Towards an Extremal Network Theory – Robust GDoF Gain of Transmitter Cooperation over TIN

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Abstract

With the emergence of aligned images bounds, significant progress has been made in the understanding of robust fundamental limits of wireless networks through Generalized Degrees of Freedom (GDoF) characterizations under the assumption of finite precision channel state information at the transmitters (CSIT), especially for smaller or highly symmetric network settings. A critical barrier in extending these insights to larger and asymmetric networks is the inherent combinatorial complexity of such networks. Motivated by other fields such as extremal combinatorics and extremal graph theory, we explore the possibility of an extremal network theory, i.e., a study of extremal networks within particular regimes of interest. As our test application, we study the GDoF benefits of transmitter cooperation over the simple scheme of power control and treating interference as Gaussian noise (TIN) for three regimes of interest where the interference is weak. The question is intriguing because while in general transmitter cooperation can be quite powerful, finite precision CSIT and weak interference favor TIN. The three regimes that we explore include a TIN regime previously identified by Geng et al. where TIN was shown to be GDoF optimal for the K user interference channel, a CTIN regime previously identified by Yi and Caire where the GDoF region achievable by TIN turns out to be convex without the need for time-sharing, and an SLS regime previously identified by Davoodi and Jafar where a simple layered superposition (SLS) scheme is shown to be optimal in the K user MISO BC, although only for $K \leq 3$. It remains an intriguing possibility that TIN may not be far from optimal in the CTIN regime, and that SLS schemes may be close to optimal even for larger networks in the SLS regime, but the curse of dimensionality is one of the obstacles that stands in the way of such generalizations. Under finite precision CSIT, appealing to extremal network theory we obtain the following results. In the TIN regime as well as the CTIN regime, we show that the extremal GDoF gain from transmitter cooperation over TIN is $\Theta(1)$, i.e., it is bounded above by a constant regardless of the number of users K. In fact, the gain is at most a factor of 2 in the CTIN regime, which automatically implies that TIN is GDoF optimal within a factor of 2 in the CTIN regime. In the TIN regime, the extremal GDoF gain of transmitter cooperation over TIN is shown to be exactly 50%, regardless of the number of users K, provided $K > 1$. However, in the SLS regime, the extremal GDoF gain of transmitter cooperation over TIN is $\Theta(\log_2(K))$, i.e., it scales logarithmically with the number of users. Remarkably, an SLS scheme suffices to demonstrate this extremal GDoF gain. Last but not the least, as a byproduct of our analysis we prove a useful cyclic decomposition property of the sum GDoF achievable by TIN in the SLS regime.

1 Introduction

Finding the capacity limits of wireless networks is one of the grand challenges of network information theory. While exact capacity characterizations remain elusive, much progress has been made on this problem within the past decade through Degrees of Freedom (DoF) and Generalized Degrees of Freedom (GDoF) studies. This includes both new achievable schemes, including those inspired by the idea of interference alignment (IA) [1–4], and new outer bounds, such as those based on the aligned images (AI) approach [5]. With these advances as stepping stones, a worthy goal at this stage is to bring the theory closer to practice by adapting the models and metrics to increasingly incorporate practical concerns. As a step in this direction, this work is motivated by three practical concerns — robustness, simplicity, and scalability.

By robustness we refer specifically to the channel state information at the transmitters (CSIT). GDoF characterizations under perfect CSIT provide important theoretical benchmarks, but often lead to fragile schemes such as asymptotic or real interference alignment [2,3] that are of little practical significance. Robustness to channel uncertainty is addressed by GDoF characterizations that limit the CSIT to finite[1] precision. Optimal schemes for such GDoF characterizations tend to be naturally robust schemes that require only a coarse knowledge of channel strength[2] parameters α_{ij} at the transmitters. Aided by advances in Aligned Images (AI) bounds [5], GDoF characterizations under finite precision CSIT have been found for a variety of wireless networks in [6–11].

The importance of simplicity is reflected in the goal of identifying parameter regimes where simple schemes are optimal in the GDoF sense [12–22]. The most relevant examples for our purpose are [12], [13] and [14]. Reference [12] identifies[3] a weak interference regime, called the TIN-regime (see Definition 1), where the simple scheme of power control and treating interference as Gaussian noise (in short, TIN[4]) is GDoF optimal for the K user interference channel (IC). A broader regime, called CTIN regime (see Definition 2, the 'C' signifies 'convex') is identified by Yi and Caire in [13] where, quite remarkably, the GDoF region achievable by TIN is shown to be **c**onvex without the need for time-sharing. It is not known whether TIN is GDoF optimal in this regime. Reference [14] identifies an even broader regime, called the SLS-regime (see Definition 3), where a simple layered superposition (SLS) scheme is GDoF optimal for the corresponding K user MISO broadcast channel (BC) under finite precision CSIT, but only for $K \leq 3$. Optimality of SLS for larger networks seems plausible, but a rapid growth in the number of parameters stands in the way of any such effort. Comparisons between the GDoF characterizations for interference and broadcast channels in these regimes are interesting because they shed light on the optimality of TIN and the benefits of transmitter cooperation. However, based on existing results, our ability to make direct comparisons is limited to very small networks. This brings us to the third practical concern, scalability.

Wireless networks often involve a large number of users. Studies of large networks have to deal with an explosion in the number of parameters. One way to limit the number of parameters is to study symmetric settings. For example, consider the symmetric setting obtained by setting $\alpha_{ij} = 1$ if $i = j$ and $\alpha_{ij} = \alpha$ if $i \neq j$, for all $i, j \in [K]$. Under finite precision CSIT, GDoF are characterized for the symmetric K user interference channel in [6], and for the symmetric K user MISO BC in [7]. Based on the symmetric settings, sum-GDoF gain of the symmetric K user MISO BC over

[1]In this work by default the term GDoF will refer to GDoF under finite precision CSIT.

[2]α_{ij} represents the channel strength from the j^{th} transmitter to the i^{th} receiver, and is measured in the db scale.

[3]While originally established under the assumption of perfect CSIT, the robustness of the TIN scheme ensures that this result carries over to finite precision CSIT.

[4]Note TIN includes optimal power control.

the symmetric K user IC is at most a factor of $3/2$ for all values of $\alpha \in [0,1]$. Furthermore, the TIN scheme can only achieve $\max(1, K(1-\alpha))$ GDoF [12] while the K user MISO BC has $\alpha + K(1-\alpha)$ GDoF [7]. Therefore, transmitter cooperation can provide an improvement over TIN by a factor of at most $3/2$ in the TIN-regime and the CTIN regime (both correspond to $\alpha \leq 1/2$), and a factor of at most 2 in the SLS-regime ($\alpha \leq 1$). Evidently the benefits of optimal transmitter cooperation over a simple scheme like TIN, are bounded for large K in both regimes.

But is this also true for *asymmetric* settings? To answer such questions, we need to venture beyond symmetric settings, and yet somehow avoid the curse of dimensionality. Other fields that face similar challenges, such as graph theory and set theory, find a path to progress through extremal analysis, i.e., the study of extremal graphs or extremal sets that satisfy various properties of interest. It follows then that a path to progress for wireless networks may be found in extremal network theory, i.e., the study of extremal networks. This is the main idea that we wish to explore in this work. Our interest in the optimality of TIN and the benefits of transmitter cooperation provides us a context within which we can test both the utility and the feasibility of the study of extremal networks.

We are interested specifically in the benefits of transmitter cooperation under *weak interference* over the simple baseline of TIN. The question is intriguing because on the one hand, we expect TIN to be a powerful scheme in weak interference regimes, but on the other hand full cooperation among all transmitters can also be quite powerful. Appealing to extremal network theory, we study the ratio,

$$\eta_K = \max_{[\alpha]_{K\times K} \in \mathcal{A}} \frac{\mathcal{D}_{\Sigma,\mathrm{BC}}}{\mathcal{D}_{\Sigma,\mathrm{TINA}}}, \tag{1}$$

where $\mathcal{D}_{\Sigma,\mathrm{TINA}}$ is the maximum sum-GDoF value *achievable* by power control and TIN in a K user IC. This is the baseline for comparison. $\mathcal{D}_{\Sigma,\mathrm{BC}}$ is the *optimal* sum-GDoF of the corresponding K user MISO BC obtained by full transmitter cooperation. The study of η_K is consistent with extremal network theory because of the maximization over $[\alpha]_{K\times K}$. Networks that maximize the ratio in (1) are extremal networks within the class of networks specified by the regime of interest, $\mathcal{A}$. The three regimes that we consider are the TIN-regime, $\mathcal{A}_{\mathrm{TIN}}$, the CTIN-regime, $\mathcal{A}_{\mathrm{CTIN}}$, and the SLS-regime, $\mathcal{A}_{\mathrm{SLS}}$. No assumption of symmetry is made within these regimes.

Our main results are threefold, corresponding to the three regimes of interest. For the CTIN and TIN regimes, we show that $\eta_K = \Theta(1)$, i.e., it is bounded by a constant regardless of the number of users, K. In fact η_K is bounded by 2 in the CTIN regime, and is exactly equal to 1.5 in the TIN regime for any number of users, $K > 1$. This result is consistent with and generalizes the insight obtained from the GDoF characterizations of symmetric IC and BC in [6,7]. It also implies that while the GDoF optimal scheme is unknown in the CTIN regime, TIN is within a factor of 2 from optimality in that regime. For the SLS regime, we show that, $\eta_K = \Theta(\log_2(K))$. Evidently, the gap to optimality of TIN schemes can be unbounded in the SLS-regime for large number of users. This is in contrast with the insights from the symmetric case where the improvement is at most by a factor of 2. The constructive proof of this result reveals a hierarchical topology that benefits greatly from transmitter cooperation. Furthermore, it is remarkable that the SLS scheme suffices to achieve the logarithmic gain from transmitter cooperation over TIN. As a by-product of our analysis we discover an important cyclic partition property of a TIN achievable region known as polyhedral TIN [12] (see Definition 8) that holds everywhere in the SLS-regime.

Notations: For positive integers X and Y, define $[X:Y] = \{X, X+1, \cdots, Y\}$, and $[X] = [1:X]$. The notation $[\alpha]_{K\times K}$ represents a $K\times K$ matrix whose $(i,j)^{th}$ element is α_{ij}. The cardinality of a set

S is denoted by $|S|$. For functions $f(K)$ and $g(K)$, denote $f(K) = \Theta(g(K))$ if $\lim_{K\to\infty} f(K)/g(K) = c$ for some finite constant $c > 0$.

2 System Model

For GDoF studies, the K user interference channel is modeled as [5, 6]

$$Y_k(t) = \sum_{i=1}^{K} \bar{P}^{\alpha_{ki}} G_{ki}(t) X_i(t) + Z_k(t) \qquad \forall k \in [K]. \tag{2}$$

During the t^{th} channel use, $X_i(t), Y_k(t), Z_k(t) \in \mathbb{C}$ are, respectively, the symbol transmitted by Transmitter i subject to a normalized unit transmit power constraint, the symbol received by User k, and the zero mean unit variance additive white Gaussian noise (AWGN) at User k. $\bar{P} \triangleq \sqrt{P}$, is a nominal parameter that approaches infinity to define the GDoF limit. The exponent $\alpha_{ki} \geq 0$ is referred to as the channel strength of the link between Transmitter i and Receiver k, and is known to all transmitters and receivers. The channel coefficients $G_{ki}(t)$ are known perfectly to the receivers but only available to finite precision at the transmitters. The finite precision CSIT assumption implies that from the transmitter's perspective, the joint and conditional probability density functions of the channel coefficients exist and the peak values of these distributions are bounded, i.e., they do not grow with P (see [5] for further description of the bounded density assumption). In the K user IC, there are K independent messages, one for each user, and each message is independently encoded by its corresponding transmitter. The definitions of achievable rate tuples and capacity region, $\mathcal{C}_{\text{IC}}(P)$ are standard, see e.g., [5]. The GDoF region of the K user interference channel is defined as

$$\mathcal{D}_{\text{IC}} = \left\{ (d_k)_{k\in[K]} \;\middle|\; \begin{array}{l} d_k = \lim_{P\to\infty} \frac{R_k(P)}{\log(P)}, \\ (R_k(P))_{k\in[K]} \in \mathcal{C}_{\text{IC}}(P). \end{array} \right\} \tag{3}$$

The maximum sum-GDoF value is denoted $\mathcal{D}_{\Sigma,\text{IC}}$.

Allowing full cooperation among the transmitters changes the problem into a K user MISO BC, where the K messages are jointly encoded by all K transmitters. The GDoF region for the MISO BC is denoted $\mathcal{D}_{\text{BC}}$ and the maximum sum-GDoF value is denoted $\mathcal{D}_{\Sigma,\text{BC}}$.

