A simple model that generates stylized facts of returns

Gawon Yoon*

Pusan National University

Abstract

This note shows that a very simple model can generate returns that resemble most of the temporal and distributional behavior of long returns surprisingly well. The model is based on the stochastic unit root process introduced in Granger and Swanson (1997).

Keywords: long returns, long memory, stochastic unit root, GARCH

JEL classifications: C22

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* Address correspondence to Gawon Yoon, Department of Economics, Pusan National University, Pusan, S. Korea, 609-735, gyoon@pusan.ac.kr, 82-51-5102581, fax: 82-51-5813187. This research was supported by the Pusan National University Research Grant. I thank Clive W.J. Granger for his comments, encouragements, and patience with me over the years.

1. Introduction

Long memory properties of returns in stock prices and foreign exchange are well documented. However, developing models that describe the phenomena turns out not to be easy. For instance, the popular models of ARCH and GARCH do not produce large enough memory in returns. Much effort has been made to explain the long memory properties in returns. See, for instance, long memory ARCH models in Ding and Granger (1996), FIGARCH model in Baillie *et al.* (1996) and Bollerslev and Mikkelsen (1996), HARCH model in Müller *et al.* (1997), and long memory stochastic volatility model in Breidt *et al.* (1998). This note presents a very simple time series model that seems to explain most of the stylized facts about the temporal and distributional properties of returns. The model is a simple generalization of a random walk process, the standard model for speculative prices since Bachelier (1900).

2. Some stylized facts of returns

Some well-known facts on long returns are reviewed in this section. The following classifications are available in Rydén *et al*. (1998), which are based mostly on the observations in Ding *et al.* (1993) and Granger and Ding (1995). First, on the temporal properties of returns:

TP1: Returns, r_t, are not autocorrelated (except possibly at lag one).

TP2: The autocorrelation functions of $|r_t|$ and r_t^2 decay slowly and $corr\left(|r_t|, \quad |r_{t-k}|\right) > corr\left(r_t^2, \quad r_{t-k}^2\right)$.

The decay is much slower than the exponential rate of a stationary AR or ARMA model. The autocorrelations remain positive for very long lags.

TP3: Autocorrelations of powers of an absolute return are highest at power one: $corr\left(|r_t|, \quad |r_{t-k}|\right) > corr\left(|r_t|^d, \quad |r_{t-k}|^d\right), \; d \neq 1$. Granger and Ding (1995) call this property the Taylor effect, following Taylor (1986).

TP4: The observed autocorrelations of $\operatorname{sign}(r_t)$ are insignificant.[1]

Rydén *et al*. (1998) also consider the following distributional properties associated with (absolute) returns:

DP1: $|r_t|$ and $\operatorname{sign}(r_t)$ are (contemporaneously) independent.

DP2: $|r_t|$ has the same mean and standard deviation.

DP3: The marginal distribution for $|r_t|$ is exponential.

A random variable with an exponential distribution has the same mean and standard deviation. Its skewness and kurtosis are 2 and 9. Granger *et al.* (2000) show that these properties hold for many economic series from various speculative markets. Ding and Granger (1996) show that an integrated GARCH(1, 1) process cannot explain the temporal properties and propose a new class of models. Rydén *et al*. (1998) use a mixture of

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[1] $\operatorname{sign}(r_t) \equiv \begin{cases} 1 & \text{if} \quad r_t > 0 \\ 0 & \text{if} \quad r_t = 0 \\ -1 & \text{if} \quad r_t < 0 \end{cases}$.

normal variables along with a hidden Markov model. See Granger *et al.* (2000) for other approaches; however, all of the models are not based on a firm theoretical foundation. In this note, a very simple model is presented that generates returns that resemble most of the temporal and distributional properties of the stylized facts surprisingly well.

3. A stochastic unit root process

The following simple model is used for y_t, which might be regarded as stock prices in logarithms:

$$y_t = (1 + a_t)\, y_{t-1} + \varepsilon_t \tag{1}$$

for $t = 1, ..., T$, where $a_t \sim i.i.N\left(0,\ \sigma_a^2\right)$ and $\varepsilon_t \sim i.i.N\left(0,\ \sigma_\varepsilon^2\right)$. a_t and ε_t are independent. Normality assumption is only for convenience. y_0 is a constant. Δy_t is the continuously compounded rate of return or simply return at time *t*. Given that $E(a_t) = 0$, y_t has a unit root only on average and is called a stochastic unit root process. It is introduced in Granger and Swanson (1997) and Leybourne, McCabe, and Tremayne (1996).[2] It is also called a random coefficient autoregressive or doubly stochastic process.[3]

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[2] They use somewhat different functional forms from (1). Readers are referred to their papers for details.

