on children's mathematical thinking. Although one outcome of this study was to discuss how the groups' dispositions differed, I used these data in my study primarily to identify and assess different aspects of the participants' productive dispositions. The rubric, then, was used to investigate these differences and found evidence for dispositions the teachers used to help them engage with a mathematical task.

The data from the STEP study was a good place to start to see if the rubric could be used to find differences in the evidence for people's productive dispositions. It did not, however, fully answer the questions of what is a productive disposition and how is it manifested when it is being enacted. In Study 2, I attempted to answer the research question ***What evidence for productive disposition is self-reported by teachers who have been identified as having strong productive dispositions?*** My goal in answering this question was to determine whether specific characteristics always arise in someone with a strong productive disposition or whether certain types or categories of people have strong productive dispositions. The rubric has possible indicators for evidence of one's productive disposition, but which do the teachers find important? Which are hidden or missing from Study 1? Do other orientations also support one in engaging in mathematics?

To start to answer these questions, I studied teachers identified as having strong productive dispositions while they engage in several mathematical tasks, including the Savannah Task. The rubric was used as a starting point to see how these people with strong productive dispositions look similar and where they differ. However, new indicators also arose from the data to give a more complete view of what constitutes a productive disposition.

In the next chapter, I provide detail about the methodology for this multipart study, building on the ideas from the constructs in this chapter. On the basis of the literature reviewed here, I consider evidence in the data collected; on the basis of researchers' suggestions from the literature for developing positive aspects of these constructs, I considered task features and modifications so that these productive disposition traits might become evident.

CHAPTER 3

METHODOLOGY

Section 3.1—Introduction

My overarching goal in undertaking this study was to better understand the construct of productive disposition. To do so, I propose to answer the following research questions:

- What differences in evidence for one's productive disposition can be found through the analysis of teachers' active engagement in a mathematical task?
- What evidence for productive disposition is self-reported by teachers who have been identified as having strong productive dispositions?

In this chapter, I discuss the methodology I used for answering these research questions. First, I discuss how I assessed for differences in the productive dispositions of 136 preservice and in-service teachers (Study 1). Then I discuss the tasks, reflections, mathematical autobiographies, and interviews I conducted with 10 in-service teachers who were found to have strong productive dispositions (Study 2).

Section 3.2—Application of the Rubric to Assess for Differences

In Study 1, I analyzed focus-group data to answer the following research question: What differences in evidence for one's productive disposition can be found through the analysis of teachers' active engagement in a mathematical task? I analyzed existing video data from the STEP Project. These data consisted of videotaped conversations by groups of 3–6 teachers engaged in discussing the mathematics in the work of a student, Savannah (see Appendix A for this task). Thirty such groups were formed from the 136 teachers in the STEP Project;[2] the rubric was first refined on the basis of the data seen and then used to categorize the evidence for the teacher's mathematical orientations in these focus-group discussions.

The reason for choosing to analyze extant focus-group data was two-fold. First, the 136 teachers included 36 preservice teachers and 100 in-service teachers. In the STEP Project, the 100 in-service teachers were placed in three groups on the basis of the number of years the teachers in the groups had engaged in sustained professional development focused on children’s mathematical thinking. The fourth group consisted of the 36 preservice teachers who were just beginning their first mathematical content course for teaching. These teachers were found to vary significantly on several measures, including their content knowledge, their beliefs about mathematics and mathematics teaching and learning, their professional noticing of children’s mathematical thinking, and their responsiveness. (For more information about the STEP study, see Philipp, Jacobs, Lamb, Bishop, Siegfried, & Schappelle, 2012.). Thus, teachers with these differences likely have a range of mathematical orientations as well. I expected that across this range I would see varied aspects of teachers’ mathematical orientations during their mathematical engagement. Second, because my goal was to identify the orientations teachers use when engaging with mathematical tasks, I welcomed the opportunity for unexpected orientations to arise from the data. Although the initial rubric was based on certain types of evidence,

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[2] There were officially 129 teachers in the STEP Project; however, the focus groups contained participants who were eventually excluded from the study. Because of their interactions with the other teachers in the focus groups, I included all 136 teachers.

these focus-group data were not biased toward having such evidence emerge. Instead, the data showed teachers engaging in a somewhat difficult task centered on mathematical content. In attempting to make sense of the task, I believed the participants were likely to display their natural mathematical orientations.

In this section, I provide background about the STEP Project, focusing on the selection of participants and the formation of the participant groups, and describe the Savannah Task, data from which were analyzed in this study. I explain how the data from the Savannah Task helped to refine the rubric from the previous section and then how the rubric was used to analyze differences in the participants' productive dispositions.

STEP Participants

The participants of the STEP Project were 136 preservice and in-service teachers (see Table 3.1). The in-service teachers were all K–3 elementary school teachers from three school districts in Southern California. The demographics for these three districts were similar: One third to one half of their students were Hispanic, one fourth of their students were classified as English Language Learners, and one fourth to one half of these students were receiving free or reduced-cost lunch. The in-service teachers for this study were grouped not by years of teaching experience but rather by years of experience in professional development focused on children's mathematical thinking. Initial Participants (IPs) were teachers who had not yet participated in this professional development but had volunteered to begin soon. Advancing Participants (APs) were teachers who had 2 years of professional development focused on children's mathematical thinking and Emerging Teacher Leaders (ETLs) had 4 or more years of this professional development. The Emerging Teacher Leaders were also beginning to engage, formally or informally, in leadership activities designed to support other teachers.

Table 3.1

Participant Groups in the STEP Project

Participant group	Description
Preservice Teachers (PSTs) (*n* = 36)	Undergraduates enrolled in a first mathematics-for-teachers content course
Experienced practicing teachers	
Initial Participants (IPs) (*n* = 35)	Experienced K–3 teachers who were about to begin sustained professional development focused on children's mathematical thinking
Advancing Participants (APs) (*n* = 31)	Experienced K–3 teachers with sustained professional development focused on children's mathematical thinking for two years
Emerging Teacher Leaders (ETLs) (*n* = 34)	Experienced K–3 teachers with sustained professional development focused on children's mathematical thinking for four or more years and who were beginning to engage in leadership activities to support other teachers

Note that all the in-service teachers in the STEP project had 4 or more years of teaching experience, with a range of 4–33 years. Moreover, in each of the three groups of in-service teachers, the average number of years of experience teaching was about the same: 14–16 years. Although we in the study acknowledge that length of teaching experience has an effect on many important aspects of teaching, our goal in the STEP study was to determine what effects professional development has on these teachers' knowledge, beliefs, and practices.

As an anchor to the study, a fourth group consisting of preservice teachers (PSTs) was added. These were university students in the region in which the teachers taught. They were generally in their first two years of study at the university and were just beginning their first mathematical content course for teaching elementary school.

Savannah Task

The participants, in focus groups of 3–6 people, discussed the Savannah Task; their discussions were videotaped and audiotaped. They were then given the work (see Figure 3.1) of a student, Savannah, described to them as follows:

> Savannah, a third grader, completed the work on the following three problems in May. After you look over Savannah's written work, please start your conversations by considering this question: What do you find noteworthy in this student's work?

The participants had hundreds charts, unifix cubes, and base-ten blocks available for their use. They were permitted to take their conversation wherever they pleased.

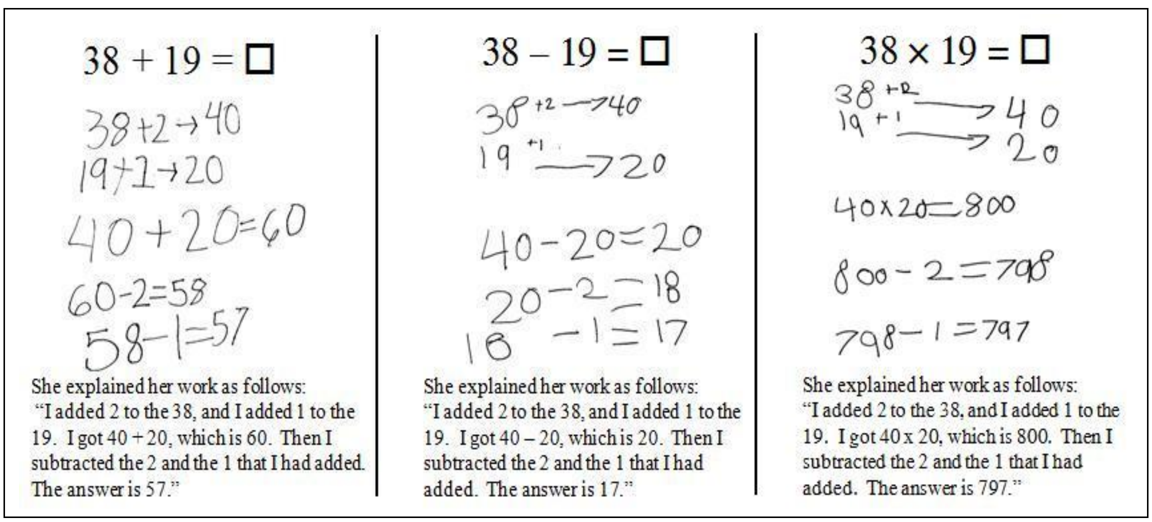

Figure 3.1. Savannah's work.

For these discussions, a member of the research staff served as facilitator but did not engage in the discussions. The participants were to direct their conversations to one another, not to the facilitator. The conversation about what the participants found noteworthy in Savannah's work was allowed to continue uninterrupted. The participants were even told that the facilitator was likely to err on the side of giving them too much time so that there might be silences from time to time. (For the full protocol, see Appendix A.)

After the conversation from the first prompt ended, the following questions were posed one at a time:

- *Pretend that you are Savannah's teacher. What problem might you pose next to Savannah? Why would you choose that problem?*
- *Savannah's approach for addition was correct. Explain why.*
- *Savannah's approach for subtraction was incorrect. Explain why.*
- *Do you have any thoughts about why Savannah's answer for the subtraction problem is off by 2?*

Only when the group indicated that they had finished discussing the previous question was a new question posed.

Mathematics in the Savannah Task

The Savannah Task, although based in K–3 mathematics, can be difficult to decipher, even for experienced teachers. Savannah's addition strategy is correct because she adds to the addends to create easy-to-add landmark numbers. She then adds the two landmark numbers together and, finally, subtracts the extras she had added. Savannah's strategy for subtraction is not mathematically correct because of the difference in the way the minuend and subtrahend function in subtraction. Adding 2 to the minuend to get 40 and then subtracting that 2 after completing the subtraction

(40 – 20) makes sense mathematically. However, when Savannah adds 1 to the subtrahend, she has the effect of subtracting 1 too many. Thus, in the end, she needs to add 1 back, not subtract an additional 1. As for the multiplication problem, Savannah is subtracting units when she should be subtracting groups of units.[3]

Analysis—Scoring

I, along with another coder (Bonnie Schappelle), watched the videotaped sessions that showed the participants thinking aloud about the Savannah Task, talking to others, and actively trying to make sense of the mathematics. We also had access to a transcript of the session and any written work done by the participants. All the data had been blinded so that we would not know the participants' group (PST, IP, AP, or ETL). We allowed ourselves to watch the video as many times as we thought was necessary. We then used the rubric to group the individual participants based on their engagement with the task. The goal of the coders during this first pass of the data was not to identify productive disposition indicators but rather to place the individual participants into groups on the basis of how much evidence they show of having positive mathematical orientations (see Table 3.2).

Because of the interval nature of the data, these groupings became the teachers' productive-disposition scores. These scores based on evidence are similar to scoring

[3] The situation in the multiplication problem is somewhat complex. Savannah's answer of 800 (before she mistakenly subtracted the 3) is 78 too large: 1 group of 38, 2 groups of 19, and 2. This discrepancy is explained in the following decomposition: $40 \times 20 = (38 + 2) \times (19 + 1) = (38 \times 19) + (38 \times 1) + (2 \times 19) + (2 \times 1)$. The first term in this expansion is Savannah's desired answer; the other terms are what she needs to subtract from 800 to compensate for the addition of 2 and 1 to 38 and 19, respectively.

of participants on the basis of other data in the STEP Project. [For more about the STEP Project, see Jacobs, Lamb, & Philipp, 2010.]

Table 3.2

Productive-Disposition-Scoring Rubric

Score	Evidence
0	The participant shows essentially no evidence of engaging with the mathematics. The participant may show resistance to engaging with the task at any level—mathematical or otherwise; or the evidence may indicate a lack of effort, either because of disinterest or lack of capability.
1	The participant shows weak evidence of engaging with the mathematics. In trying to engage, the participant may unintentionally move the group away from productive discussions. The participant may make some efforts to engage but may not persist in doing so.
2	The participant shows some evidence of engaging with the mathematics. The participant may make little progress toward understanding the mathematics or become confused but show clear evidence of willingness to engage with the mathematics of the task.
3	The participant shows strong evidence of engaging with the mathematics. Although evidence of engagement is clear, it may be somewhat shallow relative to the person's evident capacity, delayed, or limited to a brief segment of the conversation.
4	The participant shows very strong evidence of engaging with the mathematics. Effort to understand is shown until he or she does understand or throughout the focus-group session. A participant who comes to understand the mathematics, tries to help others understand by explaining and, usually, re-explaining in multiple ways to those who do not understand. One who does not understand persists in attempting to do so, trying various approaches; he or she expresses reluctance to abandon the search for understanding, often in spite of other group members' pleas to move on.

Before we coders began to group the data, we watched videos (some from each participant group, chosen by noncoders to preserve group anonymity) together to discuss and reach agreement as to how to categorize the data. During the analysis, we separately determined the scores of individuals from a focus group. We then compared our scores. When disagreement occurred, we each presented the evidence on which we based our decisions and discussed the data to reach agreement. During this time, we also introduced any potential new indicators we believed to have identified during the viewing of the videos to add to or refine the list of potential productive-disposition indicators.

Note that the participants' scores are indicative only of the evidence they showed in discussing the Savannah Task in their group. Although I believe that a participant who shows multiple positive indicators of engagement on this task would likely show high levels in other mathematical engagement, I do not make the same claim about participants who show little to no evidence of engagement with the mathematics in the Savannah Task. Other influences, such as a sick child at home, stress due to work issues, or unhelpful focus-group members, may have affected a participant's performance. Thus, one weakness of this study is that I may be underestimating the mathematical-engagement levels of some participants. Moreover, although I analyzed the participants individually, I recognize that others in the focus group may have influenced each individual. Again, the score for an individual is indicative only of the evidence shown of his or her productive disposition toward engaging in mathematics on the Savannah Task within the given focus group.

Scores for the four teacher groups were compared using 6 one-tailed pairwise *t*-test comparisons with a Type I error rate of .05. The choice of one-tailed test was based on the assumption that more years working with children or attending professional development would be unlikely to correlate with lower productive-disposition scores.

Analysis—Indicators

While we went through the Savannah task focus groups, Bonnie and I also modified the list of productive-disposition indicators based on evidence we saw from the videos. After all 136 teachers had been given productive-disposition scores, I modified the list of productive-disposition indicators, adding one new category and eight new indicators. I also broke apart some indicators and moved others to more appropriate categories. (See Chapter 4 for more information about how the list of productive-disposition indicators changed.) After these modifications to the list of productive-disposition indicators, I rewatched the teachers' focus-group videos. On this pass through the data, I looked for the specific indicators shown by each teacher while engaging with the Savannah Task. The teachers' productive-disposition scores were not used during this process so that my count would be as unbiased as possible.[4]

For each teacher, I recorded which indicators were evident. However, I did not record whether a teacher showed evidence of an indicator multiple times because for many of the indicators, I could not determine whether a teacher had shown the indicator multiple times or had simply continued to show the indicator for a long period of time. For example, when teachers asked others to clarify some part of their solution strategies multiple times, I could not tell if these additional questions were to clarify different portions of the solution or if they were a continuation of the previously asked clarifying questions. Moreover, because the purpose of this study

[4] I had scored all of the teachers previously, so I remembered a few teachers' scores. For most cases, however, I did not remember what score each teacher had received.

was to determine the types of evidence shown by teachers engaged with a mathematical task, I thought that recording *that* a teacher had shown a particular indicator of holding a strong productive disposition was more important than *how many times* that indicator was seen in an individual.

Examples of Indicators From One Focus Group

To illustrate how I analyzed the Savannah Task data for the teachers' indicators of productive disposition and to provide a sense for what I considered to be evidence as seen in the videos, I provide examples of indicators found during the analysis of data from one participant.

Jan[5] was the first person in her group to offer a possible reason that Savannah's subtraction strategy is flawed:

> **Jan:** Because she's adding 1 to the one she's taking away, so that's … you can't just subtract those extra numbers. She's added to her total [minuend], and she's added to what she's taking away [subtrahend]. And so she's taking away … essentially from her total.

I found this comment indicative of Jan's mathematical orientation for several reasons. First, Jan was engaging with the mathematics of Savannah's strategy. She offered this explanation as part of what she found noteworthy in Savannah's work, *before* she had been asked by the facilitator about why Savannah's strategy for subtraction is incorrect. Also, Jan took a risk here by offering a partial explanation. She was willing to explain what she thought she understood to that point, even though she knew that her understanding was incomplete.

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[5] All names are pseudonyms.

Jan had difficulty making sense of Savannah's subtraction strategy: "She's subtracting 1 more … because she had two extra. This should be really easy to see, and it's driving me crazy." Although Jan was struggling here, she recognized the difficulty of the task and thus seemed accepting of the fact that she was struggling. Jan seemed to have as a goal making mathematical sense of Savannah's strategy and to have the sense that she could make progress toward that goal if she persisted.

Jan was the only one in her group who seemed to understand why Savannah's subtraction strategy is incorrect and why Savannah's solution is 2 less than the correct answer. She tried several times to explain her thinking to the other members of her group, each time giving a slightly clearer and more assured explanation. Unfortunately, the other members of her group did not seem to be helped by her explanations. When the group was asked by the facilitator why Savannah's answer is off by two, the final question posed, the following exchange took place:

> **Mae:** We tried to figure that out earlier.
>
> **Jan:** We kind of figured it out, didn't we? We were saying that when you add … she added to the 19. … It's off by 2 because she needed to add to the. … Rather than take away, she needed to add, so when she took away, it ended up 2 down. If she had added it, would have been 2 higher than what she got. Does that make sense?
>
> **Sue:** I'm going with the "I really couldn't tell you why it's 2 off."
>
> **Jan:** Because she did the wrong thing. … You know what I mean? She should have added, so if she subtracted, it was minus 2 then from her starting point … I mean from where it should have been. If she added, she would have gone up 1, but she subtracted, so she went down 2.
>
> **Mae:** That sounds great (facetiously).

Although Mae and Sue had clearly given up on reasoning about why Savannah's answer is off by 2, Jan kept trying to explain and re-explain it to them. Jan tried to re-engage the others by asking questions like, "Does that make sense?" She persisted in trying to make mathematical sense not only for herself but helping the others in her group as well.

In summary, Jan consistently engaged with the mathematical content of the task. She had a sense that she could make progress on the task if given enough time. She followed Savannah's reasoning, made hypotheses about why the reasoning failed to yield the correct answer, and then tried to correct Savannah's strategy. She tried on several occasions to explain her thinking to the rest of the group. Jan showed evidence of a continual willingness to engage with the mathematics of the problem, persistence, and positive affect, which I see as *very strong* evidence of Jan's positive mathematical orientation.

The purpose of analyzing the data in this way was to determine which aspects of productive disposition are evident in those participants who actively engaged with the mathematics of the Savannah Task. Were some aspects more prevalent than others? Were some more prevalent in a particular teacher group than in others? What other constructs were identified as evidence of teachers' productive dispositions and for which group(s) of teachers did they arise? This qualitative analysis was a step toward my overarching goal of better understanding the construct of productive disposition.

Section 3.3—Evidence of Holding Strong Productive Dispositions

In Study 1, I looked for differences in individuals' productive dispositions. However, Study 1 alone does not show what a productive disposition is and how to identify it when it is being enacted. Of particular interest are people with strong productive dispositions and the indicators evident in their work on mathematical tasks. In Study 2, I propose to answer the research question ***What evidence for productive disposition is self-reported by teachers who have been identified as having strong productive dispositions?*** My goal in Study 2 then, is to determine whether specific characteristics always arise in someone with a strong productive disposition or if strong productive dispositions are conveyed in different ways. Although Study 1 helped to produce a list of indicators for evidence of one's productive disposition, several of the indicators were seen in only a few cases. Would these indicators emerge in other forms of data, such as mathematical autobiographies or reflections? Would other indicators become evident from different tasks? Most important, which indicators would resonate with teachers who were found to have strong productive dispositions?

To address these questions, I observed 10 of the in-service elementary school teachers who had participated in Study 1 while they engaged in several mathematical tasks, including the Savannah Task. These teachers were also asked to complete a mathematical autobiography and were interviewed individually. The purpose of these interviews was to verify with the participants my interpretation of the indicators they exhibited when working on the mathematical tasks and to identify others not recognized in the analysis. These indicators will be included to provide a more complete view of constituents of productive disposition.

Participants

The participants, who had participated in Study 1 five years earlier, were 10 in-service K–3 teachers from three school districts in Southern California; each had been identified as having a strong productive disposition (with a score of 3 or 4 on a 0–4 scale) in that study. The demographics for their school districts were similar: One third to one half of their students were Hispanic, one fourth of their students were classified as English Language Learners, and one fourth to one half of these students were receiving free or reduced-cost lunch. Although not grouped in this study on the basis of years of sustained professional development, the teachers were from each of the previous in-service teacher groups (2 IPs, 3 APs, 5 ETLs).

Tasks

The teachers, in two groups of 5, comprised focus groups, each of which met for 3–3.5 hours. During these sessions, the teachers took part in four mathematical tasks: the Savannah Task, the Pat Task, the Consecutive-Sums Task, and the Sam Task. After each task, the participants were asked to respond in writing to reflection questions designed to reveal aspects of their productive dispositions in relation to the task. The focus groups were both audiotaped and videotaped, and all written work from the participants was collected. To conclude the session, each participant audiorecorded her mathematical autobiography and scheduled an individual interview with me at a time and place of her choosing.

Savannah Task

The Savannah Task was conducted as it was in Study 1. (For more about the Savannah Task, see Study 1; for the full Study 2 protocol, see Appendix B.) The Savannah Task was used again for Study 2 because, in the previous study, it was found to be useful for identifying teachers with strong productive dispositions. I used it in Study 2 with the hope of obtaining results from each teacher similar to those found in Study 1, but now with the opportunity to confirm my analysis with the teacher. However, I also used other tasks to obtain a broader view of participants' productive dispositions than can be seen in one task. I next describe the three additional tasks and explain why each task was chosen for use in this study.

Pat Task

The Pat Task was originally used in the STEP Project to measure mathematical proficiency. In this study, because the Pat Task was used solely to look for indicators of productive disposition, I modified the task. For this study, I presented the Pat Task in two parts. First, the participants worked individually on the Pat Task (see Figure 3.2). The participants were asked to complete as much of the task as they thought they could and then hand in their work.

In May, a teacher provided the following situation in her third-grade class:

I was at a store, and I saw that chocolate kisses come in bags of 42. I wanted to share these kisses among 7 people. How many kisses would each person get?

Following are the steps Pat told his teacher he had performed mentally to solve the problem. The teacher's follow-up questions confirmed that Pat's steps reflected a deep understanding of the problem situation.

4 x 10 = 40
That is three 4s too many, so I have 12 left over.
12 + 2 = 14
14 ÷ 2 = 7
4 + 2 = 6. So 42 ÷ 7 = 6.

a) Please explain how each of Pat's steps makes mathematical sense in this context.

b) Use Pat's approach to solve 56 ÷ 8.

Figure 3.2. The Pat Task.

During a short break after the participants completed individual work on the task, photocopies of their work were made so that I could distinguish their individual work from work done during the group work that followed.

As a group, participants, with their work returned to them, were asked to discuss what they found noteworthy in Pat's work. As before, I allowed the discussion to continue without my input until the participants had ended their conversation. I then posed two follow-up questions, one at a time, to the group:

- Could Pat use this strategy for any division problem? Why or why not?
- Pat's work includes the statement 14 ÷ 2 = 7. What might Pat be thinking in using that fact?

Again, these conversations were allowed to continue as long as the participants wanted. At the end of the task, all the work of the participants was recollected.

Mathematics in the Pat Task

Pat's strategy for division, although complicated, is mathematically correct and applicable to all whole-number division problems.[6] To solve 42 ÷ 7, Pat finds the nearest multiple of 10 that is less than his dividend (40). Putting the extra 2 kisses aside, Pat knows that the 40 kisses can be thought of as 10 groups of 4 kisses. Furthermore, these 10 groups of 4 kisses can be seen as 7 groups of 4 kisses plus 3 groups of 4 kisses. From the 7 groups of 4 kisses, each person can get 4 kisses. However, from the 3 groups of 4 kisses leftover plus the 2 extra kisses Pat initially put aside are 14 extra kisses to be distributed.

Pat next states that 14 divided by 2 is 7. One might expect that Pat would divide 14 by 7 here, because he knows the total number of kisses he has to share (14) and the number of groups (7). However, Pat may simply use a fact family or know that dividing 14 by 2 gives the 7. Nonetheless, Pat now knows that each person receives 2 more kisses, which with the 4 kisses from the groups of 10 is 6 kisses for each of the 7 sharing the 42 kisses.

[6] For division of larger numbers, Pat's strategy must be modified slightly. Instead of starting off grouping by tens, Pat might group by hundreds, thousands, and so on, depending on the size of the dividend.

Pat might use a similar approach to solve 56 ÷ 8 as follows:

> 5 × 10 = 50
> That is two 5s too many, so I have 10 left over.
> 10 + 6 = 16
> 16 ÷ 2 = 8
> 5 + 2 = 7. So 56 ÷ 8 = 7.

Participants did not need to write their solution in this format. The important pieces of Pat's strategy are using groups of ten and then dividing the leftovers.

Consecutive-Sums Task

To determine whether a problem's context affected the productive-disposition indicators exhibited by the teachers, I chose the Consecutive-Sums Task, which, unlike the other tasks, had content not directly related to student work seen in K–3 classrooms. Like with the Pat Task, it was presented in two parts. First, the participants worked individually on the task (see Figure 3.3) to complete as much as they could before handing me their work.

The focus of this task will be on *consecutive sums*. In a consecutive sum, you can start with any positive counting number and add to it the next number and add more consecutive numbers until you reach a stopping point. The resulting number is called a consecutive sum. Here are some examples:

2 + 3 = 5,   4 + 5 = 9,   2 + 3 + 4 = 9,   1 + 2 + 3 + 4 + 5 + 6 = 21

So, these calculations show that 5, 9, and 21 are all consecutive sums. Moreover, there are two representations (4 + 5 and 2 + 3 + 4) for the sum of 9.

Your task has two parts.

- First, determine which positive counting numbers between 1 and 20 are consecutive sums and which are not. (You might find it helpful to develop a strategy to do this efficiently.)

- Second, find as many patterns as you can about consecutive sums and their representations. Try to explain your patterns.

Figure 3.3. The Consecutive-Sums Task.

Again, during a short break after the participants completed individual work on the task, photocopies of their work were made so that I could distinguish their individual work from work done during the group work that followed.

I returned their work to the participants and asked them to discuss their conjectures about consecutive sums, suggesting that they start by talking about a method they found to be helpful or an insight they had. As before, the discussion continued without my input until the participants had ended their conversation. I then posed two follow-up questions, one at a time, to the group:

- What patterns are there for sums that have only representations with two addends? Explain your reasoning.
- Why are multiples of 3 always consecutive sums? Explain your reasoning.

Again, these conversations continued as long as the participants wanted. At the end of the task, all the work of the participants was recollected.

Mathematics in the Consecutive-Sums Task

One reason for including the Consecutive-Sums Task in this study was its open-ended nature: Infinitely many patterns exist within the numbers that are consecutive sums. The follow-up questions are about two specific patterns: sums that have only representations consisting of two addends and multiples of 3. The former are all odd numbers because in any two consecutive integers, one is odd and one is even, resulting in an odd sum. In fact, every odd number can be represented by a consecutive sum with just two addends. If x is a positive odd integer, then $x = 2n + 1$, where n is a natural number. Thus, $x = n + (n + 1)$, and n and $n + 1$ are consecutive integers.

