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On Stability and Doctor-optimality of Cumulative Offer Process

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Abstract

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by

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Doctor of Philosophy in Mathematics

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I study the stability and doctor-optimality of doctors' proposing cumulative offer process in the many-to-one matching with contracts. First, I explore some conventional hospital-by-hospital conditions on each hospital's choice function, and show that unilateral substitutability is equivalent to observable substitutability across doctors combined with cumulative offer achievability, each of which is a necessary condition for cumulative offer process to be doctor-optimally stable in a sense that if a hospital does not satisfy the condition, then we could construct some choice functions for other hospitals such that cumulative offer process is not doctor-optimally stable for some doctors' preference profile. Then, I focus on the joint properties of the choice functions for the entire group of hospitals and introduce two joint conditions—independence of proposing order and group cumulative offer achievability—and show that when these conditions are satisfied, cumulative offer process is always doctor-optimally stable. And it is by far the weakest sufficient condition. Moreover, these two conditions are necessary in a sense that if not, then there exists a doctors' preference profile and a proposing order such that cumulative offer process is not doctor-optimally stable. At last, I also introduce doctor's preference monotonicity and show that when cumulative offer process is doctor-optimally stable, this condition guarantees its strategy-proofness.

To my family.

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Chapter 1

Introduction

The theory of many-to-one matching with contracts began with a paper by Hatfield and Milgrom (2005)[**8**], in which they introduced substitutability condition and showed that when every hospital satisfies substitutability, there exists a doctor-optimally stable matching outcome.[1] They also proposed the doctors' proposing cumulative offer process[2], which could be regarded as an extension of Gale and Shapley's (1962)[**5**] deferred acceptance algorithm, and showed that it can exactly produce this doctor-optimally stable outcome under the substitutability condition. Since then, there has been a lot of work trying to find weaker substitutability condition that could guarantee the existence of (doctor-optimally) stable and/or strategy-proof outcome, and cumulative offer process plays a very important role in this literature. Hatfield and Kojima (2010)[**7**] introduced two weaker conditions: bilateral substitutability and unilateral substitutability. They found that cumulative offer process could produce a stable outcome when every hospital satisfies bilateral substitutability or unilateral substitutability, and the outcome is even doctor-optimal under the unilateral substitutability condition. Hatfield and Kominers (2015)[**9**] introduced substitutable completability condition that could also ensure the stability of cumulative offer process, and it is proved to be weaker than unilateral substitutability.[3] A recent paper by Hatfield, Kominers and Westkamp (2017)[**11**] introduced a new concept, observable offer process for a hospital, which is a sequence of distinct contracts that could illustrate how doctors propose contracts in a cumulative offer process given the choice function of this hospital.[4] Based on that, they proposed some novel hospital-by-hospital conditions that could guarantee the stability and strategy-proofness of cumulative offer process. However, it still remains to be thoroughly

[1]It is shown in Aygün and Sönmez (2012)[**2**, **3**] that every hospital should also satisfy irrelevance of rejected contracts. Since then, this condition is adopted in all the important papers in this literature.

[2]Throughout this thesis, whenever I mention cumulative offer process, I refer to the doctors' proposing cumulative offer process.

[3]See more details in Kadam (2017)[**13**].

[4]It is a sequence of distinct contracts with a hospital where, for each contract in the sequence, its doctor is not currently employed with this hospital when this hospital could choose from all previous contracts in the sequence.

explored to find a "weakest"[5] condition for cumulative offer process to be doctor-optimally stable. To address this problem, I will discuss the doctor-optimal stability of cumulative offer process from two perspectives—**hospital-by-hospital conditions**, a conventional type of condition adopted by most papers in the literature that needs to be satisfied by every hospital in a group, and **joint conditions**, which, on the other hand, focus on the joint properties of a group of hospitals.

First of all, I try to explore the "weakest" hospital-by-hospital condition that guarantees the doctor-optimal stability of cumulative offer process. To begin with, I need a hospital-by-hospital condition introduced in Hatfield, Kominers and Westkamp (2017)[**11**], observable substitutability across doctors, which requires that whenever a hospital renegotiates a previously-rejected contract during an observable offer process, it was already choosing another contract with the same doctor. It is a condition weaker than unilateral substitutability and they showed that it is not only sufficient for cumulative offer process to be stable, but also "necessary" in a sense that if a hospital does not satisfy observable substitutability across doctors, then there exists a doctor's preference profile and some choice functions for other hospitals such that cumulative offer process is not stable.

In this thesis, I will introduce cumulative offer achievability, another hospital-by-hospital condition weaker than unilateral substitutability. To do so, I still focus on the hospital's observable offer process introduced in Hatfield, Kominers and Westkamp (2017)[**11**]. First, I notice that given an observable offer process, it could partially reveal some information about each doctor's preferences, since a doctor always proposes his most preferred contracts prior to less preferred ones. Based on this observation, I can define an *a hospital's observable offer process revealed preferences*, which respects the doctors' preferences revealed by this observable offer process, and *a hospital's observable offer process unblocked set of contracts*, where, if we only consider the contracts from this unblocked set and this observable offer process, it would be impossible to find another set of contracts that could be strictly preferred by this hospital and weakly preferred by all the relative doctors under the revealed preferences.[6] Then *cumulative offer achievability* says that if a doctor's contract gets rejected during an observable offer process, but it is included in a observable offer process unblocked set, then by cumulative offer process, this doctor could eventually get a contract weakly better than this rejected contract under the doctors' preferences revealed by this observable offer process[7]. Then I find that cumulative offer achievability is a necessary condition for cumulative offer process to be doctor-optimally stable in a sense that if a hospital does not satisfy cumulative offer achievability, then there exists a doctors' preference profile such that cumulative offer process is not doctor-optimally stable.

An interesting observation is that unilateral substitutability is equivalent to observable substitutability across doctors combined with cumulative offer achievability. And this equivalence implies the "necessity" of unilateral substitutability, in a sense that if a hospital does

[5]It is a sufficient condition and also a "necessary" condition in some sense.

[6]See more details in Chapter 4.

[7]That is, the contract this doctor obtains will either be the rejected contract itself, or a contract proposed prior to this rejected contract in the observable offer process.

not satisfy unilateral substitutability, then there exists a doctor's preference profile and some choice functions for other hospitals such that cumulative offer process is not doctor-optimally stable. Therefore, unilateral substitutability could be regarded as a "weakest" hospital-by-hospital condition for cumulative offer process to be doctor-optimally stable because its sufficiency has already been proved in Hatfield and Kojima (2010)[**7**].

Practically, it is not always the case that every hospital needs to satisfy the same condition. So I step out of the hospital-by-hospital convention and look for more generalized results by focusing on joint properties for an entire group of hospitals' choice functions. Hirata and Kasuya (2014)[**12**] first proposed *independence of proposing order*, which requires that when doctors' preferences are fixed, cumulative offer process always leads to the same outcome for all different proposing orders. And they also showed that hospital-by-hospital bilateral substitutability implies independence of proposing order. Therefore, independence of proposing order is also weaker than hospital-by-hospital unilateral substitutability.[8] Because of the uniqueness of doctor-optimally stable outcome, it is actually a "necessary" joint condition for cumulative offer process to be doctor-optimally stable, in a sense that if not, then there exists a doctors' preference profile and a proposing order such that cumulative offer process is not doctor-optimally stable.

I also introduce the *group observable offer process*[9] and a more generalized joint substitutability condition, *group observable substitutability*, which requires that whenever a hospital renegotiates a previously-rejected contract during a group observable offer process, either this hospital was already choosing another contract with the same doctor, or none of the contracts he proposed was chosen by any hospital in the group. Group observable substitutability guarantees the stability of cumulative offer process. Moreover, it is also a weaker joint condition than independence of proposing order, which implies that independence of proposing order could also be regarded as a sufficient condition for cumulative offer process to be stable.

Similar to hospital-by-hospital cumulative offer achievability, I also introduce *group cumulative offer achievability*, which needs to adapt the definitions of the observable offer process revealed preferences and observable offer process unblocked sets for a group of hospitals jointly. And I show that group cumulative offer achievability is still weaker than hospital-by-hospital unilateral substitutability and it is also a "necessary" condition for cumulative offer process to be doctor-optimally stable, in a sense that if not, then there exists a doctors' preference profile such that cumulative offer process is not doctor-optimally stable.

Actually, when combined with independence of proposing order, group cumulative offer achievability also guarantees the doctor-optimal stability of cumulative offer process, making it by far the "weakest" condition for cumulative offer process to be doctor-optimally stable[10].

[8]Because unilateral substitutability implies bilateral substitutability.

[9]It generalizes the idea of a single hospital's observable offer process. It is a sequence of contracts where for each contract, its doctor is not currently employed with any hospital in the group when each hospital has access to all previous contracts in this sequence.

[10]It is strictly weaker than hospital-by-hospital unilateral substitutability and I will provide an example to illustrate this in Chapter 5.

Therefore, by looking at joint properties of the entire group of hospitals, it really helps us to obtain a more generalized result that could guarantee the doctor-optimal stability of cumulative offer process. The following tables summarize the results above (Contribution of this thesis is highlighted with red.).

At last, I also briefly discuss the strategy-proofness of cumulative offer process by introducing another joint condition, *doctor's preference monotonicity*. It fulfills the greed of each doctor by delivering more contracts to a doctor via cumulative offer process if this doctor reveals more contracts on top of his current preference list. It relaxes the hospital-by-hospital law of aggregate demand, a size monotonicity condition proposed in Hatfield and Milgrom (2005)[**8**], and still guarantees its strategy-proofness when cumulative offer process is doctor-optimally stable.

The remainder of this thesis is organized as follows: Chapter 2 introduces the many-to-one matching with contracts model and cumulative offer process. Chapter 3 reviews some important results in the literature. Chapter 4 introduces hospital-by-hospital cumulative offer achievability and presents the "necessity" of hospital-by-hospital unilateral substitutability. Chapter 5 introduces two joint conditions, independence of proposing order and group cumulative offer achievability, and presents by far the weakest condition for cumulative offer process to be doctor-optimally stable. It also briefly discusses the strategy-proofness of cumulative offer process when it is always doctor-optimally stable. Chapter 6 concludes.

Hospital-by-hospital conditions	Cumulative offer process is doctor-optimally stable	
	Sufficiency	"Necessity"
Unilateral substitutability	Yes	Yes
Observable substitutability across doctors		Yes
Cumulative offer achievability		Yes
Unilateral sub. = Observable sub. across doctors + Cumulative offer achievability		

Joint conditions	Cumulative offer process is doctor-optimally stable	
	Sufficiency	"Necessity"
Independence of proposing order	Independence of proposing order + Group cumulative offer achievability	Yes
Group cumulative offer achievability		Yes
Hospital-by-hospital unilateral sub. => Independence of proposing order + Group cumulative offer achievability		

Table 1.1: Tables summarizing the contribution of this thesis.

Chapter 2

Model

2.1 Framework

Consider a matching market where there is a finite set of doctors D and a finite set of hospitals H. There is also a finite set of contracts X, with each contract $x \in X$ associated with a unique doctor $d(x) \in D$ and a unique hospital $h(x) \in H$. What makes matching with contracts different from conventional matching problems is that there might be multiple contracts between the same doctor-hospital pair. For example, we could have several contracts with different salaries or roles/positions between the same doctor and hospital. For a set of contracts $Y \subseteq X$, let $d(Y) = \cup_{y \in Y} d(y)$ be the set of doctors and $h(Y) = \cup_{y \in Y} h(y)$ be the set of hospitals in contract set Y. For any $i \in D \cup H$, let $Y_i = \{y \in Y : i \in \{d(y), h(y)\}\}$ be the set of contracts associated with i in Y. A set of contracts Y is $feasible$ if for $\forall\ d \in D$, $|Y_d| \leq 1$, which means that each doctor $d \in D$ can only sign at most one contract in a feasible set of contracts.

Each doctor $d \in D$ has a strict preference $>_d$ over $X_d \cup \emptyset$, where $\emptyset$ refers to a null contract. The null contract represents unemployment, and the contract $x \in X_d$ is acceptable to doctor d if x is more preferred than $\emptyset$ by doctor d, or $x >_d \emptyset$. Let $C_d(Y)$ be the doctor d's most preferred contract in $Y \subseteq X$, and we assume that a doctor can only sign at most one contract, which means $|C_d(Y)| \leq 1$ for $\forall\ d \in D$.

A hospital's preference is more complicated. It is not simply a preference order over doctors, as in traditional two-sided matching market. Each hospital $h \in H$ is endowed with a choice function $C^h(\cdot)$ that represents its choice from a set of contracts. That is, $C^h(Y) \subseteq Y$ for any $Y \subseteq X$. A hospital can only choose contracts that are associated with it, which means $C^h(Y) \subseteq Y_h$. A hospital can only sign at most one contract with any given doctor, which means that

$$\forall x, x' \in C^h(Y)\ with\ d(x) = d(x'),\ we\ have\ x = x'$$

We denote by $C_D(Y) = \cup_{d \in D} C_d(Y)$ the set of contracts chosen by all doctors from a set of contracts $Y \subseteq X$. Similarly, $C^H(Y) = \cup_{h \in H} C^h(Y)$ is defined as the set of contracts chosen by all hospitals from a set of contracts $Y \subseteq X$. For any $Y \subseteq X$, we denote by $R^h(Y) = Y_h \backslash C^h(Y)$ the set of contracts h rejects from Y. Hence, $R^H(Y) = \cup_{h \in H} R^h(Y)$ is the set of contracts rejected by all hospitals from a set of contracts $Y \subseteq X$.

Finally, based on the above definitions, we could see that traditional two-sided matching is a special case of a matching problem with contracts, where for each doctor-hospital pair (d, h), there exists only one contract, i.e. $|X_d \cap X_h| = 1$.

2.2 Stability and doctor-optimally stable outcome

In this model, an *outcome* is a set of contracts $Y \subseteq X$.

Definition 1. *A feasible set of contracts $Y \subseteq X$ is* ***stable*** *if*

- *Individually rational* : $C_D(Y) = C^H(Y) = Y$ *and*

- *Unblocked* : *there does not exist a hospital $h \in H$ and a set of contracts $X' \neq C^h(Y)$ such that*
$$X' = C^h(X' \cup Y) \subseteq C_D(X' \cup Y)$$

First, we need the set of contracts to be feasible, which means any doctor could not sign multiple contracts in this outcome. Then, the set of contracts should satisfy individual rationality, which means this set of contracts is acceptable to both the group of doctors and the group of hospitals, and there does not exist any unwanted contract. Finally, if a set of contracts fails to satisfy the unblocking condition, then it means that there is an alternative set of contracts that a hospital strictly prefers and its corresponding doctors weakly prefer, then this hospital and the doctors will deviate from the original matching outcome by signing this alternative set of contracts, which makes the original matching outcome unstable.

Given any doctors' preference profile and hospitals' choice functions, there may exist multiple stable outcomes for a matching with contracts market. A stable outcome X^* is doctor-optimal if each doctor weakly prefers his contract in X^* to his contract in any other stable outcome. We formalize the definition of doctor-optimally stable outcome as follows:

Definition 2. *A set of contracts X^* is a* ***doctor-optimally*** *stable outcome if for any stable outcome X', we have*
$$X^* \geq_D X'$$
That is, $X_d^* \geq_d X_d'$ for $\forall\ d \in D$.

2.3 Cumulative offer process

In matching with contracts settings, there is a popular matching algorithm that generalizes the celebrated Doctors' Proposal Deferred Acceptance[1] algorithm. It is the doctor-proposing cumulative offer process. A doctor proposing cumulative offer process is defined with respect to a strict proposing order $\triangleright$ of all contracts in X, where $x \triangleright y$ means that contract x is ranked higher than contract y. To be more specific, the doctor-proposing cumulative offer process is defined as follows[2]:

Definition 3. ***Doctor proposing cumulative offer process*** *with a strict proposing order* $\triangleright$*:*

- ***Step 0:*** *Initialize the available set of contracts that each hospital* $h \in H$ *receives at step 0 as* $\mathcal{A}^h(0) = \emptyset$.

- ***Step*** $t \geq 1$***:*** *Consider the set*
  $U_t = \{x \in X \backslash \mathcal{A}^H(t-1) : d(x) \notin d(C^H(\mathcal{A}^H(t-1)))$ *and* $\nexists z \in (X_{d(x)} \backslash \mathcal{A}^H(t-1)) \cup \{\emptyset\}$ *such that* $z >_{d(x)} x\}$.
  If U_t *is empty, then the algorithm terminates and the outcome is* $C^H(\mathcal{A}^H(t-1))$. *Otherwise, let* x_t *be the highest ranked contract in* U_t *according to* $\triangleright$, *then* $d(x_t)$ *proposes* x_t *at step* t. *And each hospital's available set will be updated as follows:* $\mathcal{A}^{h(x_t)}(t) = \mathcal{A}^{h(x_t)}(t-1) \cup \{x_t\}$ *and* $\mathcal{A}^h(t) = \mathcal{A}^h(t-1)$ *for any* $h \neq h(x_t)$.

A doctors' proposing cumulative offer process starts with no contract ever offered to hospitals, hence the available set $\mathcal{A}^h(0)$ for each hospital $h \in H$ is empty. Then, at each step $t \geq 1$, U_t refers to the set of contracts that satisfy the following conditions:

- Any contract in U_t has not been offered yet.
- Any contract in U_t is not associated to a doctor whose contract is currently held by a hospital.
- Any contract in U_t is most preferred by its doctor among all his contracts that have not yet been proposed.

If $U_t \neq \emptyset$, the highest ranked contract in U_t according to proposing order $\triangleright$ will be proposed by its doctor at step t. Then at the end of step t, the available set for each hospital will be updated as the union of the newly proposed contract and its available set from last step. The cumulative offer process will terminate at some point T when $U_T = \emptyset$ eventually. In general, we can denote by $\{x_1, ..., x_T\}$ the set of contracts proposed in a cumulative offer process, where x_t is the contract offered at step $t (1 \leq t \leq T)$.

[1]See Appendix A.

[2]Here, I adopt the definition from Hatfield, Kominers and Westkamp (2017)[**11**]

First of all, we know that doctors' preference profiles and hospitals' choice functions will affect the outcome of cumulative offer process since doctors' preference profiles will determine the new contract each rejected doctor will offer and hospitals' choice functions will determine what contracts will be chosen by hospitals in each step. Then, the following example shows that even when fixing doctors' preferences and hospitals' choice functions, proposing order $\triangleright$ can also affect the outcome of cumulative offer process[3]:

Example 1. *Suppose there are two hospitals h and h' whose preferences are as follows:*

$$h: \quad \{x, z\} > \emptyset$$
$$h': \quad \{x', z'\} > \emptyset$$

where $d(x) = d(x') = d_x$, $d(z) = d(z') = d_z$, and each contract is acceptable to its doctor. Suppose d_x prefers x to x' and d_z prefers z' to z.

If $\triangleright$ is specified by $x \triangleright z' \triangleright x' \triangleright \cdots$, then the cumulative offer process will stop at the end of step 3, where the set of contracts proposed by doctors are $\{x, z', x'\}$. Therefore, hospital h will not choose any contract, and h' will choose $\{x', z'\}$. However, if $\triangleright$ is specified by $x \triangleright z' \triangleright z \triangleright \cdots$, then the cumulative offer process will also stop at the end of step 3, where the set of contracts proposed by doctors are $\{x, z', z\}$. Then hospital h will choose $\{x, z\}$, and h' will not choose any contract.

Therefore, we could think of cumulative offer process as a function of three arguments: doctors' preference profiles $>_D$, hospitals' choice functions $C^H(\cdot)$ and a strict proposing order $\triangleright$ of contracts. And we could use $f(>_D, C^H(\cdot), \triangleright)$ to denote the cumulative offer process outcome for any given $>_D$, $C^H(\cdot)$, and $\triangleright$.

2.4 Conditions on hospitals' choice functions, sufficiency and necessity.

Mostly, researchers try to find a condition $\mathcal{C}$ such that cumulative offer process always produces a (doctor-optimally) stable outcome when every hospital satisfies this condition. We call this as a **hospital-by-hospital condition**. And we could use the following mathematical expression to represent the sufficiency:

If $C^h(\cdot)$ satisfies $\mathcal{C}$ for $\forall\ h \in H$, then $f(\cdot, \{C^h(\cdot)\}_{h\in H}, \cdot)$ is (doctor-optimally) stable.