[3] Various applications of this important and flexible class of models include Leybourne, McCabe, and Mills (1996), Gonzalo and Lee (1998), Bleaney *et al.* (1999), Sollis *et al.* (2000), Gonzalo and Montesinos (2000), and Taylor and van Dijk (2002), among others.

Market efficiency dictates that the price should be $I(1)$ and y_t is $I(1)$ on average. The process is not covariance stationary as the variance of y_t diverges to infinity as t increases. The process cannot be transformed into stationarity by taking differences.[4] This observation is in contrast to the current practice that assumes stationarity of the first differenced speculative prices. Additionally,

$$Var(\Delta y_{t+h} \mid \mathrm{I}_t) = \left(1+\sigma_a^2\right)^{h-1}\left(y_t^2\sigma_a^2 + \sigma_\varepsilon^2\right),$$

for $h = 1, 2, \ldots$, where I_t denotes an information set available at time *t*. The model is similar to the nonstationary nonlinear heteroskedasticity discussed in Park (2002) in that conditional heteroskedasticity depends on (stochastically) integrated variables.

4. Simulation results

The main results of this note are now presented. Two sets of observations, $\{\Delta y_t\}_{t=1}^{T}$, are generated with $T = 40,000$ and $2,000$, respectively. For $T = 40,000$, $\sigma_a^2 = 0.02^2$ and for $T = 2,000$, $\sigma_a^2 = 0.1^2$. Also, $\sigma_\varepsilon^2 = 1$ with $y_0 = 100$ and the first 100 observations are discarded for both sets of observations. All

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[4] With $y_0 = 0$, it is not difficult to show that $E(y_t) = 0$, $Var(y_t) = \frac{\sigma_\varepsilon^2}{\sigma_a^2}\left\{\left(1+\sigma_a^2\right)^t - 1\right\}$, $\sigma_a^2 \neq 0$, $corr(y_t,\ \ y_{t-k}) = \sqrt{\frac{\left(1+\sigma_a^2\right)^{t-k} - 1}{\left(1+\sigma_a^2\right)^t - 1}} \rightarrow \frac{1}{\sqrt{\left(1+\sigma_a^2\right)^k}}$, as $t \rightarrow \infty$. Further, $E(\Delta y_t) = 0$, $Var(\Delta y_t) = \sigma_\varepsilon^2\left(1+\sigma_a^2\right)^{t-1}$, and $corr(\Delta y_t,\ \ \Delta y_{t-k}) = 0$.

simulations in this note are done with GAUSS. Table 1 lists some descriptive statistics for Δy_t. Clearly, both data sets are far from a normal distribution. Table 2 shows descriptive statistics for $|\Delta y_t|$. Figure 1 shows the generated series in level and first difference for $T = 40,000$.

The data series $\{\Delta y_t\}_{t=1}^{T}$ can be easily confused with (G)ARCH processes. For instance, for $T = 40,000$, GARCH(1, 1) estimation results are:

$$\Delta y_t = \underset{(0.12)}{-0.0008} + \hat{\varepsilon}_t \tag{2}$$

$$h_t = \underset{(4.37)}{0.002} + \underset{(19.7)}{0.020} \times \hat{\varepsilon}_{t-1}^2 + \underset{(940)}{0.979} \times h_{t-1},$$

where $\hat{\varepsilon}_t$ is residual and $\varepsilon_t = z_t \sqrt{h_t}$, with $z_t \sim i.i.N(0,\ \ 1)$. The estimation is done with the BHHH method. Absolute *t*-values are reported in the parenthesis. Additionally, the sum of coefficients is very close to 1, so that an IGARCH(1,1) process could have been entertained. Similarly for $T = 2,000$, which is not shown to save space, the results are

$$\Delta y_t = \underset{(0.31)}{-0.0093} + \hat{\varepsilon}_t \tag{3}$$

$$h_t = \underset{(3.33)}{0.034} + \underset{(7.61)}{0.101} \times \hat{\varepsilon}_{t-1}^2 + \underset{(67.1)}{0.891} \times h_{t-1}.$$

An IGARCH model seems to be a plausible approximation.[5]

Can the simulated series generate the stylized facts of returns discussed in section 2? The following

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[5] The estimation results are not changing much if a constant is not used in (2) and (3) for Δy_t.

results are obtained with $T = 40,000$ observations:

TP1: Figure 2 shows the estimated autocorrelation functions [ACFs] of Δy_t. Approximate 95% confidence bands of $\pm 1.96/\sqrt{T}\,(= 0.0098)$ are also plotted, corresponding to the ACFs of *i.i.d.* Gaussian noise. The autocorrelations are all small in magnitude.

