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Minimum Aberration Blocking Schemes for 128-Run Designs

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Abstract: Several criteria have been proposed for ranking blocked fractional factorial designs. For large fractional factorial designs, the most appropriate minimum aberration criterion was one proposed by Cheng and Wu (2002). We justify this assertion and propose a novel construction method to overcome the computational challenge encountered in large fractional factorial designs. Tables of minimum aberration blocked designs are presented for N=128 runs and n=8–64 factors.

Key words: Blocking; Fractional factorial design; Minimum aberration; Word length pattern.

1 Introduction

Fractional factorial designs enable one to investigate many factors in an economical number of runs. In particular, for two-level factors, a regular fractional factorial design with n factors and $N = 2^{n-k}$ runs can accommodate up to $n = N - 1$ factors for resolution III designs. For a resolution IV (V) design, up to $N/2$ (roughly $N^{1/2}$) factors may be included.

Small fractional factorial designs are often performed as completely randomized designs. However, the larger the run size N, the more useful running the experiment in blocks becomes. A larger design requires more experimental units and often more time, making it likely that heterogeneous conditions will be encountered before the experiment is finished. If a 16-run experiment can be performed in a single day, while a 128-run experiment requires 7 or 8 days, then the underlying random variability encountered in the larger experiment will almost surely be greater. For this larger experiment, randomizing treatment combinations to units without restrictions implies that the error variance will be greater, and so some of the precision purportedly gained by the large design is forfeited by encountering the larger error variance.

So what is the best practice when a large design is warranted to achieve the desired resolution? First, knowledge of the nature of the underlying variation in the experimental units affords a means of partitioning the runs into subsets that are more homogeneous within sets, and consequently less homogeneous between sets. Such knowledge is essential for blocking successfully. By confounding certain factorial effects with sets of units (i.e., blocks), we make the remaining contrasts orthogonal to blocks, and so shield their estimates from the extra variability between blocks. The resulting design achieves higher precision for effects that can be estimated using within-block differences. Second, if one has enough blocks, even the effects confounded with blocks can be estimated, albeit with lower precision than if the design were completely randomized. Thus, prior knowledge of

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which experimental units are likely to be similar is utilized to enhance the precision of some effect estimates, while this benefit is achieved by sacrificing the precision of other estimates. Fisher (1950) advocated such designs, not just for the increased precision for effects of interest, but also for the wider validity that an experiment with heterogeneous experimental units provides.

The preceding discussion is familiar to statisticians, but apparently not to practitioners, since relatively few fractional factorial designs are run as blocked designs. Blocking ought to be used more frequently. Which particular incomplete block designs should be utilized is the focus of this article. In the next section we review four criteria that statisticians have suggested, and, in the context of large blocked designs, provide arguments for one criterion proposed by Cheng and Wu (2002).

Several papers address the construction of minimum aberration blocked designs. Sitter, Chen and Feder (1997) provide collections of minimum aberration blocked designs with all 8 and 16 runs, 32 runs up to 15 factors, 64 runs up to 9 factors, and 128 runs up to 9 factors. Chen and Cheng (1999) develop a theory to characterize minimum aberration blocked designs in terms of their blocked residual designs and give collections of minimum aberration blocked designs with all 8 and 16 runs, and 32 runs up to 20 factors. Cheng and Wu (2002) compare minimum aberration blocked designs with respect to different combined wordlength patterns for 8, 16, 32, 64, and 128 runs up to 9 factors; they also provide collections of minimum aberration blocked designs with all 27 runs, and 81 runs up to 10 factors. Xu (2006) and Xu and Lau (2006) further develop some theory and construct minimum aberration blocked designs with all 32 runs, all 81 runs, and 64 runs up to 32 factors. However, minimum aberration blocked designs with 128 runs are not available in the literature because one has to compare a huge number of possible blocking schemes. Indeed, for each unblocked design, there are often more than a hundred million ways to arrange it into blocks. In Section 3 we discuss this computational challenge and present a novel method to efficiently reduce the blocking schemes to be considered when the number of blocks is larger than the number of runs per block. In Section 4 we present tables of minimum aberration blocked designs with 128 runs and 8–64 factors. We conclude by analyzing a blocked fractional factorial design, based on an experiment first reported by Young, Abraham and Whitney (1991).

2 The Recommended Minimum Aberration Criterion

Suppose a regular 2^{n-k} fraction is to be conducted in 2^p blocks of size 2^{n-k-p}. Then the defining relation for this $(1/2)^k$ fraction will contain $2^k - 1$ factorial effects, and $(2^p - 1)2^k$ factorial effects will be confounded with blocks. For instance, for $n = 9$, $k = 4$, and $p = 2$, the defining relation for this 2^{9-4} fraction will contain 15 interactions, and 48 factorial effects will be confounded with blocks. The length of the shortest word in the defining relation determines the resolution R of a completely randomized 2^{n-k}. For blocked designs, one may use Bisgaard's (1994) definition of $\min\{R, r + 1\}$ for the resolution, where r is the shortest length effect confounded with blocks. For instance, if a resolution IV 2^{9-4} uses blocking that confounds two-factor interactions with blocks, then Bisgaard labels the design as resolution III.

The word length pattern of the defining relation

$$\mathrm{w}lp = (A_3, A_4, ..., A_n)$$

is used to summarize the aberration of the fractional factorial. The *minimum aberration* criterion (Fries and Hunter 1980) chooses designs that minimize $A_3, A_4, \ldots$ in a sequential order. Similarly, it is useful to create a word length pattern for the $(2^p - 1)2^k$ factorial effects confounded with blocks:

$$\text{w}lp_b = (A_{2.1}, A_{3.1}, ..., A_{n.1}),$$

where $A_{j.1}$ denotes the number of j-factor interactions confounded with blocks. Sun, Wu, and Chen (1997) declare a blocking scheme for a particular 2^{n-k} to be *admissible* if for that fractional factorial, the blocking word length pattern has minimum aberration.

Sun, Wu, and Chen's (1997) admissibility criterion excludes certain blocked designs as inferior, but does not provide any means for comparing blocked designs involving nonisomorphic fractions. However, other authors have proposed combining wlp and wlp_b into a single criterion to use in ranking blocked fractional factorial designs with different fractions. The following combined word length sequences have been suggested:

- $W_1 = [A_3,\ A_4,\ A_{2.1},\ A_5,\ A_6,\ A_{3.1},\ A_7,\ A_8,\ A_{4.1},\ ...]$
- $W_2 = [A_3,\ A_{2.1},\ A_4,\ A_5,\ A_{3.1},\ A_6,\ A_7,\ A_{4.1},\ ...]$
- $W_{CC} = [A_3 + A_{2.1}/3,\ A_4,\ A_5 + A_{3.1}/10,\ A_6,\ A_7 + A_{4.1}/35,\ ...]$
- $W_{SCF} = [A_3,\ A_{2.1},\ A_4,\ A_{3.1},\ A_5,\ A_{4.1},\ A_6,\ A_{5.1},\ ...]$

W_{SCF} was proposed by Sitter, Chen and Feder (1997). However, Chen and Cheng (1999) and Zhang and Park (2000) criticize this sequence for violating the principle of hierarchy. By placing $A_{4.1}$ before A_6, the aliasing of a single four-factor interaction with blocks is penalized more than the aliasing among three-factor interactions (and the aliasing of four-factor interactions with two-factor interactions) that results from a length-six word. Chen and Cheng (1999) mention W_2 as an improvement over W_{SCF}, but then argue for W_{CC} based on an estimation capacity perspective. However, by combining A_3 and $A_{2.1}$, W_{CC} in effect overlooks the ambiguity length-three words cause for main effects, and simply focuses on the number of eligible two-factor interactions. For this reason, we do not consider W_{CC} further. Neither do we consider the criteria proposed by Cheng and Tsai (2009, Section 3), which depend on the assumption of no three-factor and higher-order interactions. Cheng and Wu (2002) advocated both W_1 and W_2 as viable criteria. By listing A_4 second, the W_1 sequence clearly weights the word length pattern for the fractional factorial design the most heavily. We will argue that this is generally reasonable.

