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Essays on Pure and Applied Game Theory

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Abstract of the Dissertation

Essays on Pure and Applied Game Theory

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Professor Ichiro Obara, Chair

In my dissertation, I provide two models of joint contribution games that are relevant to the phenomenon of crowdfunding. I also provide a characterization of Bayes Nash equilibrium.

In the first chapter I build a model of crowdfunding. An entrepreneur finances her project with common value via crowdfunding. She chooses a funding mechanism (fixed or flexible), a price, and a funding goal. Under fixed funding money is refunded if the goal is not met; under flexible funding the entrepreneur keeps the money. Backers observe signals about the value and decide whether to contribute or postpone purchase to the retail stage. The optimal crowdfunding campaign is characterized. When the entrepreneur has commitment power, fixed funding generates more revenue than flexible funding. When the entrepreneur has no commitment power, fixed funding serves as a commitment device to eliminate moral hazard

In the second chapter I consider a dynamic contribution game under two regimes. The first regime is that all but the last rounds are cheap talk, the other is that in all rounds contribution is sunk. With binary contribution levels and a continuum of types we show that one of the monotone equilibria in the first regime implements the ex-post efficient and ex-post individually rational allocation when the cheap talk period is long enough. In contrast, when commitment is required, no equilibria achieves the same allocation. However, with a continuum of contribution levels, all equilibria of the contribution game with cheap talk will be the same as a one-shot game with no cheap talk, due to severe free riding. In this case, dynamic contribution with commitment provides credibility and can significantly

improve efficiency.

In the third chapter, coauthored with Ichiro Obara, we prove the following characterization regarding types and Bayes equilibrium actions they play across games: Given any two types in any two countable type spaces, if for all finite games, the two types have the same pure Bayes Nash equilibrium action, then there exists a bijective belief morphism between them. As an application, our result implies that the universal space for Bayes Nash equilibrium that retains non-redundancy does not exist.

The dissertation of Jen-Wen Chang is approved.

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Table of Contents

List of Figures

List of Tables

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Vita

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CHAPTER 1

The Economics of Crowdfunding

1.1 Introduction

Crowdfunding has become a popular means for small entrepreneurs to finance their projects, typically through online platforms, where the entrepreneur asks a large number of internet users to back up her project with money. Despite being a relatively new global phenomenon, it has been growing exponentially. Its global market size, which tops over 30 billion dollars in 2015, has surpassed the market size for angel funds and is expected to surpass that of venture capital in 2016. Moreover, the U.S. government has been deregulating their equity crowdfunding market, allowing non-accredited investors to join the game. It is thus of great interest and importance to understand how and why crowdfunding works and to provide a rationale for deregulation.

This paper proposes a common value model of crowdfunding that explains the success of crowdfunding despite the lack of regulation and potential moral hazard problems. We consider an entrepreneur who would like to crowdfund her project with a fixed cost from a continuum of backers. The project has a common value unknown to the entrepreneur but the backers are partially informed. We characterize optimal crowdfunding campaign when the entrepreneur has or has no commitment power, and we show that moral hazard is eliminated in the optimal campaign in the latter case. In short, the model portraits crowdfunding as a tool for the entrepreneur to learn the market value of her idea, and third party crowdfunding platforms provide a commitment device so that the entrepreneur can condition the funding of her project on the event that the value is high enough. We show that it is of the entrepreneur's interest to use such commitment device to its full extent.

In a crowdfunding campaign, the entrepreneur posts the description of her project to a third party funding platform, chooses a funding mechanism, sets a funding goal, sets a price each backer pays, and sets the reward each backer gets.[1] The reward is typically a unit of the good her project is aimed to produce, or it can be a share of the company as well. Our model admits both interpretations. There are two choices of funding mechanism, fixed or flexible. Under fixed funding, the money is refunded if the goal of the campaign is not met, while under flexible funding there is no refund, whatsoever. The dominant reward-based crowdfunding platform Kickstarter only allows fixed funding, while the campaigns from its biggest competitor, Indiegogo, are predominantly flexible funding. The backers receive conditional i.i.d. signals about the value of the project and decide whether to contribute or postpone their purchase to the retail stage, where the value is revealed and the retail price equals the value if the project is ever completed.[2] In our baseline model we assume the entrepreneur has commitment power, so the project is built if and only if the campaign outcome exceeds the funding goal. In an extension to consider moral hazard, we make the decision to build endogenous on the outcome of the campaign. Figure 1.1 provides an example of a crowdfunding campaign.[3][4]

Our first result (Theorem 1.1) shows that fixed funding generates more profit than flexible funding under the assumption that the entrepreneur has commitment power. At first, this may seem obvious: people are more willing to pay more if they are refunded when the project does not go ahead. But this logic is incorrect: Under private values, the two funding methods has been shown to raise identical revenue under the optimal price ([Cor96]). For example, suppose a project has 50% chance of being funded. Then backers are indifferent between paying \$50 under fixed funding and \$25 under flexible funding, and the expected

[1] The entrepreneur usually offers different reward levels for different prices, for example to get an album one needs to pay \$20, to get an autographed album with poster one needs to pay \$50. However, more often than not, the most popular option is the lowest price that can get the backers a unit of the good. We simplify this aspect so we do not consider price discrimination. For a treatment, see [EH15].

[2] The assumption that retail price equals to the common value is not necessary for our main results; it merely simplifies exposition.

[3] The entrepreneur in this Indiegogo project wants to raise \$50,000 to develop a temperature preserving mug. Each backer needs to contribute at least \$109 in order to receive a mug after it is produced. The entrepreneur adopts flexible funding, so the entrepreneur will receive all the funds even if the campaign fails to reach its funding goal.

[4] Link: https://www.indiegogo.com/projects/ember-temperature-adjustable-mug/

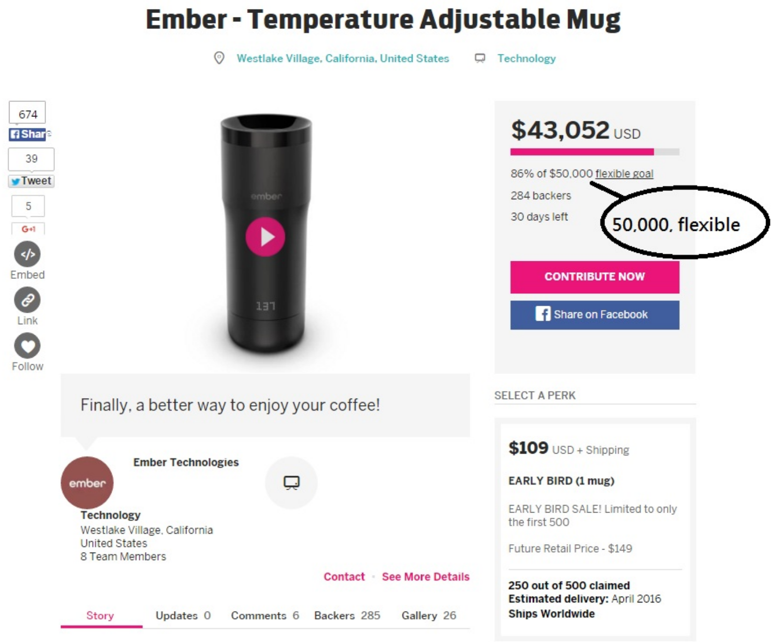

Figure 1.1: Example of crowdfunding campaign

revenue to the entrepreneur is the same in either case. When backers have common value but heterogeneous beliefs about the value– which is appropriate when quality is unknown[5] – the equivalence breaks down and fixed funding becomes preferable to the entrepreneur. Intuitively, suppose the marginal buyer thinks the project will be funded with probability 50% so is indifferent between paying $50 under fixed funding and $25 under flexible funding. A high signal backer thinks that the project will be funded with more than 50% probability, so he has higher expected payment under fixed funding than flexible funding. This logic is akin to the linkage principle in [MW82]: under fixed funding, expected payments are positively correlated with values, squeezing information rents.[6] This result is also consistent with the within comparison of projects in the online platform Indiegogo by [CLS14] and the across comparisons of Indiegogo versus Kickstarter, which are two of the top three crowdfunding sites by traffic.[7]

[5]The market value of the mug may depend on the price of the complements, say tea and coffee, or the existence of competing substitutes, like a much cheaper temperature preserving mug. In this sense the crowd can be more informed than the entrepreneur. Traditionally the entrepreneur will run surveys to focus groups to extract this information.