A similar proof can be used to show that all multiples of 3 are consecutive sums. Let y be a positive multiple of 3. Then $y = 3m$, where m is a natural number. Thus, $y = m + m + m = m + m + m + 1 - 1 = (m - 1) + m + (m + 1)$, and $m - 1$, m, and $m + 1$ are consecutive integers. Note that the teachers in this study did not need to provide mathematical proofs such as these but could, for example, give an informal proof by building three equal groups of blocks and then taking one block from the first group and adding it to the third group.

The only positive integers that cannot be expressed as consecutive sums are the powers of 2. In the case of the numbers between 1 and 20, therefore, the numbers that cannot be written as consecutive sums are 1, 2, 4, 8, and 16. The proof of this fact is difficult and, as such, is omitted here[7] and was not asked of the participants. The teachers could make hypotheses about why the powers of 2 could not be written as consecutive sums, but they were told that considering patterns among the numbers that had consecutive-sum representations might be more fruitful.

Sam Task

The last mathematical task I asked participants to complete was the Sam Task (see Figure 3.4). The participants worked individually until they had completed as much of the task as they believed that they could before handing me their work. This task had no group activity. (For the full Sam Task, see the Study 2 protocol in Appendix B.)

[7] In short, for a number to have a consecutive-sums representation, it must have at least one odd factor. The only positive integers that have no odd factor are the powers of 2.

In March, a teacher provided the following situation in his second-grade class:

On Monday, I bought a roll of 97 stickers. I gave away 42 stickers on Tuesday, 11 on Wednesday, and 23 on Thursday. How many stickers do I have left on Friday?

Following is the work of one student, Sam, along with the reasoning she told her teacher she had performed mentally to solve the problem. The teacher's follow-up questions confirmed that Sam's steps reflected a deep understanding of the problem situation.

```
 97
-42
-11
-23
---
 21
```

First, I added the ones and subtracted.
Then, I added the tens and subtracted;
7 minus 6 is 1, and 90 minus 70 is 20.
So, my answer is 21.

Sam was then asked to solve the following problem:

Last week, I bought a roll of 405 stickers. I gave away 39 stickers in the morning, 15 at lunch, and 37 in the afternoon. How many stickers did I have left at the end of the day?

Following is Sam's work on the second problem:

```
 3 8 2
 4 0 5
  -39
  -15
  -37
  ---
  314
```

Figure 3.4. The Sam Task.

The participants were asked to answer five questions related to Sam's subtraction strategy:

- Please explain how each of Sam's steps on the second problem makes mathematical sense in this context.
- Is there a second way that Sam's steps on the second problem might make mathematical sense? If so, how?
- What amount does the 2 represent in Sam's work? What amount does the 8 represent?
- Could Sam always use this method to subtract? Why or why not?
- Is there any part of Sam's strategy about which you are still unsure? What additional information might be helpful for you?

Mathematics in the Sam Task

Sam's mathematical strategy consists of adding the subtrahends and then subtracting their sum from the minuend, using place value. In the first example, Sam adds the ones of the subtrahends (2 + 1 + 3) to get 6 ones, and then subtracts that from the 7 ones in the minuend. Next, in the tens' column, she adds 4 + 1 + 2 to get 7 tens and then subtracts that from the 9 tens in the minuend. Her result is 2 tens and 1 one, or 21.

Similarly, in the second example, in the ones' column of the subtrahends, she adds 9 + 5 + 7 to get 21 ones but realizes that she has too few ones (5) in the minuend from which to subtract and so needs to regroup. Moreover, because 0 is in the tens' column of the minuend, she must regroup one group of 100. To have the ones she needs, she regroups the 100 as 8 tens and 20 ones. After the regrouping, Sam sees the minuend as 3 hundreds, 8 tens, and 25 ones, which is equivalent to her original 405.

She now can take 21 ones from 25 ones, leaving 4 ones. Next, in the tens' column, she adds 3 + 1 + 3 to get 7 tens, which she subtracts from the 8 tens in the minuend. There are no hundreds to subtract, so the 3 hundreds from the minuend remain. In total, Sam has 3 hundreds, 1 ten, and 4 ones, or 314.

Reflections

At the completion of each task, the participants wrote a brief reflection about that task in response to my questions. For example, after the Savannah Task, I posed the following three questions to the participants:

- How was this experience for you?
- I'm interested in how you feel about your understanding now that the discussion has ended. In particular, I am curious about the following:
    - About what parts of Savannah's subtraction strategy are you still unsure?
    - Is there a strategy that Savannah could apply to consistently arrive at the correct answer to the subtraction task given that she begins as she did by adding 2 and adding 1?

Through considering these questions, the participants could reflect on the experience. The reflections helped me learn how they felt about the experience and what confusions remained. In particular, participants who might be too shy or embarrassed to share these ideas with the group were able to convey them privately through their reflections. (For the full reflection sheets participants completed, see the Study 2 protocol in Appendix B.)

I used the reflections to identify evidence of mathematical orientations not discernable from the focus-group discussion. In particular, these questions were designed to show discrepancies between a participant's engagement in the discussion and her stated feelings. For example, a participant might seem confident about her answer in the discussion session but might write that she is unsure about it. The reflections, then, provide the participants' immediate and private feedback about their focus-group experiences.

Mathematical Autobiographies

At the end of the session, each participant completed a mathematical autobiography. In their mathematical autobiographies, the participants reflected on their own experiences with mathematics in their lives inside and outside the classroom. The autobiography included questions such as *How do you feel about mathematics? How have your feelings changed over time? Identify a major challenge you have faced in mathematics. How did you face, handle, or deal with this challenge?* (For a full list of the questions, see Appendix B.) The questions were designed to elicit from the participants what they enjoy about mathematics, what is frustrating, and how they see themselves in relation to mathematics. The participants were told to use the time they needed to complete the autobiography with as much detail as possible. The questions did not need to be answered in order and were meant to stimulate their thinking about their experiences with mathematics. The participants were free to talk about anything they thought was pertinent.

I had two purposes for collecting the autobiographies. First, they provided information about how the participants see themselves with respect to their mathematical orientations. In particular, I was interested to see when, if ever, the participants saw themselves as learners and doers of mathematics and if they thought that, with effort, they could succeed in making progress on mathematical problems. Although these questions were posed indirectly to avoid biasing responses, the autobiographies provided insight into participants' dispositions toward mathematics. Second, these data helped me design follow-up interviews for those participants, especially when I identified discrepancies between what a participant wrote and what she said in the focus-group discussion. For example, a participant may see herself as an academic risk taker, but in the discussion, she may avoid engaging mathematically. This apparent contradiction, then, would be discussed in the interview to learn why she seemed not to engage in the task: Perhaps I misread her actions while analyzing the video, or perhaps she found this task to be difficult or of no interest. Such instances would show that the focus-group data fail to give a complete picture of one's productive mathematical orientation.

Interviews

I conducted individual interviews with each of the 10 participants—*clinical interviews* (Bernard, 1988; Clement, 2000), meaning that I posed several starter questions that each interviewed participant would answer but created follow-up questions tailored for the individual (Ginsburg, 1997). Starter questions include *Overall, how was the experience for you?* and *How would you describe the experience to a friend?* (See Appendix C for all the preplanned questions from the clinical interviews.) Such questions were posed to learn more from the participants about their own mathematical orientations. I then compared their answers with the evidence they provided in the previous session (i.e., the evidence from the tasks, their reflections, and their autobiographies).

Using *stimulated recall* (Calderhead, 1981), I chose parts of the videotape or written work from the participant's focus group to discuss with the participant during the interview to learn whether the participant agreed with my assessment of her indicators. For example, for a participant who seemed to be unsure of a strategy she offered to the group, I might have asked how sure of her understanding she thinks she was at the time. She may actually have been quite sure of her strategy and was being modest, or she may have made a random guess because she had nothing else to offer. These distinctions, which might be unclear from viewing the video alone, were helpful in determining which evidence from the video was correctly interpreted and in what ways the story was more complicated than it seemed.

To conclude the interview, I asked the participant more direct questions about her productive disposition, still without introducing that term. I asked, for two reasons, whether the participant felt that she was different in any way from the other people in her group or from other teachers more generally. First, I was interested to learn whether participants I identified as having strong dispositions, dispositions that seem clearly different from those of many of their colleagues, see themselves as different in any way. Second, these participants had opportunities to reflect on the session and their mathematical lives more generally to see which things, if any, they feel enable them to engage more productively in mathematical endeavors. Last, I showed the participant the rubric I used when looking for evidence of teachers' productive dispositions. I asked her which of the indicators she recognized in herself, which seem implausible or unlikely to be demonstrated by her, and which she thinks are part of her disposition but may not have been visible on the videotape. These insights were used to help elaborate and refine the productive-disposition indicators.

Analysis

All 10 interviews were transcribed from their audiotapes. In looking for productive-disposition indicators in the transcripts, I identified statements in which the participants talked about what I see as productive disposition in their own voices. I categorized these statements by productive-disposition indicator. However, I was open to and found other indicators from these teachers. I also listed statements that seemed important to the teachers but were not among my initial indicators; these statements helped me create new potential productive-disposition indicators.

The purpose of the study was not to see which teacher showed what indicators but rather was to better understand what productive disposition is and how it is manifested when it is being enacted. I wanted to see which indicators were common among these 10 teachers who had previously demonstrated strong productive dispositions. In the results discussion, I first talk about the seven indicators from the list of indicators from Study 1 that were discussed most often by the 10 participants. I give examples from their interviews showing how these indicators looked similar and different among the participants. I then discuss the three new potential productive-disposition indicators. Although these 10 total indicators are not the only indicators shown by the participants, they are the ones that seemed to resonate most with these teachers.

Section 3.4—Chapter Summary

To conclude, I return to my overarching goal for this study: to better understand the construct of productive disposition. To reach this understanding, I investigated the following research questions:

- What differences in evidence for one's productive disposition can be found through the analysis of teachers' active engagement in a mathematical task?
- What evidence for productive disposition is self-reported by teachers who have been identified as having strong productive dispositions?

In this chapter, I described the methodology I used in two studies designed to help me gather data to better understand productive disposition. In the next chapter, I describe the results of both studies, including the indicators of productive dispositions seen in the participants of both studies, especially those that lead to sustained engagement in mathematical tasks. I also describe how data from the interviews, autobiographies, and mathematical tasks provided insight about participants' productive dispositions. In the final chapter, I consider approaches for assessing for these differences in future research. The results of this study helped me to better understand what productive disposition is and how it is evinced in the solving of mathematical tasks. These understandings emerged from my examining the orientations of a variety of teachers and studying in-depth the evidence from teachers who seem to hold strong productive dispositions.

CHAPTER 4

RESULTS

Section 4.1—Introduction

In this chapter, I report the results of my two studies. In Study 1, the goal was to find differences in evidence for one's productive disposition through the analysis of teachers actively engaging in a mathematical task. I first report on the differences found in the participants' productive-disposition scores. I then explain how the rubric of productive-disposition indicators changed on the basis of evidence seen in the focus-group videos. I conclude the results of Study 1 by analyzing participants' productive-disposition scores in relation to the indicators shown. In Study 2, my goal was to examine evidence for productive disposition shown by teachers identified as having strong productive dispositions in Study 1 when they engaged in multiple mathematical tasks. I chose one teacher to serve as a prototype for the 10 teachers in Study 2. I report a summary of her results to give the reader a sample of how the teachers engaged with each task. The remaining results of Study 2 are focused not on how individual teachers engaged but on commonalities across the teachers. In the individual interviews with the 10 teachers, seven traits of productive disposition surfaced. I discuss how the teachers talked about the traits (before they knew the purpose of the study and before they had heard of the indicators) and why they found them to be important. I conclude with three unexpected traits related to productive disposition that I also found in these 10 teachers.

Before reporting the results, I provide a clarification. When beginning these studies, I considered two types of productive-disposition indicators I expected to see in the participants of both studies: *productive-disposition indicators for learning* and *productive disposition indicators for teaching*. The difference between the two types of indicators is the lens through which one looks at mathematics. When trying to understand and interpret a mathematical task for oneself, one might evince his or her productive-disposition indicators for learning. However, one might instead consider how students think about mathematical tasks or what practices might help students engaging in mathematical thinking. In this case, one would be evincing his or her productive-disposition indicators for teaching. The indicators I expected to see for each of these would differ slightly. For example, consider a person who has a strong mathematics background but no teaching experience. He or she might consider alternative strategies when trying to solve a mathematical problem but would be unlikely to discuss different strategies students devise in the classroom.

I believe that both of these versions of productive-disposition indicators are important, but because of my choice of tasks, I initially expected to see, primarily, indicators of the participants' productive dispositions for learning. Although the tasks in both studies seemed to provide opportunities to view the teachers' learning productive-disposition indicators, I was concerned that the tasks, along with the questions posed, did not provide ample opportunities for me to view their teaching productive-disposition indicators. To get a sense for the participants' productive-disposition indicators for teaching, I thought one might have to view the teachers working with their students on mathematical tasks in their own classrooms or through some other method.

Instead, I found that the teachers in both studies showed indicators of their productive dispositions for both learning and teaching. Moreover, the teachers discussed issues of teaching even when engaged with mathematical tasks as learners. This teaching focus was particularly evident during the interviews in Study 2. For example, Rebecca explained that she enjoyed seeing the other teachers' organizational methods and strategies on the Consecutive-Sums Task and then talked about her student's trouble finding the total number of pieces of candy for 10 children each with 8 pieces of candy. With Rebecca's guidance, the student solved the problem after trying a new strategy—thinking of 8 as 5 and three 1s. Rebecca concluded that she appreciated seeing different strategies on mathematical tasks because of their value for her students.

The teachers in both studies came with a variety of perspectives. Some, like Rebecca, often made connections between the tasks they were solving and work they had seen from their students. Others focused on how they might make sense of the mathematical tasks for themselves, not raising issues of teaching mathematics. Most teachers, however, invoked both perspectives in their work on the tasks, and I often was unsure which perspective they were invoking at any given moment. My goal in both studies was to better understand the characteristics of teachers who are engaged with mathematical tasks. For this population of participants, I found separating the two types of indicators unhelpful (and perhaps impossible). In fact, many of these teachers employed their productive dispositions as teachers to help them make sense of and persevere on the mathematical tasks with which they were engaging as learners. In these results, especially in the discussion of the interviews in Study 2, you may note that the viewpoint of the participant fluidly moves from learner to teacher and back again, a common occurrence in both studies. As such, I believe that to draw the learner/teacher distinction in these participants' productive-disposition indicators is inappropriate for this study. Because these teachers, even in the role of mathematics learners for these tasks, thought as mathematics teachers as well, I abandoned the distinction between learning- and teaching-productive-disposition indicators. I chose, instead, to focus on any productive-disposition indicators that were evident. My goal in this paper is to better understand the construct of productive disposition, so although I believe that this learner/teacher distinction is important, I leave defining and assessing for different types of productive-disposition indicators to future research.

Section 4.2—Study 1 Results

My goal for Study 1 was to answer the question **What differences in evidence for one's productive disposition can be found through the analysis of teachers' actively engaging in a mathematical task?** To address this question, I analyzed data from focus groups of teachers engaged in discussing the mathematics in the work of the student Savannah (shown in Appendix A). Study 1 had 136 teachers in total: 36 were preservice elementary school teachers just beginning their first mathematics content course for teaching and 100 were in-service teachers who had been teaching for a minimum of 4 years.

Productive-Disposition Score

The participants' productive disposition scores (0–4) were based on their engagement with the Savannah Task, as described below. During the scoring, Bonnie Schappelle and I refined our initial list of productive-disposition indicators. I revisited the data, with participant scores blinded, and recorded the indicators each teacher showed when engaging with the task.

Before sharing the results of Study 1, I reiterate what the teachers' scores do and do not indicate. In this study, productive-disposition scores were based on the teachers' engagement with the single task according to the rubric shown in Table 3.2. A teacher who showed very strong evidence of engaging with the mathematics of the Savannah Task scored 4. A teacher who showed essentially no evidence of engaging with the mathematics of the Savannah Task was assigned a score of 0, which does not preclude the teacher's having a strong productive disposition. The score simply shows that I saw little to no evidence of productive disposition from the teacher on that task on that day. Productive disposition is a difficult trait to measure. In this study, I looked for any evidence the participants showed in one task for holding a strong productive disposition. However, it should also be noted that teachers who scored 3 or 4 on this task likely do hold strong productive dispositions, at least for tasks similar to the Savannah Task. In other words, I have more confidence in my results for teachers with high scores than for those who scored lower, and I am reluctant to conclude that those teachers who scored low on the Savannah Task have weak productive dispositions without assessing them further.

Table 3.2

Productive-Disposition-Scoring Rubric

Score	Evidence
0	The participant shows essentially no evidence of engaging with the mathematics. The participant may show resistance to engaging with the task at any level—mathematical or otherwise; or the evidence may indicate a lack of effort (either because of disinterest or lack of content knowledge).
1	The participant shows weak evidence of engaging with the mathematics. The participant may unintentionally move the group away from mathematically productive discussions. The participant may make some efforts to engage but may not persist in doing so.
2	The participant shows some evidence of engaging with the mathematics. The participant may make little progress toward understanding the mathematics or become confused but show clear evidence of willingness to engage with the mathematics of the task.
3	The participant shows strong evidence of engaging with the mathematics. Although evidence of engagement is clear, it may be somewhat delayed or limited to a brief segment of the conversation.
4	The participant shows very strong evidence of engaging with the mathematics. Effort to understand is shown until he or she does understand or throughout the focus-group session. A participant who comes to understand the mathematics, tries to help others understand by explaining and, usually, re-explaining in multiple ways to those who do not understand. One who does not understand persists in attempting to do so, trying various approaches; he or she expresses reluctance to abandon the search for understanding.

The average score for the 136 teachers on this task was 1.59. The teachers' score distribution is shown in Table 4.1. Because the scores for the preservice and in-service teachers differed greatly, however, the overall numbers may be deceptive. The average score for the preservice teachers was 0.78, whereas for the in-service teachers it was 1.88 (see the score distributions for the two groups in Table 4.2). More than half of the preservice teachers scored 0 on the Savannah Task. In fact, more than 80%

of the preservice teachers scored a 0 or 1 on the task, whereas only 42% of the in-service teachers scored 0 or 1. This difference in the preservice and in-service teachers reflects the preservice teachers' lack of engagement with the mathematics of the Savannah task.

Table 4.1

Productive-Disposition Scores

Score	**0**	**1**	**2**	**3**	**4**	**Total**
Count	38	33	30	17	18	136
Percent	27.9%	24.3%	22.1%	12.5%	13.2%	100%

Many of the preservice teachers talked about never having seen a strategy like Savannah’s before and suggested that Savannah’s teacher should have just taught her the standard algorithms. They rarely engaged in thinking about why Savannah’s addition strategy was correct or how Savannah’s subtraction strategy might be altered, even when prompted. This resistance to engaging with this task may stem from several factors. On the one hand, the task itself may not have resonated with the preservice teachers. At the time of data collection, the preservice teachers were just about to start their first mathematics content course for teaching. They may not have given much thought as to how children solve mathematical tasks. As such, they may have put forth little effort on the task because it seemed irrelevant to them. On the other hand, the preservice teachers may not have known how to engage with a mathematical task that is focused on children’s thinking. Having seen no task of this type before, they did not know how to begin to engage with it. Perhaps on a mathematical task that was not situated in the context of children’s thinking, their scores would look different, but I was surprised that even when asked the follow-up questions focused on the mathematics (e.g., Why is Savannah's answer for the subtraction problem off by 2?), the preservice teachers did not show productive moves.

Table 4.2

Comparison of Productive-Disposition Scores—Preservice Versus In-Service Teachers

Group	0	1	2	3	4	Total
Preservice	19 52.8%	10 27.8%	4 11.1%	2 5.6%	1 2.8%	36
In-Service	19 19.0%	23 23.0%	26 26.0%	15 15.0%	17 17.0%	100
Total	38 27.9%	33 24.3%	30 22.1%	17 12.5%	18 13.2%	136

The in-service teachers showed a wider range than the preservice teachers in their productive-disposition scores. In the larger STEP study, the in-service teachers were in three categories, based on participants' numbers of years of experience with professional development focused on children's mathematical thinking (see Table 4.3): IPs had 0 years, APs had 2 years, and ETLs had 4 or more years.[8]

[8] As noted in the table, ETLs differed from IPs and APs not only in years of professional development, but also in that they had begun to engage in leadership activities to support other teachers.

Table 4.3

In-Service Teachers' Groupings

Participant group	Description
Initial Participants (IPs) (*n* = 35)	Experienced K–3 teachers who were about to begin sustained professional development focused on children's mathematical thinking
Advancing Participants (APs) (*n* = 31)	Experienced K–3 teachers with sustained professional development focused on children's mathematical thinking for 2 years
Emerging Teacher Leaders (ETLs) (*n* = 34)	Experienced K–3 teachers with sustained professional development focused on children's mathematical thinking for 4 or more years and who were beginning to engage in leadership activities to support other teachers

Considered in these groupings by years of professional development, the in-service teachers' scores differed only slightly (see Table 4.4). Apart from there being a higher percentage of IPs scoring 0 than APs or ETLs, the three groups' scores are basically identical. There were subtle differences in their average scores: IPs averaged 1.71, APs averaged 1.90, and ETLs averaged 2.03.

Table 4.4

In-Service Teachers' Productive-Disposition Scores

Group	**0**	**1**	**2**	**3**	**4**	**Means (Std. Dev.)**
IP	9 25.7%	8 22.9%	7 20.0%	6 17.1%	5 14.3%	1.71 (1.41)
AP	5 16.1%	7 22.6%	10 32.3%	4 12.9%	5 16.1%	1.90 (1.30)
ETL	5 14.7%	8 23.5%	9 26.5%	5 14.7%	7 20.6%	2.03 (1.36)
Total	19 19.0%	23 23.0%	26 26.0%	15 15.0%	17 17.0%	1.88 (1.35)

I tested for statistically significant differences among the four groups (the preservice teacher group and the three in-service teacher groups) with pairwise one-tailed *t*-tests on the groups. See Table 4.5 for the results. The only statistically significant differences were between the PSTs and each other group. Of interest here is that none of the in-service teacher-group comparisons were significantly different (at a Type I error rate of 0.05). This is a noteworthy result, given that when, in the larger STEP study, these in-service teachers were individually assessed on four constructs (content knowledge, beliefs, professional noticing, and responsiveness), the teachers showed growth in line with the length of their professional development experience on each construct. (For more information about the STEP study, see Philipp, Jacobs, Lamb, Bishop, Siegfried, & Schappelle, 2012.)

Table 4.5

Comparison of p*-values*

Comparison	***p*-value**	**Comparison**	***p*-value**
PST v. IP	0.001	IP v. AP	0.286
PST v. AP	< 0.001	IP v. ETL	0.174
PST v. ETL	< 0.001	AP v. ETL	0.352

This result may seem to indicate that professional development focused on children's mathematical thinking has no effect on productive disposition. However, a major finding of the STEP project was that changes in teachers due to professional development do not all occur within the first 2 years. Some changes take 4 or more years. It may be the case that productive disposition changes *very* slowly. Note that although most of the ETLs had 4 years of sustained professional development, a subset (8 ETLs) had 7 or more years of sustained professional development. Of these 8 teachers, one scored 2, three scored 3, and four scored 4, for an average score of 3.38. Although the size of this group and the fact that the scores are based on one task are inadequate as a basis for broad conjectures, this result does leave open the possibility that teachers' productive dispositions change through professional development focused on children's mathematical thinking, but that this change occurs only after many years of such professional development.

Productive-Disposition Indicators

Having discussed differences in the productive-disposition scores of the teachers, I turn to my main goal of this study: to look for ***differences in evidence*** in the teachers' productive dispositions. To accomplish this, I created a list of individual productive-disposition indicators that might be seen in the focus-group data. However, looking for distinct indicators is problematic. Some piece of evidence from the data could potentially have several indicators associated with it. For example, a teacher might be persisting on a task because she believes progress can be made or because she has a sense that the task has not been completed. In this dissertation, though, the focus is on finding the ways in which teachers evince evidence for their productive dispositions. I, therefore, attempted to isolate individual indicators. In the examples from the focus-group data, indicators other than the one being described might also emerge. Productive disposition is a complex construct and the indicators do overlap. I attempted to choose clear examples for each indicator described, realizing that other indicators might be evident as well.

At the start of Study 1, I created a list of potential indicators of productive disposition (see Table 2.1) on the basis of the research literature and preliminary analysis of a few focus-group videos. With the help of Bonnie Schappelle, I then watched all the focus-group videos.

Table 2.1

Potential Indicators of Strong Productive Dispositions (Initial Version)

Potential categories	Evidence
1. Mathematics as a sense-making endeavor	a. Tries to make sense of the task b. Considers alternative approaches c. Asks if answer seems logical
2. Mathematics as beautiful or useful and worthwhile	a. Shows interest in the task, evidenced by one's engagement or one's comments about the task
3. Sense one can, with appropriate effort, learn mathematics	a. Indicates sense one can make progress on the task on which one is working b. Defines progress as learning through grappling, not just getting an answer
4. Approaching mathematics with particular habits of mind	a. Asks questions about the mathematics, or about one's or another's approach b. Persists c. Seeks and provides clarifications
5. Mathematical integrity	a. Has a sense for when one has completed a task (whether or not one continues) b. Appreciates one's solution
6. Does not avoid frustrations	a. Shows sense of wonder b. Shows pleasure or excitement about a particular way of reasoning c. Recognizes worthy and unworthy confusions

Each teacher received a productive-disposition score, according to the rubric discussed in the previous section. At the same time, we compared what we had seen in the videos to the list of potential indicators. Whenever we found evidence not represented among the indicators, we added to the list. We also rearranged the categories to group the indicators sensibly. The result was the list of potential indicators in Table 4.6 (with additions denoted in **bold** and changes denoted in *italics*).

The largest change to the list of indicators was the addition of a new category: self-efficacy. Self-efficacy is a person's own belief in his or her ability to take action in a particular problematic situation (Bandura, 1977). Although this belief seems important for the productive dispositions teachers need to engage with mathematical tasks, I initially did not include this category because I believed that one's self-efficacy may not be fully evident in the focus-group data. However, part of self-efficacy is confidence, which seemed evident in the data. Because I believe that self-efficacy does play a role in one's productive disposition and because evidence for participants' confidence was found in the data, I added the last category along with the two new indicators.

Table 4.6

Potential Indicators of Strong Productive Dispositions (Final Version)

Potential categories	Evidence
1. Mathematics as a sense-making endeavor	a. Tries to make sense of the task b. Considers alternative approaches c. Asks if answer seems logical **d. Is troubled by inconsistencies**
2. Mathematics as beautiful or useful and worthwhile	*a. Shows interest in the task through engagement* *b. Shows interest in the task in comments about the task* *c. Shows a sense of wonder*
3. Beliefs that one can, with appropriate effort, learn mathematics	a. Believes making progress on the task is doable *b. Persists* **c. Does not avoid frustration**
4. Mathematical habits of mind	a. Asks questions about the mathematics or about an approach (one's own or another's) *b. Shows appreciation for one's solution* c. Seeks and provides clarifications
5. Mathematical integrity and **academic risk taking**	a. Has a sense for when one has completed a task (whether or not one continues) **b. Is willing to question one's self** **c. Is willing to offer tentative ideas** *d. Recognizes worthy and unworthy confusions*
6. *Positive goals and motivation*	*a. Defines progress as learning through grappling, not just getting an answer* b. Shows pleasure or excitement about a particular way of reasoning **c. Engages longer or willingly reengages with difficult tasks** **d. Interprets frustration, when experienced, as a natural component of problem solving and not as a statement of one's mathematical competence**
7. **Self-Efficacy**	**a. Seems confident in one's own abilities and skills for solving the task** **b. Seems confident in one's knowledge**

As was noted in Chapter 2, mathematical integrity is observable only in one willing to take academic risks. As such, I added academic risk taking to the category title along with mathematical integrity. I had placed the indicator *recognizes worthy and unworthy confusions* in a category about frustration. However, this indicator seemed more about mathematical integrity than frustrations, so it was moved. Because of this change, the frustration category seemed more about goals and motivation and was renamed. The way a participant interpreted frustration, instead of being a category, became a new indicator.