As for the necessity of this hospital-by-hospital condition, they usually try to show that if $|H| > 1$ and a hospital $h \in H$ does not satisfy this condition, then there exist a doctors'

[3] Hirata and Kasuya (2014)[**12**] pointed out that proposing order really matters for cumulative offer process in general. However, as a special case, they showed that when every hospital satisfies bilateral substitutability, cumulative offer process is order-independent. Furthermore, the cumulative offer process still generates the same outcome even when rejected doctors offer new contracts simultaneously.

preference profile and some other hospitals' choice functions such that no (doctor-optimally) stable outcome exists. That is, if $C^h(\cdot)$ does not satisfy $\mathcal{C}$, then

$$\exists >_D \text{ and } C^{-h}(\cdot) \text{ such that } f(>_D, \{C^h(\cdot), C^{-h}(\cdot)\}, \cdot) \text{ is not (doctor-optimally) stable.}^4$$

And in this thesis, I will also introduce another type of condition, a **Joint condition** $\mathcal{C}^H$ that focuses on joint properties of an entire group of hospitals H, such that cumulative offer process always produces a (doctor-optimally) stable outcome when a group of hospitals satisfies this condition. The sufficiency is as follows:

$$\text{If } C^H(\cdot) \text{ satisfies } \mathcal{C}^H, \text{ then } f(\cdot, C^H(\cdot), \cdot) \text{ is (doctor-optimally) stable.}$$

As for the necessity, I would like to show that if hospitals' choice functions do not satisfy the joint condition, then there exist a doctors' preference profile and a proposing order $\triangleright$ such that the cumulative offer process is not (doctor-optimally) stable. That is, if $C^H(\cdot)$ does not satisfy $\mathcal{C}^H$, then

$$\exists >_D \text{ and } \triangleright \text{ such that } f(>_D, C^H(\cdot), \triangleright) \text{ is not (doctor-optimally) stable.}$$

[4] $C^{-h}(\cdot)$ refers to the choice functions for all the other hospitals in H.

Chapter 3

Preliminary results

In this chapter, we will go over some important results in the literature.

3.1 Irrelevance of rejected contracts

3.1.1 What is irrelevance of rejected contracts

In Aygün and Sönmez (2012)[**2**, **3**], they first pointed out the importance of irrelevance of rejected contracts in matching with contracts setting, and showed that most results in Hatfield and Kojima (2010)[**7**] would not hold under weaker substitutability conditions unless irrelevance of rejected contracts is explicitly assumed for each hospital. Irrelevance of rejected contracts is a hospital-by-hospital condition requiring that the chosen contracts of a hospital will not be affected by the removal of a rejected contract from a set of contracts. In Hatfield, Kominers and Westkamp (2017)[**11**], their analysis was also built on the foundation that every hospital satisfies irrelevance of rejected contracts. The formal definition of irrelevance of rejected contracts is as follows:

Definition 4. *A hospital $h \in H$ satisfies the* ***irrelevance of rejected contracts condition (IRC)*** *if for $Y \subseteq X$ and $\forall\ z \in X \backslash Y$:*

$$z \notin C^h(Y \cup \{z\}) \Rightarrow C^h(Y) = C^h(Y \cup \{z\})$$

An immediate result from IRC is as follows:

Lemma 1. *Given a hospital $h \in H$ and $X' \subseteq X$, let $Y = C^h(X')$, then IRC implies that for any $X'' \subseteq X'$*

$$Y = C^h(Y \cup X'')$$

Proof. Since $C^h(Y) \subseteq Y$, by removing all rejected contracts in $X' \backslash Y$ one by one, we still have $C^h(Y) = C^h(X') = Y$. Because $Y \subseteq Y \cup X'' \subseteq X'$, then by removing all rejected contracts in $X' \backslash (X'' \cup Y)$ one by one, we still have $C^h(Y) = C^h(X'' \cup Y)$. Therefore, we have $C^h(Y) = C^h(X'' \cup Y) = C^h(X') = Y$. □

Let $Y = C^h(X')$ for a set of contracts X'. This lemma shows that for any $X'' \subseteq X'$, we have

$$Y \subseteq Y \cup X'' \subseteq X'$$

and IRC will result in

$$Y = C^h(Y \cup X'') = C^h(X') = Y$$

which could be treated as a version of "squeeze theorem" in matching with contracts setting.

3.1.2 IRC, feasibility and stability of cumulative offer process

Let us start with an illustrative example to show that cumulative offer process fails to produce a feasible outcome.

Example 2. *Suppose there are two hospitals h and h' and three doctors d_x, d_y and d_z. The preferences of h and h' are as follows:*

$$\begin{aligned} h: &\quad \{x, z\} > \{y\} > \{x\} > \{z\} > \emptyset \\ h': &\quad \{z'\} > \emptyset \end{aligned}$$

where $d(x) = d_x$, $d(y) = d_y$ and $d(z) = d(z') = d_z$. Suppose each contract is acceptable to its associated doctor and d_z prefers z to z'.

Let us consider a cumulative offer process $\{z, y, z', x\}$. In the final step of this cumulative offer process, hospital h receives a set of contracts $\{z, y, x\}$ and it chooses $\{x, z\}$ according to its preference. And hospital h' only receives $\{z'\}$, then it chooses $\{z'\}$. However, in this outcome, doctor d_z is hired by two different hospitals, which makes it infeasible. Therefore, the cumulative offer process, in general, can not always produce a feasible outcome, not even a stable outcome.

What if we can find some condition on which the cumulative offer process always leads to a feasible outcome? Is this feasible outcome stable as well? As a matter of fact, the answer is Yes! It is actually implicitly implied in Hatfield and Kojima (2010)[**7**], Hatfield, Kominers and Westkamp (2017)[**11**] that if every hospital satisfies IRC, then feasibility of cumulative offer process implies its stability. Because of its importance, I would like to restate the following result:

Theorem 1. *Suppose every hospital $h \in H$ satisfies IRC. And if cumulative offer process produces a feasible outcome, then it is also stable.*

Proof. Let X^* be the feasible outcome of a cumulative offer process $\{x_1, ..., x_T\}$, which means $X^* = C^H(\{x_1, ..., x_T\})$ or $C^h(X^*) = C^h(\{x_1, ..., x_T\})$ for $\forall\ h \in H$. By definition of cumulative offer process and IRC, we could see that X^* satisfies the individual rationality. If X^* is not stable, then $\exists\ h \in H$ and a set of contracts $X' \neq C^h(X^*)$ such that

$$X' = C^h(X' \cup X^*) \subseteq C_D(X' \cup X^*)$$

If $X' \subseteq C_D(X' \cup X^*)$, then we could conclude that any $x' \in X'$ has been proposed by $d(x')$ during the cumulative offer process. It implies that $X' \subseteq \{x_1, ..., x_T\}$. Therefore,

$$X^* \subseteq X^* \cup X' \subseteq \{x_1, ..., x_T\}$$

By IRC and Lemma 1, we could conclude that

$$C^h(X^*) = C^h(X^* \cup X') = C^h(\{x_1, ..., x_T\}) = C^h(X^*)$$

Therefore, $C^h(X^*) = C^h(X^* \cup X') = X'$. However, it contradicts the fact that $X' \neq C^h(X^*)$. □

Suppose IRC is always satisfied for every hospital. Theorem 1 indicates that if we can ensure the feasibility of cumulative offer process outcome, then its stability is "automatically" guaranteed.

Since IRC plays an important role while studying many-to-one matching with contracts model, we will assume that every hospital satisfies IRC throughout this thesis.

3.2 Substitutability, unilateral substitutability and bilateral substitutability

In Hatfield and Milgrom (2005)[**8**], they proposed a hospital-by-hospital substitutability condition, which guarantees the doctor-optimal stability of cumulative offer process. The definition of substitutability is as follows:

Definition 5. *A hospital h satisfies **substitutability** condition if there do not exist contracts x, $z \in X$ and a set of contracts $Y \subseteq X$ such that $z \notin C^h(Y \cup \{z\})$, but $z \in C^h(Y \cup \{x, z\})$.*

In other words, adding a contract into a choice set will not induce a hospital to renegotiate a previously rejected contract.

Then in Hatfield and Kojima (2010)[**7**], they proposed two weaker hospital-by-hospital substitutable conditions. The first one is **bilateral substitutability**, which is defined as follows:

Definition 6. *A hospital h satisfies **bilateral substitutability** condition if there do not exist contracts $x, z \in X$ and a set of contracts $Y \subseteq X$ such that $d(x), d(z) \notin d(Y)$, $z \notin C^h(Y \cup \{z\})$ but $z \in C^h(Y \cup \{x, z\})$.*

In other words, the choice function of a hospital satisfies bilateral substitutability if whenever a contract is rejected while all available contracts involve different doctors, this contract remains rejected when a contract with new doctor is added into the choice set. As we can see, bilateral substitutability is a weaker condition than substitutability since the renegotiation of contract $z \in X$ will not happen only when other available contracts do not involve its doctor $d(z)$, and the doctor of the new added contract is not in previous available set either. An important result based on bilateral substitutability is that if every hospital's choice function satisfies bilateral substitutes and IRC, then cumulative offer process always produces a stable outcome.

Theorem 2. *(Theorem 1 in Hatfield and Kojima (2010)[**7**]) Suppose that every hospital satisfies bilateral substitutability. Then there exists a stable outcome.*

See more in Hatfield and Kojima (2010)[**7**].

The other weaker substitutable condition in that paper is unilateral substitutability, which is defined as follows:

Definition 7. *A hospital h satisfies **unilateral substitutability** condition if there do not exist contracts $x, z \in X$ and a set of contracts $Y \subseteq X$ such that $d(z) \notin d(Y)$, $z \notin C^h(Y \cup \{z\})$ but $z \in C^h(Y \cup \{x, z\})$.*

By definition, we can see that unilateral substitutability is stronger than bilateral substitutability, then the stability of cumulative offer process outcome will also be guaranteed if every hospital satisfies unilateral substitutability. In addition, it shows that if every hospital satisfies unilateral substitutability and IRC, then cumulative offer process coincides with the standard doctor-proposing deferred acceptance algorithm[1]. Then it yields a even stronger result that the outcome of cumulative offer process is doctor-optimally stable.

Theorem 3. *(Theorem 5 in Hatfield and Kojima (2010)[**7**]) Suppose that every hospital satisfies unilateral substitutability. Then there exists a doctor-optimally stable outcome. The outcome that is produced by the doctor-proposing deferred acceptance algorithm is the doctor-optimally stable outcome.*

See more in Hatfield and Kojima (2010)[**7**].

3.3 Observable substitutability and observable substitutability across doctors

For any hospital $h \in H$ whose choice function is given by $C^h(\cdot)$, we define the **offer process** for h as follows:

Definition 8. *An **offer process** for h is a finite sequence of distinct contracts $\{x_1, ..., x_T\}$ such that $x_t \in X_h$ for $\forall\ 1 \leq t \leq T$.*

[1]See doctor-proposing deferred acceptance algorithm in Appendix A.

And an **observable offer process** for $h \in H$ is an offer process $\{x_1, ..., x_T\}$ such that $d(x_t) \notin d(C^h(\{x_1, ..., x_{t-1}\}))$ for $\forall\ 1 \leq t \leq T$. Therefore, in an observable offer process, a doctor can propose a contract x_t to h at step t only when he does not have any contract held by hospital h.

Definition 9. *A choice function $C^h(\cdot)$ has an observable violation of substitutability if there exists an observable offer process $\{x_1, ..., x_t\}$ for h such that $R^h(\{x_1, ..., x_{t-1}\}) \backslash R^h(\{x_1, ..., x_t\}) \neq \emptyset$. A choice function $C^h(\cdot)$ is **observably substitutable** if it does not have an observable violation of substitutability.*

As we can see, observable substitutability rules out the possibility for contract renegotiation in an observable offer process, hence in a cumulative offer process. It weakens the traditional hospital-by-hospital substitutable conditions, such as substitutability and unilateral substitutability. See more in Hatfield, Kominers and Westkamp (2017)[**11**].

Definition 10. *A hospital's choice function $C^h(\cdot)$ is **observably substitutable across doctors** if for any observable offer process $\{x_1, ..., x_t\}$ for h, we have that if $x \in R^h(\{x_1, ..., x_{t-1}\}) \backslash R^h(\{x_1, ..., x_t\})$, then $d(x) \in d(C^h(\{x_1, ..., x_{t-1}\}))$.*

As we can see that observable substitutability across doctors is an even weaker hospital-by-hospital condition than observable substitutability, since it allows for contract renegotiation in an observable offer process if this hospital has already chosen some contract with the same doctor.

In Hatfield, Kominers and Westkamp (2017)[**11**], they have shown that if every hospital's choice function satisfies observable substitutability across doctors and IRC, then the cumulative offer process always produces a stable outcome.

Theorem 4. *(Theorem 5 in Hatfield, Kominers and Westkamp (2017)[**11**]) If every hospital satisfies observable substitutes across doctors, then the cumulative offer process is stable.*

To see this, it relies on a fact that the outcome of cumulative offer process is always stable if it is feasible and every hospital satisfies IRC, as shown in Theorem 1. Suppose every hospital's choice function satisfies observable substitutability across doctors, then if a contract is renegotiated during a cumulative offer process, then we could guarantee that the associated doctor was previously employed with the same hospital, and this renegotiation will not break the feasibility of the outcome. Therefore, observable substitutability across doctors could guarantee the feasibility of the outcome, hence the stability as well.

This paper also showed that observable substitutability across doctors is also a necessary condition for cumulative offer process to be stable in a sense that if $|H| > 1$ and some hospital $h \in H$ does not satisfy observable substitutability across doctors, then there always exists a unit-demand choice function for the other hospital $h' \neq h$ such that no cumulative offer process will produce a stable outcome.

Theorem 5. *(Theorem 6 in Hatfield, Kominers and Westkamp (2017)[**11**]) If $|H| > 1$ and that the choice function of some hospital is not observably substitutable across doctors, then there exists unit-demand choice function for the other hospitals such that no cumulative offer mechanism is stable.*

See more in Hatfield, Kominers and Westkamp (2017)[**11**].

The paper also discussed the relationship between observable substitutability across doctors and bilateral substitutability, and it showed that observable substitutability across doctors is strictly weaker than bilateral substitutability.

The following figure summarizes the relationship among all these hopital-by-hospital substitutable conditions mentioned above:

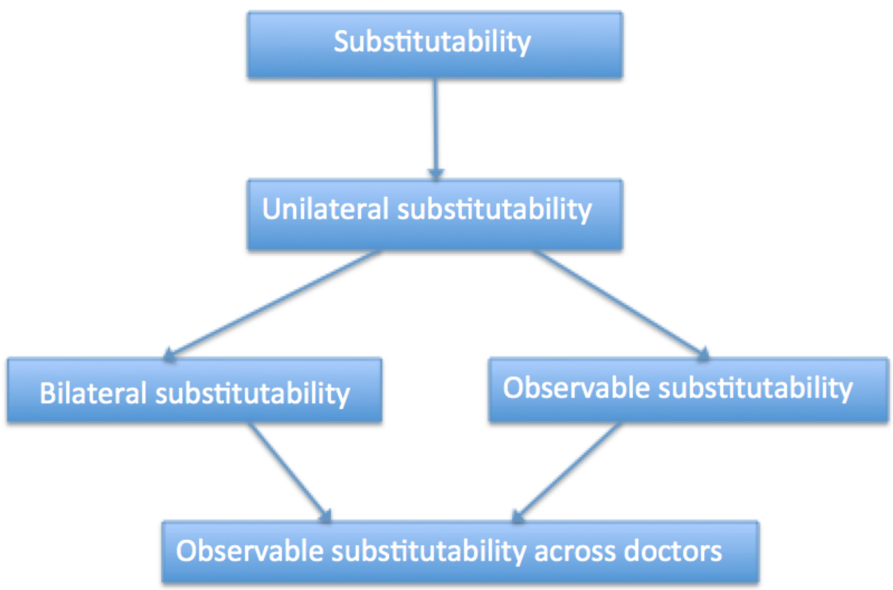

Figure 3.1: A figure summarizing hospital-by-hospital substitutable conditions.

Chapter 4

A result on a hospital-by-hospital condition: Necessity of unilateral substitutability

In this chapter, I try to explore a“ weakest” hospital-by-hospital condition that guarantees the doctor-optimal stability of cumulative offer process.

4.1 Cumulative offer achievability (C.o.achievability)

First of all, I would like to introduce cumulative offer achievability (c.o.achievability), which turns out to be a very important "necessary" condition for cumulative offer process to be doctor-optimally stable, in a sense that if a hospital does not satisfy c.o.achievability, then there exists a doctors' preference profile such that no cumulative offer process is doctor-optimally stable.

To begin with, I need to introduce some definitions relative to a single hospital's observable offer process.

4.1.1 Observable offer process revealed preferences

Given an observable offer process for a hospital h, we could partially infer doctors' preferences for contracts in X_h since a doctor always proposes his most preferred contract that has never been offered to any hospital at each step of the cumulative offer process. For example, if there is an observable offer process $\{x, z', x', z\}$ for a hospital h where $d(x) = d(x') \neq d(z) = d(z')$, then we could infer that doctor $d(x)$ prefers x to x' and $d(z)$ prefers z' to z. We can only partially figure out each doctor's preference since we have no information about his other contracts that have not been revealed in this observable offer process. However, what we do know is that those contracts should be less preferred by this doctor than revealed ones. Therefore, in the example above, if $d(x)$ has another two acceptable contracts $\hat{x}$ and $\bar{x}$ with hospital h, then we could infer that his preference could be either $x > x' > \hat{x} > \bar{x} > \emptyset$ or $x > x' > \bar{x} > \hat{x} > \emptyset$. Given an observable offer process for a hospital h, we can see that there usually exists a class of preferences that are consistent with an observable offer process revealed preferences, and the formal definition is as follows:

Definition 11. *Suppose $\{x_1, ..., x_T\}$ is an observable offer process for a hospital h. A doctors' preference profile $>_D$ is consistent with $\{x_1, ..., x_T\}$-revealed preference if it satisfies the following:*

- *Any contract $x \in \{x_1, ..., x_T\}$ is acceptable to $d(x)$.*
- *$\forall\ x_t, x_{t'} \in \{x_1, ..., x_T\}$ with $d(x_t) = d(x_{t'})$ and $t < t'$, we have $x_t >_{d(x_t)} x_{t'}$.*
- *$\forall\ x \notin \{x_1, ..., x_T\}$, if $\exists\ x' \in \{x_1, ..., x_T\}$ with $d(x) = d(x')$, then $x' >_{d(x)} x$.*

In other words, contracts in $\{x_1, ..., x_T\}$ should be acceptable to doctors, and $>_D$ respects the relative order of contracts in $\{x_1, ..., x_T\}$ for each doctor and the contracts that are not revealed will be less preferred than those revealed.

Suppose there are two observable offer processes $\{x, z, x'\}$ and $\{z, x, z'\}$ for a hospital h, where $d(x) = d(x') \neq d(z) = d(z')$. We can see that $x >_{d(x)} x'$ in $\{x, z, x'\}$ and it also respects the information revealed by the other observable offer process, which shows that x is the most preferred contract of $d(x)$. Similarly, in $\{z, x, z'\}$, doctor $d(z)$ prefers z to z', which is also consistent with fact that z is $d(z)$'s most preferred contract in $\{x, z, x'\}$. Therefore, these two observable offer processes mutually respect the information revealed by each other. Formally, when given an observable offer process $\{x_1, ..., x_T\}$ for a hospital, we could define its compatible observable offer process, which respects its revealed information, as follows:

Definition 12. *Suppose we have an observable offer process $\{x_1, ..., x_T\}$ for h. An observable offer process $\{y_1, ..., y_S\}$ for h is* ***compatible with*** *$\{x_1, ..., x_T\}$**-revealed preferences** if for any $y \in \{y_1, ..., y_S\}$, let $d = d(y)$,*

- *if $y \in \{x_1, ..., x_T\}$, let*

$$S = \{x \in \{x_1, ..., x_T\} \cap X_d : x \text{ is proposed prior to } y \text{ in } \{x_1, ..., x_T\}\}$$

  . If $S \neq \emptyset$, then $S \subseteq \{y_1, ..., y_S\}$, and $\forall\ x \in S$, x is proposed prior to y in $\{y_1, ..., y_S\}$ as well.

- *if $y \notin \{x_1, ..., x_T\}$, let $S = \{x_1, ..., x_T\} \cap X_d$. If $S \neq \emptyset$, then $S \subseteq \{y_1, ..., y_S\}$, and $\forall$ $x \in S$, x is proposed prior to y in $\{y_1, ..., y_S\}$.*

Suppose $\{x_1, ..., x_T\}$ and $\{y_1, ..., y_S\}$ are two observable offer processes for a hospital. If $\{y_1, ..., y_S\}$ is compatible with $\{x_1, ..., x_T\}$-revealed preferences, then $\{x_1, ..., x_T\}$ is also compatible with $\{y_1, ..., y_S\}$-revealed preferences. Actually, when they respect each other's revealed information, then we could find a doctors' preference profile $>_D$, which is consistent with their revealed information, such that both $\{x_1, ..., x_T\}$ and $\{y_1, ..., y_S\}$ could be treated as eligible observable offer processes generated by $>_D$ and $C^h(\cdot)$.

