Absolutely representing systems, uniform smoothness, and type

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Abstract

Absolutely representing system (ARS) in a Banach space X is a set $D \subset X$ such that every vector x in X admits a representation by an absolutely convergent series $x = \sum_i a_i x_i$ with (a_i) reals and $(x_i) \subset D$. We investigate some general properties of ARS. In particular, ARS in uniformly smooth and in B-convex Banach spaces are characterized via ε-nets of the unit balls. Every ARS in a B-convex Banach space is quick, i.e. in the representation above one can achieve $\|a_i x_i\| < cq^i \|x\|$, $i = 1, 2, \dots$ for some constants $c > 0$ and $q \in (0, 1)$.

1 Introduction

The concept of absolutely representing system (ARS) goes back to Banach and Mazur ([B], p. 109–110).

Definition 1.1 *A set D in a Banach space X is called* absolutely representing system (ARS) *if for every $x \in X$ there are scalars (a_i) and elements $(x_i) \subset D$ such that*

$$x = \sum_{i=1}^{\infty} a_i x_i \quad \text{and} \quad \sum_{i=1}^{\infty} \|a_i x_i\| < \infty.$$

It can be observed (Section 2) that if D is an ARS, then there exist a constant c such that each $x \in X$ admits a representation $x = \sum_{i=1}^{\infty} a_i x_i$ with $\sum \|a_i x_i\| \leq c\|x\|$. Then we call D a "c-ARS".

For needs of complex analysis, ARS were defined also in locally convex topological spaces [K 81]. In the theory of analytical functions such ARS happen to be a convenient tool: see [K 96], [G], [A]. Many results of general kind on ARS are obtained by Yu. Korobeĭnik and his collaborators: see, for example, [K 81], [K 86], [KK].

In the present paper we restrict ourselves to the theory of ARS in Banach spaces, which is still not quite explored. Some non-trivial examples of ARS in l_2 were found by I.Shraĭfel [S 93]. It should be noted that each example of a c-ARS in l_2^n provides by Theorem 3.1 an example of an ε-net of the n-dimensional Euclidean ball, $\varepsilon = \varepsilon(c) < 1$. See also [S 95] for results on ARS in Hilbert spaces.

Some general results concerning ARS in Banach spaces and, particularly, in uniformly smooth spaces, were obtained in [V]. There was introduced the notion of (c, q)-quick representing system, which is considerably stronger than that of ARS.

Definition 1.2 *Let $c > 0$ and $q \in (0, 1)$. A set D in a Banach space X is called (c, q)-quick representing system (or (c, q)-quick RS) if for each $x \in X$ there are scalars (a_i) and elements $(x_i) \subset D$ such that*

$$x = \sum_{i=1}^{\infty} a_i x_i \quad \text{and} \quad \|a_i x_i\| \leq c q^{i-1} \quad \text{for} \quad i \geq 1.$$

It is clear that each (c, q)-quick RS is an ARS. Despite of the strong restrictions in Definition 1.2, there exist Banach spaces X in which every ARS is, in turn, a (c_1, q)-quick RS for some c_1 and q. In [V] it was proved that this happens in each super-reflexive space X.

In the present paper we generalize this result to all B-convex Banach spaces. Suppose a space X is B-convex and Y is a subspace of X. We show that every c-ARS in Y is a (c_1, q)-quick RS for some c_1 and q depending only on c and on X. This latter statement characterizes the class of B-convex Banach spaces.

We characterize ARS and (c, q)-quick RS in uniformly smooth and B-convex Banach spaces via ε-nets of the unit balls. As a consequence, we have a theorem of B. Maurey [P] stating that the dimension of a subspace Y of l_∞^n with Y^* of a good type is at most $c \log n$.

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2 Characterizations of ARS and quick RS

Let $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ be sequences in Banach spaces X and Y respectively, and let $c > 0$. We call (x_i) and (y_i) *c-equivalent* if there is a linear operator $T : \overline{\text{span}}(x_i) \to \overline{\text{span}}(y_i)$ which maps x_i to y_i, and satisfies $\|T\|\|T^{-1}\| \le c$.

D being a non-empty set, we denote the unit vectors in $l_1(D)$ by e_d, $d \in D$.