TP2: Figure 3 compares the ACFs of $|\Delta y_t|$ and Δy_t^2. They are decaying slowly and indeed $corr\left(|\Delta y_t|,\ \ |\Delta y_{t-k}|\right) > corr\left(\Delta y_t^2,\ \ \Delta y_{t-k}^2\right)$. The autocorrelations remain outside the approximate 95% confidence bands for very long lags, up to $k = 2500$.

TP3: Figures 4 and 5 shows the ACFs corresponding to different values of *d* = 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2. The relationship $corr\left(|r_t|,\ \ |r_{t-k}|\right) > corr\left(|r_t|^d,\ \ |r_{t-k}|^d\right), d \neq 1$ appears to hold for different values of *d*.

TP4: The ACFs of $\text{sign}\left(\Delta y_t\right)$ are indeed very small. It is not shown here to save space.

Similar results are obtained for $T = 2,000$. Results are not shown to save space. For the distributional properties of returns,

DP1: The estimate correlation between $|\Delta y_t|$ and $\text{sign}\left(\Delta y_t\right)$ is very small; -0.007 for $T = 40,000$ and -0.012 for $T = 2,000$.

DP2: The estimated mean and standard deviation of $|\Delta y_t|$ are 1.411 and 1.501 for $T = 40,000$. They are 1.509 and 1.665 for $T = 2,000$. The ratio of mean and standard deviation is about 0.94 and 0.91, respectively.

DP3: The estimated skewness and kurtosis of $|\Delta y_t|$ are 2.83 and 15.93 for $T = 40,000$. They are 2.93 and 16.02 for $T = 2,000$.

Ding and Granger (1996) note that $d = 0.25$ produces the strongest long-memory property for some series like exchange rates. Figure 6 shows that the process considered in this note can generate this property as well with $T = 2,000$, for $\left\{|\Delta y_t|^d\right\}$, $d = 0.25, 0.5, 0.75, 1$. Same results are found when $d = 1, 1.25, 1.5, 1.75, 2$.

5. Conclusions

This note presents a very simple model that generates returns with similar temporal and distributional properties to those observed in stock prices and foreign exchange. The model is based on the stochastic unit root process of Granger and Swanson (1997) and is a simple generalization of a random walk model. The results should not be interpreted as implying that stock prices are generated by the simple stochastic unit root process considered here. The mere fact that a model has certain properties that are also observed in data does not necessarily imply that the model is a true data generating process. Instead, this note should serve as a warning that one should pay more attention to the time series properties of the data series under investigation before taking necessary transformations or estimating various heteroskedastic models.

Table 1. Descriptive statistics for $\{\Delta y_t\}_{t=1}^{T}$

T	mean	st.dev	minimum	maximum	skewness	kurtosis	JB
40,000	-0.0003	2.060	-17.72	17.43	-0.01	9.20	64085
2,000	-0.0040	2.247	-13.01	17.44	-0.04	9.72	3787

$\{\Delta y_t\}_{t=1}^{T}$ is generated from (1) with $y_0 = 100$. st.dev is standard deviation and JB denotes the Jarque-Bera test for normality.

Table 2. Descriptive statistics for $\{|\Delta y_t|\}_{t=1}^{T}$

T	mean	st.dev	minimum	maximum	skewness	kurtosis
40,000	1.411	1.501	0.00	17.72	2.83	15.93
2,000	1.509	1.665	0.00	17.44	2.93	16.02