Definition 13. *An **eligible** observable offer process $\{x_1, ..., x_T\}$ for h under a doctors' preference profile $>_D$ is an observable offer process for h that satisfies:*

- *Any contract in $\{x_1, ..., x_T\}$ is acceptable to its doctor.*
- $\forall\ x_t, x_{t'} \in \{x_1, ..., x_T\}$ *with* $d(x_t) = d(x_{t'})$, $t < t' \Leftrightarrow x_t >_{d(x_t)} x_{t'}$.
- *If* $x \in \{x_1, ..., x_T\}$ *and* $\exists\ x' >_{d(x)} x$, *then we have* $x' \in \{x_1, ..., x_T\}$.

Lemma 2. *Suppose $\{x_1, ..., x_T\}$ and $\{y_1, ..., y_S\}$ are two observable offer processes for a hospital h in $Y \subseteq X_h$. If $\{y_1, ..., y_S\}$ is compatible with $\{x_1, ..., x_T\}$-revealed preferences, then there exists a doctors' preference profile $>_D$, which is consistent with $\{x_1, ..., x_T\}$-revealed preferences, such that $\{x_1, ..., x_T\}$ and $\{y_1, ..., y_S\}$ are both eligible observable offer processes for h in Y under $>_D$ and $C^h(\cdot)$.*

Proof. Let $>_D$ satisfy the following conditions:

- $y \in Y$ is acceptable to $d(y)$ if and only if $y \in Y$.
- $\forall\ x$ and x' in Y with $d(x) = d(x') =: d$, we have
  - If $x, x' \in \{x_1, ..., x_T\} \cup \{y_1, ..., y_S\}$ and x is proposed prior to x' in either observable offer process, then $x >_d x'$.
  - If $x \in \{x_1, ..., x_T\} \cup \{y_1, ..., y_S\}$ and $x' \notin \{x_1, ..., x_T\} \cup \{y_1, ..., y_S\}$, then $x >_d x'$.
  - If $x, x' \notin \{x_1, ..., x_T\} \cup \{y_1, ..., y_S\}$, then we could have either $x >_d x'$ or $x' >_d x$.

Then we can verify that $>_D$ satisfies

- $y \in Y$ is acceptable to $d(y)$ if and only if $y \in Y$.
- $\forall\ x_t, x_{t'} \in \{x_1, ..., x_T\}$ with $d(x_t) = d(x_{t'}) =: d$ and $t < t'$, we have $x_t >_d x_{t'}$.
- If $x \in \{x_1, ..., x_T\}$ and $\exists\ x' >_{d(x)} x$, then we have $x' \in \{x_1, ..., x_T\}$.
- $\forall\ y_s, y_{s'} \in \{y_1, ..., y_S\}$ with $d(y_s) = d(y_{s'}) =: d$ and $s < s'$, we have $y_s >_d y_{s'}$.
- If $y \in \{y_1, ..., y_S\}$ and $\exists\ y' >_{d(y)} y$, then we have $y' \in \{y_1, ..., y_S\}$.

Therefore, both $\{x_1, ..., x_T\}$ and $\{y_1, ..., y_S\}$ are eligible observable offer processes for h in Y under $>_D$ and $C^h(\cdot)$. □

4.1.2 Observable offer process unblocked set of contracts

In this subsection, I will introduce the definition of an observable offer process unblocked set of contracts, which is another concept based on observable offer process revealed information.

Definition 14. *Suppose $\{x_1, ..., x_t\}$ is an observable offer process for a hospital h. A set of contracts Y is $\{x_1, ..., x_t\}$-**unblocked** if it satisfies the following conditions:*

- $Y = C^h(Y)$
- *$\forall\ X' \subseteq \{x_1, ..., x_t\} \cup Y$, let $Y' = C^h(X' \cup Y)$, then we have*
  - *either $Y' = Y$*
  - *or $\exists\ y' \in Y'$, $y \in Y$ with $d(y) = d(y')$ such that doctor $d(y)$ proposes y prior to y' in $\{x_1, ..., x_t\}$.*

Suppose $>_D$ is consistent with $\{x_1, ..., x_t\}$-revealed preferences and we only consider the contracts in $Y \cup \{x_1, ..., x_t\}$. Actually, the second condition says that when we add some contracts into Y, either Y is still favored by hospital h, or if Y is not, then some doctor needs to compromise by getting a less preferred contract according to $>_D$[1].

Therefore, if Y is $\{x_1, ..., x_t\}$-unblocked, then there is no way to find a different set of contracts in $Y \cup \{x_1, ..., x_t\}$ that could be strictly preferred by this hospital without harming doctors' benefits. There is always a "tradeoff" between doctors' preferences and this hospital's preference. Then we can show that Y could tentatively[2] be treated as a stable outcome in $Y \cup \{x_1, ..., x_t\}$ under $C^h(\cdot)$ and $>_D$. To see this, we can use a proof by contradiction. Suppose Y is not stable under $>_D$ and $C^h(\cdot)$. Since $Y = C^h(Y)$ and Y is acceptable to doctors, it satisfies the individual rationality. Therefore, $\exists\ X' \neq C^h(Y) = Y$ such that $X' = C^h(X' \cup Y)$ and $X' \subseteq C_D(X' \cup Y)$. $X' \subseteq C_D(X' \cup Y)$ implies that $X' \subseteq Y \cup \{x_1, ..., x_t\}$. Since $X' = C^h(X' \cup Y) \neq Y$ and Y is $\{x_1, ..., x_t\}$-unblocked, then $\exists\ x' \in X'$, $y \in Y$ with $d(y) = d(x')$ such that doctor $d(y)$ proposes y prior to x' in $\{x_1, ..., x_t\}$. However, it implies that $y >_{d(y)} x'$, which contradicts $X' \subseteq C_D(X' \cup Y)$. Therefore, Y is a stable outcome under $>_D$ and $C^h(\cdot)$.

An interesting observation of observable offer process unblocked set is as follows. Suppose Y is $\{x_1, ..., x_t\}$-unblocked for an observable offer process $\{x_1, ..., x_t\}$ for a hospital h, then it is also unblocked for any observable offer process that is a "predecessor" of $\{x_1, ..., x_t\}$:

Lemma 3. *Suppose $\{x_1, ..., x_t\}$ is an observable offer process for a hospital h, and Y is $\{x_1, ..., x_t\}$-unblocked. Then Y is also $\{x_1, ..., x_{t'}\}$-unblocked for $\forall\ 1 \leq t' \leq t$.*

Proof. If Y is not $\{x_1, ..., x_{t'}\}$-unblocked, then

[1]Because, in definition, $d(y)$ will get y', which is proposed after y by $d(y)$, and it implies that $y >_{d(y)} y'$.

[2]It is a tentatively stable outcome because we only consider the contracts in $Y \cup \{x_1, ..., x_t\}$. When more information is revealed by extending the observable offer process $\{x_1, ..., x_t\}$, Y might be unstable.

- **Case 1.** $Y \neq C^h(Y)$
  Then Y is not $\{x_1, ..., x_t\}$-unblocked either, which leads to a contradiction.

- **Case 2.** $\exists\ X' \subseteq \{x_1, ..., x_{t'}\} \cup Y$, and let $Y' = C^h(X' \cup Y)$, such that $Y' \neq Y$ and for any $y' \in Y'$, $y \in Y$ with $d(y) = d(y')$, we have either $y' = y$ or $d(y)$ proposes y' prior to y in $\{x_1, ..., x_{t'}\}$[3].
  Since $\{x_1, ..., x_t\}$ extends $\{x_1, ..., x_{t'}\}$, then $X' \subseteq \{x_1, ..., x_{t'}\} \cup Y \subseteq \{x_1, ..., x_t\} \cup Y$, and for any $y' \in Y'$, $y \in Y$ with $d(y) = d(y')$, if $d(y)$ proposes y' prior to y in $\{x_1, ..., x_{t'}\}$, then we could also say that $d(y)$ proposes y' prior to y in $\{x_1, ..., x_t\}$. Hence, Y is not $\{x_1, ..., x_t\}$-unblocked either, which leads to a contradiction.

Therefore, Y should be $\{x_1, ..., x_{t'}\}$-unblocked. □

That is, observable offer process unblocked sets are "monotone" in a sense that as an observable offer process gets longer and more information about doctors' preferences gets revealed, then less observable offer process unblocked sets could be found.

4.1.3 C.o.achievability

Suppose $\{x_1, ..., x_t\}$ is an observable offer process for a hospital h, $>_D$ is consistent with $\{x_1, ..., x_t\}$-revealed preferences, and Y is a $\{x_1, ..., x_t\}$-unblocked set. Since Y could tentatively be regarded as a stable outcome in $Y \cup \{x_1, ..., x_t\}$ under $>_D$ and $C^h(\cdot)$, if $\exists\ z \in Y$ while $d(z)$ cannot obtain a contract weakly preferred to z in $Y \cup \{x_1, ..., x_t\}$ via cumulative offer process under $>_D$ and $C^h(\cdot)$, then this cumulative offer process outcome is not doctor-optimal because $d(z)$ would prefer his contract in Y rather than his contract in this cumulative offer process outcome. I will introduce c.o.achievability to tackle this problem.

Definition 15. *Suppose $Y \subseteq X_h$ is a set of contracts. An observable offer process $\{x_1, ..., x_t\} \subseteq Y$ for h is a **complete** observable offer process for h in Y if we cannot extend this observable offer process for h by adding in more contracts from Y. That is, $\{x_1, ..., x_t, y\}$ will not be an observable offer process for h in Y for $\forall\ y \in Y$.*
In other words, a complete observable offer process for h in Y could be treated as a "maximal" sequence of contracts we got from a cumulative offer process where only contracts in Y are acceptable.

Now, we can define the c.o.achievability as follows:

Definition 16. *A choice function $C^h(\cdot)$ has a **cumulative offer unreachability (or c.o.unreachability)** in X if there exists $z \in R^h(\{x_1, ..., x_t\})$ in some observable offer process $\{x_1, ..., x_t\}$ for h, but $z \in Y$ for some Y that is $\{x_1, ..., x_t\}$-unblocked, then there does not exist any complete observable offer process $\{y_1, ..., y_s\}$ for h in $Y \cup \{x_1, ..., x_t\}$, where*

[3] Broadly speaking, if $d(y)$ proposes y' in $\{x_1, ..., x_{t'}\}$ and he has not yet proposed y in $\{x_1, ..., x_{t'}\}$, then we can also say that he proposes y' prior to y in $\{x_1, ..., x_{t'}\}$ even though $y \notin \{x_1, ..., x_{t'}\}$.

$\{y_1, ..., y_s\}$ is compatible with $\{x_1, ..., x_t\}$-revealed preferences, such that in $C^h(\{y_1, ..., y_s\})$, doctor $d(z)$ will be assigned to z or another contract z', which is proposed by $d(z)$ prior to z in $\{x_1, ..., x_t\}$.

A hospital h satisfies ***cumulative offer achievability (or c.o.achievability)*** *if there is no c.o.unreachability for h.*

To understand c.o.achievability, let us consider the following example.

Example 3. *The preference of hospital h is given by*

$$h: \qquad \{x, z\} > \{x'\} > \{x\} > \{z\} > \emptyset$$

where $d(x') = d(x) \neq d(z)$.

Consider an observable offer process $\{z, x'\}$. It will lead to a stable outcome $\{x'\}$. However, there is a c.o.unreachability. To see this, consider $z \in R^h(\{z, x'\})$ and a set of contracts $Y = \{x, z\}$. We can make the following observations:

- **$\{z, x'\}$-revealed preferences.** Since doctor $d(z)$ first proposes z in this observable offer process, then $d(z): \ z > \emptyset$. And doctor $d(x)$ first proposes x' in this observable offer process, hence we only have partial information on doctor $d(x)$'s preferences. We know that $x' >_{d(x)} x$, but we do not know whether x is acceptable or not, i.e. we have either $d(x): \ x' > x > \emptyset$ or $d(x): \ x' > \emptyset > x$.

- **$Y = \{x, z\}$ is $\{z, x'\}$-unblocked.** This is because $Y = C^h(Y)$ and for $\forall\ X' \subseteq Y \cup \{z, x'\}$, we have $Y = C^h(X' \cup Y)$. And we can see that Y is also a stable outcome in this example.

- **$d(z)$ is always unemployed in any complete observable offer process in $\{x, x', z\}$ that is compatible with $\{z, x'\}$-revealed preferences.** We could only have two possible complete observable offer processes when respecting $\{z, x'\}$-revealed preferences: $\{z, x'\}$ or $\{x', z\}$. And they both lead to the same outcome $\{x'\}$.

And this c.o.unreachability prevents the cumulative offer process outcome from being doctor-optimal since there is no way to get doctor $d(z)$ employed when respecting $\{z, x'\}$-revealed preferences and $d(z)$ will strictly prefer $Y = \{x, z\}$ to the cumulative offer process outcome $\{x'\}$.

4.1.4 "Necessity" of c.o.achievability

The following theorem shows that c.o.achievability is a "necessary" condition for cumulative offer process to be doctor-optimally stable in a sense that if a hospital h is not c.o.achievable, then there exists a doctors' preference profile such that no cumulative offer process is doctor-optimally stable.

Theorem 6. *Suppose a hospital $h \in H$ does not satisfy c.o.achievability, then there exists a doctors' preference profile such that no cumulative offer process outcome is doctor-optimally stable.*

Proof. If contracts are not c.o.achievable for hospital h, then there exists c.o.unreachability. That is, there exists a contract $z \in R^h(\{x_1, ..., x_t\})$ in some observable offer process $\{x_1, ..., x_t\}$ for h, and $z \in Y$ for some Y that is $\{x_1, ..., x_t\}$-unblocked, and there does not exist a complete observable offer process $\{y_1, ..., y_s\}$ for h in $Y \cup \{x_1, ..., x_t\}$, where $\{y_1, ..., y_s\}$ is compatible with $\{x_1, ..., x_t\}$-revealed preferences, such that in $C^h(\{y_1, ..., y_s\})$, doctor $d(z)$ is assigned to z or another contract z', which is proposed by $d(z)$ prior to z in $\{x_1, ..., x_t\}$.

Let $>_D$ be a doctors' preferences profile that satisfies the following conditions:

1. A contract x is acceptable to $d(x)$ if and only if $x \in Y \cup \{x_1, ..., x_t\}$;
2. $\forall\ x_{t_1}, x_{t_2} \in \{x_1, ..., x_t\}$ with $1 \leq t_1 < t_2 \leq t$ and $d(x_{t_1}) = d(x_{t_2})$, we have $x_{t_1} >_d x_{t_2}$ where $d := d(x_{t_1}) = d(x_{t_2})$.
3. If $x \in \{x_1, ..., x_t\}$ and $x' \in Y \backslash \{x_1, ..., x_t\}$ with $d(x) = d(x')$, then we have $x >_{d(x)} x'$.

We can see that $>_D$ is consistent with $\{x_1, ..., x_t\}$-revealed preferences. Then any complete eligible observable offer process in $Y \cup \{x_1, ..., x_t\}$ under $>_D$ and $C^h(\cdot)$ should be compatible with $\{x_1, ..., x_t\}$-revealed preferences.

If $\nexists$ a stable cumulative offer process outcome under $>_D$ and $C^h(\cdot)$, then proof is completed. Otherwise, let Y^* be an arbitrary stable cumulative offer process outcome under $>_D$ and $C^h(\cdot)$. Since Y is $\{x_1, ..., x_t\}$-unblocked, we know that Y is also stable under $>_D$ and $C^h(\cdot)$[4].

Now, consider doctor $d(z)$'s employment in the stable cumulative offer process outcome Y^*. Suppose either $z \in Y^*$ or $\exists\ z' \in Y^*$ with $d(z) = d(z')$ such that z' is proposed by $d(z)$ prior to z in $\{x_1, ..., x_t\}$. Since Y^* is the outcome of a complete observable offer process for h in $Y \cup \{x_1, ..., x_t\}$ that is compatible with $\{x_1, ..., x_t\}$-revealed preferences, it contradicts the c.o.unreachability. Therefore, $d(z)$ is assigned to a contract $z'' \neq z$ in Y^*, and z'' should not be proposed by $d(z)$ prior to z in $\{x_1, ..., x_t\}$.[5] Since $>_D$ is consistent with $\{x_1, ..., x_t\}$-revealed preferences, it implies that $z >_{d(z)} z''$. Therefore, Y^* is not a doctor-optimal stable outcome since $d(z)$ is worse-off in Y^* than in Y. □

[4]See explanation in subsection 4.1.2

[5]z'' could also be $\emptyset$, which means that $d(z)$ is unemployed in Y^*.

4.2 C.o.achievability, observable substitutability across doctors and unilateral substitutability

4.2.1 Necessity of c.o.achievability and observable substitutability across doctors.

Theorem 5 states that hospital-by-hospital observable substitutability across doctors is a "necessary" condition for cumulative offer process to be stable in a sense that if a hospital does not satisfy observable substitutability across doctors, we could find doctors' preferences and unit-demand choice function for another hospital such that no stable cumulative offer process exists. Now, Theorem 6 also provides another "necessary" condition for cumulative offer process to be doctor-optimally stable. If a hospital does not satisfy c.o.achievability, then we could find doctors' preferences such that no doctor-optimally stable cumulative offer process exists. Then hospital-by-hospital observable substitutability across doctors plus c.o.achievability is still a "necessary" condition for cumulative offer process to be doctor-optimally stable in a sense that if $|H| > 1$ and a hospital $h \in H$ does not satisfy either of these conditions, then there exist doctors' preference profiles and choice functions for other hospitals such that no cumulative offer process is doctor-optimally stable.

Corollary 1. *Suppose $|H| > 1$ and a hospital $h \in H$ does not satisfy either observable substitutability across doctors or c.o.achievability, then there exists a doctors' preference profile and some choice functions for other hospitals such that no cumulative offer process is doctor-optimally stable.*

Proof. By Theorem 5 and Theorem 6. □

4.2.2 Equivalence of unilateral substitutability to c.o.achievability combined with observable substitutability across doctors.

In Hatfield and Kojima (2010)[**7**], they proposed hospital-by-hospital unilateral substitutability which guarantees a doctor-optimal stable outcome for cumulative offer process. Actually, it turns out that unilateral substitutability is equivalent to being observable substitutable across doctors and c.o.achievable at the same time.

In Hatfield, Kominers and Westkamp (2017)[**11**], they showed that observable substitutability across doctors is a weaker condition than unilateral substitutability.[6] Then, I would like to show that unilateral substitutability also implies c.o.achievability. Finally,

[6]They showed that substitutable completability, which is first proposed in Hatfield and Kominers (2015)[**9**], implies observable substitutability (across doctors). Combined with the result in Kadam (2017)[**13**] that unilateral substitutability implies substitutable completability, it showed that observable substitutability across doctors is weaker than unilateral substitutability.

I would like to show that c.o.achievability and observable substitutability across doctors imply unilateral substitutability. Hence, it builds up the equivalence between unilateral substitutability and observable substitutability across doctors combined with c.o.achievability.

To show that unilateral substitutability implies c.o.achievability, I notice that if hospital h satisfies unilateral substitutability, once a contract z is rejected at step t in an observable offer process $\{x_1, ..., x_t\}$ for h, then it will not appear in any $\{x_1, ..., x_t\}$-unblocked set.

Lemma 4. *Suppose hospital h satisfies unilateral substitutability. Then if $z \in R^h(\{x_1, ..., x_t\}) \setminus R^h(\{x_1, ..., x_{t-1}\})$ in an observable offer process $\{x_1, ..., x_t\}$ for h, then $z \notin Y$ for $\forall\ Y$ that is $\{x_1, ..., x_t\}$-unblocked.*

Proof. See Appendix B.1. □

Now we can see that if a hospital h satisfies unilateral substitutability, then c.o.unreachability will never happen for h. This is because if $z \in R^h(\{x_1, ..., x_t\})$ in some observable offer process for h and Y is a $\{x_1, ..., x_t\}$-unblocked set, then let $t' \leq t$ be the first time z is rejected in this observable offer process, that is, $z \in R^h(\{x_1, ..., x_{t'}\}) \setminus R^h(\{x_1, ..., x_{t'-1}\})$. By Lemma 3, we know that Y is also $\{x_1, ..., x_{t'}\}$-unblocked, and by Lemma 4, we know that $z \notin Y$. Hence, c.o.unreachability will never happen if h satisfies unilateral substitutability. Therefore, c.o.achievability weakens the unilateral substitutability condition[7].

Theorem 7. *Unilateral substitutability implies c.o.achievability.*

Conversely, I can show that observable substitutability across doctors plus c.o.achievability will imply unilateral substitutability. But before I present the result, I need a preliminary result from Hatfield, Kominers and Westkamp (2017)[**11**], which shows the independence of proposing order for cumulative offer process if every hospital satisfies observable substitutability across doctors.