Five other indicators were also added because of evidence from the video data: *is troubled by inconsistencies, does not avoid frustration, is willing to question one's self, is willing to offer tentative ideas,* and *engages longer or willingly re-engages with difficult tasks*. These indicators were added to the list in their appropriate categories. Last, a few of the original indicators were moved to categories that seemed more appropriate. (The indicator *interest in the task, evidenced by one's engagement or one's comments about the task* was not moved, but rather was separated into two indicators.)

After all 30 focus-group videos had been viewed and each teacher had received a productive-disposition score, I reviewed the videos and recorded the evidence for each teacher's productive disposition according to the final list of potential indicators. Although the teachers' productive-disposition scores had been recorded, I did not consider the scores while cataloging the evidence of productive disposition shown in the focus groups. For each teacher, I recorded which indicators were evident. However, I did not record whether a teacher showed evidence of an indicator multiple times because for many of the indicators, I could not determine whether a teacher had shown the indicator multiple times or had simply continued to show the indicator over time. For example, in several instances, teachers asked others to clarify some part of their solution strategy multiple times; whether these additional questions were to clarify different portions of the solution or if they were a continuation of the previously asked clarifying questions was unclear. Given that the data were collected over only one task, the teachers' opportunities to display their productive dispositions were limited. Moreover, because the purpose of this study was to identify the types of evidence shown by teachers engaged with a mathematical task, I considered recording *that* a teacher had shown a particular indicator of holding a strong productive disposition more important than *the number of times* that indicator appeared.

The range for the number of indicators teachers evinced on the Savannah Task was 0–12, and the average was 3.40. Table 4.7 shows the distribution of these numbers of different indicators evident in their focus-group discussions for the 136 teachers. Like the scores, these numbers are skewed by the performance of the preservice teachers. The average number of indicators for the preservice teachers was 1.75. Exactly half of the preservice teachers (18 of the 36) showed no indicators whatsoever. In fact, more than 80% of the preservice teachers (29 of the 36) showed 3 or fewer indicators. More than one half of the teachers who showed 0 indicators (18 of 35) and more than one third of the teachers who showed 3 or fewer indicators (29 of 75) were from the preservice teachers' group, even though the preservice teachers comprised roughly one fourth of the teachers.

Table 4.7

Productive-Disposition Indicators per Teacher

Number of indicators evident	0	1	2	3	4	5	6	7	8	9	10	11	12
Number of teachers	35	4	22	14	20	10	9	5	7	5	3	1	1

Because the preservice teachers (PSTs) showed very few productive-disposition indicators, I report in the remaining results in this chapter data only for the 100 in-service teachers. When the preservice teachers' data are removed, the data from the previous table look slightly different, as shown in Table 4.8. The in-service teachers evinced an average of 4.00 indicators. As before, the three in-service teacher groups (IPs, APs, and ETLs) looked relatively similar in the distribution of their indicators.

Table 4.8

Productive-Disposition Indicators per In-service Teacher

Number of indicators evident	0	1	2	3	4	5	6	7	8	9	10	11	12
Number of teachers	17	3	16	10	18	7	9	4	6	5	3	1	1

Of particular interest for this study is how often each indicator was evident in the focus-group data. Table 4.9 gives the number and percentage of teachers showing each indicator. To save space, the categories are listed in brief and the indicators by letter. (For the full indicator names, see Table 4.6.) Since there were 100 in-service teachers, the count for each indicator is the same as the percent.

Table 4.9

Productive-Disposition Indicators Count

Category	1 Sense making				2 Useful			3 Can learn			4 Habits of mind		
Indicator	**a**	**b**	**c**	**d**	**a**	**b**	**c**	**a**	**b**	**c**	**a**	**b**	**c**
Count	69	30	1	7	40	4	4	15	20	4	30	12	45

Category	5 Integrity				6 Goals				7 Self-efficacy	
Indicator	**a**	**b**	**c**	**d**	**a**	**b**	**c**	**d**	**a**	**b**
Count	5	18	42	17	7	7	16	1	5	1

The following indicators were evident in fewer than 10% of the in-service teachers:

- Asks if answer seems logical (1c)
- Is troubled by inconsistencies (1d)
- Shows interest in the task in comments about the task (2b)
- Shows a sense of wonder (2c)
- Does not avoid frustration (3c)
- Has a sense for when one has completed a task (whether or not one continues) (5a)
- Defines progress as learning through grappling, not just getting an answer (6a)
- Shows pleasure or excitement about a particular way of reasoning (6b)
- Interprets frustration, when experienced, as a natural component of problem solving (6d)
- Seems confident in one's own abilities and skills for solving the task (7a)
- Seems confident in one's knowledge (7b)

Although these infrequently seen indicators may be evidence for one's holding a strong productive disposition, they may not be useful indicators of productive disposition for participants engaging with a mathematical task. For example, indicators 7a and 7b deal with confidence, which was difficult to recognize in the video data. However, confidence was a central topic of conversation during my individual interviews with a subgroup of teachers (see the results of Study 2 later in this chapter.)

Participants who showed fewer than 5 indicators generally showed similar indicators. The most common indicators among this group were *tries to make sense of the task* (1a) and *is willing to offer tentative ideas* (5c). To a lesser extent *asks questions about the mathematics or an approach* (4a), and *seeks and provides clarifications* (4c) were also common.

Each participant with more than 5 indicators had *tries to make sense of the task* (1a) as one. Several indicators were seen in more than two thirds of participants who evinced more than 5 indicators: *shows interest in the task through engagement* (2a), *seeks and provides clarifications* (4c), and *is willing to offer tentative ideas* (5c). Moreover, the indicators *believes making progress on the task is doable* (3a), *persists* (3b), *is willing to question one's self* (5b), *recognizes worthy and unworthy confusions* (5d), *shows pleasure or excitement about a particular way of reasoning* (6b), and *engages longer or willingly reengages with difficult tasks* (6c) were seen far more often in people with many indicators than in those with few.

As noted before, my purpose for this study was to find differences in evidence for one's productive disposition. In this section, I discussed which indicators appeared rarely in these teachers, which appeared often in teachers with few indicators, and which appeared more often among the teachers with many indicators. In the next section, I examine another difference in the indicators by comparing them to the teachers' productive-disposition scores.

Comparing the Indicators to the Productive-Disposition Scores

My final analysis of Study 1 data was to investigate any correlation between the teachers' productive-disposition scores (determined holistically in terms of level of engagement with the mathematics of the task) and their productive-disposition indicators (enumerated in a separate analysis of the data) in focus-group conversations. I first compared the number of indicators and the productive-disposition score for each teacher (see Table 4.10). In general, the teachers scored 0 if they showed 0 or 1 indicators, scored 1 for 2 to 3 indicators, scored 2 for 4 to 5 indicators, scored 3 for 6 to 7 indicators, and scored 4 if they showed 8 or more indicators. Again, this correlation is of interest because the productive-disposition indicators for each teacher were recorded without regard to that teacher's productive-disposition score.

Table 4.10

Distribution of Numbers of Indicators for Each Productive-Disposition Score

		Number of Indicators														
Score	***n***	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**Ave.**	**St. Dev.**
0	19	35	3												0.11	0.32
1	23		1	22	10										2.22	0.52
2	26				4	18	8								4.04	0.60
3	15					2	2	9	3	1					5.93	0.80
4	17								2	6	5	3	1	1	9.00	1.28
Total	100	35	4	22	14	20	10	9	5	7	5	3	1	1	4.00	3.01

In addition to finding this correlation between the number of indicators shown by the teachers and the teachers' productive-disposition scores, I investigated whether specific indicators correlated with scores by analyzing only indicators evident in at least 10% of the teachers. I then compared those indicators to the productive-disposition scores of the teachers. With few exceptions, these indicators were shown more often in people with higher productive-disposition scores. In other words, no indicator is more likely to be seen in someone with a lower productive-disposition score than in someone with a higher score. (It could be that some indicators show up for people with weaker productive dispositions, but people with stronger productive dispositions show these indicators less often.)

Table 4.11

Percentage of Participants, by Productive-Disposition Score, Who Evinced Each Individual Indicator

Score	*n*	**1a**	**1b**	**2a**	**3a**	**3b**	**4a**	**4b**	**4c**	**5b**	**5c**	**5d**	**6c**
0	*19*	5%	0%	0%	0%	0%	5%	0%	0%	0%	0%	0%	0%
1	*23*	57%	22%	9%	0%	9%	30%	9%	30%	0%	43%	0%	4%
2	*26*	92%	27%	42%	19%	0%	31%	19%	62%	15%	35%	15%	0%
3	*15*	93%	40%	73%	20%	47%	40%	13%	60%	33%	60%	13%	33%
4	*17*	100%	71%	94%	41%	65%	47%	18%	76%	53%	82%	65%	59%

Table 4.12 is formatted to highlight trends in the Table 4.11 data, that is, to show the relationship between high incidence of individual indicators and productive-disposition score. Although I highlight specific percentage ranges, these ranges also closely associate with the location of large jumps in the percentages. The following indicators were evident in more than 75% of the teachers who received a score of 4:

- Tries to make sense of the task (1a)
- Shows interest in the task through engagement (2a)
- Seeks and provides clarifications (4c)
- Is willing to offer tentative ideas (5c)

Table 4.12

Comparison of Productive-Disposition Score to the Indicators

Score	*n*	**1a**	**1b**	**2a**	**3a**	**3b**	**4a**	**4b**	**4c**	**5b**	**5c**	**5d**	**6c**
0	*19*	5%	0%	0%	0%	0%	5%	0%	0%	0%	0%	0%	0%
1	*23*	57%	22%	9%	0%	9%	30%	9%	30%	0%	43%	0%	4%
2	*26*	92%	27%	42%	19%	0%	31%	19%	62%	15%	35%	15%	0%
3	*15*	93%	40%	73%	20%	47%	40%	13%	60%	33%	60%	13%	33%
4	*17*	100%	71%	94%	41%	65%	47%	18%	76%	53%	82%	65%	59%

Format scheme

0% - 25%	26% - 50%	51% - 75%	76% - 100%

Figure 4.1, based on Table 4.12, shows indicators that were commonly evident in participants with a given score. The base-level indicators were often evident in teachers who received a score of 1. Teachers who received higher scores were likely to show indicators in higher levels of the diagram along with those in lower levels. The levels of the indicators are based on the score in which 25% or more of the participants receiving that score evinced that indicator. As noted above, these levels generally correlate with a large jump in the percentage of participants evincing that indicator. The one exception is *considers alternative approaches* (1b), which appears in Level 2 of Figure 4.1. I placed that indicators in Level 2 on the basis of the 25% threshold, but it would appear in Level 1 on the basis of the first large jump in its percentage. Again, this is a first attempt to classify these indicators, and further research may be needed to determine which indicators appropriately belong in which level.

In a sense, this diagram, shows, potentially, how the indicators of productive disposition build on one another. This is not to say that productive dispositions are developed in this order; these are trends seen in the 100 in-service teachers and are not meant to be predictive of any individual. One is, however, more likely to see the lower level indicators in those teachers with weaker productive dispositions and higher level indicators in those teachers with stronger productive dispositions.

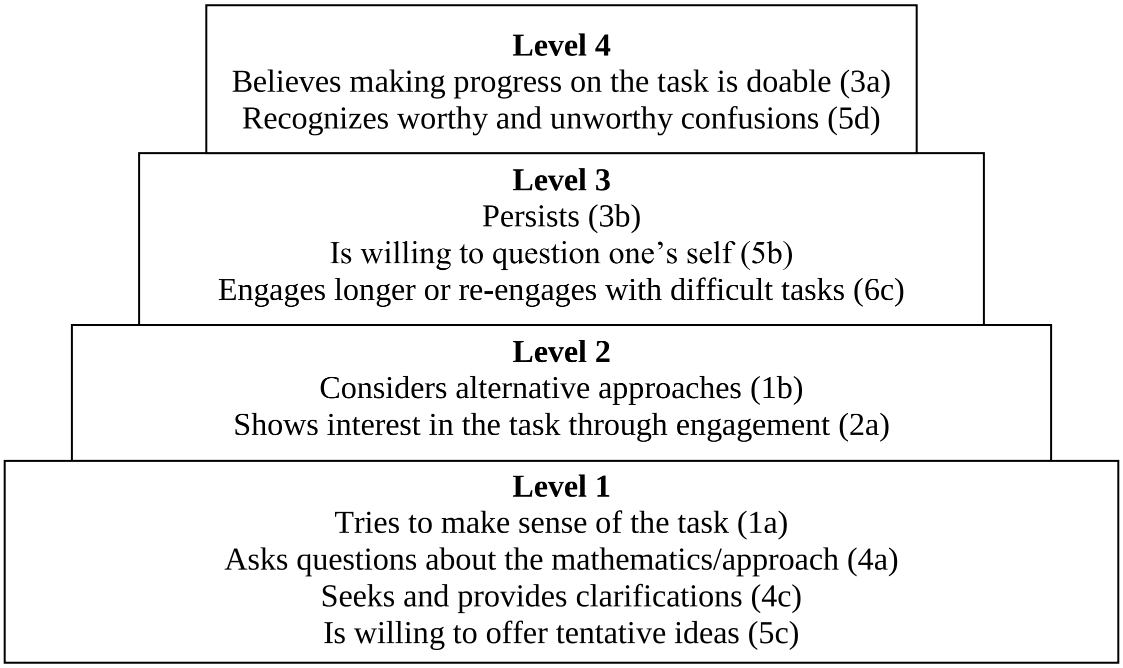

Figure 4.1. Likely indicators based on productive-disposition score.

Summary

My goal for this study was to investigate the underresearched construct of *productive disposition* by confirming predicted indicators of productive disposition, finding additional indicators if they exist, and finding differences in the evidence of the teachers' productive dispositions to discern strong from weak productive dispositions. To these ends, I analyzed the focus-group data of 136 teachers who were engaged with the Savannah Task.

Perhaps the most important result of this study is the creation and refinement of the list of productive-disposition indicators. The initial indicators grew out of the mathematics education research on constructs connected to the definition of *productive disposition.* I then used the focus-group data to add to and refine the list of indicators. All the indicators were seen in at least one of the teachers in this study, although several were seen in fewer than 10% of the teachers. (In Study 2, I investigate whether some of these indicators are evinced by teachers in work on additional tasks or during my interviews with them.)

For indictors seen in 10% or more of the in-service teachers, I found a correlation between the productive disposition-score a teacher had received and the indicators seen in his or her focus-group data. Some indicators, such as *seeks and provides clarifications*, were seen in all the teachers who showed at least some evidence of holding a strong productive disposition. Other indicators, such as *recognizes worthy and unworthy confusions*, were evident only in teachers who showed several indicators and evidence of holding a strong productive dispositions.

Several differences were found among the teachers in the study. First, the preservice teachers' scores were low, and this group showed fewer productive-disposition indicators than the in-service teachers—perhaps because stronger productive dispositions develop through teachers' working with students in the classroom or because the Savannah Task did not lead this population to invoke their productive dispositions. Second, the in-service teachers looked similar regardless of the extent of their professional development. I conjecture not that professional development fails to promote stronger productive dispositions but rather that productive disposition may be slow to change, requiring more than 4 years of professional development to change. Moreover, I cannot conclude, on the basis of this single task, that the groups would not show differences over more tasks.

Although through this single-task analysis, a first attempt to better understand the construct of productive disposition, I was able to augment and refine the list of indicators derived from the mathematics education literature, I concluded that multiple tasks might yield indicators that went unseen in the focus-group data or show that some indicators are more or less relevant to teachers. To strengthen my understanding of the construct of productive disposition, I conducted a second study focused on teachers who had shown strong productive dispositions in this first study.

Section 4.3—Study 2 Results

Ten teachers identified in Study 1 as having strong productive dispositions were invited 5 years later to participate in Study 2 by solving additional mathematical tasks, recording their mathematical autobiographies, and participating in individual interviews. My goal for Study 2 was to answer the research question **What evidence for productive disposition is self-reported by teachers who have been identified as having strong productive dispositions?** In particular, I wanted to find what indicators of productive disposition (previously or newly identified) emerged for the teachers in individual and group work in solving multiple problems and which indicators teachers themselves raised as important in our interviews. In many ways these 10 teachers were quite similar in their responses to the tasks, so I chose to summarize the responses of one teacher and then compare and contrast her responses to the responses of the other teachers.

A Summary of Sarah's Study 2 Responses

Sarah[9] was chosen as a sort of prototype for these 10 teachers for several reasons. Like all the teachers, she was more successful in completing some of the mathematical tasks than others, but she continued to participate in the discussions and sought to make mathematical connections. Further, in both her autobiography and her interview, she broached themes raised by most of the other teachers in Study 2. I first summarize Sarah's responses to each of the four mathematical tasks, mentioning the productive-disposition indicators I see exhibited. I then present highlights from her mathematical autobiography. Last, I discuss parts of my interview with Sarah and how those parts relate to Sarah's productive disposition.

Mathematical Tasks

Savannah Task

The first mathematical task all the teachers completed was the Savannah Task. [For more information about each task, see Chapter 3.] The group immediately recognized that Savannah's approach on the subtraction problem is incorrect and conjectured about why Savannah might have used the same strategy for all three operations. Sarah commented that perhaps Savannah's teacher had been using Cognitively Guided Instruction [CGI], so Savannah might have been thinking about the numbers and how to use them efficiently. Moreover, Sarah wondered whether Savannah was taught this method of compensating by working with decade numbers or was really trying to make sense of the problems. I identified two indicators of

[9] All names are pseudonyms.

Sarah’s productive disposition in these statements: She was trying to make sense of the task, and she had asked questions about Savannah’s approach.

Sarah was not the first in the group to explain why Savannah’s subtraction strategy is incorrect. However, she immediately agreed that Savannah’s strategy is not correct and helped to explain how Savannah could adjust it to get the right answer. Although Elle was the first to explain the error in Savannah's subtraction strategy, Sarah was the first to articulate that Savannah, by changing the 19 to 20, was now taking away too many, which is why she needed to add 1 instead of subtract 1 in the end. Later, when Elle tried to model the problem with blocks to determine why Savannah’s answer was off by 2, Sarah moved to her side of the table and reminded Elle that she needed to account for the 1s Savannah added (although neither was sure how to do so). Sarah not only actively engaged with the task but also showed willingness to re-engage with the task when a new confusion arose. In these ways, she demonstrated two other indicators: She was willing to offer tentative ideas and tried to provide clarifications to others’ thinking.

In her reflection on the Savannah Task, Sarah noted that she enjoyed sharing ideas with the other teachers and especially liked the opportunity to talk to teachers across grade levels. Although she found the use of the manipulatives to be helpful, she admitted that she was still unsure why Savannah subtracted the 2 and the 1 in the end of each problem. She wondered whether Savannah was repeating a strategy she had been shown or was trying to make mathematical sense. I interpreted Sarah’s willingness to admit her uncertainty about parts of the problem and her appreciation for Elle’s use of the manipulatives as two more indicators of her productive disposition.

Pat Task

On the individual portion of the Pat Task, Sarah could explain mathematically about half of Pat’s steps. She noted that the 40 is 4 groups of 10 and that Pat was probably picking a friendly number near 42. She then noted that Pat realized that because he wanted to divide by 7, 10 groups of 4 was 3 groups of 4 too many. Sarah last commented that the 12 + 2 were the leftovers for which Pat still needed to account. However, she seemed confused by the 14 ÷ 2 = 7 step and could not use Pat’s approach to solve 56 ÷ 8.

When the group began to discuss the task, Sarah was one of the first to admit that she "got lost" and did not completely understand Pat’s strategy. After the group discussed at length whether Pat was thinking about 4 groups of 10 or 10 groups of 4, Sarah commented that having Pat actually show his strategy with manipulatives to see how he was thinking about the groups would be helpful. Later, she commented that one reason for the difficulty of the task was Pat's flexibility with the number of groups and the number in each group, which is why the 14 ÷ 2 = 7 step at first was so confusing. In essence, Sarah noted the heart of the mathematics of this division problem: keeping track of the number of groups and the number in each group.

Although Sarah talked less during this task than the previous one, she still seemed engaged with the task throughout. In particular, she attended closely whenever someone in the group used manipulatives to explain her thinking. She was vocal about which parts of Pat's strategy she did not understand, and she asked the others to help explain it to her. She was able to show Amy why her approach to solving 56 ÷ 8 was different from Pat's approach and offered suggestions for modifying Amy's. Thus, in working on this problem, disposition indicators shown by Sarah include willingness to question one's self, recognizing worthy and unworthy confusions, and willingness to offer tentative ideas.

In her reflection, Sarah admitted to being frustrated by this task. She noted, "A challenge is always good. However, not being able to solve the problem was frustrating." She went on to say that she had trouble connecting the numbers to the context and would have liked to discuss the problems with others before working on it herself. She also said that using the manipulatives was very helpful to her and that after having the chance to share with others, she believed that she could "probably" explain Pat's strategy "fairly well." I took these comments as evidence that Sarah has mathematical integrity and that she is willing to work through frustration on mathematical problems.

Consecutive-Sums Task

Sarah had more written work for the Consecutive-Sums Task than for the previous tasks. She found that 1, 2, 4, 8, 16, and 20 do not have consecutive-sum representations, correct except for 20. (She later corrected this error during the group discussion by noting that 2 + 3 + 4 + 5 + 6 = 20.) She wrote several conjectures about consecutive sums and discussed why some held and others did not. For example, she proposed a two-part conjecture: Odd numbers can be represented only by the sum of 2 addends, and even numbers can be represented only by the sum of 3 or more addends. Her conjecture is reasonable and partially correct. The second half of her conjecture is true for the even numbers that have consecutive-sum representations. (The powers of 2 do not have consecutive-sum representations, as Sarah had noted earlier.) The first half of her conjecture is slightly incorrect: Every odd number can be represented by the sum of two addends, but some odd numbers also have representations with three or more addends. Sarah noticed this fact and wrote that the "entire" conjecture “didn’t prove true” when she found representations for 9 and 15 that had more than two addends. I conclude from this work that Sarah was interested in the task, persisted in finding different representations, and was able to generate tentative ideas and then test those ideas.

During the group discussion, Sarah talked more than she had in the previous discussion about the Pat Task. Whereas most people in the group worked on the problem by trying all the possible combinations of the addends, Sarah said that she started from the sum and then “worked backward” to find which numbers had consecutive-sum representations. She explained why consecutive sums with two addends are always odd and that multiples of 3 might always be consecutive sums because of the “1 + 2 + 3, 2 + 3 + 4, 3 + 4 + 5 pattern.”[10] Throughout the discussion, Sarah was focused on the mathematics of the task and (consistently) sought or provided clarifications about potential consecutive-sum patterns.

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[10] The “1 + 2 + 3, 2 + 3 + 4, 3 + 4 + 5 pattern” is as follows: The first consecutive sum with three addends is 1 + 2 + 3, which equals 6. The next sum, 2 + 3 + 4, equals 9, 3 more than the previous sum—because each addend has been increased by 1, increasing the total sum by 3. Each successive consecutive sum with 3 addends is 3 more than its predecessor for just this reason.

I was surprised when, during the discussion, Cassie asked me about the mathematical significance of consecutive sums. While I stumbled to give an acceptable response, Sarah said that this is a good problem to work on number sense with people. She mentioned that for younger children this task would be good practice with addition and would help build understanding of patterns for older children. I consider this interjection a strong indicator of Sarah's productive disposition: She explained why this task, to her at least, is useful and worthwhile. She also implied that the answer to the task itself was not as important as understanding the underlying patterns and why they hold.

On the reflection for this task, Sarah commented about her affect toward it. She mentioned several times that the task was "enjoyable." She wrote that the task "was fun and [I] see it as a good experience for children as well." Moreover, she thought that the task was "challenging, yet seemed fun (like being a detective!)." Sarah, despite the fact that she found the task challenging, indicated really enjoying making mathematical sense of this task.

Sam Task

Sarah had a bit of trouble with the Sam Task, but she was able to make sense of several portions of his mathematical strategy. On the first part of the task, she noted that Sam would likely add up the ones digits of the subtrahends to get 21 and see this 21 as "2 tens plus 1" one to take away. He would take the 1 from the 5 in the ones' column of the minuend. She saw the superscript 2 above the minuend as a reminder that Sam has two more tens to be subtracted. Adding the tens in the subtrahends along with the extra 2 tens from the ones' column, Sarah concluded that Sam would take 90 more away from the minuend and would have to regroup to carry out the calculation. Although I thought of Sam's strategy differently, Sarah's interpretation is mathematically sound and consistent with Sam's solution to the previous problem.

Sarah was unable to explain the mathematics of Sam’s strategy in a second way or to use it to solve a different subtraction problem with multiple subtrahends. In fact, she wrote, “I’m stuck!” in large letters at the bottom of that page of her work. However, she still attempted to solve all parts of this problem and was specific about what she understood and about what she still had questions. Again, Sarah exhibited her mathematical integrity by willingly admitting which parts of Sam’s strategy she did understand and which she did not.

Commenting about Sam’s mathematical thinking on this task, Sarah believed that adding all the subtrahends and then subtracting from the minuend was a much higher level of thinking than subtracting each subtrahend in turn. She also believed that Sam’s strategy could be used for any subtraction problem, especially because it applied to a problem “with zero in the middle” [of the minuend]. These comments seemed to indicate that Sarah appreciated Sam’s strategy, despite the fact that she was unable to make complete mathematical sense of it.

Mathematical Autobiography

Early in her autobiography, Sarah mentioned that she really enjoys mathematics. In grade school, she believed that she was good with formulas and basic algorithms and did relatively well in her mathematics classes. Her first unpleasant experience with mathematics was in her high school geometry class—the first time she actually had to prove things and she had a lot of difficulty doing so. Making matters worse, her father, a mathematician, told her how easy geometry is and said that he did not understand why she was struggling.

For the next few years, Sarah avoided mathematics classes whenever possible. Late in her college career, however, she had a mathematics class with a professor who she believed approached geometry in a way that was much "more spatial," which she found extremely helpful.

However, her feelings toward mathematics did not change dramatically until she was introduced to Renaissance Math and Cognitively Guided Instruction, both of which she believed helped her do and understand mathematics better; previously she knew facts, but now she believed that she could manipulate numbers better and with understanding. She noted that she does not regret her years of difficulty in understanding mathematics: “Going through times where you struggle with things is always a good experience to remember in the classroom and think about how your students are affected.”

Sarah concluded her autobiography by describing her mathematics classroom in a perfect world. She wished that she could allow the students all the time they need to discover new concepts and to have all the manipulatives they might need to fully understand the mathematical concepts. Most of all, she would like all her students to see mathematics as something exciting, interesting, and challenging, not as something to be feared.

Individual Interview

During my interview with Sarah, I asked specific questions about each of the tasks and about her autobiography. I also asked her broad questions about her relationship with mathematics and about the ways she teaches her students mathematics. A theme she mentioned several times was that mathematics should make sense. Early in the interview, Sarah said that she enjoyed the Pat Task the least of the four tasks. She explained why she continued to work on this difficult task she did not find enjoyable: “Because it’s a challenge, and I want to be able to accomplish it.” To explain why she wanted to accomplish it, she talked about the parts of the task she understood and parts that were still confusing. She believed that she was really close to understanding Pat’s strategy and that his strategy should make good mathematical sense. Before the group discussion of the task, she had said, “Oh my god, you know, I know there’s a reason for this,” referring to the parts of Pat’s mathematical strategy that she did not understand.

Sarah noted that as a student in grade school, she did not always have to make sense of mathematics. She could follow formulas or patterns, but, in retrospect, she believes that she did not make deep connections with the underlying mathematics. Now, she said, she wants to show her students as many ways to think about mathematical problems as possible. She gave examples of decomposing numbers, building to decade numbers, and using manipulatives as things she never did as a student but that she wants her students to use to make sense of mathematics for themselves.