2.1 Significance of GDoF

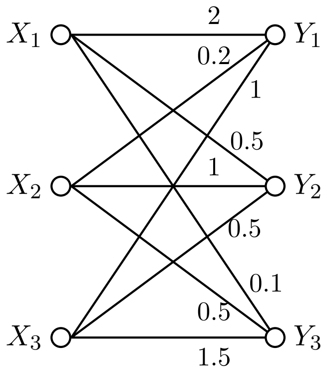
The GDoF model is essentially a generalization of the deterministic model of [23]. The significance of the GDoF model may be understood as follows. The channel strength parameters represent the arbitrary and finite values of corresponding link SNRs and INRs in the dB scale for a given network setting, i.e., $\alpha_{ii} = \log(\text{SNR}_{ii})$ and $\alpha_{ij} = \log(\text{INR}_{ij})$ (see, for example [12] for a more detailed explanation). Note that α_{ii} and α_{ij} may also be understood to be the approximate capacities of the corresponding links in isolation. Unlike the degrees of freedom (DoF) metric which linearly scales the transmit powers, the exponential scaling of powers in the GDoF model corresponds to a linear scaling of all of the corresponding link capacities by the same factor, $\log(P)$ (note that the isolated link with signal strength $P^{\alpha_{ij}}$ has capacity $\approx \alpha_{ij}\log(P)$, thus the scaling factor is $\log(P)$). The linear scaling of powers in the DoF model causes the ratios of capacities of any two non-zero links to approach 1 as $P \to \infty$. Thus, a very weak channel and a very strong channel become essentially *equally* strong in the DoF limit, thereby fundamentally changing the character of the original network of interest. The GDoF model on the other hand keeps the ratios of all capacities unchanged as $P \to \infty$, so that strong channels remain strong, and weak channels remain weak. The intuition behind GDoF is that if the capacities of all the individual links in a network are scaled by the *same* factor, then the overall network capacity region should scale by approximately the same factor as well — essentially a principle of scale invariance. If so, then normalizing by the scaling factor $\log(P)$ should produce an approximation to the capacity region of the original finite SNR network setting. This is precisely how GDoF are measured, note the normalization by $\log(P)$ in (3). Indeed, the validity of this intuition is borne out by numerous bounded-gap capacity approximations that have been enabled by GDoF characterizations (e.g., [24–28]), starting with the original result – the capacity characterization of the 2 user interference channel within a 1 bit gap in [29].

2.2 Significance of Finite Precision CSIT

Asymptotic analysis under perfect CSIT often leads to fragile schemes that are difficult to translate into practice, for example the DoF of the K user interference channel have been shown in [3, 30] to depend on whether the channels take rational or irrational values – a distinction of no practical significance. Zero forcing schemes that rely on precise channel phase knowledge to cancel signals can fail catastrophically due to relatively small phase perturbations. Robust schemes are much more valuable in practice. Restricting the CSIT to finite precision naturally shifts the focus to robust schemes that rely primarily on a coarse knowledge of channel strengths at the transmitters. While the finite precision CSIT model [5, 31] allows arbitrary fading distributions subject to bounded densities, it is instructive to consider in particular the model $G_{ki}(t) = g_{ki}^R(t) + g_{ki}^I(t)$ where $g_{ki}^R(t), g_{ki}^I(t)$ are independent and uniformly distributed over $(1-\epsilon, 1+\epsilon)$ for some arbitrarily small but positive ϵ. Interpreted this way, $G_{ki}(t)$ are arbitrarily small perturbations in the channel state that serve primarily to limit CSIT in the channel model to ϵ-precision, while the coarse knowledge of channel strengths remains available to the transmitters in the form of the parameters α_{ij}. From a GDoF perspective, these perturbations filter out fragile schemes that rely on highly precise CSIT. Indeed, the GDoF benefits of most sophisticated interference alignment and zero forcing schemes disappear under finite precision CSIT [5]. However, the benefits of robust schemes that rely only on the knowledge of channel strengths, such as rate-splitting [32], elevated multiplexing [33], layered superposition coding [23, 34], and treating interference as noise [12, 35–37] remain accessible. Thus, the finite precision CSIT model takes a conservative view that strongly favors robust schemes, the schemes with the greatest practical significance. It is therefore the natural choice for this work.

2.3 GDoF Comparisons

Comparing the GDoF of interference and broadcast channels under finite precision CSIT reveals the benefits of transmitter cooperation. As an example, consider the 3 user interference channel with the values of α_{ij} parameters as shown in Fig. 1. The channel parameters place this setting in the TIN regime [12], so its GDoF region is achieved by a TIN scheme. The GDoF region is shown both analytically and graphically in Fig. 1. Allowing transmitter cooperation under finite precision CSIT gives us a MISO BC. Since the TIN regime is included in the SLS regime, the GDoF of this MISO BC are characterized in [14]. The analytical form of the GDoF region and its graphical representation both appear in Fig. 1. Superposing the two GDoF regions we notice a significant improvement due to transmitter cooperation – 20% for this example. We would like to perform



$$\mathcal{D}_{\mathrm{IC}} = \left\{ \begin{array}{rcl} (d_1,d_2,d_3) \in \mathbb{R}_+^3 : & & \\ d_1 & \leq & 2 \\ d_2 & \leq & 1 \\ d_3 & \leq & 1.5 \\ d_1+d_2 & \leq & 2.3 \\ d_1+d_3 & \leq & 2.4 \\ d_2+d_3 & \leq & 1.5 \\ d_1+d_2+d_3 & \leq & 2.5 \end{array} \right\}, \quad \mathcal{D}_{\mathrm{BC}} = \left\{ \begin{array}{rcl} (d_1,d_2,d_3) \in \mathbb{R}_+^3 : & & \\ d_1 & \leq & 2 \\ d_2 & \leq & 1 \\ d_3 & \leq & 1.5 \\ d_1+d_2 & \leq & 2.5 \\ d_1+d_3 & \leq & 2.5 \\ d_2+d_3 & \leq & 2.0 \\ d_1+d_2+d_3 & \leq & 3.0 \end{array} \right\}$$

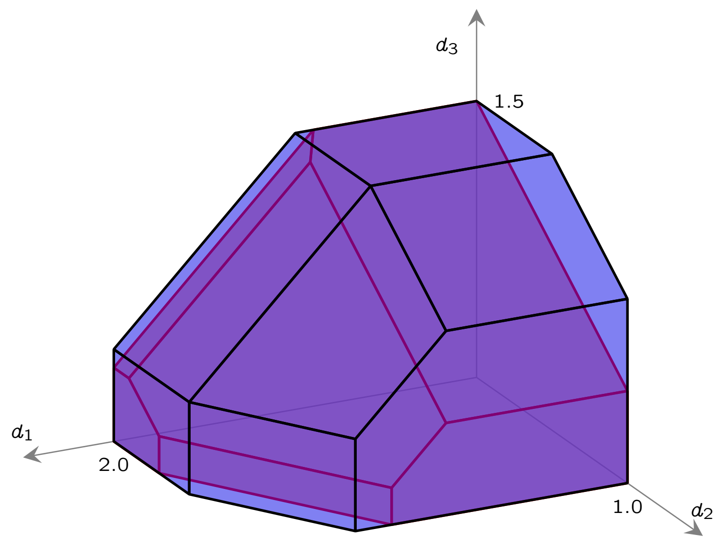


Figure 1: *The GDoF region of the 3 user interference channel in red is superimposed upon the GDoF region of the same channel with transmitter cooperation in blue. A 20% GDoF gain is seen due to transmitter cooperation for this example.*

such comparisons for larger networks, i.e., networks with more than 3 users. However, since the results of [14] are limited to 3 users, direct comparisons are not currently feasible. Instead we will explore extremal GDoF gains for large number of users. Furthermore we will limit our focus to sum-GDoF achievable by TIN and the optimal GDoF with transmitter cooperation.

3 Definitions

Definition 1 (TIN Regime) *Define*

$$\mathcal{A}_{\mathrm{TIN}} = \{[\alpha]_{K\times K} \in \mathbb{R}_+^{K\times K} : \alpha_{ii} \geq \alpha_{il} + \alpha_{mi} \ \forall i,l,m \in [K]\}. \tag{4}$$

The significance of the TIN regime is that in this regime, it was shown by Geng et al. in [12] that TIN is GDoF-optimal.

Definition 2 (CTIN Regime) *Define*

$$\mathcal{A}_{\mathrm{CTIN}} = \{[\alpha]_{K\times K} \in \mathbb{R}_+^{K\times K} : \alpha_{ii} \geq \max(\alpha_{ij} + \alpha_{ji}, \alpha_{ik} + \alpha_{ji} - \alpha_{jk}),\ \forall i,j,k \in [K], i \notin \{j,k\}\}. \tag{5}$$

The significance of the CTIN regime is that in this regime, it was shown by Yi and Caire in [13] that the GDoF region achievable with TIN (also known as $\mathcal{D}_{\text{TINA}}$, see Definition 10), is convex, without the need for time-sharing, and equal to the polyhedral TIN region over the set of all K users (see Definition 8).

Definition 3 (SLS Regime) *Define the SLS regime,*

$$\mathcal{A}_{SLS} = \{[\alpha]_{K\times K} \in \mathbb{R}_+^{K\times K} : \alpha_{ii} \geq \max(\alpha_{ij}, \alpha_{ki}, \alpha_{ik} + \alpha_{ji} - \alpha_{jk}),\ \forall i,j,k \in [K], i \notin \{j,k\}\}. \tag{6}$$

The significance of the SLS regime is that in this regime, it was shown by Davoodi and Jafar in [14] that a simple layered superposition scheme is GDoF-optimal for the MISO BC obtained by allowing transmitter cooperation in a K user interference channel. Note that the result of [14] is limited to $K \leq 3$, however the regime is defined for all K. Also note that the SLS regime includes the CTIN regime, which includes the TIN regime. Fig 2 illustrates the progressively larger regimes for TIN, CTIN and SLS in a 3 user cyclically symmetric setting parameterized by channel strengths a, b.

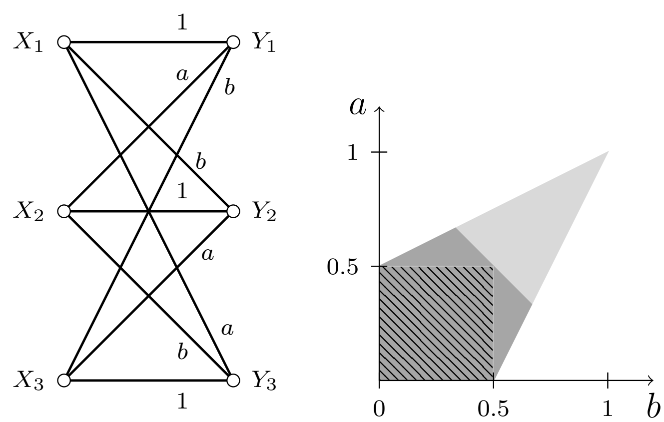


Figure 2: *For the 3 user symmetric setting shown here, the TIN regime is marked by the slanted line pattern, the CTIN regime includes the TIN regime and the region shaded in dark gray, and the SLS regime includes the CTIN regime and the region shaded in light gray.*

Definition 4 (Cycle π) *A cycle π of length $M > 1$ denoted as*

$$\pi = (i_1 \to i_2 \to \cdots \to i_M \circlearrowleft) \tag{7}$$

is an ordered collection of links in the $K \times K$ interference network, that includes the desired link between Transmitter i_m and Receiver i_m, and the interfering link between Transmitter i_m and Receiver i_{m+1}, for all $m \in [1:M]$, where we set $i_{M+1} = i_1$, and the indices $i_1, i_2, \cdots, i_M \in [K]$ are all distinct. See Fig. 3 for an example. A cycle of length $M = 1$ is called a trivial cycle, represented simply as $\pi = (i_1 \circlearrowleft)$ for some $i_1 \in [K]$, and it includes only the desired link between Transmitter i_1 and Receiver i_1.

Also define the following terms related to the cycle π.

1. *Define $\pi(1) = i_1$ as the head of the cycle. Other elements of the cycle may be similarly referenced, e.g,. $\pi(2) = i_2, \pi(3) = i_3$, and so on. Thus, the cycle may be equivalently represented as $\pi = (\pi(1) \to \pi(2) \to \cdots \to \pi(M) \circlearrowleft)$. Also note that if the cycle has length M, then the indices are interpreted modulo M, i.e., $\pi(M+i) = \pi(i)$ for all integers i. For example, if π is a cycle of length $M = 5$, then $\pi(6) = \pi(1), \pi(7) = \pi(2)$, etc.*

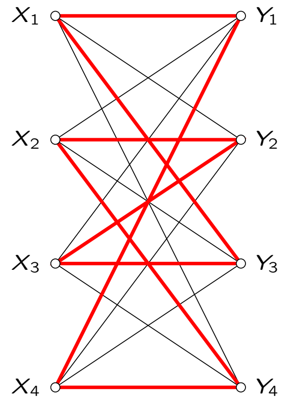


Figure 3: The links included in the cycle $\pi = (2 \to 4 \to 1 \to 3 \circlearrowleft)$ are highlighted in red.

2. *Define $\{\pi\} = \{i_1, i_2, \cdots, i_M\}$, i.e., $\{\pi\}$ represents the set of users involved in the cycle π.*

3. *Define $w(\pi)$, called the weight of the cycle π, as the sum of strengths of all interfering links included in the cycle, i.e., $w(\pi) = \sum_{m=1}^{M} \alpha_{i_{m+1}, i_m}$. The weight of a trivial cycle is zero because it includes no interfering links.*

4. *Define Π as the set of all cycles in the K user network.*

5. *Cycles $\pi_1, \pi_2, \cdots, \pi_n$ are said to be disjoint if the sets $\{\pi_1\}, \{\pi_2\}, \cdots, \{\pi_n\}$ are disjoint.*

6. *Cycles $\pi_1, \pi_2, \cdots, \pi_n$ are said to comprise a cyclic partition of the set $S \subset [K]$, if they are disjoint and $\bigcup_{i=1}^{n} \{\pi_i\} = S$.*

The significance of cycles is that they lead to bounds on the sum-GDoF of the users involved in the cycle. For the interference channel, each cycle π leads to a cycle bound $\sum_{k\in\pi} d_k \leq \Delta_\pi$ (see Definition 7) which is a bound on the GDoF region achievable by a restricted form of TIN, called polyhedral TIN (Definition 8). For the broadcast channel, each cycle π leads to a bound $\sum_{k\in\pi} d_k \leq \Delta_\pi + \alpha_{\pi(i+1)\pi(i)}$ (see Lemma 6). Unlike the interference channel, the bounds for the BC are information theoretic bounds on the optimal GDoF region. These bounds are the key to all the results in this work.