To illustrate the difference between W_1 and W_2, consider the case of nine factors in four blocks of size 8. The minimum aberration design 9-4.1 with generators $F = ABCD, G = ABE, H = ACE$, and $J = ADE$ is optimal under the W_1 criterion if one blocks on AB and $ACDE(= DH)$. As reported by Xu and Lau (2006, p. 4102), the optimal design under the W_2 criterion is obtained by blocking a higher aberration design. Table 1 contrasts these two designs, showing main effects and two-factor interactions associated with the 31 columns in Yates order. W_1 favors the design on the left because it has smaller A_4 (6 vs. 9), while W_2 favors the design on the right because it only confounds 2 two-factor interactions with blocks, $A_{2.1} = 2$ (vs. 4). However, this better confounding with blocks comes at a serious price, requiring an even resolution IV design with only (15 - 1 =) 14 degrees of freedom for two-factor interactions, whereas the minimum aberration design provides (21 - 2 =) 19 degrees of freedom for two-factor interactions, and even has eight clear two-factor interactions.

For resolution III and IV fractions, the choice between criteria W_1 and W_2 depends on whether A_4 or $A_{2.1}$ is deemed more critical; note that both pertain to the loss of information about two-factor interactions. For resolution V and VI fractions, the difference hinges on whether A_6 or $A_{3.1}$

is considered first, both of which concern three-factor interactions. If block effects are likely, while interactions among experimental factors are rare, then according to Cheng and Wu (2002) the W_2 sequence is reasonable, especially when follow-up experiments are expected, which will likely undo the worst aliasing among factorial effects.

For large fractional factorial designs, we argue for use of W_1 for the following three reasons. First and foremost, effects confounded with blocks can often be estimated. With eight or more blocks, one can usefully estimate effects confounded with blocks via inter-block contrasts, treating block effects as random. Thus, confounding with blocks does not preclude estimation; it simply lowers the precision. The real experiment analyzed in Section 5 is a good example. Thus confounding reflected in $A_{2.1}$ (and $A_{3.1}$) does not cause such confusion about active effects in the same way that $A_4 > 0$ does. On this basis alone, the W_1 criterion is more reasonable than the other three criteria listed above.

Second, in some experiments, block effects are not more likely than interactions. If blocks correspond, e.g., to different days, and not to conditions that are markedly different from block to block, then their effects may be negligible. The 2^5 experiment in four blocks by Hoang, Liauw, Allen, Fontan, and Lafuente (2004) typifies the situation where factor effects dominate. See also the potato yield experiment in Yates (1937) and the experiment by Young, Abraham and Whitney (1991), which we analyze in Section 5. While some experiments coincide with the assumption that "Block effects are more likely to be significant than treatment effects," (see Cheng and Wu 2002), for experiments where this is not the case, the loss of precision resulting from confounding with blocks can be negligible.

Finally, large experiments are less likely to require follow-up experiments, especially for high resolution fractions. Cheng and Wu (2002) point out that A_4 should be penalized less if follow-up experiments are likely, which will reduce or eliminate this aliasing. The counterpoint is also true, that if we choose a design based on W_1 rather than W_2, a follow-up design may not be required. For instance, the W_1 optimal design in Table 1, with less aliasing, is less likely to require a follow-up experiment than the W_2 optimal design. Even if there is to be a follow-up, surely the W_1 design is the preferred design. It has 19 degrees of freedom for two-factor interactions not confounded with blocks, five more than the W_2 optimal design.

For these reasons, we consider W_1 to be the most appropriate criterion proposed to date for ranking large blocked designs. While we contend that the W_1 criterion is generally preferred for fractional factorial designs with large N, there will be situations where block effects are expected to be so pronounced that no information can be gained regarding effects confounded with blocks. For these situations, one might consider both the W_1 and W_2 optimal designs, if they in fact differ, and choose the particular design that seems preferable. In the next section, we discuss the construction of the minimum aberration blocked designs for $N = 128$ according to the W_1 criterion.

3 Construction method

A regular 2^{n-k} design can be viewed as n columns of an $N \times (N-1)$ matrix which consists of $n-k$ independent columns and all possible interactions among them, where $N = 2^{n-k}$. To arrange it into 2^p blocks, one can choose p columns from the remaining columns as possible block generators. There are $\binom{N-1-n}{p}$ ways to choose p columns. Some choices lead to improper blocking schemes where some main effects are confounded with block effects. Based on coding theory, Xu and Lau (2006) develop methods to quickly screen out improper blocking schemes and to efficiently compute the treatment and block wordlength patterns without using defining contrast subgroups and alias sets. They construct minimum aberration blocked designs with 32, 64 and 81 runs according to the W_1, W_2 and W_{SCF} criteria. Xu (2006) further develops some theory and constructs minimum aberration blocked designs according to the W_{CC} criterion.

However, for $N = 128$ runs, the computation becomes cumbersome. For example, to arrange a 2^{20-13} design into $2^5 = 32$ blocks, there are $\binom{107}{5} = 106,308,566$ possible blocking schemes to be considered. This heavy burden makes it impractical to search for minimum aberration blocked designs with large N.

Here we propose an alternative construction method. Instead of choosing block generators, we consider how to partition a design into blocks directly. We present a method that works for two-level and multi-level blocked designs. Some backgrounds and notation are in order.

For a prime power s, let $GF(s)$ be the finite field of s elements. Let V_n be the n-dimensional row vector space over $GF(s)$, i.e., $V_n = \{(v_1, \ldots, v_n) : v_i \in GF(s) \text{ for } i = 1, \ldots n\}$. A regular s^{n-k} design is specified by an $(n-k) \times n$ matrix T such that T has full row rank. The design, a linear space over $GF(s)$, consists of all possible linear combinations of the rows of T.

To arrange a regular s^{n-k} design into s^p blocks of size s^q (with $q = n - k - p$), one can choose an $(n-k) \times p$ matrix B such that B has full column rank. Then a typical block of the design consists of all level combinations of the form uT, with $u \in V_{n-k}$ and $uB = v$ where v is any fixed vector in V_p. Different blocks correspond to different choices of v. Since B has full column rank p, there are s^p choices of v, leading to a division of the s^{n-k} level combinations into s^p blocks. The design is said to be *properly partitioned* into s^p blocks if no main effect is confounded with any of the block effects.

Let $L_p = (s^p - 1)/(s - 1)$. Suppose that the columns of B are $b_1, \ldots, b_p$. Let F be the $(n-k) \times L_p$ matrix whose columns are $\lambda_1 b_1 + \ldots + \lambda_p b_p$, where $\lambda_i \in GF(s)$, at least one $\lambda_i \neq 0$ and the first nonzero λ_i is 1.

The columns of T and F can be viewed as points of $PG(n-k-1, s)$, the projective geometry of dimension $n-k-1$ over $GF(s)$. In the terminology of projective geometry, F is a $(p-1)$-flat in $PG(n-k-1, s)$. It is known that a regular s^{n-k} design can be properly partitioned into s^p blocks if and only if T and F are disjoint; see Chen and Cheng (1999) and Mukerjee and Wu (1999). To avoid introducing new notation, we use T and F as both matrices and subsets of $PG(n-k-1, s)$. The meaning should be clear from the context.

Our main result is presented in the following theorem, which generalizes Theorem 4 of Xu (2006) that deals with the special case $q = 1$.

Theorem 1. *A regular s^{n-k} design D can be properly partitioned into s^p blocks of size s^q (with $q = n - k - p$) if and only if there exists a $q \times n$ submatrix V of D such that V has full row rank and every column of V is not a null vector.*

Proof. If D can be properly partitioned into s^p blocks, following the previous discussion, T and F are disjoint subsets of $PG(n-k-1, s)$. Without loss of generality, assuming $F = \binom{E}{\mathbf{0}}$, where E is a $p \times L_p$ matrix spanned by all points of $PG(p-1, s)$ and $\mathbf{0}$ is a $q \times L_p$ matrix of 0's. Further let $T = \binom{U}{V}$, where U is a $p \times n$ matrix and V is a $q \times n$ matrix. It is evident that V has full row rank because T has full row rank. Furthermore, because E consists of all points of $PG(p-1, s)$ and T and F are disjoint, every column of V must be non-null. This proves the necessity.

