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Author
Cranton Heller, Lauren

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Multigraded Regularity and Betti Numbers on Smooth Projective Toric Varieties

by

Lauren Cranton Heller

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Committee in charge:

Professor David Eisenbud, Chair
Professor Martin Olsson
Professor Line Mikkelsen

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Abstract

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Lauren Cranton Heller

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The Castelnuovo–Mumford regularity of a sheaf on projective space is an integer that describes the vanishing of higher cohomology of twists of the sheaf. Regularity can be computed from the degrees of the syzygies of the corresponding graded module. Maclagan and Smith defined an analogous invariant for sheaves on smooth projective toric varieties, where the regularity is no longer directly bounded by Betti numbers. We investigate the relationship between regularity and Betti numbers in a number of situations that generalize classical results, such as the regularity of powers of ideals and the Betti numbers of truncations of modules.

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List of Notation

$\mathbb{N}$	natural numbers $\{0, 1, 2, \ldots\}$
$\mathbf{e}_j$	jth standard basis vector in $\mathbb{Z}^r$
$\mathbf{1}$	$(1, \ldots, 1)$
$\mathbf{0}$	$(0, \ldots, 0)$
$\lvert\mathbf{d}\rvert$	sum $d_1 + \cdots + d_r$ for $\mathbf{d} \in \mathbb{Z}^r$
max	componentwise maximum in $\mathbb{Z}^r$
$\mathbb{K}$	algebraically closed field
X	smooth projective toric variety in Part I,
	product of projective spaces in Part II
$\mathbb{P}^{\mathbf{n}}$	$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ for $\mathbf{n} \in \mathbb{Z}^r$
$\mathcal{H}_t$	tth Hirzebruch surface
$\mathcal{O}$	structure sheaf of X
Cox	total coordinate ring
Pic	Picard group
$\mathcal{O}(\mathbf{d})$	line bundle corresponding to $\mathbf{d} \in \operatorname{Pic} X$
S	total coordinate ring of X, graded in $\operatorname{Pic} X$
B	irrelevant ideal of X
Nef	cone of numerically effective divisors
Eff	cone of effective divisors
$\mathbf{C}$	list $(\mathbf{c}_1, \ldots, \mathbf{c}_r)$ of minimal generators for $\operatorname{Nef} X$ as a monoid
$\lambda \cdot \mathbf{C}$	$\sum_j \lambda_j \mathbf{c}_j$
$\mathbf{d} \leq \mathbf{d}'$	partial order on $\operatorname{Pic} X$ determined by $\operatorname{Nef} X$
M	finitely generated, $\operatorname{Pic} X$-graded S-module
I	homogenous ideal in S
f_j	homogeneous generator for I
$\mathbf{P}$	list $(\mathbf{p}_1, \ldots, \mathbf{p}_s)$ of the degrees of the f_j

ann	annihilator
Fitt	Fitting ideal
$\widetilde{M}$	sheaf corresponding to M
$M_{\geq \mathbf{d}}$	truncation of M at $\mathbf{d} \in \operatorname{Pic} X$
$H^i_B(M)$	ith local cohomology of M in B
$H^i(X, \widetilde{M})$	ith sheaf cohomology of $\widetilde{M}$ on X
Γ_*	twisted global sections functor
reg	multigraded regularity
$\beta_j(M)$	set of $\mathbf{b} \in \mathbb{Z}^r$ such that $\operatorname{Tor}^S_j(M, \mathbb{K})_{\mathbf{b}} \neq 0$
Q_i, L_i	regions in $\mathbb{Z}^r$ defined in Chapter 4
p, q	projections to factors of $X \times Y$
$\mathscr{F} \boxtimes \mathscr{G}$	$p^*\mathscr{F} \otimes q^*\mathscr{G}$
π_i	projection to ith factor of $\mathbb{P}^{\mathbf{n}}$
$K_\bullet$	Koszul complex on the variables of S
$K_\bullet^{\leq \mathbf{a}}$	subcomplex of $K_\bullet$ consisting of free summands generated in degrees $\leq \mathbf{a}$
$\bigwedge^j$	jth exterior power
$\Omega^a_{\mathbb{P}^{n_i}}$	ath exterior power of the cotangent sheaf on $\mathbb{P}^{n_i}$
$\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$	$\pi_1^*\Omega^{a_1}_{\mathbb{P}^{n_1}} \otimes \cdots \otimes \pi_r^*\Omega^{a_r}_{\mathbb{P}^{n_r}}$
$\hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$	module corresponding to $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ (see Chapter 4)
$\check{C}^\bullet(B, M)$	local Čech complex of M with support in B
$\check{C}^\bullet(\mathfrak{U}_B, \mathscr{F})$	Čech complex of $\mathscr{F}$ using affine cover of X corresponding to B
Tot	total complex

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Part I

General smooth projective toric varieties

Chapter 1

Introduction

Classical algebraic geometry has established an extensive dictionary between the properties of a projective space $\mathbb{P}^n$ and the algebraic properties of polynomial rings, allowing geometric ideas to be formalized as facts in the realm of commutative algebra. More recently a similar correspondence has developed in the field of toric geometry. A *toric variety* is a variety with the action of a complex torus and an open dense orbit under the action. Cox defined a global *coordinate ring* S on a toric variety X and showed that quasicoherent sheaves on X can be described by S-modules [14]. The ring S is graded in $\operatorname{Pic} X$, the group of isomorphism classes of line bundles on X.

Toric varieties and their multigraded rings appear in representation theory, combinatorics, complex geometry, and as test cases for more general theorems, yet remarkably little is known about them from the perspective of commutative algebra. For instance, properties of modules, such as their projective dimension [5] and whether they are Cohen–Macaulay [6] or complete intersections [24], have not been well characterized in terms of multigraded algebra. Fundamental tools that could be used for this purpose, such as free resolutions and the appropriate derived functors, are not well understood.

Here we will focus on homological invariants of sheaves and their corresponding modules, particularly what was defined by Mumford (based on ideas of Castelnuovo) as the *regularity* of a coherent sheaf on projective space [37].

By Serre vanishing (e.g. [29, Prop. III.5.3]), twisting a coherent sheaf $\mathscr{F}$ by a sufficiently high power of the line bundle $\mathcal{O}(1)$ eliminates all higher sheaf cohomology. Regularity roughly measures the power of an ample line bundle that is necessary for vanishing. In [4], Bayer and Stillman interpreted it as a measure of computational complexity, particularly in describing minimal free resolutions. Uniform bounds on regularity also appear in the construction of Hilbert schemes, a type of moduli space [28].

Maclagan and Smith generalized the definition of regularity to sheaves on a smooth toric variety and their corresponding multigraded modules [36]. In this context the regularity $\operatorname{reg} M$ of a module M is a subset of $\operatorname{Pic} X$, indicating which line bundles produce sufficient vanishing. Within $\operatorname{reg} M$ the Hilbert function of M is polynomial. However, the higher sheaf cohomology of twists of M is required to vanish in a larger region, in order to facilitate inductive arguments using long exact sequences of cohomology.

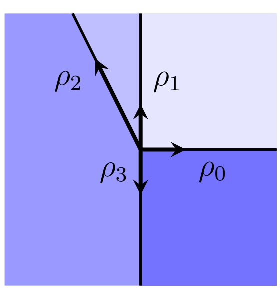
Based on the singly graded case, we might expect $\operatorname{reg} M$ to reflect the shape and complexity of free resolutions of M, i.e. its Betti numbers. We will investigate the connection between $\operatorname{reg} M$ and the Betti numbers of M in a number of contexts, all of which illustrate a more complicated relationship between algebra and geometry than for a single projective space.

1.1 Multigraded regularity

Let X be a smooth projective toric variety over an algebraically closed field $\mathbb{K}$ and determined by a fan. The total coordinate ring of X is a $\operatorname{Pic}(X)$-graded polynomial ring S over $\mathbb{K}$ with an irrelevant ideal $B \subset S$. Write $\operatorname{Eff} X$ for the monoid in $\operatorname{Pic} X$ generated by the degrees of the variables in S. Denote the structure sheaf of X by $\mathcal{O}$ and the line bundle corresponding to $\mathbf{d} \in \operatorname{Pic} X$ by $\mathcal{O}(\mathbf{d})$.

Fix minimal generators $\mathbf{C} = (\mathbf{c}_1, \dots, \mathbf{c}_r)$ for the monoid $\operatorname{Nef} X$ of classes in $\operatorname{Pic} X$ represented by numerically effective divisors. For $\lambda \in \mathbb{Z}^r$, write $\lambda \cdot \mathbf{C}$ to represent the linear combination $\lambda_1\mathbf{c}_1 + \cdots + \lambda_r\mathbf{c}_r \in \operatorname{Pic} X$, and similarly for other tuples in $\operatorname{Pic} X$. Write $|\lambda|$ for the sum $\lambda_1 + \cdots + \lambda_r$. We use a partial order on $\operatorname{Pic} X$ induced by $\operatorname{Nef} X$: given $\mathbf{a}, \mathbf{b} \in \operatorname{Pic} X$, we write $\mathbf{a} \leq \mathbf{b}$ when $\mathbf{b} - \mathbf{a} \in \operatorname{Nef} X$.

Example 1.1.1. The Hirzebruch surface $\mathcal{H}_t = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(t))$ is a smooth projective toric variety whose associated fan, shown left in Figure 1.1, has rays $(1, 0)$, $(0, 1)$, $(-1, t)$, and $(0, -1)$. For each ray there is a corresponding prime torus-invariant divisor. In particular, the total coordinate ring of $\mathcal{H}_t$ is the polynomial ring $S = \mathcal{K}[x_0, x_1, x_2, x_3]$ and its irrelevant ideal is $B = \langle x_0, x_2\rangle \cap \langle x_1, x_3\rangle$.



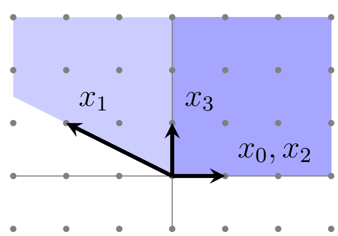


Figure 1.1: Left: fan of $\mathcal{H}_2$. Right: the cones $\operatorname{Nef} \mathcal{H}_2$ (dark blue) and $\operatorname{Eff} \mathcal{H}_2$ (blue).

Choosing a basis for $\operatorname{Pic} \mathcal{H}_t \cong \mathbb{Z}^2$, the grading on S can be given as $\deg x_0 = \deg x_2 = (1, 0)$, $\deg x_1 = (-t, 1)$, and $\deg x_3 = (0, 1)$. The effective and nef cones are illustrated on the right.

We now recall the notion of multigraded Castelnuovo–Mumford regularity introduced by Maclagan and Smith in terms of the local cohomology $H^i_B(M)$ of M with support in B. The module $H^i_B(M)$ is also $\operatorname{Pic}(X)$-graded, and we denote its degree $\mathbf{b}$ part by $H^i_B(M)_{\mathbf{b}}$.

Definition 1.1.2 (c.f. [36, Def. 1.1])**.** Let M be a graded S-module. For $\mathbf{d} \in \operatorname{Pic} X$, we say M is $\mathbf{d}$-*regular* if the following hold:

1. $H^i_B(M)_{\mathbf{b}} = 0$ for all $i > 0$ and all $\mathbf{b} \in \bigcup_{|\lambda|=i-1}(\mathbf{d} - \lambda \cdot \mathbf{C} + \operatorname{Nef} X)$ where $\lambda \in \mathbb{N}^r$.

2. $H^0_B(M)_{\mathbf{b}} = 0$ for all $\mathbf{b} \in \bigcup_j(\mathbf{d} + \mathbf{c}_j + \operatorname{Nef} X)$.

We write $\operatorname{reg} M$ for the set of $\mathbf{d}$ such that M is $\mathbf{d}$-regular.

Remark 1.1.3. Several alternate notions of Castelnuovo–Mumford regularity for the multigraded setting exist in the literature. The initial extension was introduced by Hoffman and Wang for a product of two projective spaces [32]. Following Maclagan and Smith's definition, Botbol and Chardin gave a more general definition working over an arbitrary base ring [7]. Recently, in their work on Tate resolutions on toric varieties, Brown and Erman introduced a modified notion of multigraded regularity for a weighted projective space, which they then extended to other toric varieties [8, §6.1].

1.2 Truncations and local cohomology

In this section we collect facts about truncations and local cohomology that will be used repeatedly. For a $\operatorname{Pic}(X)$-graded S-module M and $\mathbf{d} \in \operatorname{Pic} X$, denote by $M_{\geq \mathbf{d}}$ the submodule of M generated by all elements of degrees $\mathbf{d}'$ satisfying $\mathbf{d}' \geq \mathbf{d}$ (c.f. [36, Def. 5.1]). Denote by $\widetilde{M}$ the quasi-coherent sheaf on X associated to M, as in [14, §3].

For $p > 0$ and $\mathbf{a} \in \mathbb{Z}^r$ there exist natural isomorphisms

$$H^p\big(X, \widetilde{M}(\mathbf{b})\big) \cong H^{p+1}_B(M)_{\mathbf{b}},$$

and for $p = 0$ there is a $\mathbb{Z}^r$-graded exact sequence

$$0 \longrightarrow H^0_B(M) \longrightarrow M \longrightarrow \Gamma_*(\widetilde{M}) \longrightarrow H^1_B(M) \longrightarrow 0. \tag{1.2.1}$$

An important tool for computing local cohomology is the local Čech complex

$$\check{C}^\bullet(B, M)\colon 0 \longrightarrow M \longrightarrow \bigoplus M[g_i^{-1}] \longrightarrow \bigoplus M[g_i^{-1}, g_j^{-1}] \longrightarrow \cdots$$

where the g_i range over the generators of B. We index the local Čech complex so that the summands of $\check{C}^p(B, M)$ are localizations of M at p distinct generators of B. Then we have

$$H^p_B(M) \cong H^p(\check{C}^\bullet(B, M)).$$

See [34] and [16, §9] for more details.

Note that the distinguished open sets corresponding to the generators of B form an affine cover $\mathfrak{U}_B$ of X. Denote by $\check{C}^\bullet(\mathfrak{U}_B, \mathscr{F})$ the Čech complex of a sheaf $\mathscr{F}$ with respect to $\mathfrak{U}_B$:

$$\check{C}^\bullet(\mathfrak{U}_B, \mathscr{F})\colon 0 \longrightarrow \bigoplus \mathscr{F}|_{\mathfrak{U}_i} \longrightarrow \bigoplus \mathscr{F}|_{\mathfrak{U}_i \cap \mathfrak{U}_j} \longrightarrow \cdots .$$

Since $M/M_{\geq \mathbf{d}}$ is annihilated by a power of B, a module M and its truncation define the same sheaf on X. In particular $H^p_B(M) = H^p_B(M_{\geq \mathbf{d}})$ for $p \geq 2$. The long exact sequence of local cohomology applied to $0 \to M_{\geq \mathbf{d}} \to M \to M/M_{\geq \mathbf{d}} \to 0$ gives

$$0 \longrightarrow H^0_B(M_{\geq \mathbf{d}}) \longrightarrow H^0_B(M) \longrightarrow M/M_{\geq \mathbf{d}} \longrightarrow H^1_B(M_{\geq \mathbf{d}}) \longrightarrow H^1_B(M) \longrightarrow 0.$$

Hence $H^0_B(M) = 0$ implies $H^0_B(M_{\geq \mathbf{d}}) = 0$. Since $M/M_{\geq \mathbf{d}}$ is zero in degrees larger than $\mathbf{d}$ we also have $H^1_B(M_{\geq \mathbf{d}})_{\geq \mathbf{d}} = H^1_B(M)_{\geq \mathbf{d}}$. An immediate consequence is the following lemma, which we will use repeatedly to reduce to the case when $\mathbf{d} = \mathbf{0}$.

Lemma 1.2.1. *A* $\operatorname{Pic} X$*-graded* S*-module* M *is* $\mathbf{d}$*-regular if and only if* $M_{\geq \mathbf{d}}$ *is* $\mathbf{d}$*-regular.*

1.3 Outline

This dissertation is divided into two parts. Part I deals with general smooth projective toric varieties, and Part II with the specific case of products of projective spaces. Products of projective spaces are among the first examples of toric varieties outside of ordinary projective spaces. Already several new phenomena are visible, for instance the existence of modules with the same multigraded Betti numbers but different regularities (Example 5.0.1).

The general case has the added difficulty that the cones $\operatorname{Nef} X$ and $\operatorname{Eff} X$ may diverge, due to the presence of torus-invariant curves and divisors with negative intersection products. This means that there exist multiple toric varieties with the same multigraded total coordinate ring, so we would expect a less direct relationship between algebra and geometry.

In Chapter 2 we prove that the regularity of a finitely generated faithful module is bounded by a translate of the cone $\operatorname{Nef} X$ (Theorem 2.2.5). Surprisingly this is not true for all finitely generated modules (Example 2.0.1). In Chapter 3 we use our bound to generalize a classical result about the asymptotic behavior of the regularity of powers of ideals (Theorem 3.0.1).

In Chapter 4 we present a combinatorial criterion for regularity on products on projective spaces in terms of the Betti numbers of the truncations of a module (Theorem 4.2.6). In Chapter 5 we specify a subset of $\operatorname{reg} M$ based only on the Betti numbers of M itself (Theorem 5.1.3) and show that it is tight for a type of complete intersection (Theorem 5.2.2). In Chapter 6 we introduce a computational package for computing these regions.

Much of this dissertation comes from joint work with Mahrud Sayrafi and Juliette Bruce, and some with Navid Nemati. Individual papers are cited in the chapters where they appear.

Chapter 2

Bounding regularity

Most of this chapter comes from Section 3 of [10].

Since the region $\operatorname{reg} M$ is invariant under translation by the nef cone $\operatorname{Nef} X$ of X, one might expect that when M is finitely generated it could be specified by finitely many minimal elements with respect to this order. We show this is true in the case when M is faithful, meaning that $\operatorname{ann} M = 0$.

The proof produces an explicit translate of $\operatorname{Nef} X$, determined by the degrees of the generators of M (see the figure in Example 2.2.7). We use the idea that if the truncation $M_{\geq \mathbf{d}}$ is not generated in a single degree $\mathbf{d}$ then M is not $\mathbf{d}$-regular (see Theorem 2.1.1 for a simpler case).

Results from [36, 7] have excluded particular degrees from the regularity, implying that the region cannot contain degrees lying below them in the partial order determined by $\operatorname{Nef} X$. This does not preclude the existence of infinitely many incomparable elements.

It remains an interesting problem to characterize modules with torsion whose regularity is contained in a translate of the nef cone. We expect that the faithfulness hypothesis in Theorem 2.2.5 can be weakened. However, even on a Hirzebruch surface it cannot be removed, as evident in the following simple example pointed out by Daniel Erman.

Example 2.0.1. Let $M = S/\langle x_2, x_3 \rangle$ be the coordinate ring of a single point on $\mathcal{H}_t$ (see Example 1.1.1). Since $\langle x_2, x_3 \rangle$ is saturated we have $H^0_B(M) = 0$. Further, since the support of $\widetilde{M}$ has dimension 0 we must have $H^i_B(M) = 0$ for $i \geq 2$. Thus $\operatorname{reg} M$ is determined entirely by $H^1_B(M)$, which vanishes exactly where the Hilbert function of M agrees with its Hilbert polynomial.

The Hilbert function of M is equal to 1 inside $\operatorname{Eff} \mathcal{H}_t$ and 0 outside of it. Hence $\operatorname{reg} M = \operatorname{Eff} \mathcal{H}_t$. When $t > 0$ this cone does not contain finitely many minimal elements with respect to $\operatorname{Nef} X$, as illustrated in Figure 2.1.

In other words, the regularity of an arbitrary finitely generated module may fail to be contained in all translates of $\operatorname{Nef} X$. The regularity of the module in Example 2.0.1 is nevertheless contained in a translate of $\operatorname{Eff} X$. We show in Proposition 2.2.1 that this is true

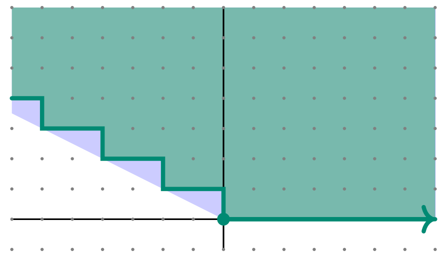


Figure 2.1: The multigraded regularity of M (green) is an infinite staircase contained in a translate of the effective cone of $\mathcal{H}_2$ (blue).

for all M. Thus the existence of a module whose regularity contains infinitely many minimal elements is a consequence of the difference between the effective and nef cones of X.

2.1 Regularity of the coordinate ring

In this section we show that the pathology seen in Example 2.0.1—a regularity region contained in no translate of Nef X—does not occur for the total coordinate ring of a smooth projective toric variety. In particular we show that $\operatorname{reg} S \subseteq \operatorname{Nef} X$.