Definition 5 (Combined Cycles) *For disjoint cycles*

$$\pi_1 = (i_1 \to \cdots \to i_{M_1} \circlearrowleft) \tag{8}$$
$$\pi_2 = (j_1 \to \cdots \to j_{M_2} \circlearrowleft) \tag{9}$$

the combined cycle, denoted $\pi_{1,2} = (\pi_1 \to \pi_2 \circlearrowleft)$, is defined as

$$\pi_{1,2} = (\pi_1 \to \pi_2 \circlearrowleft) = (i_1 \to \cdots \to i_{M_1} \to j_1 \to \cdots \to j_{M_2} \circlearrowleft) \tag{10}$$

Note that $\pi_{1,2}$ is in general different from $\pi_{2,1}$. Combinations of more than 2 cycles are similarly defined. For example, $\pi_{1,2,3} = (\pi_1 \to \pi_2 \to \pi_3 \circlearrowleft)$.

Definition 6 **(δ_{ij})** *For $i, j \in [K]$, define*

$$\delta_{ij} = \begin{cases} \alpha_{ii} - \alpha_{ji}, & i \neq j, \\ 0, & i = j. \end{cases} \tag{11}$$

Definition 7 **(Δ_π)** *For any cycle π of length M, $\pi = (i_1 \to i_2 \to \cdots \to i_M \circlearrowleft)$, define*

$$\Delta_\pi = \begin{cases} \delta_{i_1 i_2} + \delta_{i_2 i_3} + \cdots + \delta_{i_{M-1} i_M} + \delta_{i_M i_1}, & \text{if } M > 1 \\ \alpha_{i_1 i_1}, & \text{if } M = 1. \end{cases} \tag{12}$$

Definition 8 **($\mathcal{D}_{\text{P-TIN}}(S)$)** *For any subset of users, $S \subset [K]$, the polyhedral-TIN region [12] is defined as*

$$\mathcal{D}_{\text{P-TIN}}(S) = \left\{ (d_k : k \in [K]) \;\middle|\; \begin{array}{ll} 0 = d_k, & \forall k \in [K] \backslash S, \\ 0 \le d_k, & \forall k \in S, \\ \sum_{k \in \{\pi\}} d_k \le \Delta_\pi, & \forall \pi \in \Pi, \{\pi\} \subset S \end{array} \right\}. \tag{13}$$

The bounds, $\sum_{k \in \{\pi\}} d_k \le \Delta_\pi$, are called cycle-bounds. Note that these are not bounds on the general GDoF region, rather these are only bounds on the polyhedral TIN region for a given subset S. The sum-GDoF value of polyhedral-TIN over the set S is defined as

$$\mathcal{D}_{\Sigma,\text{P-TIN}}(S) = \max_{\mathcal{D}_{\text{P-TIN}}(S)} \sum_{k \in S} d_k. \tag{14}$$

If $S = [K]$, then we will simply write $\mathcal{D}_{\Sigma,\text{P-TIN}}([K]) = \mathcal{D}_{\Sigma,\text{P-TIN}}$.

A remarkable fact about the polyhedral TIN region is that even if $S_1 \subset S_2$, it is possible that the polyhedral region for S_1 is strictly larger than the polyhedral region for S_2. See the simple example at the end of this section.

Definition 9 **(P-optimal Cyclic Partition of S)** *A cyclic partition of a subset of users S, $S \subset [K]$, say into the n disjoint cycles $\pi_1, \pi_2, \cdots, \pi_n$, is said to be p-optimal if*

$$\mathcal{D}_{\Sigma,\text{P-TIN}}(S) = \Delta_{\pi_1} + \Delta_{\pi_2} + \cdots + \Delta_{\pi_n}. \tag{15}$$

In general a p-optimal cyclic partition does not exist. Reference [15] showed that such partitions exist in the TIN regime. As one of the key elements of this work, it is shown in Theorem 4 in Appendix A, that such partitions must exist in the SLS regime. Since CTIN and TIN regimes are all included in the SLS regime, these cyclic partitions exist in all three regimes.

Definition 10 **($\mathcal{D}_{\text{TINA}}$)** *The TINA region [12, 13] is defined as*

$$\mathcal{D}_{\text{TINA}} = \bigcup_{S : S \subset [K]} \mathcal{D}_{\text{P-TIN}}(S). \tag{16}$$

The sum-GDoF over the TINA region are defined as

$$\mathcal{D}_{\Sigma,\text{TINA}} = \max_{\mathcal{D}_{\text{TINA}}} \sum_{k \in [K]} d_k. \tag{17}$$

Thus the TINA region is a union of polyhedral TIN regions. In general this union does not produce a convex region. For example, consider the 2 user interference channel where all α_{ij} values are equal to 1. Incidentally this channel is in the SLS regime. For this channel, $\mathcal{D}_{\text{P-TIN}}(\{1\}) = \{(d_1, d_2) : 0 \leq d_1 \leq 1, d_2 = 0\}, \mathcal{D}_{\text{P-TIN}}(\{2\}) = \{(d_1, d_2) : d_1 = 0, 0 \leq d_2 \leq 1\}, \mathcal{D}_{\text{P-TIN}}(\{1,2\}) = \{(d_1, d_2) : 0 \leq d_1 + d_2 \leq 0\} = \{(d_1, d_2) : d_1 = 0, d_2 = 0\}$. The union of these three regions, $\mathcal{D}_{\Sigma,\text{TINA}} = \mathcal{D}_{\text{P-TIN}}(\{1\}) \bigcup \mathcal{D}_{\text{P-TIN}}(\{2\}) \bigcup \mathcal{D}_{\text{P-TIN}}(\{1,2\})$, is not convex. However, remarkably, the region $\mathcal{D}_{\text{TINA}}$ is convex for channels in the TIN regime as shown by Geng et al. in [12], and for channels in the CTIN regime as shown by Yi and Caire in [13]. This example also shows that a larger set of users ($\{1,2\}$) can produce a smaller polyhedral TIN region than a smaller set of users ($\{1\}$).

In the next section, we start presenting our results on the extremal GDoF gain from transmitter cooperation relative to TIN under the three regimes of interest – CTIN, TIN and SLS. All three regimes are weak interference regimes, where Transmitter i is the strongest possible transmitter for Receiver i, and vice versa. Before exploring the benefit of full transmitter cooperation, as a preliminary thought experiment suppose we only allow a re-assignment of transmitters to receivers in the K user interference channel, after which only the TIN scheme is used. Because the strongest transmitters and receivers are already associated with each other, intuitively we do not expect that a reassignment would be beneficial. Indeed, this intuition is confirmed by Theorem 5 that appears in Appendix B. In this sense, in all three regimes the natural association between Transmitter i and Receiver i is a *stable* association.

4 Extremal Gain from Transmitter Cooperation in the CTIN Regime

Let us start with the CTIN regime, which includes the TIN regime, i.e., $\mathcal{A}_{\text{TIN}} \subset \mathcal{A}_{\text{CTIN}}$. We will show that the extremal GDoF improvement from transmitter cooperation over TIN, is $\Theta(1)$, i.e., bounded by a constant regardless of the number of users K. In fact, as shown in the following theorem, the constant is at most 2. Remarkably, while the optimality of TIN is not known in the CTIN regime, Theorem 1 implies that TIN is within a factor of 2 from optimality in this regime.

Theorem 1 *For arbitrary number of users, K,*

$$1 \leq \max_{[\alpha]_{K\times K} \in \mathcal{A}_{\text{CTIN}}} \frac{\mathcal{D}_{\Sigma,\text{BC}}}{\mathcal{D}_{\Sigma,\text{TINA}}} \leq 2. \tag{18}$$

Proof of Theorem 1

The lower bound is trivial because cooperation cannot hurt. To prove the upper bound, let $\pi_1, \pi_2, \cdots, \pi_N$ be a p-optimal cyclic partition of $[K]$ into N disjoint cycles with at most one trivial cycle. Such a partition exists according to Theorem 4 (Appendix A) and Lemma 4 (Appendix C). If $N = 1$, then it is trivially true from Lemma 6 (Appendix C) that $\mathcal{D}_{\Sigma,\text{BC}} \leq \Delta_{\pi_1} + \mathcal{D}_{\Sigma,\text{TINA}} \leq 2\mathcal{D}_{\Sigma,\text{TINA}}$. So let us assume that $N \geq 2$. Furthermore, since there is no more than one trivial cycle in the p-optimal cyclic partition, without loss of generality, let us assume that the cycle π_1 is non-trivial, i.e., it involves $M > 1$ users. Also, without loss of generality, let us choose a representation of cycle π_1 so that the first user, User $\pi_1(1)$, has the strongest desired channel out of all the users in $\{\pi_1\}$, i.e.,

$$\alpha_{\pi_1(1)\pi_1(1)} \geq \alpha_{\pi_1(m)\pi_1(m)}, \qquad \forall m \in [1:M]. \tag{19}$$

Next, let us define π'_j as the cycle obtained by shifting π_1 by j as follows.

$$\pi'_j = \big(\pi_1(1+j) \to \pi_1(2+j) \to \cdots \to \pi_1(M+j) \circlearrowleft \big) \tag{20}$$

so that $\pi'_j(m) = \pi_1(m+j)$, $\forall m \in [1:M]$. Recall that modulo M arithmetic is used for indexing users within a cycle, e.g., $\pi_1(M+j)$ is the same as $\pi_1(j)$. Also note that the shift does not affect Δ_{π_1}, i.e., $\Delta_{\pi_1} = \Delta_{\pi'_j}$. Next, define the combined cycle

$$\pi'_{j,2,\cdots,N} = (\pi'_j \to \pi_2 \to \cdots \to \pi_N \circlearrowleft) \tag{21}$$

Since $\pi'_{j,2,\cdots,N}$ includes all K users, according to Lemma 6,

$$\begin{aligned}
\mathcal{D}_{\Sigma,\text{BC}} &\le \Delta_{\pi'_{j,2,\cdots,N}} + \alpha_{\pi_2(1)\pi'_j(M)} && (22)\\
&\le \Delta_{\pi'_j} + \Delta_{\pi_2} + \cdots + \Delta_{\pi_N} + \Delta_{(\pi'_j(1)\to\pi_2(1)\to\cdots\to\pi_N(1) \circlearrowleft)} + \alpha_{\pi_2(1)\pi'_j(M)} && (23)\\
&= \Delta_{\pi_1} + \Delta_{\pi_2} + \cdots + \Delta_{\pi_N} + \delta_{\pi'_j(1)\pi_2(1)} + \delta_{\pi_2(1)\pi_3(1)} + \cdots + \delta_{\pi_N(1)\pi'_j(1)} + \alpha_{\pi_2(1)\pi'_j(M)} && (24)\\
&= \mathcal{D}_{\Sigma,\text{TINA}} + \delta_{\pi_1(1+j)\pi_2(1)} + \delta_{\pi_2(1)\pi_3(1)} + \cdots + \delta_{\pi_N(1)\pi_1(1+j)} + \alpha_{\pi_2(1)\pi_1(M+j)} && (25)\\
&\le \mathcal{D}_{\Sigma,\text{TINA}} + \alpha_{\pi_1(1+j)\pi_1(1+j)} - \alpha_{\pi_2(1)\pi_1(1+j)} + \sum_{n=2}^{N} \alpha_{\pi_n(1)\pi_n(1)} + \alpha_{\pi_2(1)\pi_1(M+j)} && (26)\\
&\le \mathcal{D}_{\Sigma,\text{TINA}} + \sum_{n=1}^{N} \alpha_{\pi_n(1)\pi_n(1)} - \alpha_{\pi_2(1)\pi_1(1+j)} + \alpha_{\pi_2(1)\pi_1(M+j)} && (27)
\end{aligned}$$

We used Lemma 5 from Appendix C in (23). In (25) we used the fact that $\pi_1, \cdots, \pi_n$ is the p-optimal cyclic partition of $[K]$, and as shown by [13], in the $\mathcal{A}_{\text{CTIN}}$ regime, $\mathcal{D}_{\text{TINA}} = \mathcal{D}_{\text{P-TIN}}([K])$, so that $\mathcal{D}_{\Sigma,\text{TINA}} = \mathcal{D}_{\Sigma,\text{P-TIN}}([K])$. In (26) we used the fact that by definition $\delta_{ij} = \alpha_{ii} - \alpha_{ji} \ge \alpha_{ii}$. In (27) we used the assumption from (19). Writing (27) over all $j \in [1:M]$ and adding all those inequalities causes the sums of the last two terms to cancel each other perfectly, leaving us with the bound,

$$\mathcal{D}_{\Sigma,\text{BC}} \le \mathcal{D}_{\Sigma,\text{TINA}} + \sum_{n=1}^{N} \alpha_{\pi_n(1)\pi_n(1)} \tag{28}$$

