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LINEAR PROGRAMS FOR NONLINEAR HYDROLOGIC ESTIMATION[1]

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ABSTRACT: The minimization of the sum of absolute deviations and the minimization of the absolute maximum deviation (minimax) were transformed into equivalent linear programs for the estimation of parameters in a transient and linear hydrologic system. It is demonstrated that these two methods yield viable parameter estimates that are globally optimal and reproduce properly the timing and magnitude of hydrologic events and associated variables such as total runoff. The two linear estimation methods compared favorably with the popular least-squares nonlinear estimation method. The generality of the theoretical developments shows that linear program equivalents are adequate competitors of nonlinear methods of hydrologic estimation and parameter calibration.
(KEY TERMS: hydrologic model; model calibration; parameter estimation; linear programming; least-squares.)

INTRODUCTION

The process of calibrating a hydrologic model is of great importance in applied hydrologic studies. In general terms, hydrologic model calibration consists of determining the values of model parameters that satisfactorily (i.e., in some well-defined sense) reproduce sets of hydrologic observations. Such parameters must also satisfy various constraints on the possible values that they can take to comply with physical feasibility principles. The techniques for parameter estimation are numerous and range from simple eye-ball curve fitting to complex statistical estimation algorithms (Amorocho and Espildora, 1973; Mays and Taur, 1982; Unver and Mays, 1984; Patry and Marino, 1985; Loaiciga and Marino, 1987; Sorooshian, 1988). This paper presents an application of optimization theory for the estimation of hydrologic-model parameters. The objective of this work is to pose complex, nonlinear, parameter estimation problems as equivalent linear programming problems. The advantages of linear programming parameter estimation *vis a vis* nonlinear estimation and the choice of a suitable criterion (objective function) in hydrologic parameter estimation are examined in this work. The concept of reparameterization in hydrologic model calibration is introduced in this work. The developments of this paper are illustrated with the estimation of Muskingum routing parameters.

ON THE CRITERION OF HYDROLOGIC PARAMETER ESTIMATION

The choice of suitable criteria for hydrologic parameter estimation has been and still is the subject of substantial research interest. This paper focuses on the optimization (i.e., mathematical programming) approach to parameter estimation. Formally, a hydrologic (or hydraulic) model yields a single or multiple hydrologic output, denoted by $\mathbf{h}$, that depends on a set of parameters ($\mathbf{p}$) and data ($\mathbf{z}$), through an empirical or conceptual model ($\mathbf{f}$). Mathematically, this is denoted as

$$\mathbf{h} = \mathbf{f}(\mathbf{p};\mathbf{z}) \qquad (1)$$

If the measurements of the hydrologic variable are denoted by $\mathbf{b}$, then the parameter estimation problem can be expressed as

$$\text{minimize } \| \mathbf{h}-\mathbf{b} \| \qquad (2)$$

where the minimization is with respect to $\mathbf{p}$. The double vertical bars represent a norm or measure of agreement among hydrologic observations and hydrologic predictions. In general, there may be restrictions

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on the parameters in which case those must be appended to Equation (2) for a complete expression of the estimation problem.

Equation (2) is more general than what it might seem at first. Many statistical estimation problems (e.g., maximum likelihood, ordinary or generalized least-squares) can be expressed in the form of Equation (2) where the norm is the (weighted) sum of squared deviations between predicted and measured values of the hydrologic variable of interest, i.e., the least-squares (or weighted least-squares) criterion (Yeh *et al.,* 1983; Carrera and Neuman, 1986). Least-squares is one of the most commonly used criteria in hydrologic parameter estimation studies in spite of the fact that it is not necessarily adequate in some hydrologic applications. Using runoff prediction as an example, the least-squares approach places a larger weight on the larger deviations between predicted and measured flows. This uneven weighting leads in many instances to relatively inaccurate flood volume estimation (e.g., estimation errors exceeding 15 to 10 percent). Empirical evidence in this regard was presented in a classical paper by Amorocho and Espildora (1973).

Interestingly, there are attractive alternatives to least-squares estimation in hydrologic model calibration that do not suffer from the functional scaling problem cited previously. The minimization of the maximum absolute deviation and the minimization of the sum of absolute deviations between predicted and measured hydrologic variables are natural alternatives to least-squares; yet, with few exceptions (Deininger, 1969; Mays and Coles, 1980; Padmanaban and Williams, 1987), they have not been used in applied hydrologic studies. The popularity of least-squares estimation does not have a hydrologic justification. Rather, it is explained primarily by reasons of computational convenience. Chief among those reasons was the early appearance of algorithms for least-squares minimization problems, e.g., the Levenberg-Marquardt method (Levenberg, 1944; Marquardt, 1963), that provided a vital impetus for the widespread popularity of least-squares estimation.