On the other hand, suppose there exists a $q \times n$ matrix V satisfying both conditions. There exists another a $p \times n$ submatrix U of D such that $T = \binom{U}{V}$ have full row rank $p + q = n - k$

and T generates the design D. Let $F = \binom{E}{\mathbf{0}}$, where E is a $p \times L_p$ matrix spanned by all points of $PG(p-1, s)$ and $\mathbf{0}$ is a $q \times L_p$ matrix of 0's. Clearly T and F are disjoint subsets of $PG(n-k-1, s)$; therefore, T and F properly defines a blocked design. This proves the sufficiency. □

Theorem 1 is the most useful when q is small. Constructing a blocked design is equivalent to finding a matrix V satisfying both conditions. Given an unblocked 2^{n-k} design, we can choose q rows from the $N-1$ nonzero rows to form a matrix V and check whether both conditions are satisfied. There are $\binom{N-1}{q}$ choices. This number is much smaller than $\binom{N-1-n}{p}$ when $q < p$. For example, to arrange a 2^{20-13} design into $2^5 = 32$ blocks, we need to consider $\binom{127}{2} = 8,001$ possible choices of V, instead of more than 100 million ways of choosing five block generators.

When $q = 1$, V is a row vector. Theorem 1 implies that a regular 2^{n-k} design can be partitioned into 2^{n-k-1} blocks if and only if it consists of a row of n 1's, which is equivalent to the condition that the design is an even design. An even design contains entirely defining words of even length whereas an even/odd design has at least one defining word of odd length. An even design is also called a fold-over design and each block consists of a pair of mirror runs. This result was previously observed by Xu (2006).

It is not difficult to see that all linear combinations of the q rows of V form a block, called the principal block, which is a group. Other blocks are cosets of the principal block. An example will make it clear.

Example 1. Consider blocking the minimum aberration 2^{6-2} design defined by $E = ABC$ and $F = ABD$. The design is explicitly given as the first six columns in Table 2. Note that the last row does not contain any 0 (in the first six columns); therefore, it can be arranged into 8 blocks of size 2. Now consider arranging it into 4 blocks of size 4. It is easy to see that the matrix V consisting of rows 2 and 15 satisfies both conditions in Theorem 1. To determine the block generators, we need to examine the remaining nine columns in Table 2. It is sufficient to look at rows 2 and 15. There are three columns (G, H and J) where both elements are zero at rows 2 and 15. These three columns, corresponding to AB, AC and BC, are the block columns. The principal block consists of rows 1, 2, 15 and 16. Rows 3, 4, 13 and 14 form block 2; rows 5, 6, 11 and 12 form block 3; and rows 7, 8, 9 and 10 form block 4. The minimum aberration blocking scheme with four blocks is defined by the matrix V consisting of rows 4 and 14. The principal block consists of row 1, 4, 14 and 15; and the block columns are G, N and O, corresponding to AB, ACD and BCD.

Now we briefly describe the construction procedure for 128-run blocked designs. Given a 2^{n-k} design, to optimally arrange it into 2^p blocks of size 2^q, we evaluate all possible proper blocking schemes as follows. If $p > q$, we consider all possible $q \times n$ matrices V and check whether the conditions in Theorem 1 are satisfied. For those V satisfying the conditions, we compute the block wordlength patterns and determine the minimum aberration blocking scheme. If $p \leq q$, we adopt the Xu and Lau (2006) procedure by considering all possible choices of p block generators. For 128 runs we need to evaluate at most $\binom{127}{3} = 333,375$ or $\binom{127-8}{3} = 273,819$ blocking schemes depending whether or not $p > q$. This is a relatively easy task.

The minimum aberration blocked designs are determined by comparing both treatment and block wordlength patterns for all non-equivalent designs. Since the W_1 criterion minimizes A_3 and A_4 before $A_{2.1}$, it often requires one to consider only the few resolution IV designs with the smallest A_4 values, the weak minimum aberration designs given by Block and Mee (2005) and Xu (2009). This eases the computation.

The case with 64 blocks ($p = 6, q = 1$) needs special attention. Because only even designs can be arranged into 64 blocks, we need to determine minimum aberration even designs, which have minimum aberration among all possible even designs. Finding minimum aberration even designs turns out to be more difficult than finding minimum aberration designs. We use Xu's (2009) algorithm to find minimum aberration even designs from millions of resolution IV designs.

Although we consider only 128-run designs, the proposed method works for larger blocked designs such as 256 runs in principle. Given a regular 256-run design, the proposed method can be used to find the minimum aberration blocking scheme quickly. However, the challenge for 256 runs is the construction of minimum aberration unblocked designs. For example, Xu (2009) reports only minimum aberration designs up to 28 factors for 256 runs due to the difficulty of dealing with millions of resolution IV designs with a personal computer.

4 Minimum aberration blocking schemes

Table 3 gives the minimum aberration blocking schemes with 128 runs according to the W_1 criterion for $n = 8, \ldots, 64$. The first column is the unblocked design identification. Most designs are labeled as n-$k.i$, where i is the rank under the minimum aberration criterion. For n =41–44, and 50, the minimum aberration designs are not unique and they are labeled as 1a, 1b or 1c. For n=10–40, some designs are labeled as 1e and these designs have minimum aberration among all even designs. The second column is the number of blocks and the third column gives block generators in terms of the Yates order. The last two columns are the $A_{2.1}$ and $A_{3.1}$ values.

The unblocked minimum aberration designs are used except for the following cases:

- in 64 blocks of size 2, for $n = 10, \ldots, 40$: an even resolution IV design is used, not the minimum aberration designs, since otherwise the blocks would confound a main effect.
- in various 4–32 blocks, for n =12, 13, 17, 18, 21, 23, 35, a weak minimum aberration design is used.
- in 16 or 32 blocks, for $n = 8$: the resolution VII design is used.
- in 32 blocks of size 4, for $n = 25, 28, 29$: the design with second lowest A_4 is used.

In the remaining cases, the minimum aberration design provides the fractional factorial that is optimal for blocking according to the W_1 criterion. For $n = 41$ three minimum aberration designs are used and for $n = 50$ two minimum aberration designs are used. Tables 4 and 5 list the minimum aberration designs for $8 \leq n \leq 40$ and $41 \leq n \leq 64$, respectively, which are adopted from Xu (2009, Table 11) and Block and Mee (2005, Table 4). Table 6 gives the minimum aberration even designs for $n = 10, \ldots, 40$ and Table 7 lists 12 additional designs used in Table 3.

Example 2. Consider choosing 2^{12-5} designs in 8 blocks. Table 3 suggests that we shall choose the minimum aberration design 12-5.1, whose treatment generators given in Table 4 are columns 31, 103, 43, 85 and 121. Label the 12 treatment columns as $X_1, \ldots, X_{12}$, where the first seven columns are independent and correspond to columns 1, 2, 4, 8, 16, 32, and 64. Then the remaining treatment columns are

- $X_8 = X_1 * X_2 * X_3 * X_4 * X_5$ (column 31, because $31 = 2^0 + 2^1 + 2^2 + 2^3 + 2^4$)

- $X_9 = X_1 * X_2 * X_3 * X_6 * X_7$ (column 103, because $103 = 2^0 + 2^1 + 2^2 + 2^5 + 2^6$)
- $X_{10} = X_1 * X_2 * X_4 * X_6$ (column 43, because $43 = 2^0 + 2^1 + 2^3 + 2^5$)
- $X_{11} = X_1 * X_3 * X_5 * X_7$ (column 85, because $85 = 2^0 + 2^2 + 2^4 + 2^6$)
- $X_{12} = X_1 * X_4 * X_5 * X_6 * X_7$ (column 121, because $121 = 2^0 + 2^3 + 2^4 + 2^5 + 2^6$)

There are one word of length four ($A_4 = 1$) and eight words of length five ($A_5 = 8$). The three block generators given in Table 3 are columns 7, 49 and 91, which correspond to

- $X_1 * X_2 * X_3$ (column 7, because $7 = 2^0 + 2^1 + 2^2$)
- $X_1 * X_5 * X_6$ (column 49, because $49 = 2^0 + 2^4 + 2^5$)
- $X_1 * X_2 * X_4 * X_5 * X_7$ (column 91, because $91 = 2^0 + 2^1 + 2^3 + 2^4 + 2^6$)

For this blocking scheme, none of two-factor interactions are confounded with block effects ($A_{2.1} = 0$) and 16 three-factor interactions are confounded with block effects ($A_{3.1} = 16$).

