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Expanding cosmologies in brane geometries

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Five dimensional gravity coupled, both in the bulk and on a brane, to a scalar Liouville field yields a geometry confined to a strip around the brane and with time dependent scale factors for the four geometry. In various limits known models can be recovered as well as a temporally expanding four geometry with a warp factor falling exponentially away from the brane. The effective theory on the brane has a time dependent Planck mass and "cosmological constant." Although the scale factor expands, the expansion is not an acceleration.

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There is considerable interest in theories of gravity and of cosmology with extra dimensions where our world is confined to a four dimensional space-time subspace or 3-brane. All of our known fields, with the exception of gravity, are confined to the brane. The extra dimensions may have compact toroidal topologies [1] or be unbounded with a scale factor, warp, depending on the "distance" from the brane [2,3]. Branes meandering in the internal space, with the brane metric dependent on the internal coordinates, were treated in Ref. [4]. Less singular metrics we obtained in situations where the branes were thickened [5]. As many of these works concerned themselves with the hierarchy problem they restricted themselves to Minkowski metrics on the brane; specifically, the metrics were time independent. Extending these concepts to cosmology requires the introduction of time dependent scale factors on the brane and possibly in the extra dimensions. Solutions in which one posits various stress tensors on the brane and a general review of cosmology restricted to a brane in extra dimensions may be found in Ref. [6].

In this work we look at a five dimensional bulk whose dynamics is governed by a scalar Liouville field coupled to gravity in the usual way. In addition there is a coupling to the scalar field and to the tension on a thin 3-brane. As in previous works the brane tension is finely tuned to parameters of the bulk action. The general form of the metric we obtain is

$$ds^2=\left(1-\frac{|y|}{y_0}\right)^{\xi}[dt^2-a(t)^2d\mathbf{x}^2]-b(t)^2dy^2. \tag{1}$$

In the extra dimension the bulk geometry is confined to a finite strip, $y\leqslant y_0$, around the brane of interest. Although y_0 may be scaled away (set equal to one), we keep it for the convenience of limiting procedures discussed further on. It will turn out that for $\xi\geqslant 1/2$ we may ignore singularities at the edges of the strip; for $\xi<1/2$ these singularities force us either to identify the opposite edges of the strip and place a regulator brane at $y=y_0$ or place two branes at $y=\pm y_0$. As the metric vanishes on these extra branes they do not support any physics.

In addition to the trivial solution with $a(t)$ and $b(t)$ in Eq. (1) being constant in time, the ansatz $a(t)=a_0(t/t_0)^{\alpha}$, $b(t)=b_0(t/t_0)^{\beta}$ yields a solution provided

$$\alpha=\frac{2+3\xi\pm 2\sqrt{1+\xi}}{8+9\xi},\quad \beta=\frac{2\mp 6\sqrt{1+\xi}}{8+9\xi}. \tag{2}$$

These satisfy the relation $\beta=1-3\alpha$, reminiscent of one of the Kasner [7] conditions. For the upper solution, α ranges from 1/2 to 1/3 and β from $-1/2$ to 0 as ξ goes from 0 to infinity; for the lower solution α goes from 0 to 1/3 and β from 1 to 0.

There are various interesting limits. In addition to being able to recover the geometry of Refs. [2,3] we can obtain a cosmology where the four metric represents an expanding universe with a warp factor decreasing exponentially as we move away from the brane

$$ds^2=e^{-2k|y|}\left[dt^2-a_0\left(\frac{t}{t_0}\right)^{2/3}d\mathbf{x}^2\right]-dy^2. \tag{3}$$

This limit is interesting as we recover an effective four dimensional cosmology with a time dependent Planck mass. For $\xi=0$ we recover the Kasner solutions. We shall return to a discussion of these metrics further on.

The solution (1) is obtained from the action for the metric tensor and for a scalar Liouville field with contributions from the bulk and from one or two branes. Five dimensional theories with bulk scalar fields have been previously considered. A massive scalar field can determine the size of the internal dimension [8] and with intricate self couplings can thicken the branes [9,10].

The contribution from one of the branes, presumably the one we are on, will be indicated explicitly while the one from the other brane or branes will be left for later elaboration:

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$$S = S_{\rm bulk} + S_{\rm brane},$$

$$S_{\rm bulk} = \frac{1}{2k_5^2}\int d^5x\sqrt{g}\left\{R + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} + \lambda\exp(-\kappa\phi)\right\}, \tag{4}$$

$$S_{\rm brane} = \frac{h}{2k_5^2}\int d^5x\sqrt{g}\,\delta(n_\mu x^\mu)\sqrt{n_\mu n_\nu g^{\mu\nu}} \times\exp\left(-\frac{\kappa}{2}\phi\right);$$