The following useful result is more or less known: the equivallence (i)⇔(iv) goes back to S. Mazur ([B], p. 110), see also [V].

Theorem 2.1 *Given a complete normalized set D in a Banach space X, the following are equivalent:*

(i) D is an ARS;

(ii) there is a $c > 0$ such that each $x \in B(X)$ can be represented by a series $x = \sum_{i=1}^{\infty} a_i x_i$ with $\sum \|a_i x_i\| \le c$. Then we call D a "c-ARS";

(iii) there is a quotient map $q : l_1(D) \to Z$ such that the sequence $(d)_{d\in D}$ is c-equivalent to $(qe_d)_{d\in D}$;

(iv) there is a $c > 0$ such that for every $x^* \in S(X^*)$ one has $\sup_{d\in D} |x^*(d)| \ge c^{-1}$.

In (ii), (iii) and (iv) the infimums of possible constants c are equal and are attained.

Let us observe some nice consequences. The first one states that ARS are stable under fairly large perturbations. Let A and B be sets in a Banach space. By definition, put $\rho(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|$.

Corollary 2.2 *Let D and D_1 be normalized sets in X. If D is a c-ARS and $\rho(D, D_1) = \varepsilon < c^{-1}$, then D_1 is a c_1-ARS, where $c_1 = (1 - \varepsilon c)^{-1} c$.*

Proof. It follows easily from (iv) of Theorem 2.1. ■

Proposition 2.3 *Let D be a c-ARS in a Banach space X.*

(i) If X is separable, then some countable subset D_1 of D is also a c-ARS.

(ii) Let $\dim X = n$ and $c_1 > c$. Then some subset D_1 of D is a c_1-ARS and $|D_1| \le e^{an}$, where $a = 2\left(c^{-1} - c_1^{-1}\right)^{-1}$.

(iii) Let $\dim X = n$ and $\varepsilon > 0$. Then every $x \in B(X)$ can be represented by a sum $x = \sum_{i=1}^{n} a_i x_i$ with $(x_i) \subset D$ and $\sum \|a_i x_i\| \le c + \varepsilon$.

Proof. Clearly, we may assume that D is normalized. Then (i) follows in the standard way from (iv) of Theorem 2.1.

(ii). Let $\varepsilon = c^{-1} - c_1^{-1}$. Consider a maximal subset D_1 of D such that $\|x - y\| > \varepsilon$ for $x, y \in D_1$, $x \neq y$. By maximality, $\rho(D, D_1) \leq \varepsilon$. Applying Corollary 2.2, we see that D_1 is a c_1-ARS. Note that the balls $(d_1 + (\varepsilon/2)B(X))_{d_1 \in D_1}$ are mutually disjoint and are contained in $(1 + \varepsilon/2)B(X)$. By comparing the volumes we get $|D_1| \leq e^{2n/\varepsilon}$.

(iii). By (ii), we can extract from D a finite $(c + \varepsilon)$-ARS $(x_i)_{i \leq m}$. By (iii) of Theorem 2.1, there is a quotient map $q : l_1^m \to Z$ such that the sequences $(x_i)_{i \leq m}$ and $(qe_i)_{i \leq m}$ are $(c + \varepsilon)$-equivalent. Let $T : X \to Z$ be the isomorphism corresponding to this equivalence. We have $\dim Z = n$ and $B(Z) = \text{a.conv}(qe_i)_{i \leq m}$. Now we use a simple consequence of Caratheodory's theorem:

- Let K be a finite set in $\mathbf{R}^n$. Let a vector z lie on the boundary of a.conv(K). Then $z \in \text{a.conv}(z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n \in K$.

Applying this theorem to $K = (qe_i)_{i \leq m}$, we see that each $z \in S(Z)$ can be represented by a sum $z = \sum_{k=1}^n a_k(qe_{i_k})$ for some subsequence $(qe_{i_k})_{k \leq n}$ of (qe_i) and scalars (a_k) with $\sum_{k=1}^n |a_k| = 1$.