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Captions for figures

Figure 1: Simulated series with first difference, Δy_t, with $T = 40,000$

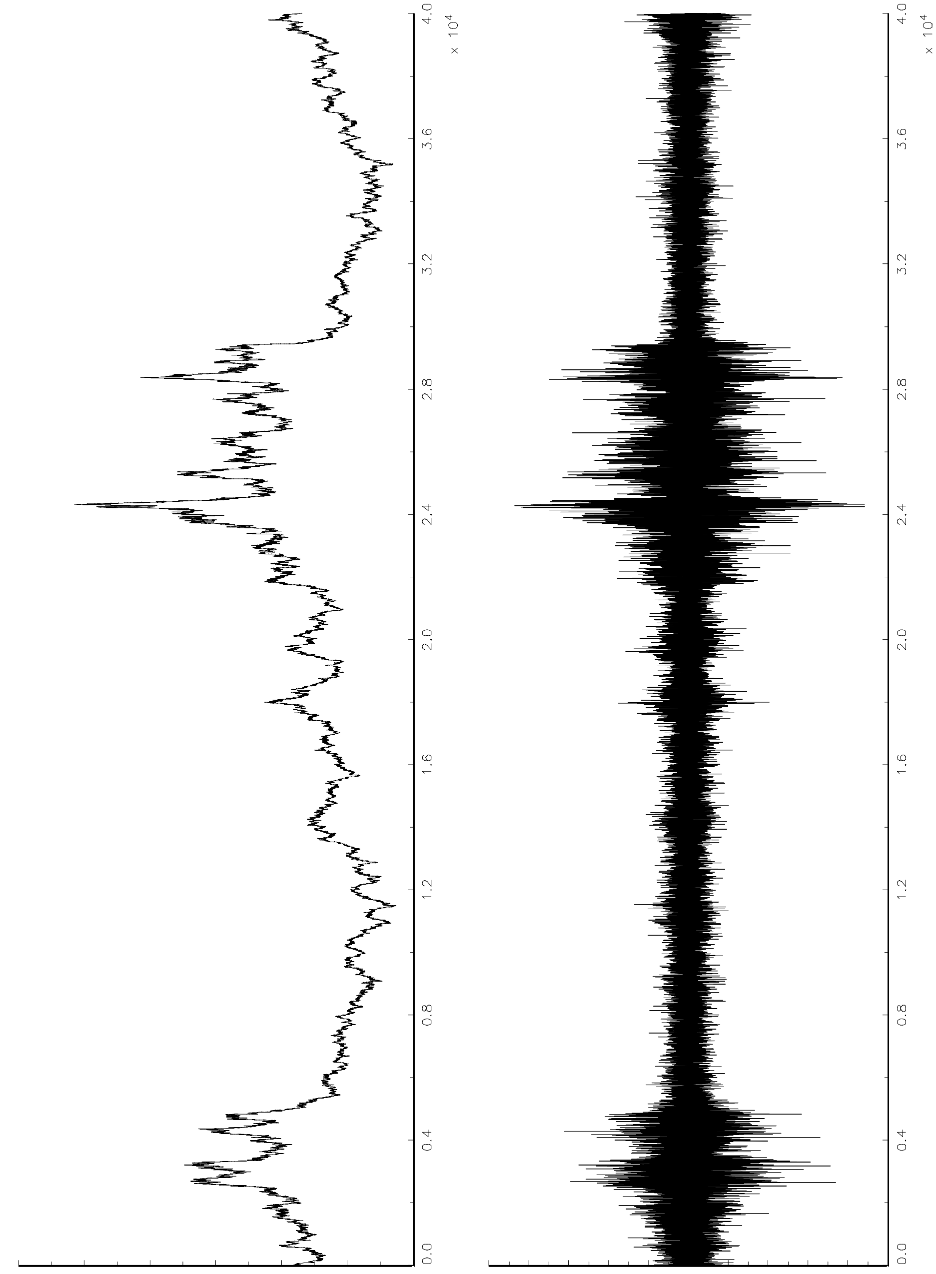
Figure 2: Estimated autocorrelation functions of Δy_t, with $T = 40,000$

Figure 3: Estimated autocorrelation functions of Δy_t and $|\Delta y_t|^d$, $d = 1,2$ with $T = 40,000$

Figure 4: Estimated autocorrelation functions of $|\Delta y_t|^d$, $d = 1, 0.75, 0.5, 0.25$, from high to low, with $T = 40,000$

Figure 5: Estimated autocorrelation functions of $|\Delta y_t|^d$, $d = 1, 1.25, 1.5, 1.75, 2$, form high to low, with $T = 40,000$

Figure 6: Estimated autocorrelation functions of $|\Delta y_t|^d$, $d = 1, 0.75, 0.5, 0.25$, from *low* to *high*, with $T = 2,000$