Lemma 5. *(Proposition 6 in Hatfield, Kominers and Westkamp (2017)[**11**]) If the choice function of every hospital is observably substitutable across doctors, then for any doctors' preference profile $>_D$ and any two proposing orders $\triangleright$, $\triangleright'$, the set of all contracts available to hospitals at the end of cumulative offer process for $\triangleright$ coincides with the set of all contracts available to hospitals at the end of cumulative offer process for $\triangleright'$.*

In other words, the cumulative offer process will always lead to the same outcome regardless of proposing orders when every hospital satisfies observable substitutability across doctors.

[7]Actually, c.o.achievability is a strictly weaker condition than unilateral substitutability. Consider the following choice function of a hospital h

$$h: \qquad \{x, z\} > \{y\} > \{x\} > \{z\} > \emptyset$$

where $d(x)$, $d(y)$ and $d(z)$ are 3 different doctors. Then h does not satisfy bilateral substitutability, not even unilateral substitutability. However, it satisfies c.o.achievability because there is no c.o.unreachability for this hospital.

Theorem 8. *Observable substitutability across doctors and c.o.achievability will imply unilateral substitutability.*

Proof. If a hospital h does not satisfy unilateral substitutability, then $\exists\ x, z$ and $Y \subseteq X$ such that $d(z) \notin d(Y)$, $z \notin C^h(Y \cup \{z\})$ but $z \in C^h(Y \cup \{x, z\})$.

Let $\bar{Y} = C^h(Y \cup \{z\})$, because $d(z) \notin d(Y)$ and $z \notin C^h(Y \cup \{z\})$, then we have $d(z) \notin d(\bar{Y})$. Therefore, $\{\bar{Y}, z\}$[8] could be treated as an observable offer process for h, and by IRC, we could conclude that $z \notin C^h(\{\bar{Y}, z\})$.

Let $Y' = C^h(Y \cup \{x, z\})$, then $z \in Y'$. By IRC, we have $Y' = C^h(Y')$. For $\forall\ X' \subseteq Y' \cup \{\bar{Y}, z\}$, we have

$$Y' \subseteq Y' \cup X' \subseteq Y \cup \{x, z\}$$

By IRC and Lemma 1, we have

$$Y' = C^h(Y' \cup X') = C^h(Y \cup \{x, z\}) = Y'$$

Hence, Y' is $\{\bar{Y}, z\}$-unblocked. Now we have $z \in R^h(\{\bar{Y}, z\})$ in this observable offer process $\{\bar{Y}, z\}$ for h, and $z \in Y'$, where Y' is $\{\bar{Y}, z\}$-unblocked.

Suppose h satisfies observable substitutability across doctors, and I would like to show that h does not satisfy c.o.achievability.

Since $z \notin C^h(\{\bar{Y}, z\})$ for an observable offer process $\{\bar{Y}, z\}$ for h, then $z \notin C^h(\{\bar{Y}, z, ..., x_t\})$ for any complete observable offer process for h that extends $\{\bar{Y}, z\}$ in $Y' \cup \{\bar{Y}, z\}$. This is because doctor $d(z)$ only has contract z in $Y' \cup \{\bar{Y}, z\}$, and if z is rejected at $\{\bar{Y}, z\}$, then it will never be renegotiated in an extending observable offer process because of observable substitutability across doctors.

Now, consider an arbitrary complete observable offer process $\{y_1, ..., y_s\} \subseteq Y' \cup \{\bar{Y}, z\}$ that is compatible with $\{\bar{Y}, z\}$-revealed preferences. By Lemma 2, we could find a doctors' preference profile $>_D$, under which $Y' \cup \{\bar{Y}, z\}$ is the only set of acceptable contracts to doctors, such that $\{\bar{Y}, z\}$ and $\{y_1, ..., y_s\}$ are both eligible observable offer processes for h in $Y' \cup \{\bar{Y}, z\}$. And if we extend $\{\bar{Y}, z\}$ in $Y' \cup \{\bar{Y}, z\}$ under $>_D$ and $C^h(\cdot)$, we could have a complete observable offer process $\{\bar{Y}, z, ..., x_t\}$ for h in $Y' \cup \{\bar{Y}, z\}$. By Lemma 5 and the fact that $z \notin C^h(\{\bar{Y}, z, ..., x_t\})$ and $d(z)$ only has contract z in $Y' \cup \{\bar{Y}, z\}$, we know that $d(z) \notin d(C^h(\{y_1, ..., y_s\}))$ either. Therefore, there does not exist any complete observable offer process $\{y_1, ..., y_s\}$ for h in $Y' \cup \{\bar{Y}, z\}$, which is compatible with $\{\bar{Y}, z\}$-revealed preferences, such that $d(z)$ will be employed in $Y' \cup \{\bar{Y}, z\}$. This is a c.o.unreachability, which means h does not satisfy c.o.achievability. □

An immediate result from Theorem 8 is that observable substitutability across doctors plus c.o.achievability for every hospital is also a sufficient condition for cumulative offer process to be doctor-optimally stable.

[8]Here, $\{\bar{Y}, z\}$ is a shorthand for $\{\bar{y_1}, ..., \bar{y_n}, z\}$, where $\{\bar{y_1}, ..., \bar{y_n}\}$ is a set of distinct contracts and $\bar{Y} = \cup_{i=1}^{n} \bar{y_i}$

Corollary 2. *Suppose every hospital satisfies observable substitutability across doctors and c.o.achievability, then cumulative offer process always produces a doctor-optimally stable outcome.*

Proof. By Theorem 3 and Theorem 8. □

4.2.3 C.o.achievability v.s. Observable substitutability across doctors.

In this subsection, I would like to show that c.o.achievability and observable substitutability across doctors are not "mutually dependent", in a sense that c.o.achievability does not imply observable substitutability across doctors and observable substitutability across doctors does not imply c.o.achievability either.

Observation 1. *C.o.achievability does not imply observable substitutability across doctors.*

To see this, let us consider the following example which is slightly different from Example 3.

Example 4. *Suppose there is a hospital h and its preference is given by:*

$$h: \qquad \{x,z\} > \{y\} > \{x\} > \{z\} > \emptyset$$

where $d(x)$, $d(y)$ and $d(z)$ are 3 different doctors.

As we can see, hospital h does not satisfy observable substitutability across doctors, because if we consider the cumulative offer process $\{z,y,x\}$, then $d(z) \notin d(C^h(\{z,y\})) = d(\{y\})$ but $z \in C^h(\{z,y,x\})$. But h satisfies c.o.achievability, because whenever x (z) is rejected from an observable offer process $\{x,y\}$ or $\{y,x\}$ ($\{z,y\}$ or $\{y,z\}$), it will be renegotiated in $Y := \{x,z\}$ when z (x) is added into this observable offer process, hence there does not exist any c.o.unreachability.

The next question is that whether observable substitutability (across doctors) implies c.o.achievability. The answer is still NO!

Observation 2. *Observable substitutability (across doctors) does not imply c.o.achievability.*

To see this, let us consider another example as follows:

Example 5. *Suppose there is a hospital h whose preference is as follows:*

$$h: \qquad \{x,y,z\} > \{y'\} > \{x,y\} > \{x,z\} > \{y,z\} > \{x\} > \{y\} > \{z\} > \emptyset$$

where $d(x)$, $d(y) = d(y')$ and $d(z)$ are 3 different doctors.

In this example, hospital h satisfies observable substitutability (across doctors) since in any cumulative offer process, either the contract will never be rejected (when $d(y)$ proposes y prior to y') or rejected contracts such as x or z will never be renegotiated (when $d(y)$ proposes y' prior to y). However, it does not satisfy c.o.achievability. To see this, let us consider an observable offer process $\{z, x, y'\}$. We have $z \notin C^h(\{z, x, y'\}) = \{y'\}$, and $z \in Y := \{x, y, z\}$ and Y is $\{z, x, y'\}$-unblocked. However, we can not find a complete observable offer process, which is compatible with $\{z, x, y'\}$-revealed preferences, such that z will be chosen. Therefore, there exists a c.o.unreachability, which implies that h does not satisfy c.o.achievability.

Thus, c.o.achievability and observable substitutability across doctors provide a "minimal" hospital-by-hospital characterization for cumulative offer process to be doctor-optimally stable, in a sense that none of them is redundant.

4.2.4 Summary

Now, we have built up the equivalence of unilateral substitutability to observable substitutability across doctors and c.o.achievability, which improves our understanding of unilateral substitutability in two ways.

- First, hospital-by-hospital unilateral substitutability is also a "necessary" condition for cumulative offer process to be doctor-optimally stable in a sense that if $|H| > 1$ and there exists a hospital that is not unilateral substitutable, then there exists a doctors' preference profile or(and) some choice functions for other hospitals such that no cumulative offer process is doctor-optimally stable. To see it, if h does not satisfy unilateral substitutability, then either it does not satisfy c.o.achievability, then by Theorem 6, there exists a doctors' preferences profile such that no cumulative offer process is doctor-optimally stable; or it does not satisfy observable substitutability across doctors, then by Theorem 5, there exists a doctors' preferences profile and unit-demand choice functions for other hospitals such that no cumulative offer process is stable.

- Second, the extension from unilateral substitutability to bilateral substitutability, or even observable substitutability across doctors needs to violate c.o.achievability (See the figure below). We could think of it as a tradeoff between "weakening substitutability" and "respecting c.o.achievability".

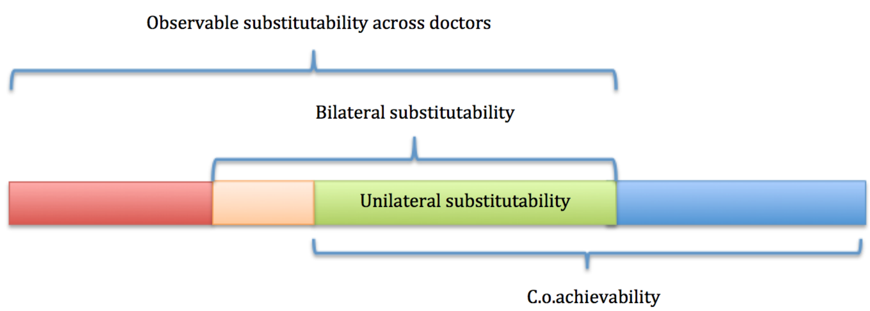

Figure 4.1: A figure that shows tradeoff between "weakening substitutability" and "respecting c.o.achievability"

Chapter 5

Joint conditions on hospital choice functions

In previous chapter, I explored the conventional hospital-by-hospital conditions such that cumulative offer process always leads to a doctor-optimally stable outcome. In this chapter, I try to further generalize the results, by looking at some joint conditions on a group of hospitals' choice functions $C^H(\cdot)$ rather than hospital-by-hospital conditions on $C^h(\cdot)$. I aim to find some sufficient and "necessary"[1] joint conditions for cumulative offer process to be doctor-optimally stable.

In this chapter, I still assume that every hospital satisfies IRC since it is the foundation of all the analysis throughout this chapter.

5.1 Independence of proposing order

In Hirata and Kasuya (2014)[**12**], they first proposed order-independence of cumulative offer process and showed that cumulative offer process is independent of proposing order if every hospital is bilaterally substitutable. In Hatfield, Kominers and Westkamp (2017)[**11**], they also showed that cumulative offer process is independent of proposing order if every hospital is observably substitutable across doctors. It turns out that the independence of proposing order is a crucial condition for cumulative offer process to be doctor-optimally stable. And it is formally defined as follows:

Definition 17. *A group of hospitals H satisfies* ***independence of proposing order*** *if for any fixed doctors' preference profile $>_D$, cumulative offer process always produces the same outcome for any two different proposing orders $\triangleright$ and $\triangleright'$.*

Since every doctor $d \in D$ has a strict preference over his contracts, if a doctor-optimally stable outcome exists, then it has to be unique. If cumulative offer process always produces

[1] Here, I need a different flavor of necessity since we can not manipulate other hospitals' choice functions anymore in a joint condition. I will explain the necessity when presenting the results in the following sections.

doctor-optimally stable outcome, then this group of hospitals H has to satisfy independence of proposing order because of the uniqueness of doctor-optimally stable outcome. Therefore, independence of proposing order could be regarded as a "necessary" condition for cumulative offer process to be doctor-optimally stable in a sense that if not, then there exist doctors' preference profile and a proposing order such that the cumulative offer process can not produce a doctor-optimally stable outcome.

Theorem 9. *Suppose a group of hospitals H does not satisfy independence of proposing order. Then there exist doctors' preference profile and a proposing order such that cumulative offer process will not produce doctor-optimally stable outcome.*

Lemma 5 shows that cumulative offer process is independent of proposing order when every hospital satisfies observable substitutability across doctors. Since observable substitutability across doctors is weaker than unilateral substitutability, we have the following corollary.

Corollary 3. *If every hospital $h \in H$ satisfies unilateral substitutability, then H satisfies independence of proposing order.*

5.2 Group observable substitutability

By Theorem 1, we can see that in order for a cumulative offer process to be stable, we need to make it feasible. Following the idea of observable substitutability across doctors, I will introduce a joint condition, group observable substitutability. To begin with, I need to define an observable offer process for a group of hospitals H as follows:

Definition 18. *Suppose $\{x_1, ..., x_T\} \subseteq X$ is a finite sequence of distinct contracts. It is said to be **an observable offer process for a group of hospitals** H if for $\forall\ 1 \leq t \leq T$, we have $d(x_t) \notin d(C^H(\{x_1, ..., x_{t-1}\}))$.*

As we can see, the definition of group observable offer process is a simple adaptation from an observable offer process for a single hospital h. The only difference is that a doctor proposes his contract at step t if he was not employed by any hospital in H at previous step. Notice that it is also the rule for a doctor to propose a new contract in the cumulative offer process, and only this kind of sequences of contracts could be observable during a cumulative offer process.

Now, we can define the group observable substitutability.

Definition 19. *A group of hospitals H satisfies **group observable substitutability** if for $\forall$ observable offer process $\{x_1, ..., x_t\}$ for H, if $z \in R^H(\{x_1, ..., x_{t-1}\}) \backslash R^H(\{x_1, ..., x_t\})$, then either $d(z) \notin d(C^H(\{x_1, ..., x_{t-1}\}))$, or if $d(z) \in d(C^h(\{x_1, ..., x_{t-1}\}))$ for some $h \in H$, then $h = h(z)$.*

In other words, when a previously-rejected contract is renegotiated with some hospital in H, then, at previous step, the doctor in this contract was either chosen by this hospital, or not chosen by any hospital in the group. The motivation for group observable substitutability is that it can guarantee the feasibility of cumulative offer process. To see it, let us take a look at the following example.

Example 6. *The preferences of h and h' are as follows:*

$$\begin{array}{ll} h: & \{x,z\} > \{y\} > \{x\} > \{z\} > \emptyset \\ h': & \{z'\} > \emptyset \end{array}$$

where $d(x)$, $d(y)$ and $d(z) = d(z')$ are 3 different doctors.

In this example, if we try a group observable offer process $\{z, y, z', x\}$, we can see that in the final step, when previously-rejected contract z is renegotiated with hospital h, its doctor has already signed another contract z' with a different hospital h' at previous step, which causes the infeasibility of the matching outcome. However, group observable substitutability can perfectly dodge this situation. As shown in the next result, group observable substitutability guarantees the feasibility of cumulative offer process, hence it also guarantees the stability by IRC.

Theorem 10. *Suppose a group of hospitals H satisfies group observable substitutability. Then cumulative offer process always produces a stable outcome.*

Proof. Consider any cumulative offer process $\{x_1, ..., x_T\}$ where T is the final step and x_t is the contract proposed at $1 \leq t \leq T$. I would like to show that $C^H(\{x_1, ..., x_t\})$ is feasible for $\forall\ 1 \leq t \leq T$. Let's prove it by induction.

- When $t = 1$, $C^H(\{x_1\})$ is always feasible, since there is only one contract.
- Suppose $C^H(\{x_1, ..., x_{t-1}\})$ is feasible. I need to show $C^H(\{x_1, ..., x_t\})$ is also feasible. If not, then there exist $z \neq z'$ with $d(z) = d(z')$ such that $z, z' \in C^H(\{x_1, ..., x_t\})$. First, we notice that $h(z) \neq h(z')$ since a hospital would not choose multiple contracts with the same doctor. Then, we notice that either $h(z) = h(x_t)$ or $h(z') = h(x_t)$. Since otherwise, $z, z' \in C^H(\{x_1, ..., x_{t-1}\})$, which contradicts the feasibility of $C^H(\{x_1, ..., x_{t-1}\})$. Without loss of generality, let $h(z) = h(x_t)$. Since $h(z') \neq h(z)$, we know that $z' \in C^H(\{x_1, ..., x_{t-1}\})$. By feasibility of $C^H(\{x_1, ..., x_{t-1}\})$, we know that $d(z) \notin d(C^{h(z)}(\{x_1, ..., x_{t-1}\}))$. And by definition of cumulative offer process, we know that $z \neq x_t$. If $z \in C^H(\{x_1, ..., x_t\})$, then it means that $z \in R^H(\{x_1, ..., x_{t-1}\}) \backslash R^H(\{x_1, ..., x_t\})$. However, $d(z)$ is employed by another hospital at previous step because $d(z) \in d(C^{h(z')}(\{x_1, ..., x_{t-1}\}))$ where $h(z') \neq h(z)$, which contradicts the group observable substitutability. Therefore, $C^H(\{x_1, ..., x_t\})$ is also feasible.

Since group observable substitutability guarantees the feasibility of cumulative offer process, then, by IRC and Theorem 1, it also guarantees the stability of cumulative offer process. □

Group observable substitutability is also a necessary condition for cumulative offer process to be stable in a sense that if not, then there exist doctors' preference profile and a proposing order such that cumulative offer process will not produce a stable outcome.

Theorem 11. *Suppose a group of hospitals H does not satisfy group observable substitutability. Then there exist doctors' preference profile and a proposing order such that cumulative offer process will not produce a stable outcome.*

Proof. Suppose $z \in R^H(\{x_1, ..., x_{t-1}\}) \backslash R^H(\{x_1, ..., x_t\})$ in some observable offer process for H and $d(z) \in d(C^h(\{x_1, ..., x_{t-1}\}))$ for some $h \neq h(z)$. Then $\exists\ z' \neq z$ with $d(z') = d(z)$ such that $z' \in C^h(\{x_1, ..., x_{t-1}\})$, and because $h(z') = h \neq h(z)$, we know that $z' \in C^H(\{x_1, ..., x_t\})$ as well. Therefore, $C^H(\{x_1, ..., x_t\})$ is infeasible because there exist $z \neq z'$ with $d(z) = d(z')$ such that $z, z' \in C^H(\{x_1, ..., x_t\})$. Let $>_D$ satisfy the following:

- A contract x is acceptable to $d(x)$ if and only if $x \in \{x_1, ..., x_t\}$.
- $\forall\ x_{t_1}, x_{t_2} \in \{x_1, ..., x_t\}$ with $d(x_{t_1}) = d(x_{t_2})$, we have $t_1 < t_2 \Leftrightarrow x_{t_1} >_{d(x_{t_1})} x_{t_2}$.

Let proposing order be $\triangleright : x_1 \triangleright x_2 \triangleright \cdots \triangleright x_t \triangleright \cdots$, i.e. $\triangleright$ is specified by $\{x_1, ..., x_t\}$. Then cumulative offer process exactly generates the outcome $C^H(\{x_1, ..., x_t\})$, which is infeasible. Hence, it is not stable either. □

In Hatfield, Kominers and Westkamp (2017)[**11**], they showed that cumulative offer process is independent of proposing order when every hospital satisfies observable substitutability across doctors. However, group observable substitutability can no longer guarantee the independence of proposing order for cumulative offer process. Let's reconsider the preferences of two hospitals h and h' in Example 1:

Example 7. *The preferences of h and h' are as follows:*

$$\begin{aligned} h: &\quad \{x, z\} > \emptyset \\ h': &\quad \{x', z'\} > \emptyset \end{aligned}$$

where $d(x) = d(x') \neq d(z) = d(z')$.

It is easy for us to verify that this group of hospitals h plus h' satisfies group observable substitutability since $d(x)$ and $d(z)$ are always bundled together in the choice set for either hospital and it is impossible for $d(x)$ or $d(z)$ to be employed by both h and h' through an observable offer process for H. However, proposing order really matters in this example. If we fix a doctors' preference profile $d(x):\ x > x' > \emptyset$ and $d(z):\ z' > z > \emptyset$. Let's consider an observable offer process $\{x, x', z'\}$, which specifies an order $\triangleright : x \triangleright x' \triangleright z' \triangleright \cdots$. It leads to an outcome $\{x', z'\}$. However, if we consider another observable offer process $\{x, z', z\}$[2],

[2]Here, $\{x, z', z\}$ is compatible with $\{x, x', z'\}$-revealed preferences, which will be defined in the following section.

which specifies a different order $\triangleright' : x \triangleright' z \triangleright' z' \triangleright' \cdots$, then it leads to a different outcome $\{x, z\}$.