In [36, Prob. 6.12], Maclagan and Smith asked for a combinatorial characterization of toric varieties X such that $\operatorname{Nef} X \subseteq \operatorname{reg} S$. Theorem 2.1.1 below shows that when X is smooth and projective, $\operatorname{Nef} X \subseteq \operatorname{reg} S$ is in fact equivalent to the a priori stronger condition that $\operatorname{reg} S = \operatorname{Nef} X$. It still remains an interesting question to characterize such toric varieties. For instance, the only Hirzebruch surface with this property is $\mathcal{H}_1$.

Theorem 2.1.1. *We have* $\operatorname{reg} S \subseteq \operatorname{Nef} X$*, so* $\operatorname{reg} S$ *contains finitely many minimal elements.*

Proof. Take $\mathbf{d} \in \operatorname{reg} S$. By [36, Thm. 5.4] the truncation $S_{\geq \mathbf{d}}$ is generated by the monomials of $S_{\mathbf{d}}$, so there is a surjection $S_{\mathbf{d}} \otimes_{\mathbb{K}} S \to S_{\geq \mathbf{d}}(\mathbf{d})$ which sheafifies to a surjection $S_{\mathbf{d}} \otimes \mathcal{O} \to \mathcal{O}(\mathbf{d})$. Hence $\mathcal{O}(\mathbf{d})$ is generated by global sections, so by [16, Thm. 6.3.11] $\mathbf{d}$ is nef.

An application of Dickson's lemma (e.g. [15, §2.4 Thm. 5]), suggested by Will Sawin [40], shows that $\operatorname{reg} S$ has finitely many minimal elements, finishing the proof.

Lemma 2.1.2. *A subset* $V \subseteq \operatorname{Nef} X$ *contains finitely many minimal elements with respect to* $\leq$ *on* $\operatorname{Pic} X$.

Elements of V can be written as linear combinations $\lambda \cdot \mathbb{C}$ of the monoid generators of $\operatorname{Nef} X$. The minimal elements of V must have coefficients $\lambda \in \mathbb{N}^r$ that are minimal in the component-wise partial order on $\mathbb{N}^r$. By Dickson's lemma only finitely many possible coefficients exist. □

Example 2.1.3. The multigraded regularity of the coordinate ring of the Hirzebruch surface $\mathcal{H}_2$ is contained in the nef cone of $\mathcal{H}_2$, as illustrated in Figure 2.2.

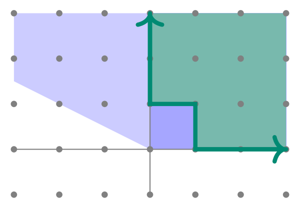


Figure 2.2: The regularity of S (dark green) is contained in $\operatorname{Nef}\mathcal{H}_2$ (dark blue).

Though we do not directly use Theorem 2.1.1 in the next section, we do rely on the idea of the proof. For an arbitrary module M, if $\mathbf{d} \in \operatorname{reg} M$ then the truncation $M_{\geq \mathbf{d}}$ is generated in a single degree $\mathbf{d}$, meaning that $\widetilde{M}(\mathbf{d})$ is globally generated. This no longer immediately implies that $\mathbf{d}$ is nef, but Lemma 2.1.4 below connects the difference between $\mathbf{d}$ and the degrees of the generators of M to monomials in truncations of S itself.

We also use the chamber complex of the rays of $\operatorname{Eff} X$, which is described in [36, §2]. By definition, this chamber complex is the coarsest fan with support $\operatorname{Eff} X$ which refines all triangulations of the degrees of the variables of S. It partitions $\operatorname{Eff} X$ into cones that govern many geometric properties of $\operatorname{Spec} S$, including its GIT quotients, birational geometry, and Hilbert polynomials (c.f. [16, Ch. 14-15], [31, §5]).

For our purposes we need only the existence of a strongly convex rational polyhedral fan that covers $\operatorname{Eff} X$ and contains $\operatorname{Nef} X$ as a cone. We will refer to the maximal cones as chambers and the codimension one cones as walls. In particular, $\operatorname{Nef} X$ is a chamber.

Lemma 2.1.4. *Let Γ be a chamber of $\operatorname{Eff} X$ other than $\operatorname{Nef} X$, and let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \operatorname{Pic} X$. If $\mathbf{a}_i \in \Gamma \setminus \operatorname{Nef} X$ for all i, then there exist monomials $m_i \in S_{\geq \mathbf{a}_i}$ such that $\prod_i m_i$ is not generated by the monomials of $S_{\sum \mathbf{a}_i}$.*

Proof. Since Γ and $\operatorname{Nef} X$ intersect at most in a wall of Γ and no $\mathbf{a}_i$ lies in $\Gamma \cap \operatorname{Nef} X$, their sum $\mathbf{b} = \sum \mathbf{a}_i$ must also be in $\Gamma \setminus \operatorname{Nef} X$. Consider the multiplication maps

$$\begin{array}{ccc} S_{\mathbf{b}} \otimes_{\mathbb{K}} S & \xrightarrow{\varphi} & S(\mathbf{b}) \\ & & \uparrow{\scriptstyle\psi} \\ & & \bigotimes_{\mathbb{K}} S_{\geq \mathbf{a}_i}(\mathbf{a}_i). \end{array}$$

Suppose the proposition is false. Then the image of ψ must be contained in the image of φ, else we could choose $(m_i) \in \bigotimes_{\mathbb{K}} S_{\geq \mathbf{a}_i}(\mathbf{a}_i)$ with image not generated by the monomials of $S_{\mathbf{b}}$.

Note that each $S_{\geq \mathbf{a}_i}(\mathbf{a}_i)$ sheafifies to $\mathcal{O}(\mathbf{a}_i)$, so sheafifying the entire diagram gives

$$\begin{array}{ccc} S_{\mathbf{b}} \otimes \mathcal{O} & \xrightarrow{\varphi} & \mathcal{O}(\mathbf{b}) \\ & & \uparrow \psi \\ & & \mathcal{O}(\mathbf{b}). \end{array}$$

In particular, the image of ψ is still contained in the image of φ. Since ψ sheafifies to an isomorphism, φ sheafifies to a surjection. This implies $\mathbf{b} \in \operatorname{Nef} X$, which is a contradiction. □

2.2 Regularity of faithful modules

The goal of this section is to prove that the multigraded regularity of a faithful module has only finitely many minimal elements.

Proposition 2.2.1 shows that the regularity of an arbitrary finitely generated module is contained in some translate of Eff X. Under the stronger assumption that M is faithful, i.e. that $\operatorname{ann} M = 0$, Proposition 2.2.2 shows that we can also eliminate degrees that are in a translate of Eff X but not Nef X.

Proposition 2.2.1. *Let M be a finitely generated graded S-module with $\widetilde{M} \neq 0$. Suppose the degrees of all minimal generators of M are contained in* Eff X*. Then* $\operatorname{reg} M \subseteq \operatorname{Eff} X$*.*

Proof. Take $\mathbf{d} \in \operatorname{reg} M$ and suppose for contradiction that $\mathbf{d} \notin \operatorname{Eff} X$. The degree $\mathbf{d}$ part $M_{\mathbf{d}}$ generates $M_{\geq \mathbf{d}}$ by [36, Thm. 5.4]. By hypothesis all elements of M have degrees inside Eff X, so $M_{\mathbf{d}} = 0$ and thus $M_{\geq \mathbf{d}} = 0$. The modules M and $M_{\geq \mathbf{d}}$ define the same sheaf by [36, Lem. 6.8], so $M_{\geq \mathbf{d}} = 0$ contradicts $\widetilde{M} \neq 0$. □

Proposition 2.2.2. *Let M be a finitely generated graded faithful S-module with $\widetilde{M} \neq 0$. Suppose Γ is a chamber of* $\operatorname{Eff} X \setminus \operatorname{Nef} X$*. If $\mathbf{d} - \deg f_i \in \Gamma \setminus \operatorname{Nef} X$ for all generators f_i of M, then M is not $\mathbf{d}$-regular.*

Proof. Assume on the contrary that M is $\mathbf{d}$-regular. Let $\mathbf{a}_i = \mathbf{d} - \deg f_i$ for each i. By choice of $\mathbf{d}$ we have $\mathbf{a}_i \in \Gamma \setminus \operatorname{Nef} X$. Hence by Lemma 2.1.4 there exist monomials $m_i \in S_{\geq \mathbf{a}_i}$ such that $\prod_i m_i$ is not generated by the monomials of $S_{\sum \mathbf{a}_i}$. Consider the elements $m_i f_i \in M_{\geq \mathbf{d}}$.

Since M is $\mathbf{d}$-regular, the degree $\mathbf{d}$ part $M_{\mathbf{d}}$ generates $M_{\geq \mathbf{d}}$ by [36, Thm. 5.4]. Let $g_1, \ldots, g_s$ with $\deg g_j = \mathbf{d}$ be generators for $M_{\geq \mathbf{d}}$. Thus we must have relations

$$m_i f_i = \sum_j b_{i,j} g_j = \sum_j b_{i,j} \left(\sum_k a_{j,k} f_k \right) = \sum_k c_{i,k} f_k$$

for some $b_{i,j}, a_{j,k}, c_{i,k} \in S$ with $\deg b_{i,j} = \deg m_i - \mathbf{a}_i$ and $\deg a_{j,k} = \mathbf{a}_k$. These relations form a partial presentation matrix

$$A = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix} - \begin{bmatrix} c_{1,1} & c_{2,1} & \cdots & c_{n,1} \\ c_{1,2} & c_{2,2} & \cdots & c_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1,n} & c_{2,n} & \cdots & c_{n,n} \end{bmatrix}. \tag{2.2.1}$$

for M. In particular, $\det(A) \in \operatorname{Fitt}_0 M \subseteq \operatorname{ann} M$ by [19, Prop. 20.7], so $\det(A)M = 0$.

Since $\operatorname{ann} M = 0$ we must have $\det(A) = 0$, but this is impossible: note that $\det(A)$ contains the monomial $m = \prod_i m_i$ and that $\det(A) \in m + I$ for $I = \prod_k \langle c_{1,k}, c_{2,k}, \ldots, c_{n,k} \rangle$, then observe that $I \subseteq \prod_k \langle a_{1,k}, a_{2,k}, \ldots, a_{n,k} \rangle \subseteq S \otimes_{\mathbb{K}} S_{\sum \mathbf{a}_k}$ since $\deg a_{j,k} = \mathbf{a}_k$. Hence $\det(A) = 0$ implies $m \in I \subseteq S \otimes_{\mathbb{K}} S_{\sum \mathbf{a}_k}$ and contradicts our choice of m_i. □

Remark 2.2.3. Example 2.0.1 shows that Theorem 2.2.5 is not true without the faithfulness hypothesis. In practice, however, we only need that the element $\det A$ from (2.2.1) does not annihilate M for some choice of m_i as in Lemma 2.1.4. Given a specific toric variety, this may be possible to verify directly in some cases where M is not faithful.

We will use the following technical lemma about the walls of $\operatorname{Nef} X$ to find a vector satisfying the hypotheses of Proposition 2.2.2.

Lemma 2.2.4. *Given* $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \operatorname{Nef} X$ *and* $\mathbf{d} \in \operatorname{Eff} X \setminus \operatorname{Nef} X$*, there exists a chamber* Γ *sharing a wall* W *with* $\operatorname{Nef} X$ *and* $\mathbf{w}$ *in the relative interior of* W *such that* $\mathbf{d} + \mathbf{w} \in \Gamma$ *and* $\mathbf{d} + \mathbf{w} \in \mathbf{a}_i + \Gamma$ *for all* i.

Proof. Consider the cone P defined by all rays of $\operatorname{Nef} X$ in addition to a primitive element along $\mathbf{d}$. Since $\operatorname{Nef} X \subsetneq P$, at least one wall W of $\operatorname{Nef} X$ must be in the interior of $P \subseteq \operatorname{Eff} X$. Let Γ be the chamber across W from $\operatorname{Nef} X$. Since $\mathbf{d} \notin \operatorname{Nef} X$, for each $\mathbf{w} \in W$ we have $\mathbf{d} + \mathbf{w} \notin \operatorname{Nef} X$.

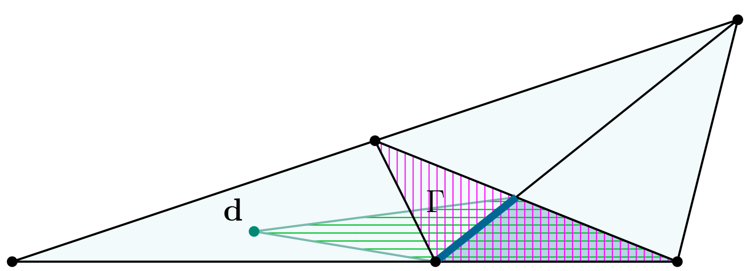


Figure 2.3: A section of a hypothetical chamber complex with P (green, horizontal) and Q (red, vertical) inside $\operatorname{Eff} X$. The chamber $\operatorname{Nef} X$ and its wall W are in blue.

Now consider the cone Q defined by all supporting hyperplanes of $\operatorname{Nef} X$ and Γ except the hyperplane containing W. Since W is in the intersection of the open half-spaces defining

Q, it lies in the interior of Q. Therefore we can find $\mathbf{w}$ in the relative interior of $W \subset Q$ so that $\mathbf{d}+\mathbf{w} \in \mathbf{a}_i + Q \subseteq \mathbf{a}_i + (\Gamma \cup \operatorname{Nef} X)$ for all i. By hypothesis $\mathbf{a}_i + \operatorname{Nef} X \subseteq \operatorname{Nef} X$ so $\mathbf{d}+\mathbf{w} \notin \mathbf{a}_i + \operatorname{Nef} X$. Hence $\mathbf{d}+\mathbf{w} \in \mathbf{a}_i + \Gamma$ for all i. □

Theorem 2.2.5. *Let M be a finitely generated graded faithful S-module with $\widetilde{M} \neq 0$. Suppose the degrees of all minimal generators of M are contained in* $\operatorname{Nef} X$. *Then* $\operatorname{reg} M \subseteq \operatorname{Nef} X$. *In particular,* $\operatorname{reg} M$ *has finitely many minimal elements.*

Proof. Suppose there exists $\mathbf{d} \in \operatorname{reg} M \setminus \operatorname{Nef} X$. Since M satisfies the hypothesis of Proposition 2.2.1, we can assume that $\mathbf{d} \in \operatorname{Eff} X$. Using Lemma 2.2.4, we can find $\mathbf{w}$ in the relative interior of a wall separating $\operatorname{Nef} X$ and an adjacent chamber Γ such that $\mathbf{d}+\mathbf{w} \in \Gamma$ and $\mathbf{d}+\mathbf{w} \in \deg f_i + \Gamma$ for all i. It follows from Proposition 2.2.2 that $\mathbf{d}+\mathbf{w} \notin \operatorname{reg} M$, which is a contradiction because $\mathbf{w} \in \operatorname{Nef} X$ and $\operatorname{reg} M$ is invariant under positive translation by $\operatorname{Nef} X$.

The conclusion that $\operatorname{reg} M$ has finitely many minimal elements follows from Lemma 2.1.2. □

Corollary 2.2.6. *Let M be a finitely-generated faithful S-module. If* $\deg f_i \in \mathbf{b} + \operatorname{Nef} X$ *for all generators f_i of M then* $\operatorname{reg} M \subseteq \mathbf{b} + \operatorname{Nef} X$.

Example 2.2.7. Consider the Hirzebruch surface $\mathcal{H}_2$, with notation from Example 1.1.1, and let M be the torsion-free module with presentation

$$S(3,-3) \oplus S(2,-2) \oplus S(1,-2) \xleftarrow{\begin{pmatrix} x_0^5 x_1 & x_1^2 x_2^6 & x_1^2 x_2^5 \end{pmatrix}^T} S(0,-4).$$

Since the degrees of the generators are contained in $(-3,2) + \operatorname{Nef} \mathcal{H}_2$, by Corollary 2.2.6 the multigraded regularity of M is contained in a translate of the nef cone, illustrated in Figure 2.4.

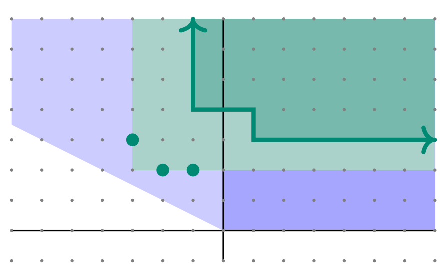


Figure 2.4: The multigraded regularity (dark green) of the module M is contained in a translate $(-3,2) + \operatorname{Nef} X$ (light green) of the nef cone of $\mathcal{H}_2$ (dark blue).

Chapter 3

Powers of ideals

Most of this chapter comes from [10], particularly from Section 4.

Building on the work of Swanson in [42], Cutkosky–Herzog–Trung in [18] and Kodiyalam in [35] described the surprisingly predictable asymptotic behavior of Castelnuovo–Mumford regularity for powers of ideals on a projective space $\mathbb{P}^r$: given an ideal $I \subset \mathbb{K}[x_0, \ldots, x_r]$, there exist $d, e \in \mathbb{Z}$ such that for $n \gg 0$ the regularity of I^n satisfies

$$\operatorname{reg}(I^n) = dn + e.$$

This phenomenon has received substantial attention [25, 11, 41, 39, 43, 3], focused mostly on projective spaces. See [12] for a survey.

A natural question is thus whether there is an analogous description for the asymptotic shape of $\operatorname{reg}(I^n) \subset \operatorname{Pic} X$ for an ideal I in the total coordinate ring of a smooth projective toric variety. Let $I = \langle f_1, \ldots, f_s \rangle \subseteq S$ be a homogeneous ideal and let $\mathbf{P}$ be the vector with coordinates $\mathbf{p}_i = \deg f_i \in \operatorname{Pic} X$.

We bound multigraded regularity by establishing regions "inside" and "outside" of $\operatorname{reg}(I^n)$ which translate linearly by a fixed vector as n increases (see the figure in Example 3.0.2). The inner bound depends on the Betti numbers of the Rees ring $S[It]$, while the outer bound depends only on the degrees of the generators of I. We continue to use the partial order defined by $\operatorname{Nef} X$.

Theorem 3.0.1. *There exists a degree $\mathbf{a} \in \operatorname{Pic} X$, depending only on I, such that for each integer $n > 0$ and each pair of degrees $\mathbf{q}_1, \mathbf{q}_2 \in \operatorname{Pic} X$ satisfying $\mathbf{q}_1 \geq \deg f_i \geq \mathbf{q}_2$ for all generators f_i of I, we have*

$$n\mathbf{q}_1 + \mathbf{a} + \operatorname{reg} S \subseteq \operatorname{reg}(I^n) \subseteq n\mathbf{q}_2 + \operatorname{Nef} X.$$

It is worth emphasizing that our result holds over smooth projective toric varieties with arbitrary Picard rank. Indeed, toric varieties of higher Picard rank introduce a wrinkle that is not present in existing asymptotic results on Castelnuovo–Mumford regularity: in general there are infinitely many possible regularity regions compatible with two given bounds. (In contrast, when $\operatorname{Pic} X = \mathbb{Z}$, inner and outer bounds correspond to upper and lower bounds, respectively, with only finitely many integers between each pair.)

Nevertheless, since regularity is invariant under positive translation by $\operatorname{Nef} X$, an outer bound of the type given in Theorem 3.0.1 cannot contain an infinite expanding chain of regularity regions. We rely on the fact that I is a torsion–free module to apply results from Chapter 2.

The following example illustrates the bounds in Theorem 3.0.1 for two monomial ideals.

Example 3.0.2. Let $I = \langle x_0x_3, x_1^2x_2^4\rangle$ and $J = \langle x_3, x_0^3x_1\rangle$ be two ideals in the total coordinate ring of the Hirzebruch surface $\mathcal{H}_2$, with notation as in Example 1.1.1. Figure 3.1 shows the multigraded regularity of powers of I and J along with the bounds from Theorem 3.0.1.