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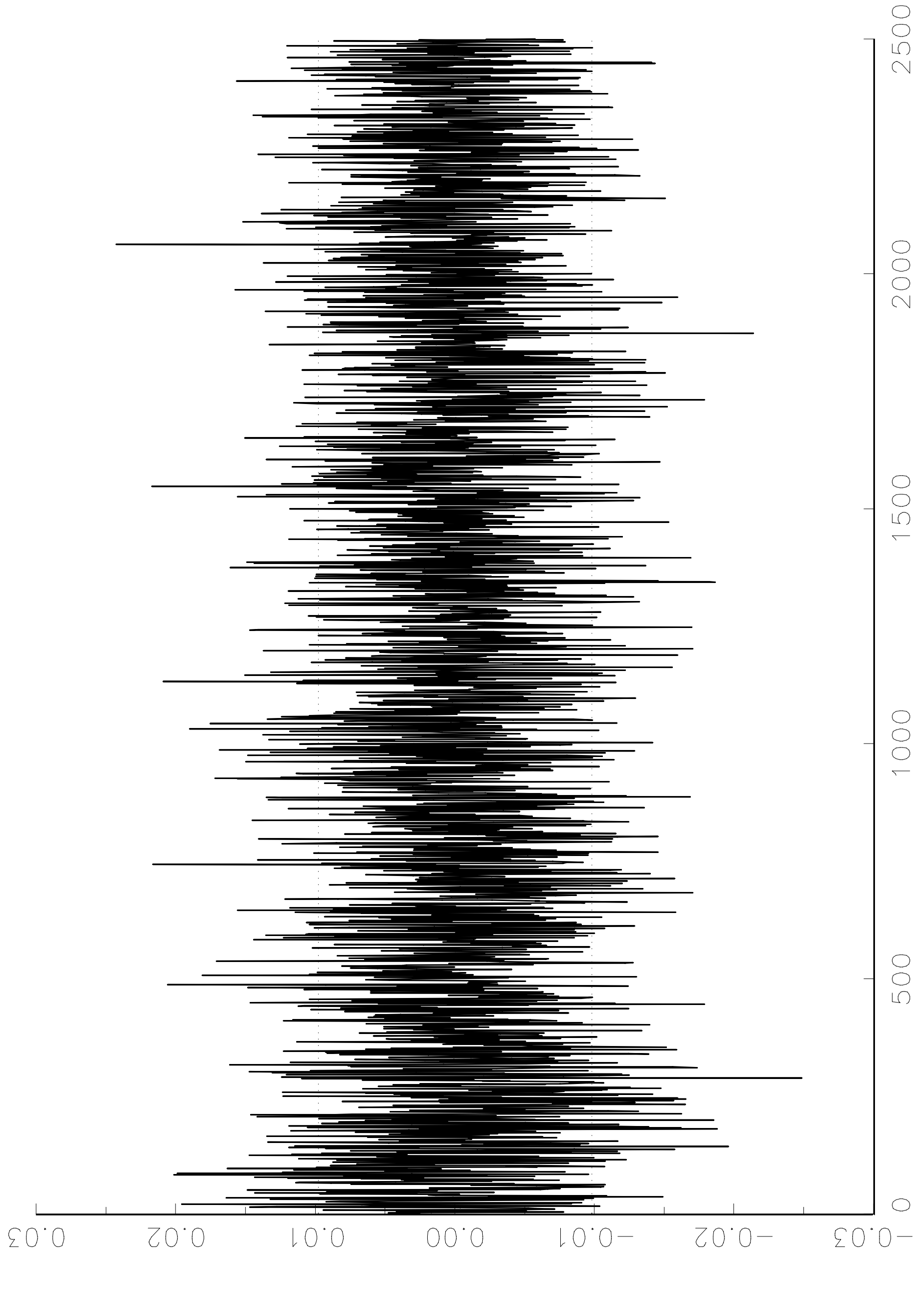
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