It will be shown below that certain nonlinear hydrologic estimation problems can be transformed into equivalent linear programming problems. The importance of these transformations cannot be overemphasized. There is reliable, well-tested, and well-documented software for solving linear programming problems. Equally important is the fact that linear programming theory provides a rich set of results for determining the nature and properties of the solution(s) to a linear programming problem. Issues concerning (1) the existence of feasible solutions to a linear programming problem, (2) whether the solution is unique and globally optimal or just one among an infinite number of solutions, and (3) the sensitivity of a solution to changes in parameters or variables in the hydrologic model can be fully resolved within a linear programming framework. The nature of the aforementioned linearizing transformations and applications to hydrologic parameter estimation follow.

MINIMAX, THE SUM OF ABSOLUTE DEVIATIONS, AND LEAST-SQUARES

The sum of absolute deviations criterion is given by

$$\text{minimize} \sum_{i=1}^{n} \left| h_i(\mathbf{p}) - b_i \right| \tag{3}$$

and the maximum absolute deviation (minimax criterion) is expressed as

$$\text{minimize} \ \left(\text{maximum} \left| h_i(\mathbf{p}) - b_i \right| \right), \ i = 1, 2, \ldots, n \tag{4}$$

In Equations (3) and (4) the subscript i denotes the ith component of either vector **h** or **b**, and the minimization is with respect to the parameter vector **p**. In addition, there may be various (linear or linearizable) constraints on **p**. In some instances, the hydrologic output **h** may be nonlinear on the parameters. At times, one can recourse to reparameterization of the hydrologic model in terms of a new set of parameters upon which the hydrologic model turns out to be linear. This powerful idea of reparameterization (in fact, a transformation of the parameter space) is illustrated in a section below that addresses the estimation of parameters in a nonlinear model.

The sum of absolute deviations is well-suited for the calibration of hydrologic models for hydrograph generation or flood routing when the key concern is runoff volume prediction for the purpose, say, of reservoir storage design. The minimax criterion seeks to minimize the largest deviation between observed and predicted runoff. Minimax provides an overall accurate reproduction of an observed hydrograph, and, in particular, it provides accurate reproduction of hydrograph peaks, where the largest deviations tend to be concentrated.

The Sum of Absolute Deviations

In order to transform Equation (3), which is a nonlinear function, into an equivalent linear programming problem, it is necessary to introduce the following constraints:

$$h_i - b_i = x_i - y_i,\ i = 1,2,...,n \tag{5}$$

or equivalently,

$$h_i - x_i + y_i = b_i,\ i = 1,2,...,n \tag{6}$$

where

$$x_i, y_i \geq 0,\ 1,2,...,n \tag{7}$$

A problem whose objective function is to

$$\text{minimize} \sum_{i=1}^{n} (x_i + y_i) \tag{8}$$

subject to Equations (6) and (7) (and possibly other linear or linearizable constraints on **p**) is equivalent to the minimization introduced in Equation (3). The minimization in Equation (8) is with respect to x_i, y_i, and **p**. The equivalence of the problems of Equations (6) to (8) and Equation (3) is proven in the Appendix. Notice that no requirement has been made on the mathematical structure of the model **f** (see Equation (1)) relating **h** and **p**, and the proof in the Appendix is rather general in this regard. However, from a practical standpoint, it is desired that the transformed problem be linear, for otherwise it would be simpler to work directly with the original problem of Equation (3). The linearity of the problem of Equations (6) to (8) occurs when the output **h** depends linearly on the parameter vector **p**. For such class of hydrologic (or hydraulic) models the estimation problem of Equations (6) to (8) is a linear programming problem, the type of estimation problem that is the subject of this study. The transformation equality of Equation (5) for h_i linearly dependent on **p** has been known for a number of years in the operations research literature (Wagner, 1975).

Hydrologic (or hydraulic) models that are linear with respect to the parameters are not uncommon and represent a fairly important spectrum of processes of interest (Eagleson *et al.*, 1966). The Muskingum routing model, applied in a following section, is an excellent example of a linear hydrologic model. Unit-hydrograph models of runoff are also classical examples of linear hydrologic models (Amorocho, 1963). Confined-aquifer flow problems (Yeh *et al.*, 1983); Loaiciga and Marino, 1987) are examples in ground water hydrology where the piezometric potential depends linearly on transmissivity.

