Absolutely representing systems, uniform smoothness, and type

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Abstract

Absolutely representing system (ARS) in a Banach space $X$ is a set $D \subset X$ such that every vector $x$ in $X$ admits a representation by an absolutely convergent series $x = \sum a_i x_i$ with $(a_i)$ reals and $(x_i) \subset D$. We investigate some general properties of ARS. In particular, ARS in uniformly smooth and in B-convex Banach spaces are characterized via $\varepsilon$-nets of the unit balls. Every ARS in a B-convex Banach space is quick, i.e. in the representation above one can achieve $\|a_i x_i\| < cq^i \|x\|$, $i = 1, 2, \ldots$ for some constants $c > 0$ and $q \in (0, 1)$.

1 Introduction

The concept of absolutely representing system (ARS) goes back to Banach and Mazur ([3], p. 109–110).

Definition 1.1 A set $D$ in a Banach space $X$ is called absolutely representing system (ARS) if for every $x \in X$ there are scalars $(a_i)$ and elements $(x_i) \subset D$ such that

$$x = \sum_{i=1}^{\infty} a_i x_i \quad \text{and} \quad \sum_{i=1}^{\infty} \|a_i x_i\| < \infty.$$

It can be observed (Section 2) that if $D$ is an ARS, then there exist a constant $c$ such that each $x \in X$ admits a representation $x = \sum_{i=1}^{\infty} a_i x_i$ with $\sum \|a_i x_i\| \leq c \|x\|$. Then we call $D$ a "$c$-ARS".
For needs of complex analysis, ARS were defined also in locally convex topological spaces [K 81]. In the theory of analytical functions such ARS happen to be a convenient tool: see [K 90], [G], [A]. Many results of general kind on ARS are obtained by Yu. Korobeinik and his collaborators: see, for example, [K 81], [K 86], [KK].

In the present paper we restrict ourselves to the theory of ARS in Banach spaces, which is still not quite explored. Some non-trivial examples of ARS in $l_2$ were found by I. Shraifel [S 93]. It should be noted that each example of a $c$-ARS in $l_2^n$ provides by Theorem 3.1 an example of an $\varepsilon$-net of the $n$-dimensional Euclidean ball, $\varepsilon = \varepsilon(c) < 1$. See also [S 95] for results on ARS in Hilbert spaces.

Some general results concerning ARS in Banach spaces and, particularly, in uniformly smooth spaces, were obtained in [V]. There was introduced the notion of $(c, q)$-quick representing system, which is considerably stronger than that of ARS.

**Definition 1.2** Let $c > 0$ and $q \in (0, 1)$. A set $D$ in a Banach space $X$ is called $(c, q)$-quick representing system (or $(c, q)$-quick RS) if for each $x \in X$ there are scalars $(a_i)$ and elements $(x_i) \subset D$ such that

$$x = \sum_{i=1}^{\infty} a_i x_i \quad \text{and} \quad \|a_i x_i\| \leq cq^{i-1} \quad \text{for} \quad i \geq 1.$$ 

It is clear that each $(c, q)$-quick RS is an ARS. Despite of the strong restrictions in Definition 1.2, there exist Banach spaces $X$ in which every ARS is, in turn, a $(c_1, q)$-quick RS for some $c_1$ and $q$. In [V] it was proved that this happens in each super-reflexive space $X$.

In the present paper we generalize this result to all B-convex Banach spaces. Suppose a space $X$ is B-convex and $Y$ is a subspace of $X$. We show that every $c$-ARS in $Y$ is a $(c_1, q)$-quick RS for some $c_1$ and $q$ depending only on $c$ and on $X$. This latter statement characterizes the class of B-convex Banach spaces.

We characterize ARS and $(c, q)$-quick RS in uniformly smooth and B-convex Banach spaces via $\varepsilon$-nets of the unit balls. As a consequence, we have a theorem of B. Maurey [P] stating that the dimension of a subspace $Y$ of $l_\infty^n$ with $Y^*$ of a good type is at most $c \log n$.

I am grateful to V. Kadets for the guidance, and to P. Terenzi for his hospitality when I was visiting Politecnic Institute of Milan.
2 Characterizations of ARS and quick RS

Let \((x_i)_{i \in I}\) and \((y_i)_{i \in I}\) be sequences in Banach spaces \(X\) and \(Y\) respectively, and let \(c > 0\). We call \((x_i)\) and \((y_i)\) \(c\)-equivalent if there is a linear operator \(T : \text{span}(x_i) \to \text{span}(y_i)\) which maps \(x_i\) to \(y_i\), and satisfies \(\|T\| \leq c\).