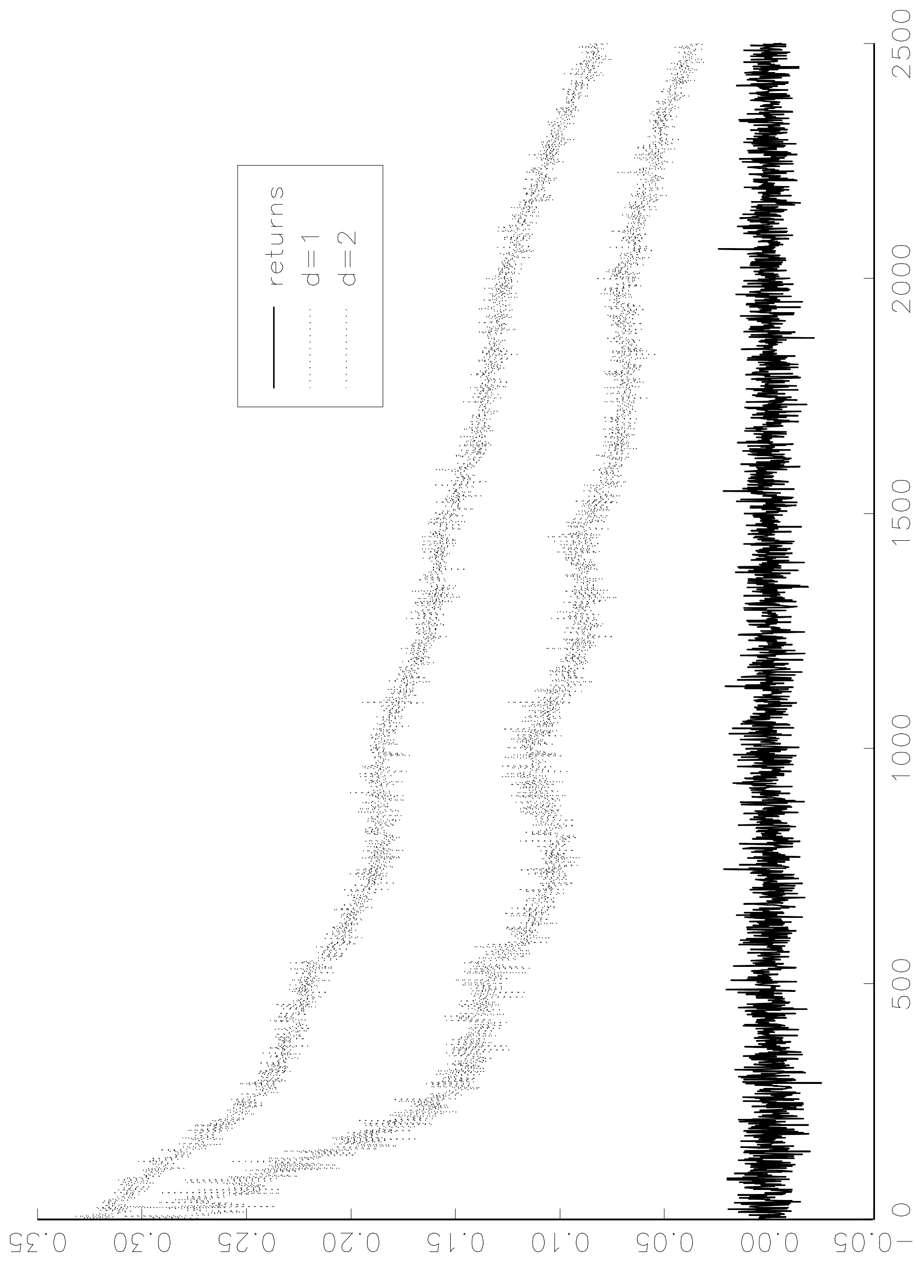
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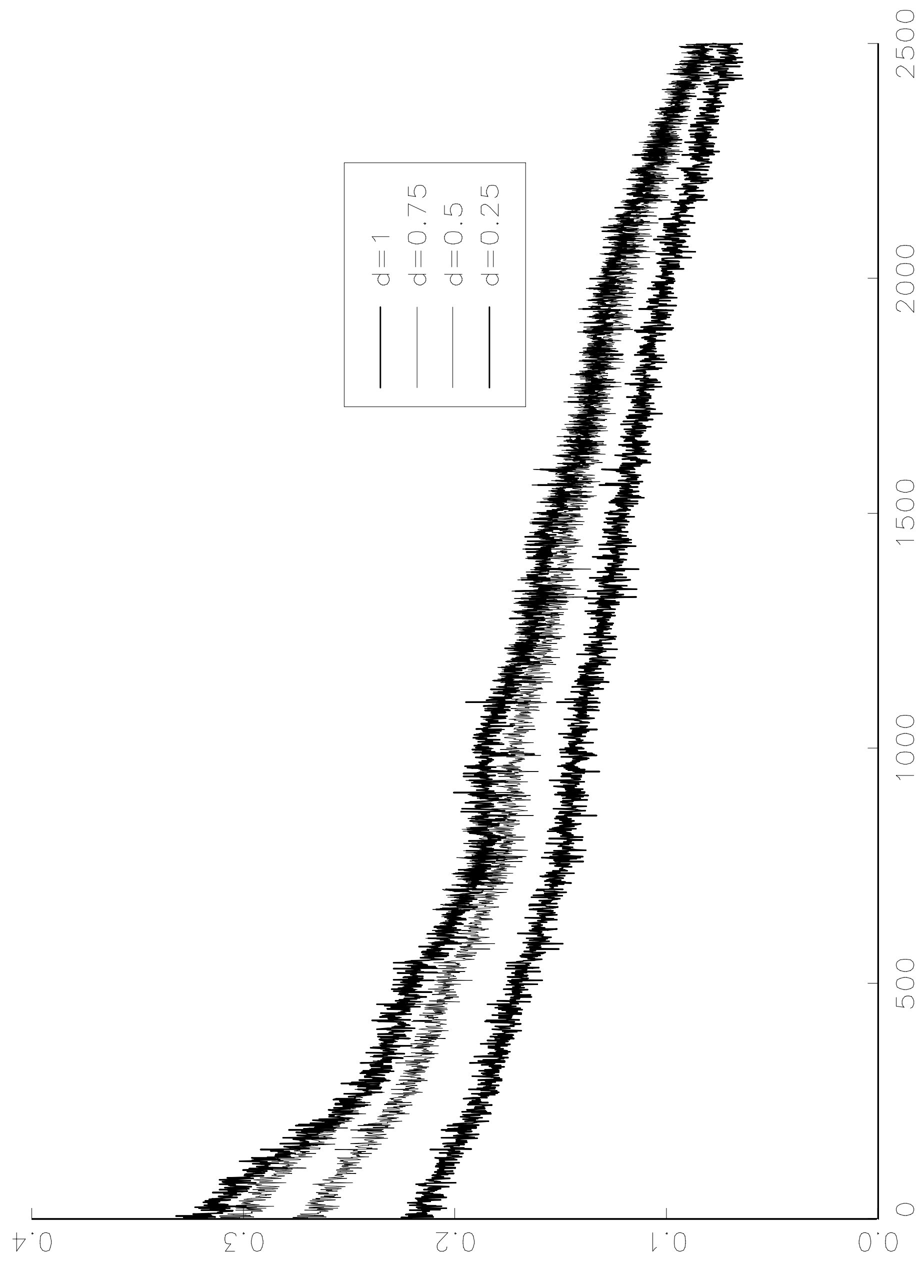