Conversely, there is a very interesting observation that whenever a group of hospitals satisfies independence of proposing order, it needs to satisfy group observable substitutability.

Theorem 12. *Suppose a group of hospitals satisfies independence of proposing order, then it also satisfies group observable substitutability.*

Proof. I use a proof by contradiction. Suppose this group of hospitals H does not satisfy group observable substitutability, then $\exists\ z \in R^H(\{x_1, ..., x_{t-1}\} \backslash R^H(\{x_1, ..., x_t\}))$ in a group observable offer process such that $d(z) \in d(C^{h'}(\{x_1, ..., x_{t-1}\}))$ for a hospital $h' \neq h(z)$. Let $h(z) = h$ and without loss of generality, let t be the first time in this observable offer process that a doctor's previously-rejected contract is renegotiated with a hospital when he is currently employed with another hospital. Then $C^H(\{x_1, ..., x_{t'}\})$ is always a feasible outcome when $1 \leq t' < t$.

Let $\{y_1, ..., y_s\}$ be h's component of $\{x_1, ..., x_t\}$. That is, $\{y_1, ..., y_s\} = \{x_1, ..., x_t\} \cap X_h$ and it keeps the same order as in $\{x_1, ..., x_t\}$[3]. Then we have $x_t = y_s$. Let $\{z_1, ..., z_r\} = C^{h'}(\{x_1, ..., x_t\})$. Consider $Y = \{y_1, ..., y_s\} \cup \{z_1, ..., z_r\}$ and let $>_D$ be a doctors' preference profile that satisfies the following conditions:

- Only contracts in Y are acceptable to their associated doctors.
- $\forall\ y, y' \in \{y_1, ..., y_s\}$ with $d(y) = d(y') =: d$, if y is proposed prior to y' in $\{y_1, ..., y_s\}$, then $y >_d y'$.
- $\forall\ y \in \{y_1, ..., y_s\}$ and $y' \in \{z_1, ..., z_r\}$ with $d(y) = d(y') =: d$, $y >_d y'$.

Claim: $\{y_1, ..., y_{s-1}, z_1, ..., z_r, y_s\}$ is a valid cumulative offer process under $>_D$.

- First, I need to show $\{y_1, ..., y_{s-1}, z_1, ..., z_r, y_s\}$ is a valid group observable offer process. First $\{y_1, ..., y_{s-1}\}$ is a valid group observable offer process since it is the first $s - 1$ contracts of h's component of $\{x_1, ..., x_t\}$. Then at step $t - 1$ in $\{x_1, ..., x_t\}$, we know $C^H(\{x_1, ..., x_{t-1}\})$ is feasible. Therefore, $d(C^h(\{x_1, ..., x_{t-1}\})) \cap d(C^{h'}(\{x_1, ..., x_{t-1}\})) = \emptyset$, which implies that $d(C^h(\{y_1, ..., y_{s-1}\})) \cap d(\{z_1, ..., z_r\}) = \emptyset$. Therefore, $\{y_1, ..., y_{s-1}, z_1, ..., z_r\}$ is a valid group observable offer process. At last, since $y_s = x_t$ and $d(x_t) \notin d(C^H(\{x_1, ..., x_{t-1}\}))$, we have $d(y_s) = d(x_t) \notin d(C^h(\{y_1, ..., y_{s-1}\}) \cup \{z_1, ..., z_r\})$. Therefore, $\{y_1, ..., y_{s-1}, z_1, ..., z_r, y_s\}$ is a valid group observable offer process.
- Then, it remains to show that $\{y_1, ..., y_{s-1}, z_1, ..., z_r, y_s\}$ is consistent with $>_D$. I only need to verify that y_s proposed after $\{z_1, ..., z_r\}$ is consistent with $>_D$. This is always true since $d(y_s) = d(x_t) \notin d(\{z_1, ..., z_r\})$.

[3] That is, y is proposed prior to y' in $\{y_1, ..., y_s\}$ if and only if y is proposed prior to y' in $\{x_1, ..., x_t\}$.

Now, we know that $z \in C^h(\{x_1, ..., x_t\}) = C^h(\{y_1, ..., y_s\})$ and $\exists\ z' \in C^{h'}(\{x_1, ..., x_t\}) = \{z_1, ..., z_r\}$. Therefore, $d(z)$ has two contracts z and z' in this cumulative offer process outcome under a proposing order specified by $y_1 \rhd \cdots \rhd y_{s-1} \rhd z_1 \rhd \cdots \rhd z_r \rhd y_s$.

However, if we consider another proposing order $y_1 \rhd \cdots \rhd y_s \rhd z_1 \rhd \cdots \rhd z_r$ under the same doctors' preferences $>_D$. Then the first s steps generates a group observable offer process $\{y_1, ..., y_s\}$ and $z \in C^h(\{y_1, ..., y_s\})$. Then we could conclude that z is the only contract $d(z)$ will obtain in the end, because from now on, $d(z)$ stays employed with hospital h and he will never propose any new contract in the subsequent cumulative offer process, then h' won't receive any contract proposed by $d(z)$. Therefore, it leads to a different cumulative offer process outcome since $d(z)$ only receive a single contract z in this outcome.

Therefore, proposing order of cumulative offer process really matters under $>_D$ and it violates the independence of proposing order, which leads to a contradiction. □

Since independence of proposing order implies group observable substitutability, an immediate result follows:

Corollary 4. *Suppose a group of hospitals satisfies independence of proposing order, then cumulative offer process always produces stable outcome.*

Proof. By Theorem 10 and 12. □

Therefore, independence of proposing order is also a sufficient condition for cumulative offer process to be stable. Actually, it improves our understanding of stability of cumulative offer process. In current literature, hospital-by-hospital observable substitutability across doctors is the weakest sufficient condition for cumulative offer process to be stable[4]. Now, we know that as long as this group of hospitals is independent of proposing order, it guarantees the stability of cumulative offer process. Then hospital-by-hospital observable substitutability across doctors is actually a special case of that. Furthermore, cumulative offer process could be stable even if a group of hospitals violates the independence of proposing order. By Theorem 10 and 12, group observable substitutability is by far the weakest sufficient condition for cumulative offer process to be stable and it is also "necessary" in a sense that if not, then there exists a doctors' preference profile and a proposing order such that cumulative offer process is not stable. The figure below summarizes the conclusions above.

5.3 Group cumulative offer achievability (Group c.o.achievability)

In this section, I will introduce another joint condition, group cumulative offer achievability (group c.o.achievability), which turns out to be a crucial necessary condition for cumulative

[4]See more in Hatfield, Kominers and Westkamp (2017)[**11**]

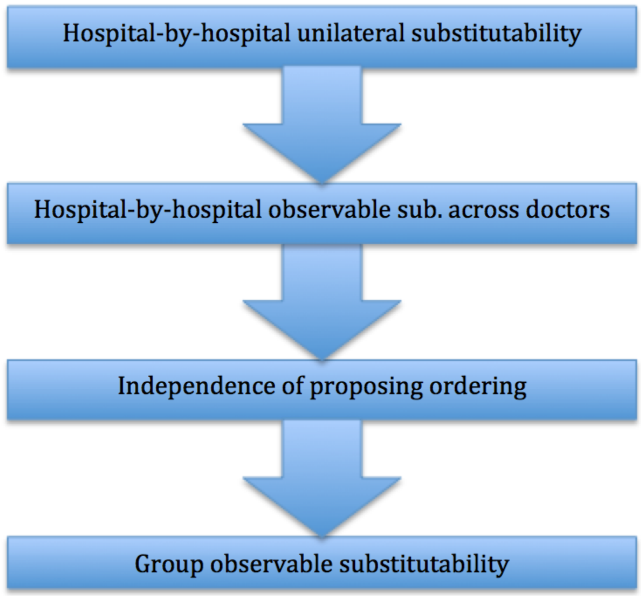

Figure 5.1: A figure showing the relationship among sufficient conditions that guarantee the stability of cumulative offer process.

offer process to be doctor-optimally stable in a sense that if not, then there exists doctors' preference profile such that no cumulative offer process is doctor-optimally stable. The idea is very similar to the hospital-by-hospital c.o.achievability in previous chapter, but everything is based on the group observable offer process. And I will also discuss the difference between group c.o.achievability and hospital-by-hospital c.o.achievability at the end of this section.

5.3.1 Group c.o.achievability and its "necessity"

First, I need to define how a doctors' preference profile $>_D$ is consistent with an observable offer process for a group of hospitals H.

Definition 20. *Suppose $\{x_1, ..., x_T\}$ is an observable offer process for a group of hospitals H. A doctors' preference profile $>_D$ is consistent with $\{x_1, ..., x_T\}$-revealed preference if it satisfies the following:*

- *Any contract $x \in \{x_1, ..., x_T\}$ is acceptable to its doctor $d(x)$.*
- *$\forall\ x_t, x_{t'} \in \{x_1, ..., x_T\}$ with $d(x_t) = d(x_{t'})$ and $t < t'$, we have $x_t >_{d(x_t)} x_{t'}$.*
- *$\forall\ x \notin \{x_1, ..., x_T\}$, if $\exists\ x' \in \{x_1, ..., x_T\}$ with $d(x) = d(x')$, then $x' >_{d(x)} x$.*

And the compatibility of two observable offer processes for a group of hospitals H is just a simple adaptation of compatibility of two observable offer processes for a single hospital h.

Definition 21. *Suppose we have an observable offer process $\{x_1, ..., x_T\}$ for H. An observable offer process $\{y_1, ..., y_S\}$ for H is **compatible with** $\{x_1, ..., x_T\}$**-revealed preferences** if for any $y \in \{y_1, ..., y_S\}$, let $d = d(y)$,*

- *if $y \in \{x_1, ..., x_T\}$, let*

$$S = \{x \in \{x_1, ..., x_T\} \cap X_d : x \text{ is proposed prior to } y \text{ in } \{x_1, ..., x_T\}\}$$

  . If $S \neq \emptyset$, then $S \subseteq \{y_1, ..., y_S\}$, and $\forall\ x \in S$, x is proposed prior to y in $\{y_1, ..., y_S\}$ as well.

- *if $y \notin \{x_1, ..., x_T\}$, let $S = \{x_1, ..., x_T\} \cap X_d$. If $S \neq \emptyset$, then $S \subseteq \{y_1, ..., y_S\}$, and $\forall$ $x \in S$, x is proposed prior to y in $\{y_1, ..., y_S\}$.*

Similarly, group observable offer process unblocked set of contracts is defined as follows:

Definition 22. *Suppose $\{x_1, ..., x_t\}$ is an observable offer process for a group of hospitals H. If a set of contracts Y is $\{x_1, ..., x_t\}$**-unblocked**, then it has to satisfy the following conditions:*

- *Y is a feasible outcome, and $Y = C^H(Y)$.*
- *For any $h \in H$, $\forall\ X' \subseteq Y \cup \{x_1, ..., x_t\}$, let $Y' = C^h(X' \cup Y)$, then*
  - *either $Y' = C^h(Y)$*
  - *or $\exists\ y' \in Y'$, $y \in Y$ with $d(y) = d(y')$ such that $d(y)$ proposes y prior to y' in $\{x_1, ..., x_t\}$.*

Suppose $>_D$ is consistent with $\{x_1, ..., x_t\}$-revealed preferences and we only consider the contracts in $Y \cup \{x_1, ..., x_t\}$. Actually, the second condition says that when we add some contracts into Y, then for any $h \in H$, either $C^h(Y)$ is still favored by hospital h, or if it is not, then some doctor needs to compromise by getting a less preferred contract according to $>_D$[5].

Therefore, if Y is $\{x_1, ..., x_t\}$-unblocked, then there is no way to find an alternative set of contracts in $Y \cup \{x_1, ..., x_t\}$ that could be strictly preferred by some hospital $h \in H$ without harming doctors' preferences. There is always a "tradeoff" between doctors' preferences and hospitals' preferences. Then Y could tentatively[6] be treated as a stable outcome in $Y \cup \{x_1, ..., x_t\}$ under $C^H(\cdot)$ and $>_D$. To see this, we can use a proof by contradiction.

[5]Because, in this definition, $d(y)$ will get y', which is proposed after y by him, and it implies that $y >_{d(y)} y'$.

[6]Here, we only consider the contracts in $Y \cup \{x_1, ..., x_t\}$. Perhaps, Y could be unstable when more information is revealed by extending this group observable offer process $\{x_1, ..., x_t\}$.

Suppose Y is not stable under $>_D$ and $C^H(\cdot)$. First, we know Y is feasible. Since $Y = C^H(Y)$ and Y is acceptable to doctors, it satisfies the individual rationality. Therefore, $\exists h \in H$ and $X' \neq C^h(Y)$ such that $X' = C^h(X' \cup Y)$ and $X' \subseteq C_D(X' \cup Y)$. $X' \subseteq C_D(X' \cup Y)$ implies that $X' \subseteq Y \cup \{x_1, ..., x_t\}$. Since $X' = C^h(X' \cup Y) \neq C^h(Y)$ and Y is $\{x_1, ..., x_t\}$-unblocked, then $\exists\ x' \in X'$, $y \in Y$ with $d(y) = d(x')$ such that doctor $d(y)$ proposes y prior to x' in $\{x_1, ..., x_t\}$. However, it implies that $y >_{d(y)} x'$, which contradicts $X' \subseteq C_D(X' \cup Y)$. Therefore, Y is a stable outcome under $>_D$ and $C^H(\cdot)$.

Now, since Y could tentatively be regarded as a stable outcome in $Y \cup \{x_1, ..., x_t\}$ under $>_D$ and $C^H(\cdot)$, then if $\exists\ z \in Y$ while $d(z)$ cannot obtain a contract weakly better than z in $Y \cup \{x_1, ..., x_t\}$ via cumulative offer process under $>_D$ and $C^H(\cdot)$, then this cumulative offer process outcome is not doctor-optimal because $d(z)$ would prefer his contract in Y rather than his contract in cumulative offer process outcome, and group c.o.achievability is to tackle this problem.

Definition 23. *Suppose $Y \subseteq X$ is a set of contracts. An observable offer process $\{x_1, ..., x_t\}$ for H is a* ***complete*** *observable offer process for H in Y if we cannot extend this observable offer process for H by adding in more contracts from Y. That is, $\{x_1, ..., x_t, y\}$ will not be an observable offer process for H in Y for $\forall\ y \in Y$.*
In other words, a complete observable offer process for H in Y could be treated as the "maximal" sequence of contracts we got from a cumulative offer process where only contracts in Y are acceptable.

Now, I would like to define the joint condition, group c.o.achievability.

Definition 24. *There is a* ***group cumulative offer unreachability (group c.o.unreachability)*** *for H if there exists $z \in R^H(\{x_1, ..., x_t\})$ in some observable offer process for H, and $z \in Y$ for some Y that is $\{x_1, ..., x_t\}$-unblocked, and there does not exist a complete observable offer process $\{y_1, ..., y_s\}$ for H in $Y \cup \{x_1, ..., x_t\}$, which is compatible with $\{x_1, ..., x_t\}$-revealed preferences, such that in $C^H(\{y_1, ..., y_s\})$, doctor $d(z)$ is assigned to z or another contract z', which is proposed by $d(z)$ prior to z in $\{x_1, ..., x_t\}$.*

A group of hospitals H satisfies ***group cumulative offer achievability (group c.o.achievability)*** *if there is no group c.o.unreachability for H.*

I would like to use the following example to illustrate group c.o.achievability.

Example 8. *Suppose there are 2 hospitals h and h' and their preferences are as follows:*

$$h: \quad \{x, z\} > \{y\} > \{x\} > \{z\} > \emptyset$$
$$h': \quad \{y'\} > \{x'\} > \emptyset$$

where $d(x) = d(x') = d_x$, $d(y) = d(y') = d_y$ and $d(z) = d_z$ are 3 different doctors.

Consider an observable offer process $\{z, y, x'\}$ for H. It leads to a stable outcome $\{h : \{y\}; h' : \{x'\}\}$. However, there is a group c.o.unreachability in this example. To see it, let's consider $z \in R^H(\{z, y, x'\})$ and a set of contracts $Y = \{x, y', z\}$, and make the following observations:

- **$\{z, y, x'\}$-revealed preferences.** Since doctor d_x proposes x' prior to x, we know that $x' >_{d_x} x$, but we do not know whether x is acceptable to d_x or not. Similarly, we have $y >_{d_y} y'$ and we do not know whether y' is acceptable to d_y or not. And the preference of d_z is $z > \emptyset$.

- **Y is $\{z, y, x'\}$-unblocked.** First, we can see that Y is feasible and $Y = C^H(Y)$. And for any $X' \subseteq Y \cup \{z, y, x'\}$, we have $C^i(Y \cup X') = C^i(Y)$ for $i \in \{h, h'\}$. Therefore, Y is $\{z, y, x'\}$-unblocked. And we can also conclude that Y is stable[7].

- **d_z is always unemployed while respecting $\{z, y, x'\}$-revealed preferences.** We can try all possible complete observable offer processes in $Y \cup \{z, y, x'\}$ which is compatible with $\{z, y, x'\}$-revealed preferences. d_z will remain unemployed since d_x always proposes x' first and he will be matched with h', and h always prefers $\{y\}$ to $\{z\}$, which results in the unemployment of d_z.

Therefore, z is always unemployed while respecting $\{z, y, x'\}$-revealed preferences and this forms the group c.o.unreachability. Because of the existence of this group c.o.unreachability, the outcome of the cumulative offer process $\{h : \{y\}; h' : \{x'\}\}$ is not doctor-optimally stable since Y is strictly preferred by d_z if $>_D$ is consistent with $\{z, y, x'\}$-revealed preferences.

Actually, this example also shows the difference between group c.o.achievability and hospital-by-hospital c.o.achievability. It is easy to verify that both h and h' are c.o.achievable[8], but they are not group c.o.achievable.[9]

The following theorem shows the "necessity" of group c.o.achievability for cumulative offer process to be doctor-optimally stable in a sense that if not, then there exists a doctors' preference profile such that no cumulative offer process is doctor-optimally stable.

Theorem 13. *Suppose a group of hospitals H does not satisfy group c.o.achievability. Then there exists a doctors' preference profile $>_D$ such that no cumulative offer process is doctor-optimally stable.*

Proof. If H does not satisfy group c.o.achievability, then there exists group c.o.unreachability. That is, there exists a contract $z \in R^H(\{x_1, ..., x_t\})$ in some observable offer process $\{x_1, ..., x_t\}$ for H, and $z \in Y$ for some Y that is $\{x_1, ..., x_t\}$-unblocked, and there does not exist a complete observable offer process $\{y_1, ..., y_s\}$ for H in $Y \cup \{x_1, ..., x_t\}$, where $\{y_1, ..., y_s\}$ is

[7] See the explanation right after the definition of $\{x_1, ..., x_t\}$-unblocked set Y.

[8] Actually, h's preference is the same as that in Example 4.

[9] Later in this section, I will show that group c.o.achievability implies that every hospital in this group should be c.o.achievable. Hence, this example shows that group c.o.achievable is a strictly stronger condition than hospital-by-hospital c.o.achievability.

compatible with $\{x_1, ..., x_t\}$-revealed preferences, such that doctor $d(z)$ is assigned to z or another contract z', which is proposed by $d(z)$ prior to z in $\{x_1, ..., x_t\}$, in $C^H(\{y_1, ..., y_s\})$.

Let $>_D$ be a doctors' preferences profile which satisfies the following conditions:

1. A contract x is acceptable to $d(x)$ if and only if $x \in Y \cup \{x_1, ..., x_t\}$;
2. $\forall\ x_{t_1}, x_{t_2} \in \{x_1, ..., x_t\}$ with $1 \leq t_1 < t_2 \leq t$ and $d(x_{t_1}) = d(x_{t_2})$, we have $x_{t_1} >_d x_{t_2}$ where $d = d(x_{t_1}) = d(x_{t_2})$.
3. If $x \in \{x_1, ..., x_t\}$ and $x' \in Y \backslash \{x_1, ..., x_t\}$ with $d(x) = d(x')$, then we have $x >_{d(x)} x'$.

Then $>_D$ is consistent with $\{x_1, ..., x_t\}$-revealed preferences. Thus, any complete observable offer process in $Y \cup \{x_1, ..., x_t\}$ under $>_D$ and $C^H(\cdot)$ should be compatible with $\{x_1, ..., x_t\}$-revealed preferences.