5 Analysis of blocked designs

Young, Abraham, and Whitney (1991) provide an excellent example of a fractional factorial experiment that benefitted from being conducted in blocks. Crankshafts are produced using an iron casting process, and reducing hardness variability was the primary objective of this experiment. Further objectives involved increasing line speed and achieving new specifications for pearlite percentage. In order to study within-run variability, all six crankshafts from each of several molds were selected, and each crankshaft was measured in two locations. Their article does not provide the actual data for any of the responses, but effect estimates are reported for mean Brinell hardness. We construct response values to coincide with these estimates.

Young et al. (1991) utilized the minimum aberration 2^{9-4} fractional factorial design for their nine factor experiment. To match Design 9-4.1 in Table 1, label the factors as A (Silicon), B (Carbon), C (Manganese), D (Tin), E (Copper), F (Line Speed), G (Compactibility), H (Chrome), and J (Temperature). The low and high levels for Line Speed were current speed and a 20% increase in speed, respectively; the other factors' levels were not reported in the article.

Rather than using four blocks as in Table 1, Young et al. used eight blocks. Each block consisted of four runs performed in an 8-hour shift. To ensure that the experiment contained a broad representation of operating conditions, the eight shifts for the experiment were spread over a two month period. In particular, between these shifts the clay levels in the sand and the batches of scrap metal used changed. By blocking on Column 5 in addition to Columns 3 and 29, the W_1 optimal block design for eight blocks is obtained. In addition to the three columns used to generate blocks, Columns 6, 24, 27, and 30 are confounded with blocks. Thus, $A_{2.1} = 12$ (refer to Table 1). The rows, arranged by blocks, are shown in Table 8, together with our reconstituted data for Brinell hardness.

Young et al. (1991) analyze the data by constructing a single normal quantile plot. However, when there are at least eight blocks, we recommend constructing two separate normal probability plots of estimates, one for effects confounded with blocks, and a second for effects orthogonal to blocks. Table 9 partitions the estimates in this way; within each partition, we sort them by magnitude to facilitate the computation of Lenth's pseudo standard error (PSE). This analysis is equivalent to the standard analysis for unreplicated split-plot designs, where separate normal plots are used for whole-plot and split-plot estimates; see Bisgaard, Fuller and Barrios (1996).

Note that the PSE is larger for effects confounded with blocks (24 versus 18). This is typical, as any between-block variation will decrease the precision of estimates confounded with blocks. Ye and Hamada (2001) provide critical values for Lenth t statistics for $m = 2^p - 1 = 7, 15, 31,$ ... contrasts, while Loeppky and Sitter (2002) add critical values for $m = 2^k - 2^p$. These critical values were obtained by simulating the null distribution of Lenth t statistics. The p-values in Table 9 were obtained via simulation as described in Edwards and Mee (2008). Three main effects are statistically significant. According to these estimates, the desirable lower hardness was achieved by decreasing Manganese and adding Copper or Silicon. The fourth largest estimate corresponds to a pair of aliased interactions confounded with blocks. This estimate is not exceptionally large, given the magnitude of the other estimates confounded with blocks. Figure 1 shows two separate normal probability plots, one for effects orthogonal to blocks and the other for effects confounded with blocks. Three main effects are identified as significant from the first plot. In the second plot, all points are close to a straight line, suggesting that none of the effects confounded with blocks appears to be large. The results are consistent with the Lenth method reported in Table 9.

Young et al. (1991) state that three factors were restricted in randomization, so that their levels were changed only once per day between the second and third run of each shift. This creates a split-plot structure, resulting in the need to partition the 24 estimates that are orthogonal to blocks into two lists: eight whole-plot contrasts and 16 split-plot contrasts. If the three whole-plot factors were A, B, and C (as the article intimates and the arrangement of rows in Table 8 suggests), then the whole-plot contrasts are these three factors plus DF, EF, FG, FH, and J. (Note that due to the blocking, a fourth factor is inadvertently constrained to switch levels only once per day.) If a PSE is calculated from these eight estimates, one obtains PSE = 13.88. Since this whole-plot PSE is smaller than the PSE for all 24 effects orthogonal to blocks, the whole-plot restriction to run order does not appear to have affected the results. Thus, we ignored that feature in our earlier analysis.

Here we have treated block effects as randomly distributed, in order to estimate effects confounded with blocks. For this foundry experiment, blocks represented a sampling of shifts spread out over time. Thus, treating blocks as a random effect seems appropriate. If any of the interactions confounded with blocks did have a large effect, this design provides opportunity to see that effect. However, no interactions appear to be active.

If a randomized block design contains four or fewer blocks, there is no point in constructing two separate normal effects plots or computing Lenth's PSE for the confounded effects. In such cases, there is no effective test for effects confounded with blocks, since there is too little information to estimate the block-to-block variability. According to Loeppky and Sitter (2002), it is best to exclude the few estimates confounded with blocks from the PSE calculation. That is the conservative approach. However, if block effects are expected to be negligible, some statisticians will interpret all the estimates using a single normal plot and a single PSE. If this is attempted, one must remain cognizant that the confounding can potentially inflate the PSE and/or make an effect confounded with blocks appear active when the true coefficient is negligible.

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Table 1: Comparison of two nine-factor designs in four blocks of size eight

Column	Design 9-4.1 blocking on Columns 3, 29 (W_1 optimal design)		Design 9-4.3 blocking on Columns 6, 26 (W_2 optimal design)	
1	A		A	
2	B		B	
3	$AB = EG$	= Block	$AB = CF = DG$	
4	C		C	
5	$AC = EH$		$AC = BF = EH$	
6	$BC = GH$		$BC = AF$	= Block
7	DF		F	
8	D		D	
9	$AD = EJ$		$AD = BG$	
10	$BD = GJ$		$BD = AG = HJ$	
11	CF		G	
12	$CD = HJ$		$CD = FG$	
13	BF			
14	AF			
15	F		$CG = DF = EJ$	
16	E		E	
17	$AE = BG = CH = DJ$		$AE = CH$	
18	$BE = AG$		$BE = FH$	
19	G			
20	$CE = AH$		$CE = AH = GJ$	
21	H		H	
22	FJ			
23	$CG = BH$		$BH = DJ = EF$	
24	$DE = AJ$		$DE = FJ$	
25	J			
26	FH			= Block
27	$DG = BJ$		$CJ = EG$	
28	FG			= Block
29	$DH = CJ$	= Block	$BJ = DH$	
30		= Block	$AJ = GH$	
31	EF		J	

Table 2: The design matrix for 16-run designs

	A	B	C	D	E	F	G	H	J	K	L	M	N	O	P
Row 1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Row 2	0	0	0	1	0	1	0	0	0	1	1	1	1	1	1
Row 3	0	0	1	0	1	0	0	1	1	0	0	1	1	1	1
Row 4	0	0	1	1	1	1	0	1	1	1	1	0	0	0	0
Row 5	0	1	0	0	1	1	1	0	1	0	1	0	0	1	1
Row 6	0	1	0	1	1	0	1	0	1	1	0	1	1	0	0
Row 7	0	1	1	0	0	1	1	1	0	0	1	1	1	0	0
Row 8	0	1	1	1	0	0	1	1	0	1	0	0	0	1	1
Row 9	1	0	0	0	1	1	1	1	0	1	0	0	1	0	1
Row 10	1	0	0	1	1	0	1	1	0	0	1	1	0	1	0
Row 11	1	0	1	0	0	1	1	0	1	1	0	1	0	1	0
Row 12	1	0	1	1	0	0	1	0	1	0	1	0	1	0	1
Row 13	1	1	0	0	0	0	0	1	1	1	1	0	1	1	0
Row 14	1	1	0	1	0	1	0	1	1	0	0	1	0	0	1
Row 15	1	1	1	0	1	0	0	0	0	1	1	1	0	0	1
Row 16	1	1	1	1	1	1	0	0	0	0	0	0	1	1	0

Note: The independent columns are A, B, C, and D. Other columns are defined by $E = ABC$, $F = ABD$, $G = AB$, $H = AC$, $J = BC$, $K = AD$, $L = BD$, $M = CD$, $N = ACD$, $O = BCD$, $P = ABCD$.