Let $x \in B(X)$. Setting $z = Tx/\|Tx\|$ in the preceding observation, we can write

$$Tx = \sum_{k=1}^n b_k(qe_{i_k}) = \sum_{k=1}^n b_k(Tx_{i_k}) \quad \text{with} \quad \sum_{k=1}^n |b_k| \leq \|T\|.$$

Thus $x = \sum_{k=1}^n b_k x_{i_k}$, and

$$\sum_{k=1}^n \|b_k x_{i_k}\| \leq \|T^{-1}\| \sum_{k=1}^n \|b_k(qe_{i_k})\| \leq \|T^{-1}\| \sum_{k=1}^n |b_k| \leq \|T^{-1}\| \|T\| \leq c + \varepsilon.$$

The proof is complete. ■

Remarks. 1. The estimate in (ii) is sharp by order: Corollary 4.2 and Theorem 2.7 show that any ARS in a B-convex Banach space X has at least exponential number of terms with respect to $\dim X$.

2. In general, one can not put $\varepsilon = 0$ in (iii). Indeed, consider $X = l_2^2$ and let D be a countable dense subset of $S(l_2^2)$. Then D is a 1-ARS. However, there are points $x \in S(l_2^2) \setminus \text{a.conv}(D)$; thus (iii) fails unless $\varepsilon > 0$.

Now we give a general characterization of (c, q)-quick RS.

Theorem 2.4 *Let D be a normalized set in a Banach space X. Suppose*

(i) D is a (c,q)-quick RS.

Then, given an $\varepsilon > 0$, there are $m = m(c,q,\varepsilon)$ and $b = c(1-q)^{-1}$ such that

(ii) the set $b \cdot \bigcup \{\text{a.conv}(D_1) : D_1 \subset D, |D_1| \le m\}$ is an ε-net of $B(X)$.

Conversely, if $\varepsilon < 1$, then (ii) implies (i) with $c = b/\varepsilon$ and $q = \varepsilon^{1/m}$.

Proof. Assume (i) holds. Let m be so that

$$\sum_{i>m} cq^{i-1} \le \varepsilon. \tag{1}$$

Let $x \in B(X)$. For some $(x_i) \subset D$ we have $x = \sum_{i=1}^{\infty} a_i x_i$ with $|a_i| \le cq^{i-1}$. Then, by (1),

$$\|x - \sum_{i \le m} a_i x_i\| = \|\sum_{i>m} a_i x_i\| \le \varepsilon,$$

while

$$\sum_{i \le m} |a_i| \le c(1-q)^{-1} = b.$$

This proves (ii).

Conversly, assume (ii) holds. Fix an $x \in B(X)$. We shall find appropriate expansion $x = \sum_i a_i x_i$ by successive iterations. S_n will denote the partial sum $\sum_{i \le n} a_i x_i$ (we assume $S_0 = 0$).

Suppose that for some $k \ge 1$ the system $(a_i)_{i \le (k-1)m}$ is constructed. By (ii), there are scalars $(a_{k,i})_{i \le m}$ and vectors $(x_{k,i})_{i \le m} \subset D$ such that $|a_{k,i}| \le b$ for $i \le m$ and

$$\left\| \frac{x - S_{(k-1)m}}{\|x - S_{(k-1)m}\|} - \sum_{i \le m} a_{k,i} x_{k,i} \right\| \le \varepsilon. \tag{2}$$

Put $a_{(k-1)m+i} = \|x - S_{(k-1)m}\| a_{k,i}$ for $1 \le i \le m$. Note that for each k

$$x - S_{km} = x - S_{(k-1)m} - \|x - S_{(k-1)m}\| \cdot \sum_{i \le m} a_{k,i} x_{k,i}.$$

Therefore, by (2), $\|x - S_{km}\| \le \|x - S_{(k-1)m}\| \cdot \varepsilon$. By the inductive argument we get $\|x - S_{km}\| \le \varepsilon^k$. Hence for $k \ge 0$ and $1 \le i \le m$,

$$|a_{km+i}| = \|x - S_{km}\| a_{k+1,i} \le \varepsilon^k b \le \varepsilon^{(km+i)/m-1} b = \varepsilon^{-1} b \cdot (\varepsilon^{1/m})^{km+i}.$$

Hence $|a_i| \le \varepsilon^{-1} b (\varepsilon^{1/m})^i$ for $i \ge 1$. This proves (i) with $c = b\varepsilon^{-1+1/m} \le b/\varepsilon$ and $q = \varepsilon^{1/m}$. ■

Theorem 2.4 yields that, actually, the tightness of the definition of (c, q)-quick RS can be substantially loosened. Let (b_i) be a scalar sequence. We say that a set D in a Banach space X is a (b_i)-*representing system*, if every $x \in B(X)$ admits a representation by a convergent series $x = \sum_i a_i x_i$ with $(x_i) \subset D$ and $(a_i) \subset \mathbf{R}$, $\|a_i x_i\| \le |b_i|$ for each i.