[6]I am grateful to Simon Board who suggests this interpretation.

[7]Lau, Jonathan. 2013. "Dollar for dollar raised, Kickstarter dominates Indiegogo SIX times over". , Au-

Our second result (Theorem 1.2) characterizes optimal campaigns and shows that crowdfunding complements traditional financing methods, such as borrowing. We show that even if the entrepreneur can borrow freely from an outside source (rich relatives, competitive capital market), she will still use crowdfunding and the funding goal will be the difference of the project cost and her outside source of funding. The reason is that crowdfunding is still costly because the backers need to be given information rents in a posted price mechanism, so the entrepreneur will not ask for more than needed if she has other free sources of money. However, crowdfunding helps her to learn the market value and to condition the building decision on the true value, so it is still beneficial to use crowdfunding.

We then turn to the setting in which the entrepreneur has no commitment power to build the project at the crowdfunding stage. After the crowdfunding campaign ends, the entrepreneur learns the value of her project, she then chooses to run away with the crowdfunded money or to invest the fixed cost and complete the project.[8] Our third result (Theorem 1.3) shows that even if the entrepreneur can not commit to build the project, under optimal pricing there will still be no moral hazard. The logic is the same as our first result. Once the entrepreneur collects funding, because our assumptions on signal structure imply that funding is increasing in the underlying value, she is able to deduce the value before investing the money into the project. She then checks whether her retail stage profit is larger than the fixed cost of the project. If not, then she will run away with the money from crowdfunding. The entrepreneur faces a similar situation as in fixed versus flexible funding: she can commit, via third party crowdfunding platforms, to get funded at a higher value. This makes the marginal backer more than happy, so she can charge a higher price while keeping the same marginal backer. This adjustment again extracts more surplus from the high signal backers. Our result is consistent with [Mol14]'s empirical observation that the default rate can be

gust 28. http://medium.com/p/2a48bc6ffd57. Wang, Dan. 2015. "The Ultimate Guide to Crowdfunding". September 30. https://www.shopify.com/guides/crowdfunding/crowdfunding-infographic.

[8]It is difficult to monitor the entrepreneur's effort and there is no auditing requirement for reward-based crowdfunding, so in theory the entrepreneur can default by claiming that she has tried her best but the project still doesn't work out and get away with money without being punished or forced to refund. Another type of default is that she uses the money to set up a new company and to complete the project but she refuses to deliver the good to her backers. This type of default has not been observed in the real world yet for various reasons (possible legal consequences), and is not considered in our model.

below 4%.

In the discussion section we compare reward crowdfunding to funding by a single investor, and show that crowdfunding can increase social welfare. We also show that the crowd funds a larger set of projects (when it is efficient to build the project) than a single investor can. We also briefly mention how our model can be used to study equity-based crowdfunding. Finally, we propose a situation where flexible funding is preferable.

Related Literature

Crowdfunding can be considered as a model in which a monopolist makes production decisions together with the financing of it through advance-purchase contracts. The case of private value models are studied in [Cor96] and more recently by [Str15] and [EH15], where crowdfunding is casted as an optimal mechanism design problem. [Str15] also takes moral hazard into account. The major distinction of this paper to those papers is that we study a common value model in which the backers are uncertain about the value of the project by the time they make their contributions. We show that fixed funding generates more expected profit then flexible funding, using a different logic than the above-mentioned models.

[NPR11] studies a monopolist screening consumers with private but noisy values using pre-order discounts so that agents with high signals purchase in advance and agents with lower signals postpone the purchase until the value realizes in the retail stage. In his model the project is already completed so there is no production decision to be made and the entrepreneur has commitment power as well. In our model, as in a typical crowdfunding situation, the production decision depends on the funding outcome, and the entrepreneur may have no commitment power at all.

The idea of fixed funding, where money is refunded if the total amount fails to reach a threshold, is simply the provision point mechanism. This has been used to study private provision of public goods ([PR88] and [BL89]). The latter showed that under a complete information, finite agent model the provision point mechanism fully implements the core of the economy. We show in our incomplete information, common value model that this mechanism can implement the efficient allocation. However, it may not be profit-maximizing, so the entrepreneur will not necessarily do so.

This paper also relates to models of information aggregation and allocation of an excludable good with common value, such as common value auction models ([MW82] and [PS00]). In particular, we demonstrate a kind of linkage principle in the crowdfunding environment. However, in their papers information aggregation and allocative efficiency have a potential conflict ([GS80]), but in our model information aggregation is automatically granted, and the distortion of efficiency comes from the entrepreneur's self interest.

For an overview of crowdfunding platforms and related economic problems, see [BOP15]. [BLS13] is the first theoretical paper to study when crowdfunding is preferred to borrowing, where they assume the backers derive altruistic utilities from the act of contribution. [HS14] considers endogenous information acquisition of backers in crowdfunding.

The chapter is organized as follows: In Section 1.2 we develop the crowdfunding model. Section 3 characterizes the equilibrium. Section 1.4 gives revenue ranking between fixed and flexible funding. Section 1.5 characterizes optimal fixed funding campaigns. Section 1.6 extends the model to include moral hazard.Section 1.7 discusses some related issues. Section 1.8 concludes. Longer proofs can be found in the appendix.

1.2 The Model

In this section we give the outline of our model.

Players An entrepreneur (she) tries to fund a project through a crowdfunding campaign. A continuum of potential backers (he) decide whether to contribute to a project in return for a unit of the good.

Project A project has a fixed cost k, which is privately known to the entrepreneur, and it generates common value v to the backers. v is unknown to both sides of the market with common prior $f(\cdot)$ on $[0,1]$. Backers privately receive i.i.d. signals, s, about v according to the conditional density $g(s|v)$ with cdf $G(s|v)$.

The following assumptions are made throughout the paper.

A1 $\{g(s|v)\}$ satisfies strict monotone likelihood ratio property.

A2 $g(s|v)$ is continuous on $[0,1]\times[0,1]$ and $g(s|v)>0$ for all $s\in[0,1], v\in[0,1]$, $f(v)>0$ for all $v\in[0,1]$.

Efficiency Benchmark Since we assume the backers eventually buy the product, only the building decision affects efficiency. The first best is to build the project whenever $v\geq k$, where the social welfare is

$$\int_k^1 (v-k)f(v)dv$$

Crowdfunding Campaign A crowdfunding campaign is a tuple (F,T,p), where

$$F\in\{Fix, Flex\}$$

denotes the funding mechanism, T is the commitment to build the project if and only if the seller gathers at least T dollars, and p is the pledge price each buyer has to pay if he is to contribute.

Under fixed funding, the entrepreneur gets money if and only if she raises at least T dollars. Under flexible funding, the entrepreneur always gets the total amount the backers contributed, even if she raises less than T. In both mechanisms, the entrepreneur commits to implement the project if and only if at least T dollars are collected.[9]

The entrepreneur chooses (T,p) from the set

$$C_a=\{(T,p): k-a\leq T\leq p\leq 1\},$$

where $a\in[0,k]$ is the funding that the entrepreneur could obtain from other sources, or simply her asset. When $a=k$, it means the entrepreneur can fund her project without crowdfunding. A natural constraint is $a=0$, which means that the entrepreneur's only source of funding is crowdfunding.[10]

Actions

[9]Here we assume default is impossible. See Section 6 for the case where default is an option.

[10]Although crowdfunding platforms do not usually set a lower bound on the funding goal, they often suggest the entrepreneurs to raise at least what is actually needed, i.e., choose from C_0.