The Minimax Criterion

The minimax estimation problem is given by Equation (4). It seeks to minimize the maximum absolute deviation between the predicted and measured values of a hydrologic variable. Equation (4) will be transformed into an equivalent linear programming problem for the class of hydrologic models that are linear on the parameters (briefly discussed earlier). First, notice that

$$|h_i - b_i| = \text{maximum}(h_i - b_i,\ b_i - h_i) \tag{9}$$

Next, the following constraints can be introduced:

$$h_i - b_i \leq y,\ \ i = 1,2,...,n \tag{10}$$

$$b_i - h_i \leq y,\ \ i = 1,2,...,n \tag{11}$$

where

$$y \geq 0 \tag{12}$$

The objective function

$$\text{minimize}\ y \tag{13}$$

subject to Equations (10) to (12) defines a minimization problem that is equivalent to Equation (4). The minimization in Equation (13) is with respect to y and **p**. The equivalence of Equation (4) and Equations (10) to (13) is obvious since the minimization of (13) ensures that the largest absolute deviation, expressed by (10) – (11), will reach a minimum (nonzero) value not exceeding y. Problem (10) – (13) is known in the operations research literature as the Chebyshev criterion (Wagner, 1975). If the hydrologic output h_i are linear on **p** then (10) – (13) is a linear programming problem. This is the type of hydrologic estimation problem of interest in this work.

Least-Squares Estimation

In a good number of hydrologic estimation problems, the objective is to minimize the sum of square deviations (i.e., the least-squares criterion)

$$\text{minimize}\ \ J = \sum_{i=1}^{n} (h_i(\mathbf{p}) - b_i)^2 \tag{14}$$

Implicit in Equation (14) is that the norm of the deviations is small (in some sense) and that n, the number of observations, is much greater than the dimension of **p**; otherwise, any arbitrary model would give a close fit to the data. This second condition also applies to the sum of deviations and minimax methods. However, the smallness of the deviations in Equation (14) plays a particularly important role in the selection of a suitable solution algorithm of the least-squares problem. Specifically, if the sum of the deviations is small, then specialized algorithms such as the Levenberg-Marquardt method are quite efficient. Otherwise, one must recourse to general unconstrained optimization algorithms, of which the BFGS (Broyden-Fletcher-Goldfarb-Shanno) method (Gill *et al.*, 1981) is a good example. For the purpose of this study, we take advantage of the fact that the hydrologic output **h** is linear on the parameters or on a reparameterized form of them. In such a case, and especially when the number of parameters is relatively small, Equation (14) can be solved by applying the Kuhn-Tucker conditions of optimality directly. The result is a set of linear equations on the parameters that can be solved regardless of the magnitude of the norm of the deviations. The set of linear equations is derived from

$$\frac{\partial J}{\partial p_j} = 0, \; j = 1, 2, .., m \tag{15}$$

This approach, embodied in Equation (15), will be used in the APPLICATIONS section below.

ESTIMATION OF MUSKINGUM ROUTING PARAMETERS

The Muskingum routing model has been chosen to illustrate the applicability of the equivalent linear programming problems of Equations (6) to (8) and Equations (10) to (13) presented above and to compare them with least-squares estimation. The background for the Muskingum routing model can be found in any book on hydrology (see e.g., Viessman *et al.*, 1989). According to the Muskingum model, the outflow hydrograph (O) at the downstream end of a channel reach is linearly related to the inflow (I) at the upstream end of the channel reach by the expression

$$O_{t+1} = C_0 I_{t+1} + C_1 I_t + C_2 O_t \tag{16}$$

for all times (t). The coefficients C_0, C_1, and C_2 are given by

$$C_0 = \frac{-Kx + 0.5\Delta t}{K - Kx + 0.5\Delta t} \tag{17}$$

$$C_1 = \frac{Kx + 0.5\Delta t}{K - Kx + 0.5\Delta t} \tag{18}$$

$$C_2 = \frac{K - Kx - 0.5\Delta t}{K - Kx + 0.5\Delta t} \tag{19}$$

where

$$C_0 + C_1 + C_2 = 1 \tag{20}$$

in which K is the storage time constant for the channel reach (units of time), and x is a weighting factor that varies between zero and one-half (dimensionless); Δt is the time step chosen in the routing simulation.

Equation (16) is a linear dynamic model in the sense that the current output is a linear combination of present and previous inflows and previous outflow. The actual model parameters are K and x, which are related to the model coefficients in Equation (16) through Equations (17) to (19). The reparameterization embodied in Equations (17) to (19) in terms of C_0, C_1, and C_2 is the key for the application via linear programming in the Muskingum routing model. First, the estimation problem will yield the coefficients C_0, C_1, and C_2 and subsequently, the parameters K and x will be obtained from the following expressions (all derived from Equations (17) to (19)):

$$Kx = \frac{0.5\Delta t \left[1 - C_0 / C_1\right]}{1 + C_0 / C_1} \tag{21}$$

where Δt is specified and the coefficients C_0, C_1, and C_2 are calculated from the linear programming problems (presented in Equations (26) to (28) and Equations (29) to (32))

$$K = \frac{Kx\left(C_2 - 1\right) - 0.5\Delta t\left(C_2 + 1\right)}{C_2 - 1} \tag{22}$$

$$x = \frac{Kx}{K} \tag{23}$$

Equations (21) to (23) are applied sequentially so that after calculating Kx in Equation (21), K can be calculated from Equation (22) and x can be derived from Equation (23).