Corollary 2.5 *Let D be a set in a Banach space X and let $\sum b_i$ be an absolutely convergent scalar series. Suppose*

(i) D is a (b_i)-representing system.

Then there are constants c and q dependent only on (b_i), such that

(ii) D is a (c,q)-quick representing system.

Conversely, (ii) implies (i) with $b_i = cq^{i-1}$.

Proof. Suppose (i) holds. Let m be so that $\sum_{i>m} |b_i| \le 1/2$. It is enough to show that (ii) of Theorem 2.4 holds for $\varepsilon = 1/2$. Fix $x \in B(X)$ and write its representation: $x = \sum_{i\ge 1} a_i x_i$ with $\|a_i x_i\| \le |b_i|$. Then

$$\|x - \sum_{i\le m} a_i x_i\| = \|\sum_{i>m} a_i x_i\| \le \sum_{i>m} \|a_i x_i\| \le \sum_{i>m} |b_i| \le 1/2.$$

Thus (ii) holds. The converse part is obvious. ∎

Like ARS, quick representing systems are also stable under fairly large perturbations. The following analogue of Corollary 2.2 can easily be derived from Theorem 2.4.

Corollary 2.6 *Let D and D_1 be normalized sets in X. If D is a (c, q)-quick RS and $\rho(D, D_1) = \varepsilon < (1-q)/c$, then D_1 is a (c_1, q_1)-quick RS, where c_1 and q_1 depend solely on c, q and ε.*

Another consequence of Theorem 2.4 states that the cardinality of every (c, q)-quick RS in a finite-dimensional space is large.

Theorem 2.7 *Let D be a (c, q)-quick RS in a n-dimensional Banach space X. Then $|D| \ge e^{an}$ for some $a = a(c, q) > 0$.*

Before we prove this result, observe that there are many spaces posessing ARS of small cardinalities. Indeed, E. Gluskin's construction [Gl] gives us n-dimensional spaces X_n and Y_n having ARS of cardinality $2n$ so that the Banach-Mazur distance between X_n and Y_n is approximately n.

Lemma 2.8 *Let X be a Banach space,* $\dim X = n$, *and E be a subspace of X,* $\dim E = m$. *For $\varepsilon \in (0,1)$ and $b > 0$, define*

$$U_{b,\varepsilon}(E) = b(E \cap B(X)) + \varepsilon B(X).$$

Then, for some $a = a(b, \varepsilon, m) > 0$,

$$\mathrm{Vol}(U_{b,\varepsilon}(E)) \leq e^{-an}\mathrm{Vol}(B(X)).$$

Proof. Fix a $\delta > 0$. Let $(z_i)_{i \leq k}$ be a δ-net of $b(E \cap B(X))$; by the standard volume argument, this can be achieved for some $k \leq e^{2bm/\delta}$ (see [MS], Section 2.6) Then $(z_i)_{i \leq k}$ is a $(\delta + \varepsilon)$-net of $U_{b,\varepsilon}(E)$. Thus

$$\mathrm{Vol}(U_{b,\varepsilon}(E)) \leq k(\delta + \varepsilon)^n \mathrm{Vol}(B(X)) \leq e^{2bm/\delta}(\delta + \varepsilon)^n \mathrm{Vol}(B(X)).$$

Now it is enough to pick δ so that $\delta + \varepsilon \leq 1$. ■

Proof of the Theorem 2.7. Let $\varepsilon = 1/2$. Theorem 2.4 implies that for some $m = m(c, q)$ and $b = b(c, q)$,

$$B(X) \subset \bigcup \{U_{b,1/2}(E) : E = \mathrm{span}(D_1), D_1 \subset D, |D_1| \leq m\}.$$

There are at most $\binom{|D|}{m}$ distinct members $U_{b,1/2}(E)$ in this union, so Lemma 2.8 gives us for some $a = a(b, m)$,

$$\mathrm{Vol}(B(X)) \leq \binom{|D|}{m} e^{-an}\mathrm{Vol}(B(X)).$$

Hence $\binom{|D|}{m} \geq e^{an}$. The desired estimate follows easily. ■

Now we shall find good renormings of a space with a given ARS or (c, q)-quick RS.