The entrepreneur chooses a crowdfunding campaign (F, T, p) where $(T, p) \in C_a$. After observing (F, T, p) and signal s, the backers jointly choose whether to *contribute* p dollars or *wait* to purchase at the retail stage. An action profile for backers is a measurable function $\sigma : [0, 1] \to [0, 1]$, where $\sigma(s)$ denotes the probability to contribute for a backer with signal s.[11] We assume that the value of the project is realized at the retail stage and the retail price equals to the value.[12]

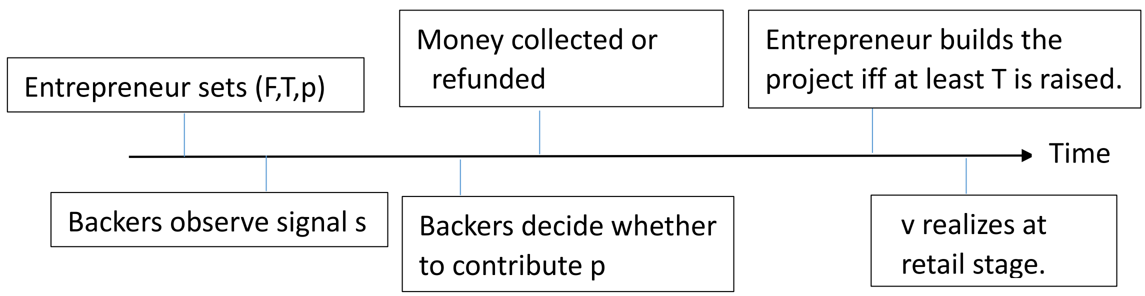

Figure 1.2: Timeline of our crowdfunding campaign model

Given a price p and an action profile ,σ, the money that will be contributed at each state v is then

$$X_0^\sigma(v) = \int_0^1 p\sigma(s)g(s|v)ds.$$

For each state, the retail stage revenue the entrepreneur gets (if the project is built) is

$$X_1^\sigma(v) = \int_0^1 v(1 - \sigma(s))g(s|v)ds.$$

The project is *funded* at state v if $X_0^\sigma(v) \geq T$.

[11]The actual crowdfunding mechanisms are mostly dynamic, where the backers can observe the current total contribution. Theoretically, backers can strategically postpone their contribution until they are sure the project is funded. However, empirical contribution dynamics are usually U-shaped with respect to time, thus only a subset of backers will strategically postpone. Their behavior, however, is not relevant to determine whether a project is funded and is thus not captured by our model.

[12]This assumption is not needed for our results but it simplifies exposition.

1.2.1 Payoffs and Equilibria in a Fixed Funding Campaign

Backers' Payoff Given (Fix, T, p) and an action profile σ, let

$$B^\sigma = \{v : X_0^\sigma(v) \geq T\}$$

be the states such that the project is funded under action profile σ.

Under fixed funding, the utility of a backer to contribute with action profile σ is

$$U^{Fix}(s;\sigma) = \int_{B^\sigma} (v-p)\beta(v|s)dv$$

where

$$\beta(v|s) = \frac{g(s|v)f(v)}{\int_0^1 g(s|v)f(v)dv}$$ [13]

is the backer's posterior about the project value if he observes signal s.

Equilibrium An action profile , σ, is a Bayes Nash equilibrium under (Fix, T, p) if for all $s \in [0,1]$,

$$U^{Fix}(s;\sigma) > 0 \Rightarrow \sigma(s) = 1$$
$$U^{Fix}(s;\sigma) < 0 \Rightarrow \sigma(s) = 0$$

For any campaign, zero contribution is always an equilibrium. That is, $\sigma(s) = 0$ for all s. However, we are interested in equilibria under which the project is fundd with positive probability. An equilibrium is funded if $\int_{B^\sigma} f(v)dv > 0$, i.e. it is funded with positive probability.

Entrepreneur's Profit The entrepreneur's profit under (Fix, T, p) and action profile σ is the sum of crowdfunded money, retail stage revenue minus the project cost

$$\Pi^{Fix}(T,p;\sigma) = \int_{B^\sigma} (X_0^\sigma(v) + X_1^\sigma(v) - k)f(v)dv$$

[13] Note that this differs from [FP97] in that the backers do not condition their posteriors on the event that they are pivotal. This is because, even in a finite agent model, the backers' utility of contributing or waiting differs whenever the project is funded (fixed funding). On the other hand, in a voting model an action leads to a difference in utility only when the voter is pivotal.

1.2.2 Payoffs and Equilibria in a Flexible Funding Campaign

Backers' Payoff Under flexible funding, the utility of a backer is

$$U^{Flex}(s;\sigma) = \int_{B^\sigma} v\beta(v|s)dv - p,$$

Equilibrium An action profile, σ, is a Bayes Nash equilibrium under $(Flex, T, p)$ if for all $s \in [0, 1]$,

$$U^{Flex}(s;\sigma) > 0 \Rightarrow \sigma(s) = 1$$
$$U^{Flex}(s;\sigma) < 0 \Rightarrow \sigma(s) = 0$$

An equilibrium is called funded if $\int_{B^\sigma} f(v)dv > 0$.

Entrepreneur's Profit The entrepreneur's profit under $(Flex, T, p)$ and action profile σ is

$$\Pi^{Fix}(T, p;\sigma) = \int_{[0,1]\setminus B^\sigma} X_0^\sigma(v)f(v)dv + \int_{B^\sigma} (X_0^\sigma(v) + X_1^\sigma(v) - k)f(v)dv$$

Remark 1. Because of the common value, full surplus is extracted by a direct incentive compatible mechanism as in [CM88]: simply ask each backer to announce s, and the good is produced and allocated with a price v whenever the aggregate distribution s is $g(s|v)$ for some $v \geq k$. We focus on indirect mechanisms that resemble the ones in reality. However, we still show in Theorem 2 that fixed funding can approximately extract full surplus when the entrepreneur has sufficient outside sources of funding.

1.3 Equilibrium Characterization

In this section we characterize the funded equilibrium under fixed and flexible funding for a given funding threshold T and pledge price p. Before that, we present an auxiliary lemma that will be used throughout this chapter.

Let $\{g(x|y)\}$ be probability densities satisfying strict monotone likelihood ratio property. For a measurable set $A \subset \mathbb{R}$, let $G(A|y) = \int_0^1 1_A(x)g(x|y)dx$ denote the conditional probability of event A under density $g(x|y)$. Let

$$g_A(x|y) = \begin{cases} g(x|y)/G(A|y) & x \in A \\ 0 & x \neq A \end{cases}$$

be the conditional density conditioning on A, and denote the conditional distribution by $G_A(x|y)$.

Lemma A 1. Let $A \subset [0,1]$ be measurable and that $G(A|y), G(A|y') > 0$. For $y < y'$ and a strictly increasing function $h(\cdot)$,

$$\int_A h(x) g_A(x|y) dx < \int_A h(x) g_A(x|y') dx.$$

Proof. MLRP implies for $y' > y$, $g_A(x|y')/g_A(x|y)$ is increasing w.r.t. $x \in A$ and since $g_A(x|y), g_A(x|y')$ are probability densities, there exists some $x^* \in A$ such that for all $0 \leq x_1 \leq x^* \leq x_2$,

$$\frac{g_A(x_1|y')}{g_A(x_1|y)} \leq 1 \leq \frac{g_A(x_2|y')}{g_A(x_2|y)},$$

with strict inequality when $x_1 < x^* < x_2$. For $x \geq x^*$, we then have

$$1 - G_A(x|y) = \int_x^1 g_A(s|y) ds \leq \int_x^1 g_A(s|y) \frac{g_A(s|y')}{g_A(s|y)} ds = \int_x^1 g_A(s|y') ds = 1 - G_A(x|y').$$

For $x \leq x^*$ we have

$$G_A(x|v) = \int_0^x g_A(s|y) ds \geq \int_0^x g_A(s|y) \frac{g_A(s|y')}{g_A(s|y)} ds = \int_0^x g_A(s|y') ds = G_A(x|y').$$

Hence, $G_A(x|y)$ satisfies FOSD, which implies that the expected value of an increasing function under $G_A(x|y')$ dominates that under $G_A(x|y)$. Since $G_A(x|y) \neq G_A(x|y')$ for $y \neq y'$ the inequality is strict. □

1.3.1 Fixed Funding

Our first observation is that the funded equilibria in fixed funding are characterized by cutoffs. Recall that contributing to a crowdfunding project is always risky because the backers can end up getting something with value less than what they paid for. However, because the signal structure satisfies MLRP and that the payment is made only when the project is funded, the backers' conditional expected payoff as a function of signal is single crossing. That is, the higher the signal, the more optimistic the backer is about $v - p$ being positive conditional on the project getting funded. Consequently, funded equilibrium is characterized by a cutoff s^*, and a project is funded if the value is above a cutoff v^*. Formally, we have the following lemma.