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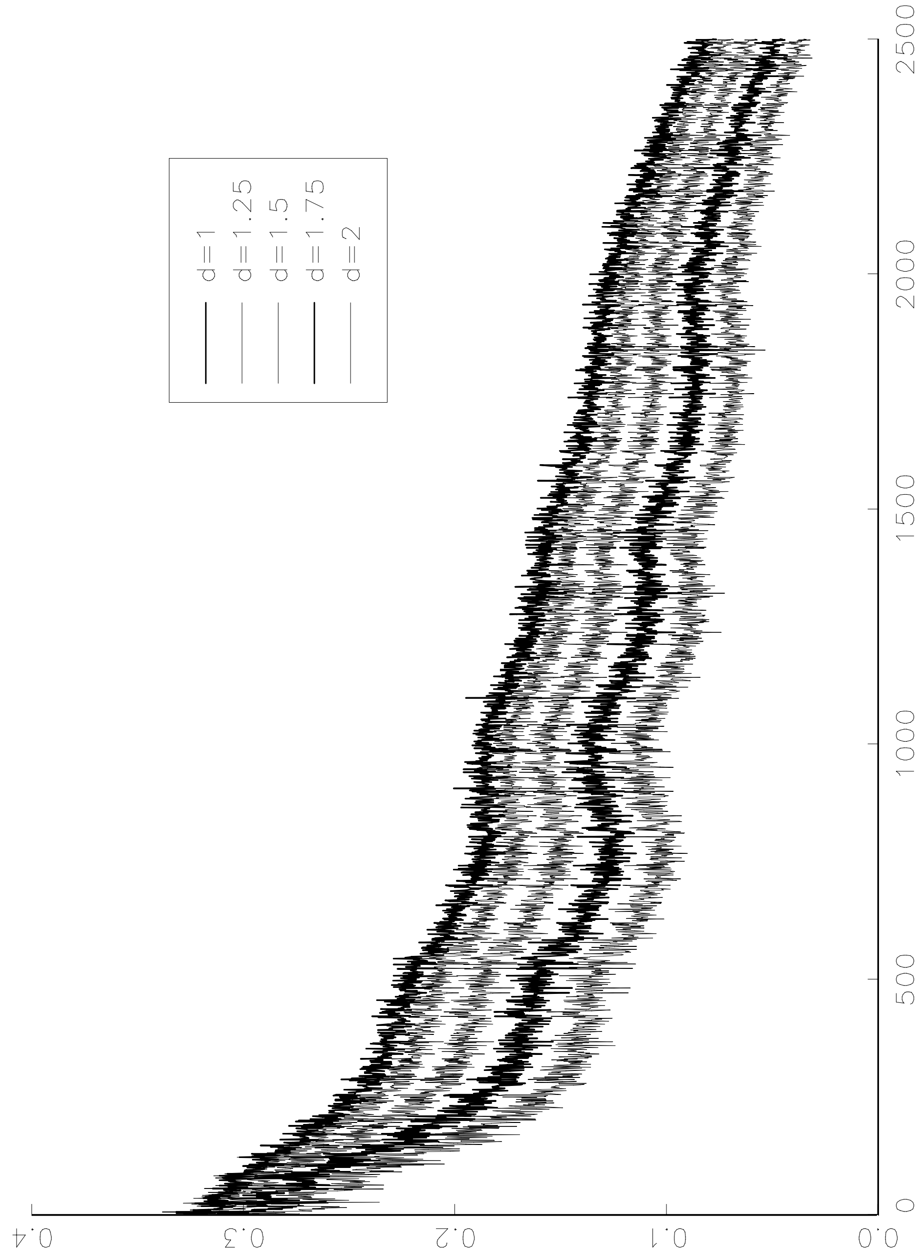