If $\nexists$ a stable cumulative offer process outcome under $>_D$ and $C^H(\cdot)$, then proof is completed. Otherwise, let Y^* be a stable cumulative offer process outcome under $>_D$ and $C^H(\cdot)$. Since Y is $\{x_1, ..., x_t\}$-unblocked, we know that Y is also stable under $>_D$ and $C^H(\cdot)$[10].

Now, consider doctor $d(z)$'s employment in the stable cumulative offer process outcome Y^*. Suppose either $z \in Y^*$ or $\exists\ z' \in Y^*$ where $d(z) = d(z')$ and z' is proposed by $d(z)$ prior to z in $\{x_1, ..., x_t\}$. Since Y^* is the outcome of a complete observable offer process for H in $Y \cup \{x_1, ..., x_t\}$ that is compatible with $\{x_1, ..., x_t\}$-revealed preferences, it contradicts the group c.o.unreachability. Therefore, $d(z)$ is assigned to a contract $z'' \neq z$ in Y^*, and z'' should be proposed by $d(z)$ after z in $\{x_1, ..., x_t\}$.[11] Since $>_D$ is consistent with $\{x_1, ..., x_t\}$-revealed preferences, it implies that $z >_{d(z)} z''$. Therefore, Y^* is not a doctor-optimal stable outcome since $d(z)$ is worse-off in Y^* than in Y. □

5.3.2 Group c.o.achievability v.s. hospital-by-hospital unilateral substitutability

In this subsection, I would like to explore the relationship between group c.o.achievability and hospital-by-hospital unilateral substitutability. First, I notice a similar "monotonicity" for group observable offer process. Suppose Y is $\{x_1, ..., x_t\}$-unblocked for a group observable offer process $\{x_1, ..., x_t\}$, then it is also unblocked by its "predecessor".

Lemma 6. *Suppose $\{x_1, ..., x_t\}$ is an observable offer process for a group of hospitals H and Y is $\{x_1, ..., x_t\}$-unblocked. Then for any $1 \leq t' \leq t$, Y is also $\{x_1, ..., x_{t'}\}$-unblocked.*

Proof. Since Y is $\{x_1, ..., x_t\}$-unblocked, then Y is feasible and $Y = C^H(Y)$. If Y is not $\{x_1, ..., x_{t'}\}$-unblocked, then $\exists\ h \in H$, $X' \subseteq Y \cup \{x_1, ..., x_{t'}\}$ and let $Y' = C^h(X' \cup Y)$, such that $Y' \neq C^h(Y)$, and $\forall\ y' \in Y$, $y \in Y$ with $d(y) = d(y')$, we have either $y' = y$ or $d(y)$

[10]See explanation right after the definition of group observable offer process unblocked set Y.

[11]z'' could also be $\emptyset$, which means that $d(z)$ is unemployed in Y^*.

proposes y' prior to y in $\{x_1, ..., x_{t'}\}$[12]. Since $t' \leq t$ and $\{x_1, ..., x_t\}$ is actually extending $\{x_1, ..., x_{t'}\}$, then we also have $X' \subseteq Y \cup \{x_1, ..., x_t\}$ with $Y' = C^h(X' \cup Y)$, such that $Y' \neq C^h(Y)$ and $\forall\ y' \in Y$, $y \in Y$ with $d(y) = d(y')$, we have either $y' = y$ or $d(y)$ proposes y' prior to y in $\{x_1, ..., x_t\}$. It contradicts the fact that Y is $\{x_1, ..., x_t\}$-unblocked. □

Then, it also holds that once a contract z is rejected in a group observable offer process $\{x_1, ..., x_t\}$, then z will not appear in any $\{x_1, ..., x_t\}$-unblocked set.

Lemma 7. *Suppose every hospital satisfies unilateral substitutability. If $z \in R^H(\{x_1, ..., x_t\}) \setminus R^H(\{x_1, ..., x_{t-1}\})$ in some observable offer process $\{x_1, ..., x_t\}$ for a group of hospitals H, then $z \notin Y$ for any Y that is $\{x_1, ..., x_t\}$-unblocked.*

Proof. See Appendix B.2. □

Therefore, hospital-by-hospital unilateral substitutability ensures that group c.o.unreachability will never happen, as shown in the following result.

Theorem 14. *If every hospital $h \in H$ satisfies unilateral substitutability, then H satisfies group c.o.achievability.*

Proof. Suppose every hospital $h \in H$ satisfies unilateral substitutability. First, I would like to show that if $z \in R^H(\{x_1, ..., x_t\})$ in some observable offer process for H, then $z \notin Y$ for any Y that is $\{x_1, ..., x_t\}$-unblocked. Suppose z is first rejected at step $t' \leq t$ in this observable offer process, i.e. $z \in R^H(\{x_1, ..., x_{t'}\}) \setminus R^H(\{x_1, ..., x_{t'-1}\})$. Since Y is $\{x_1, ..., x_t\}$-unblocked, by Lemma 6, we know that Y is also $\{x_1, ..., x_{t'}\}$-unblocked. If every hospital is unilateral substitutable, by Lemma 7, $z \notin Y$.

Now, since $z \notin Y$ for any Y that is $\{x_1, ..., x_t\}$-unblocked, the group c.o.unreachability will never happen, which means H satisfies group c.o.achievability. □

5.3.3 Group c.o.achievability v.s. hospital-by-hospital c.o.achievability

In this subsection, I will explore the relationship between group c.o.achievability and hospital-by-hospital c.o.achievability proposed in section 4.1. Example 8 illustrates that it is possible to violate group c.o.achievability even if every hospital is c.o.achievable. The following result shows that group c.o.achievability actually implies that every hospital in this group should be c.o.achievable, hence group c.o.achievability is strictly stronger than hospital-by-hospital c.o.achievability.

Theorem 15. *Suppose a group of hospitals H satisfies group c.o.achievability. Then every hospital $h \in H$ satisfies c.o.achievability.*

[12]Broadly speaking, we can also say $d(y)$ proposes y' prior to y in $\{x_1, ..., x_{t'}\}$ if he has not yet proposed y in $\{x_1, ..., x_{t'}\}$.

Proof. I will use a proof by contradiction. Suppose $\exists\, h \in H$ does not satisfy c.o.achievability. Then there exists $z \in R^h(\{x_1, ..., x_t\})$ in some observable offer process $\{x_1, ..., x_t\}$ for h and $z \in Y$ for some Y that is $\{x_1, ..., x_t\}$-unblocked, and there does not exist a complete observable offer process $\{y_1, ..., y_s\}$ for h in $Y \cup \{x_1, ..., x_t\}$, where $\{y_1, ..., y_s\}$ is compatible with $\{x_1, ..., x_t\}$-revealed preferences, such that in $C^h(\{y_1, ..., y_s\})$, $d(z)$ will be assigned to z or another contract z', which is proposed by $d(z)$ prior to z in $\{x_1, ..., x_t\}$.

First of all, $\{x_1, ..., x_t\}$ could also be regarded as an observable offer process for H. And I would like to show that Y is still $\{x_1, ..., x_t\}$-unblocked when it is regarded as a group observable offer process unblocked set. First, since Y is $\{x_1, ..., x_t\}$-unblocked when $\{x_1, ..., x_t\}$ is regarded as an observable offer process for h, then we have $Y = C^h(Y)$ and $Y \subseteq X_h$, and those imply that Y is feasible and $Y = C^H(Y)$. Second, consider any $X' \subseteq Y \cup \{x_1, ..., x_t\} \subseteq X_h$. Then for any hospital $h' \neq h$, we have $Y' := C^{h'}(X' \cup Y) = \emptyset = C^{h'}(Y)$. For hospital h, let $Y' := C^h(X' \cup Y)$, since Y is $\{x_1, ..., x_t\}$-unblocked when $\{x_1, ..., x_t\}$ is regarded as an observable offer process for h, we have either $Y' = Y = C^h(Y)$ or $\exists\, y' \in Y'$, $y \in Y$ with $d(y) = d(y')$ such that doctor $d(y)$ proposes y prior to y' in $\{x_1, ..., x_t\}$. By definition, Y is also $\{x_1, ..., x_t\}$-unblocked when $\{x_1, ..., x_t\}$ is regarded as an observable offer process for H. Now, consider any complete observable offer process $\{y_1, ..., y_s\}$ for H in $Y \cup \{x_1, ..., x_t\}$, which is compatible with $\{x_1, ..., x_t\}$-revealed preferences. Since $Y \cup \{x_1, ..., x_t\} \subseteq X_h$, $\{y_1, ..., y_s\}$ could also be treated as a complete observable offer process for h in $Y \cup \{x_1, ..., x_t\}$, which is compatible with $\{x_1, ..., x_t\}$-revealed preferences. However, by c.o.unreachability in h, we know that in $C^h(\{y_1, ..., y_s\}) = C^H(\{y_1, ..., y_s\})$[13], $d(z)$ will not be assigned to z or another contract z', which is proposed by $d(z)$ prior to z in $\{x_1, ..., x_t\}$, and this indicates a group c.o.unreachability, which contradicts group c.o.achievability for H. □

This theorem along with Theorem 14 in previous subsection show that hospital-by-hospital unilateral substitutability is stronger than group c.o.achievability and group c.o.achievability is stronger than hospital-by-hospital c.o.achievability. And Figure 5.2 summarizes their relationship.

5.4 A sufficient and necessary joint condition for cumulative offer process to be doctor-optimally stable

5.4.1 A sufficient and necessary joint condition

In previous sections, I introduced two joint conditions: independence of proposing order and group c.o.achievability, and showed that each one of them is a necessary condition for

[13]This is because $\{y_1, ..., y_s\} \subseteq Y \cup \{x_1, ..., x_t\} \subseteq X_h$

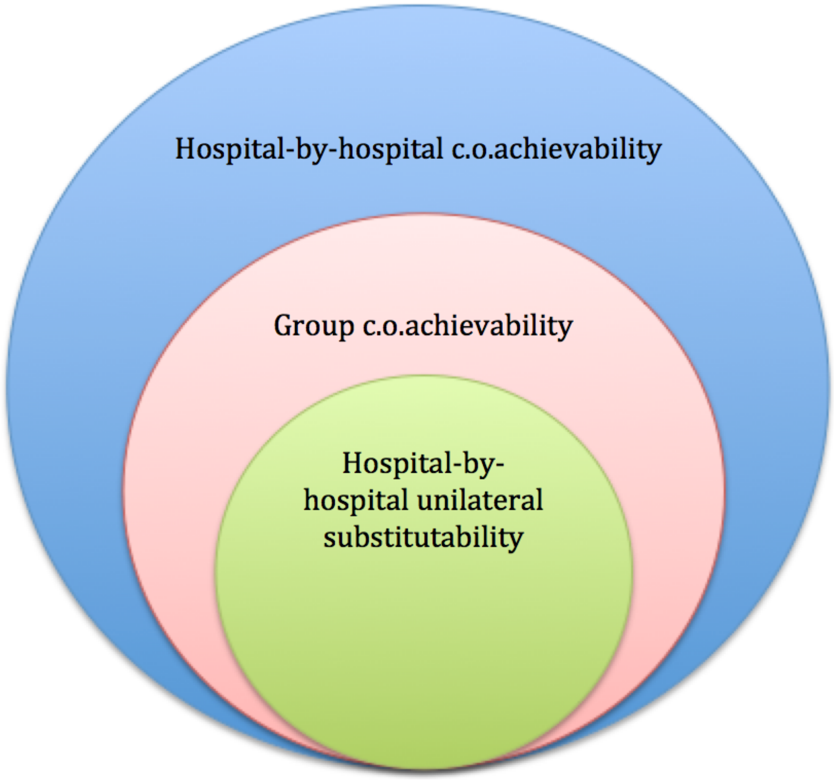

Figure 5.2: A figure showing the relationship among hospital-by-hospital unilateral substitutability, group c.o.achievability and hospital-by-hospital c.o.achievability.

cumulative offer process to be doctor-optimally stable in a sense that if not, then there exist doctors' preference profile and a proposing order such that cumulative offer process will not produce a doctor-optimally stable outcome.

It turns out that, when group c.o.achievability is combined with independence of proposing order, it provides a sufficient condition for cumulative offer process to be doctor-optimally stable. Then by Theorem 9 and 13, we know that this combined joint condition characterizes a sufficient and necessary condition for cumulative offer process to be doctor-optimally stable.

Theorem 16. *Suppose a group of hospitals H satisfies independence of proposing order and group c.o.achievability. Then cumulative offer process always produces doctor-optimally stable outcome.*

Proof. By Corollary 4, independence of proposing order guarantees the stability of cumulative offer process outcome. Now, it remains to show that cumulative offer process outcome is also doctor-optimal.

For any fixed $>_D$, let X^* be the outcome generated by a cumulative offer process $\{x_1, ..., x_T\}$ under $>_D$ and $C^H(\cdot)$. Suppose X^* is not doctor-optimal, then there exists $z^* \in X^*$, $z \in Y$, where Y is a stable outcome and $d(z) = d(z^*)$, such that $z >_{d(z)} z^*$. Therefore, $d(z)$ proposes z^* after z in $\{x_1, ..., x_T\}$ and we have $z \in R^H(\{x_1, ..., x_T\})$.
Claim 1: Y is $\{x_1, ..., x_T\}$-unblocked.
First of all, Y is a stable outcome, hence it is feasible and $Y = C^H(Y)$. If Y is not $\{x_1, ..., x_T\}$-unblocked, then $\exists\ h \in H$ and $X' \subseteq Y \cup \{x_1, ..., x_T\}$ with $Y' = C^h(X' \cup Y)$ such that $Y' \neq C^h(Y)$ and $\forall\ y' \in Y', y \in Y$ with $d(y) = d(y')$, either $y' = y$ or doctor $d(y)$ proposes y' prior to y in $\{x_1, ..., x_T\}$. Since $\{x_1, ..., x_T\}$ is a cumulative offer process under $>_D$, then we have $Y' \subseteq C_D(Y' \cup Y)$. By IRC, we also have $Y' = C^h(Y' \cup Y)$. However, $Y' \neq C^h(Y)$, and it contradicts the stability of Y. Therefore, Y is $\{x_1, ..., x_T\}$-unblocked.

Claim 2: Suppose $\{y_1, ..., y_S\}$ is a complete observable offer process for H in $Y \cup \{x_1, ..., x_T\}$, which is compatible with $\{x_1, ..., x_T\}$-revealed preferences. Then $d(z)$ is assigned to z^* in $C^H(\{y_1, ..., y_S\})$.
First of all, $\{x_1, ..., x_T\}$ is a complete observable offer process in $Y \cup \{x_1, .., x_T\}$ because otherwise, it contradicts the fact $\{x_1, ..., x_T\}$ represents an entire cumulative offer process under $>_D$[14]. If $\{y_1, ..., y_S\}$ is a complete observable offer process in $Y \cup \{x_1, ..., x_T\}$, which is compatible with $\{x_1, ..., x_T\}$-revealed preferences, then there exists $>'_D$, under which $Y \cup \{x_1, ..., x_T\}$ is the only set of acceptable contracts to doctors, such that both $\{x_1, ..., x_T\}$ and $\{y_1, ..., y_S\}$ are two eligible complete observable offer processes in $Y \cup \{x_1, ..., x_T\}$.[15] By independence of proposing order, we have $X^* = C^H(\{x_1, ..., x_T\}) = C^H(\{y_1, ..., y_S\})$, which implies that $z^* \in C^H(\{y_1, ..., y_S\})$.

Now, we have $z \in R^H(\{x_1, ..., x_T\})$ in some observable offer process for H, and by Claim 1, $z \in Y$ for some Y that is $\{x_1, ..., x_T\}$-unblocked, and by Claim 2, in any complete observable offer process $\{y_1, ..., y_S\}$ for H in $Y \cup \{x_1, ..., x_T\}$, which is compatible with $\{x_1, ..., x_T\}$-revealed preferences, doctor $d(z)$ will be assigned to z^*, which is proposed after z by $d(z)$ in $\{x_1, ..., x_T\}$. This is group c.o.unreachability, which contradicts group c.o.achievability. □

In previous chapter, I showed that hospital-by-hospital unilateral substitutability is a sufficient and "necessary" hospital-by-hospital condition for cumulative offer process to be doctor-optimally stable. Now, I find a sufficient and "necessary" joint condition for cumulative offer process to be doctor-optimally stable.[16] Corollary 3 and Theorem 14 show that hospital-by-hospital unilateral substitutability implies this combined joint condition, inde-

[14]Suppose we could add a new contract z into $\{x_1, ..., x_T\}$ such that $\{x_1, ..., x_T, z\}$ is also a group observable offer process in $Y \cup \{x_1, ..., x_T\}$, then $z \in Y \cup \{x_1, ..., x_T\}$ and it has to be proposed by some unemployed doctor $d(z)$. However, $\{x_1, ..., x_T\}$ represents an entire cumulative offer process under $>_D$, which means z is actually an unacceptable contract to $d(z)$ under $>_D$. Since Y is a stable outcome under $>_D$, then $Y \cup \{x_1, ..., x_T\}$ is a set of acceptable contracts to doctors, which implies $z \notin Y \cup \{x_1, ..., x_T\}$. It leads to a contradiction.

[15]To explain this, an adaptation of Definition 13 and Lemma 2 is needed. See Appendix B.3.

[16]However, it has a different flavor of "necessity".

pendence of proposing order plus group c.o.achievability. And the following example shows that this is actually a strict extension.

Example 9. *Suppose there are two hospitals h and h' and their preferences are as follows:*

$$h: \quad \{x,z\} > \{y\} > \{x\} > \{z\} > \emptyset$$
$$h': \quad \{y'\} > \emptyset$$

where $d(x)$, $d(y) = d(y')$ and $d(z)$ are three different doctors.

In this example, cumulative offer process always produces doctor-optimally stable outcome. We can see that h does not satisfy unilateral substitutability since z is rejected in $C^h(\{y,z\})$ but it could be renegotiated in $C^h(\{x,y,z\})$. However, if we could try all possible doctors' preference profiles and cumulative offer process will always lead to the same result when $>_D$ is fixed, hence it is independent of proposing order. And, when x (or z) is rejected in a group observable offer process $\{x,y\}$ (or $\{z,y\}$), then it can always be renegotiated by letting a new doctor $d(z)$ (or $d(x)$) propose his contract and it is also respecting $\{x,y\}$-revealed (or $\{z,y\}$-revealed) preferences. Hence, it satisfies group c.o.achievability.

So far, we have found the weakest sufficient condition that guarantees the doctor-optimal stability of cumulative offer process. And it is also a necessary condition in a sense that if not, then there exists a doctors' preference profile and a proposing order such that the cumulative offer process will not produce a doctor-optimally stable outcome.

5.4.2 A "minimal" characterization.

In this subsection, I want to show that both of the two joint conditions are very important for cumulative offer process to be doctor-optimally stable, and they do not imply each other, hence they provide a "minimal" characterization of a sufficient and necessary condition for cumulative offer process to be doctor-optimally stable.

Observation 3. *Group c.o.achievability does not imply independence of proposing order.*

Example 10. *(Example 7 revisited) The preferences of h and h' are as follows:*

$$h: \quad \{x,z\} > \emptyset$$
$$h': \quad \{x',z'\} > \emptyset$$

where $d(x) = d(x') \neq d(z) = d(z')$.

Here, I want to show H is group c.o.achievable. Without loss of generality[17], let us consider an observable offer process $\{x,z',z\}$ for H. We have $z' \in R^H(\{x,z',z\})$ and $z' \in \{x',z'\}$ which is $\{x,z',z\}$-unblocked. And if we consider another observable offer process $\{x,z',x'\}$,

[17] Because of symmetry in this example.

then we can see that this new observable offer process $\{x, z', x'\}$ is compatible with $\{x, z', z\}$-revealed preferences, and $z' \in C^H(\{x, z', x'\})$. Hence, there is no group c.o.unreachability. However, this group of hospitals does not satisfy independence of proposing order as explained in Example 7.

Observation 4. *Independence of proposing order does not imply group c.o.achievability.*

Example 11. *(Example 5 revisited) Suppose there is a hospital h whose preference is as follows:*

$$h: \qquad \{x, y, z\} > \{y'\} > \{x, y\} > \{x, z\} > \{y, z\} > \{x\} > \{y\} > \{z\} > \emptyset$$

where $d(x)$, $d(y) = d(y')$ and $d(z)$ are 3 different doctors.

In this example, we have only one hospital h and it satisfies observable substitutability (across doctors). Therefore, it is independent of proposing order. However, there exists a (group) c.o.unreachability as explained in Example 5.

5.5 Doctor's preference monotonicity and strategy-proofness of cumulative offer process

In this section, I try to find a joint condition that guarantees the strategy-proofness of cumulative offer process when cumulative offer process always produces a doctor-optimally stable outcome.