D being a non-empty set, we denote the unit vectors in \(l_1(D)\) by \(e_d\), \(d \in D\).

The following useful result is more or less known: the equivalence (i) ⇔ (iv) goes back to S. Mazur ([B], p. 110), see also [V].

Theorem 2.1 Given a complete normalized set \(D\) in a Banach space \(X\), the following are equivalent:

(i) \(D\) is an ARS;

(ii) there is a \(c > 0\) such that each \(x \in B(X)\) can be represented by a series \(x = \sum_{i=1}^{\infty} a_i x_i\) with \(\sum \|a_i x_i\| \leq c\). Then we call \(D\) a "c-ARS";

(iii) there is a quotient map \(q : l_1(D) \to Z\) such that the sequence \((d)_{d \in D}\) is \(c\)-equivalent to \((qe_d)_{d \in D}\);

(iv) there is a \(c > 0\) such that for every \(x^* \in S(X^*)\) one has \(\sup_{d \in D} |x^*(d)| \geq c^{-1}\).

In (ii), (iii) and (iv) the infimums of possible constants \(c\) are equal and are attained.

Let us observe some nice consequences. The first one states that ARS are stable under fairly large perturbations. Let \(A\) and \(B\) be sets in a Banach space. By definition, put \(\rho(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|\).

Corollary 2.2 Let \(D\) and \(D_1\) be normalized sets in \(X\). If \(D\) is a c-ARS and \(\rho(D, D_1) = \varepsilon < c^{-1}\), then \(D_1\) is a \(c_1\)-ARS, where \(c_1 = (1 - \varepsilon c)^{-1}\).

Proof. It follows easily from (iv) of Theorem 2.1.

Proposition 2.3 Let \(D\) be a c-ARS in a Banach space \(X\).

(i) If \(X\) is separable, then some countable subset \(D_1\) of \(D\) is also a c-ARS.

(ii) Let \(\dim X = n\) and \(c_1 > c\). Then some subset \(D_1\) of \(D\) is a \(c_1\)-ARS and \(|D_1| \leq e^{an}\), where \(a = 2\left(c^{-1} - c_1^{-1}\right)^{-1}\).

(iii) Let \(\dim X = n\) and \(\varepsilon > 0\). Then every \(x \in B(X)\) can be represented by a sum \(x = \sum_{i=1}^{n} a_i x_i\) with \((x_i) \subset D\) and \(\sum \|a_i x_i\| \leq c + \varepsilon\).
Proof. Clearly, we may assume that \( D \) is normalized. Then (i) follows in the standard way from (iv) of Theorem 2.1.

(ii). Let \( \varepsilon = c^{-1} - c_1^{-1} \). Consider a maximal subset \( D_1 \) of \( D \) such that \( \| x - y \| > \varepsilon \) for \( x, y \in D_1, x \neq y \). By maximality, \( \rho(D, D_1) \leq \varepsilon \). Applying Corollary 2.2, we see that \( D_1 \) is a \( c_1 \)-ARS. Note that the balls \( (d_1 + (\varepsilon/2)B(X))_{d_1 \in D_1} \) are mutually disjoint and are contained in \( (1 + \varepsilon/2)B(X) \).

By comparing the volumes we get \( \| D_1 \| \leq e^{2n/\varepsilon} \).

(iii). By (ii), we can extract from \( D \) a finite \((c + \varepsilon)\)-ARS \( \sum_{i \leq m} x_i \). By (iii) of Theorem 2.1, there is a quotient map \( q: l_1^m \rightarrow Z \) such that the sequences \( (x_i)_{i \leq m} \) and \( (qe_i)_{i \leq m} \) are \((c + \varepsilon)\)-equivalent. Let \( T: X \rightarrow Z \) be the isomorphism corresponding to this equivalence. We have \( \dim Z = n \) and \( B(Z) = a \cdot \text{conv}(qe_i)_{i \leq m} \). Now we use a simple consequence of Caratheodory’s theorem:

- Let \( K \) be a finite set in \( \mathbb{R}^n \). Let a vector \( z \) lie on the boundary of \( a \cdot \text{conv}(K) \). Then \( z \in a \cdot \text{conv}(z_1, \ldots, z_n) \) for some \( z_1, \ldots, z_n \in K \).