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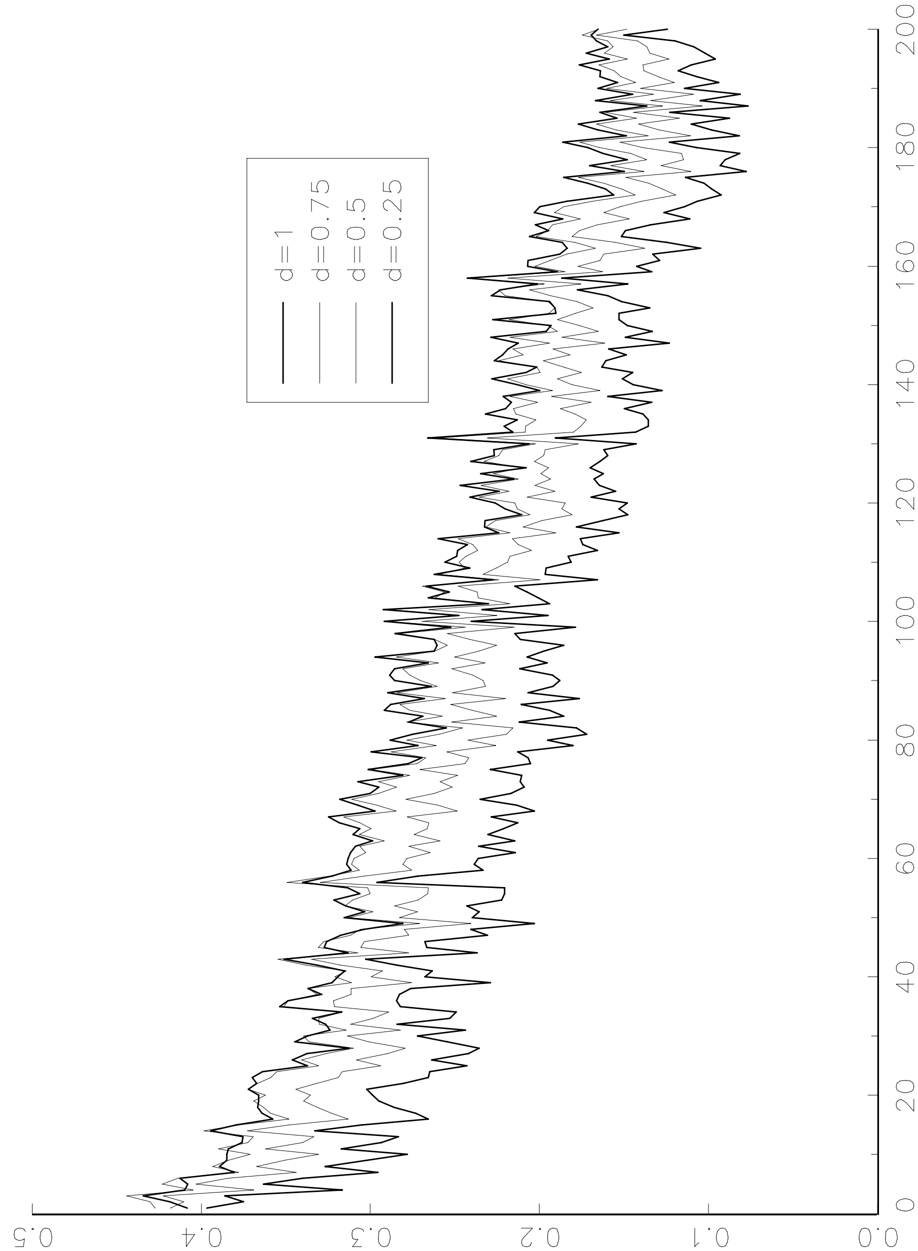
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