I concluded from Sarah's interview comments that her belief that mathematics should make sense also influenced her view of frustration in relation to work on mathematical tasks. She had previously mentioned that she somewhat disliked the Pat Task and felt frustrated that she was unable to solve it by herself. I asked her to imagine that a colleague had left the task in her mailbox with a note saying that she might find this task interesting but not to worry if she did not have time for it. She assured me that she would have tried it because the problem would bother her and she would have to try to make mathematical sense of it. I found interesting not only Sarah's use of words like *frustrating, bothered, struggled,* and *challenging,* but especially her using them in a fairly positive light. To her, being frustrated or bothered by a mathematical task motivated her to work harder on it or to seek help from others, indicating that Sarah's belief that mathematics should make sense and that she herself could make sense sustained her through her struggling. (Sarah had difficulty putting into words what motivated her to continue working on difficult problems and even commented that she had never talked to anyone about mathematics like this.)

Because I wondered whether Sarah's positive stance toward struggling influenced her teaching, I asked whether she created opportunities for her students to struggle on some of the mathematical problems she presented to them. She replied,

> Yeah. No, you know, yes and no. I don’t want to put them to the point where they’re, um, feeling bad about themselves, but I think it’s good. I mean, it’s like a lot of times you give them a problem and they want you to tell them the answer or something, and it’s like, "Well I want to see, you know, what you think about it or how you would do it," and stuff. So I think in that sense, yes, I try to get them to, to struggle enough to try it and, you know, "What would you do first? ... And after you did that, then what would you do?" You know?

This response was particularly enlightening because it showed that Sarah wanted her students to work hard and even struggle at times on mathematical problems, but, at the same time, she wanted to ensure that her students had positive affect toward learning and doing mathematics.

The subject of positive affect toward mathematics arose several times in my conversation with Sarah. Initially in the interview, I asked how she felt after completing the four mathematical tasks. She replied, “I still feel like even if I didn’t answer the questions right, I felt like I got a lot out of the experience”; she commented that she is happy any time she can try to better understand mathematics with other teachers. Later, she said that she tries to promote a positive attitude toward mathematics in students as well, by, for example, selecting students who have solved problems in different ways to share their strategies and praising students who devise unique solutions. She said that this practice not only improves their affect toward mathematics but also makes them feel safe to offer ideas about which they may not be completely confident, that is, in my terms, she encourages academic risk taking. Furthermore, students, instead of worrying about getting the right answer, focus on the strategies and making connections among the strategies.

Sarah several times commented that she believes that all of her students have the ability to understand mathematics. She wants them all to have good number sense and to be able to see that mathematics should make sense. We discussed whether she believes that certain students have this sense-making ability and others do not:

> Sarah [S]: Well, I think anybody *could* with some work, but I think certain people may be more ....
> Interviewer [I]: Inclined?
> S: Yeah, inclined to it. I mean, just like some people are more, you know, musically inclined or like that.
> I: Yeah, everybody has their strengths and weaknesses, right?
> S: Yeah.
> I: But I didn't know if you thought there may be some people who ....
> S: That never could do it?
> I: Right.
> S: Yeah, no, totally not. I think anybody can learn, whether it's math or whatever. I just think for some it may be a little easier than others.

She went on to say that all students have strengths in some areas of mathematics and that she wants to help those students build from their strengths to other areas of mathematics.

I have highlighted the major themes related to Sarah's productive disposition we discussed during the interview, but other aspects were brought to light as well. Sarah mentioned that she loves the challenge of a good mathematical problem, problems that seem like a game or that lead her to discover something new, even if the game or discovery required a bit of mathematical work on her part. She also commented that she enjoyed talking to other people to better understand mathematical problems, even if her ideas were tentative or incomplete. She went on to note that she tries to promote this academic risk taking in her students as well. Last, throughout the interview, Sarah was very open and honest about parts of the mathematical tasks she did and did not understand. She personally found this trait to be unremarkable and simply part of how one makes sense of mathematical problems.

After we had talked about several of her productive-disposition traits (although I did not use the term *productive disposition* with her until the interview was over), I asked whether these traits were specific to her teaching and learning mathematics or were part of her stance toward all aspects of her life. She said, “I don’t do that with everything in my life, for sure.” However, she went on to posit that these might be her views not about teaching just mathematics but about teaching more broadly. She believes that she has a similar stance toward all academic subjects, although these traits may play out differently depending on the subject.

Comparison of Evidence Among the Ten Teachers

Sarah’s interview touched on many of the themes other participants mentioned in their interviews, including appreciating multiple approaches, enjoying a challenge, and seeing struggling on a mathematical task in a positive light. In this section, although I discuss major themes of productive disposition that emerged in the interviews of the 10 participants in Study 2, my goal is not to talk about which teacher had which traits, but rather to discuss which traits were prominent during the interviews. I want to better understand which traits play important roles in productive disposition, at least for these 10 teachers who were found to have strong productive dispositions. The traits of productive disposition that the teachers talked about most often were as follows (in order in which they appear in the upcoming sections):

- Re-engagement and persistence with a task—an unwillingness to give up on a mathematical problem if a satisfactory solution has not yet been found
- Belief that effort is worthwhile—a general belief that spending time and energy on a mathematics task would lead to better understanding or new insights
- Appreciation of multiple approaches—an awareness that one can make sense of a mathematics problem in several ways
- Academic risk taking—engaging in behavior that may reveal one's mistakes or may make one appear less competent than others in order to build knowledge
- Mathematical integrity—knowing what one knows, knowing what was does not know, and being honest about these assessments
- Belief that mathematics should be exciting and enjoyable—the belief that mathematics should be seen as interesting and mentally fulfilling and not as something to be feared
- Belief that everyone can achieve success in mathematics—although understanding mathematics may come more easily for some learners than others, this is a belief that with effort anyone could learn and do mathematics

Figure 4.2. Traits of productive dispositions most prominent in Study 2 interview data.

In this section I focus on the teachers' comments during their individual interviews for two reasons. First, Study 1 was focused solely on my interpretation of the teachers' productive dispositions through their actions and comments working on a challenging mathematical task, and I saw many of those same indicators in the teachers' work on the four tasks in this study. Instead of again focusing on their task work, to test my findings from the tasks, I focus now on indicators that came from the teachers (who were still unaware of the term productive disposition and of my terminology for the indicators). Essentially, I wanted the teachers to talk about productive disposition in their own voices. Second, many interview questions were directly about the teacher's work on the tasks or comments in her autobiography. Thus, data drawn from the interview give a sense of both what the teacher did on the task or said in the autobiography *and* the teacher's thoughts about the action or comment.

As noted in Study 1, identifying mutually exclusive productive-disposition indicators is problematic. Some evidence from the data could potentially be associated with several indicators. For example, a teacher might persist on a task because she believes that she can make progress or because she has a sense that the task has not been completed. In this dissertation, though, I focus on the evidence teachers provide for their productive dispositions. As such, I attribute the data to specific indicators. When reading the examples from the interviews, one might see the data as associated with indicators other than the one I chose. Productive disposition is a complex construct, and the indicators do overlap. I tried to choose clear examples for each indicator described, knowing that other indicators might be evident as well. Again, I did not mention the topic of the study or any of the indicators by name to the participants until their interviews were complete. I attribute the indicator names to their comments in the results, but this is my terminology for their comments, not theirs.

Re-engagement and Persistence With a Task

During the interviews, I asked each of the 10 teachers either to re-explain why Savannah's answer was off by 2 or to explain a specific insight she had on the task. For example, I asked Beth about her suggestion that when solving 38 – 19, Savannah should add 1 only to the 19 and then subtract, leaving the 38 unchanged. I asked Hillary to explain how to apply Savannah's strategy for 39 – 17. In every case, the teachers enthusiastically re-engaged with the problem and thought more about it. Moreover, they continued to work on the problem until they found a satisfactory answer to my question, even when I suggested that we move on to another question. For example, I asked Louisa how she would have used tools to show Savannah why her subtraction strategy was incorrect (Louisa's suggestion when the group discussed the task):

> I: So how would they [mathematical tools] have helped you here?
> L: If I could've seen, okay, I'm trying to remember what she did. She added 2 and added 1, and then took the 1s away.
> I: Right.
> L: Because it's basically like you do in algebra: you got to do the same thing to both sides. So, I think if I could've built her, what she did ....
> I: Do you have blocks?
> L: I do. You know what? [Walks off and gets blocks.]
> I: No, you don't have to if you don't want to.
> L: See, this is my problem, because I still really, to be honest with you, I still really can't see it. That's why I kind of figure I'd just have to do it.

Louisa worked with blocks for the next several minutes in an attempt to understand the error in Savannah's subtraction strategy. All the teachers re-engaged with the task for several minutes. Perhaps more telling is that they persisted on the task until either they were able to answer the question I had asked or until I, because of time constraints, posed a different question to move forward with the interview. Beth, for instance, worked on the task again for nearly 10 minutes until she believed that she could explain so her students would understand why Savannah should have added the 1 back on in the end.

Elle displayed a slightly different side of this trait of productive disposition. She had commented in her reflection that Savannah’s strategy was “eye-opening.” When I started to ask her what in particular was eye opening, she asked me to remind her about the last question I had asked the group during the discussion of the task. Hearing the question (“Why is Savannah's answer off by 2?”), she then re-engaged with the task for several minutes, trying to answer the question. Although she realized that adding 1 to 18 instead of subtracting 1 in the last step would yield the correct answer, she was dissatisfied with this explanation as an answer to the question, apparently realizing that the underlying mathematical reason was not just about how to get the answer. Even after we had moved on to discuss why mathematical tasks like the Savannah Task bother her, she returned to the issue: “I can’t let it, I can’t let that go. It’s because there must be an answer, so I need to figure that out; I need to find—or somebody has got to tell me.” When I suggested that perhaps the question had no answer, she responded, “Yeah, but I mean, it’s off by 2. Why?” She returned to her earlier observation that Savannah took away 1 instead of adding 1 to explain, but when we then started to discuss the frustration she felt during this problem, she again returned to the question of *why* Savannah was off by 2. “Yes, it’s off by 2, so it’s like, 'Why is it off by 2?' What else did she—is there another reason why it’s off by 2 other than she should've added that 1, you know?" This discussion showed Elle's frustration in trying to really understand the answer to this question, while displaying her willingness to persist on a task with which she was struggling.

The teachers also re-engaged with tasks on their own. All the teachers willingly revisited the problems and reminded themselves of what they had done, but several went further. Rebecca, Beth, and Mary re-engaged with the Sam Task, Cassie and Elle re-engaged with the Pat Task, and Gina re-engaged with the Consecutive-Sums Task, all with no prompting from me. On these tasks, the teachers spent several minutes thinking and talking about portions of the problems that they still believed they did not completely understand. Moreover, many persisted on these problems even when I suggested moving to another question, perhaps showing how powerful this trait of productive disposition is in these teachers.

Finally, when I asked Hillary what she thought my goals were for the day I had collected the data from the teachers, she said that I probably was trying to see what mathematics they were bringing to the problems and what they saw when they looked at student work. However, she also thought I had another goal:

> Um, and it also almost seemed like part of it was, um, "How much stick-to-it-tiveness do you have?" you know, [laughs], especially when you throw the Consecutive-Sums problem in there. It's kind of like, "Okay, well, are we willing?" you know, "Are people willing to persevere when something doesn't seem automatically apparent to them?"

She went on to note that she had noticed different levels of perseverance by the other teachers in her group, and that those levels changed from task to task. When I asked her why she had noticed this, she stated that it was just the teacher in her and that she often checks with her students to see who is participating in conversations or who is staying engaged with mathematics problems. She went on to comment that she did not mean to notice this in the other teachers, nor was she “stalking” them in any way, but that she believed that this was such an important part of being a teacher that she was unable to "turn it off." This explanation seemed to show how important Hillary believed perseverance is for her students.

Belief That Effort Is Worthwhile

Similar to persistence, the belief that steady effort on a mathematical task will eventually lead to new ideas or better understandings was common among these 10 teachers. To clarify, the two terms differ in that the teachers saw *persistence* as the unwillingness to let a problem go if they had not found a satisfactory solution, whereas they saw *the belief that effort is worthwhile* as a more general belief that is less context- and task-based. When discussing effort, several teachers said that for a problem that at first seemed difficult, the more they thought about the problem and began to make sense of it, the easier the problem became. Louisa provided an example from her work on the Consecutive-Sums Task:

> [Initially] I couldn’t find a way to organize my thinking, but it was kind of cool, though, because I started seeing some stuff as I was writing up, and then I was like, “Oh, wait a minute! Look at that.” And then I was trying to circle the numbers to see what the pattern was. I didn’t like it at first because I couldn’t—it wasn’t making any sense to me, but the more I played with it, the more I saw.

She went on to note that while working on the task, she hated it because it seemed too confusing and she did not know of an easy way to begin. However, putting effort forth, she found her own system of organization and began to see more patterns and to think about better ways she could have organized her work; then, in turn, the task became more enjoyable for her.

Hillary had a similar experience on the Consecutive-Sums Task. Like Louisa, she found the problem overwhelming at first:

> Well, I had trouble getting started because I was having a terrible time organizing myself. Like I was trying to do something, I don't know if all those attempts are in here, but ... "That's not going to work." You know, "This," like okay, "Now," like, "Now I'm going to try to do it" [makes screaming noise]. I just needed a system where I could see it more, watch the patterns unfold or not.

Eventually, she believed that she found a system that helped her organize the consecutive sums, but doing so took a substantial amount of effort. She was able to figure out why all multiples of 3 are consecutive sums, but she still believed that she could have found even more patterns. She commented that she could find herself working on tasks like this one for days, putting it down for periods of time and then returning to work more. Hillary believed that she would find an answer satisfactory to her only by continuing to work on the problem. This desire to find an answer and making the effort to do so seemed to be a strong part of Hillary's productive disposition.

Rebecca talked about not only the effort she had to put forth on some of the tasks but also her belief that her students need to do the same on difficult mathematical tasks. She initially commented that all the tasks I gave her were more difficult than she thought they were going to be. She admitted to not normally seeing patterns easily, but after working to understand the tasks, she began to see patterns and to make sense of the tasks. She said, "When I figured it out, then I felt better about it." This newfound confidence motivated her to exert more effort on the other tasks, which, in turn, helped her to make even more sense of the tasks. Rebecca later connected this experience to her difficulty with a student, who was, in her opinion, “kind of lazy” and generally unwilling to try new problems, because of a lack of effort, not mathematical skills: “She doesn’t want to try to understand the problem, and she wants somebody to help her, but she wants that all the time.” For Rebecca, effort was the missing part of this student’s productive disposition, and she believed that with effort, the student could become much better in mathematics.

Two other teachers talked about their students' need to put forth effort to achieve in mathematics. Beth first mentioned that she still consciously exerts effort on mathematical tasks to continue learning new mathematical ideas for herself and that she wants her students to recognize the need to put effort forth as well. Some of her students still failed to see 17 as 1 ten and 7 ones, but she believed that with effort and practice they would begin to make sense:

> B: They *know* that they have to practice. So for those kids right now who today, you know, they had no idea what the number 17 was, it’s okay. We’re still practicing that, and he says, “I’ll get it.”
> I: So in other words, if they put effort forth, they’ll get it.
> B: They'll get it. Yeah, they’ll get it. He’s not going to—not a single child in the class feels that there’s something wrong with them if they don’t get it.

She then noted that she had difficulty with some of the tasks I gave her, but she tried and believed that she had made progress on all of them. Similarly, she believed that if her students put forth effort, although they might not find the right answer, they would all make progress and make new mathematical connections.

Sarah similarly believed that if her students put forth the effort, they could be more mathematically productive and make better sense of mathematical tasks. Sarah said that she wants her students to see mathematics as a challenge or a game (more on this later in the chapter), but for her students to enjoy the challenge, they need to put forth effort and try to make sense for themselves. She commented that the only way students will enjoy mathematics and see it as interesting as she does is for them to try to solve problems, even if the problems seem too difficult at first. I asked her more about her stance:

> I: So would you say that if your students put in effort toward mathematics, they could start to see it as interesting, as challenging, as this game, as this ... or is there something more that you'd have to do?
>
> S: No, I think if they put in some effort, they could see it. And I mean that would be what I'm trying to project, too. So I mean if they see me going, "Wow, look at this number. Isn't it interesting? What could we do with this number?" kind of a thing as opposed to, "Okay, what's 2 plus 2?"

She explained that her students' efforts could take various forms, such as using manipulatives, drawing a diagram, or decomposing the numbers, but that they needed to first exert effort before she could help them make connections across these different mathematical approaches. Sarah seemed to believe that her students have to be willing to put effort forth before she can help them develop other traits of productive disposition.

Appreciation of Multiple Approaches

As noted in the last section, Sarah encourages her students' efforts on mathematical tasks, leading to their developing stronger productive dispositions, by prompting the students to try different strategies. This practice seems fitting because Sarah enjoys problems that can be solved in multiple ways. In her reflection on the Consecutive-Sums Task, she wrote, “Enjoyed this task!” and she gave two reasons when asked to explain: She felt capable on the task, and the task “was a little more open-ended so you could kind of, you know, again, you could do it in several different ways, like whatever kind of went with your thinking.” She later added, “I liked, again, that you could solve it in different ways and still get the same answer.” Part of Sarah's enjoyment of the mathematical tasks was seeing how others approached them and how the methods were similar or different.

Several teachers commented on approaches that they found surprising and intriguing from either the students in the tasks or their own students. Gina found the Sam Task to be enjoyable partially because Sam’s method was novel to her. She said, “I liked this problem because it was fairly different, different from the way my students would’ve [solved it], I think. I think my students would’ve added them together and then subtracted the whole bunch.” Similarly, Cassie commented about a student's showing her the Lattice Method for multiplication, which was new to her; she remarked, “I mean, what a great method.” She liked that it could be connected both to representations and to the standard multiplication algorithm. Both of these teachers enjoyed seeing new methods that made mathematical sense.

Amy commented that because during her education she had learned only one way to solve problems, she disliked mathematics for many years. When I asked her when her opinions about mathematics changed, she credited her work in professional development:

> I think that's what I really enjoyed about this approach ... is that it's really exciting to see that there are other ways to approach problem solving and that I don't know them all. So I'm just amazed at how, when given the opportunity, that children or other people solve math in different ways, and it's just really exciting.

Amy said that now she gets excited by the opportunity to share her mathematical strategies or see the strategies that others used. For Amy, learning that multiple approaches can be used to solve any mathematical problem helped to change her disposition toward mathematics.

Elle professed a love of "mental challenges" because they force her to think "outside of the box." Like Amy, she had grown up learning mathematics traditionally and now relished the freedom to use multiple strategies to solve problems instead of the single method she had been taught. Her outlook toward mathematics is now very different from her earlier view: "I challenge myself to think about solving problems in a different way than I would have done previously." She noted that part of what she enjoyed most from the experience was seeing how other people used approaches that differed from her approach to solve the problems. She then recounted being amazed by a student's mathematical strategy:

> It was like you just took away one; you just did this, and it was like, "Wow," you know? It's one of those things, and we had to put it up on the board just so we could look at it from a different perspective, you know? I'd just go, "Wow, I would have never thought of solving the problem that way," and that's what I love. I love when I can see a way that it's like, "Wow, I never would've done that."

She concluded that seeing multiple approaches to solving mathematical problems completely changed the way she views mathematics, and because of that change she now enjoys teaching and learning mathematics more than she did before.

Both Rebecca and Mary also described promoting and sharing the strategies that their students devise in the classroom. Rebecca commented that different strategies help her reach more of her students. One student, as mentioned, struggled to find the total number of candies if 10 children each had 8 candies, but after Rebecca reminded him of a classmate's strategy of seeing 8 as 5 and 3 leftover, he counted by 5s and then by 1s successfully. Because she sees such students, first unable to solve a problem and then successful with a different strategy, she promotes the use of multiple approaches in mathematics. Mary's approach is to select in advance 2–4 of her children's strategies appropriate for the next mathematics problem; she shares those using her overhead projector and helps the students make connections among the strategies before the students are asked to solve a similar problem using someone else's strategy. Mary claimed that this approach helps her students develop more efficient strategies and more willingly share new strategies they create. Mary claimed that promoting in her students one trait of productive disposition, appreciating alternative approaches, led to their developing another trait, which I call *academic risk taking*.

Academic Risk Taking

As was noted in chapter 2, *academic risk taking* is the willingness to ask questions or share ideas that may expose one's misconceptions or weaknesses. Like Mary, Sarah tries to promote this trait of productive disposition in her students, as she described when I asked if being a risk taker is an innate trait:

> No, I'd like to think that I try to instill that in my students, because when you're sharing problems and doing them on the board, um, sometimes you pick kids that didn't do it right so you can talk about where—or that they can; you want that child to see where their thinking didn't quite go where it needed to.

Sarah explained that she is evidence that risk taking is not inborn: She did not take many risks as a child but was a "good girl" who never raised her hand in class or asked questions. Although unable to pinpoint the moment when she started to change, she thought that it was after she had children of her own and began to advocate for them that she became more willing to take risks and ask questions when she was confused. She said that whereas in the past she did what everyone else wanted her to do, in the present she was driven to make sense of those things she wanted to understand.

Although the other teachers described their efforts to develop other traits of productive disposition in their students, they generally talked about ways their own stance toward taking risks had changed. Hillary, for example, explained that she had changed her stance after becoming a teacher to ensure that she was "imparting knowledge on little people":

> H: Where before I just—I'm fine to just be quiet and in the background, and if I got it, I got it, and if I didn't, [whispers] shh, don't tell. But then, when I think as a teacher, it's like, "Well, I kind of need to know why so I can do a good job at teaching."
> I: So do you feel like you're not the "Shh, don't tell" anymore if you don't know something?
> H: I'm much more apt to ask or say, "What?! Wait, wait, wait." ... I'm not usually afraid to say, "Wait. I didn't understand that. Back up." You know, "Let's do that over again for me."

Hillary said that this trait now carries over to other parts of her life. She said that at staff meetings she is often thanked by other teachers for asking questions that they themselves had.

Rebecca did not become a risk taker after becoming a teacher but, instead, noticed this change occurring during her professional development. In her interview, she explained this change:

> Well, because I think a lot of it had to do with the way we were trained doing this. At first it was a little—I was a little nervous, but when you share failures with each other, you are more vulnerable, and then you're—you can learn more if you're not so frightened to show that you don't care to be wrong.

After her training, she not only stopped fearing being wrong in front of other teachers but also began sharing her mistakes with her students, not only in mathematics, but across all subjects. Risk taking has now become just another part of Rebecca's personality.

Like Rebecca, Beth believed that her stance toward taking risks had changed because of her professional development. In her group's discussion of the Pat Task, Beth was specific about the parts of his strategy she did and did not understand: At his $14 \div 2$ step, she "got lost." When I inquired about her willingness to share her lack of understanding with this group of teachers she did not know, she responded, “I really valued their input, but I am one that wants to learn, so if I don’t get something, I’m really okay with saying, 'I don’t get it.' That’s just the kind of person I am.” Beth explained the change in her stance resulting from professional development:

> I used to be the person who when [the professional developer] would say, “Okay, think about this and give me something to share,” I used to not raise my hand. And now I raise my hand, and usually she’ll say, “Oh my gosh, that’s a great point.” So I’ve had a lot of validation that makes me feel like I’m not totally off .... I’m not afraid to take risks. I’m much more comfortable now.

For Beth, this comfort with taking risks along with ability to recognize which parts of Pat’s strategy she did and did not understand helped her eventually make sense of Pat’s mathematical strategy.

Mathematical Integrity

Mathematical integrity is knowing what one knows, knowing what one does not know, and being honest about these assessments. As Beth illustrated, mathematical integrity may be coupled with academic risk taking in people with strong productive dispositions. Beth's mathematical integrity on the Pat Task was evident because she knew that she understood the beginning of his strategy but was unable to interpret his 14 ÷ 2 line, and because of her risk taking, she was willing to share this confusion with the rest of her focus group. Beth’s approach was similar on most of the other tasks, including the task she enjoyed most, the Consecutive-Sums Task. To explain her enjoyment, she said, “I can tell you what I did, and I know what I did, and whether I did it right or wrong,” but when I described her mathematical integrity as strong and as a defining feature of her problem solving, she said, “That sounds really deep, much deeper than I thought I was." Beth believed that being honest with herself and others about what she did and did not know was important, but she was unable to explain the origin of this trait or why she believed that it was important.

Determining whether people have mathematical integrity while they are working on a mathematical task is difficult because to make a valid assessment, one would have to know their thoughts and judge whether they were making honest assessments of themselves. However, during the interviews, the teachers were willing to say, often without prompting from me, what they did and did not understand. Cassie admitted that at first she thought that Savannah's subtraction strategy was correct but after viewing her multiplication strategy, realized that the subtraction was incorrect. When I asked Louisa why she thought blocks might be helpful to explain Savannah's subtraction strategy, she informed me that she still was not quite sure why the strategy was incorrect and immediately found blocks to help her thinking. In both cases, the teachers were willing to tell me what they did or did not understand about the problems.

As mentioned previously, Sarah clearly displayed her mathematical integrity on the Sam Task. Although she correctly explained Sam's strategy for the subtraction problem with regrouping, when she could not use his strategy to solve a similar subtraction problem, she wrote, "I'm stuck!" in large letters beneath that portion of the task. To explain her written comment, she said that initially she believed that she understood Sam's strategy and in using what she believed was Sam's strategy, she got the same numbers as Sam, but when she tried to check her answer a different way, she got different numbers. She said, “I’d have a moment of clarity and then start to write it, and then something wouldn’t come out just right.” Her written work showed several attempts to make sense of Sam’s strategy, but Sarah noted that after her second or third attempt, she gave up and wrote, “I’m stuck!” She explained that Sam had added the ones' and tens' columns of the subtrahends separately, but she did not know what to do with the 2 tens from the ones' column. Sarah believed that she should be honest and simply note that she was stuck.

Gina showed her mathematical integrity on the Pat Task. When trying to use Pat’s strategy to solve a similar multiplication problem, Gina gave two solutions. Next to one of her answers she wrote, “‘Proving the answer’ way”; next to the other she wrote, “2nd try (better match to Pat’s work).” She explained those comments:

> Hmm, because I knew the answer was right, but then as I was doing it, I kept thinking, “Well, he switches the order here [14 ÷ 2 = 7] and knows the fact, but the part that’s getting divided has to be different.” I don’t know. I just wasn’t satisfied at that point. I felt like I needed to try it again, and so I wrote “second try,” and then after I finished the whole thing, I said, that’s why I wrote, “better match to Pat’s,” because I felt like it was a better solution.

After her first try, she thought that she had “proved the answer” but had not really solved the problem, a practice she found common among her students who cared more about getting an answer that matched their neighbor’s than solving the problem using a valid mathematical approach. Gina said that she would be hypocritical to expect her students to admit when they lacked understanding of a mathematical problem if she herself did not do so.

Elle similarly believed that she needed to have mathematical integrity if she expected her students to develop this trait. Elle tried over several minutes to re-explain why Savannah's subtraction answer was off by 2, but even after providing a clear explanation for the discrepancy, she was dissatisfied:

> It just makes me think of some kids [who], when you ask them how they got their right answer, and they basically will just do anything to match the answer, which doesn't go ... with what their thinking was. And that's sort of what I feel like I'm doing right now. I'm like, "I just add," you know. I see that you're adding the 1, but I'm still not completely comprehending.

Elle tells her students when she is puzzled--whether at the board or looking at their mathematical work—because she wants the students to know when their thinking challenges her and that she too is still learning more mathematics every day. My interpretation is that Elle wants to be a model of mathematical integrity for her students so that they might learn to develop mathematical integrity in themselves as well.