Table 3: Minimum Aberration Blocking for 128-Run Designs

Design ID	No. Blocks	Block Generators	$A_{2.1}$	$A_{3.1}$
8-1.1	2	15	0	0
8-1.1	4	15 51	0	0
8-1.1	8	15 51 85	0	0
8-1.2	16	7 25 42 65	1	10
8-1.2	32	3 5 24 40 73	7	18
8-1.1	64	3 5 9 17 33 65	28	0
9-2.1	2	43	0	0
9-2.1	4	43 85	0	0
9-2.1	8	7 41 82	0	6
9-2.1	16	3 13 37 84	2	14
9-2.1	32	7 9 18 33 66	9	27
9-2.1	64	3 5 9 17 33 65	36	0
10-3.1	2	85	0	0
10-3.1	4	44 81	0	2
10-3.1	8	7 49 74	0	9
10-3.1	16	3 12 53 69	3	19
10-3.1	32	3 9 17 36 69	12	36
10-3.1e	64	3 5 9 17 33 65	45	0
11-4.1	2	121	0	1
11-4.1	4	14 115	0	4
11-4.1	8	7 49 91	0	12
11-4.1	16	6 11 49 67	4	25
11-4.1	32	3 9 20 36 69	15	48
11-4.1e	64	3 5 9 17 33 65	55	0
12-5.1	2	13	0	2
12-5.1	4	13 49	0	6
12-5.1	8	7 49 91	0	16
12-5.2	16	3 13 52 69	5	34
12-5.1	32	3 9 20 36 69	18	64
12-5.1e	64	3 5 9 17 33 65	66	0
13-6.1	2	88	0	2
13-6.1	4	25 105	0	8
13-6.2	8	13 55 67	0	22
13-6.2	16	3 13 52 69	6	44
13-6.1	32	5 11 18 35 67	22	80
13-6.1e	64	3 5 9 17 33 65	78	0
14-7.1	2	67	0	4
14-7.1	4	13 67	0	12
14-7.1	8	13 55 67	0	28
14-7.1	16	13 21 34 67	7	56
14-7.1	32	3 5 17 33 72	26	100
14-7.1e	64	3 5 9 17 33 65	91	0
15-8.1	2	78	0	4
15-8.1	4	13 116	0	12
15-8.1	8	13 55 86	0	35
15-8.1	16	13 21 34 70	9	68
15-8.1	32	3 5 17 33 73	30	125
15-8.1e	64	3 5 9 17 33 65	105	0

Design ID	No. Blocks	Block Generators	$A_{2.1}$	$A_{3.1}$
16-9.1	2	19	0	6
16-9.1	4	19 97	0	18
16-9.1	8	9 19 97	2	40
16-9.1	16	3 13 20 101	12	80
16-9.1	32	3 5 9 49 65	36	144
16-9.1e	64	3 5 9 17 33 65	120	0
17-10.1	2	121	0	5
17-10.2	4	58 86	0	20
17-10.1	8	7 25 97	3	49
17-10.3	16	3 13 20 101	13	99
17-10.1	32	3 5 17 33 73	40	180
17-10.1e	64	3 5 9 17 33 65	136	0
18-11.1	2	7	0	11
18-11.2	4	24 100	1	24
18-11.2	8	13 21 100	5	56
18-11.2	16	6 9 19 66	16	116
18-11.2	32	3 5 17 33 73	46	210
18-11.1e	64	3 5 9 17 33 65	153	0
19-12.1	2	91	0	9
19-12.1	4	13 100	2	30
19-12.1	8	3 37 89	7	66
19-12.1	16	5 10 19 99	19	134
19-12.1	32	3 5 17 33 73	51	252
19-12.1e	64	3 5 9 17 33 65	171	0
20-13.1	2	77	0	16
20-13.1	4	3 105	3	36
20-13.1	8	3 12 101	9	76
20-13.1	16	6 9 19 35	22	160
20-13.1	32	3 12 21 37 68	62	256
20-13.1e	64	3 5 9 17 33 65	190	0
21-14.1	2	112	0	14
21-14.1	4	17 97	2	42
21-14.2	8	9 19 97	6	96
21-14.2	16	6 9 18 98	22	186
21-14.1	32	3 12 20 37 68	64	336
21-14.1e	64	3 5 9 17 33 65	210	0
22-15.1	2	124	0	20
22-15.1	4	13 113	2	48
22-15.1	8	13 23 102	8	100
22-15.1	16	7 10 51 66	24	216
22-15.1	32	6 9 17 33 67	71	384
22-15.1e	64	3 5 9 17 33 65	231	0
23-16.1	2	73	0	24
23-16.2	4	34 89	3	49
23-16.2	8	9 48 66	11	110
23-16.2	16	6 9 18 98	29	238
23-16.1	32	5 10 18 34 67	77	448
23-16.1e	64	3 5 9 17 33 65	253	0

Table 3: Continued

Design ID	No. Blocks	Block Generators	$A_{2.1}$	$A_{3.1}$
24-17.1	2	62	0	24
24-17.1	4	15 112	2	64
24-17.1	8	7 58 74	10	136
24-17.1	16	5 11 35 81	30	280
24-17.1	32	3 5 9 49 65	84	512
24-17.1e	64	3 5 9 17 33 65	276	0
25-18.1	2	3	2	19
25-18.1	4	3 13	6	64
25-18.1	8	3 9 17	16	140
25-18.1	16	3 5 9 17	38	294
25-18.2	32	7 9 18 33 66	92	576
25-18.1e	64	3 5 9 17 33 65	300	0
26-19.1	2	112	0	28
26-19.1	4	15 112	2	84
26-19.1	8	15 22 101	13	176
26-19.1	16	5 11 35 81	36	356
26-19.1	32	3 13 21 36 65	121	504
26-19.1e	64	3 5 9 17 33 65	325	0
27-20.1	2	127	0	30
27-20.1	4	7 123	5	100
27-20.1	8	7 17 106	15	200
27-20.1	16	5 10 34 81	39	400
27-20.1	32	7 11 18 35 66	135	540
27-20.1e	64	3 5 9 17 33 65	351	0
28-21.1	2	124	0	56
28-21.1	4	5 121	6	112
28-21.1	8	5 10 115	18	224
28-21.1	16	5 10 34 81	42	448
28-21.2	32	7 11 18 35 66	150	576
28-21.1e	64	3 5 9 17 33 65	378	0
29-22.1	2	5	3	31
29-22.1	4	5 10	9	93
29-22.1	8	5 10 34	21	217
29-22.1	16	3 5 9 17	69	369
29-22.2	32	3 13 21 36 65	191	475
29-22.1e	64	3 5 9 17 33 65	406	0
30-23.1	2	91	0	56
30-23.1	4	9 70	5	138
30-23.1	8	3 20 76	19	276
30-23.1	16	3 12 49 84	50	548
30-23.1	32	3 5 9 17 65	183	648
30-23.1e	64	3 5 9 17 33 65	435	0
31-24.1	2	51	0	61
31-24.1	4	9 70	5	153
31-24.1	8	3 61 84	20	305
31-24.1	16	5 10 51 82	54	605
31-24.1	32	6 9 17 33 65	189	756
31-24.1e	64	3 5 9 17 33 65	465	0