Proposition 2.9 *Let D be a c-ARS in a Banach space X. Then there is a norm $|||\cdot|||$ on X which satisfies $\|\cdot\| \leq |||\cdot||| \leq c\|\cdot\|$ and such that D is a 1-ARS in $(X, |||\cdot|||)$.*

Proof. Set $|||x||| = \inf \sum_i \|a_i x_i\|$, where the infimun is taken over all representations $x = \sum_i a_i x_i$ with $(x_i) \subset D$. Then it is enough to apply (ii) of Theorem 2.1. ■

For (c, q)-quick RS, only an equivalent quasi-norm can be constructed.

Proposition 2.10 *Let D be a normalized (c, q)-quick RS in X. Then there is a quasi-norm $|||\cdot|||$ on X which satisfies $(1-q)\|\cdot\| \leq |||\cdot||| \leq c\|\cdot\|$ and such that*

(i) $D \subset B(X, |||\cdot|||)$.

(ii) D is a $(1, q)$-quick RS in $(X, |||\cdot|||)$;

(iii) the set $\cup\{tD : |t| \leq c\}$ is a q-net of $B(X, |||\cdot|||)$;

Proof. For an $x \in X$, define

$$|||x||| := \inf\{\sup_{i\geq 1} |a_i|/q^{i-1}\}, \tag{3}$$

where the infimum is taken over all sequences $(x_i) \subset D$ such that

$$x = \sum_{i=1}^{\infty} a_i x_i. \tag{4}$$

The homogenity of $|||\cdot|||$, (i) and (ii) follow easily.

Now we show that $1 - q \leq |||x||| \leq c$ for every $x \in S(X)$. The right hand side follows from (3). Conversely, let (4) be a representation of x such that $\sup_i |a_i|/q^{i-1} = \lambda < \infty$. Then

$$1 = \|\sum_{i=1}^{\infty} a_i x_i\| \leq \sum_{i=1}^{\infty} |a_i| \leq \sum_{i=1}^{\infty} \lambda q^{i-1} = \lambda(1-q)^{-1}.$$

Thus $\lambda \geq 1 - q$; therefore $|||x||| \geq 1 - q$.

It remains to prove (iii). Pick any $x \in X$ with $|||x||| \leq 1$ and $\varepsilon > 0$. Let (4) be any expansion with $|a_i|/q^{i-1} \leq 1 + \varepsilon$ for $i \geq 1$. Write

$$x - a_1 x_1 = \sum_{i=1}^{\infty} a_{i+1} x_{i+1}.$$

Then $|||x - a_1 x_1||| \leq \sup_i |a_{i+1}|/q^{i-1} \leq (1+\varepsilon)q$. This proves (iii). ■

Remarks. 1. The statement (iii) of Proposition 2.10 means that in the new norm one can take $\varepsilon = q$, $b = c$ and $m = 1$ in Theorem 2.4 (ii).

2. In general, there is no equivalent *norm* $|||\cdot|||$ satisfying (ii) or (iii) of Proposition 2.10. Indeed, take $X = l_2^2$ and $D = \{(1,0),(0,1)\}$. Then D is a $(4, 1/4)$-quick RS, but D cannot be $(1, 1/4)$-quick RS in any norm $|||\cdot|||$ on X, nor can the set $\cup\{tD : t \in \mathbf{R}\}$ be a $1/4$-net of $B(X, |||\cdot|||)$.

3 Absolutely representing systems in uniformly smooth spaces

We recall the notion of uniform smoothness (see [DGZ]). Let X be a Banach space. The *modulus of smoothness* of X is the function defined for $\tau > 0$ by

$$\rho(\tau) = \sup\{(\|x+y\| + \|x-y\|)/2 - 1 : x, y \in X, \|x\| = 1, \|y\| \leq \tau\}.$$

X is called *uniformly smooth* if $\lim_{\tau\to 0} \rho(\tau)/\tau = 0$.