The idea of (1) reparameterization (see Equations (17) to (19)), (2) linear estimation, and (3) subsequent inversion (Equations (21) to (23)) to obtain actual model parameters is quite powerful. An otherwise highly nonlinear model and parameter estimation problem can be reduced to the most basic estimation (i.e., linear programming) and closed-form inversion to yield the hydrologic parameters of interest. Conceptually, an estimation problem in terms of a parameter set (**p**) expressed by

$$\text{minimize } \|\mathbf{h}(\mathbf{p})-\mathbf{b}\| \tag{24}$$

subject to all pertinent constraints on **p** is reparameterized and rewritten as follows,

$$\text{minimize } \|\mathbf{h}(\mathbf{r})-\mathbf{b}\| \tag{25}$$

where **r** is the reparameterized (parameter) set related to **p** by some suitable transformations such as those in Equations (17) to (19). In the Muskingum routing model, **r** is composed of the coefficients C_0, C_1, and C_2 that relate outflow and inflow hydrographs via Equation (16). The parameter set **p** is composed of K and x. The equivalent linear programming estimation problems (see Equations (6) to (8) and Equations (10) to (13)) are applied to the Muskingum model below.

The Sum of Absolute Deviations

According to Equations (6) to (8) the minimization of absolute deviations between observed and model-estimated outflows is given by (the sub-index t replaces the sub-index i in the following equations)

$$\text{minimize } \sum_{t=1}^{n} (x_t + y_t) \tag{26}$$

subject to

$$(C_0 I_{t+1} + C_1 I_t + C_2 O_t) - x_{t+1} + y_{t+1} = O_{t+1}, \quad t = 0, 1, .., n-1 \tag{27}$$

$$C_0 + C_1 + C_2 = 1 \tag{28}$$

where x_t, $y_t \geq 0$, $t = 1,2,...,n$. In Equations (27) to (28) the index t runs from zero to n–1 (I_0 and O_0 are the initial values). In Equation (26) the minimization is with respect to x_t, $(t = 1,2,...,n)$, C_0, C_1, and C_2 (the latter three coefficients are unrestricted in sign). The outflows (O) and inflows (I) constitute the data available for parameter estimation. The linear programming problem of Equations (26) to (28) is equivalent to the (nonlinear) minimization of the sum of absolute deviations between predicted and observed outflow ordinates. Upon estimation of C_0, C_1, and C_2, the Muskingum routing parameters K and x are obtained from Equations (21) to (23). A computational example is given below.

The Minimax Criterion

The equivalent linear programming problem that minimizes the maximum absolute deviation between observed and predicted outflow hydrograph ordinates according to the Muskingum routing model is to

$$\text{minimize } y \tag{29}$$

subject to

$$C_0 I_{t+1} + C_1 I_t + C_2 O_t - O_{t+1} \leq y, \quad t = 0, 1, .., n-1 \tag{30}$$

$$O_{t+1} - C_0 I_{t+1} - C_1 I_t - C_2 O_t \leq y, \quad t = 1, 2, .. n-1 \tag{31}$$

$$C_0 + C_1 + C_2 = 1 \tag{32}$$

where $y \geq 0$ and C_0, C_1, and C_2 are unrestricted in sign. The minimization in Equation (30) is with respect to y, C_0, C_1, and C_2. Upon solution of Equations (29) to (32), the calculated values of C_0, C_1, and C_2 are used to obtain the Muskingum routing parameters using Equations (21) to (23). An application of the minimax approach and a comparison with the results corresponding to the sum of absolute deviations formulation follow in the next section.

Least-Squares Estimation

Using Equation (15), where

$$J = \sum_{t=1}^{n} \left[C_0 I_t + C_1 I_{t-1} + C_2 O_{t-1} - O_t \right]^2 \tag{33}$$

and realizing that the C's are linearly dependent through the expression

$$C_0 + C_1 + C_2 = 1 \quad (34)$$

one can write the partial derivatives of J with respect to any two of three coefficients C to obtain a two-by-two linear system of equations. Upon its solution, K and x follow from Equations (21) to (23). Results from least squares estimation and a comparison with the minimax and sum of deviations methods follows next.

APPLICATION

Example 1: Calibration

The sum of absolute deviations and minimax problems (Equations (26) to (28)) and Equations (29) to (31), respectively) were solved by linear programming. The least-squares problem was solved by means of Equation (15) applied to Equation (33). The outflow and inflow values were obtained from the data set presented in Table 13.3 of Viessman *et al.* (1989). The streamflow hydrograph (input and output in the channel reach) is bimodal, with two discernable peaks occurring at times t = 11 and t = 18, possibly caused by a multievent precipitation pattern dominated by two predominant periods of high-intensity rain. Model calibration in this example is tantamount to the estimation of the Muskingum parameters K and x according to the sum of deviations, minimax, and least-squares methods.