Now, following similar reasoning as in [13], we show that in the $\mathcal{A}_{\text{CTIN}}$ regime, $\alpha_{\pi_n(1)\pi_n(1)} \le \Delta_{\pi_n}$, as follows. If the cycle π_n is a trivial cycle, then $\Delta_{\pi_n} = \alpha_{\pi_n(1)\pi_n(1)}$. So let us now assume the cycle is non-trivial, i.e., it involves $M_n > 1$ users. Recall that in this regime, $\alpha_{jj} \ge \alpha_{ij} + \alpha_{ji}$, and $\delta_{ij} + \delta_{jk} \ge \delta_{ij}$. Therefore,

$$\begin{aligned}
\alpha_{\pi_n(1)\pi_n(1)} &\le \alpha_{\pi_n(1)\pi_n(1)} + \alpha_{\pi_n(2)\pi_n(2)} - \alpha_{\pi_n(1)\pi_n(2)} - \alpha_{\pi_n(2)\pi_n(1)} && (29)\\
&= \Delta_{(\pi_n(1)\to\pi_n(2) \circlearrowleft)} && (30)\\
&= \Delta_{(\pi_n(1)\to\pi_n(2)\to\pi_n(3) \circlearrowleft)} + \delta_{\pi_n(2)\pi_n(1)} - \delta_{\pi_n(2)\pi_n(3)} - \delta_{\pi_n(3)\pi_n(1)} && (31)\\
&\le \Delta_{(\pi_n(1)\to\pi_n(2)\to\pi_n(3) \circlearrowleft)} && (32)\\
&\ \vdots && (33)\\
&\le \Delta_{(\pi_n(1)\to\cdots\to\pi_n(m) \circlearrowleft)} && (34)\\
&= \Delta_{(\pi_n(1)\to\cdots\to\pi_n(m)\to\pi_n(m+1) \circlearrowleft)} + \delta_{\pi_n(m)\pi_n(1)} - \delta_{\pi_n(m)\pi_n(m+1)} - \delta_{\pi_n(m+1)\pi_n(1)} && (35)
\end{aligned}$$

$$\leq \Delta_{(\pi_n(1)\to\cdots\to\pi_n(m+1)\ \circlearrowleft\ )} \tag{36}$$

$$\vdots \tag{37}$$

$$\leq \Delta_{(\pi_n(1)\to\pi_n(2)\cdots\to\pi_n(M_n)\ \circlearrowleft\ )} \tag{38}$$

$$= \Delta_{\pi_n} \tag{39}$$

Substituting $\alpha_{\pi_n(1)\pi_n(1)} \leq \Delta_{\pi_n}$ into (28) we have,

$$\mathcal{D}_{\Sigma,\mathrm{BC}} \leq \mathcal{D}_{\Sigma,\mathrm{TINA}} + \sum_{n=1}^{N} \Delta_{\pi_n} \tag{40}$$

$$= 2\mathcal{D}_{\Sigma,\mathrm{TINA}} \tag{41}$$

because $\pi_1, \cdots, \pi_n$ is the p-optimal cyclic partition and we are in the $\mathcal{A}_{\mathrm{CTIN}}$ regime. This completes the proof of Theorem 1. □

5 Extremal Gain from Transmitter Cooperation in the TIN Regime

Next, let us consider the TIN regime. Note that Theorem 1 already shows that the extremal improvement from transmitter cooperation over TIN is at most a constant factor of 2 regardless of the number of users, in the CTIN regime, which includes the TIN regime. In this section, specifically for the TIN regime, we will further sharpen the result and show that the constant is exactly 1.5 for arbitrary number of users K, provided $K > 1$. Note that $K = 1$ is a degenerate case because there can be no cooperation among transmitters when there is only one transmitter.

Theorem 2 *For $K \geq 2$ users,*

$$\max_{[\alpha]_{K\times K}\in\mathcal{A}_{\mathrm{TIN}}} \frac{\mathcal{D}_{\Sigma,\mathrm{BC}}}{\mathcal{D}_{\Sigma,\mathrm{IC}}} = \max_{[\alpha]_{K\times K}\in\mathcal{A}_{\mathrm{TIN}}} \frac{\mathcal{D}_{\Sigma,\mathrm{BC}}}{\mathcal{D}_{\Sigma,\mathrm{TINA}}} = \frac{3}{2}. \tag{42}$$

Proof of Theorem 2

Note that in the TIN regime, the GDoF of the K user interference channel are achieved by TIN as shown in [12], so $\mathcal{D}_{\Sigma,\mathrm{IC}} = \mathcal{D}_{\Sigma,\mathrm{TINA}}$. First, let us prove the upper bound, i.e, in the TIN-regime, $\mathcal{D}_{\Sigma,\mathrm{BC}} \leq 1.5\mathcal{D}_{\Sigma,\mathrm{IC}}$. Let $\pi = (i_1 \to i_2 \cdots \to i_M \ \circlearrowleft\ )$ by any cycle of length $M > 1$, and consider the corresponding IC cycle bound, which is an information theoretic bound on $\mathcal{D}_{\Sigma,\mathrm{IC}}(\{\pi\})$, i.e., the sum-GDoF of the IC restricted to just the users that are involved in the cycle,

$$\mathcal{D}_{\Sigma,\mathrm{IC}}(\{\pi\}) \leq \delta_{i_1i_2} + \delta_{i_2i_3} + \cdots + \delta_{i_{M-1}i_M} + \delta_{i_Mi_1} = \Delta_\pi \tag{43}$$

Note that $\Delta_\pi \geq \alpha_{i_1i_1}$ because α_{i_1,i_1} GDoF are trivially achievable by simply allowing only user i_1 to transmit. For the same M users, by Lemma 6 the sum-GDoF in the BC are bounded in two ways as,

$$\mathcal{D}_{\Sigma,\mathrm{BC}}(\{\pi\}) \leq \delta_{i_1i_2} + \delta_{i_2i_3} + \cdots + \delta_{i_{M-1}i_M} + \delta_{i_Mi_1} + \alpha_{i_1i_M} = \Delta_\pi + \alpha_{i_1i_M} \tag{44}$$

$$\mathcal{D}_{\Sigma,\mathrm{BC}}(\{\pi\}) \leq \delta_{i_1i_2} + \delta_{i_2i_3} + \cdots + \delta_{i_{M-1}i_M} + \delta_{i_Mi_1} + \alpha_{i_2i_1} = \Delta_\pi + \alpha_{i_2i_1} \tag{45}$$

$$\Rightarrow 2\mathcal{D}_{\Sigma,\mathrm{BC}}(\{\pi\}) \leq 2\Delta_\pi + \alpha_{i_2i_1} + \alpha_{i_1i_M} \leq 2\Delta_\pi + \alpha_{i_1i_1} \leq 3\Delta_\pi. \tag{46}$$

In (46) we made use of the fact that in the TIN-regime, $\alpha_{i_2i_1} + \alpha_{i_1i_M} \leq \alpha_{i_1i_1} \leq \Delta_\pi$. Also for a trivial cycle, π, of length $M = 1$, say comprised of only user m, we have $\mathcal{D}_{\Sigma,\text{IC}}(\{\pi\}) = \mathcal{D}_{\Sigma,\text{BC}}(\{\pi\}) = \alpha_{mm} = \Delta_\pi$, so here also $\mathcal{D}_{\Sigma,\text{BC}}(\{\pi\}) \leq 1.5\Delta_\pi$. Therefore for every cycle π we have $\mathcal{D}_{\Sigma,\text{BC}}(\{\pi\}) \leq 1.5\Delta_\pi$. Now, let us consider the total GDoF of all K users. Since $[\alpha]_{K\times K} \in \mathcal{A}_{\text{TIN}}$, from [15] we know that $\mathcal{D}_{\Sigma,\text{IC}}$ is given by a cycle partition, comprised of, say the N cycles $\pi_1, \pi_2, \cdots, \pi_N$. Note that the cycles are disjoint and $\bigcup_{i=1}^{n}\{\pi_i\} = [K]$.

$$\mathcal{D}_{\Sigma,\text{IC}} = \sum_{n=1}^{N} \Delta_{\pi_n}, \tag{47}$$

$$\mathcal{D}_{\Sigma,\text{BC}} \leq \sum_{n=1}^{N} \mathcal{D}_{\Sigma,\text{BC}}(\{\pi_n\}) \leq \sum_{n=1}^{N} 1.5\Delta_{\pi_n} = 1.5\mathcal{D}_{\Sigma,\text{IC}}. \tag{48}$$

Next, let us prove that for any $K \geq 2$, there exist $[\alpha]_{K\times K} \in \mathcal{A}_{\text{TIN}}$, such that $\mathcal{D}_{\Sigma,\text{BC}} \geq 1.5\mathcal{D}_{\Sigma,\text{IC}}$. For $K = 2$ users consider the channel with $\alpha_{11} = \alpha_{22} = 1, \alpha_{12} = \alpha_{21} = 0.5$, for which $\mathcal{D}_{\Sigma,\text{IC}} = 1$ according to [12] but $\mathcal{D}_{\Sigma,\text{BC}} = 1.5$ according to [38]. For $K \geq 3$ it is trivial to generate such $[\alpha]_{K\times K} \in \mathcal{A}_{\text{TIN}}$ simply by adding trivial users $k \in [3 : K]$ such that all α_{ij} associated with these additional users are zero. This completes the proof of Theorem 2. $\square$

6 Extremal Gain from Transmitter Cooperation in the SLS Regime

Theorem 3

$$\max_{[\alpha]_{K\times K}\in\mathcal{A}_{\text{SLS}}} \frac{\mathcal{D}_{\Sigma,\text{BC}}}{\mathcal{D}_{\Sigma,\text{TINA}}} = \Theta(\log(K)). \tag{49}$$

Proof of Theorem 3: Upper Bound

Let us describe an iterative procedure. Stage λ of the procedure, $\lambda \in [0 : \Lambda]$, is characterized by a subset of users, $S_\lambda \subset [K]$, a cyclic partition of S_λ into N_λ disjoint cycles $\pi_1^{S_\lambda}, \pi_2^{S_\lambda}, \cdots, \pi_{N_\lambda}^{S_\lambda}$, and a cyclic partition of $[K]$ into N_λ disjoint cycles $\pi_1^\lambda, \pi_2^\lambda, \cdots, \pi_{N_\lambda}^\lambda$. The procedure stops in stage $\lambda = \Lambda$ as soon as we find $N_\lambda = 1$.

Stage 0 is the initialization stage. The procedure is initialized with the set $S_o = [K]$, the set of all users. Let $\pi_1^{S_o}, \pi_2^{S_o}, \cdots, \pi_{N_o}^{S_o}$ be a p-optimal cyclic partition of S_o with at most one trivial cycle. Such a partition exists and produces the tight sum-GDoF bound for polyhedral TIN over S_o so that

$$\mathcal{D}_{\Sigma,\text{P-TIN}}(S_o) = \Delta_{\pi_1^{S_o}} + \Delta_{\pi_2^{S_o}} + \cdots + \Delta_{\pi_{N_o}^{S_o}}. \tag{50}$$

Choose $(\pi_1^o, \pi_2^o, \cdots, \pi_{N_o}^o) = (\pi_1^{S_o}, \pi_2^{S_o}, \cdots, \pi_{N_o}^{S_o})$. This completes the initialization stage. Note that because the p-optimal cyclic partition cannot have more than one trivial cycle, we must have $N_o \leq (K+1)/2$. If $N_o = 1$, then $\Lambda = 0$ and the procedure stops here. If not, then we move to the next stage.

Stage 1 begins by defining the set of users,

$$S_1 = \{\pi_1^o(1), \pi_2^o(1), \cdots, \pi_{N_o}^o(1)\} \tag{51}$$

Let $\pi_1^{S_1}, \pi_2^{S_1}, \cdots, \pi_{N_1}^{S_1}$ be a p-optimal cyclic partition of S_1 with at most one trivial cycle, so that

$$\mathcal{D}_{\Sigma,\text{P-TIN}}(S_1) = \Delta_{\pi_1^{S_1}} + \Delta_{\pi_2^{S_1}} + \cdots + \Delta_{\pi_{N_1}^{S_1}}. \tag{52}$$

Note that these cycles only span S_1. For each of these cycles, $\pi_n^{S_1}$, $n \in [1 : N_1]$, we will create a combined cycle, π_n^1 such that the N_1 combined cycles will be a cyclic partition of $[K]$. This is done as follows. Let us write the n^{th} cycle, $\pi_n^{S_1}$, explicitly as,

$$\pi_n^{S_1} = (\pi_{n_1}^o(1) \to \pi_{n_2}^o(1) \to \cdots \to \pi_{n_{m_n}}^o(1) \circlearrowleft) \tag{53}$$

Then the corresponding combined cycle is defined as

$$\pi_n^1 = (\pi_{n_1}^o \to \pi_{n_2}^o \to \cdots \to \pi_{n_{m_n}}^o \circlearrowleft) \tag{54}$$

for $n \in [1 : N_1]$. Now note that $\pi_1^1, \pi_2^1, \cdots, \pi_{N_1}^1$ span $[K]$, in fact they constitute a cyclic partition of $[K]$. This completes Stage 1.