Lemma 1.1. Consider a fixed funding campaign with $(T, p) \in C_a$. Suppose σ is a funded equilibrium, then σ is characterized by a cutoff $s^* \in [0, 1)$ such that

$$\sigma(s) = 0, \quad s < s^*$$
$$\sigma(s) = 1, \quad s > s^*.$$

Under such an equilibrium there is a threshold $v^* < 1$ such that the project will be funded if and only if $v \geq v^*$.

Proof. Let σ be any action profile with $\int_{B^\sigma} f(v)dv > 0$. Then there exists $s < 1$ such that $U(s) \geq 0$. Since $\{g(s|v)\}$ satisfies strict MLRP, the set of posteriors $\{\beta(v|s)\}$ also satisfies strict MLRP. Therefore, by Lemma A.1, if for some $s^* < 1$,

$$U(s^*; \sigma) = \int_{B^\sigma} (v - p)\beta(v|s^*)dv = \mathbb{P}(B^\sigma|s^*) \int_{B^\sigma}^1 (v - p) \frac{\beta(v|s^*)}{\mathbb{P}(B^\sigma|s^*)} dv = 0,$$

then $U(s; \sigma) > 0$ when $s > s^*$ and $U(s; \sigma) < 0$ when $s < s^*$. Hence σ is characterized by cutoff $s^* < 1$.

Accordingly,

$$X_0^\sigma(v) = p(1 - G(s^*|v)),$$

which is increasing in v by MLRP, so B^σ is of the form $[v^*, 1]$ for some v^*. □

We are now ready to show existence and uniqueness of funded equilibrium

Proposition 1.1. Given (Fix, T, p) with $(T, p) \in C_a$ and $a < k$. Suppose

$$T < p < 1 \tag{1.1}$$

then an unique funded equilibrium exists. Conversely, if a funded equilibrium exists then $T \leq p < 1$.

Proof. By Lemma 1.1 we only need to focus on cutoff strategies.

Let σ_s be the cutoff strategy with cutoff s. For each $s \in [0,1]$, let

$$v^f(s) = \min\{v : p(1 - G(s|v)) \geq T\}$$

be the minimum value v above which the project is funded, given that backers use σ_s. Note that since $p > T$, $v^f(\cdot)$ is continuous. Also, $v^f(\cdot)$ is increasing.

Define $\Phi : [0,1] \to [0,1]$ as

$$\Phi(s) = \min\left\{s' : \int_{v^f(s)}^{1} (v - p)\beta(v|s')dv \geq 0\right\}.$$

Note that whenever $v^f(s) < 1$, by Lemma A.1 the conditional expected utility $\int_{v^f(s)}^{1}(v - p)\beta(v|s')dv$ is strictly single-crossing in s'. This implies that funded equilibria is completed characterized by the fixed points $s^* < 1$ of Φ.[14]

Claim 1 $\Phi(\cdot)$ is continuous. $\Phi(s) = 0$ when $v^f(s) = 1$.

Given any $s \in [0,1]$. Suppose $v(s) < 1$. Suppose $\Phi(s) \in (0,1)$. Then for all $\epsilon > 0$,

$$\int_{v^f(s)}^{1} (v - p)\beta(v|\Phi(s) + \epsilon)dv > 0$$

and

$$\int_{v^f(s)}^{1} (v - p)\beta(v|\Phi(s) - \epsilon)dv < 0,$$

Then by the continuity of $v^f(\cdot)$ and the integral with respect to the lower limit of integration, there exists a neighborhood of s, $B_\delta(s)$ such that when $s' \in B_\delta(s)$,

$$\Phi(s') \in B_\delta(\Phi(s)).$$

The case $\Phi(s) \in \{0,1\}$ is treated similarly.

Suppose $v^f(s) = 1$. Then by definition of $\Phi(\cdot)$, $\Phi(s) = 1$. If there exists $\epsilon > 0$ such that $v^f(s - \epsilon) = 1$, then $\Phi(s') = 1$ for all $s' > s - \epsilon$. If $v^f(s - \epsilon) < 1$ for all $\epsilon > 0$, since $p < 1$ and $v^f(\cdot)$ is continuous, choose δ such that for all $s' > s - \delta$, $v(s') > p$. Then $\Phi(s') = 1$ for all $s' > s - \delta$. [15]

[14] There may be multiple unfunded equilibria, which are not captured by Φ.

[15] It is essential that $p < 1$. If $p \geq 1$, $\Phi(\cdot)$ will be discontinuous: the backers will not contribute whenever $v^f(s) < 1$ ($\Phi(s) = 1$), and will contribute ($\Phi(s) = 0$, by definition of Φ) whenever $v^f(s) = 1$.

Claim 2 $\Phi(\cdot)$ is decreasing.

Let $s_1 < s_2$. When $\Phi(s_1) = 1$ then trivially $\Phi(s_2) \leq \Phi(s_1)$. When $\Phi(s_1) \in (0,1)$, then by definition of Φ we have

$$\int_{v^f(s_1)}^{1} (v-p)\beta(v|\Phi(s_1))dv = 0$$

and that $v^f(s_1) < 1$. This also implies $v^f(s_1) < p$ (else $\Phi(s_1) = 0$ because the backer is never going to lose.).

Hence

$$\int_{v^f(s_2)}^{1} (v-p)\beta(v|\Phi(s_1)))dv \geq 0.$$

So by definition $\Phi(s_2) \leq \Phi(s_1)$.

If $\Phi(s_1) = 0$ then, since $v^f(s_2) \geq v^f(s_1)$, $\int_{v^f(s_2)}^{1}(v-p)\beta(v|0) \geq 0$. So $\Phi(s_2) = 0$ as well.

Claim 3 $\Phi(s) = 0$ when s is sufficiently high. Since $T \geq k - a > 0$, whenever s is sufficiently high $v^(s) = 1$.

By Claim 1,2,3, $\Phi(\cdot)$ has a fixed point $s^* \in [0,1)$. The funded equilibrium is given by σ_{s^*}, and the project is funded when $v \in B^\sigma = [v^f(s^*), 1]$. □

Note that if $p = T + \epsilon$ where ϵ is small, nearly everyone must contribute to fund the project with positive probability. Hence if a funded equilibrium does exist (which it does by Prop. 1.1), s^* will be very small. However, the existence of funded equilibrium at $T = p$ is not guaranteed.

Note also that under fixed funding, equilibrium existence is independent of the prior $f(v)$ or the signal structure $\{g(s|v)\}$. Precisely, as long as there is positive probability that the value of the project is larger than p, the project can be funded with positive probability. This is because the backers can adopt a higher cutoff s to ensure the project is funded only when v is high enough, and the refund policy offered by fixed funding will in tern incentivize them to contribute.

1.3.2 Flexible Funding

For flexible funding we restrict to the class of equilibria in cutoff strategies. For any cutoff strategy σ, $U^{Flex}(s;\sigma)$ is single-crossing because of MLRP.[16] Therefore, best response to any cutoff strategies σ is also characterized by a cutoff $s^* \in [0,1)$ such that the backers contribute when $s > s^*$ and postpone purchase whenever $s < s^*$. However, existence of equilibrium is subject to a more restrictive condition than fixed funding.

Proposition 1.2. Given $(Flex, T, p)$ with $(T,p) \in C_a$. A funded equilibrium in cutoff strategies exists if and only if

$$T \leq \max_v p(1 - G(s(v)|v)),$$

where

$$s(v) = \min\left\{s : \int_v^1 \tilde{v}\beta(\tilde{v}|s)d\tilde{v} - p \geq 0\right\}$$

is the backers' cutoff if they expect the project to be funded when value is above v.