Mathematics Should Be Exciting and Enjoyable

For many of the teachers, an important part of their mathematics lessons was ensuring that their students were excited to learn mathematics and that they enjoyed working on mathematical problems. For example, Sarah commented in her autobiography that she hoped students "see math as something exciting, interesting, challenging, and not something fearful [*sic*]." Similarly, Louisa said that she hated the way mathematics was taught to her, and she does not want her students to have that same experience:

> I want my kids—my overall goal is when we say, “Ooh, it’s time to do math,” they all jump up and down and say “Yay” instead of “Ooh, my gosh.” You know, I didn’t ever want to hear the groans. So, more than anything, even if I don’t teach a lot of great math, um, my personal teaching, you know, I just want them to enjoy it. When we count, whatever we’re doing, I want them to like math and feel successful.

She went on to say that she thinks that children who enjoy doing mathematics become better problem solvers, and, in turn, want to try to understand new mathematical concepts. In other words, Louisa seemed to think that enjoying mathematics is a way for her students to start building more productive dispositions.

Beth, like Louisa, stated that students who enjoy and are excited by mathematics will be more successful on mathematical tasks, but she understood why many students disliked mathematics. She believed that too many teachers "stand at overhead projectors just teaching the algorithm" and hoping students will do well on the standardized tests but that those students are not being taught how to think mathematically. She was confident that she could get her students to enjoy mathematics:

> Yes, every kid in my class, if you were to ask them now and you were to ask them on the last day of school, I think that they would tell you that they like math, and they feel good about math, and they can do math. ... If we’re just totally just thinking about like number sense or mathematical reasoning, they’re going to have success in some way in this classroom. They’re going to have a positive association with math in this classroom. They will. I guarantee it.

She went on to say that she wants her students to find mathematics to be fun and not scary, and if they do make a positive association with mathematics, they will, with effort, be able to become better at mathematics.

Rebecca, who now enjoys mathematics because she finds it to be meaningful and wants that for her students, explained how important she thinks enjoyment in mathematics is: “I think it’s *huge*.” She described the relationship between motivation and enjoyment when I asked how she got her students to enjoy mathematics:

> I don’t. I think they enjoy it because when they see that they have some power over their own thinking and can explain it, they feel motivated and confident and that makes them want to do more. The better they get, the more they want to do it.

In essence, Rebecca believes that many traits of productive disposition develop through allowing children to think, explain, and enjoy mathematics.

The teachers spoke of enjoyment and excitement in mathematics in relation to their students, but also when talking personally. Several talked about mathematics as a puzzle or a game and enjoyed trying to solve the puzzle. (More on this later in this chapter.) Amy said that she found several of the mathematical tasks I posed to be enjoyable. When I asked why she seemed to prefer the Consecutive-Sums Task, she replied, “Because there was—I just think that it’s really fun [laughs] because I think patterns are fun.” She talks to students and parents about how enjoyable finding patterns and understanding math can be for everyone. Her students record in their math journals any pattern they discover, and later they discuss why the pattern holds true, which Amy said is fun for the students. “This is kind of why I think math is a lot of fun, because this is—it’s showing and really understanding the way.”

Cassie also spoke about why she believed mathematics should be enjoyable, stating in her autobiography, “In a perfect world, people would see the pleasures of manipulating mathematical ideas.” When I asked her to explain, she replied, “I was thinking, in a perfect world, people would love math. Kids would; they would see it as—it wouldn't be such a horrible thing." She explained that her son who has strong mathematical skills dislikes mathematics and so avoids taking mathematics classes and makes little effort to solve difficult problems. She believes that too many people, like her son, see mathematics as "a bunch of equations and algorithms that we have to use." She noted that algorithms are important, but if the purpose of mathematics were only to learn the algorithms, then "it wouldn't be fun. You know, I can't have a lot of fun doing that." For Cassie, finding enjoyment in mathematics is essential for being productive in mathematics.

Belief That Everyone Can Do Mathematics

The definition of productive disposition from *Adding It* Up includes the phrase "see oneself as an effective learner and doer of mathematics" (NRC, 2001). Teachers relate this aspect of productive disposition to their students instead of themselves. The 10 teachers in this study said that they believed all their students could learn and do mathematics, although they admitted that understanding mathematics comes more easily for some children than others. When Louisa mentioned how amazing some of her students were mathematically and how easy numbers seemed to them, I asked whether for some students perhaps mathematics is just too difficult. She replied, "No, probably not, not unless they are just disinterested in everything." She described a former student who had difficulty counting and making one-to-one connections with numbers and objects, but even this student was able to build stronger mathematical skills though through a slow process.

Both Sarah and Rebecca had similar stories about former students who had real difficulties understanding mathematics. Sarah's student initially had extreme difficulty explaining his thinking, but now she often brings his work to workshops to share his unique mathematical strategies. She anticipated my question about whether all students could be reached.

> I: But I didn't know if you thought there may be some people who ....
> S: That never could do it?
> I: Right.
> S: Yeah, no, totally not. I think anybody can learn, whether it's math or whatever. I just think for some it may be a little easier than others.

Like Sarah, Rebecca described a former student, one she said did not try to understand mathematical problems but immediately sought help and did no work if she did not receive the help. Rebecca believes that this student can eventually be stronger in mathematics:

> R: Given time, yes.
> I: ... that she could even...?
> R: Yes.
> I: She's not unreachable.
> R: Yes. Yes, yes.

Even though both Sarah and Rebecca had experiences with students who had difficulties in mathematics, they believed that every student could become a learner and doer of mathematics.

Cassie thought that not only her students but also their parents could come to better understand mathematics. She seeks out students who say that they hate mathematics or that they are bad in mathematics and helps them to see mathematics more as a puzzle or a game, to enjoy finding patterns, and to realize that mathematics is not simply memorizing equations and algorithms. Regarding parents, she recounted the story of a mother having a difficult time helping her daughter with her homework:

> [The student's] mother called me or e-mailed me and said, "I have no idea. I can't help her with her math." And I said, "Well, you're welcome, you know, to come in the morning. I'll be teaching this." So she brought—she made her husband take off of work; he's a Marine. So we have this big Marine in there and the mother, and they are just amazed, and they got it, and they loved it.

She sometimes wishes that she taught adults instead of children so that the adults could see that they too could become more proficient in mathematics.

Gina credits her professional development in Cognitively Guided Instruction (CGI) for helping form her belief that all students can learn mathematics. Gina discussed students who were "less mathy/sciency" and whether one could help these students become more interested in and better at mathematics and science:

> Isn't that the goal of CGI in a way? I kind of think so because I've taken some kids who thought they were really bad at math, and I've given them at least the step into the world, and while they are not going to become mathematicians as their job, at least they're not afraid of it. It definitely is part of our goal is to make it just more accessible and understandable to everybody.

She added the caveat that getting students to see themselves as learners and doers of mathematics is much easier if you start with younger students than if you wait until they are older.

Beth mentioned several times in our interview that she whole-heartedly believes that all students can learn and do mathematics. Moreover, she gets very upset with parents and other teachers who believe that certain students do not already know some good mathematics and are not able to learn even more.

> So for people who—that say, “Well, my Rebecca from last year, she doesn’t know math,” [I say], "No, no, no. Sit with her. Talk to her. She knows math." ... So I do believe that anybody—I never encountered a child that when set up appropriately has not been able to do math. So I think every kid, my perception is they can all do math.

She added that all students should also be shown that mathematics is enjoyable, exciting, and doable, and that she works hard to ensure that mathematics is not scary for her students. She concluded our interview by quite vehemently saying, “I have never felt that math was anything but a positive thing for my students, ever in my 18 years. I have never felt that, anything different, ever. Yeah, and I will stand by that.”

Parting Thoughts

I have given examples of traits from my list of productive-disposition indicators that the teachers discussed during our individual interviews—their views of mathematics as sense making, the importance of interest in a mathematical task, their motivation to complete mathematical tasks, and their persistence on mathematical problems. These are the traits the teachers discussed most of their own volition. Note that, so I would not bias the teachers, I avoided all mention of productive disposition or my list of potential indicators before or during the interviews. I described the purpose and subject of my study only after the interviews had concluded,

In the next section, I discuss three findings from the interview data that relate to productive disposition but that were not among my indicators. These findings may change the way I, and others, research and assess for the productive disposition of teachers in the future.

Frustration, Confidence, and Puzzles

To conclude this chapter, I present three findings from Study 2 that I found to be somewhat surprising. Through my interviews with the 10 teachers who showed evidence of holding strong productive dispositions, I learned, first, that they find frustration with a mathematical problem to be highly motivating. Second, their confidence in learning mathematics themselves differs greatly from their confidence in teaching mathematics to their students. Third, all the teachers viewed solving mathematical problems as working on a puzzle, playing a game, or solving a mystery. Because these findings about teachers with strong productive disposition are unexpected, I find them to be noteworthy. They may be characteristics of teachers with strong productive dispositions generally but not of other populations of mathematics learners and doers. I expect these results to change my further research on teachers' productive dispositions.

Frustration

Two productive-disposition indicators relate to frustration: *does not avoid frustration* and *interprets frustration, when experienced, as a natural component of problem solving*. However, the teachers often explained that frustration on a mathematical task motivated them to complete the task or at least to continue to try and make progress. Moreover, an unexpected finding was that the teachers differentiated two kinds of frustration, referred to by Amy as "frustrated good" and "frustrated bad." For many of these teachers, the good kind of frustration was both enjoyable and motivating.

Several teachers explained that frustration motivated them on the tasks I had posed. Elle worked for several minutes during the interview attempting to explain why Savannah’s subtraction strategy yields an answer off by 2. When I asked why she chose to persist on this task, she noted that she felt "stumped and frustrated," but she was smiling the entire time. In explaining how she could be both frustrated and smiling, she said, “Because it’s a mental challenge.” She went on to explain that she loves mental challenges, especially trying to challenge herself to solve problems in novel ways. In spite of her difficulty in solving the problem, she found the frustration motivating.

Similarly, Hillary felt frustrated on the Savannah Task, and like Elle, she was motivated by this frustration to continue her work on the problem. Later, in a seeming contradiction, Hillary also admitted that she often will not attempt mathematical tasks that she feels are frustrating. She then noted that she was using frustration in different ways. She attempted to clarify her distinction:

> With the Savannah thing, I felt like, "Okay, I’m seeing what’s happening here; I just can’t get my brain totally around it yet." But I didn’t feel like "I’m not going to be able to figure this out," whereas a problem that I don’t have any access to, even for—"I don’t even know how to start doing that," um, would be like be a totally different kind of frustrating because I wouldn’t even want to try it. So maybe frustrating is not the right word in that instance. Maybe the word should be, um, just *put off*. I mean, because I’m not even going to try that because I can’t even begin to see how to do it.

She went on to comment that she likes mathematical tasks about which she feels frustrated as long as she also feels that she can make progress on the task.

The task that caused the most frustration for the participants was the Pat Task. Sarah, Beth, and Elle each talked about wanting to truly understand Pat’s strategy. They noted that Pat’s strategy was challenging, which in turn got them frustrated. When asked about this frustration, Elle groaned, Beth said it “drove me nuts,” and Sarah repeatedly said that she "really wanted to figure it out." This frustration motivated these three teachers to continue their work on the Pat Task, but none could explain why they did not just give up—apart from noting that they knew there was more to understand. Mary and Amy, however, described this task as simultaneously enjoyable and frustrating. Mary wrote in her reflection that this task was frustrating at first but then enjoyable once she understood and explained this apparent dichotomy during the interview:

> Well, initially I’m looking at the problem and I have no idea what he’s done, and I kind of feel stuck and then I’m kind of defeated, like, "What do I do?" And then because it was such a challenge and I figured it out, I solved it and I was feeling good about myself and good about the strategy he has used, good about his problem.

Similarly, Amy noted that this task was “both interesting/frustrating” on her reflection. During the interview she elaborated on that comment:

> I think on one level I can feel my brain kind of getting it, but I couldn’t really articulate what he was doing, so that’s what was frustrating. But it was enjoyable because grappling with how he was doing it was really fun. ... I really wanted to keep working on it. I really wanted to because I really wanted to figure out what—I wanted to crack his code.

Mary and Amy (and perhaps other teachers as well), felt frustrated while trying to understand Pat’s mathematical strategy, but they knew that if they could make sense of it, they would enjoy the accomplishment.

When the teachers talked about the frustration they felt on the tasks, I often noted that the frustration they were describing seemed different from the frustration one might experience in a traffic jam and asked whether they could articulate the difference. I found their replies interesting. Beth believed that being frustrated on mathematical tasks like the ones I had given her was like a boxer being knocked down to the canvas. She wondered how many times she would allow herself to get knocked down before she would just lie there, but then she reminded herself that she could not win if she did not get up.

Amy referred to the frustration during problem solving as “frustrated good,” as opposed to “frustrated bad.” To describe “frustrated good,” she said, “It was that state of, you know, before you have the aha moment, that disequilibrium, yeah. Like I was on the verge of getting it and I *really* wanted to get it.” She said that she felt the good kind of frustration whenever she was being intellectually challenged.

Sarah referred to this kind of frustration as “that little thing that like kind of bugs me,” which she feels when she thinks that she can figure out something more. Although it bothers her greatly, she "kind of likes" that kind of frustration and now sees it as a natural part of doing mathematics. Moreover, as mentioned previously, she wants her students to feel frustrated and struggle on mathematical tasks, at least up to a point:

> I don’t want to put them to the point where they’re, um, feeling bad about themselves, but I think it’s good. I mean, it’s like a lot of times you give them a problem and they want you to tell them the answer or something and it’s like, "Well, I want to see, you know, what you think about it or how you would do it" and stuff. So I think in that sense, yes, I try to get them to, to struggle enough to try it and, you know, "What would you do first?" you know. "And after you did that, then what would you do?"

Sarah did not limit her students' problems to problems they could solve quickly because she believed that they would not be learning from that experience. She saw struggling on thoughtful mathematical tasks in a positive light, and she wanted to promote that positive view of struggling with her students as well.

Amy also wanted her students to view frustration in a positive light. She commented that students are generally not given sufficient time to work on problems to really understand the mathematics and feel comfortable solving problems. Although she does not want to distress her students, she does want them to experience good frustration. She explained how she conveys to her students that frustration on mathematical problems is good:

> Now good frustration, I do say to kids that when you’re learning, you will get frustrated because you do have to have that; that’s part of the learning process. You feel frustrated, and kind of there’s that sense of disequilibrium because you have your old information and you’re getting new information and it kind of—that’s part of learning and we talk about that.

Amy went on to say that struggling and frustration are normal parts of problem solving, but fear is not. She wants her students to work through their frustration, not simply try to avoid it.

In her autobiography, Elle made a similar comment about wanting her students to struggle—that teachers need to allow children to “have a bit of a struggle with figuring things out, because I think that’s where it is.” At my request she explained what *it* is and why she thought that letting students struggle was important:

> They’re learning. That’s where the learning is, and I want kids to have that aha moment. You know, I think when you struggle, just as I’m struggling with this or I struggled over here, it’s making me learn more about something that I would not have even thought of a way of solving the problem. So when I let kids struggle, you know, I don’t want to give them the answers ever, you know, right away, or give them a suggestion right away, like, "Try this." I’ll let them struggle, you know; I’ll suggest they use manipulatives or do this, but I do feel that that’s where their learning is. That’s where they’re going to have their aha moment, so they can then transfer it into other problem solving and grow in their strategies.

Elle described encouraging her own students to willingly struggle with mathematical tasks. Although it was a difficult process for some students, eventually they all came to see the benefit of persisting on frustrating mathematical tasks. She concluded, “So I think they need to—we all need to struggle a little bit. I think it teaches us a lesson [laughs].”

The 10 teachers saw frustration as a natural part of the problem-solving process, but more interesting is that they saw frustration in certain circumstances, such as problem solving, as positive, not just negative (or as Amy said, “frustrated good” and “frustrated bad”). As shown above, several of the teachers liked working on *frustrating* mathematical tasks for their sense of enjoyment in completing those tasks. Others wanted their students to experience this frustration and come to associate it with the normal process of solving mathematical problems and not as something to fear or avoid. Perhaps *frustration*, with its negative connotations, is a poor choice of words, however, neither the teachers nor I found an appropriate term for this positive kind of frustration apart from the aforementioned *good frustration.*

I believe this distinction in types of frustration is noteworthy for two reasons. First, as a researcher, I must clearly define the constructs under investigation. In this case, I intended to look for indictors of participants' frustration. Initially, I did not consider whether this frustration might be a motivation or an impediment to engaging with mathematical tasks. As such, I oversimplified the construct and might have looked for the participants' avoiding frustration and not for their using frustration productively. Fortunately, these teachers distinguished clearly these two types of frustration. Second, because these teachers were the ones who highlighted the existence of “frustrated good” and “frustrated bad,” this distinction is shown to be important to teachers, not just to researchers. By raising the distinction when working with teachers, one might help them come to associate “frustrated good” with the normal process of solving mathematical problems, not something for their students to avoid.

Confidence

As with frustration, I had two productive disposition indicators on my list related to confidence: *seems confident in one’s own abilities and skills for solving tasks* and *seems confident in one’s knowledge*. Through the interviews, I realized that *confident in one’s knowledge* is too broad as a category and that the teachers’ confidence in their knowledge for learning and doing mathematics differ from their confidence in their knowledge for teaching mathematics. Only Mary professed robust confidence in her knowledge for both learning and teaching when I asked how confident she was in mathematics:

> I would say I'm pretty confident teaching mathematics, um, and I get it. I get what I need to, I get how to talk with students about strategies. I can get, if I sit and think and work at it, I can get their strategies and see if they're making sense or what their next steps needs to be.

I then asked her whether her confidence in learning mathematics for herself was similar to or different from her confidence in teaching mathematics. She replied, "It's similar. It seems like my mathematics world is just teaching." Mary explained that she wants her students to feel the same confidence in their abilities to learn and do mathematics as she feels.

The other 9 teachers expressed moderate confidence in their knowledge for teaching mathematics but much less confident in the knowledge for learning mathematics. For example, Gina first brought up confidence in relation to the Consecutive-Sums Task. In her written work, Gina first found that 4, 8, and 16 could not be written as consecutive sums, but she later erased 16 and added 13. When I told her that her initial answer was correct, she did not believe me. When I asked her why she had changed her answer, she simply replied, "I didn't have any confidence." Later she mentioned that she finds "funny" the fact that she is seen by some of her colleagues as being strong in mathematics, and she elaborated:

> It is funny. I'm the math expert. I'm like, "What? Are you kidding me?" ... I think there's this world of math understanding out there that has passed me, and I'm not good at that and it's most—I could be confident at that if I gave it time, I think, but I'm too far removed from it.

In comparing herself to another teacher considered a mathematics expert, Gina claimed that the other teacher had earned the title because she had earned her master's degree, whereas she, Gina, had not earned the title because she had only "attended all these meetings," her professional development experience. After further discussion of her confidence in learning mathematics, she noted, "You know, a lot of questions you're asking are about adult solutions, which is a different level of confidence for us." She went on to explain that she sees herself as a competent and successful teacher and that she wants all her students to have confidence that they can learn mathematics.

Louisa also made this distinction between her confidence in teaching mathematics and confidence in learning or doing mathematics. When I failed to make the distinction in the question I asked, she was quick to correct me:

> I: Do you feel now that you're confident in math?
> L: I feel confident in teaching math to young kids.
> I: So there's a difference here in learning/doing math for you versus teaching math to your students?
> L: Yes, I feel more confident now; say if I took a math class now, I think I'd be better at it, and I think I'd do better because I have more of an attitude, "Hey, I can figure this out; I can get it."

In spite of her prediction that she would probably be successful in a mathematics class now, Louisa did not see herself as being strong in understanding mathematics: "Maybe it's just about the way you think about yourself and the subject, you know? I don't think of myself as being a smart math person. Um, yeah, and it bugs me. You know, I can't do—algebra's hard." She told me that she did not feel confident in her solution to the Savannah Task and was unsure if she had really figured out anything of importance on that task. However, Louisa does see herself as a strong mathematics teacher at the elementary school level, and she is very confident in the teaching of her mathematics content and in understanding her students' strategies.

Rebecca also talked about the difference in her confidence in mathematics teaching versus her confidence in mathematics learning. While discussing the Pat Task, Rebecca commented that she was surprised that she was able to make sense of the task because she is not a "strong math student." I then ask her about her confidence in teaching.

> I: You said you don't think you're a strong math student. Do you think you're a strong math teacher?
> R: For the young kids?
> I: Yeah, for your kids.
> R: Yes, yes.
> I: That's a strange ....
> R: No, because I don't have to work with fractions, and I don't have to work with division problems that are too hard for me, and so on and so forth.

We then talked about her solution to the first part of the Pat Task. I told her that her explanation of Pat's strategy was one of the best that I had seen, so I was surprised when she wrote on her reflection that she believed that she had not explained his strategy very well. When I asked her to explain her view, she said, "I don't know, maybe just confidence." She then admitted that she has very low confidence when doing mathematics, but feels very confident in her teaching of mathematics. She explained the difference between these two: In teaching, she is the one to "control the variables" such as what problems to give, what questions to ask, and so on, whereas she is less confident in learning mathematics because “I always think maybe I’m doing stuff wrong.”

Although Gina, Mary, and Rebecca said that their confidence in teaching and in learning mathematics differed, in Beth this difference was more striking. She mentioned in her autobiography that she "never" saw herself as a strong mathematics student and that she "felt really embarrassed" because of her limited participation in the focus-group conversations and because she did not "do well on any of the tasks." I had drawn different conclusions in watching the video and reading her written work: Beth was fairly quiet on the Savannah Task but asked specific appropriate questions and had thoughtful insights on the other tasks. She disagreed with my assessment. When I showed her examples of her good thinking in her work on the tasks, she expressed her feeling that she had “just faked it.” However, in talking through her work with me, she was surprised by how much she really did understand. She commented that her self-perceived poor performance on all the tasks that day made her initially doubt her skills as a mathematics teacher because she saw her performance on the tasks as “a reflection of me as a teacher, that I couldn’t do, that I couldn’t help these children.” However, on the drive home after completing the tasks, Beth realized that her performance was not a reflection of her teaching abilities, and she quickly regained her confidence in her mathematics teaching. She characterized this confidence at the end of our interview:

> I think I’m better than most, as far as math teacher, but part of the reason I think I’m better than most is that I care deeply to be better. I really, really, really, really, really want these children to get a really good start in school.

Even though she does not feel confident in her ability to learn mathematics, she works hard to ensure that all her students feel confident that they can learn and do mathematics.

I was surprised by these teachers' lack of confidence in their knowledge for learning mathematics. I had watched them work on and discuss four fairly challenging mathematical tasks, and although my focus was on qualities of their productive dispositions, I was impressed by their mathematical content knowledge. Thus, to see that 9 of the 10 teachers viewed their knowledge differently than I did, especially because they had been selected because they had displayed strong productive dispositions, was startling. I attribute this difference to the difference in our definitions of *being good in mathematics.* The teachers are using a traditional definition: Someone who is good in mathematic has taken and done well in many upper-level classes in the subject and, perhaps, has attained a bachelor’s degree or higher. I believe that these teachers had strong mathematical knowledge for teaching, having thought deeply about the mathematical subjects they teach their students, such as operation, place value, and number sense. They are able to find alternative approaches and make connections between strategies. I conclude that confidence does play an important role in these teachers’ productive dispositions, but this confidence may be observed only if one assesses for the teachers’ confidence in mathematical knowledge for teaching.

Puzzles

On my list of potential indicators of strong productive dispositions is finding mathematics to be beautiful, useful, or worthwhile. I was surprised that all 10 teachers in Study 2 referred to seeing mathematics as a puzzle, a game, or a mystery. In fact, 9 of the 10 teachers used this terminology to describe mathematics in a positive light. The only exception was Rebecca, who does not like mathematical tasks that seem like puzzles to her. When I asked why she disliked such tasks, she commented, “It takes me long to figure things out, and I don’t like that. I’m not a puzzle person. I don’t—I’ve tried a Sudoku; I can’t do them. I don’t do well.” She stated that the Consecutive-Sums Task to her was "too much like solving a puzzle," and, therefore, she did not find it enjoyable. However, Rebecca went on to note that although she “can’t see patterns easily,” she was able to make sense of the Consecutive-Sums Task and actually had more patterns than most of the other teachers.

The other 9 teachers spontaneously mentioned enjoying the puzzlelike aspect of mathematics; it motivated them to persevere and solve problems. Amy, for example, explained her desire for more time alone to work on the Pat Task: “Because I really wanted to figure it out [laughs]. It’s like a puzzle to me.” She went on to note that she enjoys mathematical tasks that feel like puzzles. Elle gave a similar explanation for continuing to put forth effort on difficult mathematical tasks:

> Because it’s a mental challenge. This is a mental challenge that, you know, it’s like Sudoku [laughs]. You’re there and you’re like, “I know this number’s going to fit in here,” you know; or “it had got to fit, and that’s why,” you know?

She said that these types of mathematical tasks, which she found to be mental challenges, motivated her because they are both fun and frustrating. Like Elle, Gina believed that solving good mathematical tasks was like trying to “figure out puzzles,” whereas Sarah said that they were “more like a challenge, like a discovery, being a detective.”

Cassie mentioned that in the past she had audited mathematics classes at a local university and that she sometimes logs online as her son to "play around" with his mathematics and statistics homework problems. She explained her motivation: “I love it because it’s a puzzle. It’s like solving a problem.” When I later asked if she believes that mathematical ability is innate, she replied, “I think everybody can be good in math because math is so much—it’s like a game, you know? It’s like, as I said, patterns, puzzle.” She mentioned that she enjoyed the Consecutive-Sums Task because “it was like a real game.” She contrasted her husband's stance on mathematics with hers; he uses some mathematics in his career but does not enjoy it or find it as intriguing as she does.

Beth believed that understanding her students’ mathematical strategies was like trying to solve a Sudoku puzzle. In both, she knew there was a right answer and she refused to give up until she had solved the puzzle. She went on to clarify that this metaphor probably extends to all mathematical thinking, even that in calculus. Three times during our interview Beth likened understanding her students’ mathematical thinking to working Sudoku puzzles, and each time she talked about enjoying the puzzle-solving process. She also noted that novel mathematical problems seem “more like games to me." She appreciated the multiple ways one could play the game by using different mathematical strategies to solve the problems. Most of all, she believed that these game or puzzle qualities in mathematics problems motivated her to spend more time on them and try to finish them, even if they initially seemed rather difficult.

Louisa also found making sense of children's mathematical strategies to be like solving a puzzle:

> The challenge is kind of fun because it's kind of a puzzle, you know? It's always, "Ooh, can I, can I figure out what this kid was thinking? Can I find the thread?" You know, because usually if you look long or think about it enough, you can kind of figure it out.

She later noted that teaching mathematics is "kind of a game. How am I going to help this kid get to the next level? Picking numbers—that's a huge challenge and that's fun. You know, that's a game." In the past, Louisa found solving mathematical problems daunting, and she explained the change in her thinking:

> Now I look at math more like a puzzle; it's a game. It's not, it shouldn't be scary. It's just, you know, it's more of a game. I often think about, "Boy, I wish I could take"—maybe I should. It'd be good for my brain.

She had started to say that she might consider taking more college-level mathematics courses in the future just for her own enjoyment.

Last, several teachers commented that the view of mathematics as being like a game or a puzzle is abnormal relative to the views of their friends and colleagues. Amy noted that even though she is not a "big math person," she enjoys tasks that are puzzlelike or for which she needs to find patterns, causing her to think that others see her as a “different teacher” or “odd.” Hillary compared doing mathematics to solving the *New York Times* crossword puzzle in that she enjoys both and wants to complete both, but sometimes she has to walk away from both and come back to them later. Although her husband finds the way she solves problems and enjoys problem solving to be “weird,” she remarked, “I don’t care if he thinks it’s weird.” Mary referred to herself as a “dork” for enjoying mathematics as much as she does, whereas Gina thought that she sounded “goofy” when she talked about her excitement in watching her students solving mathematical puzzles. Beth mentioned several times that mathematics is like a puzzle or game to her but that this stance is different from that of many of the other teachers at her school.