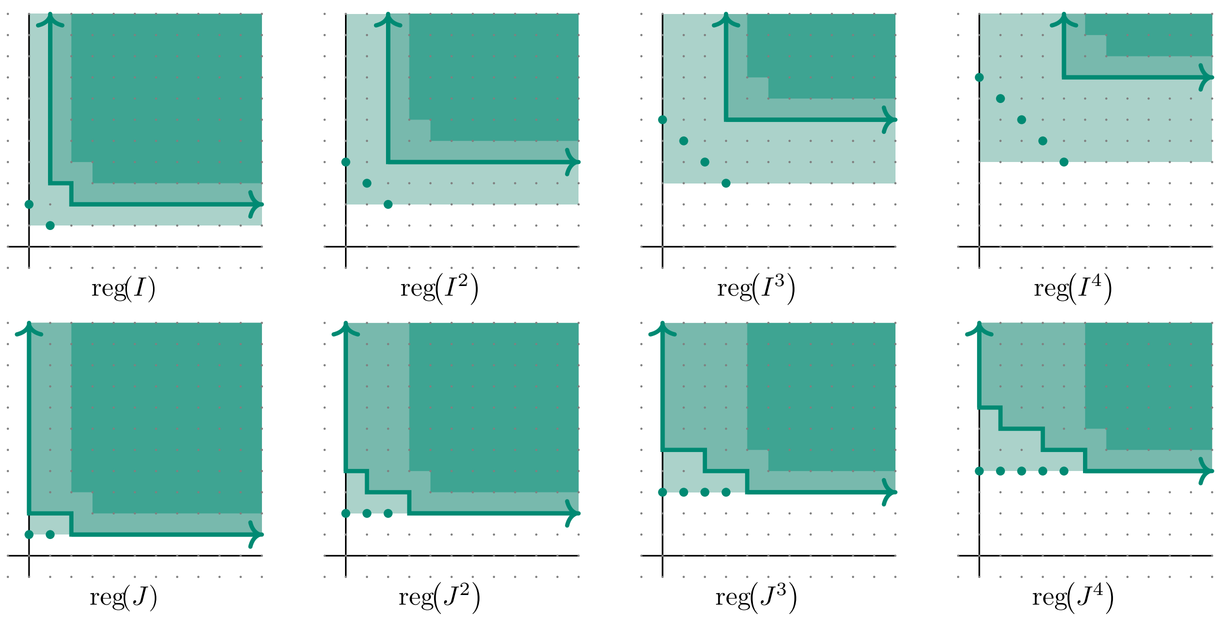


Figure 3.1: The inner (dark green) and outer (light green) bounds for powers of I and J. The circles correspond to the degrees of the generators of each power.

Remark 3.0.3. If $\mathbf{q}_2$ is not nef, then the bounds in Theorem 3.0.1 will not increase with n in the partial order on $\operatorname{Pic} X$. We can see that this behavior is necessary by taking I to be a principal ideal generated outside of $\operatorname{Nef} X$.

We prove half of Theorem 3.0.1 in Proposition 3.1.2 and half in Proposition 3.3.3.

3.1 Finite generation of regularity

We begin by constructing an outer bound for the regularity of I^n—a subset of $\operatorname{Pic} X$ that contains $\operatorname{reg}(I^n)$. In [35], Kodiyalam constructs this from a bound on the degrees of the generators of I^n. However, more nuanced behavior can occur in the multigraded setting. The following example shows that the degree of a minimal generator of an ideal does not bound its regularity on an arbitrary toric variety.

Example 3.1.1. Let $I = \langle x_0x_3, x_0x_2, x_1x_2\rangle$ be an ideal in the coordinate ring of the Hirzebruch surface $\mathcal{H}_t$, with notation as in Example 1.1.1. A local cohomology computation verifies that I is $(1,1)$-regular. However x_0x_2 is a minimal generator with $\deg(x_0x_2) = (2,0) \not\leq (1,1)$.

The existence of a similar example with $H^0_B(M) \neq 0$ was noted by Macalagan and Smith, who asked whether B-torsion was necessary in [36, §5]. Example 3.1.1 shows that it is not.

We saw in Example 2.0.1 that it is also possible for the regularity of an arbitrary finitely generated module to contain infinitely many minimal elements with respect to $\operatorname{Nef} X$. Since ideals are faithful modules this behavior cannot occur in the current situation.

Proposition 3.1.2. *For each integer $n > 0$ and degree $\mathbf{q} \in \operatorname{Pic} X$ satisfying $\mathbf{q} \leq \deg f_i$ for all homogeneous generators f_i of I we have*

$$\operatorname{reg}(I^n) \subseteq n\mathbf{q} + \operatorname{Nef} X.$$

Proof. The ideal I^n is generated by all products of n choices of generators of I, where $\deg \prod_{j=1}^n f_{i_j} = \sum_{j=1}^n \mathbf{p}_{i_j} \geq n\mathbf{q}$. Thus the proposition follows from Theorem 2.2.5. □

3.2 The Rees ring

One way to find a subset of the regularity of a module is by using its multigraded Betti numbers. In order to describe $\operatorname{reg}(I^n)$, we would thus like a uniform description of the Betti numbers of I^n for all n. For this purpose, consider the multigraded Rees ring of I:

$$S[It] \coloneqq \bigoplus_{n\geq 0} I^n t^n \subseteq S[t],$$

which is a $\operatorname{Pic}(X) \times \mathbb{Z}$-graded noetherian ring with $\deg ft^k = (\deg f, k)$ for $f \in S$. Let $R = S[T_1, \ldots, T_s]$ be the $\operatorname{Pic}(X) \times \mathbb{Z}$-graded ring with $\deg(T_i) = (\deg f_i, 1) = (\mathbf{p}_i, 1)$. Notice that there is a surjective map of graded S-algebras:

$$\begin{aligned} R &\longrightarrow\!\!\!\!\!\rightarrow S[It] \\ T_i &\longmapsto f_i\, t \end{aligned}$$

Since R is a finitely generated standard graded algebra over S, taking a single degree of a finitely generated R-module in the auxiliary $\mathbb{Z}$ grading yields a finitely generated S-module.

Definition 3.2.1. For a $\operatorname{Pic}(X)\times\mathbb{Z}$-graded R-module M, define $M^{(n)}$ to be the $\operatorname{Pic}(X)$-graded S-module

$$M^{(n)} \coloneqq \bigoplus_{\mathbf{a}\in \operatorname{Pic} X} M_{(\mathbf{a},n)}.$$

Following [35], we record three important properties of this operation.

Lemma 3.2.2. *Consider the functor* $-^{(n)}\colon M \mapsto M^{(n)}$ *from the category of* $\operatorname{Pic}(X)\times\mathbb{Z}$*-graded* R*-modules to the category of* $\operatorname{Pic}(X)$*-graded* S*-modules.*

(i) $-^{(n)}$ *is an exact functor.*

(ii) $S[It]^{(n)} \cong I^n$.

(iii) $R(-\mathbf{a}, -b)^{(n)} \cong R^{(n-b)}(-\mathbf{a}) \cong \bigoplus_{|\nu|=n-b} S(-\nu\cdot\mathbf{P} - \mathbf{a})$ *where* $\nu \in \mathbb{N}^s$.

Since $S[It]$ is a finitely generated module over the polynomial ring R, it has a finite free resolution. Applying $-^{(n)}$ gives a resolution by (i), which has cokernel I^n by (ii) and whose terms are finitely generated free S-modules by (iii). Thus we can constrain the Betti numbers of I^n in terms of those of $S[It]$.

3.3 Regularity of powers of ideals

Given a description of the Betti numbers of I^n in terms of n, we obtain an inner bound on $\operatorname{reg}(I^n)$ using the following lemma.

Lemma 3.3.1. *If* $F_\bullet$ *is a finite free resolution for* M *with* $F_j = \bigoplus_i S(-\mathbf{a}_{i,j})$ *and* $H^0_B(M) = 0$ *then*

$$\bigcap_{i,j} \bigcup_{|\lambda|=j} (\mathbf{a}_{i,j} - \lambda\cdot\mathbb{C} + \operatorname{reg} S) \subseteq \operatorname{reg} M \tag{3.3.1}$$

where $\mathbb{C} = (\mathbf{c}_1, \dots, \mathbf{c}_r)$ *is the sequence of nef generators for* X *and the union is over* $\lambda \in \mathbb{N}^r$.

Remark 3.3.2. This result amounts to switching the union and intersection in the statement of [36, Cor. 7.3] for modules with $H^0_B(M) = 0$, which increases the size of the subset by allowing a different choice of λ for each i, j.

Proof. Fix $\mathbf{d}$ in the left hand side of (3.3.1) and consider the hypercohomology spectral sequence for $F_\bullet$ (see [9, Thm. 4.14] for a description of this spectral sequence). We must show that M is $\mathbf{d}$-regular, meaning that $H^k_B(M)_{\mathbf{d}-\mu\cdot\mathbb{C}} = 0$ for all k and all μ with $|\mu| = k-1$. Since $F_\bullet$ is a resolution for M, a diagonal of our spectral sequence converges to $H^k_B(M)$. Thus it is sufficient to prove that this entire diagonal vanishes in degree $\mathbf{d} - \mu\cdot\mathbb{C}$, i.e. that

$$H^{k+j}_B(F_j)_{\mathbf{d}-\mu\cdot\mathbb{C}} = \bigoplus_i H^{k+j}_B(S(-\mathbf{a}_{i,j}))_{\mathbf{d}-\mu\cdot\mathbb{C}} = 0 \tag{3.3.2}$$

for all j. This is satisfied for $k = 0$ by hypothesis. Now fix $k > 0$, μ, j, and i. By choice of $\mathbf{d}$ we have $\mathbf{d} \in \mathbf{a}_{i,j} - \lambda\cdot\mathbb{C} + \operatorname{reg} S$ for some λ with $|\lambda| = j$, so that $\mathbf{d} - \mathbf{a}_{i,j} + \lambda\cdot\mathbb{C} \in \operatorname{reg} S$. Call this degree $\mathbf{d}'$, and let $\mathbf{c}' = (\lambda+\mu)\cdot\mathbb{C}$, where $|\lambda+\mu| = k+j-1$. Then by the definition of the regularity of S we have $H_B^{k+j}(S)_{\mathbf{d}'-\mathbf{c}'} = 0$ where

$$\mathbf{d}' - \mathbf{c}' = \mathbf{d} - \mathbf{a}_{i,j} + \lambda\cdot\mathbb{C} - (\lambda+\mu)\cdot\mathbb{C} = \mathbf{d} - \mu\cdot\mathbb{C}.$$

Hence each summand in (3.3.2) is zero for $k > 0$, as desired. □

Proposition 3.3.3. *There exists a degree $\mathbf{a} \in \operatorname{Pic} X$, depending only on the Rees ring of I, such that for each integer $n > 0$ and degree $\mathbf{q} \in \operatorname{Pic} X$ satisfying $\mathbf{q} \geq \deg f_i$ for all homogeneous generators f_i of I, we have*

$$n\mathbf{q} + \mathbf{a} + \operatorname{reg} S \subseteq \operatorname{reg}(I^n).$$

Proof. Let $F_\bullet$ be a minimal $\operatorname{Pic}(X)\times\mathbb{Z}$-graded free resolution of $S[It]$ as an R-module, and write $F_j = \bigoplus_i R(-\mathbf{a}_{i,j}, -b_{i,j})$ for $\mathbf{a}_{i,j} \in \operatorname{Pic} X$ and $b_{i,j} \in \mathbb{Z}$. By Lemma 3.2.2, applying the $-^{(n)}$ functor to $F_\bullet$ yields a (potentially non-minimal) resolution of $S[It]^{(n)} \cong I^n$ consisting of free S-modules

$$F_j^{(n)} \cong \bigoplus_i R(-\mathbf{a}_{i,j}, -b_{i,j})^{(n)} \cong \bigoplus_i \left[\bigoplus_{|\nu| = n - b_{i,j}} S(-\nu\cdot\mathbf{P} - \mathbf{a}_{i,j}) \right],$$

where $\mathbf{P} = (\deg f_1, \ldots, \deg f_s)$ is the sequence of degrees of the homogeneous generators f_i of I. From this Lemma 3.3.1 gives the following bound on the regularity of I^n:

$$\bigcap_{\substack{i,j \\ |\nu| = n-b_{i,j}}} \bigcup_{|\lambda|=j} [\nu\cdot\mathbf{P} + \mathbf{a}_{i,j} - \lambda\cdot\mathbb{C} + \operatorname{reg} S] \subseteq \operatorname{reg}(I^n). \tag{3.3.3}$$

Note that $b_{0,0} = 0$, as $S[It]$ is a quotient of R, and thus $b_{i,j} \geq 0$ for all i, j, as R is positively graded in the $\mathbb{Z}$ coordinate.

Take $\mathbf{a} \in \operatorname{Pic} X$ so that $\mathbf{a} \geq \mathbf{a}_{i,j}$ for all i, j. There are only finitely many $\mathbf{a}_{i,j}$ because $S[It]$ is a finitely generated R-module and R is noetherian. We may now simplify the left hand side of (3.3.3) by noting three things: (i) for all $|\lambda| = j$ and all j we have $\operatorname{reg} S \subseteq -\lambda\cdot\mathbb{C} + \operatorname{reg} S$, (ii) if $|\nu| = n - b_{i,j}$ then $(n - b_{i,j})\mathbf{q} \in \nu\cdot\mathbf{P} + \operatorname{reg} S$, and (iii) for all i and all j we have $n\mathbf{q} + \mathbf{a} \in (n - b_{i,j})\mathbf{q} + \mathbf{a}_{i,j} + \operatorname{reg} S$. Combining these facts gives that

$$\begin{aligned}
\operatorname{reg}(I^n) &\supseteq \bigcap_{\substack{i,j \\ |\nu| = n-b_{i,j}}} \bigcup_{|\lambda|=j} [\nu\cdot\mathbf{P} + \mathbf{a}_{i,j} - \lambda\cdot\mathbb{C} + \operatorname{reg} S] \\
&\supseteq \bigcap_{\substack{i,j \\ |\nu| = n-b_{i,j}}} [\nu\cdot\mathbf{P} + \mathbf{a}_{i,j} + \operatorname{reg} S] \\
&\supseteq \bigcap_{i,j} [(n - b_{i,j})\mathbf{q} + \mathbf{a}_{i,j} + \operatorname{reg} S] \\
&\supseteq n\mathbf{q} + \mathbf{a} + \operatorname{reg} S.
\end{aligned}$$

□

A similar problem is to characterize the asymptotic behavior of regularity for symbolic powers of I. Note that the symbolic Rees ring of I is not necessarily noetherian (see [27], for instance), so our argument for the existence of the degree $\mathbf{a}$ in the proof of Proposition 3.3.3 does not work in this case. More generally, if $\mathcal{I} = \{I_n\}$ is a filtration of ideals, then one may ask for sufficient conditions so that $\operatorname{reg}(I_n)$ is uniformly bounded.

Part II

Products of projective spaces

Chapter 4

Criterion for regularity

Most of this chapter comes from Sections 3, 4, and 6 of [9].

Let S be the polynomial ring on $n+1$ variables over an algebraically closed field $\mathbb{K}$ and $\mathfrak{m}$ its maximal homogeneous ideal. A coherent sheaf $\mathscr{F}$ on the projective space $\mathbb{P}^n = \operatorname{Proj} S$ is d-regular for $d \in \mathbb{Z}$ if

1. $H^i(\mathbb{P}^n, \mathscr{F}(b)) = 0$ for all $i > 0$ and all $b \geq d - i$.

The Castelnuovo–Mumford regularity of $\mathscr{F}$ is then the minimum d such that $\mathscr{F}$ is d-regular. In [23], Eisenbud and Goto considered the analogous condition on the local cohomology of a finitely generated graded S-module M, proving the equivalence of the following:

2. $H^i_{\mathfrak{m}}(M)_b = 0$ for all $i \geq 0$ and all $b > d - i$;
3. the truncation $M_{\geq d}$ has a linear free resolution;
4. $\operatorname{Tor}_i(M, \mathbb{K})_b = 0$ for all $i \geq 0$ and all $b > d + i$.

In particular, if $M = \bigoplus_p H^0(\mathbb{P}^n, \mathscr{F}(p))$ is the graded S-module corresponding to $\mathscr{F}$ (so that $H^0_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(M) = 0$) then conditions (1) through (4) are equivalent (c.f. [20, Prop. 4.16]).

Maclagan and Smith's definition of regularity in [36] (Definition 1.1.2) is essentially a generalization of condition (2). When $X = \mathbb{P}^n$ the minimum element of the multigraded regularity recovers the classical Castelnuovo–Mumford regularity. However, when X has higher Picard rank, translating the geometric definition of Maclagan and Smith into algebraic conditions like (3) and (4) above is an open problem. We now focus on the case when X is a product of projective spaces and explore the relationship between multigraded regularity, truncations, Betti numbers, and virtual resolutions.

The obvious way one might hope to generalize Eisenbud and Goto's result to products of projective spaces is false: the truncation $M_{\geq \mathbf{d}}$ of a $\mathbf{d}$-regular $\operatorname{Pic}(X)$-graded module M can have nonlinear maps in its minimal free resolution (see Example 4.2.2). We show that under a mild saturation hypothesis, multigraded Castelnuovo–Mumford regularity is determined by a slightly weaker linearity condition, which we call *quasilinearity* (see Definition 4.2.3).

Let S be the $\mathbb{Z}^r$-graded Cox ring of $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and B the corresponding irrelevant ideal. The following complex contains all allowed twists for a quasilinear resolution generated in degree zero on a product of 2 projective spaces:

$$0 \longleftarrow S \longleftarrow \begin{matrix} S(-1,\ \ 0) \\ \oplus \\ S(\ \ 0,-1) \end{matrix} \oplus S(-1,-1) \longleftarrow \begin{matrix} S(-2,\ \ 0) \\ \oplus \\ S(-1,-1) \\ \oplus \\ S(\ \ 0,-2) \end{matrix} \oplus \begin{matrix} S(-2,-1) \\ \oplus \\ S(-1,-2) \end{matrix} \longleftarrow \cdots .$$

Within each term, the summands in the left column (green) are linear syzygies while those in the right column (pink) are nonlinear syzygies. In general, for twists $-\mathbf{b}$ appearing in the i-th step of a quasilinear resolution, the sum of the positive components of $\mathbf{b} - \mathbf{d} - \mathbf{1}$ is at most $i-1$, where $\mathbf{d}$ is the degree of all generators.

The main theorem of this chapter characterizes multigraded regularity of modules on products of projective spaces in terms of the Betti numbers of their truncations.

Theorem (4.2.6)**.** *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $H^0_B(M) = 0$. Then M is $\mathbf{d}$-regular if and only if $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ such that F_0 is generated in degree $\mathbf{d}$.*

In [5, Thm. 2.9] Berkesch, Erman, and Smith established a similar result characterizing multigraded regularity of modules on products of projective spaces in terms of the existence of short virtual resolutions of a certain shape, which they construct by taking the Fourier–Mukai transform of $\widetilde{M}$ with Beilinson's resolution of the diagonal as the kernel.

The proof of Theorem 4.2.6 is based in part on a Čech–Koszul spectral sequence that relates the Betti numbers of $M_{\geq \mathbf{d}}$ to the Fourier–Mukai transform of $\widetilde{M}(\mathbf{d})$. Precisely, if M is $\mathbf{d}$-regular and $H^0_B(M) = 0$ then

$$\dim_{\mathbb{K}} \operatorname{Tor}^S_j(M_{\geq \mathbf{d}}, \mathbb{K})_{\mathbf{a}} = h^{|\mathbf{a}|-j}\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{d}) \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\big) \quad \text{for } |\mathbf{a}| \geq j \geq 0, \tag{4.0.1}$$

where the $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ are cotangent sheaves on $\mathbb{P}^{\mathbf{n}}$. The regularity of M implies certain cohomological vanishing for $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$, which, using (4.0.1), implies quasilinearity of the resolution of $M_{\geq \mathbf{d}}$.

Conversely, building on [5, Thm. 2.9], a computation of $H^i_B(S)$ shows that the cokernel of a quasilinear resolution generated in degree $\mathbf{d}$ is $\mathbf{d}$-regular. In Corollary 4.2.14 we show that the resolution of $M_{\geq \mathbf{d}}$ is in fact quasi-isomorphic to the Fourier–Mukai transform, giving an explicit construction of this complex and a computable criterion for regularity.

Since a linear resolution is necessarily quasilinear, Theorem 4.2.6 implies that having a linear truncation at $\mathbf{d}$ is strictly stronger than being $\mathbf{d}$-regular. That is to say, when $H^0_B(M) = 0$:

$$\begin{matrix} M_{\geq \mathbf{d}} \text{ has a linear resolution} \\ \text{generated in degree } \mathbf{d} \end{matrix} \Longrightarrow \begin{matrix} M_{\geq \mathbf{d}} \text{ has a quasilinear resolution} \\ \text{generated in degree } \mathbf{d} \end{matrix} \iff M \text{ is } \mathbf{d}\text{-regular.}$$

Using (4.0.1), we also get a cohomological characterization of when $M_{\geq \mathbf{d}}$ has a linear resolution.

Notation

Fix a Picard rank $r \in \mathbb{N}$ and dimension vector $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. We denote by $\mathbb{P}^{\mathbf{n}}$ the product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ of r projective spaces over a field $\mathbb{K}$. Given $\mathbf{b} \in \mathbb{Z}^r$ we let

$$\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{b}) := \pi_1^* \mathcal{O}_{\mathbb{P}^{n_1}}(b_1) \otimes \cdots \otimes \pi_r^* \mathcal{O}_{\mathbb{P}^{n_r}}(b_r)$$

where π_i is the projection of $\mathbb{P}^{\mathbf{n}}$ to $\mathbb{P}^{n_i}$. This gives an isomorphism $\operatorname{Pic} \mathbb{P}^{\mathbf{n}} \cong \mathbb{Z}^r$, which we use implicitly throughout.