5.5.1 Doctor's preference monotonicity

Let's start this subsection by looking at the following example:

Example 12. *There are two hospitals h and h' and their preferences are as follows:*

$$h: \qquad \{z\} > \{x, y\} > \{x\} > \{y\} > \emptyset$$
$$h': \qquad \{x'\} > \{y'\} > \{z'\} > \emptyset$$

where $d(x) = d(x')$, $d(y) = d(y')$ and $d(z) = d(z')$ are three different doctors.

Suppose the preferences of $d(x)$ is $x > x' > \emptyset$ and preferences of $d(z)$ is $z' > z > \emptyset$. If $d(y)$ submits his preference as $y > \emptyset$, then the cumulative offer process outcome is $\{h : \{x, y\}; h' : \{z'\}\}$ and $d(y)$ will be assigned to contract $\{y\}$. However, if $d(y)$ submits a longer preference $y' > y > \emptyset$ by reversely extending $y > \emptyset$, then he will receive nothing since the cumulative offer process outcome will be $\{h : \{z\}; h' : \{x'\}\}$. Therefore, cumulative offer process is not strategy-proof in this example because if $d(y)$'s true preference is the longer one $y' > y > \emptyset$, he could shorten his own preference by removing y' and get a better result.

Now I would like to introduce a joint condition based on this observation.

Definition 25. *Suppose* $>_d$: $x_1 > x_2 > \cdots > x_n > \emptyset$ *and* $>'_d$: $y_1 > y_2 > \cdots > y_m > \emptyset$ *are two preference lists of the same doctor* $d \in D$. *We say* $>_d$ ***reversely extends*** $>'_d$ *if they satisfy the following conditions:*

- $\{y_1, ..., y_m\} \subseteq \{x_1, ..., x_n\}$.
- *For any* $x \in \{x_1, ..., x_n\} \backslash \{y_1, ..., y_m\}$, $x >_d y$ *for any* $y \in \{y_1, ..., y_m\}$.
- *For any* $y, y' \in \{y_1, ..., y_m\}$, $y >_d y' \Leftrightarrow y >'_d y'$.

In other words, $>'_d$ *is just the "tail" part of* $>_d$.

Definition 26. *A group of hospitals satisfies **doctor's preference monotonicity**, if for any* $d \in D$ *and any pair of doctors' preference profiles* $(>_d, >_{-d})$ *and* $(>'_d, >_{-d})$ *where* $>_d$ *reversely extends* $>'_d$, *when fixing the proposing order and let* Y *and* Y' *be the outcomes of the cumulative offer process under* $(>_d, >_{-d})$ *and* $(>'_d, >_{-d})$ *respectively, then* $|Y_d| \geq |Y'_d|$.

In other words, when fixing proposing order and other doctors' preferences, if a doctor reversely extends his own preference, then he could get at least the same number of contracts via cumulative offer process. Therefore, doctors' preference monotonicity fulfills doctor's greed in a way that the more he reveals on top of his current preference list, the more he will get eventually.

Suppose the cumulative offer process outcome is always doctor-optimally stable. Then its necessity for cumulative offer process to be strategy-proof is straightforward. Since otherwise, if there exists a doctor $d \in D$ who gets less contracts when submitting $>_d$ that reversely extends $>'_d$, then in many-to-one matching settings, d is unemployed when submitting $>_d$. Suppose $>_d$ is his true preference, then he could be strictly better-off by just shortening his true preference and submitting $>'_d$ instead. And this violates the strategy-proofness of cumulative offer process. Therefore, we have the following result.

Theorem 17. *Suppose cumulative offer process is always doctor-optimally stable. If a group of hospitals does not respect doctor's preference monotonicity, then cumulative offer process is not strategy-proof.*

The following example shows that doctor's preference monotonicity alone cannot guarantee the strategy-proofness of cumulative offer process.

Example 13. *(Example 7 revisited) There are two hospitals* h *and* h' *and their preferences are as follows:*

$$h: \quad \{x, z\} > \emptyset$$
$$h': \quad \{x', z'\} > \emptyset$$

where $d(x) = d(x') \neq d(z) = d(z')$.

It is easy to check that this group of hospitals satisfies doctor's preference monotonicity. However, cumulative offer process is not always strategy-proof. To see this, suppose $d(x)$'s preference is $x > x' > \emptyset$ and $d(z) : z' > z > \emptyset$. And proposing order is $x \triangleright z' \triangleright z \triangleright x'$. Then $d(z)$ will get contract z in this cumulative offer process. However, if $d(z)$ submits $z' > \emptyset$, then he will get z' instead, in which $d(z)$ is strictly better-off.

5.5.2 Sufficiency result

Conversely, in this subsection, I would like to show that when cumulative offer process is doctor-optimally stable, doctor's preference monotonicity is also a sufficient condition for cumulative offer process to be strategy-proof.

To show its sufficiency, I need a result from Hatfield and Milgrom (2005)[**8**], which shows that when cumulative offer process always produces doctor-optimally stable result, if there exists a preference list for a doctor $d \in D$ such that d obtains contract x by submitting this preference list, then d also gets x if he submits a preference list that includes only contract x.

Lemma 8. *(Theorem 10 in Hatfield and Milgrom (2005)[**8**]) Suppose the matching algorithm always produces the doctor-optimal stable outcome. Fixing the preferences of hospitals and of doctors besides d, let x be the contract that d receives by reporting preference list* $>_d$: $z_1 > \cdots > z_n > x$. *Then d also receives x by reporting another preference list* $>'_d$: x.

Proof. Let $>_{-d}$ be the preferences of other doctors. Suppose Y is the matching outcome under $(>_d, >_{-d})$. The idea is to show that Y is also stable under $(>'_d, >_{-d})$. Since matching algorithm produces doctor-optimally stable outcome, then d still gets x. See more details in Hatfield and Milgrom (2005)[**8**]. □

Here comes the sufficiency result.

Theorem 18. *Suppose a group of hospitals H satisfies doctor's preference monotonicity and cumulative offer process always produces doctor-optimally stable outcome. Then fixing the preferences of hospitals and doctors besides d, let x be the contract d receives when submitting* $>_d$: $z_1 > \cdots > z_n > x$. *Then if he submits* $>^*_d$: $y_1 > \cdots > y_m > x > y_{m+1} > \cdots > y_M$, *he will receive a contract that is* $>^*_d$*-preferred or indifferent to x.*

Proof. Let $>_{-d}$ be the preferences of other doctors. By Lemma 8, we know that d receives x when submitting $>'_d$: x. Now, let us consider another preference list for d: $>''_d$: $y_1 > \cdots > y_m > x$. Then $>''_d$ reversely extends $>'_d$. Therefore, by doctor's preference monotonicity, d will get a contract that is $>''_d$-preferred or indifferent to x when submitting $>''_d$[18]. And this cumulative offer process outcome under $(>''_d, >_{-d})$ is still the doctor-optimally stable

[18] If $x = \emptyset$, then by doctor's preference monotonicity, if d receives nothing under $>''_d$, then it is indifferent to x; if d receives a contract under $>''_d$, then it is $>''_d$-preferred to x. If $x \neq \emptyset$, then d will receive a contract under $>''_d$ and it is $>''_d$-preferred or indifferent to x by the definition of $>''_d$ itself.

outcome under $(>_d^*, >_{-d})$, as d's contracts that are less preferred than x could not be used to block a matching result where he receives a contract weakly preferred to x. □

According to this theorem, if a doctor d's true preference is $>_d^*$, then d can never be better-off by reporting a different preference. Therefore, it guarantees the strategy-proofness of cumulative offer process.

Corollary 5. *Suppose a group of hospitals H satisfies independence of proposing order, group c.o.achievability and doctor's preference monotonicity. Then the cumulative offer process is doctor-optimally stable and strategy-proof.*

5.5.3 Revisit hospital-by-hospital law of aggregate demand

In Hatfield and Milgrom (2005)[**8**], they introduced a hospital-by-hospital condition, law of aggregate demand, and showed that when cumulative offer process produces a doctor-optimal stable outcome under substitutability condition, the law of aggregate demand guarantees its strategy-proofness. In Hatfield and Kojima (2010)[**7**], they re-examined this condition when unilateral substitutability was first proposed, and they also showed that it still guarantees the strategy-proofness of cumulative offer process. The law of aggregate demand is defined as follows:

Definition 27. *(Hatfield and Milgrom (2005)[**8**]) A hospital $h \in H$'s choice function satisfies the **law of aggregate demand** if for all $X' \subseteq X'' \subseteq X$, $|C^h(X')| \leq |C^h(X'')|$.*

In other words, a hospital's choice function is monotone, in a sense that when there is a weakly larger set of contracts to choose from, it will choose at least the same number of contracts.

In the following result, I would like to show that, under the condition that cumulative offer process always produces doctor-optimal stable outcome, doctor's preference monotonicity generalizes hospital-by-hospital law of aggregate demand.

Theorem 19. *Suppose cumulative offer process always produces a doctor-optimal stable outcome for a group of hospitals H. If any $h \in H$ satisfies law of aggregate demand, then H satisfies doctor's preference monotonicity.*

Proof. Let me prove it by contradiction. Suppose H violates doctor's preference monotonicity. Then there exists a doctor $d \in D$ and a pair of doctors' preference profiles $(>_d, >_{-d})$ and $(>_d', >_{-d})$ where $>_d$ reversely extends $>_d'$, and let Y and Y' be the cumulative offer process outcomes under $(>_d, >_{-d})$ and $(>_d', >_{-d})$ respectively, we have $|Y_d| < |Y_d'|$. It implies that d is unemployed in Y whereas d is employed in Y'.
Claim: Y is also a stable outcome under $(>_d', >_{-d})$ and $C^H(\cdot)$.
Since Y is a cumulative offer process outcome under $(>_d, >_{-d})$ and d is unemployed in Y, then

Y still satisfies individual rationality under $(>'_d, >_{-d})$. If Y is not stable under $(>'_d, >_{-d})$, then there exists a $h \in H$ and a set of contracts X' such that $X' \neq C^h(Y)$ and

$$X' = C^h(X' \cup Y) \subseteq C'_D(X' \cup Y)$$

where $C'_D(\cdot)$ is determined by $(>'_d, >_{-d})$.

Now, we consider $C_D(\cdot)$ that is determined by $(>_d, >_{-d})$. Since $Y_d = \emptyset$, then $X'_d = C_d(X' \cup Y)$. And because $>_{-d}$ stays the same, then we still have $X' \subseteq C_D(X' \cup Y)$. Therefore, X' and h still blocks Y under $(>_d, >_{-d})$. However, it contradicts the fact that Y is a stable outcome generated by cumulative offer process under $(>_d, >_{-d})$.
Therefore, Y is also stable under $(>'_d, >_{-d})$ and $C^H(\cdot)$. Let $\{x_1, ..., x_t\}$ and $\{y_1, ..., y_s\}$ represent cumulative offer processes under $(>_d, >_{-d})$ and $(>'_d, >_{-d})$ respectively, i.e. $C^H(\{x_1, ..., x_t\}) = Y$ and $C^H(\{y_1, ..., y_s\}) = Y'$. Since Y' is doctor-optimally stable under $(>'_d, >_{-d})$, we have $Y' \geq_D Y$ under $(>'_d, >_{-d})$, which implies $\{y_1, ..., y_s\} \subseteq \{x_1, ..., x_t\}$[19]. And it also indicates that $Y_D \subseteq Y'_D$. However, since $d \in Y'_D \backslash Y_D$, we have $|Y| < |Y'|$. Therefore, there exists a $h \in H$ such that $|Y_h| < |Y'_h|$, i.e. $|C^h(\{x_1, ..., x_t\})| < |C^h(\{y_1, ..., y_s\})|$. It violates the law of aggregate demand for hospital h. □

The following example shows that doctor's preference monotonicity strictly extends hospital-by-hospital law of aggregate demand when cumulative offer process always produces doctor-optimally stable outcome.

Example 14. *There are two hospitals h and h' and their preferences are as follows:*

$$\begin{aligned} h: &\quad \{z\} > \{x, y\} > \{x\} > \{y\} > \emptyset \\ h': &\quad \{x'\} > \{z'\} > \emptyset \end{aligned}$$

where $d(x) = d(x')$, $d(y)$ and $d(z) = d(z')$ are three different doctors.

In this example, both h and h' satisfy unilateral substitutability. Therefore, cumulative offer process is always doctor-optimally stable. We also notice that h violates the law of aggregate demand because $|C^h(\{x, y, z\})| < |C^h(\{x, y\})|$. However, this group of hospitals respects doctor's preference monotonicity. First, it is easy to check that doctor's preference monotonicity holds by only reversely extending $d(y)$'s preference since $d(y)$ only has contract y. Then, for doctor $d(x)$, we just need to check two cases while fixing an arbitrary $(>_y, >_z)$:

- $x > \emptyset \Rightarrow x' > x > \emptyset$
  We know that when $d(x)$ submits $x' > x > \emptyset$, he will directly receive x' since it is also the most preferred contract for h'. Therefore, doctor's preference monotonicity holds for $d(x)$ in this case because $d(x)$ can not get more than one contract by submitting $x > \emptyset$.

[19] When $Y' \geq_D Y$ under $(>'_d, >_{-d})$, for any $d' \neq d$, d' proposes more(or equal number of) contracts in $\{x_1, ..., x_t\}$ than in $\{y_1, ..., y_s\}$ because $>_{d'}$ does not change and $Y'_{d'} \geq'_{d'} Y_{d'}$. For doctor d, since $Y_d = \emptyset$, then d proposes not only all contracts that reversely extends $>'_d$, but also all contracts in $>'_d$. Therefore, we have $\{y_1, ..., y_s\} \subseteq \{x_1, ..., x_t\}$.

- $x' > \emptyset \Rightarrow x > x' > \emptyset$
  We know that $d(x)$ always receives x' by submitting $x' > \emptyset$. Now, it remains to show that he will also receive a contract by submitting $x > x' > \emptyset$. This is true because either he receives x and he will not propose any contract anymore, or his x gets rejected at some point, then he will propose x' and x' will be secured for him afterwards because it is the most preferred contract of h'.

Similarly, for doctor $d(z)$, we just need to check two cases while fixing an arbitrary $(>_x, >_y)$:

- $z > \emptyset \Rightarrow z' > z > \emptyset$
  We know that $d(z)$ always receives z by submitting $z > \emptyset$ because z is the most preferred contract of h. Now, it remains to show that he will also receive a contract by submitting $z' > z > \emptyset$. This is true because either he receives z' and he will not propose any contract anymore, or his z' gets rejected at some point, then he will propose z and z will be secured for him afterwards because it is the most preferred contract of h.

- $z' > \emptyset \Rightarrow z > z' > \emptyset$
  We know that $d(z)$ always receives z by submitting $z > z' > \emptyset$ because z is the most preferred contract of h. Therefore, doctor's preference monotonicity holds for $d(z)$ in this case because $d(z)$ can not get more than one contract by submitting $z' > \emptyset$.

5.6 Summary

In this chapter, I stepped out of the traditional hospital-by-hospital framework and focused on the joint properties instead. Independence of proposing order combined with group c.o.achievability is a sufficient condition for cumulative offer process to be doctor-optimally stable. It strictly extends hospital-by-hospital unilateral substitutability in current literature and is by far the weakest sufficient condition. It is also "necessary" in a sense that if not, then there exists a doctors' preference profile and a proposing order such that cumulative offer process is not doctor-optimally stable. A byproduct is that independence of proposing order also guarantees the stability of cumulative offer process, which generalizes some hospital-by-hospital substitutable conditions that ensure the stability of cumulative offer process. Finally, I briefly explored another joint condition, doctor's preference monotonicity, and showed that it guarantees the strategy-proofness of cumulative offer process when cumulative offer process is always doctor-optimally stable. It is strictly weaker than hospital-by-hospital law of aggregate demand, a size monotonicity condition in current literature.

Chapter 6

Conclusion

In this thesis, I studied the cumulative offer process in many-to-one matching with contracts model. I assumed that each hospital satisfies irrelevance of rejected contracts and then mainly discussed the stability and doctor-optimality of cumulative offer process from two perspectives.

- First, I sticked to the traditional hospital-by-hospital condition on each hospital's choice function and found that unilateral substitutability is both observable substitutable across doctors and c.o.achievable. And it deepens our understanding of unilateral substitutability because it is not only sufficient for cumulative offer process to be doctor-optimally stable but also "necessary" for in a sense that if $|H| > 1$ and a hospital $h \in H$ is not unilateral substitutable, then there exists a doctors' preference profile and a unit demand choice function for other hospitals such that cumulative offer process can not produce doctor-optimally stable outcomes.

- Then, I stepped out of the hospital-by-hospital framework and hoped to find a more generalized characterization of the doctor-optimal stability of cumulative offer process. A combined joint condition, independence of proposing order plus group c.o.achievability, turns out to be sufficient for cumulative offer process to be doctor-optimally stable. It is also "necessary" in a sense that if not, then there exists a doctors' preference profile and a proposing order such that cumulative offer process is not doctor-optimally stable. This condition is by far the weakest sufficient condition since it strictly extends the hospital-by-hospital unilateral substitutability. In the end, I also briefly discussed another joint condition, doctor's preference monotonicity, and showed that it guarantees the strategy-proofness of cumulative offer process when it is always doctor-optimally stable. It also strictly extends the hospital-by-hospital law of aggregate demand, a size monotonicity condition in current literature.

An important "takeaway" is that stepping out of the traditional hospital-by-hospital framework really helps us to gain a better understanding of the stability and doctor-optimality of cumulative offer process, since it "broadens our horizon" by focusing on more complex joint properties of an entire group of hospitals' choice functions. It also generalizes the sufficient condition for the existence of doctor-optimally stable outcome in current literature. However, this combined joint condition is not easy to check in practice. And this thesis does not provide a necessary condition[1] for the existence of doctor-optimally stable outcome because I only focused on cumulative offer process and did not discuss some other potential matching algorithms. I will leave them for my future research.

[1]Perhaps a necessary condition in some sense.

Bibliography

[1] Afacan, M. O. and B. Turhan (2015). On relationships between substitutes conditions. *Economics Letters 126, 10-12.*

[2] Aygün, O. and T. Sönmez (2012). The Importance of Irrelevance of Rejected Contracts in Matching under Weakened Substitutes Conditions. *Boston College Working papers in Economics 805, Boston College Department of Economics.*

[3] Aygün, O. and T. Sönmez (2013). Matching with Contracts: Comment. *American Economic Review, 103 (5): 2050-51.*

[4] Flanagan, F. (2014). Relaxing the substitutes condition in matching markets with contracts. *Economics Letters 123, 113-117.*

[5] Gale, D. and L. Shapley (1962). College Admission and the Stability of Marriage. *American Mathematical Monthly, 69, 9-15.*

[6] Hatfield, J. W., and F. Kojima (2008). Matching with Contracts: Comment. *American Economic Review, 98 (3): 1189-94.*

[7] Hatfield, J. W. and F. Kojima (2010). Substitutes and stability for matching with contracts. *Journal of Economic Theory 145(5), 1704-1723.*

[8] Hatfield, J. W., and P. Milgrom (2005). Matching with Contracts. *American Economic Review, 95 (4): 913-935.*

[9] Hatfield, J. W. and S. D. Kominers (2015). Hidden Substitutes. *Working paper, Harvard University.*

[10] Hatfield, J. W. and S. D. Kominers (2017). Contract Design and Stability in Many-to-Many Matching. *Games and Economic Behavior 101 (January 2017): 78-97.*

[11] Hatfield, J. W., S. D. Kominers, and A. Westkamp (2017). Stability, Strategy-Proofness, and Cumulative Offer Mechanisms. *In Proceedings of the 2017 ACM Conference on Economics and Computation (EC '17). ACM, New York, NY, USA, 673-674.* `https://doi.org/10.1145/3033274.3084085`

[12] Hirata, D. and Y. Kasuya (2014). Cumulative offer process is order-independent. *Economics Letters 124, 37-40.*

[13] Kadam, S. V. (2017). Unilateral substitutability implies substitutable completability in many-to-one matching with contracts. *Games and Economic Behavior, Elsevier, vol. 102(C), pages 56-68.*

[14] Kominers, S. D. and T. Sönmez (2016). Matching with slot-specific priorities: Theory. *Theoretical Economics 11 (2016), 683-710.*

[15] Zhang, J. (2016). On sufficient conditions for the existence of stable matchings with contracts. *Economics Letters 145, 230-234.*

Appendix A

Doctors' proposal deferred acceptance algorithm

Suppose $\triangleright$ is a strict order of all the contracts in X[1]. The following is the doctors' proposing deferred acceptance algorithm with $\triangleright$:

- **Step 0:** Initialize the set of contracts offered by each doctor d as $\mathcal{F}_d(0) = \emptyset$ and the set of offers that each hospital $h \in H$ holds at step 0 as $\mathcal{W}^h(0) = \emptyset$.