Design ID	No. Blocks	Block Generators	$A_{2.1}$	$A_{3.1}$
32-25.1	2	70	0	66
32-25.1	4	3 88	5	168
32-25.1	8	3 20 76	21	334
32-25.1	16	3 12 49 84	58	664
32-25.1	32	3 5 9 17 65	195	882
32-25.1e	64	3 5 9 17 33 65	496	0
33-26.1	2	79	0	71
33-26.1	4	12 87	5	184
33-26.1	8	3 20 76	23	368
33-26.1	16	3 12 49 84	62	730
33-26.1	32	3 5 9 17 65	213	931
33-26.1e	64	3 5 9 17 33 65	528	0
34-27.1	2	93	0	76
34-27.1	4	3 88	6	203
34-27.1	8	5 10 82	25	404
34-27.1	16	3 12 49 84	66	800
34-27.1	32	3 5 9 17 65	232	980
34-27.1e	64	3 5 9 17 33 65	561	0
35-28.1	2	62	0	91
35-28.1	4	3 88	7	224
35-28.2	8	3 61 84	27	444
35-28.1	16	3 12 49 84	70	875
35-28.1	32	3 5 9 17 65	252	1029
35-28.1e	64	3 5 9 17 33 65	595	0
36-29.1	2	87	0	98
36-29.1	4	3 88	7	245
36-29.1	8	3 12 84	28	483
36-29.1	16	3 5 17 65	121	588
36-29.1	32	3 5 9 17 65	259	1176
36-29.1e	64	3 5 9 17 33 65	630	0
37-30.1	2	88	0	105
37-30.1	4	3 88	7	266
37-30.1	8	3 49 88	29	525
37-30.1	16	3 5 24 104	124	672
37-30.1	32	3 5 24 41 65	266	1344
37-30.1e	64	3 5 9 17 33 65	666	0
38-31.1	2	91	0	112
38-31.1	4	49 91	7	288
38-31.1	8	3 9 96	45	427
38-31.1	16	3 5 24 104	134	704
38-31.1	32	3 5 24 41 65	287	1408
38-31.1e	64	3 5 9 17 33 65	703	0
39-32.1	2	106	0	119
39-32.1	4	3 105	16	231
39-32.1	8	3 9 96	48	455
39-32.1	16	3 5 24 41	145	736
39-32.1	32	3 5 24 41 65	309	1472
39-32.1e	64	3 5 9 17 33 65	741	0

Table 3: Continued

Design ID	No. Blocks	Block Generators	$A_{2.1}$	$A_{3.1}$
40-33.1	2	3	8	64
40-33.1	4	3 9	24	192
40-33.1	8	3 5 9	68	384
40-33.1	16	3 5 9 17	156	768
40-33.1	32	3 5 9 17 65	332	1536
40-33.1e	64	3 5 9 17 33 65	780	0
41-34.1a	2	94	0	174
41-34.1a	4	22 102	12	352
41-34.1c	8	22 42 76	38	708
41-34.1b	16	15 22 33 65	100	1404
41-34.1a	32	3 5 9 48 80	400	0
41-34.1a	64	3 5 9 17 33 65	820	0
42-35.1a	2	121	0	187
42-35.1a	4	25 97	13	380
42-35.1a	8	6 10 112	40	764
42-35.1a	16	3 9 20 101	201	0
42-35.1a	32	3 5 9 17 96	421	0
42-35.1a	64	3 5 9 17 33 65	861	0
43-36.1a	2	11	0	205
43-36.1a	4	3 61	14	410
43-36.1a	8	3 60 69	43	820
43-36.1a	16	3 5 9 65	213	0
43-36.1a	32	3 5 9 17 65	443	0
43-36.1a	64	3 5 9 17 33 65	903	0
44-37.1a	2	31	0	216
44-37.1a	4	14 83	14	439
44-37.1a	8	7 9 83	43	882
44-37.1a	16	3 5 17 33	223	0
44-37.1a	32	3 5 9 33 80	463	0
44-37.1a	64	3 5 9 17 33 65	946	0
45-38.1	2	14	0	235
45-38.1	4	9 84	15	470
45-38.1	8	7 9 83	45	945
45-38.1	16	3 5 9 80	234	0
45-38.1	32	3 5 9 17 65	486	0
45-38.1	64	3 5 9 17 33 65	990	0
46-39.1	2	19	0	250
46-39.1	4	19 103	15	500
46-39.1	8	7 9 83	48	1008
46-39.1	16	3 5 17 96	243	0
46-39.1	32	3 5 9 17 96	507	0
46-39.1	64	3 5 9 17 33 65	1035	0
47-40.1	2	19	0	265
47-40.1	4	7 19	16	535
47-40.1	8	7 9 83	51	1074
47-40.1	16	3 5 17 96	253	0
47-40.1	32	3 5 9 17 96	529	0
47-40.1	64	3 5 9 17 33 65	1081	0
48-41.1	2	7	0	286
48-41.1	4	3 21	18	572
48-41.1	8	3 21 72	54	1144
48-41.1	16	3 5 17 96	264	0
48-41.1	32	3 5 9 17 96	552	0
48-41.1	64	3 5 9 17 33 65	1128	0
49-42.1	2	7	0	304
49-42.1	4	7 18	18	608
49-42.1	8	3 21 72	57	1216
49-42.1	16	3 5 17 96	276	0
49-42.1	32	3 5 9 17 96	576	0
49-42.1	64	3 5 9 17 33 65	1176	0
50-43.1a	2	28	0	321
50-43.1a	4	29 70	19	644
50-43.1c	8	3 21 72	60	1291
50-43.1a	16	5 10 17 33	288	0
50-43.1a	32	5 10 17 33 65	600	0
50-43.1a	64	3 5 9 17 33 65	1225	0
51-44.1	2	19	0	343
51-44.1	4	9 112	20	686
51-44.1	8	3 5 121	64	1372
51-44.1	16	3 9 20 33	301	0
51-44.1	32	3 9 20 33 65	625	0
51-44.1	64	3 5 9 17 33 65	1275	0
52-45.1	2	7	0	364
52-45.1	4	3 21	21	728
52-45.1	8	7 49 89	66	1456
52-45.1	16	3 5 9 33	312	0
52-45.1	32	3 5 9 17 33	650	0
52-45.1	64	3 5 9 17 33 65	1326	0
53-46.1	2	7	0	385
53-46.1	4	6 88	22	770
53-46.1	8	7 49 89	69	1540
53-46.1	16	3 5 9 33	325	0
53-46.1	32	3 5 9 17 33	676	0
53-46.1	64	3 5 9 17 33 65	1378	0
54-47.1	2	14	0	407
54-47.1	4	3 21	23	815
54-47.1	8	14 21 67	72	1628
54-47.1	16	3 12 17 33	338	0
54-47.1	32	3 9 20 33 65	702	0
54-47.1	64	3 5 9 17 33 65	1431	0
55-48.1	2	7	0	431
55-48.1	4	7 17	24	862
55-48.1	8	3 5 40	163	0
55-48.1	16	3 5 9 96	351	0
55-48.1	32	3 5 9 17 96	729	0
55-48.1	64	3 5 9 17 33 65	1485	0