When referring to vectors in $\mathbb{Z}^r$ we use a bold font. Given a vector $\mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r$ we denote the sum $v_1 + \cdots + v_r$ by $|\mathbf{v}|$. For $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^r$ we write $\mathbf{v} \leq \mathbf{w}$ when $v_i \leq w_i$ for all i, and use $\max\{\mathbf{v}, \mathbf{w}\}$ to denote the vector whose i-th component is $\max\{v_i, w_i\}$. We reserve $\mathbf{e}_1, \ldots, \mathbf{e}_r$ for the standard basis of $\mathbb{Z}^r$ and for brevity we write $\mathbf{1}$ for $(1, 1, \ldots, 1) \in \mathbb{Z}^r$ and $\mathbf{0}$ for $(0, 0, \ldots, 0) \in \mathbb{Z}^r$.

Let S be the $\mathbb{Z}^r$-graded Cox ring of $\mathbb{P}^{\mathbf{n}}$, which is isomorphic to the polynomial ring $\mathbb{K}[x_{i,j} \mid 1 \leq i \leq r,\ 0 \leq j \leq n_i]$ with $\deg(x_{i,j}) = \mathbf{e}_i$. Further, let $B = \bigcap_{i=1}^r \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle$ be the irrelevant ideal in S.

The twisted global sections functor Γ_* given by $\mathscr{F} \mapsto \bigoplus_{\mathbf{p} \in \mathbb{Z}^r} H^0(\mathbb{P}^{\mathbf{n}}, \mathscr{F}(\mathbf{p}))$ takes coherent sheaves on $\mathbb{P}^{\mathbf{n}}$ to S-modules. Given a $\mathbb{Z}^r$-graded S-module M, let $\beta_i(M)$ denote the set $\{\mathbf{b} \in \mathbb{Z}^r \mid \operatorname{Tor}_i^S(M, \mathbb{K})_{\mathbf{b}} \neq 0\}$ of multidegrees of i-th syzygies of M.

As in the case of a single projective space, the truncation of a graded module on a product of projective spaces at multidegree $\mathbf{d}$ (according to [36, Def. 5.1]) contains all elements of degree at least $\mathbf{d}$.

Definition 4.0.1. For $\mathbf{d} \in \mathbb{Z}^r$ and M a $\mathbb{Z}^r$-graded S-module, the *truncation* of M at $\mathbf{d}$ is the $\mathbb{Z}^r$-graded S-submodule $M_{\geq \mathbf{d}} := \bigoplus_{\mathbf{d}' \geq \mathbf{d}} M_{\mathbf{d}'}$.

In this setting the following lemma is immediate.

Lemma 4.0.2. *The truncation map $M \mapsto M_{\geq \mathbf{d}}$ is an exact functor of $\mathbb{Z}^r$-graded S-modules.*

Remark 4.0.3. Since truncation is exact, if $F_\bullet$ is graded free resolution of a module M then the term by term truncation $(F_\bullet)_{\geq \mathbf{d}}$ is a resolution of $M_{\geq \mathbf{d}}$. However, in general the truncation of a free module is not free, so $(F_\bullet)_{\geq \mathbf{d}}$ is generally not a free resolution of $M_{\geq \mathbf{d}}$.

Multigraded regularity

In order to streamline our definitions of regions inside the Picard group of $\mathbb{P}^{\mathbf{n}}$, we introduce the following subsets of $\mathbb{Z}^r$: for $\mathbf{d} \in \mathbb{Z}^r$ and $i \in \mathbb{N}$ let

$$L_i(\mathbf{d}) := \bigcup_{|\lambda| = i} (\mathbf{d} - \lambda_1 \mathbf{e}_1 - \cdots - \lambda_r \mathbf{e}_r + \mathbb{N}^r) \quad \text{for } \lambda_1, \ldots, \lambda_r \in \mathbb{N}$$

$$Q_i(\mathbf{d}) := L_{i-1}(\mathbf{d} - \mathbf{1}) \quad \text{for } i > 0 \quad \text{and} \quad Q_0(\mathbf{d}) = \mathbf{d} + \mathbb{N}^r.$$

Note that for fixed $\mathbf{d} \in \mathbb{Z}^r$ we have $L_i(\mathbf{d}) \subseteq Q_i(\mathbf{d})$ for all i.

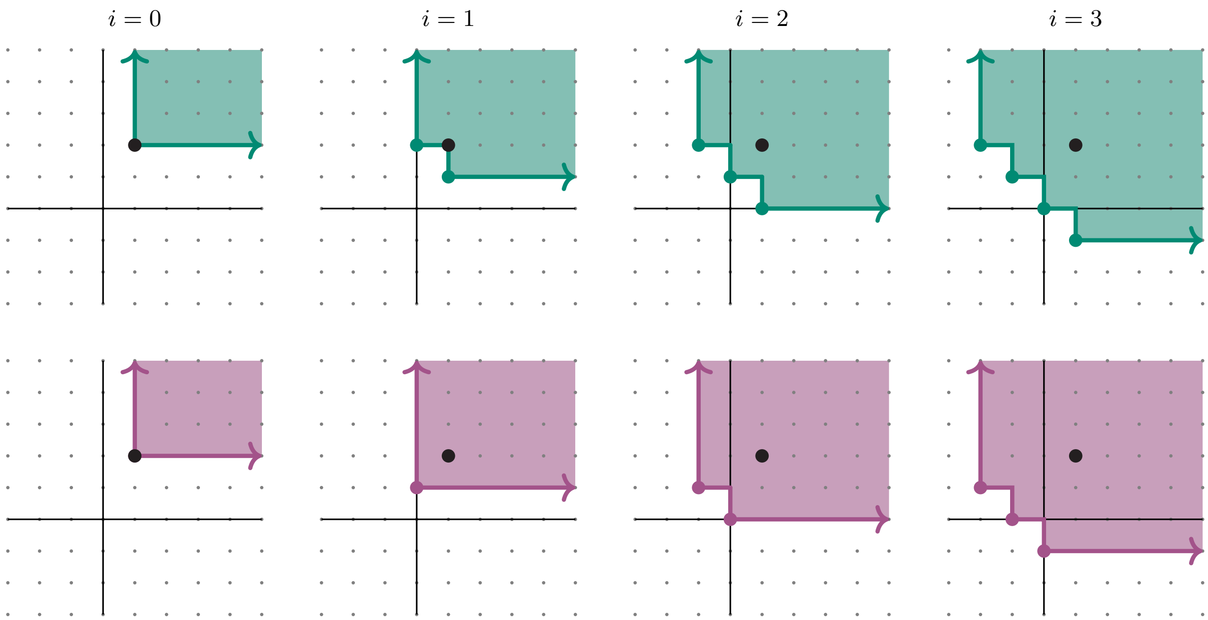


Figure 4.1: The top row shows the regions $L_i(1, 2)$ in green, and the bottom row $Q_i(1, 2)$ in pink, for $i = 0, 1, 2, 3$, from left to right, as defined in Section 4.

Example 4.0.4. When $r = 2$ the regions $L_i(\mathbf{d})$ and $Q_i(\mathbf{d})$ can be visualized as in Figure 4.1. For $i > 1$ they are shaped like staircases with $i + 1$ and i "corners," respectively; in other words $L_i(\mathbf{d})$ contains $i + 1$ minimal elements and $Q_i(\mathbf{d})$ contains i.

Remark 4.0.5. An alternate description of $L_i(\mathbf{d})$ will also be useful: it is the set of $\mathbf{b} \in \mathbb{Z}^r$ so that the sum of the positive components of $\mathbf{d} - \mathbf{b}$ is at most i. (This ensures that we can distribute the λ_j so that $\mathbf{b} + \sum_j \lambda_j \mathbf{e}_j \geq \mathbf{d}$.)

With this notation in hand we can recall Definition 1.1.2 in this setting.

Definition 4.0.6. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. We say that M is $\mathbf{d}$-*regular* for $\mathbf{d} \in \mathbb{Z}^r$ if the following hold:

1. $H^0_B(M)_\mathbf{p} = 0$ for all $\mathbf{p} \in \bigcup_{1 \leq j \leq r}(\mathbf{d} + \mathbf{e}_j + \mathbb{N}^r)$,
2. $H^i_B(M)_\mathbf{p} = 0$ for all $i > 0$ and $\mathbf{p} \in L_{i-1}(\mathbf{d})$.

Koszul complexes and cotangent sheaves

For each factor $\mathbb{P}^{n_i}$ of $\mathbb{P}^{\mathbf{n}}$, the Koszul complex on the variables of $S_i = \operatorname{Cox}\mathbb{P}^{n_i}$ is a resolution of $\mathbb{K}$:

$$K^i_\bullet \colon 0 \leftarrow S_i \leftarrow S_i^{n_i+1}(-1) \leftarrow \bigwedge^2 \big[S_i^{n_i+1}(-1)\big] \leftarrow \cdots \leftarrow \bigwedge^{n_i+1} \big[S_i^{n_i+1}(-1)\big] \leftarrow 0. \qquad (4.0.2)$$

The Koszul complex $K_\bullet$ on the variables of S is the tensor product of the complexes $\pi_i^* K^i_\bullet$.

For $1 \le a \le n$ let $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$ be the kernel of $\bigwedge^{a-1}\big[S_i^{n_i+1}(-1)\big] \leftarrow \bigwedge^{a}\big[S_i^{n_i+1}(-1)\big]$ and let $\Omega^a_{\mathbb{P}^{n_i}}$ denote its sheafification. The minimal free resolution of $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$ then consists of the terms of $K^i_\bullet$ with homological index greater than a. Write $\hat{\Omega}^0_{\mathbb{P}^{n_i}}$ for the kernel of $\mathbb{K} \leftarrow S_i$ (so that $\Omega^0_{\mathbb{P}^{n_i}} = \mathcal{O}_{\mathbb{P}^{n_i}}$) and take $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$ to be 0 otherwise. For $\mathbf{a} \in \mathbb{Z}^r$ with $\mathbf{0} \le \mathbf{a} \le \mathbf{n}$ define

$$\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}} \coloneqq \pi_1^* \Omega^{a_1}_{\mathbb{P}^{n_1}} \otimes \cdots \otimes \pi_r^* \Omega^{a_r}_{\mathbb{P}^{n_r}}$$

and write $\hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ for the analogous tensor product of the modules $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$.

Given a free complex $F_\bullet$ and a multidegree $\mathbf{a} \in \mathbb{Z}^r$, denote by $F_\bullet^{\le \mathbf{a}}$ the subcomplex of $F_\bullet$ consisting of free summands generated in degrees at most $\mathbf{a}$.

Lemma 4.0.7. *Fix $\mathbf{a} \in \mathbb{Z}^r$ and let $K_\bullet$ be the Koszul complex on the variables of S. The subcomplex $K_\bullet^{\le \mathbf{a}}$ is equal to $K_\bullet$ in degrees $\le \mathbf{a}$, and its sheafification is exact except at homological index $|\mathbf{a}|$, where it has homology $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$.*

Proof. The first statement follows from the fact that the terms appearing in $K_\bullet$ but not $K_\bullet^{\le \mathbf{a}}$ have no elements in degrees $\le \mathbf{a}$.

Note that $K_\bullet^{\le \mathbf{a}}$ is a tensor product of pullbacks of subcomplexes of the $K^i_\bullet$ in (4.0.2):

$$K_\bullet^{\le \mathbf{a}} = \pi_i^*(K^1_\bullet)^{\le a_1} \otimes \cdots \otimes \pi_r^*(K^r_\bullet)^{\le a_r}.$$

After sheafification, each complex $\pi_i^*(K^i_\bullet)^{\le \mathbf{a}}$ is exact away from its kernel $\pi_i^* \Omega^{a_i}_{\mathbb{P}^{n_i}}$, which appears at homological index a_i. Thus $\widetilde{K}_\bullet^{\le \mathbf{a}}$ has homology $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$, appearing in index $|\mathbf{a}|$. □

4.1 Constructing virtual resolutions

While a minimal free resolution $F_\bullet$ of a multigraded module can be easily computed using Gröbner methods, it does not always reflect the geometry of the toric variety. For example, when the Picard rank of X is greater than one, the length of $F_\bullet$ may exceed the dimension of the space. To bridge this gap, Berkesch, Erman, and Smith introduced *virtual resolutions* [5].

Definition 4.1.1. A $\operatorname{Pic}(X)$-graded complex of free S-modules $G_\bullet$ is a *virtual resolution* of M if the complex $\widetilde{G}_\bullet$ of locally free sheaves on X is a resolution of the sheaf $\widetilde{M}$.

Despite more faithfully capturing the geometry of X, virtual resolutions are often less rigid than minimal free resolutions. For example, a module M generally has many non-isomorphic virtual resolutions. Virtual resolutions will appear in our proof of Theorem 4.2.6. Inspired by the work of Berkesch, Erman, and Smith, we use a Fourier–Mukai construction to give a virtual resolution of M whose Betti numbers are computable in terms of certain cohomology groups. In Section 4.2 we then equate this virtual resolution with the minimal free resolution of $M_{\geq \mathbf{d}}$ using the following technical proposition.

Proposition 4.1.2. *Suppose $G_\bullet$ is a virtual resolution of a module M on a product $\mathbb{P}^{\mathbf{n}}$ of projective spaces and $F_\bullet$ is the minimal free resolution of M. If*

1. *the differentials of $G_\bullet$ have no degree $\mathbf{0}$ components;*
2. *the terms of $G_\bullet$ and $F_\bullet$ are generated in degrees between $\mathbf{0}$ and $\mathbf{n}$; and*
3. *the quasi-isomorphism $\widetilde{G}_\bullet \to \widetilde{M}$ is represented by a map of complexes $G_\bullet \to M$*

then $G_\bullet$ is isomorphic to $F_\bullet$.

Proof. By the free-to-acyclic lemma [19, Lem. 20.3] the map $G_\bullet \to M$ lifts to a map $\psi\colon G_\bullet \to F_\bullet$, which induces an isomorphism between $H_0(\widetilde{G}_\bullet)$ and $H_0(\widetilde{F}_\bullet)$ by hypothesis. For $i > 0$ we have $H_i(\widetilde{G}_\bullet) = H_i(\widetilde{F}_\bullet) = 0$, so $\widetilde{\psi}$ is a quasi-isomorphism.

We will show that ψ is an isomorphism of complexes by showing that it is an isomorphism on generators of degree $\mathbf{a}$ for all $\mathbf{a} \in \mathbb{Z}^r$. Fix $\mathbf{a}$. By condition (2) we have $\Omega^{\mathbf{a}}(\mathbf{a}) \neq 0$. By lifting $\widetilde{\psi}$ to resolutions of $\widetilde{G}_\bullet$ and $\widetilde{F}_\bullet$ and applying $\operatorname{Hom}((\Omega^{\mathbf{a}}(\mathbf{a}))^*, -)$, we can compute complexes of vector spaces representing $\mathbf{R}\operatorname{Hom}\Big((\Omega^{\mathbf{a}}(\mathbf{a}))^*, \widetilde{G}_\bullet\Big)$ and $\mathbf{R}\operatorname{Hom}\Big((\Omega^{\mathbf{a}}(\mathbf{a}))^*, \widetilde{F}_\bullet\Big)$, as well as an induced quasi-isomorphism Ψ between them.

The proof of [21, Thm. 3.1] uses condition (1) and $H^p(\Omega^{\mathbf{a}}(\mathbf{a}) \otimes \mathcal{O}(-\mathbf{c}))$ to compute the homology of $\mathbf{R}\operatorname{Hom}((\mathcal{O}(-\mathbf{c}))^*, -)$ applied to a complex whose terms are direct sums of the $\Omega^{\mathbf{a}}(\mathbf{a})$ (written as $U^{-\mathbf{a}}$ in their notation). The same sheaf cohomology calculation shows that the homology of $\mathbf{R}\operatorname{Hom}\Big((\Omega^{\mathbf{a}}(\mathbf{a}))^*, \widetilde{G}_\bullet\Big)$ and $\mathbf{R}\operatorname{Hom}\Big((\Omega^{\mathbf{a}}(\mathbf{a}))^*, \widetilde{F}_\bullet\Big)$ gives the degree $\mathbf{a}$ coefficients of $G_\bullet$ and $F_\bullet$.

The restriction of ψ agrees with the map of homology induced by Ψ and is therefore an isomorphism. □

Fourier–Mukai transforms

The sheafification of a virtual resolution of M is a resolution of $\widetilde{M}$ by direct sums of line bundles. More generally, following [21, §8], we define a *free monad* of a coherent sheaf $\mathscr{F}$ to be a finite complex

$$\mathscr{L}\colon 0 \leftarrow \mathscr{L}_{-s} \leftarrow \cdots \leftarrow \mathscr{L}_{-1} \leftarrow \mathscr{L}_0 \leftarrow \mathscr{L}_1 \leftarrow \cdots \mathscr{L}_t \leftarrow 0$$

whose terms are direct sums of line bundles and whose homology is $H_\bullet(\mathscr{L}) = H_0(\mathscr{L}) \simeq \mathscr{F}$.

In this section we introduce a type of geometric functor between derived categories known as a Fourier–Mukai transform. We will use a particular instance in Section 4.1 to prove that a complex constructed from the Beilinson spectral sequence is a free monad. See [33, §5] for background and further details.

Let X and Y be smooth projective varieties and consider the two projections

$$\begin{array}{ccc} & X \times Y & \\ {}^{q}\swarrow & & \searrow^{p} \\ X & & Y. \end{array}$$

A *Fourier–Mukai transform* is a functor

$$\Phi_{\mathcal{K}} \colon \mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(Y)$$

between the derived categories of bounded complexes of coherent sheaves. It is represented by an object $\mathcal{K} \in \mathcal{D}^{\mathrm{b}}(X \times Y)$ and constructed as a composition of derived functors

$$\mathscr{F} \mapsto \mathbf{R}p_*\big(\mathbf{L}q^*\mathscr{F} \otimes^{\mathbf{L}} \mathcal{K}\big).$$

Here $\mathbf{L}q^*$, $\mathbf{R}p_*$, and $-\otimes^{\mathbf{L}}\mathcal{K}$ are the derived functors induced by q^*, p_*, and $-\otimes\mathcal{K}$, respectively. Moreover, since q is flat $\mathbf{L}q^*$ is the usual pull-back, and if $\mathcal{K}$ is a complex of locally free sheaves $-\otimes^{\mathbf{L}}\mathcal{K}$ is the usual tensor product. In fact, all equivalences between $\mathcal{D}^{\mathrm{b}}(X)$ and $\mathcal{D}^{\mathrm{b}}(Y)$ arise in this way.

A special case of the Fourier–Mukai transform occurs when $Y = X$ and $\mathcal{K} \in \mathcal{D}^{\mathrm{b}}(X \times X)$ is a resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal subscheme $\iota\colon \Delta \to X \times X$. Such $\mathcal{K}$ is referred to as a *resolution of the diagonal*.

Using the projection formula, one can see that the Fourier–Mukai transform $\Phi_{\mathcal{O}_\Delta}$ is simply the identity in the derived category; that is to say, replacing $\mathcal{O}_\Delta$ with $\mathcal{K}$ produces quasi-isomorphisms. We will use this fact in the proof of Proposition 4.1.4.

The Beilinson spectral sequence

Returning to the case of products of projective spaces, we consider coherent sheaves on $X = \mathbb{P}^{\mathbf{n}}$. We construct a free monad for $\widetilde{M}$ from the Beilinson spectral sequence on $\mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}$ and describe its Betti numbers. When M is $\mathbf{0}$-regular it is a virtual resolution, which we will use in Sections 4.2 and 4.3. See [38, §3.1] for a geometric exposition and [33, §8.3] or [2, §3] for an algebraic exposition on a single projective space.

For sheaves $\mathscr{F}$ and $\mathscr{G}$ on $\mathbb{P}^{\mathbf{n}}$, denote $p^*\mathscr{F} \otimes q^*\mathscr{G}$ by $\mathscr{F} \boxtimes \mathscr{G}$. Consider the vector bundle

$$\mathcal{W} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i) \boxtimes \mathcal{T}^{\mathbf{e}_i}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{e}_i),$$

where $\mathcal{T}^{\mathbf{e}_i}_{\mathbb{P}^{\mathbf{n}}}$ is the pullback of the tangent bundle, as in the Euler sequence on the factor $\mathbb{P}^{n_i}$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n_i}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n_i}}^{n_i+1}(\mathbf{e}_i) \longrightarrow \mathcal{T}_{\mathbb{P}^{n_i}} \longrightarrow 0. \qquad (4.1.1)$$

There is a canonical section $s \in H^0(\mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}, \mathcal{W})$ whose vanishing cuts out the diagonal subscheme $\Delta \subset \mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}$ (see [5, Lem. 2.1]), giving a Koszul resolution of $\mathcal{O}_\Delta$:

$$\mathcal{K}\colon 0 \longleftarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \longleftarrow \mathcal{W}^\vee \longleftarrow \bigwedge^2 \mathcal{W}^\vee \longleftarrow \cdots \longleftarrow \bigwedge^n \mathcal{W}^\vee \longleftarrow 0. \quad (4.1.2)$$

The terms of $\mathcal{K}$ can be written as

$$\mathcal{K}_j = \bigwedge^j \left(\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{e}_i) \boxtimes \Omega^{\mathbf{e}_i}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i) \right) = \bigoplus_{|\mathbf{a}|=j} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \boxtimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}), \quad \text{for } 0 \le j \le |\mathbf{n}|. \quad (4.1.3)$$

As in Section 4.1, we are interested in the derived pushforward of $q^*\widetilde{M} \otimes \mathcal{K}$, which we will compute by resolving the second term of each box product with a Čech complex to obtain a spectral sequence. Since $\mathcal{K}$ is a resolution of the diagonal, the pushforward will be quasi-isomorphic to $\widetilde{M}$.