Applying this theorem to \( K = (qe_i)_{i \leq m} \), we see that each \( z \in S(Z) \) can be represented by a sum \( z = \sum_{k=1}^n a_k(qe_i) \) for some subsequence \( (qe_i)_{k \leq N} \) of \( (qe_i) \) and scalars \( (a_k) \) with \( \sum_{k=1}^n |a_k| = 1 \).

Let \( x \in B(X) \). Setting \( z = T x/\|T x\| \) in the preceding observation, we can write

\[
Tx = \sum_{k=1}^n b_k(qe_i) = \sum_{k=1}^n b_k(Tx_k) \quad \text{with} \quad \sum_{k=1}^n |b_k| \leq \|T\|.
\]

Thus \( x = \sum_{k=1}^n b_kx_k \), and

\[
\sum_{k=1}^n \|b_kx_k\| \leq \|T^{-1}\| \sum_{k=1}^n \|b_k(qe_i)\| \leq \|T^{-1}\| \sum_{k=1}^n |b_k| \leq \|T^{-1}\| \|T\| \leq c + \varepsilon.
\]

The proof is complete. \( \square \)

Remarks. 1. The estimate in (ii) is sharp by order: Corollary 4.2 and Theorem 2.7 show that any ARS in a B-convex Banach space \( X \) has at least exponential number of terms with respect to \( \dim X \).

2. In general, one cannot put \( \varepsilon = 0 \) in (iii). Indeed, consider \( X = l_2^2 \) and let \( D \) be a countable dense subset of \( S(l_2^2) \). Then \( D \) is a 1-ARS. However, there are points \( x \in S(l_2^2) \setminus a \cdot \text{conv}(D) \); thus (iii) fails unless \( \varepsilon > 0 \).
Theorem 2.4 Let $D$ be a normalized set in a Banach space $X$. Suppose
(i) $D$ is a $(c, q)$-quick RS.
Then, given an $\varepsilon > 0$, there are $m = m(c, q, \varepsilon)$ and $b = c(1 - q)^{-1}$ such that
(ii) the set $b \cdot \bigcup \{a.\text{conv}(D_1) : D_1 \subset D, |D_1| \leq m\}$ is an $\varepsilon$-net of $B(X)$.
Conversely, if $\varepsilon < 1$, then (ii) implies (i) with $c = b/\varepsilon$ and $q = \varepsilon^{1/m}$.

Proof. Assume (i) holds. Let $m$ be so that
$$\sum_{i>m} cq^{i-1} \leq \varepsilon. \quad (1)$$
Let $x \in B(X)$. For some $(x_i) \subset D$ we have $x = \sum_{i=1}^{\infty} a_i x_i$ with $|a_i| \leq cq^{i-1}$. Then, by (1),
$$\|x - \sum_{i \leq m} a_i x_i\| = \|\sum_{i > m} a_i x_i\| \leq \varepsilon,$$
while
$$\sum_{i \leq m} |a_i| \leq c(1 - q)^{-1} = b.$$
This proves (ii).

Conversely, assume (ii) holds. Fix an $x \in B(X)$. We shall find appropriate expansion $x = \sum_i a_i x_i$ by successive iterations. $S_n$ will denote the partial sum $\sum_{i \leq n} a_i x_i$ (we assume $S_0 = 0$).

Suppose that for some $k \geq 1$ the system $(a_i)_{i \leq (k-1)m}$ is constructed. By (ii), there are scalars $(a_{k,i})_{i \leq m}$ and vectors $(x_{k,i})_{i \leq m} \subset D$ such that $|a_{k,i}| \leq b$ for $i \leq m$ and
$$\left\| \frac{x - S_{(k-1)m}}{\|x - S_{(k-1)m}\|} - \sum_{i \leq m} a_{k,i} x_{k,i} \right\| \leq \varepsilon. \quad (2)$$
Put $a_{(k-1)m+i} = \|x - S_{(k-1)m}\| a_{k,i}$ for $1 \leq i \leq m$. Note that for each $k$
$$x - S_{km} = x - S_{(k-1)m} - \|x - S_{(k-1)m}\| \sum_{i \leq m} a_{k,i} x_{k,i}.$$
Therefore, by (2), $\|x - S_{km}\| \leq \|x - S_{(k-1)m}\| \cdot \varepsilon$. By the inductive argument we get $\|x - S_{km}\| \leq \varepsilon^k$. Hence for $k \geq 0$ and $1 \leq i \leq m$,
$$|a_{km+i}| = \|x - S_{km}\| a_{k+1,i} \leq \varepsilon^k b \leq \varepsilon^{(km+i)/m} - 1 b = \varepsilon - 1 b \cdot (\varepsilon^{1/m})^{km+i}.$$
Hence $|a_i| \leq \varepsilon^{-1} b (\varepsilon^{1/m})^i$ for $i \geq 1$. This proves (i) with $c = b\varepsilon^{-1+1/m} \leq b/\varepsilon$ and $q = \varepsilon^{1/m}$. \(\blacksquare\)
Theorem 2.4 yields that, actually, the tightness of the definition of \((c,q)\)-quick RS can be substantially loosened. Let \((b_i)\) be a scalar sequence. We say that a set \(D\) in a Banach space \(X\) is a \((b_i)\)-representing system, if every \(x \in B(X)\) admits a representation by a convergent series \(x = \sum a_i x_i\) with \(x_i \subset D\) and \((a_i) \subset \mathbb{R}, \|a_i x_i\| \leq |b_i|\) for each \(i\).