Note that S_1 has N_o users, and the p-optimal cyclic partition does not have more than one trivial cycle, so we must have $N_1 \leq (N_o + 1)/2$. Furthermore, it follows from Lemma 5 that

$$\Delta_{\pi_n^1} \leq \Delta_{\pi_{n_1}^o} + \Delta_{\pi_{n_2}^o} + \cdots + \Delta_{\pi_{n_{m_n}}^o} + \Delta_{\pi_n^{S_1}} \tag{55}$$

Summing over all $n \in [1 : N_1]$ we have

$$\begin{aligned}\Delta_{\pi_1^1} + \Delta_{\pi_2^1} + \cdots + \Delta_{\pi_{N_1}^1} &\leq \Delta_{\pi_1^o} + \Delta_{\pi_2^o} + \cdots + \Delta_{\pi_{N_o}^o} + \Delta_{\pi_1^{S_1}} + \Delta_{\pi_2^{S_1}} + \cdots + \Delta_{\pi_{N_1}^{S_1}} &(56)\\ &= \Delta_{\pi_1^o} + \Delta_{\pi_2^o} + \cdots + \Delta_{\pi_{N_o}^o} + \mathcal{D}_{\Sigma,\text{P-TIN}}(S_1) &(57)\\ &\leq \Delta_{\pi_1^o} + \Delta_{\pi_2^o} + \cdots + \Delta_{\pi_{N_o}^o} + \mathcal{D}_{\Sigma,\text{TINA}} &(58)\end{aligned}$$

If $N_1 = 1$, then we set $\Lambda = 1$ and the procedure stops here. If not, then we proceed to the next stage.

The procedure now simply repeats, so that at the $(\lambda + 1)^{th}$ stage we have the set of users

$$S_{\lambda+1} = \{\pi_1^\lambda(1), \pi_2^\lambda(1), \cdots, \pi_{N_\lambda}^\lambda(1)\}. \tag{59}$$

A p-optimal cyclic partition of $S_{\lambda+1}$ with at most one trivial cycle produces $N_{\lambda+1}$ disjoint cycles, $\pi_1^{S_{\lambda+1}}, \pi_2^{S_{\lambda+1}}, \cdots, \pi_{N_{\lambda+1}}^{S_{\lambda+1}}$, such that the l^{th} cycle in this partition,

$$\pi_l^{S_{\lambda+1}} = (\pi_{l_1}^\lambda(1) \to \pi_{l_2}^\lambda(1) \to \cdots \to \pi_{l_{m_l}}^\lambda(1) \circlearrowleft) \tag{60}$$

produces the l^{th} combined cycle

$$\pi_l^{\lambda+1} = (\pi_{l_1}^\lambda \to \pi_{l_2}^\lambda \to \cdots \to \pi_{l_{m_l}}^\lambda \circlearrowleft) \tag{61}$$

for $l \in [1 : N_{\lambda+1}]$. This completes Stage $\lambda + 1$. Since $S_{\lambda+1}$ has N_λ users, and the p-optimal cycle cannot have more than one trivial cycle, we must have $N_{\lambda+1} \leq (N_\lambda + 1)/2$. Furthermore, it follows from Lemma 5 that

$$\Delta_{\pi_1^{\lambda+1}} + \Delta_{\pi_2^{\lambda+1}} + \cdots + \Delta_{\pi_{N_{\lambda+1}}^{\lambda+1}} \leq \Delta_{\pi_1^\lambda} + \Delta_{\pi_2^\lambda} + \cdots + \Delta_{\pi_{N_\lambda}^\lambda} + \mathcal{D}_{\Sigma,\text{TINA}}. \tag{62}$$

If $N_{\lambda+1} = 1$, then the procedure stops and $\Lambda = \lambda + 1$, otherwise the procedure continues. This completes the description of the procedure.

Λ can be bounded by using $N_{\lambda+1} \leq (N_\lambda + 1)/2$, $N_o \leq (K+1)/2$ and $N_{\Lambda-1} \geq 2$, as follows. $N_{\Lambda-1} \geq 2 \Rightarrow N_{\Lambda-2} \geq 3 \Rightarrow N_{\Lambda-3} \geq 5 \Rightarrow \cdots \Rightarrow N_o \geq 2^{\Lambda-1} + 1 \Rightarrow K \geq 2^\Lambda + 1 \Rightarrow \Lambda \leq \log_2(K-1)$.

Finally, we complete the proof of the upper bound as follows.

$$\begin{aligned}
\mathcal{D}_{\Sigma,\text{TINA}} &\geq \mathcal{D}_{\Sigma,\text{P-TIN}}(S_o) && (63)\\
&= \Delta_{\pi_1^o} + \Delta_{\pi_2^o} + \cdots + \Delta_{\pi_{N_o}^o} && (64)\\
&\geq \Delta_{\pi_1^1} + \Delta_{\pi_2^1} + \cdots + \Delta_{\pi_{N_1}^1} - \mathcal{D}_{\Sigma,\text{TINA}} && (65)\\
&\geq \Delta_{\pi_1^2} + \Delta_{\pi_2^2} + \cdots + \Delta_{\pi_{N_2}^2} - 2\mathcal{D}_{\Sigma,\text{TINA}} && (66)\\
&\ \ \vdots \\
&\geq \Delta_{\pi_1^\Lambda} - \Lambda\mathcal{D}_{\Sigma,\text{TINA}} && (67)\\
&\geq \mathcal{D}_{\Sigma,\text{BC}} - \mathcal{D}_{\Sigma,\text{TINA}} - \Lambda\mathcal{D}_{\Sigma,\text{TINA}} && (68)
\end{aligned}$$

where in the last step we used Lemma 6. Substituting the bound for Λ we obtain

$$\begin{aligned}
\frac{\mathcal{D}_{\Sigma,\text{BC}}}{\mathcal{D}_{\Sigma,\text{TINA}}} &\leq 2 + \log_2(K-1) && (69)\\
&= \Theta(\log_2(K)) && (70)
\end{aligned}$$

and the proof of the upper bound is complete. □

Proof of Theorem 3: Lower Bound

For the lower bound, let us define a class of interference networks, $\mathcal{N}^{[n,\nu]}$, that is parameterized by the two numbers, $n \in \mathbb{N}, \nu \in \mathbb{R}$, $0 \leq \nu \leq 1$. The number of users $K(n) = 2^n$, all desired channel strengths $\alpha_{kk} = 1$, and cross-channel strengths satisfy $\alpha_{ij}^{[n,\nu]} = \alpha_{ji}^{[n,\nu]}$ for all $i, j, k \in [K(n)]$. Since $\alpha_{ij}^{[n,\nu]} = \alpha_{ii} - \delta_{ji}^{[n,\nu]} = 1 - \delta_{ji}^{[n,\nu]} = 1 - \delta_{ij}^{[n,\nu]}$ it suffices to specify the $\delta_{ij}^{[n,\nu]}$ values instead of the $\alpha_{ij}^{[n,\nu]}$ values. To specify the $\delta_{ij}^{[n,\nu]}$ values it will be useful to represent $\mathcal{N}^{[n,\nu]}$ as a full binary tree of depth n. The 2^n leaf nodes of this tree represent the 2^n users. The value of $\delta_{ij}^{[n,\nu]} = \delta_{ji}^{[n,\nu]} = \left(\frac{2^{p-1}}{2^n}\right)\nu$ if the closest common ancestor of user i and user j is p levels above them. For example, $\delta_{ij}^{[n,\nu]} = \frac{\nu}{2^n}$ if user i and j are siblings (share a common parent), $\frac{2\nu}{2^n}$ if they share the same grandparent (but not the same parent), and the largest possible value of $\delta_{ij}^{[n,\nu]}$ in $\mathcal{N}^{[r,\nu]}$ is $\nu/2$, between users whose closest common ancestor is the root node. An interference network with these parameter values is said to be an $\mathcal{N}^{[n,\nu]}$ network. Fig. 4 shows the binary tree for the network $\mathcal{N}^{[3,1]}$, and Fig. 5 shows another representation of the same network. We are primarily interested in the network for $\nu = 1$.

Let us first prove that an $\mathcal{N}^{[n,\nu]}$ network is indeed in the SLS regime. From the definition of $\delta_{ij}^{[n,\nu]} = 1 - \alpha_{ij}^{[n,\nu]}$, we have,

$$\begin{aligned}
\alpha_{ij}^{[n,\nu]} &= 1 - \left(\frac{2^{p_{ij}-1}}{2^n}\right)\nu && (71)\\
\alpha_{ki}^{[n,\nu]} &= 1 - \left(\frac{2^{p_{ki}-1}}{2^n}\right)\nu && (72)
\end{aligned}$$

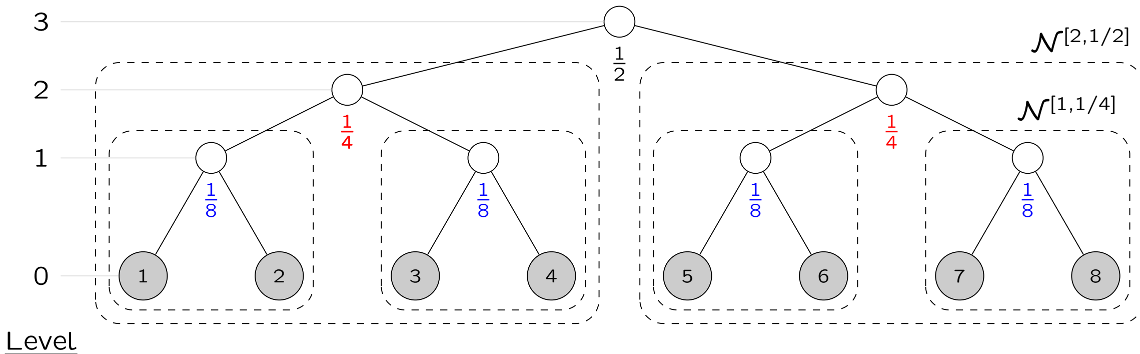


Figure 4: *The binary tree representation of the network $\mathcal{N}^{[3,1]}$, and its subnetworks. The value of $\delta_{ij}^{[n,1]} = 1 - \alpha_{ij}^{[n,1]}$ between users i and j is given by the number indicated under their closest common ancestor. For example, $\delta_{78} = 1/8, \delta_{14} = 1/4, \delta_{37} = 1/2$.*

Since $\alpha_{ii} = 1$ and $\nu \geq 0$, it is trivially verified that $\alpha_{ii} \geq \max(\alpha_{ij}(\nu), \alpha_{ki}(\nu))$ for all $i, j, k \in [K(n)]$. Now, if users i, j have their closest common ancestor p_{ij} levels above them, and if users i, k have their closest common ancestor p_{ki} levels above them, then the users j, k must have a common ancestor no more than $\max(p_{ij}, p_{ki})$ levels above them. Therefore,

$$\alpha_{jk}^{[n,\nu]} \geq 1 - \left(\frac{2^{\max(p_{ij},p_{ki})-1}}{2^n}\right)\nu \tag{73}$$

$$\implies \alpha_{ii} + \alpha_{jk}^{[n,\nu]} \geq 1 + 1 - \left(\frac{2^{\max(p_{ij},p_{ki})-1}}{2^n}\right)\nu \tag{74}$$

$$\geq 1 + 1 - \left(\frac{2^{p_{ij}-1}}{2^n} + \frac{2^{p_{ki}-1}}{2^n}\right)\nu \tag{75}$$

$$= \alpha_{ij}^{[n,\nu]} + \alpha_{ki}^{[n,\nu]} \tag{76}$$

Thus the SLS condition is satisfied.

Next we will prove that the TINA region for this network does not allow more than 2 sum-GDoF. For this let us go through the following three steps.

1. The main argument for this proof is recursive, where we repeatedly reduce a network into its subnetworks. In particular, we are interested in the *left* and *right* subnetworks of $\mathcal{N}^{[n,\nu]}$, as described next. Consider the root node of the binary tree representation of $\mathcal{N}^{[n,\nu]}$. It has two child nodes, say labeled as 'left' and 'right'. If the root node is eliminated, then the tree splits into two binary trees, and each of those original child nodes becomes the root node of one of those trees. Let us denote these two networks as Left($\mathcal{N}^{[n,\nu]}$) and Right($\mathcal{N}^{[n,\nu]}$). Let us show that each of the networks Left($\mathcal{N}^{[n,\nu]}$) and Right($\mathcal{N}^{[n,\nu]}$) is an $\mathcal{N}^{[n-1,\nu/2]}$ network, as follows. Since the original root node is eliminated, it is obvious that the binary tree representation of each of these subnetworks has depth $n-1$, and correspondingly each subnetwork has 2^{n-1} users. The channel strengths are the same as before, but since the value of n has changed to $n-1$, the value of ν needs to change to $\nu/2$ to preserve the channel strengths, so in the new subnetworks we have

$$\delta_{ij}^{[n-1,\nu/2]} = \left(\frac{2^{p-1}}{2^{n-1}}\right)\left(\frac{\nu}{2}\right) = \left(\frac{2^{p-1}}{2^n}\right)\nu = \delta_{ij}^{[n,\nu]}. \tag{77}$$

where either both i, j belong to the left subnetwork or both belong to the right subnetwork.

2. Next we show that $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{TINA}} \leq \max\left(1, \frac{1}{2}\mathcal{D}^{[n,2\nu]}_{\Sigma,\text{TINA}}\right)$, where $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{TINA}}$ represents the optimal sum-GDoF value over the $\mathcal{D}^{[n,\nu]}_{\text{TINA}}$ region for $\mathcal{N}^{[n,\nu]}$. This is proved as follows. From Definition 10, we know that $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{TINA}}$ is equal to $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{P-TIN}}(S)$ for some subset of users, $S \subset [K(n)]$. From Theorem 4 we know that $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{P-TIN}}(S)$ is determined by the cycle bounds corresponding to a p-optimal cyclic partition of S. There are two possibilities — either the cyclic partition includes a trivial cycle, or it does not, and we will consider them one by one.