Consider the double complex

$$C^{-s,t} = \bigoplus_{|\mathbf{a}|=s} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \boxtimes \check{C}^t\Big(\mathfrak{U}_B, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\Big),$$

with vertical maps from the Čech complexes and horizontal maps from $\mathcal{K}$. Since taking Čech complexes is functorial and exact we have $\mathrm{Tot}(C) \sim q^*\widetilde{M} \otimes \mathcal{K}$, which is a resolution of $q^*\widetilde{M} \otimes \mathcal{O}_\Delta$ because $\mathcal{K}$ is locally free. Moreover, since the first term of each box product in $q^*\widetilde{M} \otimes \mathcal{K}$ is locally free, the columns of C are p_*-acyclic (c.f. [30, Prop. 3.2], [2, Lem. 3.2]). Hence the pushforward

$$E_0^{-s,t} = p_*(C^{-s,t}) = \bigoplus_{|\mathbf{a}|=s} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \otimes \Gamma\Big(\mathbb{P}^{\mathbf{n}}, \check{C}^t\Big(\mathfrak{U}_B, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\Big)\Big) \quad (4.1.4)$$

satisfies $\mathrm{Tot}(E_0) = \Phi_{\mathcal{K}}(\widetilde{M}) \sim \widetilde{M}$. With this notation, the *Beilinson spectral sequence* is the spectral sequence of the double complex E_0, whose (vertical) first page has terms

$$E_1^{-s,t} = \bigoplus_{|\mathbf{a}|=s} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \otimes H^t\Big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\Big) = \mathbf{R}^t p_*(q^*\widetilde{M} \otimes \mathcal{K}_s). \quad (4.1.5)$$

Beilinson's resolution of the diagonal and the associated spectral sequence are crucial ingredients in constructions of Beilinson monads, Tate resolutions, and virtual resolutions [22, 21, 5]. Recently, Brown and Erman [8] expanded these constructions to toric varieties using a noncommutative analogue of a Fourier–Mukai transform. More generally, Costa and Miró-Roig [13] have introduced a Beilinson type spectral sequence for a smooth projective variety under certain conditions on its derived category.

The main result of this section is the next proposition, which describes the Betti numbers of a free monad constructed from the Beilinson spectral sequence (c.f. [5, Thm. 2.9]). A key component of the construction is the following lemma.

Lemma 4.1.3. *Let $(C_\bullet, d_\bullet)$ be a bounded above complex of free S-modules and let $B_i = \operatorname{im} d_{i-1}$ and $Z_i = \ker d_i$. If every homology module Z_i/B_i of $C_\bullet$ is free then there is a splitting $f_i \colon C_i \to B_i \oplus Z_i/B_i \oplus C_i/Z_i$ such that f and d commute on each summand.*

Proof. Since $C_\bullet$ is bounded above, there is some k such that $B_i = 0$ for all $i > k$, so in particular $B_{k+1} \cong C_k/Z_k$ is free. Since C_k is free, the exact sequence $0 \to Z_k \to C_k \to C_k/Z_k \to 0$ implies that Z_k is free and $C_k \cong Z_k \oplus C_k/Z_k$. Since Z_k/B_k is free by assumption, the exact sequence $0 \to B_k \to Z_k \to Z_k/B_k \to 0$ implies that B_k is free and $Z_k \cong B_k \oplus Z_k/B_k$. Together, we get $C_k \cong B_k \oplus Z_k/B_k \oplus C_k/Z_k$, and the freeness of B_k means that we can induct backwards on the whole complex. □

Proposition 4.1.4. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. There is a free monad $\mathscr{L}$ for $\widetilde{M}$ with terms*

$$\mathscr{L}_k = \bigoplus_{|\mathbf{a}| \geq k} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \otimes H^{|\mathbf{a}|-k}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}))$$

so that

1. *the free complex $G_\bullet = \Gamma_*(\mathscr{L})$ has Betti numbers $\beta_{k,\mathbf{a}}(G_\bullet) = h^{|\mathbf{a}|-k}\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\big)$;*
2. *if $H^i\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\big) = 0$ for $i > |\mathbf{a}|$ then $G_\bullet$ is a virtual resolution for M whose differentials have no degree $\mathbf{0}$ components.*

Proof. Let $\mathcal{K}$ be the resolution of the diagonal from (4.1.3) and let $\Phi_{\mathcal{K}}$ be the corresponding Fourier–Mukai transform. The Beilinson spectral sequence has (vertical) first page $E_1^{-s,t}$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\mathbf{R}^2 p_*(q^*\widetilde{M} \otimes \mathcal{K}_0) & \longleftarrow & \mathbf{R}^2 p_*(q^*\widetilde{M} \otimes \mathcal{K}_1) & \longleftarrow & \mathbf{R}^2 p_*(q^*\widetilde{M} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
\mathbf{R}^1 p_*(q^*\widetilde{M} \otimes \mathcal{K}_0) & \longleftarrow & \mathbf{R}^1 p_*(q^*\widetilde{M} \otimes \mathcal{K}_1) & \longleftarrow & \mathbf{R}^1 p_*(q^*\widetilde{M} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
p_*(q^*\widetilde{M} \otimes \mathcal{K}_0) & \longleftarrow & p_*(q^*\widetilde{M} \otimes \mathcal{K}_1) & \longleftarrow & p_*(q^*\widetilde{M} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
k=0 & & k=1 & & k=2 & &
\end{array} \tag{4.1.6}$$

Since both (4.1.4) and (4.1.5) have locally free terms, by Lemma 4.1.3 the vertical differential of E_0 satisfies the splitting hypotheses of [22, Lem. 3.5], which implies that the total complex of E_0 is homotopy equivalent to a complex $\mathscr{L}$ with terms $\mathscr{L}_k = \bigoplus_{s-t=k} E_1^{-s,t}$. Hence

$$\mathscr{L} \sim \operatorname{Tot}(E_0) = \Phi_{\mathcal{K}}(\widetilde{M}) \sim \widetilde{M}.$$

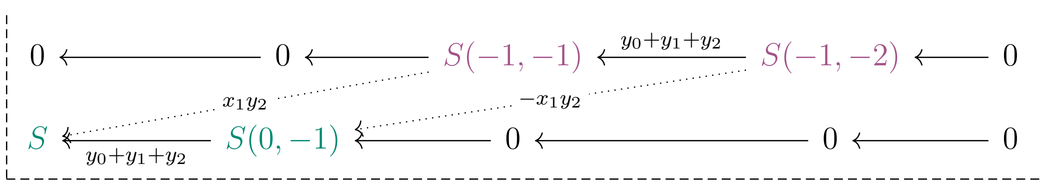
Since the terms of E_1 are direct sums of line bundles, the complex $\mathscr{L}$ is a free monad for $\widetilde{M}$.

Observe that the only terms with twist $\mathbf{a}$ appear in $\mathcal{K}_s$ for $s = |\mathbf{a}|$ and that the Betti numbers in homological index k come from the higher direct images $E_1^{-s,t}$ on diagonals with $s - t = k$. Hence $\beta_{k,\mathbf{a}}(G_\bullet)$ is the rank of $\mathcal{O}_{\mathbb{P}^\mathbf{n}}(-\mathbf{a})$ in $E_1^{-|\mathbf{a}|,|\mathbf{a}|-k}$ which is $h^{|\mathbf{a}|-k}(\mathbb{P}^\mathbf{n}, \widetilde{M} \otimes \Omega^\mathbf{a}(\mathbf{a}))$.

Lastly, note that the hypothesis of part (2) implies that the terms of (4.1.6) on diagonals with $k < 0$ vanish; hence the free monad $\mathscr{L}$ is a locally free resolution. Since each map in the construction from [22, Lem. 3.5] increases the index $-s$, the differentials in $G_\bullet$ have no degree $\mathbf{0}$ components. □

Remark 4.1.5. In the proof of [5, Prop. 1.2], Berkesch, Erman, and Smith show that if M is sufficiently twisted so that all higher direct images of $\widetilde{M} \otimes \Omega^\mathbf{a}_{\mathbb{P}^\mathbf{n}}(\mathbf{a})$ vanish, then the E_1 page will be concentrated in one row, which results in a linear virtual resolution. Similarly in [21, Prop. 1.7], Eisenbud, Erman, and Schreyer prove that for sufficiently positive twists, the truncation of M has a linear free resolution. However, in both cases the positivity condition is stronger than $\mathbf{0}$-regularity for M, as illustrated by the following example.

Example 4.1.6. Write $S = \mathbb{K}[x_0, x_1, y_0, y_1, y_2]$ for the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2$ and consider the ideal $I = (y_0 + y_1 + y_2, x_0y_0 + x_0y_1 + x_0y_2 + x_1y_0 + x_1y_1)$. Then $M = S/I$ is a bigraded, $(0,0)$-regular S-module. The global sections of the Beilinson spectral sequence for $\widetilde{M}$ has first page



where the dotted diagonal maps are lifts of maps from the second page of the spectral sequence, which agree with the maps from [22, Lem. 3.5].

In the next section we state and prove Theorem 4.2.6 by illustrating the restrictions on the virtual resolution above that follow from the regularity of $\widetilde{M}$ and using them to bound the shape of the minimal free resolution of a truncation of M.

4.2 A criterion for multigraded regularity

To investigate the relationship between multigraded regularity and resolutions of truncations we first need to establish a definition of linearity for a multigraded resolution. We would like the differentials to be given by matrices with entries of total degree at most 1. However, we will examine only the twists in the resolution, requiring that they lie in the L regions from Section 4. In particular, we will identify a complex with a map of degree > 1 as nonlinear even if that map is zero.

Definition 4.2.1. Let $F_\bullet$ be a $\mathbb{Z}^r$-graded free resolution. We say $F_\bullet$ is *linear* if F_0 is generated in a single multidegree $\mathbf{d}$ and the twists appearing in F_j lie in $L_j(-\mathbf{d})$.

We require F_0 to be generated in a single degree so that the truncation of a module with a linear resolution also has a linear resolution (see Proposition 4.2.5). Otherwise, for instance, the minimal resolution of M in the following example would be considered linear, yet the resolution of its truncation $M_{\geq(1,0)}$ would not.

Example 4.2.2. Write $S = \mathbb{K}[x_0, x_1, y_0, y_1]$ for the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ and let M be the module with resolution $S(-1,0)^2 \oplus S(0,-1)^2 \leftarrow S(-1,-1)^4 \leftarrow 0$ given by the presentation matrix

$$\begin{bmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_1 & x_0 \\ -y_0 & 0 & -y_0 & 0 \\ 0 & -y_1 & 0 & -y_1 \end{bmatrix}.$$

A *Macaulay2* computation shows that M is $(1,0)$-regular. However, the minimal graded free resolution of the truncation $M_{\geq(1,0)}$ is

$$0 \longleftarrow S(-1,0)^2 \longleftarrow S(-2,-1)^2 \longleftarrow 0$$

which is not linear because $(-2,-1) \notin L_1(-1,0)$.

This example shows that a module can be $\mathbf{d}$-regular yet have a nonlinear resolution for $M_{\geq \mathbf{d}}$. Thus in order to characterize regularity in terms of truncations we need to weaken the definition of linear. We will use the larger Q regions from Section 4 in order to allow some maps of higher degree.

Definition 4.2.3. Let $F_\bullet$ be a $\mathbb{Z}^r$-graded free resolution. We say $F_\bullet$ is *quasilinear* if F_0 is generated in a single multidegree $\mathbf{d}$ and for each j the twists appearing in F_j lie in $Q_j(-\mathbf{d})$.

Example 4.2.4. Unlike on a single projective space, the resolution of S/B for the irrelevant ideal B on a product of projective spaces is not linear. However it is quasilinear. On $\mathbb{P}^1 \times \mathbb{P}^2$, for instance, S/B has resolution

$$0 \longleftarrow S \longleftarrow S(-1,-1)^6 \longleftarrow \begin{matrix} S(-1,-2)^6 \\ \oplus \\ S(-2,-1)^3 \end{matrix} \longleftarrow \begin{matrix} S(-1,-3)^2 \\ \oplus \\ S(-2,-2)^3 \end{matrix} \longleftarrow S(-2,-3) \longleftarrow 0,$$

which has generators in degree $(0,0)$ and relations in degree $(1,1)$. Thus the resolution is not linear, since $(-1,-1) \notin L_1(0,0)$. However $(-1,-1) \in Q_1(0,0)$ is compatible with quasilinearity.

This condition is inspired by [5, Thm. 2.9], which characterized regularity in terms of the existence of virtual resolutions with Betti numbers similar to those of S/B—see Corollary 4.2.14 and Section 4.2 for a more complete discussion. Note that both linear and quasilinear reduce to the standard definition of linear on a single projective space. As one might expect from that setting, they satisfy the property below, which will follow from Theorems 5.1.2 and 5.1.3.

Proposition 4.2.5. *Let M be a $\mathbb{Z}^r$-graded S-module. If $M_{\geq \mathbf{d}}$ has a linear (respectively quasilinear) resolution and $\mathbf{d}' \geq \mathbf{d}$ then $M_{\geq \mathbf{d}'}$ has a linear (respectively quasilinear) resolution.*

A linear resolution for $M_{\geq \mathbf{d}}$ implies that M is $\mathbf{d}$-regular when $H^0_B(M) = 0$. To obtain a converse that generalizes Eisenbud–Goto's result one should instead check that the resolution is quasilinear. This gives a criterion for regularity that does not require computing cohomology.

Theorem 4.2.6. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $H^0_B(M) = 0$. Then M is $\mathbf{d}$-regular if and only if $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ such that F_0 is generated in degree $\mathbf{d}$.*

Example 4.2.7. A smooth hyperelliptic curve of genus 4 can be embedded into $\mathbb{P}^1 \times \mathbb{P}^2$ as a curve of degree $(2, 8)$. An example of such a curve is given explicitly in [5, Ex. 1.4] as the B-saturation I of the ideal

$$\left\langle x_0^2 y_0^2 + x_1^2 y_1^2 + x_0 x_1 y_2^2,\, x_0^3 y_2 + x_1^3 (y_0 + y_1) \right\rangle.$$

Using Theorem 4.2.10 it is relatively easy to check that S/I is not $(2, 1)$-regular: the minimal, graded, free resolution of $(S/I)_{\geq (2,1)}$ is

$$0 \longleftarrow S(-2,-1)^9 \longleftarrow \begin{matrix} S(-3,-1)^7 \\ \oplus \\ S(-2,-2)^{10} \\ \oplus \\ S(-2,-3)^2 \end{matrix} \longleftarrow \begin{matrix} S(-3,-2)^6 \\ \oplus \\ S(-2,-3)^3 \\ \oplus \\ S(-3,-3)^3 \end{matrix} \longleftarrow S(-3,-3)^2 \longleftarrow 0$$

which is not quasilinear because $(-2, -3) \notin Q_1(-2, -1)$.

We prove one direction of Theorem 4.2.6 in Section 4.2 (Theorem 4.2.10) and the other in Section 4.2 (Theorem 4.2.16).

Regularity implies quasilinearity

In Proposition 4.1.4 we constructed a virtual resolution with Betti numbers determined by the sheaf cohomology of $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$. By resolving the $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$ in terms of line bundles and tensoring with $\widetilde{M}$, we can relate the cohomological vanishing in the definition of multigraded regularity to the shape of this virtual resolution. The following lemma implies that when M is $\mathbf{d}$-regular the virtual resolution is quasilinear, i.e., the coefficients of twists outside of $Q_i(-\mathbf{d})$ are zero. The lemma is a variant of [5, Lem. 2.13] (see Section 4.2).

Lemma 4.2.8. *If a $\mathbb{Z}^r$-graded S-module M is $\mathbf{0}$-regular then $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) = 0$ for all $-\mathbf{a} \notin Q_i(\mathbf{0})$ and all $i > 0$.*

Proof. Fix i and $\mathbf{a} \in \mathbb{Z}^r$ with $-\mathbf{a} \notin Q_i(\mathbf{0})$, and suppose that $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \neq 0$. We will show that M is not $\mathbf{0}$-regular. We must have $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$, else $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}) = 0$. Let ℓ be the number of nonzero coordinates in $\mathbf{a}$.

A tensor product of locally free resolutions for the factors $\pi_i^*(\Omega^{a_i}_{\mathbb{P}^{n_i}})$ gives a locally free resolution for $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$. Since $\Omega^0_{\mathbb{P}^{n_i}} = \mathcal{O}_{\mathbb{P}^{n_i}}$ we can use $r-\ell$ copies of $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}$ and ℓ linear resolutions, each generated in total degree 1, to obtain such a resolution $\mathcal{F}$ (see Section 4). Thus the twists in $\mathcal{F}_j$ have nonpositive coordinates and total degree $-j-\ell$, so they are in $L_{j+\ell}(\mathbf{0})$.

Since $\mathcal{F}$ is locally free the cokernel of $\widetilde{M} \otimes \mathcal{F}$ is isomorphic to $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$. By a standard spectral sequence argument, explained in the proof of Theorem 4.2.16, the nonvanishing of $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}))$ implies the existence of some j such that $H^{|\mathbf{a}|-i+j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{F}_j) \neq 0$.

If $i = 0$ then

$$|\mathbf{a}| - i + j \geq \ell - i + j = j + \ell.$$

If $i > 0$ then $\mathbf{a} - \mathbf{1}$ has ℓ nonnegative coordinates that sum to $|\mathbf{a}| - \ell$. Thus $|\mathbf{a}| - \ell > i - 1$, since $-\mathbf{a} \notin Q_i(\mathbf{0}) = L_{i-1}(-\mathbf{1})$ (see Remark 4.0.5). This also gives

$$|\mathbf{a}| - i + j \geq (\ell + i) - i + j = j + \ell.$$

so in either case $L_{j+\ell}(\mathbf{0}) \subseteq L_{|\mathbf{a}|-i+j}(\mathbf{0})$. Therefore $H^{|\mathbf{a}|-i+j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{F}_j) \neq 0$ for $\mathcal{F}_j$ with twists in $L_{j+\ell}(\mathbf{0})$ implies that M is not $\mathbf{0}$-regular. □

See [13, Thm. 5.5] for a similar result relating Hoffman and Wang's definition of regularity [32] to a different cohomology vanishing for $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$.

Motivated by the quasilinearity of the virtual resolution in Proposition 4.1.4, we will prove that the $\mathbf{d}$-regularity of M implies that the minimal free resolution of $M_{\geq \mathbf{d}}$ is quasilinear. Let K be the Koszul complex from Section 4 and $\check{C}^p(B, \cdot)$ the Čech complex as in Section 1.2. We will use the spectral sequence of a double complex with rows from subcomplexes of K and columns given by Čech complexes in order to relate the Betti numbers of $M_{\geq \mathbf{d}}$ to the sheaf cohomology of $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$. We will also need the following lemma about Čech complexes.

Lemma 4.2.9. *Given a complex of graded S-modules $L \to M \to N$ such that $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact, the complex $\check{C}^p(B, L) \to \check{C}^p(B, M) \to \check{C}^p(B, N)$ is exact for each $p \geq 0$.*

Proof. Fix p. Then $\check{C}^p(B, L) \to \check{C}^p(B, M) \to \check{C}^p(B, N)$ splits as a direct sum of complexes

$$L[g_1^{-1}, \ldots, g_p^{-1}] \to M[g_1^{-1}, \ldots, g_p^{-1}] \to N[g_1^{-1}, \ldots, g_p^{-1}]$$

each of which can be obtained by applying $\Gamma(U, -)$ to $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$, where U is the complement of $V(g_1, \ldots, g_p)$. Since U is affine they are exact. □

Theorem 4.2.10. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $H^0_B(M)_{\mathbf{d}} = 0$. If M is $\mathbf{d}$-regular then $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ with F_0 generated in degree $\mathbf{d}$.*

Proof. Without loss of generality we may assume $\mathbf{d} = \mathbf{0}$ and $M = M_{\geq \mathbf{0}}$ (see Lemma 1.2.1).