Theorem 3.1 *Let D be a normalized set in a Banach space X and $c > 1$. Suppose $\rho(\tau)/\tau \leq (4c)^{-1}$ for some $\tau \in (0,1)$. Suppose*

(i) D is a c-ARS in X.

Then letting $t = 2\tau/3$ and $\varepsilon = 1 - \tau/3c$, we have

(ii) the set $\pm tD$ is an ε-net of $B(X)$.

Conversely, if $\varepsilon < 1$, then (ii) implies (i) with $c = c(t, \varepsilon)$.

Remark. The converse part of Theorem 3.1 holds in every Banach space X. Indeed, it is enough to apply Theorem 2.4 and note that each (c, q)-quick RS is a c_1-ARS for $c_1 = c(1-q)^{-1}$.

An immediate consequence follows:

Corollary 3.2 *Let D be a normalized c-ARS in a uniformly smooth space X. Then there are constants $t > 0$ and $\varepsilon < 1$ depending solely on c and on the modulus of smoothness of X so that the set $\pm tD$ is an ε-net of $B(X)$.*

Recall that each superreflexive space X has an equivalent norm $|||\cdot|||$ such that $(X, |||\cdot|||)$ is a uniformly smooth space (see [DGZ]). Therefore, for each super-reflexive space X the conclusion of Corollary 3.2 will be true after an equivalent renorming.

Moreover, this property characterizes the class of super-reflexive spaces. Indeed, let X be not super-reflexive; then X is not super-reflexive in any equivalent norm. Let $\delta > 0$. Then there are almost square sections of $B(X)$ (see [DGZ]). More precisely, there is a system of two vectors (z_1, z_2) in $S(X)$ which is $(1+\delta)$-equivalent to the canonical vector basis of l^2_∞. Let $Z = \text{span}(z_1, z_2)$. Then Z is $(1+\delta)$-isomorphic to l^2_∞ and hence is $(1+\delta)$-complemented in X; write $X = Z \oplus Y$ for an corresponding complement Y in X. Put $D = \{z_i + y : y \in Y,\ i = 1, 2\}$. Now it is not hard to check that D is a 3-ARS in X, but the set $\cup\{tD : t \in \mathbf{R}\}$ is not an ε-net of $B(X)$ unless $\varepsilon > 1 - \delta/2$. This argument was shown to me by V. Kadets.

The proof of Theorem 3.1 requires some $(\varepsilon < 1)$-net tools.

Lemma 3.3 *Let $\lambda \in [0, 1]$ and $A \subset \lambda \cdot B(X)$. Suppose that A is a λ-net for $S(X)$. Then A is a λ-net for $B(X)$.*

Proof. For each $x \in B(X)$, there exists an $y \in A$ such that $\|x/\|x\| - y\| \leq \lambda$. Hence

$$\begin{aligned} \|x - y\| &= \|\|x\|(x/\|x\| - y) - (1 - \|x\|)y\| \\ &\leq \|x\|\lambda + (1 - \|x\|)\lambda = \lambda. \end{aligned}$$

This completes the proof. ∎

Lemma 3.4 *Let $A \subset X$ be a $(1-\delta)$-net for $S(X)$ with $\delta \in (0, 1)$. Then, for each $\gamma \in [0, 1]$, the set γA is a $(1 - \gamma\delta)$-net for $S(X)$.*

Proof. For any $x \in S(X)$ there exists an $y \in A$ such that $\|x - y\| \leq 1 - \delta$. Hence

$$\begin{aligned} \|x - \gamma y\| &= \|\gamma(x - y) + (1 - \gamma)x\| \\ &\leq \gamma(1 - \delta) + (1 - \gamma) = 1 - \gamma\delta, \end{aligned}$$

which concludes the proof. ∎

Corollary 3.5 *Let $\tau > 0$, $\delta \in (0, 1)$ and let $A \subset \tau \cdot B(X)$ be a $(1 - \delta)$-net for $S(X)$. Then, for each $0 \leq \gamma \leq \min(1, \frac{1}{\tau+\delta})$, the set γA is a $(1 - \gamma\delta)$-net for $B(X)$.*

Proof. By Lemma 3.4, γA is a $(1-\gamma\delta)$-net for $S(X)$. On the other hand, $\gamma\tau \leq 1-\gamma\delta$, so that $\gamma A \subset (1-\gamma\delta)\cdot B(X)$. Then, by Lemma 3.3, γA is a $(1-\gamma\delta)$-net for $B(X)$. ∎

Now, we establish a "locally equivalent norm" on X.