Table 3: Continued

Design ID	No. Blocks	Block Generators	$A_{2.1}$	$A_{3.1}$
56-49.1	2	7	0	455
56-49.1	4	3 13	25	910
56-49.1	8	15 51 85	168	0
56-49.1	16	3 12 48 85	364	0
56-49.1	32	3 5 9 48 80	756	0
56-49.1	64	3 5 9 17 33 65	1540	0
57-50.1	2	14	0	480
57-50.1	4	10 84	26	960
57-50.1	8	3 5 40	175	0
57-50.1	16	3 5 9 33	378	0
57-50.1	32	3 5 9 17 33	784	0
57-50.1	64	3 5 9 17 33 65	1596	0
58-51.1	2	22	0	506
58-51.1	4	10 84	27	1012
58-51.1	8	3 5 40	182	0
58-51.1	16	3 5 9 33	392	0
58-51.1	32	3 5 9 17 33	812	0
58-51.1	64	3 5 9 17 33 65	1653	0
59-52.1	2	47	0	533
59-52.1	4	10 84	28	1066
59-52.1	8	3 5 33	189	0
59-52.1	16	3 5 9 33	406	0
59-52.1	32	3 5 9 17 33	841	0
59-52.1	64	3 5 9 17 33 65	1711	0
60-53.1	2	49	0	561
60-53.1	4	10 84	29	1122
60-53.1	8	3 5 33	196	0
60-53.1	16	3 5 33 89	420	0
60-53.1	32	3 5 9 33 80	870	0
60-53.1	64	3 5 9 17 33 65	1770	0
61-54.1	2	67	0	590
61-54.1	4	10 84	30	1180
61-54.1	8	3 5 33	203	0
61-54.1	16	3 5 33 65	435	0
61-54.1	32	3 5 9 33 65	900	0
61-54.1	64	3 5 9 17 33 65	1830	0
62-55.1	2	84	0	620
62-55.1	4	10 84	31	1240
62-55.1	8	3 5 17	210	0
62-55.1	16	3 5 17 33	450	0
62-55.1	32	3 5 17 33 65	930	0
62-55.1	64	3 5 9 17 33 65	1891	0
63-56.1	2	94	0	651
63-56.1	4	3 5	93	0
63-56.1	8	3 5 9	217	0
63-56.1	16	3 5 9 17	465	0
63-56.1	32	3 5 9 17 33	961	0
63-56.1	64	3 5 9 17 33 65	1953	0
64-57.1	2	3	32	0
64-57.1	4	3 5	96	0
64-57.1	8	3 5 9	224	0
64-57.1	16	3 5 9 17	480	0
64-57.1	32	3 5 9 17 33	992	0
64-57.1	64	3 5 9 17 33 65	2016	0

Table 4: Minimum Aberration 128-Run Designs for $8 \leq n \leq 40$

Design ID	Treatment Generators	A_4	A_5	Res.
8-1.1	127	0	0	VIII
9-2.1	31 103	0	0	VI
10-3.1	31 103 43	0	3	V
11-4.1	31 103 43 85	0	6	V
12-5.1	31 103 43 85 121	1	8	IV
13-6.1	31 103 43 85 44 86	2	16	
14-7.1	31 103 43 85 46 61 114	3	24	
15-8.1	31 103 43 85 46 61 114 67	7	32	
16-9.1	31 103 43 85 44 86 88 53 110	10	48	
17-10.1	31 103 43 85 46 61 114 67 78 116	15	60	
18-11.1	31 103 43 85 46 61 114 67 78 116 121	20	80	
19-12.1	31 103 43 85 46 61 114 67 78 55 58 86	27	120	
20-13.1	31 103 43 85 46 61 114 67 78 55 58 86 91	36	152	
21-14.1	31 103 43 85 44 82 54 56 88 78 123 125 104 25	51	200	
22-15.1	31 103 43 85 44 86 88 53 78 58 83 97 28 104 114	65	248	
23-16.1	31 103 43 85 44 82 54 56 88 78 123 125 104 25 112 49	83	316	
24-17.1	31 103 43 85 44 86 88 53 110 19 28 57 67 98 100 26 105	102	384	
25-18.1	31 103 43 85 44 86 88 53 38 58 79 83 110 124 97 104 114 123	124	482	
26-19.1	31 103 43 85 44 86 88 53 110 19 28 57 67 98 100 26 105 62 77	152	568	
27-20.1	31 103 43 85 44 86 88 53 110 19 28 57 67 98 100 26 105 62 77 112	180	690	
28-21.1	31 103 43 85 44 86 88 53 110 19 28 57 67 98 100 26 105 62 77 112 127	210	840	
29-22.1	31 103 43 85 44 86 88 53 110 19 28 57 67 98 100 26 105 62 77 112 127 124	266	945	
30-23.1	31 103 43 81 45 26 114 127 22 67 56 94 116 7 38 108 14 69 53 25 73 121 28	335	972	
31-24.1	same as design 30-23.1, plus 91	391	1134	
32-25.1	same as design 30-23.1, plus 51 97	452	1322	
33-26.1	same as design 30-23.1, plus 51 97 70	518	1543	
34-27.1	same as design 30-23.1, plus 51 97 70 79	589	1800	
35-28.1	same as design 30-23.1, plus 51 97 70 79 93	665	2100	
36-29.1	same as design 30-23.1, plus 51 97 70 79 93 62	756	2401	
37-30.1	same as design 30-23.1, plus 51 97 70 79 93 62 87	854	2744	
38-31.1	same as design 30-23.1, plus 51 97 70 79 93 62 87 88	959	3136	
39-32.1	same as design 30-23.1, plus 51 97 70 79 93 62 87 88 91	1071	3584	
40-33.1	same as design 30-23.1, plus 51 97 70 79 93 62 87 88 91 106	1190	4096	

Note: Adopted from Xu (2009, Table 11).

Table 5: Minimum Aberration 128-Run Designs for $41 \leq n \leq 64$

Design ID	Treatment Generators	A_4	A_5
41-34.1a	11 13 14 19 21 26 28 31 35 38 41 49 52 56 59 61 62 67 69 73 74 76 82 84 88 97 98 100 109 110 115 117 118 122	1648	0
41-34.1b	11 13 14 19 21 26 28 31 35 38 41 49 52 59 61 62 67 69 70 73 74 81 82 84 93 98 100 103 107 109 110 115 117 127	1648	0
41-34.1c	same as design 41-34.1b, except replace 31 with 56	1648	0
42-35.1a	same as design 41-34.1b, plus 56	1822	0
43-36.1a	7 13 19 22 25 26 31 37 38 41 42 47 49 50 52 55 56 59 73 74 76 82 84 88 93 97 100 103 104 107 109 110 112 115 118 124	2009	0
44-37.1a	11 13 25 26 28 35 37 38 41 42 44 50 52 55 56 59 61 62 69 70 73 74 76 79 81 87 91 97 98 100 107 110 117 118 121 122 124	2214	0
45-38.1	same as design 44-37.1a, plus 31	2430	0
46-39.1	same as design 44-37.1a, plus 31 115	2665	0
47-40.1	same as design 44-37.1a, plus 31 115 103	2915	0
48-41.1	same as design 44-37.1a, plus 31 115 103 19	3180	0
49-42.1	same as design 44-37.1a, plus 31 115 103 19 127	3466	0
50-43.1a	7 11 13 19 22 26 31 35 37 38 41 42 44 47 49 50 52 55 56 59 73 74 76 81 82 84 88 93 94 97 98 100 103 104 107 109 110 112 115 117 118 124 127	3770	0
50-43.1c	same as design 49-42.1, plus 112	3770	0
51-44.1	7 11 13 22 25 26 28 31 37 38 41 42 44 47 49 50 52 55 56 61 62 69 70 73 74 76 79 81 82 84 87 88 93 94 97 98 100 103 107 109 110 115 117 118	4091	0
52-45.1	same as design 50-43.1c, plus 82 93	4433	0
53-46.1	same as design 50-43.1c, plus 82 93 109	4797	0
54-47.1	same as design 50-43.1c, plus 82 93 109 104	5182	0
55-48.1	same as design 50-43.1c, plus 82 93 109 104 88	5589	0
56-49.1	19 21 22 25 26 28 35 37 38 41 42 44 49 50 52 55 56 59 61 62 67 69 70 73 74 76 81 82 84 87 88 91 93 94 97 98 100 103 104 107 109 110 115 117 118 121 122 124 127	6020	0
57-50.1	same as design 55-48.1, plus 7 21	6475	0
58-51.1	same as design 55-48.1, plus 7 21 14	6955	0
59-52.1	same as design 55-48.1, plus 7 21 14 22	7461	0
60-53.1	same as design 55-48.1, plus 7 21 14 22 47	7994	0
61-54.1	same as design 55-48.1, plus 7 21 14 22 47 49	8555	0
62-55.1	same as design 55-48.1, plus 7 21 14 22 47 49 67	9145	0
63-56.1	same as design 55-48.1, plus 7 21 14 22 47 49 67 84	9765	0
64-57.1	same as design 55-48.1, plus 7 21 14 22 47 49 67 84 94	10416	0

Note: Adopted from Block and Mee (2005, Table 4). For n = 41, 42, 43, 44 and 50, minimum aberration designs are not unique.