By Proposition 4.1.4 there exists a free monad $G_\bullet$ of M with j-th Betti number given by $h^{|\mathbf{a}|-j}(\widetilde{M \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}}(\mathbf{a}))$. Since M is $\mathbf{0}$-regular the vanishing of these cohomology groups results in a quasilinear virtual resolution by Lemma 4.2.8 and (2) from Proposition 4.1.4. Let $F_\bullet$ be the minimal free resolution of M. We will show that the Betti numbers of $F_\bullet$ are equal to those of $G_\bullet$, so that $F_\bullet$ is also quasilinear and $F_0 = G_0$ is generated in degree $\mathbf{d}$. (In fact this is enough to show that $F_\bullet$ and $G_\bullet$ are isomorphic, as we will do in Corollary 4.2.14.)

Fix a degree $\mathbf{a} \in \mathbb{Z}^r$. Construct a double complex $E^{\bullet,\bullet}$ by taking the Čech complex of each term in $M \otimes K^{\leq \mathbf{a}}_\bullet$ and including the Čech complex of $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ as an additional column. Index $E^{\bullet,\bullet}$ so that

$$E^{s,t} = \begin{cases} \check{C}^t\Big(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|+1-s}\Big) & \text{if } s > 0, \\ \check{C}^t\Big(B, M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}\Big) & \text{if } s = 0. \end{cases}$$

We will compare the vertical and horizontal spectral sequences of $E^{\bullet,\bullet}$ in degree $\mathbf{a}$. By Lemma 4.0.7 and the fact that $K^{\leq \mathbf{a}}_\bullet$ is locally free, the sheafification of the 0-th row $E^{\bullet,0}$ is exact. Thus by Lemma 4.2.9 the rows of $E^{\bullet,\bullet}$ are exact for $t \neq 0$.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\check{C}^2(B, M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}) & \to & \check{C}^2(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|}) & \to & \check{C}^2(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|-1}) & \to & \cdots & \to & \check{C}^2(B, M \otimes K^{\leq \mathbf{a}}_0) \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\check{C}^1(B, M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}) & \to & \check{C}^1(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|}) & \to & \check{C}^1(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|-1}) & \to & \cdots & \to & \check{C}^1(B, M \otimes K^{\leq \mathbf{a}}_0) \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}} & \longrightarrow & M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|} & \longrightarrow & M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|-1} & \longrightarrow & \cdots & \longrightarrow & M \otimes K^{\leq \mathbf{a}}_0
\end{array}$$

Since the elements of M have degrees $\geq \mathbf{0}$, the elements of degree $\mathbf{a}$ in $M \otimes K_\bullet$ come from elements of degree $\leq \mathbf{a}$ in $K_\bullet$. Thus by Lemma 4.0.7 the homology of $M \otimes K^{\leq \mathbf{a}}_\bullet$ in degree $\mathbf{a}$ is the same as that of $M \otimes K_\bullet$. Hence the cohomology of the 0-th row $E^{\bullet,0}$ in degree $\mathbf{a}$ computes the degree $\mathbf{a}$ Betti numbers of F_j for $0 \leq j \leq |\mathbf{a}|$, i.e., for $s > 0$,

$$H^s(E^{\bullet,0})_{\mathbf{a}} = \operatorname{Tor}_{|\mathbf{a}|+1-s}(M, \mathbb{K})_{\mathbf{a}}. \tag{4.2.1}$$

The vertical cohomology of $E^{\bullet,\bullet}$ gives the local cohomology of the terms of $M \otimes K^{\leq \mathbf{a}}_\bullet$ along with $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$. Consider the degree $\mathbf{a}$ part of this double complex. The cohomology coming from $M \otimes K^{\leq \mathbf{a}}_\bullet$ has summands of the form $H^i_B(M(-\mathbf{b}))_{\mathbf{a}} = H^i_B(M)_{\mathbf{a}-\mathbf{b}}$ where $\mathbf{b} \leq \mathbf{a}$. These vanish because M is $\mathbf{0}$-regular, except possibly $H^0_B(M)_{\mathbf{0}}$ which vanishes by hypothesis, so the only nonzero terms come from $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$.

Since $K_\bullet^{\leq \mathbf{a}}$ is a resolution of $\mathbb{K}$ in degrees $\leq \mathbf{a}$, there are no elements of degree $\mathbf{a}$ in $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$. Hence, using (1.2.1),

$$H^1_B\Big(M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}\Big)_{\mathbf{a}} = H^0\Big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\Big).$$

Therefore the cohomology of the 0-th column $E^{0,\bullet}$ in degree $\mathbf{a}$ is

$$H^t(E^{0,\bullet})_{\mathbf{a}} = H^t_B(M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}})_{\mathbf{a}} = H^{t-1}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \tag{4.2.2}$$

for $t > 0$, i.e., the Betti numbers of $G_\bullet$ indexed differently.

Since both spectral sequences of the double complex $E^{\bullet,\bullet}$ converge after the first page, their total complexes agree in degree $\mathbf{a}$, so by equating the dimensions of (4.2.1) and (4.2.2) in total degree $|\mathbf{a}| + 1 - j$ we get

$$\dim_{\mathbb{K}} \operatorname{Tor}_j(M, \mathbb{K})_{\mathbf{a}} = \dim_{\mathbb{K}} H^{|\mathbf{a}|-j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \tag{4.2.3}$$

for $|\mathbf{a}| \geq j \geq 0$. When $j > |\mathbf{a}|$, neither $F_\bullet$ nor $G_\bullet$ has a nonzero Betti number for degree reasons, and when $\mathbf{a}$ has $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}} = 0$ the argument above still holds. Hence the Betti numbers of $G_\bullet$ and $F_\bullet$ are equal in degree $\mathbf{a}$. □

To check that a module M is $\mathbf{d}$-regular directly from Definition 4.0.6, condition (2) requires one to show that $H^i_B(M)_{\mathbf{p}}$ vanishes for all $i > 0$ and all

$$\mathbf{p} \in \bigcup_{|\lambda|=i} (\mathbf{d} - \lambda_1\mathbf{e}_1 - \cdots - \lambda_r\mathbf{e}_r + \mathbb{N}^r)$$

with $\lambda \in \mathbb{N}^r$. The proof of Theorem 4.2.10, when combined with Theorem 4.2.6 and Lemma 4.2.8, shows that on a product of projective spaces the full strength of this condition is unnecessary. In particular, one only needs to consider λ_j with $\lambda_j \leq n_j + 1$.

Proposition 4.2.11. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If*

1. $H^0_B(M)_{\mathbf{p}} = 0$ *for all* $\mathbf{p} \geq \mathbf{d}$
2. $H^i_B(M)_{\mathbf{p}} = 0$ *for all* $i > 0$ *and all* $\mathbf{p} \in \bigcup_{|\lambda|=i}(\mathbf{d} - \sum_1^r \lambda_j\mathbf{e}_j + \mathbb{N}^r)$ *where* $0 \leq \lambda_j \leq n_j + 1$

then M is $\mathbf{d}$-regular.

Proof. The only difference between (2) above and condition (2) in Definition 4.0.6 is the restriction to $\lambda_j \leq n_j + 1$. By the proof of Theorem 4.2.10, if $H^0_B(M)_{\mathbf{b}} = 0$ and M satisfies the hypotheses of Proposition 4.1.4 and Lemma 4.2.8 then M has a quasilinear resolution generated in degree $\mathbf{d}$ and is thus $\mathbf{d}$-regular by Theorem 4.2.6. In the proof of Lemma 4.2.8 it is sufficient for the cohomology of $M(\mathbf{d})$ to vanish in degrees appearing in the resolution of some $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$, which excludes those with coordinates not $\leq \mathbf{n} + \mathbf{1}$. □

Example 4.2.12. On $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, to show that a module M is $\mathbf{0}$-regular using Definition 4.0.6 one must check that $H^3_B(M)_{\mathbf{p}} = 0$ for $\mathbf{p}$ in the region with minimal elements

$$(-3,0,0),(-2,-1,0),(-2,0,-1),\ldots,(0,-3,0),\ldots,(0,0,-3).$$

However, Proposition 4.2.11 implies that a smaller region is sufficient. For instance, we need not check that $H^3_B(M)_{\mathbf{p}} = 0$ for $\mathbf{p}$ equal to each of $(-3,0,0)$, $(0,-3,0)$, and $(0,0,-3)$.

Remark 4.2.13. One may also deduce Proposition 4.2.11 from the proofs in [5] without the hypothesis that $H^0_B(M)_{\mathbf{d}} = 0$.

The proof of Theorem 4.2.10 also implies that when M is $\mathbf{d}$-regular the resolution of $M_{\geq \mathbf{d}}$ is isomorphic to the virtual resolution constructed in Proposition 4.1.4. In other words, the minimal free resolution of $M_{\geq \mathbf{d}}$ is a splitting of the Beilinson spectral sequence for $M(\mathbf{d})$, giving a concrete construction of the abstractly defined virtual resolutions used in [5, Thm. 2.9] to witness the regularity of $M(\mathbf{d})$.

Corollary 4.2.14. *The complexes $F_\bullet$ and $G_\bullet$ in the proof of Theorem 4.2.10 are isomorphic.*

Proof. The complex $G_\bullet$ satisfies conditions (1) and (2) of Proposition 4.1.2 by Proposition 4.1.4 and the fact that $\Omega^{\mathbf{a}}_{\mathbb{P}^n}$ is nonzero only for $0 \leq \mathbf{an}$. Since the Betti numbers of $F_\bullet$ agree with those of $G_\bullet$ it must also satisfy (2).

Since $H^0_B(M)_{\mathbf{d}} = H^1_B(M) = 0$ we have $M_{\mathbf{d}} = H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{d}))$ by (1.2.1). The map from $G_0 = S \otimes H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{d}))$ to $M_{\geq \mathbf{d}}$ required by condition (3) is given by multiplication. □

Quasilinearity implies regularity

We will now prove the reverse implication of Theorem 4.2.6, namely that a quasilinear resolution generated in degree $\mathbf{d}$ for $M_{\geq \mathbf{d}}$ implies that M is $\mathbf{d}$-regular. We use a hypercohomology spectral sequence argument, which relates the local cohomology of M to that of the terms in a resolution for $M_{\geq \mathbf{d}}$.

The following lemma will show that entire diagonals in our spectral sequence vanish when the resolution is quasilinear. Thus the local cohomology modules $H^i_B(M)$ to which the diagonals converge also vanish in the same degrees.

Lemma 4.2.15. *If $i, j \in \mathbb{N}$ then $H^{i+j+1}_B(S)_{\mathbf{a}+\mathbf{b}} = 0$ for all $\mathbf{a} \in L_i(\mathbf{0})$ and all $\mathbf{b} \in Q_j(\mathbf{0})$.*

Proof. Note that $L_i(\mathbf{0}) + Q_j(\mathbf{0}) = L_i(\mathbf{0}) + L_{j-1}(-\mathbf{1}) = L_{i+j-1}(-\mathbf{1})$ as sets. We also have $H^0_B(S) = H^1_B(S) = 0$, so it suffices to show that $H^{k+1}_B(S)_{\mathbf{c}} = H^k(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{c})) = 0$ for $k \geq 1$ and $\mathbf{c} \in L_{k-1}(-\mathbf{1})$.

The cohomology of $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}$ is given by the Künneth formula. Fix a nonempty set of indices $J \subseteq \{1,\ldots,r\}$ and consider the term

$$\left[\bigotimes_{j\in J} H^{n_j}(\mathbb{P}^{n_j}, \mathcal{O}_{\mathbb{P}^{n_j}}(d_j))\right] \otimes \left[\bigotimes_{j\notin J} H^0(\mathbb{P}^{n_j}, \mathcal{O}_{\mathbb{P}^{n_j}}(d_j))\right],$$

which contributes to $H^k(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{c}))$ for $k = \sum_{j \in J} n_j$. It will be nonzero if and only if $d_j \leq -n_j - 1$ for $j \in J$ and $d_j \geq 0$ for $j \notin J$. If $\mathbf{c} \in L_{k-1}(-\mathbf{1})$ then

$$\mathbf{c} \geq -\mathbf{1} - \lambda_1 \mathbf{e}_1 - \cdots - \lambda_r \mathbf{e}_r$$

for some λ_i with $\sum \lambda_i = k - 1 = -1 + \sum_{j \in J} n_j$. It is not possible for the right side to have components $\leq -n_j - 1$ for all $j \in J$. Since all cohomology of $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}$ arises in this way, the lemma follows. □

In [5, Thm. 2.9] Berkesch, Erman, and Smith show for M with $H^0_B(M) = H^1_B(M) = 0$ that M is $\mathbf{d}$-regular if and only if M has a virtual resolution $F_\bullet$ so that the degrees of the generators of $F(\mathbf{d})_\bullet$ are at most those appearing in the minimal free resolution of S/B. This Betti number condition is stronger than quasilinearity, but the additional strength is not used in their proof, so the existence of such a virtual resolution is equivalent to the existence of a quasilinear one.

Since a resolution of $M_{\geq \mathbf{d}}$ is a type of virtual resolution, the reverse implication of Theorem 4.2.6 mostly reduces to this result. We present a modified proof for completeness. In particular, we do not need to require $H^1_B(M) = 0$ because we have more information about the cokernel of our resolution.

From this perspective Theorem 4.2.6 says that the regularity of M is determined not only by the Betti numbers of its virtual resolutions, but by the Betti numbers of only those virtual resolutions that are actually minimal free resolutions of truncations of M. Thus we provide an explicit method for checking whether M is $\mathbf{d}$-regular.

Theorem 4.2.16. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $H^0_B(M) = 0$. If $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ with F_0 generated in degree $\mathbf{d}$, then M is $\mathbf{d}$-regular.*

Proof. Without loss of generality we may assume $\mathbf{d} = \mathbf{0}$ and $M = M_{\geq \mathbf{0}}$ (see Lemma 1.2.1).

Let $F_\bullet$ be a quasilinear resolution of M, so that the twists of F_j are in $Q_j(\mathbf{0})$. Then the spectral sequence of the double complex $E^{\bullet,\bullet}$ with terms

$$E^{s,t} = \check{C}^t(B, F_{-s})$$

converges to the cohomology $H^i_B(M)$ of M in total degree i. The first page of the vertical spectral sequence has terms $H^t_B(F_{-s})$, so $H^{i+j}_B(F_j)_{\mathbf{a}} = 0$ for all j (i.e., for all $(s,t) = (-j, i+j)$) implies $H^i_B(M)_{\mathbf{a}} = 0$.

Therefore it suffices to show that $H^{i+j}_B(S(\mathbf{b}))_{\mathbf{a}} = 0$ for $i \geq 1$ and all $\mathbf{a} \in L_{i-1}(\mathbf{0})$ and $\mathbf{b} \in Q_j(\mathbf{0})$, as is done in Lemma 4.2.15. □

4.3 Truncations with linear resolutions

As demonstrated by Example 4.2.2, in general $\mathbf{d}$-regularity is a stronger condition than having a linear resolution for $M_{\geq \mathbf{d}}$. Still, linear truncations have been independently studied in the literature [21, 5].

Our main result in this section is a cohomological vanishing condition that specifies when $M_{\geq \mathbf{d}}$ has a linear resolution. Our arguments largely mimic those for the analogous statements about quasilinear resolutions by switching the roles of L and Q.

Lemma 4.3.1. *Let M be a $\mathbb{Z}^r$-graded S-module. If $H^i(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{b})) = 0$ for all $i > 0$ and all $\mathbf{b} \in Q_i(\mathbf{0})$, then $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) = 0$ for all $i \geq 0$ and all $-\mathbf{a} \notin L_i(\mathbf{0})$.*

Proof. We will modify the argument from Lemma 4.2.8.

Suppose that $-\mathbf{a} \notin L_i(\mathbf{0})$ and $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \neq 0$. Since $\mathbf{a} \geq 0$ we have $|\mathbf{a}| > i$. There must exist j such that $H^{|\mathbf{a}|-i+j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{F}_j) \neq 0$, where the twists $\mathbf{b}$ in $\mathcal{F}_j$ have total degree $-j - \ell$ for ℓ the number of nonzero coordinates in $\mathbf{a}$. Each twist has ℓ negative coordinates, so that the positive coordinates of $-\mathbf{1} - \mathbf{b}$ sum to $j + \ell - \ell = j$. Hence $H^{|\mathbf{a}|-i+j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{b})) \neq 0$ for some $\mathbf{b} \in L_j(-\mathbf{1}) = Q_{j+1}(\mathbf{0}) \subseteq Q_{|\mathbf{a}|-i+j}(\mathbf{0})$ with $|\mathbf{a}|-i+j > 0$. □

As in our main theorem, the conclusion of this lemma ensures the vanishing of certain Betti numbers of $M_{\geq \mathbf{d}}$.

Theorem 4.3.2. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module with $H^0_B(M) = 0$. Then $M_{\geq \mathbf{d}}$ has a linear resolution $F_\bullet$ with F_0 generated in degree $\mathbf{d}$ if and only if $H^i_B(M)_{\mathbf{b}} = 0$ for all $i > 0$ and all $\mathbf{b} \in Q_{i-1}(\mathbf{d})$.*

Proof. The proof of the forward implication is analogous to the proof of Theorem 4.2.16, switching the roles of L and Q. For the reverse, notice that the proof of Theorem 4.2.10 shows that the virtual resolution of M from Proposition 4.1.4 has the same Betti numbers as the minimal free resolution of $M_{\geq \mathbf{d}}$, i.e.,

$$\dim_{\mathbb{K}} \operatorname{Tor}_j(M_{\geq \mathbf{d}}, \mathbb{K})_{\mathbf{a}} = \dim_{\mathbb{K}} H^{|\mathbf{a}|-j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}))$$

for $|\mathbf{a}| \geq j \geq 0$ and both are 0 otherwise. The vanishing of the right hand side for $-\mathbf{a} \notin L_j(\mathbf{0})$, given by Lemma 4.3.1, then implies that the minimal free resolution of $M_{\geq \mathbf{d}}$ is linear. □

Corollary 4.3.3. *The minimal free resolution of $S(-\mathbf{b})_{\geq \mathbf{d}}$ is linear for all $\mathbf{b}, \mathbf{d} \in \mathbb{Z}^r$.*

Proof. By adjusting $\mathbf{d}$ we may assume that $\mathbf{b} = \mathbf{0}$. Note that $S_{\geq \mathbf{d}} = S_{\geq \mathbf{d}'}$ for $\mathbf{d}' = \max\{\mathbf{d}, \mathbf{0}\} \in \mathbf{0} + \mathbb{N}^r$. Thus by Theorem 4.3.2 and Proposition 4.2.5 it suffices to show that $H^i_B(S)_{\mathbf{b}} = 0$ for all $i > 0$ and all $\mathbf{b} \in Q_{i-1}(\mathbf{0})$, which follows from Lemma 4.2.15. □

Chapter 5

Multigraded regularity and Betti numbers

Most of this chapter comes from Sections 5 and 7 of [9].

Unlike in the single graded setting, it is possible for two modules on a product of projective spaces to have the same multigraded Betti numbers but different multigraded regularities.

Example 5.0.1. Let M be the module on $\mathbb{P}^1 \times \mathbb{P}^1$ with resolution

$$S(-1,0)^2 \oplus S(0,-1)^2 \leftarrow S(-1,-1)^4 \leftarrow 0$$

given in Example 4.2.2. Computation shows that M is $(1,0)$-regular but not $(0,1)$-regular. Notice that all of the twists appearing in the minimal resolution of M are symmetric with respect to the factors of $\mathbb{P}^1 \times \mathbb{P}^1$. Hence the cokernel N given by exchanging x and y in the presentation matrix has the same multigraded Betti numbers as M. However N is not $(1,0)$-regular because M was not $(0,1)$-regular.

Remark 5.0.2. Example 5.0.1 answers a question of Botbol and Chardin [7, Ques. 1.2].

Hence the Betti numbers of M also do not determine the Betti numbers of $M_{\geq \mathbf{d}}$. Still, we can intersect combinatorially the regions $L_i(\mathbf{b})$ and $Q_i(\mathbf{b})$ (see Figure 4.1) to specify a subset of the degrees $\mathbf{d} \in \mathbb{Z}^r$ where $M_{\geq \mathbf{d}}$ has a linear or quasilinear resolution generated in degree $\mathbf{d}$.

On a single projective space we recover condition (4) of Eisenbud–Goto from Chapter 4. Our proof is based on the observation that we can construct a possibly nonminimal free resolution of $M_{\geq \mathbf{d}}$ from the truncations of the terms in the minimal free resolution of M.

A number of inner[1] bounds on the multigraded regularity of a module in terms of its Betti numbers exist in the literature. For example, [36, Cor. 7.3] used a local cohomology long exact sequence argument to deduce such a bound. These methods were extended in [7,

[1]We use the terms inner and outer bound since in general there is no total ordering on $\operatorname{reg} X$ when $\operatorname{Pic} X \neq \mathbb{Z}$. For a single projective space an inner bound corresponds to an upper bound and an outer bound to a lower bound.