Why is this finding important? First, I personally was surprised to hear this language from teachers, but I was especially astonished that *all* the teachers used *puzzle*, *game*, or *mystery* to describe mathematics. Moreover, all the teachers except Rebecca considered this puzzlelike aspect of mathematics to be positive. (Rebecca still saw mathematics in a positive light; she just believes that she is not a “puzzle person.”) Second, the teachers wanted their students to see mathematics as a puzzle or a game and hoped that students would find enjoyment in solving the puzzle of mathematical problems. Last, this view of mathematics as a puzzle, game, or mystery may not be common in people who do not have strong productive dispositions. None of my several friends who have described themselves as having low productive dispositions toward mathematics believe that mathematics is puzzlelike. This difference in attitude may partially account for the teachers' thinking that they are weird or odd for viewing mathematics this way. Although this evidence is from only 10 teachers, because all 10 used *puzzle* or *game* terminology, I consider it a noteworthy finding and a potential indicator of productive disposition to investigate in future research.

Summary

My goal for this study was to learn which productive-disposition indicators are evinced by teachers previously identified as having strong productive dispositions and to discover which indicators resonate with them personally. Thus, I asked 10 Study 1 teachers with strong productive dispositions to complete four tasks, write reflections on those tasks, complete a mathematical autobiography, and participate in an individual interview. I was not attempting to assess the individual teachers but sought commonalities across the teachers to learn more about the construct of productive disposition itself.

I analyzed the data to determine which traits of productive dispositions the participants discussed most often, focusing mainly on the interview data. The traits that were common among these 10 teachers are listed in Figure 4.2.

- Re-engagement and persistence with a task—unwillingness to abandon a mathematical problem if a satisfactory solution had not yet been found
- Belief that effort is worthwhile—a general belief that spending time and energy on a mathematics task would lead to better understanding or new insights
- Appreciation of multiple approaches—an awareness that one can make sense of a mathematics problem in several ways
- Academic risk taking—engaging in behavior that may reveal one's mistakes or may make one appear less competent than others in order to build knowledge
- Mathematical integrity—knowing what one knows, knowing what was does not know, and being honest about these assessments
- Belief that mathematics should be exciting and enjoyable—the belief that mathematics should be seen as interesting and mentally fulfilling, not as something to be feared
- Belief that everyone can do math—while understanding mathematics may come easier for some learners than others, this is a belief that with effort anyone could learn and do mathematics

Figure 4.2. Traits of productive dispositions most prominent in Study 2 interview data.

The teachers discussed other traits during the interviews, but the traits of productive disposition listed in Figure 4.2 are traits the teachers discussed to the greatest extent of their own volition.

Three other traits not included as such in my list of productive disposition indicators emerged as common among these teachers. First, the teachers identified two types of frustration: one that is highly motivating and another that impedes progress. Second, their confidence in learning mathematics for themselves differs greatly from their confidence in teaching mathematics to their students. Third, all the teachers considered solving mathematical problems similar to working a puzzle, playing a game, or solving a mystery. These findings about teachers with strong productive disposition were unexpected, so I found them particularly noteworthy. These three traits are perhaps indicative of teachers with strong productive dispositions more broadly, but further research is needed to determine whether they also indicate strong productive dispositions for other populations of mathematics learners and doers.

Section 4.4—Chapter Summary

To conclude, I return to my overarching goal for this study: to better understand the construct of productive disposition. To reach this understanding, I investigated the following research questions:

- What differences in evidence for one's productive disposition can be found through the analysis of teachers' active engagement in a mathematical task?
- What evidence for productive disposition is self-reported by teachers who have been identified as having strong productive dispositions?

Answering these questions helped me better understand the productive dispositions that support teachers in their engagement with mathematics. In this chapter, I reported the results of the two studies I conducted. In Study 1, I analyzed focus-group data from 136 preservice and in-service teachers who worked on the Savannah Task. I found differences in their indicators of holding strong productive dispositions. In Study 2, for 10 teachers identified as having strong productive dispositions, I looked for commonalities about their productive dispositions across the transcripts of my individual interviews of the teachers.

In the next chapter, I summarize the major findings from these two studies and discuss implications of the studies for research and for teaching. I also describe limitations of this study and, finally, consider potential avenues of future research related to productive disposition.

CHAPTER 5

CONCLUSION, IMPLICATIONS, AND FUTURE RESEARCH

To conclude my dissertation, I first highlight the major findings of my studies and relate them to the research questions. This section will serve as a summary and review of the previous chapter. In particular, I highlight the significant differences in the strong productive dispositions of two teachers. In the remainder of this chapter, I present the implications of this research and suggest possible future studies building from this research. I also discuss limitations of my research and avenues of potential future research on productive disposition. I end by discussing implications for teaching, including how my own practice has changed.

Section 5.1—Conclusion

To begin this chapter, I summarize and review the results of my two studies, presented in detail in chapter 4. To help the reader I highlight the major findings.

Study 1

My goal in Study 1 was to answer the question **What differences in evidence for one's productive disposition can be found through the analysis of teachers' actively engaging in a mathematical task?** To answer this question, I analyzed data from focus groups of teachers engaged with discussing the mathematics in the work of a student, Savannah (see Appendix A). The teachers were each assigned a score (0–4) on the basis of the evidence they showed for holding a strong productive disposition for mathematics.

Productive-Disposition Score

Of the 136 teachers in the study, 100 were in-service teachers and 36 were preservice teachers. Results showed that the majority of the preservice teachers showed little evidence of holding strong productive dispositions. More than 80% of the preservice teachers (29 of 36) were assigned a score of 0 or 1, indicating that they showed little to no evidence of holding a strong productive disposition while engaging with the Savannah Task. To be clear, I do not claim that the majority of preservice teachers do not hold strong productive dispositions. I simply found little evidence about the preservice teachers' productive dispositions on this task, perhaps because it did not resonate with the preservice teachers or because they did not know how to engage with a mathematical task focused on children's thinking. (I discuss additional issues of task selection later in this chapter.) Nonetheless, this finding was surprising because although these preservice teachers lacked experiences working with children before, I was disappointed that they did not engage with the mathematics of the task.

The in-service teachers were grouped by their years of professional development focused on children's mathematical thinking. Of the 100 in-service teachers in this study, 35 had yet to begin professional development, 31 had completed 2 years, and 34 had completed 4 or more years. These three groups differed little on their productive-disposition scores, either in the mean group scores or the spreads in their scores. One might interpret this result as showing that professional development focused on children's mathematical thinking has no effect on productive disposition. However, it may instead be that productive disposition changes *very* slowly. In the study were 8 teachers who had 7 or more years of professional development, and these teachers' average score was 3.38, well above the 1.88 average for all the in-service teachers. A sample size of 8 teachers is too small from which to draw concrete conclusions, but this finding might be a site for future studies about productive dispositions.

Productive-Disposition Indicators

The heart of this study was in determining which of the potential indicators for strong productive disposition emerged when the teachers were actively engaged with a mathematical task. The list of indicators was originally derived from the definition of productive disposition presented in *Adding It Up* (NRC, 2001) and from eight constructs from the literature that I believed to be connected to productive disposition, namely *affect*, *beliefs*, *identity*, *mathematical integrity*, *risk taking*, *goals*, *motivation*, and *self-efficacy*. The list of indicators was augmented and modified on the basis of evidence from the focus-group discussions to create the list in Table 4.6.

Table 4.6

Potential Indicators of Strong Productive Dispositions (Final Version)

Potential categories	Evidence
1. Mathematics as a sense-making endeavor	a. Tries to make sense of the task b. Considers alternative approaches c. Asks if answer seems logical d. Is troubled by inconsistencies
2. Mathematics as beautiful or useful and worthwhile	a. Shows interest in the task through engagement b. Shows interest in the task in comments about the task c. Shows a sense of wonder
3. Beliefs that one can, with appropriate effort, learn mathematics	a. Believes making progress on the task is doable b. Persists c. Does not avoid frustration
4. Mathematical habits of mind	a. Asks questions about the mathematics or about an approach (one's own or another's) b. Shows appreciation for one's solution c. Seeks and provides clarifications
5. Mathematical integrity and academic risk taking	a. Has a sense for when one has completed a task (whether or not one continues) b. Is willing to question one's self c. Is willing to offer tentative ideas d. Recognizes worthy and unworthy confusions
6. Positive goals and motivation	a. Defines progress as learning through grappling, not just getting an answer b. Shows pleasure or excitement about a particular way of reasoning c. Engages longer or willingly reengages with difficult tasks d. Interprets frustration, when experienced, as a natural component of problem solving and not as a statement of one's mathematical competence
7. Self-Efficacy	a. Seems confident in one's own abilities and skills for solving the task b. Seems confident in one's knowledge

Although each of these indicators was evinced by at least one teacher in this study, several of the indicators were identified in fewer than 10% of the teachers. These infrequently observed indicators may indicate a strong productive disposition but may not be evident in focus-group conversations. (In fact, in Study 2, some of these indicators were identified often during individual interviews.) The indicators identified in 10% or more of the teachers showed interesting trends. On the one hand, teachers who exhibited only a few indicators generally displayed the same three or four. On the other hand, some of the indicators were evident only in those teachers who exhibited many indicators of holding strong productive dispositions. These trends are diagramed in Figure 4.1.

Level 4
Believes making progress on the task is doable (3a)
Recognizes worthy and unworthy confusions (5d)

Level 3
Persists (3b)
Is willing to question one’s self (5b)
Engages longer or re-engages with difficult tasks (6c)

Level 2
Considers alternative approaches (1b)
Shows interest in the task through engagement (2a)

Level 1
Tries to make sense of the task (1a)
Asks questions about the mathematics/approach (4a)
Seeks and provides clarifications (4c)
Is willing to offer tentative ideas (5c)

Figure 4.1. Likely indicators based on productive-disposition score.

A tension in discussing these indicators is that the indicators were identified in the teachers’ comments about the Savannah Task. One view of these results might be that one can use Figure 4.1 to associate specific comments of the teachers to their levels of productive disposition, but this is not my intent. The teachers’ comments were assessed holistically to determine their productive-disposition scores. Scores were not based on specific types or a specific number of comments but rather were assigned on the basis of moves made in conjunction with aspects of the problem under discussion or comments of other group members. The diagram shows the indicators that emerged during teachers' work on the Savannah Task. A teacher assigned a score of 4 was more likely than others to show evidence of believing progress on the task was doable or recognizing worthy or unworthy confusions than those teachers who received a lower score, but these indicators were not required for a score of 4.

Indicators from teachers who evinced only a few were likely to be those in the lowest level in Figure 4.1, and when the number of indicators from a given teacher increased, indicators at higher levels were more likely to be evident *along* with those at the lower levels. I do not, however, conjecture that productive disposition is developed in this order. Still to consider is how professional developers who are trying to facilitate development of strong productive dispositions in teachers could use these findings, but professional developers are more likely to see the lower level indicators in those teachers with weaker productive dispositions and might initially focus on both strengthening those indicators and promoting other indicators to support growth. The professional developer may introduce indicators from the higher levels or note them when they are evinced by teachers. However, the indicators I have identified in the teachers' comments may reflect changes in their productive dispositions. In other words, I am unsure whether promoting the indicators would help to change the teachers' productive dispositions or if their indicators change only after their productive dispositions are strengthened. In fact, the indicators and the productive dispositions may change together. I caution, however, that I base these conjectures solely on the Study 1 data as sites for professional developers to initiate conversations about productive disposition.

Study 2

Five years after Study 1 and building on that study, I invited 10 teachers identified as having strong productive dispositions to participate in Study 2 by engaging with additional mathematical tasks, recording their mathematical autobiographies, and participating in individual interviews. My goal in Study 2 was to answer the research question **What evidence for productive disposition is self-reported by teachers who have been identified as having strong productive dispositions?** I designed this study to investigate whether specific indicators are characteristic of teachers with strong productive dispositions or if strong productive dispositions are manifested in multiple ways. In particular, I planned to explore the indicators (from my existing list or new) of productive disposition the teachers themselves discussed in our interviews about their work on the mathematical tasks and their comments in their autobiographies to gauge the indicators' relative importance to the teachers. My goal was not to determine which teachers had which traits but rather to identify, as a result of traits that surfaced during the interviews, indicators important for strong productive dispositions, at least for these 10 teachers.

Evidence Among the Ten Teachers

I focused on the interview data for this study for two reasons: first, to confirm or refute my interpretation of the indicators in the teachers' engagement in the tasks by eliciting their comments and reactions to stimulated recall of their work (on video) and to their reflections on that work, and, second, to enable the teachers to elaborate on aspects of their work on the tasks and comments in their autobiographies to provide further evidence of their productive dispositions. The prominent traits of productive disposition that emerged from these interviews follow:

- Re-engagement and persistence with a task
- Belief that effort is worthwhile
- Appreciation of multiple approaches
- Academic risk taking
- Mathematical integrity
- Importance of mathematics that is exciting and enjoyable
- Belief that everyone can be successful in mathematics

Again, the teachers did not use these terms; these are my interpretations of their comments and actions based on the research literature.

All 10 teachers consistently re-engaged with the mathematical tasks and continued to work on a given problem until they found what they believed to be a satisfactory answer, even when I offered the option to end the discussion. This persistence was perhaps due in part to their strong belief that putting effort forth on mathematical tasks is worthwhile. When discussing effort, many teachers said that they had initially found a problem difficult but found it easier and were able to make sense of the problem after giving it more thought. This effort included trying multiple approaches on a given task. Because of their work with children, the teachers seemed to appreciate multiple approaches. Several teachers commented that seeing children use different approaches to solve problems was initially surprising and intriguing; they are excited to share their own mathematical strategies or see the strategies that others have used.

Mathematical integrity is knowing what one knows, knowing what one does not know, and being honest about these assessments. Mathematical integrity is visible only when accompanied by *academic risk taking*, the willingness to ask questions or share ideas that may expose one's misconceptions or weaknesses. (For more about this relationship, see the Indicators section in Chapter 2.) Their stances on these constructs changed for many of the teachers in Study 2 as a result of their participation in professional development. Several Study 2 teachers commented that they now admit what they do not know and ask questions because they want their students to adopt these practices.

An important component of mathematics lessons for these teachers is ensuring that their students are excited to learn mathematics and that they enjoy working on mathematical problems. To enjoy mathematics, though, these teachers believe that students must believe that they can be successful in mathematics. The 10 teachers believe that all their students, by actively engaging with mathematical tasks, become increasingly successful in mathematics and enjoy mathematics more, and will, in turn, want to continue to learn more mathematics.

Frustration, Confidence, and Puzzles

Three surprising traits, not specifically on the list of productive-disposition indicators, were prominently displayed during my individual interviews of these 10 teachers. First, the teachers described two types of frustration, articulated as "good frustrations" and "bad frustrations"; solving a difficult but doable mathematical problem was, for them, a good (and often enjoyable) frustration. Moreover, they find the good kind of frustration on a mathematical problem to be highly motivating. Second, their confidence in learning mathematics for themselves differs greatly from their confidence in teaching mathematics to their students. Only 1 of the 10 teachers feels confident in both her teaching and learning of mathematics; the others are less confident in their learning of mathematics, which I found surprising inasmuch as all 10 were quite successful on the challenging mathematical tasks posed in this study. Third, all the teachers think of solving mathematical problems as doing a puzzle, playing a game, or solving a mystery. Although they enjoy this aspect of mathematics, they see this view as “weird” and different from their colleagues’ thinking. (For more detail on each of these three traits, see the section *Frustration, Confidence, and Puzzles* in the previous chapter.) These findings were unexpected and could lead to lines of further research about productive disposition.

Connecting the Two Studies

In both studies, I found that having a strong productive disposition does not take a single form. Although those teachers who showed less evidence of holding a strong productive disposition often displayed similar indicators, the teachers who displayed a large number of the indicators showed greater variation in their indicators. For example, consider the two teachers from Study 1 who evinced the greatest number of indicators. Although half of their indicators were common, their approaches to solving the Savannah Task were very different. One teacher seemed strong in her content knowledge, working by herself but offering tentative ideas. When she found a solution, she explained her thinking to the group and then re-explained until most of the members seemed to understand. However, in the end, she was not confident in her answer, even though it was correct. In contrast, the second teacher seemed weaker in her content knowledge, and although she had trouble getting started, she believed throughout that progress was doable. She constantly asked questions of her group members and refused to give up. At one point she pushed her paper away saying she was "done," only to quickly pull it back and continue working on the problem. Although she was confident in her progress and pushed the entire group to persist, she did not completely solve the problem. Each teacher used her own strengths in engaging with the Savannah Task.

Similarly, in Study 2, although all 10 teachers were previously found to have strong productive dispositions, the evidence varied greatly. The results for Study 2 in the previous chapter focused on the commonalities among the teachers,[11] but these teachers' productive dispositions differ. For example, Mary is confident in her mathematical content knowledge. She has felt successful in mathematics for most of her life and wants her students to experience similar success. As such, she enjoys challenging problems and is confident that, if given enough time, she can solve almost any mathematical problem based on K–12 content. Unlike Mary, Beth lacks confidence in her mathematical content knowledge. She believes, however, that she is a strong mathematics teacher and wants to ensure that her students are more comfortable solving mathematical problems than she was. Willing to expose

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[11] In Study 2, I discussed indicators common among the 10 participants but also ways the participants differed in displaying and discussing these indicators. Other indicators were evinced by these 10 teachers as well; I focus, however, not on the individual differences but on better understanding the construct of *productive disposition.*

mathematical weaknesses to others, Beth asks questions to better understand problems and shares her own insights. Although Mary and Beth differed in the indicators they exhibited, they both used their strengths to help them actively engage with the mathematical tasks. Simply put, the traits for holding strong productive disposition vary depending on the individual person's own mathematical strengths and weaknesses.

Section 5.2—Furthering Productive Disposition Research

As noted before, this research was a first attempt to better understand the construct of productive disposition. My hope is that other researchers will find these two studies helpful and will build from the results. In this section, I state implications of this research to both assess productive disposition and better understand the construct itself. I then note limitations of these two studies. I conclude with potential avenues for future research on productive disposition.

Implications for Assessing Productive Disposition

Although the assessments in these studies were in service of the creation and refinement of a list of productive-disposition indicators, implications for assessing for productive dispositions became evident. First, the choice and number of tasks are important. Use of a single task in Study 1 as a first attempt to assess for productive disposition gave the participants only one opportunity to show evidence for their productive dispositions. The use of multiple tasks provides a greater chance for participants to manifest their productive-disposition indicators. In Study 2, I devised some tasks in K–3 contexts and others more focused on the mathematics and less on children's thinking (perhaps a better choice for preservice teachers). Some tasks were more open-ended than others to allow for multiple answers and strategies. As was shown in Study 2, no one task was best for assessing. In fact, no one task was preferred or enjoyed most by the teachers, and their indicators differed across the four tasks. Perhaps the primary criteria are that the tasks should be challenging but doable and of interest to the participants.

Assessing for productive disposition through focus-group discussions depends on ensuring that the participants in the group work well together; that is, the choice of focus-group participants is important. I derived the most information about the teachers' productive dispositions from the focus-group tasks (along with the interviews for confirmation and elaboration); my only totally individual task, the Sam Task, yielded less information about productive disposition than the other tasks (but perhaps another individual task would be more useful). The number of participants in a group may affect the indicators seen. In large groups, some participants may have little chance to reveal their productive dispositions, and in groups with six (sometimes five) members, multiple simultaneous conversations may create assessment difficulty. In groups with too few participants, productive conversations are less likely to arise. In Study 1, focus groups with three members often had conversations that ended fairly quickly, making assessment difficult.

Rich conversations about the mathematics of a task are less likely to take place in focus groups consisting mainly of people with weak productive dispositions, whereas focus groups with too many strong participants may mask the indicators of those with moderate productive dispositions. For example, in one focus group, all four teachers seemed to have strong productive dispositions, but one of them said little because the others quickly explained and clarified all portions of the Savannah Task before she had a chance, or was ready, to speak. Although she did contribute to the group, she may have had a better opportunity to show more evidence in a group of teachers with differing levels of productive disposition. One may not, of course, know, even vaguely, the levels of participants' productive disposition in advance for the purpose of creating mixed groups.

Implications for the Construct of Productive Disposition

I believe that to develop better assessments for productive disposition, researchers will need to further study the construct itself. One outcome of this research is the creation of a list of productive-disposition indicators, which I believe describe the look of productive disposition being enacted by people engaged with a mathematical task. However, not all the indicators were seen with equal frequency. Moreover, some indicators resonated more than others with the teachers in Study 2. Figure 4.1 shows how some productive-disposition indicators can potentially build on others. Perhaps looking for all these indicators at once is a less useful way to research productive disposition than looking for the relationship between the indicators or examining a subset of the indicators in depth. For example, earlier I noted the relationship between mathematical integrity and academic risk taking. An interesting line of research might be to research how often these two traits appear together in mathematics learners or whether other indicators of productive disposition tend to accompany these two. The results from Studies 1 and 2 show which indicators may be most closely related to the construct of productive disposition, but more research is needed to better understand the interplay of the indicators.

I think that for any future research on productive disposition, whether it is assessment, watching its enactment in classrooms, or studying the construct further, the list of indicators, albeit incomplete, is a useful basis for the research. All the indicators were derived from the mathematics education literature and based on evidence from teachers engaging with challenging mathematical tasks. As such, I believe that this list may be a major contribution of this study to the field of mathematics education. My hope is that future research, including my own, will contribute to augmenting and refining this list so that we, as mathematics education researchers, will create a more robust definition for the construct and, in turn, find ways to identify and build strong productive dispositions in as many mathematics learners as possible.

Limitations

My overarching goal in this research was to better understand the construct of productive disposition. Because only K–3 teachers participated in the studies, the participants could think and discuss both as mathematics learners and teachers. As noted before, teachers were chosen for the studies because their dispositions greatly affect the dispositions of their students:

> The teacher of mathematics plays a critical role in encouraging students to maintain positive attitudes toward mathematics. How a teacher views mathematics and its learning affects the teacher's teaching practice, which ultimately affects not only what the students learn but how they view themselves as mathematics learners. (NRC, 2001, p. 132)

I designed the study to learn more about the productive dispositions of teachers, including ways to assess their dispositions. In doing so, I also learned about the teachers' views of their own students' dispositions. The teachers often reflected on their own thinking and compared or contrasted it with the thinking of one or more of their students. Because mathematics educators care about the productive dispositions of both students *and* teachers and because teachers' productive dispositions influence their students' productive dispositions, studying the teachers seemed important.

Although teachers' productive dispositions affect those of other learners, the study results may not generalize to other learners because these teachers differ from other populations of mathematics learners in several ways. For example, the language used by these teachers may differ from the language used by a population of college mathematics majors. Further, productive disposition is likely to look different in young learners and seasoned teachers. These teachers could reflect on both their own learning and the learning of their students. They had built their productive dispositions over the spans of their careers as students and teachers. Young students, meanwhile, are learning mathematical content for the first time, and so their engagement with mathematical problems is likely to differ from that of teachers. Although the list of productive disposition indicators may apply in part to young students, it may need to be altered for this population, or students' indicators may be subtler than those of the teachers. In either case, the list of indicators is at least a starting point for studying the productive dispositions of students.

Moreover, these teachers were special in that all of the in-service teachers in Study 1 had at least volunteered to begin professional development focused on children's mathematical thinking, and all teachers in Study 2 had completed at least 2 years of professional development, with some having more than 7 years of professional development experience. Because through this professional development, the teachers had opportunities to reflect on their own thinking and participate in conversations with other teachers about students' thinking, these teachers may be more likely than teachers generally to have extensively considered the mathematics taught in their classroom and ways to help all students learn mathematics. This stance might also lead them to have thought about their own productive dispositions or the productive dispositions of their students, even if they had never heard of the term itself before. Again, these teachers were chosen specifically because of these qualities so that I would have the best chance to work with teachers who truly hold strong productive dispositions. As such, these teachers were extremely helpful in my attaining the research goal of better understanding the construct of productive disposition, but their results may look significantly different from those of teachers more generally. Further study is needed to determine whether these results are applicable even for all teachers.

The finding that people are influenced by what they think others think or expect of them is a widely documented social science result (e.g., Blanck, 1993;

Rosenthal, 2002). This phenomenon is known as the *expectancy effect*: one person's expectation for another person's behavior helps to cause that behavior to be enacted. I acknowledge the role that the expectancy effect could have had in my study. In the case of the interviews in Study 2, the teachers could have been telling me what they thought I wanted to hear, perhaps even unconsciously. In both studies, the teachers were clearly concerned about doing well because they believed that their performance reflected on themselves as teachers. For example, Beth, during her interview in Study 2, said that she saw her performance on the tasks as "a reflection of me as a teacher, that I couldn't do, that I couldn't help these children." Beth seemed to have believed that proficient teachers should be able to complete the tasks, and since she had difficulty with the tasks, she felt less confident in her own teaching abilities. This concern about not appearing to be a proficient teacher could have affected the teachers consciously or unconsciously as they spoke with me in the interviews. While the teachers appeared to me to be forthright and honest in their responses in the interviews, there may have an expectancy effect. In other words, what the teachers said may have been affected by what they believed I expected from them as teachers. If so, then their evidence may not validly capture their true stance on learning and teaching mathematics.

Similarly, I may have experienced a researcher expectancy effect when analyzing the data for the teachers in Study 2. I had watched the teachers work on the mathematical tasks in both studies. I assessed their performance in the videos for productive disposition indicators. I read their task reflections and listened to their mathematical autobiographies. Most of all, I knew they had shown evidence of holding strong productive dispositions. This knowledge may have influenced the questions I asked to the teachers and the ways in which I interacted with them, which, in turn, may have influenced their responses to me. My expectations of their strong dispositions may have led the teachers to say things that supported my own expectations. Moreover, knowing that these teachers had been found to hold strong productive dispositions may have influenced what I took as evidence from the teachers from their interviews. As a researcher, I strive to be as impartial and unbiased as possible when interviewing the teachers and analyzing their data, but I have to acknowledge the role that the expectancy effect could have had in my study.

Last, because of its importance, I repeat once more that the scores of participants in Study 1 are indicative only of the evidence they showed in discussing the Savannah Task in their group. Although I believe that a participant who manifests multiple positive indicators of engagement on this task would likely show high levels in other mathematical engagement, I do not make the same claim about participants who show little to no evidence of engagement with the mathematics in the Savannah Task. Other influences, such as a sick child at home, stress due to work issues, or unhelpful focus-group members, may have affected a participant’s performance. Thus, one weakness of this study is that I may be underestimating the mathematical-engagement levels of some participants. Moreover, although I analyzed the participants individually, I recognize that others in the focus group may have influenced each individual. Again, the score for an individual is indicative only of the evidence shown of his or her productive disposition toward engaging in mathematics on the Savannah Task within the given focus group. I believe that Study 2 is a better assessment of the productive disposition of the teachers. I recommend that in future studies to assess for productive disposition researchers consider participant selection for each focus group and use multiple tasks on multiple days.

Future Research

This research was a first attempt to better understand the construct of productive disposition. I see several sites for further research productive for the field of mathematics education. First, I believe that the list of indicators is a particularly fertile starting point for future studies. I looked in depth in Study 2 for indicators in teachers who held strong productive disposition, whereas in future studies, researchers might delve into the productive-disposition indicators of teachers more generally or of students, either broadly, at specific grade levels, or engaged with specific kinds of tasks. In such research, the list of indicators can be refined.

Second, in Study 1, I assessed for differences in the productive dispositions of teachers. My hope is that future research will be built on this work and that researchers (and potentially teachers) will find ways to assess for the productive dispositions of students. My hypothesis is that many students will have productive-disposition indicators similar to those of teachers, but the evidence shown by a given student may be more subtle.

I used the list of indicators to guide my search for evidence of productive disposition in teachers actively engaging with a mathematical task, but this list might be extended in looking for evidence in a mathematics classroom. In this case, the focus would be not on assessing any one student but rather on identifying indicators evinced by the students (in small groups or whole group) and examining the teachers' efforts to build on such evidence. The researcher's goal would be to learn how the traits of productive disposition are developed and reinforced in the classroom. Furthermore, I believe that studying the effects of a teacher's productive disposition on the dispositions of the students would be fruitful. If my hypothesis that teachers with stronger productive dispositions are more likely to emphasize those traits in their students, even without necessarily having heard the term *productive disposition*, perhaps in the future, professional developers will focus on facilitating strong productive dispositions in teachers.