Proof. Assumptions on $\{g(s|v)\}$ implies $s(\cdot)$ is continuous when $p < 1$. Moreover, when $v = 1$, $s(v) = 0$ by definition, so $p(1 - G(s(1)|1)) = 0$. Since $T \leq \max_v p(1 - G(s(v)|v))$, by continuity of $p(1 - G(s(v)|v))$ with respect to v there exists $v^* < 1$ such that

$$T = p(1 - G(s(v^*)|v^*)).$$

Then by construction $\sigma_{s(v^*)}$ is a funded equilibrium. Conversely, for any funded equilibrium with cutoff s^* and v^*, it must be that

$$T = p(1 - G(s(v^*)|v^*))$$

and that $s(v^*) = s^* < 1$, which also implies $p < 1$. □

The condition of Proposition 1.2 implies that there exists v such that the expectation is correct and backers are responding optimally to the expectation. The solutions to the

[16] For an arbitrary σ, single-crossing could fail: the other backers can coordinate to contribute only when s is low. Then a backer who contributes when observing a high signal will end up paying p while getting nothing in return because the project fails to be funded.

equation

$$T = p(1 - G(s(v)|v))$$

will then be equilibrium cutoffs.

A direct implication of Proposition 1.1 and 1.2 is that for any given (T, p) the existence of funded equilibrium under flexible funding implies existence under fixed funding.

Corollary 1. Compared to the funded equilibrium under $(Flex, T, p)$, the funded equilibrium under (Fix, T, p) achieves a higher probability of getting funded, attracts a larger number of contributors, and gets a higher amount of funds in each state than flexible funding.

Proof. Suppose funded equilibrium σ_s exists for $(Flex, T, p)$. Then a funded eqilibrium $\sigma_{s'}$ also exists for (Fix, T, p). If $s = 0$ then $\sigma_{s'} = \sigma_s$, so the funded equilibrium is identical for the two mechanisms. Otherwise,

$$\int_{v^*}^{1} v\beta(v|s^*)dv - p = 0$$

implies

$$\int_{v^*}^{1} (v - p)\beta(v|s^*)dv > 0$$

This implies $s' < s$ so the project is funded when v belongs to a strictly larger subset of $[0, 1]$ and by more backers, under fixed funding. □

This is because given the same (T, p), backers contribute more aggressively under fixed funding.

This comparison, however, ignores the fact that in fixed funding the entrepreneur needs to fully refund the backers if the project fails to be funded. What we will show next is that entrepreneurs who use fixed funding can earn more profit in expectation even with such trade-off.

1.4 Revenue Ranking

The goal of this section is to show that fixed funding generates higher profit than flexible funding. Our result is robust to variations in profit functions (see Remark 4.1).

Given (Fix, T, p), by Lemma 1.1, the profit under the unique funded equilibrium is

$$\Pi^{Fix}(T,p) = \int_{v^*}^{1} (p(1 - G(s^*|v)) - k) + G(s^*|v)vdF(v). \tag{1.2}$$

for some equilibrium cutoff (v^*, s^*). Given $(Flex, T, p)$, by Lemma 3.2, the profit under a funded equilibrium, σ, is

$$\Pi^{Flex}(T,p;\sigma) = \int_{0}^{1} (p(1 - G(s^*|v)) - k)dF(v) + \int_{v^*}^{1} (p(1 - G(s^*|v)) - k) + G(s^*|v)vdF(v) \tag{1.3}$$

for some equilibrium cutoff (v^*, s^*). Note that the equilibrium cutoff v^* and s^* under fixed funding or flexible funding with the same (T, p) need not be the same.

When the entrepreneur switches from flexible funding to fixed funding, she attracts buyers that would otherwise purchase at the retail stage to contribute at the crowdfunding stage instead. Thus her retail stage revenue decreases. Moreover, she needs to refund the money to the backers if the project is not funded. Therefore, it is not readily obvious that fixed funding has a higher expected profit.

Before stating the main result, we need the following lemmas. First, we will express the profits (1.2) and (1.3) as the difference between social surplus and consumer surplus. For each (F, T, p) and each corresponding equilibrium cutoff (v^*, s^*), the social surplus is given by

$$SS^F = \int_{v^*}^{1} ((1 - G(s^*|v))v + G(s^*|v)v - k)dF(v)$$

The (ex-ante) consumer surplus under fixed funding and flexible are

$$CS^{Fix}(T,p) = \int_{0}^{1} \left(\int_{s^*}^{1} \left(\int_{v^*}^{1} (\tilde{v} - p)\beta(\tilde{v}|s)d\tilde{v} \right) g(s|v)ds \right)$$
$$CS^{Flex}(T,p;\sigma) = \int_{0}^{1} \left(\int_{s^*}^{1} \left(\int_{v^*}^{1} \tilde{v}\beta(\tilde{v}|s)d\tilde{v} - p \right) g(s|v)ds \right)$$

Lemma A 2. For any funded equilibrium, σ, under (F, T, p),

$$\Pi^{Fix}(T,p) = SS^{Fix}(T,p) - CS^{Fix}(T,p) \tag{1.4}$$

$$\Pi^{Flex}(T,p;\sigma) = SS^{Flex}(T,p) - CS^{Flex}(T,p;\sigma) \tag{1.5}$$

Proof. We give a proof of (1.4), the computation leading to (1.5) is along the same line. Let σ be a funded equilibrium under (Fix, T, p).

Note that

$$\begin{aligned}
&SS^{Fix} - \Pi^{Fix}(T, p; \sigma) \\
&= \int_{v^*}^{1} (1 - G(s^*|v))vf(v)dv - \int_0^1 p(1 - G(s^*|v))f(v)dv
\end{aligned} \tag{1.6}$$

The consumer surplus can be written as

$$CS^{Fix} = \int_0^1 \left(\int_{s^*}^1 \left(\int_{v^*}^1 (\tilde{v} - p)\beta(\tilde{v}|s)d\tilde{v} \right) g(s|v)ds \right) f(v)dv \tag{1.7}$$

It then suffices to show that (1.6) and (1.7) are equal. To this end, recall the definition of β, we have

$$\begin{aligned}
&\int_0^1 \left(\int_{s^*}^1 \left(\int_{v^*}^1 \tilde{v}\beta(\tilde{v}|s)d\tilde{v} \right) g(s|v)ds \right) f(v)dv \\
&= \int_0^1 \int_{s^*}^1 \int_{v^*}^1 \frac{1}{\int_0^1 g(s|v)f(v)dv} \tilde{v}g(s|\tilde{v})f(\tilde{v})g(s|v)f(v)d\tilde{v}dsdv \\
&= \int_{s^*}^1 \int_{v^*}^1 \frac{\tilde{v}g(s|\tilde{v})f(\tilde{v})}{\int_0^1 g(s|v)f(v)dv} \left(\int_0^1 g(s|v)f(v)dv \right) d\tilde{v}ds \\
&= \int_{s^*}^1 \int_{v^*}^1 \tilde{v}g(s|\tilde{v})f(\tilde{v})d\tilde{v}ds \\
&= \int_{v^*}^1 \tilde{v}f(\tilde{v})(1 - G(s^*|\tilde{v}))d\tilde{v}.
\end{aligned}$$

This completes the proof. □

We show that the entrepreneur can decrease consumer surplus while keeping the marginal buyer indifferent by switching to fixed funding.

Lemma A 3. Suppose a project can be funded by $(Flex, T, p)$ with equilibrium cutoff (s^*, v^*). Suppose $v^* > 0$. Then there exists (T', p') with $T' > T, p' > p$ such that the project can be funded by (Fix, T', p') with the same equilibrium cutoff (s^*, v^*). Suppose instead $v^* = 0$, then for any $\epsilon > 0$ there exists (T', p') with $T' > T, p' > p$ such that the project is funded by (Fix, T', p') with cutoff (s^*, ϵ).

Proof. Let (v^*, s^*) be the equilibrium cutoff under $(Flex, T, p)$.

For each $\tilde{p} \geq p$, define the function $s : [p, 1] \to [0, 1]$ as

$$s(\tilde{p}) = \min \left\{ s : \int_{v^*}^{1} (v - \tilde{p})\beta(v|s)dv \geq 0 \right\},$$

which is the signal making a buyer indifferent between contributing or wait when they expect the project is built if $v \geq v^*$. Note that $s(\cdot)$ is continuous and increasing.