Thm. 4.14] using a local cohomology spectral sequence. Our bounds in Theorems 5.1.3 and 5.1.2 are generally larger and thus closer to the actual regularity than these results.

Moreover, they are sharp in a number of examples. For instance, we use Theorem 4.2.6 to show that the containment in Corollary 5.1.4 is equal to the regularity for all saturated *ample* complete intersections, meaning those determined by ample hypersurfaces.

Note that on a product of projective spaces the intermediate cohomology of a complete intersection does not necessarily vanish. Even the local cohomology of a hypersurface in a product of projective spaces is not determined by its degree [7, Sec. 4.5]. Thus computing the multigraded regularity of complete intersections on products of projective spaces is more complicated than in the case of a single projective space.

5.1 Inner bound from Betti numbers

While the multigraded Betti numbers of a module do not determine its regularity, in this section we show that they do determine a subset of the regularity. In particular, the following lemma restricts the possible Betti numbers of a truncation of M given the Betti numbers of M. Intuitively, it states that the degrees of Betti numbers of $M_{\geq \mathbf{d}}$ come from the maximum of $\mathbf{d}$ and the degrees of Betti numbers of M, possibly after adding some linear terms.

Lemma 5.1.1. *Let M be a $\mathbb{Z}^r$-graded S-module. If $M_{\geq \mathbf{d}}$ has $\operatorname{Tor}^S_{m'}(M_{\geq \mathbf{d}}, \Bbbk)_{\mathbf{b}'} \neq 0$ for some $\mathbf{b}' \in \mathbb{Z}^k$ then there exist $\mathbf{b} \leq \mathbf{b}'$ and $m \leq m'$ such that $\operatorname{Tor}^S_m(M, \Bbbk)_{\mathbf{b}} \neq 0$ and $|\mathbf{b}' - \mathbf{c}| \leq m' - m$ where $\mathbf{c} = \max\{\mathbf{b}, \mathbf{d}\}$.*

Proof. Let $0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ be the minimal free resolution of M. Then the termwise truncation $0 \leftarrow M_{\geq \mathbf{d}} \leftarrow (F_0)_{\geq \mathbf{d}} \leftarrow (F_1)_{\geq \mathbf{d}} \leftarrow \cdots$ is also exact by Lemma 4.0.2. For each i, let $G^i_\bullet$ be a minimal free resolution of $(F_i)_{\geq \mathbf{d}}$.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & G^0_1 & & G^1_1 & & G^2_1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & G^0_0 & & G^1_0 & & G^2_0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & (F_0)_{\geq \mathbf{d}} & \longleftarrow & (F_1)_{\geq \mathbf{d}} & \longleftarrow & (F_2)_{\geq \mathbf{d}} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We will see in Corollary 4.3.3 that $S(-\mathbf{b})_{\geq \mathbf{d}}$ has a linear resolution for all $\mathbf{b} \in \mathbb{Z}^k$. Thus the $G^i_\bullet$ are linear. By taking iterated mapping cones we can construct a free resolution of $M_{\geq \mathbf{d}}$ with terms

$$0 \leftarrow G^0_0 \leftarrow G^0_1 \oplus G^1_0 \leftarrow G^0_2 \oplus G^1_1 \oplus G^2_0 \leftarrow \cdots . \tag{5.1.1}$$

Then $\mathbf{b}'$ corresponds to the degree of a generator of some G^i_j with $i + j = m'$. Since $G^i_\bullet$ is linear, there is a minimal generator of $(F_i)_{\geq \mathbf{d}}$ with degree $\mathbf{c}$ such that $|\mathbf{b}' - \mathbf{c}| = j$.

However the generators of $(F_i)_{\geq \mathbf{d}}$ have degrees equal to $\max\{\mathbf{b}, \mathbf{d}\}$ for degrees $\mathbf{b}$ of generators of F_i. These correspond to $\mathbf{b} \in \mathbb{Z}^k$ such that $\mathrm{Tor}^S_i(M, \Bbbk)_\mathbf{b} \neq 0$. Thus the lemma holds for $m = i$, so that $m' - m = j = |\mathbf{b}' - \mathbf{c}|$ as desired. □

Lemma 5.1.1 shows that each Betti number of $M_{\geq \mathbf{d}}$ comes from a Betti number of M in a predictable way. Note that the process cannot be reversed—not all Betti numbers of M produce minimal Betti numbers of $M_{\geq \mathbf{d}}$. However, the Betti numbers of M limit the degrees where a nonlinear truncation could exist. The following theorem identifies such degrees.

Theorem 5.1.2. *Let M be a $\mathbb{Z}^r$-graded S-module. For all $\mathbf{d} \in \bigcap L_m(\mathbf{b})$, the truncation $M_{\geq \mathbf{d}}$ has a linear resolution generated in degree $\mathbf{d}$, where the intersection is over all m and all $\mathbf{b}$ with $\mathrm{Tor}^S_m(M, \Bbbk)_\mathbf{b} \neq 0$.*

Proof. We may assume that $\mathbf{d} = \mathbf{0}$. Suppose instead that $M_{\geq \mathbf{0}}$ does not have a linear resolution generated in degree $\mathbf{0}$. Then there exist $\mathbf{b}' \in \mathbb{N}^k$ and $m' \in \mathbb{Z}$ such that $\mathrm{Tor}^S_{m'}(M_{\geq \mathbf{0}}, \Bbbk)_{\mathbf{b}'} \neq 0$ and $|\mathbf{b}'| > m'$.

By Lemma 5.1.1 there exist $\mathbf{b}$ and m so that $\mathrm{Tor}^S_m(M, \Bbbk)_\mathbf{b} \neq 0$ and $|\mathbf{b}' - \mathbf{c}| \leq m' - m$ where $\mathbf{c} = \max\{\mathbf{b}, \mathbf{0}\}$. The sum of the positive components of $\mathbf{b}$ is

$$|\mathbf{c}| = |\mathbf{b}'| - |\mathbf{b}' - \mathbf{c}| > m' - (m' - m) = m$$

so $\mathbf{0} \notin L_m(\mathbf{b})$ (see Remark 4.0.5). □

An analogous statement to Theorem 5.1.2 exists for truncations with quasilinear resolutions. By Theorem 4.2.6 it also gives a subset of the multigraded regularity. We will see in Section 5.2 that this inner bound is sharp.

Theorem 5.1.3. *Let M be a $\mathbb{Z}^r$-graded S-module. For all $\mathbf{d} \in \bigcap Q_m(\mathbf{b})$, the truncation $M_{\geq \mathbf{d}}$ has a quasilinear resolution generated in degree $\mathbf{d}$, where the intersection is over all m and all $\mathbf{b}$ with $\mathrm{Tor}^S_m(M, \Bbbk)_\mathbf{b} \neq 0$.*

Proof. Assume $\mathbf{d} = \mathbf{0}$ and suppose instead that $M_{\geq \mathbf{0}}$ does not have a quasilinear resolution generated in degree $\mathbf{0}$. If $M_{\geq \mathbf{0}}$ is not generated in degree $\mathbf{0}$ then some generator of M has a degree $\mathbf{b}$ with a positive coordinate, so that $\mathbf{0} \notin \mathbf{b} + \mathbb{N}^r = Q_0(\mathbf{b})$.

Otherwise there exist $\mathbf{b}' \in \mathbb{N}^k$ and $m' \in \mathbb{Z}$ such that $\mathrm{Tor}^S_{m'}(M_{\geq \mathbf{0}}, \Bbbk)_{\mathbf{b}'} \neq 0$ and $|\mathbf{b}'| > m' + \ell' - 1$ where ℓ' is the number of nonzero coordinates in $\mathbf{b}'$. Thus by Lemma 5.1.1 there exist $\mathbf{b}$ and m so that $\mathrm{Tor}^S_m(M, \Bbbk)_\mathbf{b} \neq 0$ and $|\mathbf{b}' - \mathbf{c}| \leq m' - m$ for $\mathbf{c} = \max\{\mathbf{b}, \mathbf{0}\}$.

Let ℓ be the number of coordinates for which $\mathbf{c}$ differs from $\mathbf{c}' = \max\{\mathbf{b}, \mathbf{1}\}$. Then $|\mathbf{c}'| = |\mathbf{c}| + \ell$, so the sum of the positive components of $\mathbf{b} - \mathbf{1}$ is

$$\begin{aligned} |\mathbf{c}' - \mathbf{1}| &= |\mathbf{c}| + \ell - r \\ &= |\mathbf{b}'| - |\mathbf{b}' - \mathbf{c}| - r + \ell \\ &> (m' + \ell' - 1) - (m' - m) - r + \ell \\ &= m - 1 + \ell' - (r - \ell). \end{aligned}$$

Note that $r - \ell$ is the number of nonzero coordinates in $\mathbf{c}$. Since $\mathbf{b}' \geq \mathbf{0}$ and $\mathbf{b}' \geq \mathbf{b}$ we have $\mathbf{b}' \geq \mathbf{c} \geq \mathbf{0}$, so $\ell' \geq r - \ell$. Hence the right side of the inequality is $\geq m - 1$, so $\mathbf{0} \notin L_{m-1}(\mathbf{b} - \mathbf{1}) = Q_m(\mathbf{b})$ (see Remark 4.0.5). $\square$

Corollary 5.1.4. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $H^0_B(M) = 0$, then*

$$\bigcap_{i \in \mathbb{N}} \bigcap_{\mathbf{b} \in \beta_i(M)} Q_i(\mathbf{b}) \subseteq \operatorname{reg}(M).$$

We can now prove Proposition 4.2.5.

Proof of Proposition 4.2.5. Suppose that $M_{\geq \mathbf{d}}$ has a linear resolution. We will apply Theorem 5.1.2 to $M_{\geq \mathbf{d}}$ to show that $M_{\geq \mathbf{d}'}$ has a linear resolution for $\mathbf{d}' \geq \mathbf{d}$ as desired. We may assume that the intersection contains all possible terms that could arise from a linear resolution:

$$\bigcap_{i \in \mathbb{N}} \bigcap_{-\mathbf{b} \in L_i(-\mathbf{d})} L_i(\mathbf{b})$$

Note that $-\mathbf{b} \in L_i(-\mathbf{d})$ if and only if $\mathbf{d} \in L_i(\mathbf{b})$. Thus $\mathbf{d} \in L_i(\mathbf{b})$ for all $\mathbf{b}$, so $\mathbf{d}'$ is in the intersection as well. For quasilinear resolutions replace L with Q. $\square$

Other bounds on the multigraded regularity of a module in terms of its Betti numbers exist in the literature. For example, Maclagan and Smith use a long exact sequence argument to bound regularity in [36, Thm. 1.5, Cor 7.2]. While our theorem has the added hypothesis that $H^0_B(M) = 0$, it is often sharper than Maclagan and Smith's.

Example 5.1.5. In [36, Ex. 7.6] Maclagan and Smith consider the B-saturated ideal $I = \langle x_{1,0} - x_{1,1}, x_{2,0} - x_{2,1}, x_{3,0} - x_{3,1} \rangle \cap \langle x_{1,0} - 2x_{1,1}, x_{2,0} - 2x_{2,1}, x_{3,0} - 2x_{3,1} \rangle$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. They show that the regularity of S/I is

$$\operatorname{reg}(S/I) = \big((1,0,0) + \mathbb{N}^3\big) \cup \big((0,1,0) + \mathbb{N}^3\big) \cup \big((0,0,1) + \mathbb{N}^3\big)$$

and their bound from the Betti numbers of S/I is

$$\big((2,2,1) + \mathbb{N}^3\big) \cup \big((2,1,2) + \mathbb{N}^3\big) \cup \big((1,2,2) + \mathbb{N}^3\big) \subset \operatorname{reg}(S/I).$$

However, Corollary 5.1.4 implies that $(1,1,1) + \mathbb{N}^r \subseteq \operatorname{reg}(S/I)$, giving a larger inner bound.

5.2 Regularity of complete intersections

As an application of Theorems 4.2.6 and 5.1.3, in this section we compute the multigraded regularity of a saturated complete intersection satisfying minor hypotheses on its generators. To do this we make the bound from Corollary 5.1.4 explicit in the case of complete intersections. We then use our characterization of regularity to prove that the resulting bound is sharp by explicitly constructing truncations outside this region that do not have quasilinear resolutions.

Lemma 5.2.1. *If* $\mathbf{b}, \mathbf{c} \in \mathbb{N}^r$ *with* $b_j, c_j > 0$ *for all* j *then* $Q_{i+1}(\mathbf{b}+\mathbf{c}) \subseteq Q_i(\mathbf{b})$ *for all* $i > 0$.

Proof. By definition the minimal elements of $Q_{i+1}(\mathbf{b}+\mathbf{c})$ are of the form $\mathbf{b}+\mathbf{c}-\mathbf{1}-\mathbf{v}$ where $\mathbf{v} \in \mathbb{N}^r$ and $|\mathbf{v}| = i$. It is enough to show that each $\mathbf{b}+\mathbf{c}-\mathbf{1}-\mathbf{v}$ is in $Q_i(\mathbf{b})$. Since $|\mathbf{v}| = i$ it has at least one nonzero coordinate, say v_j. From this we have

$$\mathbf{b}+\mathbf{c}-\mathbf{1}-\mathbf{v} = (\mathbf{b}-\mathbf{1}-(\mathbf{v}-\mathbf{e}_j)) + (\mathbf{c}-\mathbf{e}_j).$$

The desired containment follows from the above equality given that $|\mathbf{v}-\mathbf{e}_j| = i-1$ and that by assumption $\mathbf{c}-\mathbf{e}_j$ is in $\mathbb{N}^r$. □

Theorem 5.2.2. *Let* $I = \langle f_1, \ldots, f_c \rangle \subset B$ *be a saturated complete intersection of codimension* c *in* S, *meaning that the* f_i *form a regular sequence of elements from* B *and* $H^0_B(S/I) = 0$. *Then*

$$\operatorname{reg}(S/I) = Q_c\left(\sum_{i=1}^{c} \deg f_i\right).$$

Proof. Write $\mathbf{a} = \sum_{i=1}^{c} \deg f_i$. By Theorem 4.2.6 it suffices to show that $(S/I)_{\geq \mathbf{d}}$ has a quasilinear resolution generated in degree $\mathbf{d}$ if and only if $\mathbf{d} \in Q_c(\mathbf{a})$. We will prove one direction by showing that $Q_c(\mathbf{a})$ is the bound from Corollary 5.1.4, i.e., that

$$\bigcap_{j \in \mathbb{N}} \bigcap_{\mathbf{b} \in \beta_j(S/I)} Q_j(\mathbf{b}) = Q_c(\mathbf{a})$$

By hypothesis the minimal free resolution $F_\bullet$ of S/I is a Koszul complex, so the elements of $\beta_j(S/I)$ are sums of j choices of $\deg f_i$. In particular $\beta_0(S/I) = \{\mathbf{0}\}$ and $\beta_c(S/I) = \{\mathbf{a}\}$. We have $Q_c(\mathbf{a}) \subset \mathbb{N}^r = Q_0(\mathbf{0})$, so it suffices to show that

$$Q_{j+1}(\deg f_{i_1} + \cdots + \deg f_{i_j} + \deg f_{i_{j+1}}) \subseteq Q_j(\deg f_{i_1} + \cdots + \deg f_{i_j})$$

for all $0 < j < c$ and all $1 \leq i_1 < \cdots < i_{j+1} \leq c$, since each of the other sets in the intersection can be obtained from $Q_c(\mathbf{a})$ in this way. Note that since $I \subset B$, all coordinates of each $\deg f_i$ are positive; therefore the inclusion follows from Lemma 5.2.1.

Now we need that $(S/I)_{\geq \mathbf{d}}$ does not have a quasilinear resolution if $\mathbf{d} \notin Q_c(\mathbf{a})$. Specifically, we will show that the resolution of $(S/I)_{\geq \mathbf{d}}$ has a c-th syzygy in degree $\mathbf{a}' = \max\{\mathbf{d}, \mathbf{a}\}$. If $\mathbf{d} \notin Q_c(\mathbf{a})$ then $\mathbf{d} \notin Q_c(\mathbf{a}')$ and thus $-\mathbf{a}' \notin Q_c(-\mathbf{d})$, so this will complete our argument.

The proof of Lemma 5.1.1 constructs a possibly nonminimal free resolution (5.1.1) of $(S/I)_{\geq \mathbf{d}}$ from resolutions of truncations of the F_j. Since $(F_c)_{\geq \mathbf{d}}$ has a generator of degree $\mathbf{a}'$, the minimal resolution of $(S/I)_{\geq \mathbf{d}}$ will contain a c-th syzygy of degree $\mathbf{a}'$ unless there is a nonminimal map from the generators G^c_0 of $(F_c)_{\geq \mathbf{d}}$ to $G^{c-1}_0 \oplus \cdots \oplus G^0_{c-1}$. Suppose for contradiction that this is true.

The degrees of the summands in G^{c-1-i}_i have the form $\max\{\mathbf{d}, \mathbf{b}\} + \mathbf{v}$ where $\mathbf{b}$ is the sum of the degrees of $c-1-i$ choices of the generators f_j and some $\mathbf{v} \in \mathbb{N}^r$ with $|\mathbf{v}| = i$. In order to have a degree $\mathbf{0}$ map we need $\max\{\mathbf{d}, \mathbf{b}\} + \mathbf{v} = \mathbf{a}' = \max\{\mathbf{d}, \mathbf{a}\}$ for some $\mathbf{b}$ and $\mathbf{v}$. Since all coordinates of each $\deg f_j$ are positive $b_j + i + 1 \leq a_j$ for each j, so $b_j + v_j \neq a_j$. Thus $\mathbf{d} \geq \mathbf{b}$, so $\mathbf{d} + \mathbf{v} = \mathbf{a}'$, contradicting the fact that $\mathbf{d} \notin Q_c(\mathbf{a}')$. □

Note the assumption that $H^0_B(S/I) = 0$ is automatically satisfied if $\operatorname{codim}(P) \neq \operatorname{codim}(I)$ for all minimal primes P over B. However, based on a number of examples it seems that a weaker saturation hypothesis may be sufficient.

Example 5.2.3. Write $S = \Bbbk[x_0, x_1, x_2, y_0, y_1, y_2]$ and consider the saturated complete intersection $I = (x_0y_0, x_1y_1^2)$ that defines a surface in $\mathbb{P}^2 \times \mathbb{P}^2$. Then Theorem 5.2.2 implies

$$\operatorname{reg}(S/I) = Q_2\big((2,3)\big) = \big((0,2) + \mathbb{N}^2\big) \cup \big((1,1) + \mathbb{N}^2\big).$$

5.3 Generalizing Eisenbud–Goto

Recall Eisenbud–Goto's conditions (2) through (4) from the start of Chapter 4. As we have seen, these conditions diverge substantially for products of projective spaces. However, they can each be generalized to give interesting, albeit different, regions inside $\operatorname{Pic}\mathbb{P}^{\mathbf{n}}$.

If M is a finitely generated $\mathbb{Z}^r$-graded S-module, then (2) defines the multigraded regularity region $\operatorname{reg}(M) \subset \operatorname{Pic}\mathbb{P}^{\mathbf{n}}$ of Maclagan and Smith. On the other hand condition (3) naturally generalizes to two truncation regions. First, the obvious generalization gives the linear truncation region:

$$\operatorname{trunc}^L(M) \coloneqq \{\mathbf{d} \in \mathbb{Z}^r \mid M|_{\geq \mathbf{d}} \text{ has a linear resolution generated in degree } \mathbf{d}\}.$$

Second, our characterization of regularity gives the quasilinear truncation region:

$$\operatorname{trunc}^Q(M) \coloneqq \{\mathbf{d} \in \mathbb{Z}^r \mid M|_{\geq \mathbf{d}} \text{ has a quasilinear resolution generated in degree } \mathbf{d}\}.$$

Finally, condition (4) on the Betti numbers of M also naturally generalizes to two Betti regions; the L-Betti region as in Theorem 5.1.2 and the Q-Betti region as in Theorem 5.1.3:

$$\operatorname{betti}^L(M) \coloneqq \bigcap_{i\in\mathbb{N}} \bigcap_{\mathbf{d}\in\beta_i(M)} L_i(\mathbf{d}), \qquad \operatorname{betti}^Q(M) \coloneqq \bigcap_{i\in\mathbb{N}} \bigcap_{\mathbf{d}\in\beta_i(M)} Q_i(\mathbf{d}).$$

Theorem 4.2.6 now states that $\operatorname{reg}(M) = \operatorname{trunc}^Q(M)$ when $H^0_B(M) = 0$. Moreover, since all linear resolutions are quasilinear we get $\operatorname{trunc}^L(M) \subseteq \operatorname{trunc}^Q(M)$. Similarly, since $L_i(\mathbf{d}) \subseteq Q_i(\mathbf{d})$, by definition $\operatorname{betti}^L(M) \subseteq \operatorname{betti}^Q(M)$.