Last, although *productive disposition* in isolation is a little-researched construct, connection of productive disposition to the other four strands of mathematical proficiency (*conceptual understanding*, *procedural fluency*, *strategic competence*, and *adaptive reasoning*) is completely unresearched. The authors of *Adding It Up* emphasized that “the five strands are interwoven and interdependent in the development of proficiency in mathematics” (NRC, 2001, p. 116). I believe that an important research topic is how productive disposition interacts with the other strands of proficiency. The authors of *Adding It Up* seemed to believe that all five strands should grow simultaneously, but the strands may interrelate in such a way that some strands develop before others. For example, perhaps the other four strands are easier to develop after students have strong productive-disposition bases on which to build, or perhaps students need a degree of conceptual understanding and procedural fluency before their mathematical dispositions begin to change. I believe that research into how productive disposition might be interwoven with the other four strands of mathematical proficiency could potentially not only heighten awareness of this important strand but also lead to new ways to help students engage more productively in mathematics.

Section 5.3—Implications for Teaching

I conclude this chapter by briefly describing how this dissertation has changed my teaching practices. Promoting strong productive dispositions in students can be a challenging task, especially inasmuch as the term is unknown to many teachers. Even teachers who actively try to build productive dispositions in their students face many challenges. I, for example, feel somewhat hypocritical about giving my students “standard” exams. Although I do my best to assess for the four other strands of proficiency, I admit that I generally do not assess for their productive dispositions. I do account for their productive dispositions to some extent when I assign the class-participation portions of their final grades, but these efforts seem subjective and arbitrary. For promoting strong productive dispositions in my students, I believe that I am only "touching the surface" of what I can do.

Thankfully, several teachers with whom I have talked are considering how to promote productive dispositions in their students, even without having heard the term. Teachers make two common suggestions: Highlight and compliment a student's display of productive disposition, and emulate the traits you want to see in your students. One Study 2 teacher leapt out of her seat when I began to describe productive disposition after her interview. She ran across her classroom to share a sheet she gives her students. It lists what she referred to as scholarly characteristics:

- Scholars are curious. They ask thoughtful questions.
- Scholars spend time pondering ideas and problems.
- Scholars are risk-takers. They are willing to try something new and difficult.
- Scholars exercise their talents by trying challenging tasks.
- Scholars consider themselves one-half full. They exercise academic humility by understanding that they have so much more to learn.

She said that she was pleased to learn that she had been promoting something researchers cared about, because she considers developing these traits in all her students important. (She tries to promote these traits in all subjects, but she thinks that these traits are particularly important for learning mathematics.)

Researching productive disposition and talking with these teachers caused me to reflect on the question *What can I do in my classes to help promote the productive dispositions of my students?* Because of this reflection, I have begun to change my practice in both significant and subtle ways. For example, I now start each class with what my students call "The Cool Math Thing of the Day," which varies from a clever mathematical proof they may not have seen (like the proof that there are infinitely primes) to an interesting open problem in mathematics (like the Biham-Middleton-Levine traffic model) to a counterintuitive mathematics problem (like the Monty Hall probability problem). Although I devote only about 5 minutes of class to this activity, the students have become more interested in mathematics and see that mathematics can be exciting, surprising, and beautiful at the same time. Further, they see that mathematics is more than what they see in the classroom. In fact, some students have brought in their own “cool math thing” (such as why *e* is an important mathematical number) or have asked questions about mathematics from the news (such as why mathematics shows that the Higgs boson should exist).

I also want my students to realize that they can solve seemingly difficult mathematical problems themselves. Instead of rushing to ensure that each student's question is answered by the end of class, as before, I may now purposely leave some questions unanswered or ask the students to think about a question I pose at the end of class. For several such problems, when I did not ask for the students’ solutions, they let me know that they had solved them and wanted to share their strategies. I am working to engage more of my students in thinking about these open questions, but students who do engage seem to be strengthening their belief that with effort they can solve even difficult mathematical problems.

Last, I am attempting to shift my focus to effort more than right answers. For example, I pose problems for which I tell the students the correct answer at the outset, which shifts the focus from getting a right answer to finding an appropriate strategy. To engage the students in the practice of doing mathematics more than finding the right answer to a mathematics problem, I devote class time to discussing the mathematics underlying a strategy or how a strategy might be altered so that it is more efficient or would be applicable in more mathematical cases.

As was noted before, promoting strong productive dispositions is difficult, but important. Additional research is needed to determine effective ways to develop productive dispositions in students. I hope that researchers will first examine what teachers are already doing to help promote these traits in their own classrooms. For now, I will continue to promote strong productive disposition in my students in ways I think will be effective for them and hope that future research on productive disposition will help me accomplish this goal with even more students.

APPENDICES

Appendix A: Study 1 Protocol

Before introducing the student work, share the following background information:

[Italicized words are to be read aloud.] *Next I'll be giving you a set of student work that Savannah, a third grader, completed in May. After you look over Savannah's work, we would like for you to start your conversation by discussing what you find noteworthy in this work, but feel free to take the conversation in any direction you choose. Some tools—the hundreds charts, unifix cubes, and base-ten blocks—are available, and you are welcome to use them. When you have completed your discussion, I will ask a few questions. I'll wait to ask these questions until you tell me that you are ready to move on.*

[Hand out the student work.]

[If the participants have not started discussing the student work after 5 minutes, read the following:]
Does anyone need more time? I want to remind you that I won't be saying much, but I do have some questions at the end, so let me know when you feel as though you have completed your conversation. I also want to remind you of the initial question: What do you find noteworthy in Savannah's work? You are welcome to take the conversation in other directions as you see fit.

After the teachers feel as though they have finished talking about the student work (or 15 minutes ± 5 minutes), say the following*:*
Is there anything else anyone would like to add before we go on? I have several questions for you. You may feel as though you have already answered some of these questions. If so, you may tell me that you have already addressed the question, and we will move on.

1) *Pretend that you are Savannah's teacher. What problem might you pose next to Savannah? Why would you choose that problem?* [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

2) *Savannah's approach for addition was correct. Explain why.* [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

3) *Savannah's approach for subtraction was incorrect. Explain why*. [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

4) *Do you have any thoughts about why Savannah's answer for the subtraction problem is off by 2?* [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

5) *Those are all the questions we have for you. Is there anything else you want to say about Savannah's work?*

We'd like to end this task by reminding you that while using this task with other teachers, we have found that it differs from what many teachers have considered previously. Thank you for being so thoughtful in analyzing and discussing Savannah's work!

[Note. Do not collect the student work. Participants need access to the work during the group discussion.]

Name ______________________________
Date ____________________

Savannah's Student-Work Discussion

Savannah, a 3rd grader, completed the work on the following three problems in May.

After you look over Savannah's written work, please start your conversation by considering this question:

What do you find noteworthy in this student's work?

You may write on these pages if you wish.

$$38 + 19 = \square$$

38 + 2 → 40

19 + 1 → 20

40 + 20 = 60

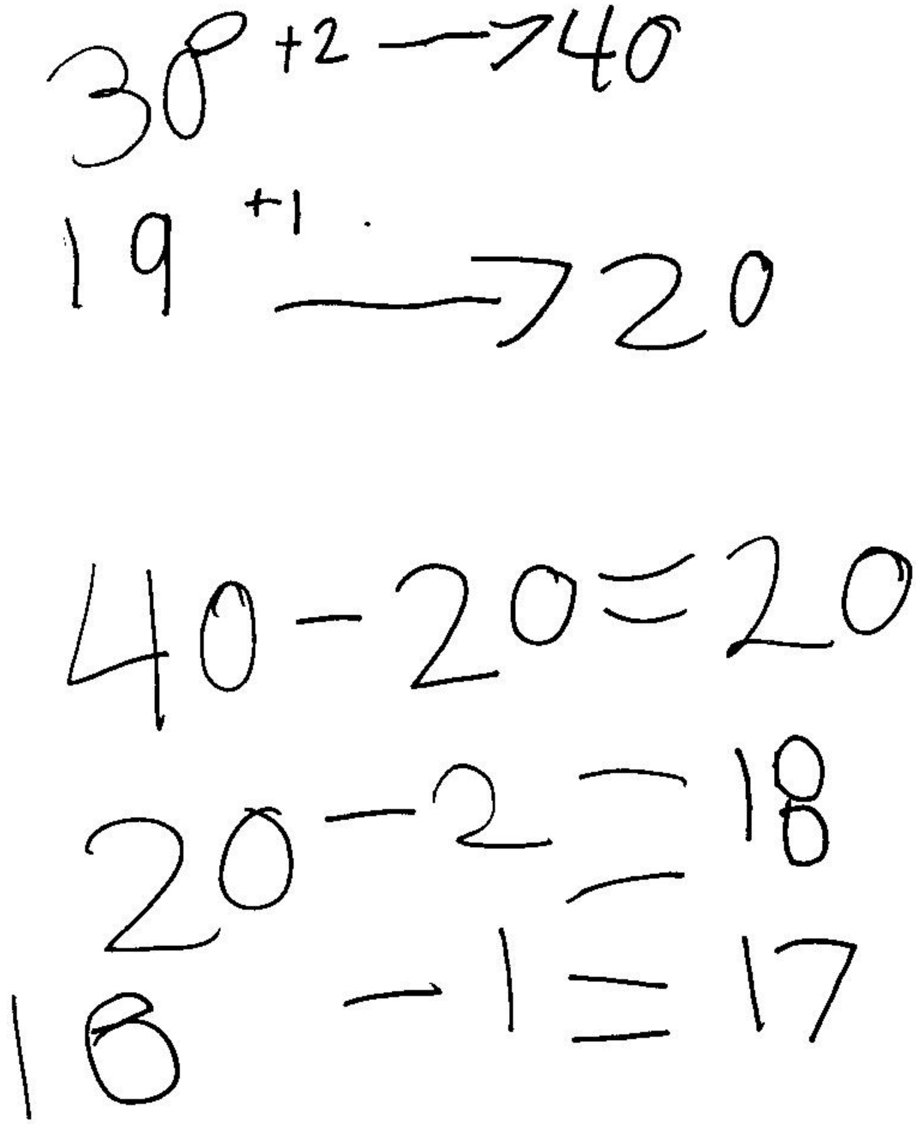
60 - 2 = 58

58 - 1 = 57

She explained her work as follows:

> "I added 2 to the 38, and I added 1 to the 19. I got 40 + 20, which is 60. Then I subtracted the 2 and the 1 that I had added. The answer is 57."

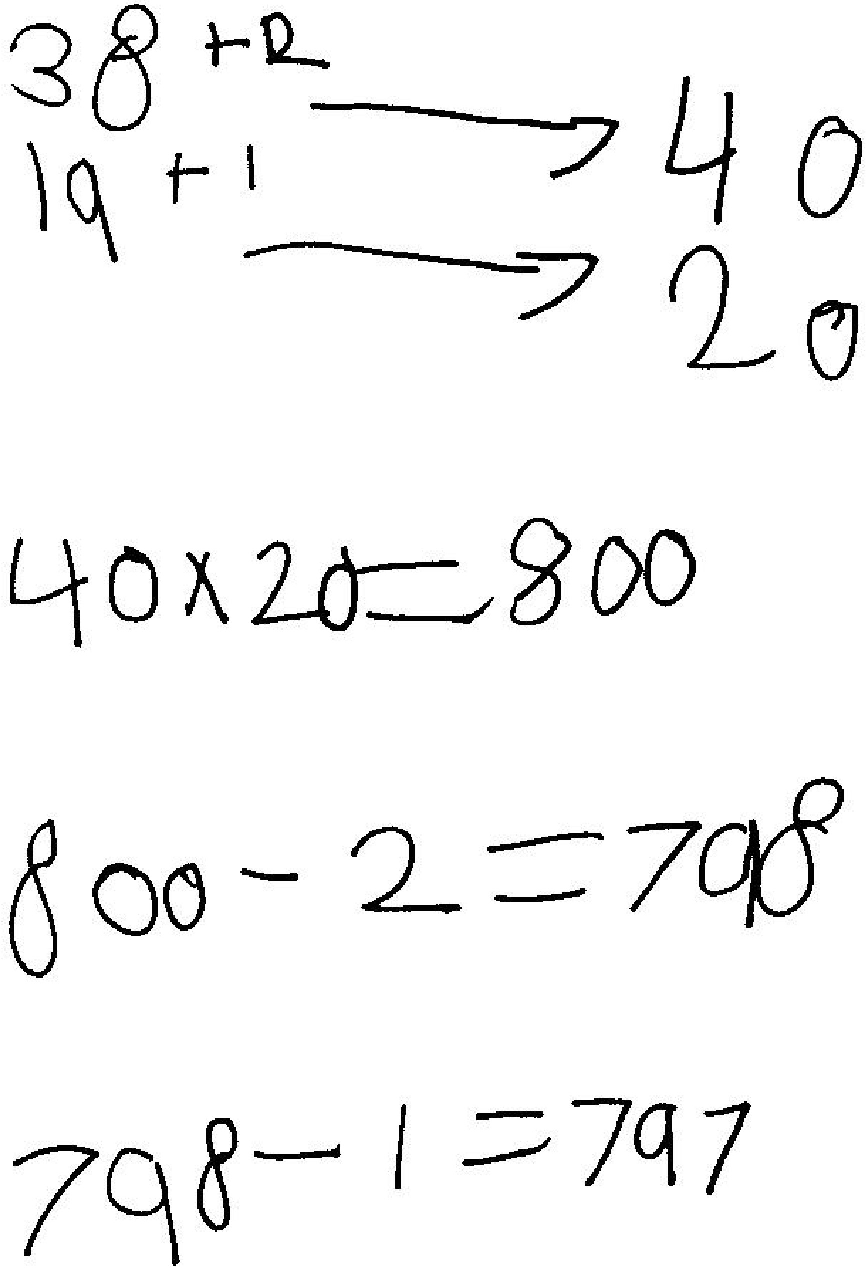
$$38 - 19 = \square$$



She explained her work as follows:

"I added 2 to the 38, and I added 1 to the 19. I got 40 – 20, which is 20. Then I subtracted the 2 and the 1 that I had added. The answer is 17."

$$38 \times 19 = \square$$



She explained her work as follows:

> "I added 2 to the 38, and I added 1 to the 19. I got 40 x 20, which is 800. Then I subtracted the 2 and the 1 that I had added. The answer is 797."

Appendix B: Study 2 Protocol

Introduction
(Agenda included at end of this document)

[Check for nametags; introduce yourself.]

Hi, my name is John (Zig) Siegfried, and I am a graduate student working on my doctorate in mathematics education. I met several of you a few years ago when you participated in the STEP project with Dr. Randy Philipp and Dr. Vicki Jacobs here at San Diego State. For part of my dissertation, I am doing small follow-up study with some of the participants of the previous STEP project, which is what you have volunteered to help me do today. Thank you so much for participating in this study. I could not do this work and complete my dissertation without your participation, and I again thank you for your efforts!

Today you will respond to a variety of tasks. Nothing you say or do will be shared with any administrators. In fact, because this is research, the university has specific rules that prevent me from sharing data about individual teachers with administrators. Furthermore, this is not an evaluative study, and I am not trying to grade you or label you. My focus is not on how *individual* teachers respond. Instead, I am interested in the way teachers look at their worlds. These are not timed tasks, and although I will keep you busy today, I want each of you to feel that you may take the time you need to respond to each question.

The mathematical *concepts* contained in all the tasks you will see today were designed to be accessible to K–3 teachers. Although all the tasks are related to mathematics teaching and learning, some may feel more mathematical to you whereas others may feel more teaching related. Some of the tasks I have selected may be ones that most teachers have not seen before, so do the best you can, without worrying if you are not happy with all your responses. A deep understanding of issues related to mathematics teaching and learning is complex, and I am not looking for single right answers! **<u>I really want to understand your perspectives when you engage in these tasks.</u>**

So, what specifically will happen today? I will give you an agenda that will provide you a sense of what you will be doing today. *(Hand out agenda. Describe rest-rooms locations when mentioning the breaks listed in the agenda.)* You will be given four paper-and-pencil tasks, one at a time. Some of these tasks you will work on individually, while others you will work on as a group. After each task you will complete a short reflection. Last, you will complete a mathematical autobiography.

After completing the last task today, please plan to talk for a few minutes with me. During that time you will have a chance to share your thoughts about the session and the process. Any questions?

I hope that you find the work today and during the follow-up interview to be thought provoking and enjoyable. Again, I want to thank you, in advance, for your efforts!

SESSION AGENDA

- Introduction
- Savannah Task
- Pat Task

BREAK

- Consecutive-Sums Task
- Sam Task

STRETCH BREAK

- Mathematical Autobiographies
- Short One-on-One Debriefing Session

Scheduling for one-hour follow-up interview

I will be following up on this session with a one hour one-on-one interview with you at a place and time of your choosing. The only requirement is that the place be relatively quiet so that I can audiorecord our session. Below, please list the potential date, time, and place for this follow-up interview, along with a back-up date. (You can hold on to this form for now. I will talk with you about schedule at the end of today's session.)

Primary Date ______________________ Primary Time ______________________

Location ____

Secondary Date ____________________ Secondary Time ____________________

Location ____

If something comes up and the date, time, and/or location need to be changed, please contact me as soon as possible. (If something occurs on the day we are scheduled to meet, please call me directly.)

Protocol for Savannah's Student-Work Discussion
Before introducing the student work, share the following background information:

First I'll be giving you a set of student work that Savannah, a third grader, completed in May. After you look over Savannah's work, I would like for you to start your conversation by discussing what you find noteworthy in this work, but feel free to take the conversation in any direction you choose. Some tools—the hundreds charts, unifix cubes, and base-ten blocks—are available, and you are welcome to use them. When you have completed your discussion, I will ask a few questions. I'll wait to ask these questions until you tell me that you are ready to move on.

[Hand out the student work.]

[If the participants have not started discussing the student work after 5 minutes, read the following:] *Does anyone need more time? I want to remind you that I won't be saying much, but I do have some questions at the end, so let me know when you feel as though you have completed your conversation. I also want to remind you of the initial question: What do you find noteworthy in Savannah's work? You are welcome to take the conversation in other directions as you see fit.*

After the teachers feel as though they have finished talking about the student work (or 15 minutes ± 5 minutes), say the following:
Is there anything else anyone would like to add before we go on? I have several questions for you. You may feel as though you have already answered some of these questions. If so, you may tell me that you have already addressed the question, and we will move on.

1) *Pretend that you are Savannah's teacher. What problem might you pose* next to *Savannah? Why would you choose that problem?* [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

2) *Savannah's approach for addition was correct. Explain why.* [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

3) *Savannah's approach for subtraction was incorrect. Explain why*. [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

4) *Do you have any thoughts about why Savannah's answer for the subtraction problem is off by 2?* [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

5) *Those are all the questions we have for you. Is there anything else you want to say about Savannah's work?*

I'd like to end this task by reminding you that while using this task with other teachers, I have found that it differs from what many teachers have considered previously. Thank you for being so thoughtful in analyzing and discussing Savannah's work!

[Collect the student work.]

Name ___________________________
Date _______________

Savannah's Student-Work Discussion

Savannah, a 3rd grader, completed the work on the following three problems in May.

After you look over Savannah's written work, please start your conversation by considering this question:

What do you find noteworthy in this student's work?

You may write on these pages if you wish.

$$38 + 19 = \square$$

$38 + 2 \rightarrow 40$

$19 + 1 \rightarrow 20$

$40 + 20 = 60$

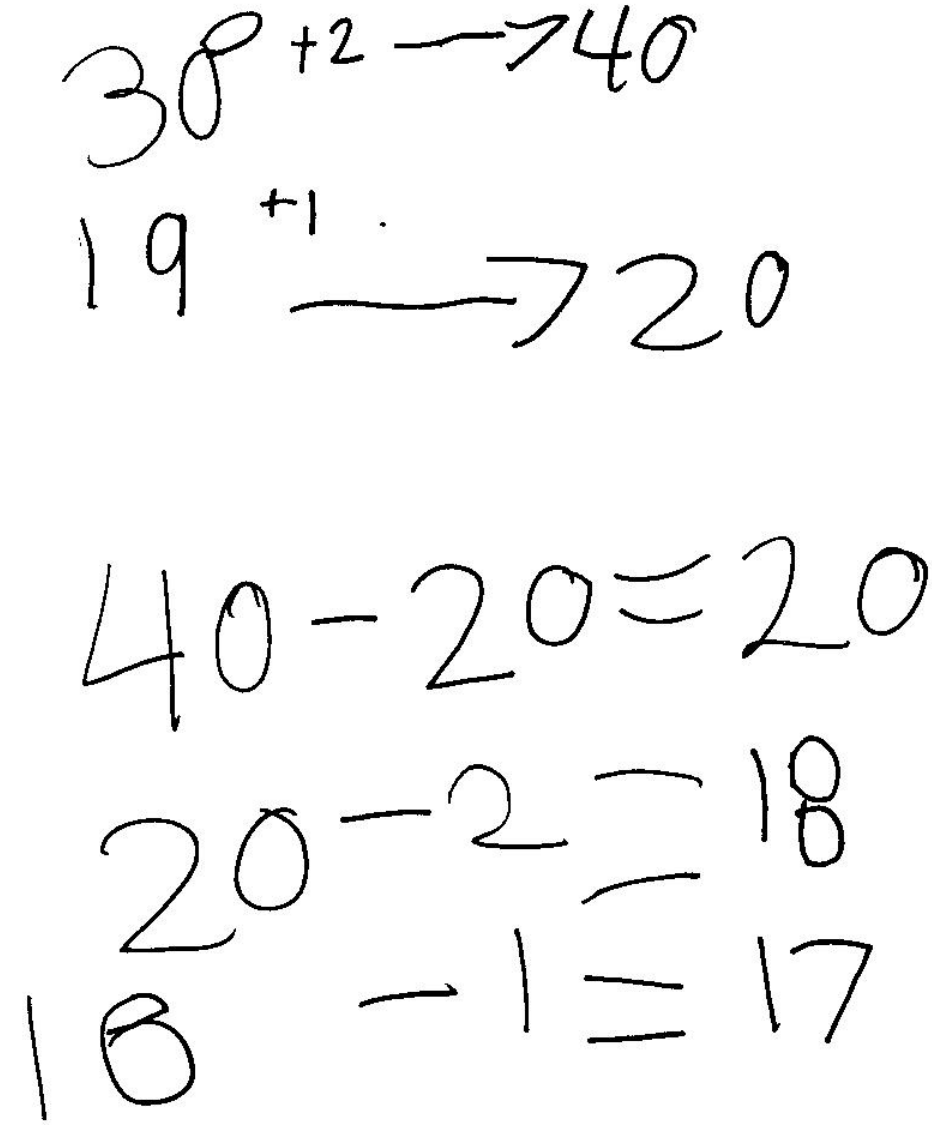
$60 - 2 = 58$

$58 - 1 = 57$

She explained her work as follows:

> "I added 2 to the 38, and I added 1 to the 19. I got 40 + 20, which is 60. Then I subtracted the 2 and the 1 that I had added. The answer is 57."

$$38 - 19 = \square$$



She explained her work as follows:

> "I added 2 to the 38, and I added 1 to the 19. I got 40 – 20, which is 20. Then I subtracted the 2 and the 1 that I had added. The answer is 17."

$$38 \times 19 = \square$$

38 +2 → 40

19 +1 → 20

40 x 20 = 800

800 - 2 = 798

798 - 1 = 797

She explained her work as follows:

> "I added 2 to the 38, and I added 1 to the 19. I got 40 x 20, which is 800. Then I subtracted the 2 and the 1 that I had added. The answer is 797."

Protocol for Savannah Task – Reflection

Now that you have completed this task, I am interested in your reaction to this task. Please answer the following questions, reflecting on your experience. Please be succinct but provide specifics whenever possible. Feel free to use the back of the page if you need more space.

[Hand out Reflection #1.]

Name ______________________________
Date ________________

Savannah-Task Reflection

Please take a few minutes to reflect on the previous task and answer the following questions:

1. How was this experience for you?

2. I'm interested in how you feel about your understanding now that the discussion has ended. In particular, I am curious about the following two questions:

    a. About what parts of Savannah's subtraction strategy are you still unsure?

    b. Is there a strategy that Savannah could apply to consistently arrive at the correct answer to the subtraction task given that she begins as she did by adding 2 to 38 and adding 1 to 19?

Please include as much detail as possible. Feel free to use the back of this page if you need more space. After completing your reflection, please give it to me.

Protocol for Pat Task (Individual)
Before introducing the student work, share the following background information:

Next I'll be giving you a set of student work that Pat, a third grader, completed in May. For this task, you will be working individually.

[Hand out Pat Task.]

There are two questions related to Pat's strategy. The mathematics tools—the hundreds charts, unifix cubes, and base-ten blocks—are still available, and you are welcome to use them. When you have finished this task, please hand it to me and then you are welcome to take a break. [Remind where the restrooms are located.]

Any questions?

[Check in on the participants if 15 minutes have passed. At that point, let them know they will have time to talk about this task with others shortly.]

[When participants have finished, collect papers. Make copies before beginning group portion.]

Protocol for Pat Task (Group)
Before handing back the participant's individual work on the task, share the following information:

I'll be giving you back your work from the previous task about Pat's strategy. You have had time to think about Pat's strategy alone, but now I'd like to give you a chance to discuss it with others. I would like for you as a group to start your conversation by discussing what you find noteworthy in this work. For example you might start by talking about Pat's mathematical reasoning. This is just a suggestion as to where to start you conversation, and feel free to take the conversation in any direction you choose. The tools—the hundreds charts, unifix cubes, and base-ten blocks—are still available, and you are welcome to use them. When you have completed your discussion, I will ask a few questions. I'll wait to ask these questions until you tell me that you are ready to move on.

[Hand back the participants' work on the Pat Task.]

[If the participants have not started discussing the student work after 30 seconds, read the following:] *Does anyone need more time? I want to remind you that I won't be saying much, but I do have some questions at the end, so let me know when you feel as though you have completed your conversation. I also want to remind you of the initial question: What do you find noteworthy in Pat's work? You are welcome to take the conversation in other directions as you see fit.*

After the teachers feel as though they have finished talking about the student work (or 15 minutes ± 5 minutes), say the following:
Is there anything else anyone would like to add before we go on? I have a few questions for you. You may feel as though you have already answered some of these questions. If so, you may tell me that you have already addressed the question, and we will move on.

1) *Could Pat use this strategy for any division problem? Why or why not?* [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

2) *Pat's work includes the statement 14 ÷ 2 = 7. What might Pat be thinking here?* [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

3) *Those are all the questions I have for you. Is there anything else you want to say about Pat's work?*

I'd like to end this task by reminding you that while using this task with other teachers, I have found that it differs from what many teachers have considered previously. Thank you for being so thoughtful in analyzing and discussing Pat's work!

[Collect the all work, even if nothing has been added or changed.]

Name ____________________________

Date ________________

In May, a teacher provided the following situation in her third-grade class:

I was at a store, and I saw that chocolate kisses come in bags of 42. I wanted to share these kisses among 7 people. How many kisses would each person get?

Following are the steps Pat told his teacher he had performed mentally to solve the problem. The teacher's follow-up questions confirmed that Pat's steps reflected a deep understanding of the problem situation.

$4 \times 10 = 40$ That is three 4s too many, so I have 12 left over. $12 + 2 = 14$ $14 \div 2 = 7$ $4 + 2 = 6$. So $42 \div 7 = 6$.

a) Please explain how each of Pat's steps makes mathematical sense in this context.

b) Use Pat's approach to solve $56 \div 8$.

Please give this packet to me when you have finished it.

Protocol for Pat-Task Reflection

As with the previous task, I am interested in your reaction to this task. Please answer the following questions, reflecting on your experience. Please be succinct but provide specifics whenever possible. Feel free to use the back of the page if you need more space.

[Hand out Reflection #2.]