Lemma 3.6 *Let $x \in S(X)$ and $x^* \in S(X^*)$ be such that $x^*(x) = 1$. Then for each $z \in X$ we have:*

$$x^*(z) \leq \|z\| \leq x^*(z) + 2\rho(\|z-x\|).$$

Proof. Put $y = x - z$. Then

$$\begin{aligned} 2\rho(\|y\|) &\geq \|x+y\| + \|x-y\| - 2 \\ &\geq x^*(x+y) + \|x-y\| - 2 \\ &\geq 1 + x^*(y) + \|x-y\| - 2 \\ &= \|x-y\| - x^*(x-y) = \|z\| - x^*(z). \end{aligned}$$

Hence the right inequality is proved while the left one is trivial. ∎

Proof of the Theorem 3.1. Assume (i) holds. We claim that the set $\pm\tau D$ is a $(1-\tau/2c)$-net of $S(X)$. Indeed, given an $x \in S(X)$, one can pick a functional $x^* \in S(X^*)$ such that $x^*(x) = 1$. Then, by Theorem 2.1, we have

$$\theta x^*(x) \geq c^{-1}$$

for some $x \in D$ and some $\theta \in \{-1, 1\}$. Now apply Lemma 3.6 with $z = x - \theta\tau x$:

$$\begin{aligned} \|x - \theta\tau x\| &\leq x^*(x-\theta\tau x) + 2\rho(\tau) \\ &\leq 1 - \tau c^{-1} + 2\rho(\tau) \\ &\leq 1 - \tau c^{-1} + 2\cdot\tau/4c = 1 - \tau/2c. \end{aligned}$$

This proves our claim.

Then apply Corollary 3.5: $A = \pm\tau D$, $\delta = \tau/2c$ and $\gamma = 2/3$ will satisfy its conditions. We get that $\frac{2}{3}A$ turns to be a $(1-\tau/3c)$-net of $B(X)$, proving (ii).

The converse part follows from the remark above. ∎

4 Absolutely representing systems and type of Banach spaces

The theory of type and cotype for normed spaces can be found in [MS] or [LeT]. By (ε_i) we denote a sequence of independent random variables with the distibution $\mathbf{P}\{\varepsilon_i = 1\} = \mathbf{P}\{\varepsilon_i = -1\} = 1/2$. Consider a Banach space X of type $p > 1$, i.e. such that there is a $c > 0$ such that the inequality

$$\mathbf{E}\|\sum_{i\leq n} \varepsilon_i x_i\|^p \leq c^p \sum_{i\leq n} \|x_i\|^p \tag{5}$$

holds for each $n > 0$ and each sequence $(x_i)_{i\leq n}$ in X. By $T_p(X)$ we denote the least constant c for which the inequality (5) always holds. For $p > 1$, we denote by p^* the conjugate number: $1/p + 1/p^* = 1$.

The following result is contained implicitly in [P] and is known as a "dimension-free variant of Caratheodory's theorem". For the sake of completeness, we include its proof.

Theorem 4.1 *Let D be a normalized set in a Banach space X of type $p > 1$. Suppose that for some $c > 1$*

(i) D is a c-ARS.

Let $k > 0$. Put $c_1 = c$ and $\varepsilon = 4cT_p(X)k^{-1/p^*}$. Then

(ii) the set $\{c_1 k^{-1} \sum_{i\leq k} \pm x_i : (x_i) \subset D\}$ is an ε-net of $B(X)$.

Conversely, (ii) implies (i) with $c = c(c_1, k)$.

Applying Theorem 2.4, we obtain

Corollary 4.2 *Let D be a normalized set in a Banach space X of type $p > 1$. Suppose that for some $c > 1$*

(i) D is a c-ARS.