   First, suppose the p-optimal cyclic partition of S does not include any trivial cycles. In that case, let $\pi = (i_1 \to \cdots \to i_M \circlearrowleft)$ be any cycle from the p-optimal cyclic partition of S. By assumption, the length of π is $M > 1$. The cycle bound corresponding to π for $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{TINA}}$ is

$$\sum_{k\in\{i_1,\cdots,i_M\}} d_k \leq \delta^{[n,\nu]}_{i_1i_2} + \cdots + \delta^{[n,\nu]}_{i_{M-1}i_M} + \delta^{[n,\nu]}_{i_Mi_1} \tag{78}$$

$$= \frac{1}{2}\left(\delta^{[n,2\nu]}_{i_1i_2} + \cdots + \delta^{[n,2\nu]}_{i_{M-1}i_M} + \delta^{[n,2\nu]}_{i_Mi_1}\right) \tag{79}$$

   Therefore, all the non-trivial cycle bounds $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{TINA}}$ are exactly half as large as the corresponding cycle bounds in $\mathcal{D}^{[n,2\nu]}_{\Sigma,\text{TINA}}$, proving that in this case $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{TINA}} = \frac{1}{2}\mathcal{D}^{[n,2\nu]}_{\Sigma,\text{TINA}}$.

   Now consider the remaining alternative, that the p-optimal cyclic partition of S includes a trivial cycle. We claim that in this case $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{TINA}} = 1$. This is shown as follows. Suppose $\pi = \{i\}$ is a trivial cycle included in the p-optimal cyclic partition of S. Since the trivial cycle bound is active we must have $d_i = \alpha_{ii} = 1$. Now, let User j be any other user in S. We immediately have the bound $d_i + d_j \leq \delta_{ij} + \delta_{ji} \leq 1$ (because in $\mathcal{N}^{[n,\nu]}$, all $\delta_{ij} \leq \nu/2$ and $\nu \leq 1$). Since $d_i = 1$, we must have $d_i + d_j = 1$ and therefore, $d_j = 0$. This is true for every user in S besides user i. Therefore, $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{TINA}} = 1$ in this case.

3. The final step is to prove that $\mathcal{D}^{[n,\nu]}_{\Sigma,\text{TINA}} \leq 2$. Based on previous steps, this is proved as follows. Isolating the left and right subnetworks of $\mathcal{N}^{[n,\nu]}$ from each other's interference does not hurt either of them, therefore,

$$\mathcal{D}^{[n,\nu]}_{\Sigma,\text{TINA}} \leq \mathcal{D}^{[n-1,\nu/2]}_{\Sigma,\text{TINA}} + \mathcal{D}^{[n-1,\nu/2]}_{\Sigma,\text{TINA}} \tag{80}$$

$$= 2\mathcal{D}^{[n-1,\nu/2]}_{\Sigma,\text{TINA}} \tag{81}$$

$$\leq 2\max\left(1, \frac{1}{2}\mathcal{D}^{[n-1,\nu]}_{\Sigma,\text{TINA}}\right) \tag{82}$$

$$= \max(2, \mathcal{D}^{[n-1,\nu]}_{\Sigma,\text{TINA}}) \tag{83}$$

$$\leq \max(2, \max(2, \mathcal{D}^{[n-2,\nu]}_{\Sigma,\text{TINA}})) \tag{84}$$

$$= \max(2, \mathcal{D}^{[n-2,\nu]}_{\Sigma,\text{TINA}}) \tag{85}$$

$$\leq \vdots \tag{86}$$

$$\leq \max(2, \mathcal{D}^{[1,\nu]}_{\Sigma,\text{TINA}}) \tag{87}$$

$$= 2. \tag{88}$$

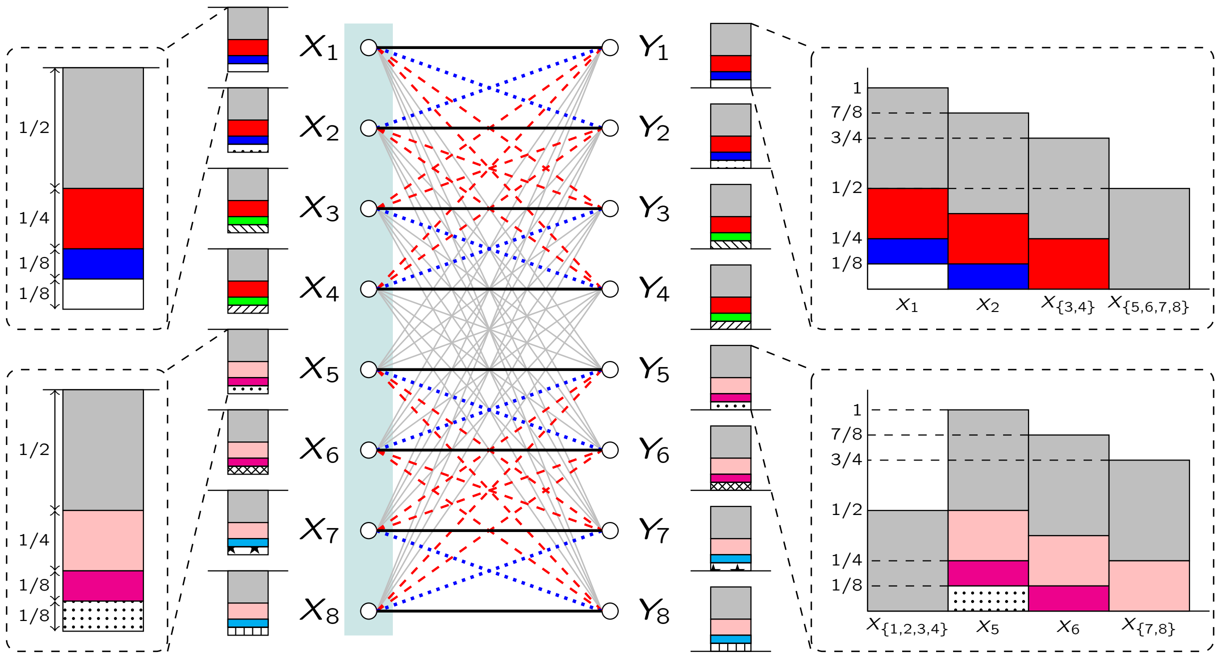


Figure 5: *The SLS scheme for $\mathcal{N}^{[3,1]}$ that achieves $1+\frac{1}{2}\log_2(K) = \frac{5}{2}$ sum-GDoF by transmitter cooperation.*

Thus, TIN cannot achieve more than 2 sum-GDoF for our network.

Henceforth we will set $\nu = 1$ and prove that by allowing transmitter cooperation in this network, a sum-GDoF value of $1 + \frac{1}{2}\log_2(K)$ is achievable (and optimal). Recall that in a GDoF model, if Transmitter j sends a message W with power level $-\gamma_j$ to Receiver i over a channel with strength α_{ij}, then the received signal strength level is $\alpha_{ij} - \gamma_j$. The power levels are additive because these are exponents of P, or equivalently because they are being measured in dB scale. If the effective noise floor, i.e., the maximum power level of noise and interference from other messages heard by Receiver i is μ_i, and W carries d_W GDoF, then W can be decoded successfully while treating all other signals as noise if $d_W \leq \alpha_{ij} - \gamma_j - \mu_i$. Once a message is decoded it can be subtracted from the received signal before decoding other messages. This is the basic principle of successive decoding, and we will use it for the achievability proof.

Before a detailed presentation of the achievable scheme for $\mathcal{N}^{[n,1]}$ networks, let us start with a sketch of the achievable scheme for the example network $\mathcal{N}^{[3,1]}$, as shown in Fig. 5. We saw the binary tree representation of this network earlier in Fig. 4. Recall that for this example, all direct links are of strength $\alpha_{ii} = 1$. For the cross links, in Fig. 5 the dotted blue lines are links of strength $\alpha_{ij} = 7/8$, the dashed red lines are of strength $\alpha_{ij} = 3/4$, and the gray lines are links of strength $\alpha_{ij} = 1/2$. The same gray common message at the top level is sent from all antennas to all users and carries 1/2 sum GDoF. The dashed red links are in two separate clusters of 4 users each, representing 2 subnetworks, each of the type $\mathcal{N}^{[2,1/2]}$ containing 4 users. A red common message is sent for the first cluster and a pink common message is used for the second cluster, each carrying 1/4 GDoF. Similarly, the dotted blue links are in 4 separate clusters of 2 users each, representing 4 subnetworks, each of the type $\mathcal{N}^{[1,1/4]}$ containing 2 users. The corresponding blue, green, magenta and cyan power levels represent separate common messages for each of the 4 subnetworks, carrying 1/8 GDoF each. Finally, at the bottom level there is an independent message carrying 1/8 GDoF for each user. The total sum-GDoF value thus achieved is $\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + 8\left(\frac{1}{8}\right) = 5/2$. For the decoding, consider User 5 as an example. The gray message which carries 1/2 GDoF, is seen with power level 1 and noise floor due to interference from other messages is at power level 1/2 so it is successfully decoded and subtracted. Then the pink message, which carries 1/4 GDoF, is seen with power level 1/2 and effective noise floor 1/4, so it is also decoded and subtracted. Next, the magenta message which carries 1/8 GDoF is seen with power level 1/4 and noise floor 1/8, so it is also decoded and subtracted successfully. Finally, only the dotted white message, which carries 1/8 GDoF is seen with power levels 1/8 and noise floor 0, so it is decoded as well.

Now, let us explain the scheme for arbitrary $\mathcal{N}^{[n,1]}$. As in the example, the achievable scheme is also hierarchical where we will start with a common message for all users in $\mathcal{N}^{[n,\nu]}$ and then progressively include additional messages for its subnetworks while maintaining the successive decodability of all messages. For ease of reference, let us call the common message for the users in a $\mathcal{N}^{[n,\nu]}$ network a level n message.

The same level-n message, is sent from every transmitter with strength $\gamma = 0$, so that it is received at every receiver with strength $\gamma + \alpha_{ii} = 1$. It carries 0.5 GDoF. The power levels of all other messages are set to $-1/2$ or less so that all other messages are received with strength no more than $-1/2 + 1 = 1/2$. Since the noise floor from other messages is at 1/2, the common message is received at strength level 1, and it carries only 1/2 GDoF, it is decodable at every receiver, After decoding it, every receiver subtracts out the codeword due to the level n message.

There are two different level $n-1$ sub-networks. Within each of these two networks a different level $n-1$ message is sent with power level $-1/2$, so it is received at power level 1/2 at each receiver within the sub-network. Signals from one sub-network are not heard by the other subnetwork because the channel strength between the users in different sub-networks is 1/2 and the transmit power of the level $n-1$ message is $-1/2$. All lower level messages are sent with power levels less than $-3/4$, so the noise floor due to lower level messages at each receiver is at power level 1/4. Thus, the level $n-1$ message is able to achieve $1/2 - 1/4 = 1/4$ GDoF. Since there are 2 such messages corresponding to the 2 subnetworks, the total sum GDoF value contributed by level $n-1$ messages is $1/4 + 1/4 = 1/2$. After decoding each receiver subtracts out the codeword due to level $n-1$ message from its own subnetwork.

Next, there are 4 level $n-2$ sub-networks. A different common message is sent within each subnetwork with power level $-1 + (1/2)^2 = -3/4$, so it is received at power level 1/4, while all lower level messages are sent with power no more than $-1 + (1/2)^3 = -7/8$, so the noise floor due to lower level messages is $1 - 7/8 = 1/8$. The sub-networks do not interfere with each other because the cross-subnetwork channel strengths are $1 - 1/2^2 = 3/4$ so the received signals from other subnetworks are below the noise floor. Thus, each of the 4 of the $(n-2)$-level messages is able to achieve $1/4 - 1/8 = 1/8$ GDoF for a total of $4 \times 1/8 = 1/2$. The decoded messages are subtracted.

This pattern continues, so that for each $i \in [0:n]$, there are 2^i different level-$(n-i)$ subnetworks. Within each of these subnetworks, a different common message is sent with power level $-1+(1/2)^i$ so it is received at power level $(1/2)^i$ while all lower level messages are sent with power level no more than $-1 + (1/2)^{i+1}$ so that the noise floor due to lower level messages is $(1/2)^{i+1}$ at each receiver. Thus each of the 2^i subnetworks achieves $1/2^i - 1/2^{i+1} = 1/2^{i+1}$ GDoF for a total of $2^i/2^{i+1} = 1/2$ sum GDoF.

Adding these values across all n levels we achieve a total of $n/2$ sum-GDoF. In fact, it is possible to do a little bit better. At level 0, there are 2^n subnetworks comprised of individual users, and since there are no more lower level messages, the noise floor is 0, so it is possible to achieve

$1/2^n - 0 = 1/2^n$ GDoF per user for a total of 1 GDoF instead of just $1/2$ GDoF for level 0 messages. Thus, the total sum-GDoF value achieved is $1 + n/2 = 1 + \frac{1}{2}\log_2(K)$ sum-GDoF.