Name ______________________________
Date ________________

Pat-Task Reflection

Please take a few minutes to reflect on the previous task and answer the following questions:

1. Did you prefer working on this task alone or with others? Why?

2. If you were now asked to explain Pat's division strategy to someone else, how well do you think you could explain it?

3. Did you find any particular part of this task to be enjoyable or frustrating?

Please include as much detail as possible. Feel free to use the back of this page if you need more space. After completing your reflection, please give it to me.

Protocol for Consecutive-Sums Task (Individual)
Before introducing the task, share the following background information:

Next I'll be asking you to complete a task about consecutive sums. For this task, you will be working individually.

[Hand out Consecutive-Sums Task.]

There are two questions related to consecutive sums. The mathematics tools—the hundreds charts, unifix cubes, and base-ten blocks—are still available, and you are welcome to use them. When you have finished this task, please hand it to me and then you are welcome to take a break. [Remind where the restrooms are located.]

Take a moment and read the instructions of the task.

Are there any questions about what consecutive sums are? Any other questions?

[When participants have finished, collect papers. Make copies before beginning group portion.]

Protocol for Consecutive-Sums Task (Group)
Before handing back the participant's individual work on the task, share the following information:

I'll be giving you back your work from the previous task about consecutive sums. You have had time to think about consecutive sums alone, but now I'd like to give you a chance to discuss it with others. I would like for you as a group to start your conversation by discussing what conjectures you have made about consecutive sums. For example you might start by talking about a method you found to be helpful or an insight you had. These are just suggestions as to where to start you conversation, and feel free to take the conversation in any direction you choose. The tools—the hundreds charts, unifix cubes, and base-ten blocks—are still available, and you are welcome to use them. When you have completed your discussion, I will ask a few questions. I'll wait to ask these questions until you tell me that you are ready to move on.

[Hand back the participants' work on the Consecutive-Sums Task.]

[If the participants have not started discussing the student work after 30 seconds, read the following:] *Does anyone need more time? I want to remind you that I won't be saying much, but I do have some questions at the end, so let me know when you feel as though you have completed your conversation. I also want to remind you of the initial question: What conjectures did you make about consecutive sums? You are welcome to take the conversation in other directions as you see fit.*

After the teachers feel as though they have finished talking about the student work (or 15 minutes ± 5 minutes), say the following:
Is there anything else anyone would like to add before we go on? I have a few questions for you. You may feel as though you have already answered some of these questions. If so, you may tell me that you have already addressed the question, and we will move on.

1) *What patterns are there for sums with representations that have only two addends? Explain your reasoning.* [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

2) *Why are multiples of 3 always consecutive sums? Explain your reasoning.* [Continue after the conversation appears to have ended.] *Is there anything else anyone would like to add before we go on?*

3) *Those are all the questions I have for you. Is there anything else you want to say about consecutive sums?*

Thank you for all of your efforts on this task.

[Collect the all work, even if nothing has been added or changed.]

Name ____________________________
Date _______________

The focus of this task will be on *consecutive sums*. In a consecutive sum, you can start with any positive counting number and add to it the next number and add more consecutive numbers until you reach a stopping point. The resulting number is called a *consecutive sum.* Here are some examples:

2 + 3 = 5,	4 + 5 = 9,	2 + 3 + 4 = 9,	1 + 2 + 3 + 4 + 5 + 6 = 21

So, these calculations show that 5, 9, and 21 are all consecutive sums. Moreover, there are two representations (4 + 5 and 2 + 3 + 4) for the sum of 9.

Your task has two parts.

- First, determine which positive counting numbers between 1 and 20 are consecutive sums and which are not. (You might find it helpful to develop a strategy to do this efficiently.)

- Second, find as many patterns as you can about consecutive sums and their representations. Try to explain your patterns.

Protocol for Consecutive-Sums-Task Reflection

As with the previous task, I am interested in your reaction to this task. Please answer the following questions, reflecting on your experience. Please be succinct but provide specifics whenever possible. Feel free to use the back of the page if you need more space.

[Hand out Reflection #3.]

Name ______________________________
Date ________________

Consecutive-Sums-Task Reflection

Please take a few minutes to reflect on the previous task and answer the following questions:

1. Compare this task to the previous one (the Pat Task). Did it feel fairly similar or somewhat different? Why?

2. Were there any parts of this task that you found particularly enjoyable or frustrating?

Please include as much detail as possible. Feel free to use the back of this page if you need more space. After completing your reflection, please give it to me.

Protocol for Sam Task

Before introducing the student work, share the following background information:

Next I'll be giving you a set of student work that Sam, a second grader, completed in March. For this task, you will be working individually. This time, however, there will be no group follow-up.

[Hand out Sam Task.]

There are four questions related to Sam's strategy. The mathematics tools—the hundreds charts, unifix cubes, and base-ten blocks—are still available, and you are welcome to use them. When you have finished this task, please hand it to me. After you complete this task, I will ask you to complete only one additional task.

Any questions?

Name ____________________________
Date ________________

In March, a teacher provided the following situation in his second-grade class:

On Monday, I bought a roll of 97 stickers. I gave away 42 stickers on Tuesday, 11 on Wednesday, and 23 on Thursday. How many stickers do I have left on Friday?

Following is the work of one student, Sam, along with the reasoning she told her teacher she had performed mentally to solve the problem. The teacher's follow-up questions confirmed that Sam's steps reflected a deep understanding of the problem situation.

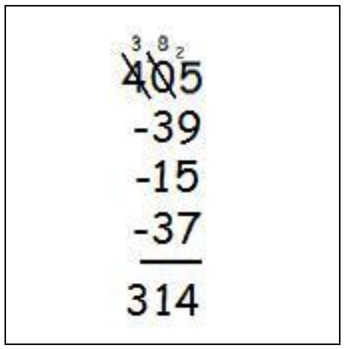
```
 97
-42
-11
-23
---
 21
```

First, I added the ones and subtracted.
Then, I added the tens and subtracted;
7 minus 6 is 1, and 90 minus 70 is 20.
So, my answer is 21.

Sam was then asked to solve the following problem:

Last week, I bought a roll of 405 stickers. I gave away 39 stickers in the morning, 15 at lunch, and 37 in the afternoon. How many stickers did I have left at the end of the day?

Following is Sam's work on the second problem:



a) Please explain how each of Sam's steps on the second problem makes mathematical sense in this context.

b) Is there a second way that Sam's steps on the second problem might make mathematical sense? If so, how?

c) What amount does the 2 represent in Sam's work? What amount does the 8 represent?

d) Consider the following problem:

Last week, I bought one last roll of 598 stickers. I gave away 64 stickers to my first class, 38 stickers to my second class, 56 stickers to my third class, and 75 stickers to my last class. How many stickers did I have left at the end of the day?

Use Sam's strategy to solve this problem.

e) Are there other circumstances under which Sam could correctly use this method to subtract? Are there cases where Sam's method could not be used to obtain a correct result? Explain.

f) Think back on the four tasks you completed today (the Savannah Task, the Pat Task, the Consecutive-Sums Task, and this one. If I were going to work with another group of teachers and had time for only one task, which would you recommend I use (or perhaps, which would you recommend I do *not* use)? Why?

Protocol for Mathematical Autobiographies
Before introducing the autobiographies form, share the following background information:

I have one last task for you today. I would like to learn a little bit about your past experiences with mathematics by having you record a short mathematical autobiography *reflecting on your own experiences with mathematics—as a student, as a teacher, and in life.*

[Hand out Autobiography Sheet.]

Here is a list of questions about your previous mathematical experiences. You might answer some or all of these in your autobiography, but do not feel limited by them. You may talk about whatever you think is pertinent. When talking about your past experiences, please mention ***specific experiences and events (****instead of just generalities***)** *that you remember. The questions are meant to stimulate thinking, but I am not looking for series of answers to these questions. Instead, this should be a narrative and read somewhat like a story. These questions do not need to be answered in order.*

I will start the tape for you. You may want to sketch a short outline first or jot down some notes. Do NOT worry about any silences that may occur. Please push stop only when you are finished. Take as much time as you need. When you have completed your autobiography, please come find me, and then you are free to leave.

I will be the only one listening to this tape. No one but me will ever hear what you say (unless you choose to share what you have said with others).

Before you start, I want to thank you again for participating in this study. I sincerely appreciate your taking the time for this. If you have any other questions, feel free to ask me. You may begin whenever you wish.

[When the participant has finished, ask him/her the following:] *This experience of recording your mathematical autobiography was probably an unusual experience for you. Did you surprise yourself by anything you said? Did anything in particular stand out for you?*

Name ______________________________
Date ________________

Please take a few moments to record a short *mathematics life story*, reflecting on your own experiences with mathematics—as a student, as a teacher, and in life. For each question, think about specific experiences and events **(instead of just generalities) that you remember. The questions are meant to stimulate thinking, so feel free to talk about additional experiences you remember. These questions do not need to be answered in order and are meant as a jumping-off point. Feel free to talk about anything you feel is pertinent.**

Mathematical-Autobiography-Reflection Questions

- How do you feel about mathematics? How have your feelings changed over time?

- What do you think mathematics is? How would you set up a mathematics class to embody this vision of mathematics? In other words, describe your conception of a mathematics class in a perfect world.

- Can you recall any high points in your mathematics experience (person, event, class, et cetera)? If so, describe one. What happened and who was involved? Did you have a low point in your mathematics experience? What happened and who was involved?

- Identify a major challenge you have faced in mathematics. How did you face, handle, or deal with this challenge?

Feel free to talk about mathematical experiences you had as a student, as a teacher, or outside the classroom entirely.

Appendix C: Clinical-Interview Questions

Greet each participant and reintroduce yourself (if needed). Thank the participant for coming in and make small talk to make the participant feel comfortable. When ready, share the following background information:

Recently you were part of a research session in which you worked on some tasks individually and some in groups. You also recorded a mathematical autobiography. To help you remember the session, I am giving you back your work. [Give the person all four tasks.] *Today, I am going to ask you a few questions about your experiences during the session. At times, I may show you snippets of video from the group to help you remember specific events and to ask you about specific moments. My goal here is not to test your mathematical knowledge but to see the experience through your eyes. You are welcome at any time to skip a question or say that you do not remember. Do you have any questions before we begin?*

Start with the following questions, asking them one at a time. Give the participant plenty of time to answer. Ask for more details when they are needed.

Overall, how was the experience for you?

How would you describe the experience to a friend?

How did the experience compare with what you expected?

Next, move on to the specific questions for this participant. Some may include showing segments of the video from the focus group. For any segment, ask the participant if he or she wants to view it again after having seen it. Remind the participant that the focus is not on him or her but is on how he or she **experienced** the activity.

After all the specific questions have been asked, conclude with the following questions.

Did any of the tasks stand out as being as the most (or perhaps least) relevant or interesting to you? Which one(s)? Why?

What makes a particular mathematical problem interesting *to you?*

Was there a moment that for you was particularly helpful in solving the problem? If so, when?

Was there a moment that was a hindrance to you? If so, when? Why was this unhelpful to you?

Was there a time during the session when you got stuck in the mathematics? If so, how did you get unstuck?

Last, how did you decide to participate in this experience?

Thank the participant again for his or her time. Remind the participant that the responses will not be revealed to any professors and that in any publications or presentations, responses will be anonymous or a pseudonym will be used. All comments will be kept in a secure location.

Affect
Do you generally enjoy mathematics? Why or why not?
What do you enjoy most about mathematics? What do you enjoy least?
How would you describe your feelings when you are asked to complete a difficult mathematical task?
Beliefs
Are there people who cannot learn mathematics, or can anyone who tries learn?
How would you characterize the process of doing mathematics?
What do you think mathematicians do when they do mathematics?
How quickly do you think that you should be able to solve a mathematics problem?
Where do you use mathematics in your everyday life?
Identity
Could you categorize what type of mathematics person you are?
What was your peak experience in mathematics? Did you have a low point?
Was there a turning point in your life that made you look at mathematics differently?
Do you feel as though your mathematical experiences were steady or more like a roller coaster?
Mathematical Integrity
How do you know when you have satisfactorily completed a problem?
Risk Taking
How willing are you to share new ideas, if, in doing so, you may expose mistakes you made?
For which parts of the tasks did you feel most certain about your solutions? Least certain?
How comfortable do you generally feel about asking questions about someone else's solution?
Goals
Do you like trying/learning new things in mathematics or doing more of things you can do already?
How much does effort play a part in learning mathematics?
What did you think the goal of the _____ task was? How did you determine that?

Motivation
How challenging did you find the tasks to be? How successful did you think you were?
During any of the tasks, did you want to give up yet decided to keep going? Why?
When do you choose to give up on a problem? How do you make that choice?
Self-Efficacy
How confident did you feel about your solution on the ______ task? Why?
How confident are you in your own mathematical abilities?
What would make you feel more confident (in either your solutions or your abilities)?

REFERENCES

Ames, C. (1992). Classrooms: Goals, structures, and student motivation. *Journal of Educational Psychology*, *84*, 261–271.

Ames, C., & Archer, J. (1988). Achievement goals in the classroom: Students' learning strategies and motivation process. *Journal of Educational Psychology*, *80*, 260–267.

Anderman, E. M., & Maehr, M. L. (1994). Motivation and schooling in the middle grades. *Review of Educational Research*, *64*, 287–309.

Anderson, R. (2007). Being a mathematics learner: Four faces of identity. *Mathematics Educator*, *17*(1), 7–14.

Bandura, A. (1977). Self-efficacy: Toward a unifying theory of behavioral change. *Psychological Review*, *84*, 191–215.

Bandura, A. (1986). *Social foundations of thought and action: A social cognitive theory*. Englewood Cliffs, NJ: Prentice-Hall.

Bandura, A. (1993). Perceived self-efficacy in cognitive development and functioning. *Educational Psychologist*, *28*, 117–148.

Bandura, A., & Schunk, D. H. (1981). Cultivating competence, self-efficacy, and intrinsic interest through proximal self-motivation. *Journal of Personality and Social Psychology*, *41*, 586–598.

Beghetto, R. A. (2009). Correlates of intellectual risk taking in elementary school science. *Journal of Research in Science Teaching*, *46*, 210–223.

Bernard, H. R. (1988). Research methods in cultural anthropology. Newbury Park, CA: Sage.

Blanck, P. D. (Ed.). (1993). *Interpersonal expectations: Theory, research, and applications*. Cambridge, England: Cambridge University Press.

Boaler, J. (2002). The development of disciplinary relationships: Knowledge, practice and identity in mathematics classrooms. *For the Learning of Mathematics*, *22*(1), 42–47.

Bong, M. (1996). Problems in academic motivation research and advantages and disadvantages of solutions. *Contemporary Educational Psychology*, *21*, 149-165.

REFERENCES

Brown, C. A., Carpenter, T. P., Kouba, V. L., Lindquist, M. M., Silver, E. A., & Swafford, J. O. (1988). Secondary school results for the fourth NAEP mathematics assessment: Algebra, geometry, mathematical methods, and attitudes. *Mathematics Teacher*, *81*, 337–347, 397.

Calderhead, J. (1981). Simulated recall: A method for research on teaching. *British Journal of Educational Psychology*, *51*, 211–217.

Carey, D. A., & Franke, M. L. (1993, April). *Young children's perceptions of mathematics in cognitively guided instruction classrooms.* Paper presented at the annual meeting of the American Educational Research Association, Atlanta.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics*, *58*, 45–75.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–590). Mahwah, NJ: Erlbaum.

Clifford, M. M. (1988). Failure tolerance and academic risk taking in ten to twelve year-old students. *British Journal of Educational Psychology*, *58*, 15–27.

Clifford, M. M. (1991). Risk taking: Theoretical, empirical, and educational considerations. *Educational Psychologist*, *26*, 263–297.

Clifford, M. M., Lan, W. Y., Chou, F. C., & Qi, Y. (1989). Academic risk-taking: Developmental and cross-cultural observations. *Journal of Experimental Education*, *57*, 321–338.

Csikszentmihalyi, M. (1978). Intrinsic rewards and emergent motivation. In M. R. Lepper & D. Greene (Eds.), *The hidden costs of rewards.* Hillsdale, NJ: Erlbaum.

DeBellis, V. A., & Goldin, G. A. (1999). Aspects of affect: Mathematical intimacy, mathematical integrity. In O. Zaslavsky (Ed.), *Proceedings of the 23rd conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 249–256). Haifa, Israel.

REFERENCES

DeBellis, V. A., & Goldin, G. A. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics*, *63*, 131–147.

Diaz-Obando, E., Plasencia-Cruz, I., & Solano-Alvarado, A. (2003). The impact of beliefs in students' learning: An investigation with students of two different contexts. *International Journal of Mathematics Education in Science and Technology*, *34*, 161–173.

Drake, C., Spillane, J. P., & Hufferd-Ackles, K. (2001). Storied identities: Teacher learning and subject-matter context. *Journal of Curriculum Studies*, *33*(1), 1–23.

Dweck, C. S. (1986). Motivational processes affecting learning. *American Psychologist*, *41*, 1040–1048.

Dweck, C. S., & Elliot, E. S. (1983). Achievement motivation. In P. H. Mussen (Gen. Ed.), & E. M. Hetherington (Vol. Ed.), *Handbook of child psychology: Vol. IV. Social and personality development* (pp. 643–691). New York: Wiley.

Dweck, C. S., & Leggett, E. L. (1988). A social-cognitive approach to motivation and personality. *Psychological Review*, *95*, 256–273.

Eccles, J. S., & Wigfield, A. (2002). Motivational beliefs, values, and goals. *Annual Review of Psychology, 53*, 109–132.

Fennema, E. & Franke, M. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.) *Handbook of Research on Mathematics Teaching and Learning* (pp. 147–164). New York: Macmillian.

Ferla, J., Valcke, M., & Cai, Y. (2009). Academic self-efficacy and academic self-concept: Reconsidering structural relationships. *Learning and Individual Differences*, *19*, 499–505.

Fleener, M. J. (1996). Scientific world building on the edge of chaos: High school students' beliefs about mathematics and science. *School Science and Mathematics*, *96*, 312–321.

Ford, M. E. (1992). *Human motivation: Goals, emotions, and personal agency beliefs*. Newbury Park, CA: Sage.

REFERENCES

Frank, M. L. (1988). Problem solving and mathematical beliefs. *Arithmetic Teacher*, *35*(5), 32–34.

Gabriele, A. J., & Montecinos, C. (2001). Collaborating with a skilled peer: The influence of achievement goals and perceptions of partners' competence on the participation and learning of low-achieving students. *Journal of Experimental Education*, *69*, 152–179.

Garofalo, J. (1989). Beliefs and their influence on mathematical performance. *Mathematics teacher, 82*, 502–505.

Gee, J. P. (2001). Identity as an analytic lens for research in education. *Review of Research in Education*, *25*, 99–125.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice.* New York: Cambridge University Press.

Gómez-Chazón, I. M. (2000). Affective influences in the knowledge of mathematics. *Educational Studies in Mathematics*, *43*, 149–168.

Gottfried, A. E. (1990). Academic intrinsic motivation in young elementary school children. *Journal of Educational Psychology*, *82*, 525–538.

Haladyna, T., Shaughnessy, J., & Shaughnessy, J. M. (1983). A causal analysis of attitude toward mathematics. *Journal for Research in Mathematics Education*, *14*, 19–29.

Hannula, M. S. (2002). Attitude towards mathematics: Emotions, expectations, and values. *Educational Studies in Mathematics*, *49,* 25–46.

Hannula, M. S. (2006). Motivation in mathematics: Goals reflected in emotions. *Educational Studies in Mathematics*, *63*, 165–178.

Harter, S. (1978). Pleasure derived from challenge and the effects of receiving grades on children’s difficulty level choices. *Child Development*, *49*, 788–799.

Hatch, J. A. (2002). Doing qualitative research in education settings. Albany: State University of New York Press.

REFERENCES

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Erlbaum.

Holland, D., & Lave, J. (Eds.). (2001). *History in person: Enduring struggles, contentious practice, intimate identities*. Santa Fe, NM: School of American Research Press.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*, 169–202.

Kloosterman, P. (2002). Beliefs about mathematics and mathematics learning in the secondary school: Measurement and implications for motivation. In G. Leder, E. Pehkonen, & G. Törher (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 247–269). Dordrecht, The Netherlands: Kluwer.

Kloosterman, P., Raymond, A. M., & Emenaker, C. (1996). Students' beliefs about mathematics: A three-year study. *Elementary School Journal*, *97*, 39–56.

Kouba, V. L., & McDonald, J. L. (1986). Children's and teachers' perceptions and beliefs about the domain of elementary mathematics. In G. Lappan & R. Even (Eds.), *Proceedings of the eighth annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 250–255). East Lansing: Michigan State University.

Lent, R. W., Brown, S. D., & Larkin, K. C. (1984). Relation of self-efficacy expectations to academic achievement and persistence. *Journal of Counseling Psychology*, *31*, 356–362.

Lepper, M. R. (1988). Motivational considerations in the study of instruction. *Cognition and Instruction*, *5*, 289–309.

Ma, X. (2006). Cognitive and affective changes as determinants for taking advanced mathematics courses in high school. *American Journal of Education*, *113*, 123–149.

MacIver, D., Stipek, D., & Daniels, D. (1991). Explaining within-semester changes in student effort in junior and senior high school courses. *Journal of Educational Psychology*, *83*, 201–211.

REFERENCES

Matsumoto, D., & Sanders, M. (1988). Emotional experiences during engagement in intrinsically and extrinsically motivated tasks. *Motivation and Emotion*, *12*, 353–369.

Mawhinney, H., & Xu, F. (1997). Restructuring the identity of foreign-trained teachers in Ontario schools. *TESOL Quarterly*, *31*(3), 632–639.

Mayer, R. E., & Hegarty, M. (1996). The process of understanding mathematical problems. In R. J. Sternberg & T. Ben-Zeev (Eds.), *The nature of mathematical thinking*. Hillsdale, NJ: Erlbaum.

McLeod, D. B. (1992). Research in affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York: Macmillan.

Meyer, D. K., Turner, J. C., & Spencer, C. A. (1997). Challenge in a mathematics classroom: Students' motivation and strategies in project-based learning. *Elementary School Journal*, *97*, 501–521.

Middleton, J. A. (1995). A study of intrinsic motivation in the mathematics classroom: A personal constructs approach. *Journal for Research in Mathematics Education*, *26*, 254–279.

Middleton, J. A., & Spanias, P. A. (1999). Motivation for achievement in mathematics: Findings, generalizations, and criticisms of the research. *Journal for Research in Mathematics Education*, *30*, 65–88.

Middleton, M. J., & Midgley, C. (1997). Avoiding the demonstration of lack of ability: An underexplored aspect of goal theory. *Journal of Educational Psychology*, *89*, 710–718.

Middleton, M. J., & Midgley, C. (2002). Beyond motivation: Middle school students' perception of press for understanding. *Contemporary Educational Psychology*, *27*, 373–391.

Midgley, C., Feldlaufer, H., & Eccles, J. S. (1989). Change in teacher efficacy and student self- and task-related beliefs in mathematics during the transition to junior high school. *Journal of Educational Psychology*, *81*, 247–258.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis* (2nd ed.). Thousand Oaks, CA: Sage.

REFERENCES

Miller, D. C., & Byrnes, J. P. (1997). The role of contextual and personal factors in children's risk taking. *Developmental Psychology*, *33*, 814–823.

Morrone, A. S., Harkness, S. S., D'Ambrosio, B, & Caulfield, R. (2004). *Educational Studies in Mathematics*, *56*, 19–38.

Muis, K. R. (2004). Personal epistemology and mathematics: A critical review and synthesis of research. *Review of Educational Research*, *74,* 317–377.

Multon, K. D., Brown, S. D., & Lent, R. W. (1991). Relation of self-efficacy beliefs to academic outcomes: A meta-analytic investigation. *Journal of Counseling Psychology*, *38,* 30–38.

National Assessment of Educational Progress. (1999). *The nation's report card*. Retrieved January 18, 2010, from http://nces.ed.gov/nationsreportcard/pdf/main1996/1999452.pdf

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Nicholls, J. G. (1984). Achievement motivation: Conceptions of ability, subjective experience, task choice, and performance. *Psychological Review*, *91*, 328–346.

Pajares, F. (1996). Self-efficacy beliefs in academic settings. *Review of Educational Research*, *66*, 543–578.

Patrick, H, Anderman, L. H., Ryan, A. M., Edelin, K. C., & Midgley, C. (2001). Teachers' communication of goal orientations in four fifth-grade classrooms. *Elementary School Journal*, *102*, 35–58.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257–315). Charlotte, NC: Information Age.

Raymond, A. M. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education*, *28*, 550–576.

REFERENCES

Resnick, L. B. (1987). *Education and learning to think.* Washington, DC: National Academy Press.

Rosenthal, R. (2002). Covert communication in classrooms, clinics, courtrooms, and cubicles. *American Psychologist, 57*(11), 839-849.

Ruffel, M., Mason, J, & Allen, B. (1998). Studying attitude to mathematics. *Educational Studies in Mathematics*, *35*, 1–18.

Schunk, D. H. (1983). Developing children's self-efficacy and skills: The roles of social comparative information and goal setting. *Contemporary Educational Psychology*, *8*, 76–86.

Schunk, D. H. (1984). Enhancing self-efficacy and achievement through rewards and goals: Motivational and informational effects. *Journal of Educational Research*, *78*, 29-34.

Schunk, D. H. (1990). Goal setting and self-efficacy during self-regulated learning. *Educational Psychologist*, *25*, 71–86.

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher*, *34*(4), 14–22.

Silver, E. A. (1985). Research on teaching mathematical problem solving: Some underrepresented themes and needed directions. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 247–266). Hillsdale, NJ: Erlbaum.

Spangler, D. A. (1992). Assessing students' beliefs about mathematics. *Mathematics Educator*, *3*, 19–23.

Stevens, T., Olivarez, A., Jr., Lan, W., & Tallent-Runnels, M. K. (2004). The role of mathematics self-efficacy and motivation in mathematics performance across ethnicity. *The Journal of Educational Research*, *97*, 208–221.

Stevenson, H. W., Lee, S. Y., & Stigler, J. W. (1986). Mathematics of Chinese, Japanese, and American children. *Science*, *231*, 693–698.

Stodolsky, S. S. (1985). Telling math: Origins of math aversion and anxiety. *Educational Psychologist*, *20*, 125–133.

REFERENCES

Streitmatter, J. (1997). An exploratory study of risk taking and attitudes in a girls-only middle school math class. *The Elementary School Journal*, *98*, 15–26.

Sugrue, C. (1997). Student teachers' lay theories and teaching identities: Their implications for professional development. *European Journal of Teacher Education*, *20*(3), 213–225.

Third International Mathematics and Science Study. (1996). *Pursuing excellence: A study of U.S. eighth-grade mathematics and science teaching, learning, curriculum, and achievement in international context: Initial findings from the third international mathematics and science study*. Washington, DC: Government Printing Office.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York: Macmillan.

Townsend, B., Lannin, J., & Barker, D. (2009). Promoting efficient strategy use. *Mathematics Teaching in the Middle School*, 14, 542.

Tsamir, P., & Tirosh, D. (2002). Intuitive beliefs, formal definitions, and undefined operations: Cases of division by zero. In G. Leder, E. Pehkonen, & G. Törher (Eds.), *Beliefs: A hidden variable in mathematics education?* Dordrecht: Kluwer.

Urdan, T., & Maehr, M. L. (1995). Beyond a two-goal theory of motivation: A case for social goals. *Review of Educational Research*, *65*, 213–244.

Wentzel, K. R. (1989). Adolescent classroom goals, standards for performance, and academic achievement: An interactionist perspective. *Journal of Educational Psychology, 81,* 131–142.

Wilkins, J. L. M., & Ma, X. (2003). Modeling change in student attitude toward and beliefs about mathematics. *Journal of Educational Research, 97*, 52–63.

Zimmerman, B. J., Bandura, A., & Martinez-Pons, M. (1992). Self-motivation for academic attainment: The role of self-efficacy beliefs and personal goal setting. *American Educational Research Journal, 29*, 663–676.