The globally optimal values of the coefficients C_0, C_1, and C_2, and the Muskingum routing parameters K and x are shown in Table 1. Evidently, the solutions obtained from the minimization of the sum of absolute deviations and the minimax linear programs differ but are within the same order of magnitude in their respective values. In spite of the radically different formulations of these two alternative estimation problems, the hydrologic estimates K and x are robust. In contrast, the least-squares estimate of x is an order of magnitude less than that obtained by the linear estimation methods. The least-squares estimate of the storage time constant K is in close agreement with the estimates derived from the two other methods. Viessman *et al.* (1989), used an iterative (visual) graphical procedure to approximate the Muskingum routing parameters, and their calculated values were K = 2.0 days and x = 0.3, in good agreement with the linearly optimized values presented in Table 1. The choice of an estimator amongst a set of alternative values depends on a variety of hydrologic factors. In the context of hydrograph routing, one must consider individual factors such as the timing and magnitude of flood peaks, as well as generalized parameters such as total runoff volume. These hydrologic criteria are discussed next.

Inflow data were routed with the two alternative sets of parameter estimates and the outflows are presented in Table 2 with observed outflow values. The results in Table 2 indicate that: (1) observed outflows are consistently underestimated in the rising limb of the hydrograph and at the two peaks of the bimodal hydrograph; (2) observed flows are consistently overestimated in the falling limb of the hydrograph; (3) the timing of the first hydrograph peak was correctly predicted by all of the estimation approaches, whereas the time to the second peak was predicted two days earlier than its real occurrence; and (4) the largest flow prediction deviation was produced with the sum of absolute deviations method (prediction at time t=16), but this method yielded a better overall hydrograph reproduction than that achieved with minimax and least squares. A summary of statistics is given in Table 3, from which it is apparent that: (1) there was a good runoff volume prediction by all estimation methods, with the method of sum of absolute deviations predicting total runoff within –1.0 percent of the observed value; (2) the prediction of flow peaks showed estimates within –10 percent of actual values by the sum of deviations and minimax methods, while least-squares peak estimates were within –11 percent; and (3) the overall hydrograph fit

TABLE 1. Results of Parameter Estimates: Calibration Results.

Estimate (1)	Sum of Deviations (2)	Minimax (3)	Least-Squares (4)	Viessman *et al.*, 1989 (5)
C0	0.109710	0.092841	0.152962	–0.052632
C1	0.329771	0.238153	0.225840	0.578947
C2	0.560519	0.669006	0.621261	0.473684
K (day)	2.026	3.241	2.237	2.0
x	0.124	0.222	0.043	0.3

TABLE 2. Predicted Values and Deviations: Calibration Results.

Day (1)	Observed Flows (2)	Predicted Flows SD (3)	Minimax (4)	Least Squares (5)	Deviations SD (6)	Minimax (7)	Least Squares (8)
1	4,180	– –	– –	– –	– –	– –	– –
2	6,970	5,300	5,058	5,346	–1,670	–1,912	–1,624
3	7,560	8,392	8,078	8,405	832	518	845
4	14,200	10,741	10,033	10,744	–3,459	–4,167	–3,456
5	18,300	17,024	16,332	16,663	–1,276	–1,968	–1,637
6	18,500	19,750	19,416	19,690	1,250	916	1,190
7	21,300	21,651	20,967	21,747	351	–333	447
8	29,300	28,735	27,133	29,043	–565	–2,167	–257
9	39,700	41,571	38,617	40,301	1,871	–1,083	601
10	48,700	48,700	46,375	46,867	0	–2,325	–1,833
11	53,300	49,383	49,118	48,739	–3,917	–4,182	–4,561
12	48,700	46,886	48,286	46,722	–1,814	–414	–1,978
13	37,100	38,109	40,699	39,433	1,009	3,599	2,333
14	35,800	32,834	34,075	34,497	–2,966	–1,725	–1,303
15	35,800	39,183	38,441	39,252	3,383	2,641	3,452
16	35,800	41,657	40,159	40,550	5,857	4,359	4,750
17	42,700	39,262	38,341	38,404	–3,438	–4,359	–4,296
18	44,100	39,691	40,314	39,419	–4,409	–3,786	–4,681
19	35,400	35,400	37,437	35,972	0	2,037	572
20	25,200	26,545	28,692	27,547	1,345	3,492	2,347
21	16,400	19,333	20,759	20,016	2,933	4,359	3,616
22	11,500	13,443	14,151	13,728	1,943	2,651	2,228
23	9,380	9,770	10,176	9,888	390	796	508
24	7,860	7,794	8,175	7,957	–66	315	97

Notes: 1. All flows and deviations in cfs (1 cfs = 0.0238 m^3/s).
2. Column (2) adapted from Table 13.3 from Viessman *et al.*, 1989.
3. Deviations are predicted minus observed flows (Columns (5) and (6), respectively).
4. SD in Columns (3) and (6) stands for sum of deviations.