$k_5 = 8\pi/M_5^3$ where M_5 is the five dimensional Planck mass. κ is a free parameter and although λ is included for convenience it can be scaled by any positive number through a shift in the field ϕ. In $S_{\rm brane}$ n_μ is a spacelike vector normal to the brane and the product $\delta(n_\mu x^\mu)\sqrt{n_\mu n_\nu g^{\mu\nu}}$ is independent of the magnitude of this vector. Varying the combination $\sqrt{g}\sqrt{n_\mu n_\nu g^{\mu\nu}}$ with respect to $g^{\mu\nu}$ yields terms proportional to $(-g_{\mu\nu} + n_\mu n_\nu / n_\alpha n_\beta g^{\alpha\beta})$, namely depending only on the metric along the brane; this procedure leads to the same results as one would get by using the Israel junction conditions [11]. As in all previous works we will take n_μ to be along x_5 for which we will use the symbol y. Note that the ϕ coupling in $S_{\rm brane}$ is $\kappa/2$ as opposed to κ in $S_{\rm bulk}$. The magnitude of the brane tension, determined by h, as in previous works, is related to bulk parameters; our solutions require

$$h = \sqrt{\frac{2\lambda}{\kappa^2(8/3\kappa^2 - 1)}}; \tag{5}$$

this restricts $\lambda \geqslant 0$ for $\kappa^2 \leqslant 8/3$ and $\lambda < 0$ otherwise.

The solution for the equations of motion obtained by varying Eq. (4) with respect to $g^{\mu\nu}$ and ϕ we seek have a metric given in Eq. (1) and the scalar field of the form

$$\phi = A\ln\left[\left(1 - \frac{|y|}{y_0}\right)b(t)\right] - C. \tag{6}$$

It is straightforward to check that for $y_0 > |y|$ these are indeed solutions provided

$$A = 2/\kappa,$$

$$C = \frac{1}{\kappa}\ln\left[\frac{2}{\lambda\kappa^2 y_0^2}\left(\frac{8}{3\kappa^2} - 1\right)\right], \tag{7}$$

$$\xi = \frac{4}{3\kappa^2}$$

and the scale factors $a(t)$ and $b(t)$ satisfy

$$4\left(\frac{\dot a}{a}\right)^2 + 4\left(\frac{\dot a}{a}\right)\left(\frac{\dot b}{b}\right) - \xi\left(\frac{\dot b}{b}\right)^2 = 0,$$

$$4\frac{\ddot a}{a} + 4\left(\frac{\dot a}{a}\right)^2 + \xi\left(\frac{\dot b}{b}\right)^2 = 0, \tag{8}$$

$$8\left(\frac{\ddot a}{a}\right)^2 + 4\left(\frac{\ddot b}{b}\right)^2 + 4\left(\frac{\dot a}{a}\right)^2 + 8\left(\frac{\dot a}{a}\right)\left(\frac{\dot b}{b}\right) + 3\xi\left(\frac{\dot b}{b}\right)^2 = 0;$$

the overdot represents differentiation with respect to time.

With the choice of metric in Eq. (1), of the twenty five equations for the components of the Einstein tensor, $R_{\mu\nu} - \cdots = 0$ and the equation of motion for the field ϕ only five are nontrivial and independent. These may be chosen to be the equation of motion for ϕ and for the *tt*, *ty*, *yy* and any of the diagonal space-space component of the Einstein tensor along the brane. The relations between A, κ and ξ in Eq. (7) solve the *ty* equation while Eq. (8) takes care of the other four. That, in the bulk, these *four* equations yield only the *three* conditions in Eq. (8) is not surprising as the equation of motion for the field and the Einstein equations for the metric are related by the conservation of the energy-momentum tensor. What is pleasant is that all the four independent equations on the brane, the ones involving $\delta(y)$ terms, where the energy-momentum tensor is not conserved, are also satisfied.

We now turn to possible singularities at $|y| = y_0$. For $\xi > 1/2$ or equivalently $\kappa < \sqrt{3/8}$ we can restrict the bulk to the strip $|y| \leqslant y_0$ as the solutions discussed above may be continued to the end points. For $\xi \leqslant 1/2$ singularities develop at these points and the solutions are no longer valid there. As in many previous discussions of bulk-brane geometries the cure consists of either identifying $y = y_0$ with $y = -y_0$ (orbifolding) and introducing a brane at $|y| = y_0$ or introducing independent branes at $y = \pm y_0$. In the first case, the action on the $y = y_0$ brane is

$$S'_{\rm brane} = -\frac{h}{2k_5^2}\int d^5x\sqrt{g}\,\delta(n_\mu x^\mu - y_0)\sqrt{n_\mu n_\nu g^{\mu\nu}} \times\exp\left(-\frac{\kappa}{2}\phi\right). \tag{9}$$

If one wishes to place branes at both ends of the strip, the action contributed is one half that of Eq. (9) on each of the two branes. Since the four-metric in Eq. (1) vanishes at $|y| = y_0$, these branes or brane cannot support any physics. The explicit forms for $a(t)$ and $b(t)$ are given in Eq. (2). For $\xi \geqslant 2$ the edge $|y| = y_0$ is at the horizon in that it takes an infinite time to reach it.