**Corollary 2.5** Let \(D\) be a set in a Banach space \(X\) and let \(\sum b_i\) be an absolutely convergent scalar series. Suppose

(i) \(D\) is a \((b_i)\)-representing system.

Then there are constants \(c\) and \(q\) dependent only on \((b_i)\), such that

(ii) \(D\) is a \((c,q)\)-quick representing system.

Conversely, (ii) implies (i) with \(b_i = cq^{i-1}\).

**Proof.** Suppose (i) holds. Let \(m\) be so that \(\sum_{i>m} |b_i| \leq 1/2\). It is enough to show that (ii) of Theorem 2.4 holds for \(\epsilon = 1/2\). Fix \(x \in B(X)\) and write its representation: \(x = \sum_{i \geq 1} a_i x_i\) with \(\|a_i x_i\| \leq |b_i|\). Then

\[
\|x - \sum_{i \leq m} a_i x_i\| = \| \sum_{i > m} a_i x_i\| \leq \sum_{i > m} \|a_i x_i\| \leq \sum_{i > m} |b_i| \leq 1/2.
\]

Thus (ii) holds. The converse part is obvious. \(\blacksquare\)

Like ARS, quick representing systems are also stable under fairly large perturbations. The following analogue of Corollary 2.2 can easily be derived from Theorem 2.4.

**Corollary 2.6** Let \(D\) and \(D_1\) be normalized sets in \(X\). If \(D\) is a \((c,q)\)-quick RS and \(\rho(D, D_1) = \epsilon < (1-q)/c\), then \(D_1\) is a \((c_1,q_1)\)-quick RS, where \(c_1\) and \(q_1\) depend solely on \(c, q\) and \(\epsilon\).

Another consequence of Theorem 2.4 states that the cardinality of every \((c,q)\)-quick RS in a finite-dimensional space is large.

**Theorem 2.7** Let \(D\) be a \((c,q)\)-quick RS in a \(n\)-dimensional Banach space \(X\). Then \(|D| \geq e^{an}\) for some \(a = a(c,q) > 0\).
Before we prove this result, observe that there are many spaces possessing ARS of small cardinalities. Indeed, E. Gluskin’s construction [Gl] gives us \( n \)-dimensional spaces \( X_n \) and \( Y_n \) having ARS of cardinality \( 2^n \) so that the Banach-Mazur distance between \( X_n \) and \( Y_n \) is approximately \( n \).

**Lemma 2.8** Let \( X \) be a Banach space, \( \dim X = n \), and \( E \) be a subspace of \( X \), \( \dim E = m \). For \( \varepsilon \in (0, 1) \) and \( b > 0 \), define

\[
U_{b,\varepsilon}(E) = b(E \cap B(X)) + \varepsilon B(X).
\]

Then, for some \( a = a(b, \varepsilon, m) > 0 \),

\[
\text{Vol}(U_{b,\varepsilon}(E)) \leq e^{-an}\text{Vol}(B(X)).
\]

**Proof.** Fix a \( \delta > 0 \). Let \( (z_i)_{i \leq k} \) be a \( \delta \)-net of \( b(E \cap B(X)) \); by the standard volume argument, this can be achieved for some \( k \leq e^{2bm/\delta} \) (see [MS], Section 2.6). Then \( (z_i)_{i \leq k} \) is a \( (\delta + \varepsilon) \)-net of \( U_{b,\varepsilon}(E) \). Thus

\[
\text{Vol}(U_{b,\varepsilon}(E)) \leq k(\delta + \varepsilon)^n\text{Vol}(B(X)) \leq e^{2bm/\delta}(\delta + \varepsilon)^n\text{Vol}(B(X)).
\]