TABLE 3. Summary of Hydrologic Estimates: Calibration Results.

Estimate (1)	Sum of Deviations (2)	Minimax (3)	Least-Squares (4)
Maximum Absolute Deviation (cfs)	5,857 (14.0)	4,359 (26.5)	4,750 (13.3)
Total Volume (cfs-day)	641,154 (99.0)	640,832 (98.9)	640,930 (98.9)
Deviation at Peak (cfs)			
Peak 1	–3,917 (7.3)	–4,182 (7.8)	–4,561 (8.6)
Peak 2	–4,409 (10.0)	–3,786 (8.6)	–4,681 (11)
Sum of Absolute Deviations (cfs-day)	44,744 (6.9)	54,104 (8.4)	48,612 (7.5)

Notes: 1. 1 cfs = 0.0238 m^3/s; 1 cfs-day = 2,447 m^3.
2. Deviations are predicted minus observed flows (see Table 2).
3. The numbers within parentheses denote the percentage that the estimate represents from the corresponding observed flow (in cfs) or from the total observed volume (= 647,775 cfs-day).
4. The hydrograph is bimodal with two peaks.
5. The observed times to the hydrograph peaks are 11 and 18 days for the first and second peaks, respectively. The three estimation methods yielded times to peak of 11 and 16 days for the first and second peaks, respectively.

provided by the method of sum of absolute deviations (and measured by the sum of absolute deviations) was superior to that obtained with the minimax and least-squares methods, with the sum of absolute deviations being only 6.9 percent (of total actual volume) for the sum of deviations method, and 8.4 percent and 7.5 percent, for the minimax and least-squares methods, respectively. Based on the calibration results of this application the sum of absolute deviations method has yielded the better hydrologic estimators according to the full spectrum of criteria previously examined. It shall be noticed, however, that the previous results were obtained by fitting the flow data with parameters that were calibrated with that same data set. In order to compare the estimation methods within a more realistic framework, we proceed to verify the model with data different to those used in the calibration procedure.

Example 2: Simulation Experiment

Calibration is a procedure by which parameters are estimated based on observed data. Verification, on the other hand, consists of evaluating the performance of a calibrated model with data different to those used in the calibration process. In order to test the three estimation methods we conducted a simulation experiment in which flows were generated by means of the following equation:

$$O_{t+1} = C_0 I_{t+1} + C_1 I_t + C_2 O_t + v_t \quad (35)$$

where C_0, C_1, and C_2, have values −0.0526532, 0.578947, and 0.473684, respectively (see Table 1), that correspond to the parameters K = 2.0 and x = 0.3 of the channel reach used in the example given by Viessman *et al.* (1989); v_t is a Gaussian, uncorrelated, error term used to simulate the outflows (**O**). The error term v_t has zero mean and a standard deviation of 2,000 cfs. The value of the standard deviation of v_t was chosen to avoid generating negative outflows, and it was set equal to roughly 15 percent of the standard deviation of observed flows. With the standard deviation of the error term v_t at a level of 2,000 cfs, the probability that v_t exceeds the absolute value of the smallest observed outflow (see Column (2) of Table (2) is less than 10 percent. The inflows (I) in Equation (35) were the same as those specified in Column 2 of Table 13.3 of Viessman *et al.* (1989). The simulation experiment was conducted by generating variates v_t and using Equation (35) to produce 100 synthetic outflow hydrographs, each of a length of 24 days (to match the length of the hydrograph specified in Table 13.3 of Viessman *et al.*, 1989). Parameter estimates (C_0, C_1, and C_2, K and x) were derived from each synthetic hydrograph using the sum of absolute deviations, minimax, and least-squares methods. The simulation experiment, therefore, produced 100 sets of parameters (C_0, C_1, and C_2, K and x) for each estimation method. Representative parameter values for each estimation method were calculated by means of an arithmetic average taken over the 100 sets of estimates. For instance, the representative storage time constant for, say, the minimax method was calculated by the equation,

$$\bar{K} = (1/100) \sum_{1}^{100} K_i \quad (36)$$

where the overbar denotes arithmetic average and K_i is the ith estimate of K using the minimax method. Analogous definitions hold for the other representative parameter estimates in all estimation methods.