Certain limits of these solutions are interesting. The case $\xi = 0$ corresponds to Kasner's [7] solutions. For the $a(t)$ and $b(t)$ constant case the limit $\xi \to \infty$ with $y_0 = \xi/(2k)$ yields the Randall-Sundrum solution [2,3]. Equation (5) is equivalent to their relation between the bulk cosmological constant and brane tension. In the same limit, but with α and β given by either solution in Eq. (2) we obtain the metric (3) and

$$\phi(t,y) = 2\frac{\ln(t)}{\sqrt{3}}. \tag{10}$$

The above solution may be obtained independently from the action $S=S_{\rm bulk}+S_{\rm brane}$, where

$$S_{\rm bulk}=\frac{1}{2k_5^2}\int d^5x\sqrt{g}\left\{R+\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}+3k^2\right\},$$

$$S_{\rm brane}=\frac{3k}{4k_5^2}\int d^5x\sqrt{g}\,\delta(n_\mu x^\mu)\sqrt{n_\mu n_\nu g^{\mu\nu}}. \tag{11}$$

How well gravity on the brane is described by $ds^2=dt^2-a(t)^2d\mathbf{x}^2$ depends on solution of the equation

$$-\left(1-\frac{|y|}{y_0}\right)^{-2\xi}\partial_y\left[\left(1-\frac{|y|}{y_0}\right)^{2\xi}\partial_y\right]h(y,t)=m^2b(t)^2h(y,t), \tag{12}$$

with $h(y,t)$ describing fluctuations around the metric. Fluctuation equations for nonzero m are quite complex (see, for example, Refs. [12,13]) and here we shall restrict our study to $m=0$; it is necessary to have an acceptable solution for this case and it is easy to exhibit such a solution

$$h_{m=0}(y,t)\sim\epsilon(y)\left[\left(1-\frac{|y|}{y_0}\right)^{1-2\xi}-1\right]. \tag{13}$$

Four dimensional gravity on the brane appears after integrating the action (4) with the metric

$$ds^2=\left(1-\frac{|y|}{y_0}\right)^{\xi}\,{}^{(4)}g_{ij}(x)dx^idx^j-b(t)^2dy^2 \tag{14}$$

over y; ${}^{(4)}g_{ij}(x)$ is the four metric on the brane. This may be accomplished by using the ADM reduction with $b(t)$ playing the role of the lapse function and conformally transforming the resulting four dimensional metric by the factor $(1-|y|/y_0)^{-\xi}$. The result is

$$S=\frac{1}{2k_5^2}\int d^4x\,dy\sqrt{-{}^{(4)}g}\left(1-\frac{|y|}{y_0}\right)^{2\xi}b(t)\left[\frac{R({}^{(4)}g_{ij}(x))}{\left(1-\frac{|y|}{y_0}\right)^{\xi}}+\frac{3\xi}{2\left(1-\frac{|y|}{y_0}\right)^{\xi}}\left(\frac{\dot b}{b}\right)^2\right],$$

$$=\frac{1}{2k_5^2}\int d^4x\frac{2y_0b(t)}{\xi+1}\sqrt{-{}^{(4)}g}\left[R({}^{(4)}g_{ij}(x))+\frac{3\xi}{2}\left(\frac{\dot b}{b}\right)^2\right]. \tag{15}$$

The four dimensional theory has a time dependent Planck mass

$$M_4(t)^2=M_5^3\frac{2y_0b(t)}{\xi+1} \tag{16}$$

and a time dependent "cosmological constant"

$$\frac{\Lambda(t)}{M_4(t)^2}\sim\xi\beta^2\left(\frac{t}{t_0}\right)^{\beta-1}. \tag{17}$$

In cosmologies with small extra dimensions, when known physics is not restricted to a brane, time dependence of the internal dimensions is severely restricted by limits on the temporal variations of all fundamental constants [14]; in contrast for theories with most of known physics restricted to a brane, only limits on the time evolution of the Planck mass may come into play. The solutions discussed here can accommodate any such limits as by choosing κ small and equivalently β small we can make this variation as soft as necessary. The most stringent present limit on $\dot G/G\leqslant 8\times10^{-12}$ [15] translates into a limit on β of $\beta\leqslant 0.1$. Temporal variations of the cosmological constant, or more generally of the dark energy are coming into consideration [16].

The solutions presented have $\alpha\leqslant 1/2$ and thus represent decelerating cosmologies. In line with recent observations [16] we would like to accommodate an accelerating, expanding scale factor. Having a time dependent scale factor for the external dimensions circumvents some no-go theorems [17,18] and such cosmologies have been found in M theories can achieve accelerating scale factors by analytically continuing the solutions to negative ξ. This, however, corresponds to an imaginary exponent in the Liouville action. Whether this difficulty can be circumvented is under investigation. Difficulty in finding accelerating solution was noted in Refs. [18,19].

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