- **Step $t \geq 1$:** Consider the set

  $$U_t = \{x \in X \backslash \mathcal{F}_D(t-1) : d(x) \notin d(\mathcal{W}^H(t-1)) \; and \; \nexists z \in (X_{d(x)} \backslash \mathcal{F}_D(t-1)) \cup \{\emptyset\} \; such \; that \; z >_{d(x)} x\}$$

  If U_t is empty, then the algorithm terminates and the outcome is $\mathcal{W}^H(t-1)$. Otherwise, let x_t be the highest ranked contract in U_t according to $\triangleright$, then $d(x_t)$ proposes x_t at step t, and $\mathcal{W}^{h(x_t)}(t) = C^{h(x_t)}(\mathcal{W}^{h(x_t)}(t-1) \cup \{x_t\})$ and $\mathcal{W}^h(t) = \mathcal{W}^h(t-1)$ for any $h \neq h(x_t)$.

A doctors' proposing deferred acceptance algorithm starts with no contract offered to hospitals and no contract held by hospitals. Then, at each step t, U_t refers to the set of contracts that satisfy the following conditions:

- Any contract in U_t has not been offered yet.
- Any contract in U_t is not associated to a doctor whose contract is currently held by a hospital.
- Any contract in U_t is most preferred by its doctor among all his contracts that have not yet been proposed.

[1]Here, I adopt the definition from Hatfield, Kominers and Westkamp (2017)[**11**].

And the highest ranked contract in U_t according to proposing order $\triangleright$ will be proposed by its doctor at step t. Eventually, U_t will be empty at some point T and the final outcome will be $\mathcal{W}^H(T-1)$.

We notice that in deferred acceptance algorithm, at each step t, each hospital's choice set is the union of a newly proposed contract and the contracts held in last step, not the available set of contracts received by this hospital from the very beginning. And this leads to the difference between deferred acceptance algorithm and cumulative offer process. The following example helps illustrate this difference.

Example 15. *Suppose a hospital h's preference is as follows:*

$$h: \quad \{x,z\} > \{z\} > \{y\} > \{x\} > \emptyset$$

where $d(x)$, $d(y)$, $d(z)$ are three different doctors.

Suppose every contract is acceptable to its doctor. Consider this proposing order $x \triangleright y \triangleright z$. Both cumulative offer process and deferred acceptance algorithm terminates at step 3. At step 3, h will choose $\{x,z\}$ in cumulative offer process because its available set is $\{x,y,z\}$. However, in deferred acceptance algorithm, h holds $\{y\}$ at the end of step 2. Then at step 3, h will choose $C^h(\{y,z\}) = \{z\}$. Therefore, it leads to a different outcome because it does not consider contract x that was rejected at step 1.

Appendix B

Omitted proofs

B.1 Proof of Lemma 4.

Proof. Let me use a proof by contradiction. Suppose $z \in Y$ for some Y that is $\{x_1, ..., x_t\}$-unblocked. Without loss of generality, let t be the first time in this observable offer process when a contract $z \in R^h(\{x_1, ..., x_t\}) \backslash R^h(\{x_1, ..., x_{t-1}\})$ and there exists a $\{x_1, ..., x_t\}$-unblocked set Y such that $z \in Y$. Actually, t is also the first time when z is rejected in this observable offer process, since if z is first rejected at some $t' < t$, and by Lemma 3, we know that Y is also $\{x_1, ..., x_{t'}\}$-unblocked, then it contradicts the fact that t is the first time when such a contract is rejected in this observable offer process.

Claim 1: For $\forall\ y \in Y$, $x \in \{x_1, ..., x_t\}$ with $d(x) = d(y)$, we have either $x = y$ or $d(x)$ proposes x prior to y in $\{x_1, ..., x_t\}$[1].
This is because if $\exists\ y \in Y$, $x \in \{x_1, ..., x_t\}$ with $d(x) = d(y)$ and $d(x)$ proposes y prior to x in $\{x_1, ..., x_t\}$, then we have $y \in \{x_1, ..., x_t\}$ and it has been first rejected at some $t' < t$. However, $y \in Y$, and by Lemma 3, if Y is $\{x_1, ..., x_t\}$-unblocked, then Y is also $\{x_1, ..., x_{t'}\}$-unblocked. Therefore, we found a contract $y \in R^h(\{x_1, ..., x_{t'}\}) \backslash R^h(\{x_1, ..., x_{t'-1}\})$ and it is also in a $\{x_1, ..., x_{t'}\}$-unblocked set Y. It contradicts the fact that t is the first time when such a contract is rejected in this observable offer process.

Claim 2: $d(z) \notin d(C^h(\{x_1, ..., x_t\}))$.
If not, $\exists\ z_1 \in \{x_1, ..., x_t\}$ with $d(z) = d(z_1)$ such that $z_1 \in C^h(\{x_1, ..., x_t\})$. Since t is the first time when z is rejected, then $d(z)$ proposes z_1 prior to z in this observable offer process. Let $t_1 < t$ be the first time when z_1 is rejected in this observable offer process, i.e. $z_1 \in R^h(\{x_1, ..., x_{t_1}\}) \backslash R^h(\{x_1, ..., x_{t_1-1}\})$. Then

- **Case 1.** $d(z_1) \notin d(C^h(\{x_1, ..., x_{t_1}\}))$.
  Let $Y_1 = C^h(\{x_1, ..., x_{t_1}\})$, then we have $d(z_1) \notin d(Y_1)$, $z_1 \notin C^h(\{z_1\} \cup Y_1)$ by IRC,

[1]Broadly speaking, if $d(x)$ proposes x in $\{x_1, ..., x_t\}$ and he has not yet proposed y in $\{x_1, ..., x_t\}$, then we can also say that he proposes x prior to y in $\{x_1, ..., x_t\}$ even though $y \notin \{x_1, ..., x_t\}$.

but $z_1 \in C^h(\{x_1, ..., x_t\})$ where $\{z_1\} \cup Y_1 \subseteq \{x_1, ..., x_t\}$. It violates the unilateral substitutability of h.

- **Case 2.** $d(z_1) \in d(C^h(\{x_1, ..., x_{t_1}\}))$.
  Then $\exists\ z_2 \in \{x_1, ..., x_{t_1}\}$ with $d(z_2) = d(z_1) = d(z)$ such that $z_2 \in C^h(\{x_1, ..., x_{t_1}\})$. Since t_1 is the first time when z_1 is rejected, then $d(z)$ proposes z_2 prior to z_1 in this observable offer process. Let $t_2 < t_1$ be the first time when z_2 is rejected in this observable offer process, i.e. $z_2 \in R^h(\{x_1, ..., x_{t_2}\}) \backslash R^h(\{x_1, ..., x_{t_2-1}\})$. Then
  - **Case 2.1.** $d(z_2) \notin d(C^h(\{x_1, ..., x_{t_2}\}))$.
    Let $Y_2 = C^h(\{x_1, ..., x_{t_2}\})$, then we have $d(z_2) \notin d(Y_2)$, $z_1 \notin C^h(\{z_2\} \cup Y_2)$ by IRC, but $z_2 \in C^h(\{x_1, ..., x_{t_1}\})$ where $\{z_2\} \cup Y_2 \subseteq \{x_1, ..., x_{t_1}\}$. It violates the unilateral substitutability of h.
  - **Case 2.2** $d(z_2) \in d(C^h(\{x_1, ..., x_{t_2}\}))$.
    Then $\exists\ z_3 \in \{x_1, ..., x_{t_2}\}$ with $d(z_3) = d(z_2) = d(z_1) = d(z)$ such that $z_3 \in C^h(\{x_1, ..., x_{t_2}\})$. Since t_2 is the first time when z_2 is rejected, then $d(z)$ proposes z_3 prior to z_2 in this observable offer process. Let $t_3 < t_2$ be the first time when z_3 is rejected in this observable offer process, i.e. $z_3 \in R^h(\{x_1, ..., x_{t_3}\}) \backslash R^h(\{x_1, ..., x_{t_3-1}\})$. Then
    - **Case 2.2.1** $d(z_3) \notin d(C^h(\{x_1, ..., x_{t_3}\}))$
    - **Case 2.2.2** $d(z_3) \in d(C^h(\{x_1, ..., x_{t_3}\}))$

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In each case during the process above, we can see that if $d(z)$ is not in the choice set for h, then it contradicts unilateral substitutability; if $d(z)$ is in the choice set for h, then it will result in two other sub-cases and goes back to the "loop" again. However, $d(z)$ only has finite contracts, which implies that eventually we can find a contract z_n such that doctor $d(z_n) = \cdots = d(z_1) = d(z)$ is not in the choice set for h and it will contradict the unilateral substitutability for h.

Let $\bar{Y} = C^h(\{x_1, ..., x_t\})$. By IRC, we have

$$z \notin \bar{Y} = C^h(\bar{Y} \cup \{z\})$$

Now, we have $d(z) \notin d(\bar{Y})$ by Claim 2, $z \notin C^h(\bar{Y} \cup \{z\})$. Then, I would like to show that $z \in C^h(\{x_1, ..., x_t\} \cup Y)$. Let $\hat{Y} = C^h(\{x_1, ..., x_t\} \cup Y)$. If $z \notin \hat{Y}$, then $\hat{Y} \neq Y$, and by IRC, we have

$$\hat{Y} = C^h(\hat{Y} \cup Y) \neq Y$$

Since Y is $\{x_1, ..., x_t\}$-unblocked, then $\exists\ \hat{y} \in \hat{Y}$, $y \in Y$ with $d(y) = d(\hat{y})$ such that $d(y)$ proposes y prior to $\hat{y}$ in $\{x_1, ..., x_t\}$. However, since $\hat{Y} \subseteq Y \cup \{x_1, ..., x_t\}$, it contradicts Claim 1.

Therefore, we found z, $\bar{Y}$ and $d(z) \notin d(\bar{Y})$ such that $z \notin C^h(\bar{Y} \cup \{z\})$, but $z \in C^h(Y \cup \{x_1, ..., x_t\})$ where $\bar{Y} \subseteq \{x_1, ..., x_t\} \subseteq \{x_1, ..., x_t\} \cup Y$. It violates the unilateral substitutability of h. □

B.2 Proof of Lemma 7.

The idea is very similar to the proof of Lemma 4, and some modification is needed.

Proof. Let me prove it by contradiction. Without loss of generality, let t be the first time in this group observable offer process when $\exists\ z \in R^H(\{x_1, ..., x_t\}) \backslash R^H(\{x_1, ..., x_{t-1}\})$ and $z \in Y$ for some Y that is $\{x_1, ..., x_t\}$-unblocked. Actually, t is also the first time when z is rejected in this observable offer process, since if z is first rejected at some $t' < t$, and by Lemma 6, we know that Y is also $\{x_1, ..., x_{t'}\}$-unblocked, then it contradicts the fact that t is the first time when such a contract is rejected in this observable offer process.

Claim 1: For $\forall\ y \in Y$, $x \in \{x_1, ..., x_t\}$ with $d(x) = d(y)$, we have either $x = y$ or $d(x)$ proposes x prior to y in $\{x_1, ..., x_t\}$[2].
This is because if $\exists\ y \in Y$, $x \in \{x_1, ..., x_t\}$ with $d(x) = d(y)$ and $d(x)$ proposes y prior to x in $\{x_1, ..., x_t\}$, then we have $y \in \{x_1, ..., x_t\}$ and it has been rejected at some $t' < t$. However, $y \in Y$, and by Lemma 6, if Y is $\{x_1, ..., x_t\}$-unblocked, then Y is also $\{x_1, ..., x_{t'}\}$-unblocked. Therefore, we found a contract $y \in R^H(\{x_1, ..., x_{t'}\}) \backslash R^H(\{x_1, ..., x_{t'-1}\})$, and it is also in a $\{x_1, ..., x_{t'}\}$-unblocked set Y. It contradicts the fact that t is the first time when such a contract is rejected in this observable offer process.

Claim 2: $d(z) \notin d(C^H(\{x_1, ..., x_t\}))$.
If not, $\exists\ z_1 \in \{x_1, ..., x_t\}$ with $d(z) = d(z_1)$ such that $z_1 \in C^H(\{x_1, ..., x_t\})$. Since t is the first time when z is rejected, then $d(z)$ proposes z_1 prior to z in this observable offer process. Let $t_1 < t$ be the first time when z_1 is rejected in this observable offer process, i.e. $z_1 \in R^H(\{x_1, ..., x_{t_1}\}) \backslash R^H(\{x_1, ..., x_{t_1-1}\})$. Then

- **Case 1.** $d(z_1) \notin d(C^H(\{x_1, ..., x_{t_1}\}))$.
  Let $h_1 = h(z_1)$ and $Y_1 = C^{h_1}(\{x_1, ..., x_{t_1}\})$, then we have $d(z_1) \notin d(Y_1)$, $z_1 \notin C^{h_1}(\{z_1\} \cup Y_1)$ by IRC, but $z_1 \in C^{h_1}(\{x_1, ..., x_t\})$ where $\{z_1\} \cup Y_1 \subseteq \{x_1, ..., x_t\}$. It violates the unilateral substitutability of h_1.

- **Case 2.** $d(z_1) \in d(C^H(\{x_1, ..., x_{t_1}\}))$.
  Then $\exists\ z_2 \in \{x_1, ..., x_{t_1}\}$ with $d(z_2) = d(z_1) = d(z)$ such that $z_2 \in C^H(\{x_1, ..., x_{t_1}\})$. Since t_1 is the first time when z_1 is rejected, then $d(z)$ proposes z_2 prior to z_1 in this observable offer process. Let $t_2 < t_1$ be the first time when z_2 is rejected in this observable offer process, i.e. $z_2 \in R^H(\{x_1, ..., x_{t_2}\}) \backslash R^H(\{x_1, ..., x_{t_2-1}\})$. Then

[2] Broadly speaking, if $d(x)$ proposes x in $\{x_1, ..., x_t\}$ and he has not yet proposed y in $\{x_1, ..., x_t\}$, then we can also say that he proposes x prior to y in $\{x_1, ..., x_t\}$ even though $y \notin \{x_1, ..., x_t\}$.

- **Case 2.1.** $d(z_2) \notin d(C^H(\{x_1, ..., x_{t_2}\}))$.
  Let $h_2 = h(z_1)$ and $Y_2 = C^{h_2}(\{x_1, ..., x_{t_2}\})$, then we have $d(z_2) \notin d(Y_2)$, $z_1 \notin C^{h_2}(\{z_2\} \cup Y_2)$ by IRC, but $z_2 \in C^{h_2}(\{x_1, ..., x_{t_1}\})$ where $\{z_2\} \cup Y_2 \subseteq \{x_1, ..., x_{t_1}\}$. It violates the unilateral substitutability of h_2.
- **Case 2.2** $d(z_2) \in d(C^H(\{x_1, ..., x_{t_2}\}))$.
  Then $\exists\ z_3 \in \{x_1, ..., x_{t_2}\}$ with $d(z_3) = d(z_2) = d(z_1) = d(z)$ such that $z_3 \in C^H(\{x_1, ..., x_{t_2}\})$. Since t_2 is the first time when z_2 is rejected, then $d(z)$ proposes z_3 prior to z_2 in this observable offer process. Let $t_3 < t_2$ be the first time when z_3 is rejected in this observable offer process, i.e. $z_3 \in R^H(\{x_1, ..., x_{t_3}\}) \backslash R^H(\{x_1, ..., x_{t_3-1}\})$. Then
  * **Case 2.2.1** $d(z_3) \notin d(C^H(\{x_1, ..., x_{t_3}\}))$
  * **Case 2.2.2** $d(z_3) \in d(C^H(\{x_1, ..., x_{t_3}\}))$

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In each case during the process above, we can see that if $d(z)$ is not in the choice set for H, then it contradicts unilateral substitutability for some hospital; if $d(z)$ is in the choice set for h, then it will result in two other sub-cases and go back to the loop again. However, $d(z)$ only has finite contracts, which implies that eventually we can find a contract z_n such that doctor $d(z_n) = \cdots = d(z_1) = d(z)$ is not in the choice set for H and it will contradict the unilateral substitutability for some hospital $h(z_n)$.

Let $h = h(z)$ and $\bar{Y} = C^h(\{x_1, ..., x_t\})$. By IRC, we have

$$z \notin \bar{Y} = C^h(\bar{Y} \cup \{z\})$$

Now, we have $d(z) \notin d(\bar{Y})$ by Claim 2, $z \notin C^h(\bar{Y} \cup \{z\})$. Then, I would like to show that $z \in C^h(\{x_1, ..., x_t\} \cup Y)$. Let $\hat{Y} = C^h(\{x_1, ..., x_t\} \cup Y)$. If $z \notin \hat{Y}$, then $\hat{Y} \neq C^h(Y)$, and by IRC, we have

$$\hat{Y} = C^h(\hat{Y} \cup Y) \neq C^h(Y)$$

Since Y is $\{x_1, ..., x_t\}$-unblocked, then $\exists\ \hat{y} \in \hat{Y}$, $y \in Y$ with $d(y) = d(\hat{y})$ such that $d(y)$ proposes y prior to $\hat{y}$ in $\{x_1, ..., x_t\}$. However, since $\hat{Y} \subseteq Y \cup \{x_1, ..., x_t\}$, it contradicts Claim 1.

Therefore, we found z, $\bar{Y}$ and $d(z) \notin d(\bar{Y})$ such that $z \notin C^h(\bar{Y} \cup \{z\})$, but $z \in C^h(Y \cup \{x_1, ..., x_t\})$ where $\bar{Y} \subseteq \{x_1, ..., x_t\} \subseteq \{x_1, ..., x_t\} \cup Y$. It violates the unilateral substitutability of h. □

B.3 Two compatible group observable offer processes could share a doctors' preferences $>_D$.

A simple adaptation of Definition 13 and Lemma 2 yields the following definition and result relative to two compatible group observable offer processes:

Definition 28. *An **eligible** observable offer process $\{x_1, ..., x_T\}$ for a group of hospitals H under a doctors' preference profile $>_D$ satisfies the following conditions:*

- *Any contract in $\{x_1, ..., x_T\}$ is acceptable to its doctor.*
- $\forall\ x_t, x_{t'} \in \{x_1, ..., x_T\}$ *with* $d(x_t) = d(x_{t'})$, $t < t' \Leftrightarrow x_t >_{d(x_t)} x_{t'}$.
- *If $x \in \{x_1, ..., x_T\}$ and $\exists\ x' >_{d(x)} x$, then we have $x' \in \{x_1, ..., x_T\}$.*

Lemma 9. *Suppose $\{x_1, ..., x_T\}$ and $\{y_1, ..., y_S\}$ are two observable offer processes for a group of hospitals H in $Y \subseteq X$. If $\{y_1, ..., y_S\}$ is compatible with $\{x_1, ..., x_T\}$-revealed preferences, then there exists a doctors' preference profile $>_D$, which is consistent with $\{x_1, ..., x_T\}$-revealed preferences, such that $\{x_1, ..., x_T\}$ and $\{y_1, ..., y_S\}$ are both eligible group observable offer processes in Y under $>_D$ and $C^H(\cdot)$.*

Proof. Let $>_D$ satisfy the following conditions:

- $y \in Y$ is acceptable to $d(y)$ if and only if $y \in Y$.
- $\forall\ x$ and x' in Y with $d(x) = d(x') =: d$, we have
  - If $x, x' \in \{x_1, ..., x_T\} \cup \{y_1, ..., y_S\}$ and x is proposed prior to x' in either group observable offer process, then $x >_d x'$.
  - If $x \in \{x_1, ..., x_T\} \cup \{y_1, ..., y_S\}$ and $x' \notin \{x_1, ..., x_T\} \cup \{y_1, ..., y_S\}$, then $x >_d x'$.
  - If $x, x' \notin \{x_1, ..., x_T\} \cup \{y_1, ..., y_S\}$, then we could have either $x >_d x'$ or $x' >_d x$.

Then we can verify that $>_D$ satisfies

- $y \in Y$ is acceptable to $d(y)$ if and only if $y \in Y$.
- $\forall\ x_t, x_{t'} \in \{x_1, ..., x_T\}$ with $d(x_t) = d(x_{t'}) =: d$ and $t < t'$, we have $x_t >_d x_{t'}$.
- If $x \in \{x_1, ..., x_T\}$ and $\exists\ x' >_{d(x)} x$, then we have $x' \in \{x_1, ..., x_T\}$.
- $\forall\ y_s, y_{s'} \in \{y_1, ..., y_S\}$ with $d(y_s) = d(y_{s'}) =: d$ and $s < s'$, we have $y_s >_d y_{s'}$.
- If $y \in \{y_1, ..., y_S\}$ and $\exists\ y' >_{d(y)} y$, then we have $y' \in \{y_1, ..., y_S\}$.

Therefore, both $\{x_1, ..., x_T\}$ and $\{y_1, ..., y_S\}$ are eligible group observable offer processes in Y under $>_D$ and $C^H(\cdot)$. □