The results of parameter estimation in the simulation experiment are reported in Table 4, that also contains the parameter values used in the simulation (via Equation (35)) in Column (5). In addition, Table 4 reports the relative error ($\overline{RE}$) of parameter estimates for each estimation method. The relative error in Table 4 is also an (arithmetic) average value obtained over a sample of 100 estimates as suggested by Equation (36). For each synthetic hydrograph the relative error (RE) is defined as follows:

$$\overline{RE} = \| \mathbf{p} - \mathbf{p}^* \| / \| \mathbf{p}^* \| \quad (37)$$

The numerator in Equation (37) is the Euclidean norm of the difference between the estimated vector of parameter (**p**=(C_0, C_1, and C_2, K, x)) and the "true" parameter vector, **p***, given in column (5) of Table 4; The denominator in Equation (37) is the Euclidean norm (or vectorial magnitude) of the parameter vector **p***. The results of Table 4 show that the sum of absolute-deviations method provided the most accurate parameter estimates with a relative error of 5.83 percent, well below the relative errors of 17.4 percent and 32.6 percent corresponding to the minimax and least-squares method. The higher accuracy of the method of sum of absolute deviations revealed by the simulation experiment corroborates the findings obtained in the calibration process. However, the simulation experiment strongly suggests a much sharper ranking of the estimation methods, with the sum of absolute deviations given the best fit and least-squares ranking at the bottom in terms of its performance, with a dismal relative error of 32.6 percent, almost six times as large as that of the sum of absolute deviations.

TABLE 4. Results of Parameter Estimates: Simulation Results.

Estimate (1)	Sum of Deviations (2)	Minimax (3)	Least-Squares (4)	Viessman *et al.*, 1989 (5)
$\bar{C}0$	0.005723	-0.003588	0.155647	-0.052632
$\bar{C}1$	0.488942	0.435983	0.191110	0.578947
$\bar{C}2$	0.505335	0.567605	0.653243	0.473684
$\bar{K}$ (day)	2.010	2.321	2.435	2.0
$\bar{x}$	0.243	0.219	0.021	0.3
$\overline{RE}$ (percent)	5.83	17.4	32.6	

Notes: 1. $\overline{RE}$ = relative error (in percentage).
2. The overbar in Column (1) denotes an arithmetic average taken over 100 estimates.

TABLE 5. Summary of Hydrologic Estimates: Simulation Results.

Estimate (1)	Sum of Deviations (2)	Minimax (3)	Least-Squares (4)
Maximum Absolute Deviation (cfs)	7,840 (15.0)	6,133 (18.0)	7,546 (14.0)
Total Volume (cfs-day)	642,317 (99.0)	641,832 (97.3)	638,715 (97.0)
Deviation at Peak (cfs)			
Peak 1	-3,843 (7.1)	-4,215 (8.0)	-4,833 (9.1)
Peak 2	-4,003 (8.2)	-3,700 (8.1)	-5.018 (12.3)
Sum of Absolute Deviations (cfs-day)	45,639 (6.7)	55,787 (8.7)	49,657 (8.9)

Notes: 1. 1 cfs = 0.0238 m^3/s; 1 cfs-day = 2,447 m^3.
2. Deviations are predicted minus synthetic flows.
3. The numbers within parentheses denote the percentage that the estimate represents from the corresponding synthetic flow (in cfs) or from the total synthetic volume (in cfs-day).
4. All figures in Table 5 are averages over 100 synthetic hydrographs.
5. All of the estimation methods predicted the times to peaks one and two correctly for all simulations.

Table 5 provides a summary of the performance of the estimation method based on their capability to reproduce certain hydrograph characteristics. Hydrograph characteristics examined in Table 5 include the maximum absolute deviation, total hydrograph volume, time(s) to hydrograph peak(s), and the sum of absolute deviations. The results in Table 5 were derived from the simulation experiment previously described. The parameter estimates obtained from each synthetically generated hydrograph were used to reconstruct the hydrograph according to Equation (16). The differences between the generated and reconstructed hydrographs provided a basis for calculating the "mismatch" statistics, such as the maximum absolute deviation, percent difference in total hydrograph volume, and so forth. The procedure was repeated for each of the 100 synthetic hydrographs, and for all the estimation methods. Finally, representative hydrograph characteristics were obtained by averaging the statistics (arithmetically) over the 100 synthetic realizations. Those representative characteristics are reported in Table 5. The figures in Table 5 are much more definitive in revealing the relative merits of the three estimation methods than the calibration results of Table 3. In fact, Table 5 reaffirms unambiguously some of the problems with the least squares method already seen in Table 3. Specifically, Table 5 shows that: (1) least squares provides the largest bias in estimating peak flow values, and (2) it underestimates the total hydrograph volume more than the two other estimation methods. The peak flow and total flow are important hydrograph characteristics, especially with regards to flood control. The larger biases in peak and total flow estimation by least squares can be explained by the uneven scaling introduced by squaring the differences between predicted and observed flows, as discussed previously. The results of Table 5 also indicate that the method of sum of absolute deviations provides the more accurate and consistent (i.e., over all simulated hydrographs) reproduction of the key hydrograph characteristics, except for the maximum absolute deviation. The latter variable, however, is of minor relevance since it does not impact neither directly, nor significantly, the timing and magnitude of peak flows or the total flow volume.