Now it is enough to pick \( \delta \) so that \( \delta + \varepsilon \leq 1 \). \( \square \)

**Proof of the Theorem 2.7**. Let \( \varepsilon = 1/2 \). Theorem 2.4 implies that for some \( m = m(c, q) \) and \( b = b(c, q) \),

\[
B(X) \subset \bigcup\{U_{b,1/2}(E) : E = \text{span}(D_1), D_1 \subset D, |D_1| \leq m\}.
\]

There are at most \( \binom{|D|}{m} \) distinct members \( U_{b,1/2}(E) \) in this union, so Lemma 2.8 gives us for some \( a = a(b, m) \),

\[
\text{Vol}(B(X)) \leq \left( \binom{|D|}{m} \right) e^{-an}\text{Vol}(B(X)).
\]

Hence \( \binom{|D|}{m} \geq e^{an} \). The desired estimate follows easily. \( \square \)

Now we shall find good renormings of a space with a given ARS or \((c, q)\)-quick RS.
Proposition 2.9 Let $D$ be a $c$-ARS in a Banach space $X$. Then there is a norm $||| \cdot |||$ on $X$ which satisfies $\| \cdot \| \leq ||| \cdot ||| \leq c \| \cdot \|$ and such that $D$ is a 1-ARS in $(X, ||| \cdot |||)$.

Proof. Set $||| x ||| = \inf \sum_i \| a_i x_i \|$, where the infimum is taken over all representations $x = \sum_i a_i x_i$ with $(x_i) \subset D$. Then it is enough to apply (ii) of Theorem 2.1.

For $(c, q)$-quick RS, only an equivalent quasi-norm can be constructed.

Proposition 2.10 Let $D$ be a normalized $(c, q)$-quick RS in $X$. Then there is a quasi-norm $||| \cdot |||$ on $X$ which satisfies $(1 - q) \| \cdot \| \leq ||| \cdot ||| \leq c \| \cdot \|$ and such that

(i) $D \subset B(X, ||| \cdot |||)$.
(ii) $D$ is a $(1, q)$-quick RS in $(X, ||| \cdot |||)$;
(iii) the set $\cup \{ tD : |t| \leq c \}$ is a $q$-net of $B(X, ||| \cdot |||)$.

Proof. For an $x \in X$, define

$$||| x ||| := \inf \{ \sup_{i \geq 1} |a_i| / q^{i-1} \},$$

where the infimum is taken over all sequences $(x_i) \subset D$ such that

$$x = \sum_{i=1}^{\infty} a_i x_i.$$  (4)

The homogeneity of $||| \cdot |||$, (i) and (ii) follow easily.

Now we show that $1 - q \leq ||| x ||| \leq c$ for every $x \in S(X)$. The right hand side follows from (3). Conversely, let (4) be a representation of $x$ such that $\sup_i |a_i| / q^{i-1} = \lambda < \infty$. Then

$$1 = \| \sum_{i=1}^{\infty} a_i x_i \| \leq \sum_{i=1}^{\infty} |a_i| \leq \sum_{i=1}^{\infty} \lambda q^{i-1} = \lambda (1 - q)^{-1}.$$  

Thus $\lambda \geq 1 - q$; therefore $||| x ||| \geq 1 - q$.

It remains to prove (iii). Pick any $x \in X$ with $||| x ||| \leq 1$ and $\varepsilon > 0$. Let (4) be any expansion with $|a_i| / q^{i-1} \leq 1 + \varepsilon$ for $i \geq 1$. Write

$$x - a_1 x_1 = \sum_{i=1}^{\infty} a_{i+1} x_{i+1}.$$  

Then $||| x - a_1 x_1 ||| \leq \sup_i |a_{i+1}| / q^{i-1} \leq (1 + \varepsilon)q$. This proves (iii).
Remarks.  1. The statement (iii) of Proposition 2.10 means that in the new norm one can take \( \varepsilon = q, b = c \) and \( m = 1 \) in Theorem 2.4 (ii).

2. In general, there is no equivalent norm \( ||| \cdot ||| \) satisfying (ii) or (iii) of Proposition 2.10. Indeed, take \( X = l^2_2 \) and \( D = \{(1,0),(0,1)\} \). Then \( D \) is a \( (4,1/4) \)-quick RS, but \( D \) cannot be \( (1,1/4) \)-quick RS in any norm \( ||| \cdot ||| \) on \( X \), nor can the set \( \bigcup \{tD : t \in \mathbb{R} \} \) be a \( 1/4 \)-net of \( B(X, ||| \cdot |||) \).