Suppose first that $v^* > 0$. This will imply $s^* > 0$.[17] Therefore,

$$\int_{v^*}^{1} (v - p)\beta(v|s^*)dv > \int_{v^*}^{1} v\beta(v|s^*)dv - p = 0.$$

Thus $s(p) < s^*$. On the other hand, $\lim_{p \to 1} s(p) = 1$, hence the intermediate value theorem guarantees the existence of $p' \in (p^*, 1)$ with $s(p') = s^*$. Now define $T' = p'(1 - G(s^*|v^*))$, which ensures the expectation is correct, that is, when the cutoff on signal is s^* and price is p', the project is funded if and only if $v \geq v^*$. Then, (s^*, v^*) is the equilibrium cutoff for fixed funding given (T', p'). Finally note that since $p' > p$,

$$T' > p(1 - G(s^*|v^*)) = T.$$

So $(T', p') \in C_a$.

Suppose $v^* = 0$. Pick an $\epsilon \in (0, k)$, let $p(\epsilon)$ be such that

$$\int_{\epsilon}^{1} (v - p(\epsilon))\beta(v|s^*)dv = 0$$

It is straightforward to see that $p(\epsilon) > p$. Define $T(\epsilon) = p(\epsilon)(1 - G(s^*|\epsilon))$. Then the equilibrium cutoff under $(Fix, T(\epsilon), p(\epsilon))$ is by construction (s^*, ϵ). Again note that $(T(\epsilon), p(\epsilon)) \in C_a$. □

Our main result in this section shows that there is a profitable adjustment after switching from flexible funding to fixed funding.

Theorem 1.1. For any $a \in [0, k]$ and for any funded equilibrium, σ, under $(Flex, T, p)$, where $(T, p) \in C_a$, there exists $(T', p') \in C_a$ such that

$$\Pi^{Fix}(T', p') > \Pi^{Flex}(T, p; \sigma).$$

[17] If $s^* = 0$ then everyone contributes, implying $v^* = 0$.

Proof. Let (s^*, v^*) be the cutoff of a funded equilibrium σ for $(Flex, T, p)$. By Lemma A2, it suffices to show that $SS^{Fix}(T', p') = SS^{Flex}(T, p; \sigma)$ and $CS^{Fix}(T', p') < CS^{Flex}(T, p; \sigma)$.

Suppose $v^* > 0$. By Lemma A3 there exists $(T', p') \in C_a$ that supports the same cutoffs as a funded equilibrium in fixed funding.

First observe that since the cutoffs are the same, the social surplus remain the same. Second,

$$\begin{aligned} & CS^{Fix}(T', p') - CS^{Flex}(T, p; \sigma) \\ = & \int_0^1 \int_{s^*}^1 \left(p - \int_{v^*}^1 p' \beta(\tilde{v}|s) d\tilde{v} \right) g(s|v) ds f(v) dv \end{aligned}$$

Since by construction $U^{Fix}(s^*) = U^{Flex}(s^*)$, and $\beta(\tilde{v}|s)$ satisfies MLRP,

$$p - \int_{v^*}^1 p' \beta(\tilde{v}|s) d\tilde{v} < 0$$

as $s > s^*$. So $CS^{Fix}(T', p') \le CS^{Flex}(T, p; \sigma)$.

Suppose $v^* = 0$. Consider an ϵ such that

$$\int_\epsilon^1 (v - k) f(v) dv > \int_0^1 (v - k) f(v) dv.$$

Let $(T', p') = (T(\epsilon), p(\epsilon))$ be given by Lemma A.3. Then by construction $SS^{Fix}(T', p') > SS^{Flex}(T, p; \sigma)$ and by the same argument as above $CS^{Fix}(T', p') < CS^{Flex}(T, p; \sigma)$. $\square$

The intuition is as follows. Suppose under campaign $(Flex, T, p)$ the marginal backer, s^*, thinks the project is funded with probability 0.5, and a backer with higher signal $s > s^*$ thinks the project will be funded with probability 0.75. Their expected utilities for contributing are, respectively,

$$U^{Flex}(s^*) = 0.5\mathbb{E}[value|s^*, \text{funded}] - p = 0$$
$$U^{Flex}(s) = 0.75\mathbb{E}[value|s, \text{funded}] - p > 0$$

Suppose the entrepreneur now switches to fixed funding but doubles the price p to $2p$, and

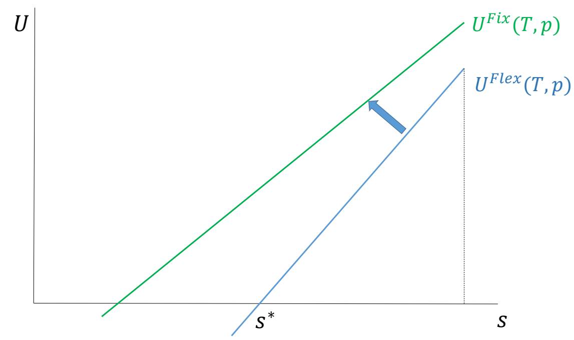

Figure 1.3: Switch to fixed funding

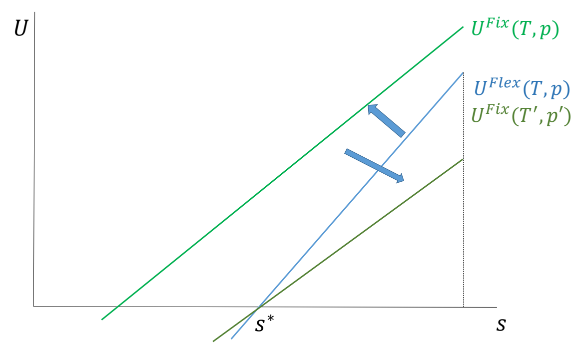

Figure 1.4: And increase the price

adjusts the corresponding T properly so that v^* stays unchanged.[18] We then have

$$U^{Fix}(s^*) = 0.5\mathbb{E}[value|s^*, \text{funded}] - 0.5\mathbb{E}[2p|s^*, \text{funded}] = 0$$
$$U^{Fix}(s) = 0.75\mathbb{E}[value|s, \text{funded}] - 0.75\mathbb{E}[2p|s, \text{funded}] < U^{Flex}(s)$$

This adjustment makes the marginal backer as happy as before. Moreover, it extracts more surplus from high signal backers; hence it leads to a higher profit. Figure 1.3, 1.4 summarizes the argument graphically.

To further explain, backers with different signals have different preference intensities between fixed funding and flexible funding. A backer with a high signal thinks the project is very likely to be successful, so he doesn't prefer fixed funding that much. On the other hand, a backer with a lower signal worries about funding failures more and thus prefers fixed funding. Hence, if we simultaneously switch from flexible funding to fixed funding and

[18]Technically, the entrepreneur sets $T' = 2p(1 - G(s^*|v^*)) = 2T$.

increase the price, the high signal backers will suffer more than backers with lower signal. A carefully tailored adjustment then allows us to keep the same cutoff v^* as before switching while extracting more surplus from high signal backers. This leads to higher profit. The underlying logic is akin to the the linkage principle. Under fixed funding, each backer's expected payment is

$$\mathbb{P}(v \geq v^*|s)p$$

which positively correlates with the underlying value because the signal structure satisfies MLRP, and is also increasing in the backer's "type" s. So a higher type has a higher expected payment, raising expected revenue.

Note also that the adjustment we use keeps the retail stage profits the same across the two mechanisms, so the increase in expected profit comes entirely from crowdfunding.[19]

Remark 4.1 [Robustness to Variations of Profit Function] Our method of proof is not based on the comparison of profits under optimal pricing. Instead, it is based on a feasibility argument. For fixed funding the entrepreneur can achieve the same allocation (decision to build v^* and who contributes s^*) with less consumer surplus and a higher funding goal. Hence the revenue ranking is robust to many possible modifications of the profit function.[20]

1.5 Characterization of Optimal Fixed Campaign

Having shown that fixed funding generates more expected profit, we now characterize optimal pricing. The entrepreneur's problem is given by

$$\max_{(T,p)\in C_a} \Pi^{Fix}(T,p),$$

where the profit for any choice of $(T,p) \in C_a := \{(T,p) : k - a \leq T \leq p \leq 1\}$ is defined to be the one given by the unique funded equilibrium when there exists one, and zero when it does not exist.