SUMMARY AND CONCLUSIONS

Three nonlinear methods for hydrologic parameter estimations have been presented and compared in this work. The minimization of the sum of absolute deviations and the minimization of the maximum absolute deviations (minimax) were transformed into equivalent linear programs for the estimation of Muskingum routing parameters. It was found that based on various hydrologic criteria such as time to flood peak, peak-flood magnitude and total runoff prediction, these two estimation methods yielded adequate parameter estimates that compared very favorably with the third estimation method, the popular least-squares estimator. Calibration results and an extensive simulation experiment indicate that the method of least squares is less accurate than the sum of absolute deviations and minimax problems in reproducing hydrograph characteristics such as peak flow and hydrograph volume. This is attributed to the scaling imposed by the method of least squares, where the weights attached to larger deviations is in many instances orders of magnitude larger than those imposed on medium to small deviations.

The approach followed in this work indicates that the sum of absolute deviations and minimax methods applied in this work are suitable for linear hydrologic systems, both steady-state and transient. The linear programming formulation of these two methods (i.e., sum of absolute deviations and minimax) overcomes the problem of having to specify initial parameter estimates when these are unknown, and bounds on estimated parameter values are straightforwardly included. In addition, parameter estimates from the linear programs are globally optimal. These advantageous features should provide enough incentive for more widespread application of the presented methodology to other types of linear hydrologic estimation problems.

APPENDIX A

It is shown herein that (see Equation (3))

$$\text{minimize} \sum_{t=1}^{n} \left| h_t(\mathbf{p}) - b_t \right| \tag{A1}$$

is equivalent to (see Equations (6) to (8))

$$\text{minimize} \sum_{t=1}^{n} (x_t + y_t) \tag{A2}$$

subject to

$$h_t(\mathbf{p}) - x_t + y_t = b_t,\ t = 1, 2, \ldots, n \tag{A3}$$

$$x_t, y_t \geq 0,\ t = 1, 2, .., n \tag{A4}$$

The proof below is applicable to hydrologic outputs h_t that are either linear or nonlinear on the parameters ($\mathbf{p}$). In order to prove the equivalence of Equation (A1) and Equations (A2) to (A4) it is sufficient to show that, at the minimum, the following equality holds,

$$\sum_{t=1}^{n} \left| h_t(\mathbf{p}) - b_t \right| = \sum_{t=1}^{n} (x_t + y_t) \tag{A5}$$

so that the minimization of the left and right hand sides of Equation (A5) leads to the same value for the optimal $\mathbf{p}$.

The proof proceeds by examining the following cases.

Case 1

$h_t - b_t > 0$

Assume:

(1) $x_t > 0$ and $y_t = 0$

From Equation (5),

$$|h_t - b_t| = |x_t - y_t|$$

$$= x_t$$

$$= x_t + y_t \qquad \text{(A6)}$$

(2) $x_t > 0$ and $y_t > 0$

From Equation (5),

$$|h_t - b_t| = |x_t - y_t|$$

$$< |x_t| + |y_t|$$

$$= x_t + y_t \qquad \text{(A7)}$$

From Equations (A6) and (A7) above, it is seen that when $h_t - b_t$ is positive then $x_t + y_t$ is bounded below by $|h_t - b_t|$ with equality holding when $x_t > 0$ and $y_t = 0$.

Case 2

$$h_t - b_t < 0$$

Assume:

(1) $x_t = 0$ and $y_t > 0$

From Equation (5),

$$|h_t - b_t| = |x_t - y_t|$$

$$= |y_t|$$

$$= x_t + y_t \qquad \text{(A8)}$$

(2) $x_t > 0$ and $y_t > 0$

From Equation (5),

$$|h_t - b_t| = |x_t - y_t|$$

$$< |x_t| + |y_t|$$

$$= x_t + y_t \qquad \text{(A9)}$$

From Equations (A8) and (A9) above indicate that if $h_t - b_t$ is negative then $x_t + y_t$ is bounded below by $|h_t - b_t|$ with equality holding at $x_t = 0$ and $y_t > 0$.

Case 3

$$h_t - b_t = 0$$

Assume:

(1) $x_t = y_t = 0$

From Equation (5),

$$|h_t - b_t| = |x_t - y_t|$$

$$= x_t + y_t \qquad \text{(A10)}$$

(2) $x_t = y_t \geq 0$

From Equation (5),

$$|h_t - b_t| = |x_t - y_t|$$

$$< |x_t| + |y_t|$$

$$= x_t + y_t \qquad \text{(A11)}$$

From Equations (A10) and (A11) it follows that if $h_t - b_t$ equals zero then $x_t + y_t$ is bounded below by $|h_t - b_t|$ with equality holding at x_t and y_t being equal to 0.

Cases 1 to 3 examined above showed that, given the equality constraint (Equation 5), $x_t + y_t$ is bounded below by $|h_t - b_t|$ for all times t with equality holding when either x_t or y_t , or both, equal zero. Therefore Equation (A5) holds and the minimization of Equation (3) is equivalent to the minimization of Equations (6) to (8) as hypothesized earlier. Q.E.D.

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