Table 6: Minimum Aberration Even 128-Run Designs

Design ID	Treatment Generators	A_4	A_5
10-3.1e	31 103 41	1	0
11-4.1e	31 103 41 82	2	0
12-5.1e	31 103 41 82 124	3	0
13-6.1e	31 103 41 82 44 93	6	0
14-7.1e	31 103 41 82 44 93 7	11	0
15-8.1e	31 103 41 82 124 7 52 94	15	0
16-9.1e	31 103 41 82 124 7 52 94 109	20	0
17-10.1e	31 103 41 82 124 7 52 94 109 11	30	0
18-11.1e	31 103 41 82 124 7 52 94 109 11 49	40	0
19-12.1e	31 103 41 82 124 7 52 94 109 11 49 122	51	0
20-13.1e	31 103 41 82 44 93 7 19 97 107 61 25 87	68	0
21-14.1e	31 103 41 82 44 93 7 19 97 107 61 25 87 117	84	0
22-15.1e	31 103 41 82 44 93 7 19 97 107 61 25 87 117 11	107	0
23-16.1e	31 103 41 82 44 93 7 19 97 107 61 25 87 117 11 37	133	0
24-17.1e	31 103 41 82 44 93 7 19 97 107 61 25 87 117 11 37 70	162	0
25-18.1e	31 103 41 82 44 93 7 19 97 107 22 49 74 50 87 110 117 11	194	0
26-19.1e	31 103 41 82 124 7 52 94 11 49 67 110 84 121 13 19 37 73 59	230	0
27-20.1e	31 103 41 82 124 7 52 94 11 49 67 110 84 121 13 19 37 73 59 97	270	0
28-21.1e	31 103 41 82 124 7 52 94 11 49 67 110 84 121 13 19 37 73 59 97 127	315	0
29-22.1e	31 103 41 82 124 7 52 94 11 49 67 110 84 121 13 19 37 73 59 97 127 14	371	0
30-23.1e	31 103 41 82 124 7 52 94 11 49 67 110 84 121 13 19 37 73 59 97 127 14 21	433	0
31-24.1e	same as design 30-23.1e, plus 35	501	0
32-25.1e	same as design 30-23.1e, plus 35 70	575	0
33-26.1e	same as design 30-23.1e, plus 35 70 56	656	0
34-27.1e	same as design 30-23.1e, plus 35 70 56 93	744	0
35-28.1e	same as design 30-23.1e, plus 35 70 56 93 107	840	0
36-29.1e	same as design 30-23.1e, plus 35 70 56 93 107 112	945	0
37-30.1e	same as design 30-23.1e, plus 35 70 56 93 107 112 22	1065	0
38-31.1e	same as design 30-23.1e, plus 35 70 56 93 107 112 22 25	1195	0
39-32.1e	same as design 30-23.1e, plus 35 70 56 93 107 112 22 25 50	1335	0
40-33.1e	same as design 30-23.1e, plus 35 70 56 93 107 112 22 25 50 81	1486	0

Table 7: Additional 128-Run Designs Used in Table 3

Design ID	Treatment Generators	A_4	A_5	Res.
8-1.2	63	0	0	VII
12-5.2	31 103 43 85 44	1	10	VI
13-6.2	31 103 43 85 46 61	2	16	
17-10.2	31 103 43 85 46 61 114 67 78 55	15	66	
17-10.3	31 103 43 85 44 86 88 53 38 58	15	68	
18-11.2	31 103 43 85 46 61 114 67 78 55 58	20	92	
21-14.2	31 103 43 85 44 86 88 53 38 58 79 83 110 124	51	202	
23-16.2	31 103 43 85 44 86 88 53 38 58 79 83 110 124 97 104	83	318	
25-18.2	31 103 43 85 44 86 88 53 123 54 56 97 104 98 112 79 83 124	125	504	
28-21.2	31 103 43 85 44 86 88 53 110 19 28 57 67 98 100 26 105 62 77 112 124	230	780	
29-22.2	31 103 43 85 44 86 88 53 78 62 19 114 26 28 57 100 105 67 113 127 77 91	287	823	
35-28.2	same as design 30-23.1, plus 51 97 70 79 91	665	2101	

Table 8: Reconstructed Design and Data for Young, Abraham, and Whitney (1991) Foundry Experiment

Block	A	B	C	D	E	F	G	H	J	y
1	1	1	0	1	0	0	0	1	0	392
	1	1	0	0	1	1	1	0	0	196
	0	0	1	1	0	1	0	1	1	652
	0	0	1	0	1	0	1	0	1	410
2	1	0	1	1	1	0	0	1	1	367
	1	0	1	0	0	1	1	0	1	605
	0	1	0	1	1	1	0	1	0	497
	0	1	0	0	0	0	1	0	0	425
3	0	1	1	1	1	0	0	0	0	497
	0	1	1	0	0	1	1	1	0	683
	1	0	0	1	1	1	0	0	1	305
	1	0	0	0	0	0	1	1	1	389
4	0	0	0	1	0	0	0	0	1	616
	0	0	0	0	1	1	1	1	1	344
	1	1	1	1	0	1	0	0	0	514
	1	1	1	0	1	0	1	1	0	356
5	0	0	1	0	0	0	0	1	0	824
	0	0	1	1	1	1	1	0	0	734
	1	1	0	0	0	1	0	1	1	532
	1	1	0	1	1	0	1	0	1	444
6	0	1	0	0	1	0	0	1	1	369
	0	1	0	1	0	1	1	0	1	601
	1	0	1	0	1	1	0	1	0	531
	1	0	1	1	0	0	1	0	0	677
7	1	0	0	0	1	0	0	0	0	419
	1	0	0	1	0	1	1	1	0	267
	0	1	1	0	1	1	0	0	1	555
	0	1	1	1	0	0	1	1	1	717
8	1	1	1	0	0	0	0	0	1	556
	1	1	1	1	1	1	1	1	1	410
	0	0	0	0	0	1	0	0	0	646
	0	0	0	1	1	0	1	1	0	470

Table 9: Lenth t Statistics for Young, Abraham, and Whitney (1991) Foundry Example

	Effects	Estimate	PSE	Lenth t	p-value
Between Blocks					
	$DG = BJ$	47.0	24	1.96	0.073
	$DH = CJ$	-26.0	24	-1.08	0.237
	$DE = AJ$	24.0	24	1.00	0.269
	$BC = GH$	-16.0	24	-0.67	0.500
	$BCDE = AEF = \cdots$	-13.5	24	-0.56	0.640
	$AB = EG$	6.0	24	0.25	0.833
	$AC = EH$	-1.0	24	-0.04	0.971
Orthogonal to Blocks					
	E	-68.5	18	-3.81	0.004
	C	68	18	3.78	0.005
	A	-65	18	-3.61	0.006
	$AD{=}EJ$	-23	18	-1.28	0.197
	$CG{=}BH$	23	18	1.28	0.197
	AF	-19.5	18	-1.08	0.268
	FD	-17	18	-0.94	0.330
	G	-17	18	-0.94	0.330
	$CE{=}AH$	-17	18	-0.94	0.330
	B	-16	18	-0.89	0.358
	$BD{=}GJ$	15	18	0.83	0.388
	CF	13	18	0.72	0.456
	H	-12.5	18	-0.69	0.478
	$BG{=}AE{=}CH{=}DJ$	12	18	0.67	0.500
	EF	10.5	18	0.58	0.582
	BF	10	18	0.56	0.601
	D	10	18	0.56	0.601
	J	-8	18	-0.44	0.674
	GF	-7.5	18	-0.42	0.694
	$CD{=}HJ$	-7	18	-0.39	0.713
	F	4.5	18	0.25	0.812
	FJ	4	18	0.22	0.833
	FH	-2.5	18	-0.14	0.895
	$AG{=}BE$	0	18	0.00	1.000

Figure 1: Normal Probability Plots for the Foundry Example