Then, for some $c_1 = c_1(c, p, T_p(X))$ and $q = q(c, p, T_p(X))$, we have:

(ii) D is a (c_1, q)-quick RS.

Conversely, (ii) implies (i) with $c = c(c_1, q)$.

Before the proof of Theorem 4.1, let us give some comments. A Banach space X is called *B-convex* if it does not contain l_1^n uniformly. X is B-convex iff X is of some type $p > 1$. It follows that if X is a B-convex Banach space and

D is a c-ARS in some subspace of X, then D is a (c_1, q)-quick RS, where the constants c_1 and q depend only on c and X.

Moreover, the latter property characterizes B-convex Banach spaces. Indeed, fix a space X which is not B-convex. Then, for each positive integer n, there is a sequence $(x_{n,i})_{i\le n}$ in X which is 2-equivalent to the canonical vector basis of l_1^n. Take $D_n = (x_{n,i})_{i\le n}$ and $Y_n = \mathrm{span}(D_n)$. Then D_n is a 2-ARS in Y_n. However, letting $n \to \infty$, we see that D_n cannot be a (c_1, q)-quick RS for fixed c_1 and q.

One exciting problem remains unsolved. We have got that each ARS in a B-convex space X is a (c, q)-quick RS for some c and q. Does this happen only in B-convex spaces?

Proof of Theorem 4.1. Fix any $x \in B(X)$. Then, for some $(x_i) \subset D$, there is a representation $x = \sum_{i=1}^{\infty} a_i x_i$ with $\sum |a_i| \le c$.

Then there is a sequence $(\xi_j)_{j\ge 1}$ of independent random variables with the following distribution for every $i, j \ge 1$:

$$\begin{aligned}&\mathbf{P}\{\xi_j = \mathrm{sign}(a_i) c x_i\} = c^{-1}|a_i|,\\ &\mathbf{P}\{\xi_j = 0\} = 1 - c^{-1}\sum_n |a_n|.\end{aligned}$$

Therefore $\mathbf{E}\xi_j = x$ for each j. Now, since ξ_j are independent, we have

$$\mathbf{E}\|\sum_{j\le k}(\xi_j - \mathbf{E}\xi_j)\|^p \le (2T_p(X))^p \sum_{j\le k}\mathbf{E}\|\xi_j - \mathbf{E}\xi_j\|^p$$

(see [LeT], Chapter 9). Note that $\mathbf{E}\|\xi_j - \mathbf{E}\xi_j\|^p \le (c+1)^p$; hence

$$\mathbf{E}\|k^{-1}\sum_{j\le k}(\xi_j - \mathbf{E}\xi_j)\|^p \le (2T_p(X))^p k^{-p}\cdot k(c+1)^p.$$

Therefore

$$\mathbf{E}\| - x + k^{-1}\sum_{j\le k}\xi_j\|^p \le \Big(2T_p(X)(c+1)k^{-1/p^*}\Big)^p.$$

In particular, there is one realization of the random variable $(-x + k^{-1}\sum_{j\le k}\xi_j)$ so that

$$\| - x + k^{-1}\sum_{j\le k}\xi_j\| \le 2T_p(X)(c+1)k^{-1/p^*}.$$

This concludes the proof. ∎

In conclusion, let us show how these results provide an estimate from above on the dimension of nice sections of the cube. The following result due to B. Maurey is proved in [P].

Theorem 4.3 *(B. Maurey). Let X be a finite dimensional space, $p > 1$ and $T_{p^*}(X^*) \leq C$. Suppose that X is c-isomorphic to some subspace of l_∞^n. Then, for some $a = a(p, C, c)$, we have*

$$\dim X \leq a \log n.$$

Proof. By duality, X^* is c-isomorphic to some quotient space of l_1^n. Then, Theorem 2.1 gives us a c-ARS D in X^* with $|D| = n$. By Corollary 4.2, D is a (c_1, q)-quick RS in X^* for some $c_1 = c_1(p, C, c)$ and $q = q(p, C, c)$. Then Theorem 2.7 yields $n \geq e^{a \dim X}$ for some $a = a(c_1, q) > 0$. ∎

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