3 Absolutely representing systems in uniformly smooth spaces

We recall the notion of uniform smoothness (see [DGZ]). Let \( X \) be a Banach space. The modulus of smoothness of \( X \) is the function defined for \( \tau > 0 \) by

\[
\rho(\tau) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| \leq \tau \right\}.
\]

\( X \) is called uniformly smooth if \( \lim_{\tau \to 0} \rho(\tau)/\tau = 0 \).

Theorem 3.1  Let \( D \) be a normalized set in a Banach space \( X \) and \( c > 1 \). Suppose \( \rho(\tau)/\tau \leq (4c)^{-1} \) for some \( \tau \in (0,1) \). Suppose

(i) \( D \) is a \( c \)-ARS in \( X \).

Then letting \( t = 2\tau/3 \) and \( \varepsilon = 1 - \tau/3c \), we have

(ii) the set \( \pm tD \) is an \( \varepsilon \)-net of \( B(X) \).

Conversely, if \( \varepsilon < 1 \), then (ii) implies (i) with \( c = c(t, \varepsilon) \).

Remark.  The converse part of Theorem 3.1 holds in every Banach space \( X \). Indeed, it is enough to apply Theorem 2.4 and note that each \((c, q)\)-quick RS is a \( c_1 \)-ARS for \( c_1 = c(1-q)^{-1} \).

An immediate consequence follows:

Corollary 3.2  Let \( D \) be a normalized \( c \)-ARS in a uniformly smooth space \( X \). Then there are constants \( t > 0 \) and \( \varepsilon < 1 \) depending solely on \( c \) and on the modulus of smoothness of \( X \) so that the set \( \pm tD \) is an \( \varepsilon \)-net of \( B(X) \).

Recall that each superreflexive space \( X \) has an equivalent norm \( ||| \cdot ||| \) such that \((X, ||| \cdot |||)\) is a uniformly smooth space (see [DGZ]). Therefore, for each super-reflexive space \( X \) the conclusion of Corollary 3.2 will be true after an equivalent renorming.
Moreover, this property characterizes the class of super-reflexive spaces. Indeed, let $X$ be not super-reflexive; then $X$ is not super-reflexive in any equivalent norm. Let $\delta > 0$. Then there are almost square sections of $B(X)$ (see [DGZ]). More precisely, there is a system of two vectors $(z_1, z_2)$ in $S(X)$ which is $(1 + \delta)$-equivalent to the canonical vector basis of $l^2_\infty$. Let $Z = \text{span}(z_1, z_2)$. Then $Z$ is $(1 + \delta)$-isomorphic to $l^2_\infty$ and hence is $(1 + \delta)$-complemented in $X$; write $X = Z \oplus Y$ for an corresponding complement $Y$ in $X$. Put $D = \{z_i + y : y \in Y, i = 1, 2\}$. Now it is not hard to check that $D$ is a 3-ARS in $X$, but the set $\cup \{tD : t \in \mathbb{R}\}$ is not an $\varepsilon$-net of $B(X)$ unless $\varepsilon > 1 - \delta/2$. This argument was shown to me by V. Kadets.

The proof of Theorem 3.1 requires some $(\varepsilon < 1)$-net tools.

**Lemma 3.3** Let $\lambda \in [0, 1]$ and $A \subset \lambda \cdot B(X)$. Suppose that $A$ is a $\lambda$-net for $S(X)$. Then $A$ is a $\lambda$-net for $B(X)$.

**Proof.** For each $x \in B(X)$, there exists an $y \in A$ such that $\|x/\|x\| - y\| \leq \lambda$. Hence

$$
\|x - y\| = \|\|x\|(x/\|x\| - y) - (1 - \|x\|)y\|
\leq \|x\|\lambda + (1 - \|x\|)\lambda = \lambda.
$$

This completes the proof. \[\blacksquare\]

**Lemma 3.4** Let $A \subset X$ be a $(1 - \delta)$-net for $S(X)$ with $\delta \in (0, 1)$. Then, for each $\gamma \in [0, 1]$, the set $\gamma A$ is a $(1 - \gamma \delta)$-net for $S(X)$.