Theorem 5.1.2 shows that the L-Betti region $\operatorname{betti}^L(M)$ is a subset of the linear truncation region $\operatorname{trunc}^L(M)$. Similarly, Theorem 5.1.3 shows that the Q-Betti region $\operatorname{betti}^Q(M)$ is a subset of the quasilinear truncation region $\operatorname{trunc}^Q(M)$.

$$\begin{array}{ccccc}
\operatorname{betti}^L(M) & \xhookrightarrow{5.1.2} & \operatorname{trunc}^L(M) & & \\
\big\downarrow & & \big\downarrow & & \\
\operatorname{betti}^Q(M) & \xhookrightarrow{5.1.3} & \operatorname{trunc}^Q(M) & \overset{4.2.6}{=\!=\!=} & \operatorname{reg}(M)
\end{array}$$

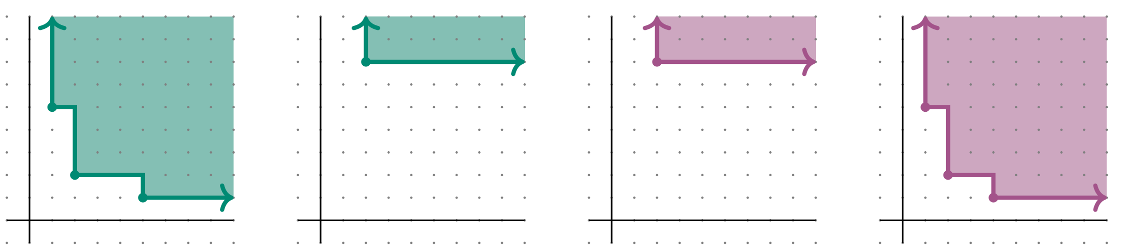
We saw in Section 4.3 that we can switch the roles of Q and L in the proof of Theorem 4.2.6 to complete the upper right corner of this diagram. The resulting cohomological characterization of $\operatorname{trunc}^L(M)$ in Theorem 4.3.2 is related to the positivity conditions described in Remark 4.1.5. We suspect that the reversal of Q and L between the Betti number and cohomological conditions has a deeper explanation in terms of the BGG correspondence.

We illustrate the four regions above in the following example.

Example 5.3.1. Let I be the B-saturated ideal in Example 4.2.7, defining a smooth hyperelliptic curve of genus 4 embedded into $\mathbb{P}^1 \times \mathbb{P}^2$ as a curve of degree $(2, 8)$. As noted in [5, Ex. 1.4], using *Macaulay2* one finds that the minimal graded free resolution of I is:

$$S \longleftarrow \begin{array}{c} S(-3,-1) \\ \oplus \\ S(-2,-2) \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-1,-5)^3 \\ \oplus \\ S(0,-8) \end{array} \longleftarrow \begin{array}{c} S(-3,-3)^3 \\ \oplus \\ S(-2,-5)^6 \\ \oplus \\ S(-1,-7) \\ \oplus \\ S(-1,-8)^2 \end{array} \longleftarrow \begin{array}{c} S(-3,-5)^3 \\ \oplus \\ S(-2,-7)^2 \\ \oplus \\ S(-2,-8) \end{array} \longleftarrow S(-3,-7) \longleftarrow 0.$$

From this we can calculate $\operatorname{betti}^L(S/I)$ and $\operatorname{betti}^Q(S/I)$. Figure 5.1 also depicts $\operatorname{trunc}^L(S/I)$ and $\operatorname{trunc}^Q(S/I)$, which equals $\operatorname{reg}(S/I)$ as I is saturated, inside the box $[0, 9]^2$.



$\operatorname{trunc}^L(S/I)$ $\quad$ $\operatorname{betti}^L(S/I)$ $\quad$ $\operatorname{betti}^Q(S/I)$ $\quad$ $\operatorname{trunc}^Q(S/I) = \operatorname{reg}(S/I)$

Figure 5.1: The four regions for Example 4.2.7 inside $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^2)$.

Chapter 6

Computation

Most of this chapter comes from [17].

We introduce the *LinearTruncations* package for [26], which provides tools for studying the resolutions of truncations of modules over rings with standard multigradings. Given a module and a bounded range of multidegrees, our package can identify all linear truncations in the range. The algorithm uses a search function that is also applicable to other properties of modules described by sets of degrees. The examples here were computed using version 1.18 of *Macaulay2* and version 1.0 of *LinearTruncations*.

In section 6.1 we describe the main algorithms, `findRegion` and `linearTruncations`. In Section 6.2, we introduce `regularityBound` and `linearTruncationsBound` as faster methods for calculating subsets of the multigraded regularity and linear truncation regions, respectively.

The function `multigradedPolynomialRing` produces standard multigraded rings:

```
i1 : needsPackage "LinearTruncations"
o1 = LinearTruncations
o1 : Package
i2 : S = multigradedPolynomialRing {1,2}
o2 = S
o2 : PolynomialRing
i3 : degrees S
o3 = {{1, 0}, {1, 0}, {0, 1}, {0, 1}, {0, 1}}
o3 : List
```

The function `isLinearComplex` checks whether a multigraded complex is linear according to Definition 4.2.1. To print the degrees appearing in the complex use `supportOfTor`.

```
i4 : B = irrelevantIdeal S
o4 = ideal (x    x   , x    x   , x    x   , x    x   , x    x   , x    x   )
             0,1 1,2   0,0 1,2   0,1 1,1   0,0 1,1   0,1 1,0   0,0 1,0
o4 : Ideal of S
i5 : F = res comodule B
```

```
      1      6      9      5      1
o5 = S  <-- S  <-- S  <-- S  <-- S  <-- 0

     0      1      2      3      4      5
o5 : ChainComplex
i6 : netList supportOfTor F

     +------+------+
o6 = |{0, 0}|      |
     +------+------+
     |{1, 1}|      |
     +------+------+
     |{2, 1}|{1, 2}|
     +------+------+
     |{2, 2}|{1, 3}|
     +------+------+
     |{2, 3}|      |
     +------+------+
i7 : isLinearComplex F
o7 = false
```

6.1 Finding linear truncations

Eisenbud, Erman, and Schreyer proved in [21] that the linear truncation region of M is non-empty. In particular it contains the output of the function `coarseMultigradedRegularity` from their package *TateOnProducts* `TateOnProducts`. However, in general this degree is neither a minimal element itself nor greater than all the minimal elements. (See Example 6.1.1.)

The function `linearTruncations` searches for multidegrees where the truncation of M has a linear resolution by calling the function `findRegion`, which implements Algorithm 1. Since we do not know of a bound on the total degree of the minimal elements in the linear truncation region given the Betti numbers of M, `linearTruncations` is not guaranteed to produce all generators as a module over the semigroup $\mathbb{N}^r$. By default it searches above the componentwise minimum of the degrees of the generators of M and below the degree with all coordinates equal to $c + 1$, where c is the output of `regularity`. Otherwise the range is taken as a separate input.

Example 6.1.1. Let $S = k[x_{0,0}, x_{0,1}, x_{0,2}, x_{1,0}, x_{1,1}, x_{1,2}, x_{1,3}]$ be the Cox ring of $\mathbb{P}^2 \times \mathbb{P}^3$. For each $d \geq 2$, let $\phi_d \colon S(-d,-d)^6 \to S(0,-d)^2 \oplus S(-d,0)^4$ be given by

$$\begin{pmatrix} x_{0,0}^d & x_{0,1}^d & x_{0,2}^d & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{0,1}^d & x_{0,0}^d & x_{0,2}^d \\ x_{1,0}^d & 0 & 0 & x_{1,0}^d & 0 & 0 \\ 0 & x_{1,1}^d & 0 & 0 & x_{1,1}^d & 0 \\ 0 & 0 & x_{1,2}^d & 0 & 0 & x_{1,2}^d \\ 0 & 0 & 0 & x_{1,3}^d & 0 & 0 \end{pmatrix},$$

and define $M^{(d)} := \operatorname{coker} \phi_d$. The `coarseMultigradedRegularity` of $M^{(3)}$ is $\{3,3\}$, the `regularity` of $M^{(3)}$ is 5, and $\{3,3\}$ and $\{8,2\}$ are minimal elements of the linear truncation region. Since $\{8,2\}$ is not below $\{5+1, 5+1\}$ it will not be returned by the `linearTruncations` function with the default options:

```
i8 : (S,E) = productOfProjectiveSpaces{2,3};
i9 : d = 3;
i10 : M = coker(map(S^{{0,-d},{0,-d},{-d,0},{-d,0},{-d,0},{-d,0}},
        S^{{-d,-d},{-d,-d},{-d,-d},{-d,-d},{-d,-d},{-d,-d}},
        {{x_(0,0)^d,x_(0,1)^d,x_(0,2)^d,0,0,0},
        {0,0,0,x_(0,1)^d,x_(0,0)^d,x_(0,2)^d},
        {-x_(1,0)^d,0,0,-x_(1,0)^d,0,0},
        {0,-x_(1,1)^d,0,0,-x_(1,1)^d,0},
        {0,0, -x_(1,2)^d,0,0, -x_(1,2)^d},
        {0,0,0, -x_(1,3)^d,0,0}}));
i11 : linearTruncations M
o11 = {{3, 3}}
o11 : List
i12 : linearTruncations({{0,0},{8,6}},M)
o12 = {{3, 3}, {8, 2}}
o12 : List
```

Based on the computations from $M^{(d)}$ for $2 \leq d \leq 10$ we expect that for $d \geq 2$ the module $M^{(d)}$ will have `coarseMultigradedRegularity` equal to $\{d,d\}$, with $\{d,d\}$ and $\{3d-1, d-1\}$ both minimal elements of the linear truncation region.

At each step of Algorithm 1 the set A contains degrees satisfying f and the set K contains the minimal degrees remaining to be checked. There are options to initialize A and K differently—degrees in A will be assumed to satisfy f, and degrees below those in K will be excluded from the search (and thus assumed not to satisfy f). Supplying such prior knowledge can decrease the length of the computation by limiting the number of times the algorithm calls f.

The pseudocode in Algorithm 1 masks the fact that A and K are stored as monomial ideals in a temporary singly graded polynomial ring. Similarly, the function `findMins` will

Input : a module M, a Boolean function f that takes M as input, and a range $(\mathbf{a}, \mathbf{b})$
Output : minimal elements between $\mathbf{a}$ and $\mathbf{b}$ where M satisfies f

```
A := ∅;
K := {a};
while K ≠ ∅ do
    d := first element of K;
    K = K \ {d};
    if d ∉ A + ℕ^r then
        if M satisfies f at d then
            A = A ∪ {d};
        else
            for 1 ≤ i ≤ r do
                if d + e_i ≤ b then
                    K = K ∪ {d + e_i};
                end
            end
        end
    end
end
return minimal elements of A
```

Algorithm 1: `findRegion`

convert a list of multidegrees to a monomial ideal in order to calculate its minimal elements via a Gröbner basis.

6.2 Estimating regularity

As discussed above, the multigraded Betti numbers of M do not determine either its regularity or its linear truncations. The functions `regularityBound` and `linearTruncationsBound` compute subsets of these regions using only the twists appearing in the minimal free resolution of M, based on Theorems 5.1.3 and 5.1.2. In many examples they produce the same outputs as `multigradedRegularity` (from the package *VirtualResolutions* [1]) and `linearTruncations`, respectively, without computing sheaf cohomology or truncating the module.

The function `partialRegularities` calculates the Castelnuovo–Mumford regularity in each component of a multigrading.

Remark 6.2.1. In the bigraded case, Theorem 5.1.2 implies that $\mathbf{d}$ is in `linearTruncations M` if $\mathbf{d} \geq$ `partialRegularities M` and $|\mathbf{d}| \geq$ `regularity M`.

In some cases `linearTruncationsBound` gives a proper subset of the linear truncations:

```
i13 : S = multigradedPolynomialRing 2;
i14 : M = coker(map(S^{{-1,0},{0,-1},{0,-1}},S^{{-1,-1},{-1,-1}},
        {{x_(1,0),x_(1,1)},{-x_(0,0),0},{0,-x_(0,1)}}));
i15 : multigraded betti res M
            0   1
o15 = 1: a+2b   .
      2:    . 2ab
i16 : linearTruncations M
o16 = {{0, 2}, {1, 1}}
i17 : linearTruncationsBound M
o17 = {{1, 1}}
```

An affirmative answer to the following open question would reduce the process of finding all minimal elements to a finite computation.

Question 6.2.2. Can the minimal elements of the regularity of M be bounded in terms of S and the Betti numbers of M?

Bibliography

[1] Ayah Almousa et al. "The Virtual Resolutions Package for Macaulay2". In: *Journal of Software for Algebra and Geometry* 10.1 (2020), pp. 50–60.

[2] Vincenzo Ancona and Giorgio Ottaviani. "An introduction to the derived categories and the theorem of Beilinson". In: *Atti Accad. Peloritana Pericolanti* 67 (Jan. 1989), pp. 99–110.

[3] Amir Bagheri, Marc Chardin, and Huy Tài Hà. "The eventual shape of Betti tables of powers of ideals". In: *Math. Res. Lett.* 20.6 (2013), pp. 1033–1046. ISSN: 1073-2780. DOI: 10.4310/MRL.2013.v20.n6.a3.

[4] David Bayer and Michael Stillman. "On the complexity of computing syzygies". In: vol. 6. 2-3. Computational aspects of commutative algebra. 1988, pp. 135–147. DOI: 10.1016/S0747-7171(88)80039-7.

[5] Christine Berkesch, Daniel Erman, and Gregory G. Smith. "Virtual resolutions for a product of projective spaces". In: *Algebraic Geometry* 7.4 (July 2020), pp. 460–481.

[6] Christine Berkesch et al. "Homological and combinatorial aspects of virtually Cohen–Macaulay sheaves". In: (2021). arXiv: 2012.14047 [math.AC].

[7] Nicolás Botbol and Marc Chardin. "Castelnuovo–Mumford regularity with respect to multigraded ideals". In: *Journal of Algebra* 474 (Mar. 2017), pp. 361–392.

[8] Michael K. Brown and Daniel Erman. "Tate resolutions on toric varieties". In: (2021). arXiv: 2108.03345 [math.AG].

[9] Juliette Bruce, Lauren Cranton Heller, and Mahrud Sayrafi. *Characterizing Multigraded Regularity on Products of Projective Spaces*. 2022. arXiv: 2110.10705 [math.AC].

[10] Juliette Bruce, Lauren Cranton Heller, and Mahrud Sayrafi. *Bounds on Multigraded Regularity*. 2022. arXiv: 2208.11115 [math.AC].

[11] Karen A. Chandler. "Regularity of the powers of an ideal". In: *Comm. Algebra* 25.12 (1997), pp. 3773–3776. ISSN: 0092-7872. DOI: 10.1080/00927879708826084.

[12] Marc Chardin. "Powers of ideals: Betti numbers, cohomology and regularity". In: *Commutative algebra*. Springer, New York, 2013, pp. 317–333. DOI: 10.1007/978-1-4614-5292-8_9.

[13] L. Costa and R. M. Miró-Roig. "m-blocks collections and Castelnuovo–Mumford regularity in multiprojective spaces". In: *Nagoya Math. J.* 186 (2007), pp. 119–155.

[14] David A. Cox. "The homogeneous coordinate ring of a toric variety". In: *J. Algebraic Geom.* 4.1 (1995), pp. 17–50.

[15] David A. Cox, John Little, and Donal O'Shea. *Ideals, varieties, and algorithms*. Fourth. Undergraduate Texts in Mathematics. An introduction to computational algebraic geometry and commutative algebra. Springer, Cham, 2015, pp. xvi+646. DOI: `10.1007/978-3-319-16721-3`.

[16] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.

[17] Lauren Cranton Heller and Navid Nemati. "Linear truncations package for Macaulay2". In: *J. Softw. Algebra Geom.* 12.1 (2022), pp. 11–16. DOI: `10.2140/jsag.2022.12.11`.

[18] S. Dale Cutkosky, Jürgen Herzog, and Ngô Viêt Trung. "Asymptotic behaviour of the Castelnuovo-Mumford regularity". In: *Compositio Math.* 118.3 (1999), pp. 243–261. ISSN: 0010-437X. DOI: `10.1023/A:1001559912258`.

[19] David Eisenbud. *Commutative Algebra with a View toward Algebraic Geometry*. Vol. 150. Graduate Texts in Mathematics. Springer, 1995.

[20] David Eisenbud. *The Geometry of Syzygies*. Vol. 229. Graduate Texts in Mathematics. Springer, 2005.

[21] David Eisenbud, Daniel Erman, and Frank-Olaf Schreyer. "Tate resolutions for products of projective spaces". In: *Acta Math. Vietnam.* 40.1 (2015), pp. 5–36.

[22] David Eisenbud, Gunnar Fløystad, and Frank-Olaf Schreyer. "Sheaf cohomology and free resolutions over exterior algebras". In: *Trans. Amer. Math. Soc.* 355.11 (July 2003), pp. 4397–4426.

[23] David Eisenbud and Shiro Goto. "Linear free resolutions and minimal multiplicity". In: *Journal of Algebra* 88.1 (May 1984), pp. 89–133.

[24] Jiyang Gao et al. "Virtual complete intersections in $\mathbb{P}^1 \times \mathbb{P}^1$". In: *J. Pure Appl. Algebra* 225.1 (2021), Paper No. 106473, 15. ISSN: 0022-4049. DOI: `10.1016/j.jpaa.2020.106473`.

[25] Anthony V. Geramita, Alessandro Gimigliano, and Yves Pitteloud. "Graded Betti numbers of some embedded rational n-folds". In: *Math. Ann.* 301.2 (1995), pp. 363–380. ISSN: 0025-5831. DOI: `10.1007/BF01446634`.

[26] Daniel R. Grayson and Michael E. Stillman. *Macaulay2, a software system for research in algebraic geometry*. Available at `https://faculty.math.illinois.edu/Macaulay2/`.

[27] Eloísa Grifo and Alexandra Seceleanu. "Symbolic Rees algebras". In: *Commutative algebra*. Springer, Cham, 2021, pp. 343–371. DOI: `10.1007/978-3-030-89694-2\_11`.

[28] Alexander Grothendieck. "Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert". In: *Séminaire Bourbaki, Vol. 6.* Soc. Math. France, Paris, 1995, Exp. No. 221, 249–276.

[29] Robin Hartshorne. *Algebraic Geometry.* Vol. 52. Graduate Texts in Mathematics. Springer, 1977.

[30] Robin Hartshorne. *Residues and Duality.* Vol. 20. Lecture Notes in Mathematics. Springer, 1966.

[31] M. Hering, A. Küronya, and S. Payne. "Asymptotic cohomological functions of toric divisors". In: *Adv. Math.* 207.2 (2006), pp. 634–645. ISSN: 0001-8708. DOI: 10.1016/j.aim.2005.12.007.

[32] J. William Hoffman and Hao Hao Wang. "Castelnuovo–Mumford regularity in biprojective spaces". In: *Adv. Geom.* 4.4 (2004), pp. 513–536.

[33] Daniel Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry.* Oxford Mathematical Monographs. Clarendon Press, 2006.

[34] Srikanth B. Iyengar et al. *Twenty-four hours of local cohomology.* Vol. 87. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2007.

[35] Vijay Kodiyalam. "Asymptotic behaviour of Castelnuovo-Mumford regularity". In: *Proc. Amer. Math. Soc.* 128.2 (2000), pp. 407–411. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-99-05020-0.

[36] Diane Maclagan and Gregory G. Smith. "Multigraded Castelnuovo–Mumford regularity". In: *J. Reine Angew. Math.* 2004.571 (Jan. 2004), pp. 179–212.

[37] David Mumford. *Lectures on Curves on an Algebraic Surface.* Vol. 59. Ann. of Math. Studies. Princeton University Press, 1966.

[38] Christian Okonek, Michael Schneider, and Heinz Spindler. *Vector bundles on complex projective spaces.* Vol. 3. Progress in Mathematics. Birkhäuser, 1980.

[39] Tim Römer. "Homological properties of bigraded algebras". In: *Illinois J. Math.* 45.4 (2001), pp. 1361–1376. ISSN: 0019-2082.

[40] Will Sawin. *Generalization of Dickson's Lemma.* MathOverflow. eprint: https://mathoverflow.net/q/383015.

[41] Karen E. Smith and Irena Swanson. "Linear bounds on growth of associated primes". In: *Comm. Algebra* 25.10 (1997), pp. 3071–3079. ISSN: 0092-7872. DOI: 10.1080/00927879708826041.

[42] Irena Swanson. "Powers of ideals. Primary decompositions, Artin-Rees lemma and regularity". In: *Math. Ann.* 307.2 (1997), pp. 299–313. ISSN: 0025-5831. DOI: 10.1007/s002080050035.

[43] Ngô Viêt Trung and Hsin-Ju Wang. "On the asymptotic linearity of Castelnuovo-Mumford regularity". In: *J. Pure Appl. Algebra* 201.1-3 (2005), pp. 42–48. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2004.12.043.