As a final remark, the sum-GDoF $1 + \frac{n}{2} = 1 + \frac{1}{2}\log_2(K)$ is optimal for the BC obtained by allowing transmitter cooperation in $\mathcal{N}^{[n,1]}$. Applying Lemma 6 with cycle $\pi = (1 \to 2 \to \cdots \to K \circlearrowleft)$, we have the sum-GDoF in the BC bounded above by

$$\mathcal{D}_{\Sigma,\mathrm{BC}}^{[n,1]}([K]) \leq \Delta_\pi + \alpha_{1K}^{[n,1]} \tag{89}$$

$$= \sum_{k=1}^{K-1} \delta_{k,k+1}^{[n,1]} + \delta_{K1}^{[n,1]} + \alpha_{1K}^{[n,1]} \tag{90}$$

$$= \sum_{\ell=1}^{n} \frac{1}{2^{n-\ell+1}} 2^{n-\ell} + \frac{1}{2} + \frac{1}{2} \tag{91}$$

$$= \frac{1}{2}\log_2 K + 1, \tag{92}$$

which matches the achieved sum-GDoF.

7 Conclusion

The results presented here open the door to a number of open questions where extremal analysis could be useful to gain a deeper understanding of the benefits of transmitter cooperation. For example, is it possible to achieve more than logarithmic GDoF gain by transmitter cooperation over TIN in a *general* weak interference regime where the only constraint is that the direct channels are stronger than cross channels? What is the maximum possible sum-GDoF gain of a K user MISO BC over the corresponding K user IC in the general weak interference regime? Or, even in the SLS-regime? In general, it seems extremal analysis may be useful to gauge the relative benefits of a myriad of factors such as multiple antennas, power control, rate-splitting, space-time multiplexing and network coherence – all intriguing issues for which the current understanding is extremely limited. Indeed, the main message of this work is to underscore the importance of extremal analysis in order to advance our understanding of fundamental limits of large wireless networks beyond symmetric settings, where the curse of dimensionality stands in the way. In particular, extremal analysis used in conjunction with the GDoF metric under finite precision CSIT, as exemplified by this work, appears to be a promising research avenue to bridge the gap between theory and practice.

Appendix

A Optimality of Cyclic Partition for Polyhedral TIN in SLS Regime

Theorem 4 *If $[\alpha]_{K\times K} \in \mathcal{A}_{\mathrm{SLS}}$, then for any subset of users, S, $S \subset [K]$, there exists a p-optimal cyclic partition of S.*

Proof of Theorem 4

Without loss of generality we will prove the lemma for $S = [K]$, since the same proof works for any $S \subset [K]$ as well. Let us start with arbitrary $[\alpha]_{K\times K}$, i.e., not necessarily in the SLS regime. The sum-GDoF value in the polyhedral region, $\mathcal{D}_{\Sigma,\text{P-TIN}}$ is the solution to the following linear program,

$$(LP_1) \qquad \mathcal{D}_\Sigma = \max\ d_1 + d_2 + \cdots + d_K \tag{93}$$

$$\text{such that} \sum_{k\in\{\pi\}} d_k \le \sum_{k\in\{\pi\}} \alpha_{kk} - w(\pi), \quad \forall \pi \in \Pi \tag{94}$$

$$d_k \ge 0, \quad \forall k \in [K] \tag{95}$$

and can be equivalently expressed by the following dual linear program.

$$(LP_2) \qquad \mathcal{D}_\Sigma = \min \sum_{\pi\in\Pi} \lambda_\pi \left(\sum_{k\in\{\pi\}} \alpha_{kk} - w(\pi) \right) \tag{96}$$

$$\text{such that} \sum_{\pi\in\Pi} \lambda_\pi 1(k \in \{\pi\}) \ge 1, \quad \forall k \in [K] \tag{97}$$

$$\lambda_\pi \ge 0, \quad \forall \pi \in \Pi \tag{98}$$

where $1(\cdot)$ is the indicator function that returns the values 1 or 0 when the argument to the function is true or false, respectively.

For all $\pi \in \Pi$, let us define λ_π^* as the optimizing values of λ_π for LP_2. Let the corresponding optimal values for LP_1 be d_k^* for all $k \in [K]$. Because a solution must exist, by the strong-duality of linear programming, the optimal D_Σ for LP_2 is the same as the optimal D_Σ for LP_1. Therefore, the following conditions are implied.

$$D_\Sigma = d_1^* + d_2^* + \cdots + d_K^* = \sum_{\pi\in\Pi} \lambda_\pi^* \left(\sum_{k\in\{\pi\}} \alpha_{kk} - w(\pi) \right) \tag{99}$$

$$\sum_{k\in\{\pi\}} \alpha_{kk} - w(\pi) \ge \sum_{k\in\{\pi\}} d_k^* \qquad \forall \pi \in [\Pi] \tag{100}$$

$$\lambda_\pi^* \ge 0, \qquad \forall \pi \in [\Pi] \tag{101}$$

$$d_k^* \ge 0, \qquad \forall k \in [K] \tag{102}$$

Definition 11 (Set of Active Cycles, Π^*) *Based on the optimizing solution to LP_2, define*

$$\Pi^* = \{\pi \in \Pi : \lambda_\pi^* > 0\} \tag{103}$$

This is called the set of active cycles, because the corresponding cycle bounds are active (i.e., tight) in the solution to LP_2 (see Lemma 1).

Definition 12 (Set of Inactive Users, $\mathcal{K}_o$) *Define $\mathcal{K}_o \subset [K]$ as the set of all users k for which the inequality in* (97) *is strict. Thus,*

$$\mathcal{K}_o = \{k \in [K] : \sum_{\pi\in\Pi} \lambda_\pi^* 1(k \in \{\pi\}) > 1\} \tag{104}$$

This is called the set of inactive users because for each of these users, we must have $d_k^* = 0$ (see Lemma 1).

Lemma 1

$$\forall k \in \mathcal{K}_o \qquad \textit{we must have} \qquad d_k^* = 0, \tag{105}$$

$$\text{and } \forall \pi \in \Pi^* \qquad \textit{we must have} \qquad \sum_{k \in \{\pi\}} d_k^* = \sum_{k \in \{\pi\}} \alpha_{kk} - w(\pi). \tag{106}$$

Note that the conditions are simply complementary slackness conditions, therefore Lemma 1 holds for arbitrary channel parameters, i.e., even if $[\alpha]_{K\times K} \notin \mathcal{A}_{\text{SLS}}$. For the sake of completeness, a proof of Lemma 1 appears in Appendix A.1.

Henceforth, let us restrict our attention to the SLS regime. In fact, let us define a *strict* SLS regime as

$$\bar{\mathcal{A}}_{\text{SLS}} = \{[\alpha]_{K\times K} \in \mathbb{R}_+^{K\times K} : \alpha_{ii} > \max(\alpha_{ij}, \alpha_{ki}, \alpha_{ik} + \alpha_{ji} - \alpha_{jk}),\ \forall i, j, k \in [K], i \notin \{j, k\}\}. \tag{107}$$

Note that the only difference between $\mathcal{A}_{\text{SLS}}$ and $\bar{\mathcal{A}}_{\text{SLS}}$ is that the defining inequalities in the latter are all strict inequalities. Note that all α_{ii} and δ_{ij} are strictly positive in the strict SLS regime. Following the same reasoning as the proof of Lemma 3, in the strict SLS regime, for distinct $i, j, k \in [K]$, we must have

$$[\alpha]_{K\times K} \in \bar{\mathcal{A}}_{\text{SLS}} \qquad \Rightarrow \qquad \delta_{ki} + \delta_{ij} > \delta_{kj} \tag{108}$$

note that the inequality is strict here as well. This is important for the proof.

We will first prove Theorem 4 for the strict SLS regime and later use a continuity argument to show that the result holds even when the inequalities are relaxed to include equalities. The shell of the proof is identical to the proof of Theorem 3 in [15]. The main step that connects the two proofs is Lemma 2 in this paper.
Now define the following linear program.

$$(LP_3) \qquad \mathcal{D}_\Sigma = \min \sum_{\pi \in \Pi} \lambda_\pi \left(\sum_{k \in \{\pi\}} \alpha_{kk} - w(\pi) \right) \tag{109}$$

$$\text{such that} \sum_{\pi \in \Pi} \lambda_\pi 1(k \in \{\pi\}) = 1, \quad \forall k \in [K] \tag{110}$$

$$\lambda_\pi \geq 0, \quad \forall \pi \in \Pi \tag{111}$$

Note that the only difference between LP_2 and LP_3 is that the inequality in (97) has been replaced with the equality in (110). The following lemma is the most critical part of the proof, as it shows that this change does not matter in the strict SLS regime, thereby reducing the problem to another problem that is already solved in [15].

Lemma 2

$$[\alpha]_{K\times K} \in \bar{\mathcal{A}}_{SLS} \qquad \Rightarrow \qquad LP_2 \equiv LP_3 \tag{112}$$

The proof of Lemma 2 appears in Appendix A.2.
Following Lemma 2, LP_3 is identical to the LP_3 in [15] and the rest of the proof is identical to the proof of Theorem 3 in in [15]. Thus, the proof of Theorem 4 is complete. $\square$

A.1 Proof of Lemma 1

$$0 = \sum_{\pi \in \Pi} \lambda_\pi^* \left(\sum_{k \in \{\pi\}} \alpha_{kk} - w(\pi) \right) - D_\Sigma \tag{113}$$

$$\geq \sum_{\pi \in \Pi} \lambda_\pi^* \left(\sum_{k \in \{\pi\}} d_k^* \right) - D_\Sigma \tag{114}$$

$$= \sum_{k \in [K]} \left(d_k^* \left(\sum_{\pi \in [\Pi]} \lambda_\pi^* 1(k \in \{\pi\}) \right) \right) - D_\Sigma \tag{115}$$

$$= \sum_{k \in [K] \setminus \mathcal{K}_o} d_k^* + \sum_{k \in \mathcal{K}_o} c_k d_k^* - D_\Sigma \tag{116}$$

$$= \sum_{k \in [K] \setminus \mathcal{K}_o} d_k^* + \sum_{k \in \mathcal{K}_o} c_k d_k^* - \sum_{k \in [K]} d_k^* \tag{117}$$

$$= \sum_{k \in \mathcal{K}_o} (c_k - 1) d_k^* \tag{118}$$

$$\geq 0 \tag{119}$$

because $c_k \triangleq \sum_{\pi \in \Pi} \lambda_\pi^* 1(k \in \pi) > 1$ for all $k \in \mathcal{K}_o$, and $d_k^* \geq 0$ for all $k \in [K]$. Since we started and ended with 0, all steps from (113) to (119) must be equalities. Thus, the proof of Lemma 1 is complete. $\square$

A.2 Proof of Lemma 2

We need to prove that the set $\mathcal{K}_o$ is empty. Suppose, on the contrary, that there exists $k_o \in \mathcal{K}_o$. According to Lemma 1 the user k_o must be inactive, i.e., $d_{k_o}^* = 0$. Let $\pi_o = (i_1 \to i_2 \to \cdots \to i_M \circlearrowleft)$ be an active cycle that includes User k_o. Without loss of generality, suppose $k_o = i_M$. We will consider 3 cases.

1. **Case 1:** $(M > 2)$
   Suppose the length of the cycle is greater than 2. Since π_o is an active cycle, according to Lemma 1,

$$\sum_{k \in \pi_o} d_k^* = \sum_{k \in \pi_o} \alpha_{kk} - w(\pi_o) \tag{120}$$

$$\Rightarrow d_{i_1}^* + d_{i_2}^* + \cdots + d_{i_M}^* = \delta_{i_1 i_2} + \delta_{i_2 i_3} + \cdots + \delta_{i_{M-1} i_M} + \delta_{i_M i_1}. \tag{121}$$

   But since $k_o \in \mathcal{K}_o$, according to Lemma 1 we must have $d_{k_o}^* = d_{i_M}^* = 0$. Therefore,

$$d_{i_1}^* + d_{i_2}^* + \cdots + d_{i_{M-1}}^* = \delta_{i_1 i_2} + \delta_{i_2 i_3} + \cdots + \delta_{i_{M-2} i_{M-1}} + \delta_{i_{M-1} i_M} + \delta_{i_M i_1}. \tag{122}$$

   But now consider the cycle $\pi' = (i_1 \to i_2 \cdots \to i_{M-1} \circlearrowleft)$. This may or may not be an active cycle. Regardless, the following bound must hold.

$$d_{i_1}^* + d_{i_2}^* + \cdots + d_{i_{M-1}}^* \leq \delta_{i_1 i_2} + \delta_{i_2 i_3} + \cdots + \delta_{i_{M-2} i_{M-1}} + \delta_{i_{M-1} i_1} \tag{123}$$

Subtracting (122) from (123) we have

$$0 \le \delta_{i_{M-1}i_1} - \delta_{i_{M-1}i_M} - \delta_{i_M i_1} \tag{124}$$

$$\Rightarrow \delta_{i_{M-1}i_M} + \delta_{i_M i_1} \le \delta_{i_{M-1}i_1} \tag{125}$$

But this is a contradiction because under strict SLS condition, according to (108),

$$\delta_{i_{M-1}i_M} + \delta_{i_M i_1} > \delta_{i_{M-1}i_1}. \tag{126}$$

2. **Case 2:** $(M = 1)$
   The length of the cycle, M, cannot be 1 because then Lemma 1 would imply that the single user bound is active, i.e., $d^*_{k_o} = \alpha_{k_o k_o}$, but $\alpha_{k_o k_o} > 0$ in the strict SLS regime, so user k_o must be active, i.e., we would have a contradiction. This leaves us with the only possibility, $M = 2$.