**Proof.** For any $x \in S(X)$ there exists an $y \in A$ such that $\|x - y\| \leq 1 - \delta$. Hence

$$
\|x - \gamma y\| = \|\gamma(x - y) + (1 - \gamma)x\|
\leq \gamma(1 - \delta) + (1 - \gamma) = 1 - \gamma \delta,
$$

which concludes the proof. \[\blacksquare\]

**Corollary 3.5** Let $\tau > 0$, $\delta \in (0, 1)$ and let $A \subset \tau \cdot B(X)$ be a $(1 - \delta)$-net for $S(X)$. Then, for each $0 \leq \gamma \leq \min(1, \frac{\tau}{\tau+\delta})$, the set $\gamma A$ is a $(1 - \gamma \delta)$-net for $B(X)$.
Proof. By Lemma 3.4, $\gamma A$ is a $(1 - \gamma \delta)$-net for $S(X)$. On the other hand, $\gamma \tau \leq 1 - \gamma \delta$, so that $\gamma A \subset (1 - \gamma \delta) \cdot B(X)$. Then, by Lemma 3.3, $\gamma A$ is a $(1 - \gamma \delta)$-net for $B(X)$. 

Now, we establish a "locally equivalent norm" on $X$.

Lemma 3.6 Let $x \in S(X)$ and $x^* \in S(X^*)$ be such that $x^*(x) = 1$. Then for each $z \in X$ we have:

$$x^*(z) \leq \|z\| \leq x^*(z) + 2\rho(\|z - x\|).$$

Proof. Put $y = x - z$. Then

$$2\rho(\|y\|) \geq \|x + y\| + \|x - y\| - 2 \geq x^*(x + y) + \|x - y\| - 2 \geq 1 + x^*(y) + \|x - y\| - 2 = \|x - y\| - x^*(x - y) = \|z\| - x^*(z).$$

Hence the right inequality is proved while the left one is trivial. 

Proof of the Theorem 3.1 Assume (i) holds. We claim that the set $\pm \tau D$ is a $(1 - \tau / 2c)$-net of $S(X)$. Indeed, given an $x \in S(X)$, one can pick a functional $x^* \in S(X^*)$ such that $x^*(x) = 1$. Then, by Theorem 2.1, we have

$$\theta x^*(x) \geq c^{-1}$$

for some $x \in D$ and some $\theta \in \{-1, 1\}$. Now apply Lemma 3.8 with $z = x - \theta \tau x$:

$$\|x - \theta \tau x\| \leq x^*(x - \theta \tau x) + 2\rho(\tau) \leq 1 - \tau c^{-1} + 2\rho(\tau) \leq 1 - \tau c^{-1} + 2 \cdot \tau / 4c = 1 - \tau / 2c.$$

This proves our claim.

Then apply Corollary 3.3: $A = \pm \tau D$, $\delta = \tau / 2c$ and $\gamma = 2/3$ will satisfy its conditions. We get that $\frac{2}{3} A$ turns to be a $(1 - \tau / 3c)$-net of $B(X)$, proving (ii).

The converse part follows from the remark above. 

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4 Absolutely representing systems and type of Banach spaces

The theory of type and cotype for normed spaces can be found in \[\text{MS}\] or \[\text{LeT}\]. By \((\varepsilon_i)\) we denote a sequence of independent random variables with the distribution \(\mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = 1/2\). Consider a Banach space \(X\) of type \(p > 1\), i.e. such that there is a \(c > 0\) such that the inequality

\[
\mathbb{E}\|\sum_{i \leq n} \varepsilon_i x_i\|^p \leq c^p \sum_{i \leq n} \|x_i\|^p
\]  

(5)

holds for each \(n > 0\) and each sequence \((x_i)_{i \leq n}\) in \(X\). By \(T_p(X)\) we denote the least constant \(c\) for which the inequality (5) always holds. For \(p > 1\), we denote by \(p^*\) the conjugate number: \(1/p + 1/p^* = 1\).

The following result is contained implicitly in \[\text{P}\] and is known as a ”dimension-free variant of Caratheodory’s theorem”. For the sake of completeness, we include its proof.

Theorem 4.1 Let \(D\) be a normalized set in a Banach space \(X\) of type \(p > 1\). Suppose that for some \(c > 1\)

(i) \(D\) is a \(c\)-ARS.

Let \(k > 0\). Put \(c_1 = c\) and \(\varepsilon = 4cT_p(X)k^{-1/p^*}\). Then

(ii) the set \(\{c_1 k^{-1} \sum_{i \leq k} \pm x_i : (x_i) \subset D\}\) is an \(\varepsilon\)-net of \(B(X)\).