[19]The result is thus independent of the choice of retail price, as long as the retail price is non-decreasing in v and that it is higher than p with positive probability. The result is also independent of the existence of a retail market.

[20]See the discussion after Theorem 2 for examples of the modifications.

Theorem 1.2.

(a) Optimal Campaign

For every $0 \le a < k$, the entrepreneur's problem has a solution. Moreover, at the optimum, $T = k - a < p < 1$.

(b) Approximate Full Surplus Extraction

For $a = k$, there exists a sequence of $\{(T_n, p_n)\} \subset C_k$ such that funded equilibrium exists for each n and that

$$\lim_{n\to\infty} \Pi^{Fix}(T_n, p_n) = \int_k^1 v - kf(v)dv. \tag{1.8}$$

Moreover, $\lim_{n\to\infty} T_n = 0$, $\lim_{n\to\infty}(s_n^*, v_n^*) = (0, k)$, where s^*, v^* are corresponding equilibrium cutoffs.

Proof. Let $v^*(T,p), s^*(T,p)$ denote the cutoff of the funded equilibrium under $(T,p) \in C_a$ whenever the equilibrium exists. We establish the result for $a \in [0,k)$ by three claims.

Claim 1 Suppose $(T,p) \in C_a$ maximizes Π^{Fix}, then $v^*(T,p) \ge k$.

Suppose $v^*(T,p) < k$. The entrepreneur can pick some $p' > p$ such that $v^*(T,p') \in (v^*(T,p), k)$ and that $s^*(T,p') > s^*(T,p)$. Such p' exists because $v^*(T,p)$ is continuous on $p \in (T,1)$, by the construction of this fixed point in Proposition 3.1. Also, whenever the equilibrium v^* becomes higher due to price increase, corresponding s^* must rise (Otherwise v^* will be lower due to increased price and increased number of contributors).

This process raises social surplus and decreases consumer surplus. To see this, let $U'(s) = \int_{v^*(T,p')}^1 (v-p')\beta(v|s)dv$ and $U(s) = \int_{v^*(T,p)}^1 (v-p)\beta(v|s)dv$. Let

$$h(v) = \begin{cases} p^*, & v > v^*(T,p') \\ v, & v^*(T,p) \le v \le v^*(T,p'). \end{cases}$$

We thus have $U'(s^*(T,p')) - U(s^*(T,p')) < 0$.[21] Moreover, for all $s > s^*(T,p')$, because

[21] The first term is zero, the second term is the expected utility for contribution evaluated at a signal higher than the equilibrium cutoff $s^*(T,p)$.

$h(\cdot)$ is increasing and $\beta(\cdot|s)$ satisfies MLRP,

$$
\begin{aligned}
U'(s) - U(s) &= \int_{v^*(T,p')}^{1} (v - p')\beta(v|s)dv - \int_{v^*(T,p)}^{1} (v - p)\beta(v|s)dv \\
&= -\int_{v^*(T,p)}^{1} (h(v) - p)\beta(v|s)dv < 0.
\end{aligned}
$$

Finally note that consumer surplus is

$$
\begin{aligned}
CS(T, p') &= \int_0^1 \int_{s^*(T,p')}^{1} U'(s)g(s|v)dsf(v)dv \\
CS(T, p) &= \int_0^1 \int_{s^*(T,p)}^{1} U(s)g(s|v)dsf(v)dv.
\end{aligned}
$$

Hence $CS(T, p) - CS(T, p') > 0$.

This shows Claim 1.

Claim 2 Suppose $(T, p) \in C_a$ maximizes Π^{Fix}, then $T = k - a$.

If $s^* = 0$ then $v^* = 0$ and (T, p) is not optimal by Claim 1. So assume $s^* > 0$.

Suppose $T > k - a$. Choose $T' = k - a =< T$. For each $\tilde{p} \in [p, 1)$, define

$$s(\tilde{p}) = \min\{s : \tilde{p}(1 - G(s|v^*(T, p))) = T'\}.$$

Then $s(\cdot)$ is continuous and $s(p) > s^*$ because $T' < T$ and $s^* > 0$. Since $T' > 0$, $s(\cdot) < 1$.

Define the continuous function $J : [p, 1) \to [0, 1]$ as

$$J(\tilde{p}) = \int_{v^*}^{1} (v - \tilde{p})\beta(v|s(\tilde{p}))dv.$$

Since $s(p) > s^*$ and s^* is an equilibrium cutoff, $J(p) > 0$. Since $s(1) < 1$, $J(1) < 0$. Hence intermediate value theorem implies the existence of some $p' \in (p, 1)$ such that $J(p') = 0$.

By construction, funded equilibrium under (T', p') exists, with cutoff $(v^*, s(p'))$, and $s(p') > s^*(T, p)$ because $p' > p$ and $T' < T$. This implies that the social surplus under (T', p') is the same as that under (T, p) while the consumer surplus decreases due to a higher price. Hence $T > k - a$ is non-optimal.

Claim 3 For $T = k - a$, there exists ϵ such that if $T \leq p < T + \epsilon$, then $(k - a, p)$ is not optimal.

To see this, we show the following:

$$\limsup_{p\to T} v^*(T,p) < T.$$

Suppose there exists a sequence $\{p_n\}$ with $p_n > T$ and $\lim_{n\to\infty} p_n = T$ such that $\lim_{n\to\infty} v^*(T,p_n) \geq T$. Then there exists some N such that $p_n < v^*(T,p_n)$ for $n > N$. By Proposition 3.1, $v^*(T,p_n) < 1$. Hence for $n > N$,

$$\int_{v^*(T,p_n)}^{1} (v-p_n)\beta(v|s)dv \geq 0$$

for all s, which implies $v^*(T,p_n) > 0$ is not an equilibrium cutoff (since everyone would like to contribute), a contradiction.

Suppose instead $\lim_{n\to\infty} v^*(T,p_n) = T < 1$, dominated convergence implies

$$\lim_{n\to\infty} \int_{v^*(T,p_n)}^{1} (v-p_n)\beta(v|s^*(T,p_n))dv = \int_{v^*}^{1} (v-T)\beta(v|0)dv > 0$$

where the last inequality follows from the assumptions on $g(s|v)$. This again implies existence of some n such that

$$\int_{v^*(T,p_n)}^{1} (v-p_n)\beta(v|s)dv \geq 0$$

for all s, so the cutoff v^* can not be positive, a contradiction.

By the three claims,

$$\max_{(T,p)\in C_a} \Pi^{Fix}(T,p) = \max_{p\in[k-a+\epsilon,1]} \Pi^{Fix}(k-a,p)$$

Since Π^{Fix} as a function of p is continuous over $[k-a+\epsilon,1]$, maximum exists.

This finishes the proof when $a \in [0,k)$.

Now we show how to approximately extract full surplus when the entrepreneur has asset k to fund the project.

Let $\{s_n\}$ be a sequence converging to 1. Define p_n to be such that

$$\int_{k}^{1} (v-p_n)\beta(v|s_n)dv = 0,$$

so $\{p_n\}$ is a bounded sequence. Define $T_n = p_n(1-G(s_n|k))$.

By construction funded equilibrium under (T_n, p_n) exists, with cutoffs $(v^*, s^*) = (k, s_n)$ for each n. Moreover, seller's profit is

$$\int_k^1 (p_n(1 - G(s_n|v)) + G(s_n|v)v - k)f(v)dv.$$

Since $s_n \to 1$ and $\{p_n\}$ is bounded, the profit converges to the maximal social surplus. $\square$

When $a < k$, the entrepreneur chooses $T = k - a$. This is because asking money from the crowd incurs information rents, so the entrepreneur will never ask for more than what is needed. When $a = k$, (b) says that the entrepreneur will use all of her asset and minimize the amount of money raised from crowdfunding to zero. However, the entrepreneur will still use crowdfunding to gauge the market value of her idea. Full surplus extraction when the entrepreneur has sufficient assets is a feature of the continuum agent model, where even a small number of agents can provide accurate information. It is not uncommon for the entrepreneur to first prove that his product is profitable by selling to a smaller audience and then acquire the larger portion of funding needed for the project by venture capital. A famous example is Oculus, which is a virtual reality headgear, that raised 2.5 million through nearly $10,000$ backers on Kickstarter and was acquired by Facebook subsequently for 2 billion.