3. **Case 3:** $(M = 2)$
   Now suppose the length of the cycle π_o is $M = 2$. Then we have

   $$d^*_{i_1} + d^*_{i_M} = \delta_{i_1 i_M} + \delta_{i_M i_1} \tag{127}$$

   and since $k_o = i_M \in \mathcal{K}_o$ according to Lemma 1 we have $d^*_{i_M} = 0$. Therefore,

   $$d^*_{i_1} = \delta_{i_1 i_M} + \delta_{i_M i_1} \tag{128}$$

   Consider the following two subcases.

   (a) **Subcase 1: π_o is the only active bound that includes user k_o**
   Then $\lambda_{\pi_o} > 1$. But this would mean that user i_1 also belongs to $\mathcal{K}_o$, because the sum of weights of active cycles that include user i_1 must be greater than 1 as well. However, if both user i_1 and user i_M are in $\mathcal{K}_o$, then they must both be inactive. This is a contradiction, because $d^*_{i_1} + d^*_{i_M} = \delta_{i_1 i_M} + \delta_{i_M i_1} > 0$.

   (b) **Subcase 2: There is another active bound, $\pi_1 \neq \pi_o$ that includes user k_o**
   Now, π_1 must also have length $M = 2$ because, as we have already established, any other possibility leads to a contradiction. Since π_1 is different from π_o it must involve a user other than i_1 in addition to user i_M. Let's call this user i_2. Then, proceeding similarly as in the case of π_o we find that we must have

   $$d^*_{i_2} = \delta_{i_2 i_M} + \delta_{i_M i_2} \tag{129}$$

   But we also know that the following bound must hold.

   $$d^*_{i_1} + d^*_{i_2} \le \delta_{i_1 i_2} + \delta_{i_2 i_1} \tag{130}$$

   Subtracting (128) and (129) from (130) we have,

   $$0 \le \delta_{i_1 i_2} + \delta_{i_2 i_1} - \delta_{i_1 i_M} - \delta_{i_M i_1} - \delta_{i_2 i_M} - \delta_{i_M i_2} \tag{131}$$

   $$< (\delta_{i_1 i_M} + \delta_{i_M i_2}) + (\delta_{i_2 i_M} + \delta_{i_M i_1}) - \delta_{i_1 i_M} - \delta_{i_M i_1} - \delta_{i_2 i_M} - \delta_{i_M i_2} \tag{132}$$

   $$= 0 \tag{133}$$

   which is a contradiction. Note that we used (108) in (131).

Thus, we have a contradiction in every case, so there cannot be any such $k_o \in \mathcal{K}_o$, which implies that $\mathcal{K}_o$ is empty, and the proof is complete. □

B Transmitter Reassignment does not help in the SLS Regime

Consider a K user interference network with $[\alpha]_{K\times K} \in \mathcal{A}_{\text{SLS}}$. Full cooperation between the transmitters would produce the corresponding K user BC which allows joint coding of messages. A more restricted but simpler form of cooperation is to allow re-assignment of transmitters to receivers. Suppose the original assignment, where Transmitter i is the desired transmitter for Receiver i, is denoted as σ_o. Let σ be a permutation on $[K]$ representing a new assignment, such that with σ, Transmitter $\sigma(i)$ is the desired transmitter for Receiver i. Note that following the reassignment, we have a different K user interference channel. Recall that under the original assignment, the achievable TIN region is denoted $\mathcal{D}_{\text{TINA}}$. Let the corresponding region under σ be denoted as $\mathcal{D}^{\sigma}_{\text{TINA}}$. The following theorem shows that such a reassignment does not help.

Theorem 5 *If $[\alpha]_{K\times K} \in \mathcal{A}_{\text{SLS}}$, then*

$$\mathcal{D}^{\sigma}_{\text{TINA}} \subset \mathcal{D}^{\sigma_o}_{\text{TINA}}. \tag{134}$$

Proof of Theorem 5

Recall that $\mathcal{D}_{\text{TINA}}$ is the union of polyhedral TIN regions, which are described by cycle bounds. Consider an arbitrary cycle, $\pi = (i_1 \to \cdots \to i_M \hookleftarrow)$, and let us compare the cycle bounds for σ_o versus σ.

$$\text{With } \sigma_o, \qquad \sum_{k\in\{\pi\}} d_k \leq \sum_{m=1}^{M} \alpha_{i_m i_m} - \alpha_{i_{m+1} i_m} = \Delta^{\sigma_o}_{\pi}. \tag{135}$$

$$\text{With } \sigma, \qquad \sum_{k\in\{\pi\}} d_k \leq \sum_{m=1}^{M} \alpha_{i_m \sigma(i_m)} - \alpha_{i_{m+1} \sigma(i_m)} = \Delta^{\sigma}_{\pi}. \tag{136}$$

But since $[\alpha]_{K\times K} \in \mathcal{A}_{\text{SLS}}$, we must have $\alpha_{i_m\sigma(i_m)} - \alpha_{i_{m+1}\sigma(i_m)} \leq \alpha_{i_m i_m} - \alpha_{i_{m+1} i_m}$, and therefore $\Delta^{\sigma}_{\pi} \leq \Delta^{\sigma_o}_{\pi}$ for every cycle π. Therefore, $\mathcal{D}^{\sigma}_{\text{TINA}} \subset \mathcal{D}^{\sigma_o}_{\text{TINA}}$ and the proof of Theorem 5 is complete. $\square$

C Other Useful Lemmas

C.1 A condition on δ_{ij} in the SLS Regime

Lemma 3 *For all $i, j, k \in [K]$,*

$$[\alpha]_{K\times K} \in \mathcal{A}_{SLS} \qquad \Rightarrow \qquad \delta_{ki} + \delta_{ij} \geq \delta_{kj} \tag{137}$$

Proof of Lemma 3

Proof: $\delta_{ki} + \delta_{ij} - \delta_{kj} = \alpha_{kk} - \alpha_{ik} + \alpha_{ii} - \alpha_{ji} - \alpha_{kk} + \alpha_{jk} = \alpha_{ii} + \alpha_{jk} - \alpha_{ik} - \alpha_{ji}$ which, by definition, is non-negative in $\mathcal{A}_{\text{SLS}}$. $\square$

C.2 Trivial cycles in the SLS Regime

Lemma 4 *If $[\alpha]_{K\times K} \in \mathcal{A}_{\text{SLS}}$, then for every $\mathcal{S} \subset [K]$ there exists a p-optimal cyclic partition containing at most one trivial cycle.*

Proof of Lemma 4

Let $\{\pi_i\}_{i=1}^N$ be a p-optimal cyclic partition for $\mathcal{S}$. Suppose there is more than one trivial cycle in a p-optimal cyclic partition, we claim that they can be combined into one cycle, and the resulting partition is still p-optimal and free of trivial cycles. Let $\pi_1 = (i_1 \circlearrowleft), \pi_2 = (i_2 \circlearrowleft), \cdots, \pi_j = (i_j \circlearrowleft)$, $2 \leq j \leq N$, be all the trivial cycles in $\{\pi_i\}_{i=1}^N$. These trivial cycles can be combined into $\pi_{1,2,\cdots,j} = (\pi_1 \to \pi_2 \to \cdots \to \pi_j \circlearrowleft)$. Since $\pi_{1,2,\cdots,j}$ and all the other cycles are disjoint, $\{\pi_{1,2,\cdots,j}, \pi_{j+1}, \cdots, \pi_N\}$ is a cyclic partition. Moreover,

$$\Delta_{\pi_{1,2,\cdots,j}} = \sum_{m=1}^{j} \delta_{i_m,i_{m+1}} \leq \sum_{m=1}^{j} \alpha_{i_m,i_m} = \sum_{m=1}^{j} \Delta_{\pi_m}, \tag{138}$$

where $\delta_{i_j,i_{j+1}} = \delta_{i_j,i_1}$. As a result, $\{\pi_{1,2,\cdots,j}, \pi_{j+1}, \cdots, \pi_N\}$ is also p-optimal, and contains no trivial cycles. $\square$

C.3 Combining Disjoint Cycles in the SLS Regime

Lemma 5 *If $[\alpha]_{K\times K} \in \mathcal{A}_{\text{SLS}}$, $\pi_1, \pi_2, \cdots, \pi_n$ are $n > 1$ disjoint cycles, and*

$$\pi_{1,2,\cdots,n} = (\pi_1 \to \pi_2 \to \cdots \to \pi_n \circlearrowleft) \tag{139}$$

is their combination, then

$$\Delta_{\pi_{1,2,\cdots,n}} \leq \Delta_{\pi_1} + \Delta_{\pi_2} + \cdots + \Delta_{\pi_n} + \Delta_{\pi} \tag{140}$$

where $\pi = (\pi_1(1) \to \pi_2(1) \to \cdots \to \pi_n(1) \circlearrowleft)$.

Proof of Lemma 5

Let us represent the cycles explicitly as

$$\begin{aligned}
\pi_1 &= (i_{1,1} \to \cdots \to i_{1,m_1} \circlearrowleft) && (141)\\
\pi_2 &= (i_{2,1} \to \cdots \to i_{2,m_2} \circlearrowleft) && (142)\\
&\vdots\\
\pi_n &= (i_{n,1} \to \cdots \to i_{n,m_n} \circlearrowleft) && (143)\\
\pi_{1,2,\cdots,n} &= (i_{1,1} \to \cdots \to i_{1,m_1} \to i_{2,1} \to \cdots \to i_{2,m_2} \to \cdots \to i_{n,m_n} \circlearrowleft) && (144)
\end{aligned}$$

Then we have

$$\begin{aligned}
\Delta_{\pi_{1,2,\cdots,n}} &\leq (\Delta_{\pi_1} - \delta_{i_{1,m_1} i_{1,1}}) + (\Delta_{\pi_2} - \delta_{i_{2,m_2} i_{2,1}}) + \cdots + (\Delta_{\pi_n} - \delta_{i_{n,m_n} i_{n,1}})\\
&\qquad + \delta_{i_{1,m_1} i_{2,1}} + \delta_{i_{2,m_2} i_{3,1}} + \cdots + \delta_{i_{n-1,m_{n-1}} i_{n,1}} + \delta_{i_{n,m_n} i_{1,1}} && (145)\\
&\leq \Delta_{\pi_1} + \delta_{i_{1,1} i_{2,1}} + \Delta_{\pi_2} + \delta_{i_{2,1} i_{3,1}} + \cdots + \Delta_{\pi_n} + \delta_{i_{n,1} i_{1,1}} && (146)\\
&= \Delta_{\pi_1} + \Delta_{\pi_2} + \cdots + \Delta_{\pi_n} + \Delta_{\pi}. && (147)
\end{aligned}$$

Note that in (146) we used the fact that since $[\alpha]_{K\times K} \in \mathcal{A}_{\text{SLS}}$, we must have $\delta_{ij} + \delta_{jk} \geq \delta_{ik}$. $\square$

C.4 Connecting BC Bounds to Cycle Bounds in the SLS Regime

Lemma 6 *In the SLS regime, for any cycle* $\pi \in \Pi$,

$$\pi = (i_1 \to i_2 \to \cdots \to i_M \circlearrowleft) \tag{148}$$

we have the following bound on the sum-GDoF of the BC restricted to the users involved in the cycle π,

$$\mathcal{D}_{\Sigma,\mathrm{BC}}(\{\pi\}) \leq \Delta_\pi + \alpha_{i_{m+1}i_m} \tag{149}$$

for any $m \in [1:M]$, *with* $i_{M+1} = i_1$. *Furthermore,*

$$\mathcal{D}_{\Sigma,\mathrm{BC}}(\{\pi\}) \leq \Delta_\pi + \mathcal{D}_{\Sigma,\mathrm{TINA}}. \tag{150}$$

Proof of Lemma 6

To obtain an outer bound for the sum GDoF of the BC with the users in $\{\pi\}$ involved, we start with the integer-valued deterministic model as in [5],

$$\bar{Y}_k = \sum_{i \in \{\pi\}} \left\lceil \bar{P}^{\bar{\alpha_{ki}}} G_{ki}(t) \bar{X}_i(t) \right\rceil \qquad \forall k \in \{\pi\} \tag{151}$$

where $\bar{\alpha_{ki}} \triangleq \alpha_{ki} - \alpha_{ii}$ and $\bar{X}_i(t) \in \{0, 1, \cdots, \lceil \bar{P}^{\alpha_{ii}} \rceil\}$. The GDoF region for the deterministic model (151) is an outer bound on the GDoF region of the original model in (2) as shown in [5]. In the SLS regime, an outer bound associated with the cycle π with i_m as its head, is found for the sum GDoF of (151) based on the aligned image sets argument in [14, Theorem 2]:

$$\mathcal{D}_{\Sigma,\mathrm{BC}}(\{\pi\}) \leq \alpha_{i_m,i_m} + \sum_{l=1, l\neq m}^{M} \delta_{i_l,i_{l+1}} = \sum_{l=1}^{M} \delta_{i_l,i_{l+1}} + \alpha_{i_{m+1},i_m} = \Delta_\pi + \alpha_{i_{m+1},i_m} \tag{152}$$

Furthermore, seeing that $\alpha_{ij} \leq \alpha_{ii} \leq \mathcal{D}_{\Sigma,\mathrm{TINA}}(\{\pi\})$ for all $i,j \in \{\pi\}$ in the SLS regime, we have

$$\mathcal{D}_{\Sigma,\mathrm{BC}}(\{\pi\}) \leq \Delta_\pi + \mathcal{D}_{\Sigma,\mathrm{TINA}}(\{\pi\}). \tag{153}$$

This completes the proof of Lemma 6. □

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