Conversely, (ii) implies (i) with \(c = c(c_1, k)\).

Applying Theorem 2.4, we obtain

Corollary 4.2 Let \(D\) be a normalized set in a Banach space \(X\) of type \(p > 1\). Suppose that for some \(c > 1\)

(i) \(D\) is a \(c\)-ARS.

Then, for some \(c_1 = c_1(c, p, T_p(X))\) and \(q = q(c, p, T_p(X))\), we have:

(ii) \(D\) is a \((c_1, q)\)-quick RS.

Conversely, (ii) implies (i) with \(c = c(c_1, q)\).

Before the proof of Theorem 4.1, let us give some comments. A Banach space \(X\) is called B-convex if it does not contain \(l_1^n\) uniformly. \(X\) is B-convex iff \(X\) is of some type \(p > 1\). It follows that if \(X\) is a B-convex Banach space and
$D$ is a $c$-ARS in some subspace of $X$, then $D$ is a $(c_1, q)$-quick RS, where the constants $c_1$ and $q$ depend only on $c$ and $X$.

Moreover, the latter property characterizes B-convex Banach spaces. Indeed, fix a space $X$ which is not B-convex. Then, for each positive integer $n$, there is a sequence $(x_{n,i})_{i \leq n}$ in $X$ which is 2-equivalent to the canonical vector basis of $l_1^n$. Take $D_n = (x_{n,i})_{i \leq n}$ and $Y_n = \text{span}(D_n)$. Then $D_n$ is a 2-ARS in $Y_n$. However, letting $n \to \infty$, we see that $D_n$ cannot be a $(c_1, q)$-quick RS for fixed $c_1$ and $q$.

One exciting problem remains unsolved. We have got that each ARS in a B-convex space $X$ is a $(c, q)$-quick RS for some $c$ and $q$. Does this happen only in B-convex spaces?

**Proof of Theorem 4.1** Fix any $x \in B(X)$. Then, for some $(x_i) \subset D$, there is a representation $x = \sum_{i=1}^{\infty} a_i x_i$ with $\sum |a_i| \leq c$.

Then there is a sequence $(\xi_j)_{j \geq 1}$ of independent random variables with the following distribution for every $i, j \geq 1$:

$$P\{\xi_j = \text{sign}(a_i)cx_i\} = c^{-1}|a_i|,$$

$$P\{\xi_j = 0\} = 1 - c^{-1} \sum_n |a_n|.$$

Therefore $E\xi_j = x$ for each $j$. Now, since $\xi_j$ are independent, we have

$$E\| \sum_{j \leq k} (\xi_j - E\xi_j) \|^p \leq (2T_p(X))^p \sum_{j \leq k} E\| \xi_j - E\xi_j \|^p$$

(see [LeT], Chapter 9). Note that $E\|\xi_j - E\xi_j\|^p \leq (c + 1)^p$; hence

$$E\| k^{-1} \sum_{j \leq k} (\xi_j - E\xi_j) \|^p \leq (2T_p(X))^p k^{-p} \cdot k(c + 1)^p.$$

Therefore

$$E\| - x + k^{-1} \sum_{j \leq k} \xi_j \|^p \leq \left(2T_p(X)(c + 1)k^{-1/p'}\right)^p.$$

In particular, there is one realization of the random variable $(-x + k^{-1} \sum_{j \leq k} \xi_j)$ so that

$$\| - x + k^{-1} \sum_{j \leq k} \xi_j \| \leq 2T_p(X)(c + 1)k^{-1/p'}.$$

This concludes the proof.
In conclusion, let us show how these results provide an estimate from above on the dimension of nice sections of the cube. The following result due to B. Maurey is proved in [P].

**Theorem 4.3 (B. Maurey).** Let $X$ be a finite dimensional space, $p > 1$ and $T_p(X^*) \leq C$. Suppose that $X$ is $c$-isomorphic to some subspace of $l_n^\infty$. Then, for some $a = a(p, C, c)$, we have

$$\dim X \leq a \log n.$$  

**Proof.** By duality, $X^*$ is $c$-isomorphic to some quotient space of $l_1^n$. Then, Theorem 2.1 gives us a $c$-ARS $D$ in $X^*$ with $|D| = n$. By Corollary 1.2, $D$ is a $(c_1, q)$-quick RS in $X^*$ for some $c_1 = c_1(p, C, c)$ and $q = q(p, C, c)$. Then Theorem 2.7 yields $n \geq e^{a \dim X}$ for some $a = a(c_1, q) > 0$. □

**References**


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