Moreover, under some regularity conditions, it can be shown that in optimum, $v^* > k$. That is, the entrepreneur gets less funding relative to the efficiency benchmark, which requires $v^* = k$. To see this, suppose under some crowdfunding campaign (Fix, T, p), $v^* = k$. Then raising p by a little has no first order effect on social surplus since $v^* = k$ maximizes social surplus, while it has a negative first order effect on consumer surplus because every backer that contributes suffer from an increase in price.

There are several ways to prevent full surplus extraction in the model when $a = k$. For one thing, we can make it costly to use the outside asset a, by imposing a high interest rate. For another, the number of backers may have an advertisement effect and thus the number of retail consumers is increasing in the number of backers. Formally,

$$\Pi^{Fix}(T, p) = \int_{v^*}^1 (p(1 - G(s^*|v)) + H(1 - G(s^*|v))v - k)f(v)dv$$

where $H(\cdot)$ is an increasing function that represents the measure of buyers at the retail stage.

One can also assume that the backers have a positive rate of not going to the retail stage if they choose not to contribute in the crowdfunding stage. Their attention can shift elsewhere, as there may be substitutes for the product. For the entrepreneur, these customers are forever lost if they are not attracted to contribute in the first place.

1.6 Moral Hazard

In previous sections we assumed that the entrepreneur has commitment power, so moral hazard problem is assumed away. This is equivalent to imposing a large enough penalty on default.

Empirically, [Mol14] shows that in the 381 projects they analyze, which consists of the Design and Technology category in Kickstarter, only 14 projects failed to deliver, and 3 of them even offered refunds. The other 11 can be considered frauds,[22] and account for 3.6% of all projects in the sample. In spite of the lack of legal consequences of defaulting, the default rate is surprising low compared to what many people would initially expect. We will see in this section that the entrepreneurs can use third party crowdfunding platforms as commitment devices to avoid moral hazard, and it is of their interest to do so.

We take the moral hazard problem into account by (1). making the entrepreneur's decision of whether to invest the funds raised for the project endogenous and (2). explicitly assuming that there is no penalty to default. Crowdfunding, in general, is conducted on third-party funding platforms, so the only commitment the entrepreneur can make is the collection of money when total funding reaches a certain threshold. Under this setting, a fixed funding campaign is a pair (T, p) where $T > 0$ means the commitment to collect money only if the total contribution exceeds T. "Flexible Funding" is then equivalent to setting $T = 0$. The campaign (T, p) is thus silent on when the project will be built, which is determined by a sequential rationality constraint and a feasibility constraint formulated

[22] The entrepreneurs took money away but stopped reporting any progress of their projects. In the end nothing is delivered.

below.

In this section, we assume that the entrepreneur has no other assets so she needs to fund at least k to kickstart the project, and that this is common knowledge.[23] We will focus on equilibrium in cutoff strategies. The entrepreneur learns the market value of its project, v, perfectly after the crowdfunding stage ends whenever the backers use a cutoff in $(0, 1)$. because MLRP implies $X_0^\sigma(v) = p(1 - G(s|v))$ is increasing in v. The entrepreneur then faces a choice of running away with the money or investing k to build the project and getting revenue from selling at the retail stage. In particular, the entrepreneur has an incentive to build at state v if and only if the retail stage revenue is larger than the cost:

$$X_1^\sigma(v) \geq k. \tag{RB}$$

Under a cutoff strategy σ, $X_1^\sigma(v) = G(s|v)v$, which may be decreasing in v when v is large because $G(s|v)$ is decreasing in v The reason is that the higher the value, the more people would choose to contribute via crowdfunding so the less buyers there would be at retail. To prevent this distortion to the entrepreneur's incentive (which may lead to non-existence of funded equilibria), we assume that there is a measure μ of buyers showing up only at the retail stage[24] so

$$X_1^\sigma(v) = G(s|v)v + \mu v.$$

Moreover, since there is no penalty to default, setting $T < k$ and running the risk of default because of infeasibility becomes an option for the entrepreneur. It is feasible for the entrepreneur to build the project at state v if

$$X_0^\sigma(v) \geq k \tag{FB}$$

We add two additional assumptions.

A3 There is a measure μ of buyers at the retail stage such that

$$\mu > k \quad \& \quad G(s|v)v + \mu v \text{ is increasing in } v \text{ for all } s.$$

[23] Making the information of k private to the entrepreneur will introduce a signalling game, which is of interest to study as well, but is tangent to the intuition we want to illustrate in this section.

[24] The addition of such μ does not change the results in previous sections.

A4

$$\int_0^1 v\beta(v|0)dv < k$$

A3 says that when the state is high enough, the entrepreneur always has incentive to build the project because the sales at the retail stage can cover the cost. In addition, the retail stage profit of the entrepreneur is increasing in v for all s. This guarantees that the entrepreneur's incentive to build is increasing in the value of her project.[25] A4 is a technical assumption that precludes equilibrium in which everyone contributes regardless of the signal.

Denote $\Pi^{MH}(T,p)$ to be the profit under the funded equilibrium given (T,p) if such equilibrium exists. In Proposition 6.1 we show that funded equilibrium is unique when it exists. The entrepreneur's problem is then

$$\max_{(T,p)\in C} \Pi^{MH}(T,p).$$

where $C = \{(T,p) : 0 \le T \le p \le 1\}$.

Our main result is a characterization of optimal campaign.

Theorem 1.3. The entrepreneur's problem has a solution. Moreover, at the optimum, $k = T < p < 1$, and the probability of default is zero.

The intuition behind the result is as follows. Default is like an unfunded project with flexible funding, both involving the entrepreneur taking money away while doing nothing. This possibility of default is taken into account when backers make contribution decisions. Thus a similar logic to Theorem 1 could apply. In particular, the entrepreneur can commit to fund her project, thus taking money away, only when the value is high enough, by setting a campaign (T,p) so that the project can be funded only when the value is high. This ensures that the entrepreneur has the incentive to build the project whenever the project is funded (Assumption A3). The entrepreneur can then charge a higher price while retaining the marginal backer.

[25] Without this assumption of retail stage buyers, imagine a situation where value is high and thus backers mostly receive high signal and choose to contribute. After receiving the funding the entrepreneur knows that the size of the retail market is small because most people have already paid at the crowdfunding stage, so the entrepreneur has incentive to abandon the project. This in turn makes high signal backers less likely to contribute.

To formally analyze the case with moral hazard, we proceed as follows. Fix (T, p) and a cutoff action profile σ with cutoff s. Define

$$F^\sigma(s) = \{v : p(1 - G(s|v)) \geq T\}$$

to be the set of states such that the project is funded. Since σ is a cutoff strategy, $F^\sigma(s)$ is of the form $[v^f(s), 1]$, where $v^f(s) = \min\{v : p(1 - G(s|v)) \geq T\}$. Define

$$RB^\sigma(s) = \{v : vG(s|v) + \mu v - k \geq 0\},$$

to be the set of states where the seller has incentive to carry out the project. Assumption A3 guarantees that RB^σ is of the form $[v^{rb}(s), 1]$. Define

$$FB^\sigma(s) = \{v : p(1 - G(s|v)) \geq k\}$$

to be the set of states where it is feasible to build the project, which is of the form $[v^{fb}, 1]$.[26]

The set of states the entrepreneur will default under σ is then

$$[v^f, \max\{v^{rb}, v^{fb}\}].$$

which is when the value is high enough so that the project is funded but the value is not high enough so that either there is not enough funding or that the value is too low to incentivize the entrepreneur to actually invest it.

The backers' expected utility to contribute, conditional on receiving signal s, under some (T, p) and a strategy σ with cutoff s', is

$$U(s;\sigma) = \int_{\max\{v^f(s'), v^{rb}(s'), v^{fb}(s')\}}^{1} v\beta(v|s)dv - \int_{v^f(s')} p\beta(v|s)dv.$$

Using Lemma A.1, we see that the best response to a cutoff strategy is again a cutoff strategy.

Entrepreneur's profit under an action profile σ with cutoff $s^* > 0$ is

$$\Pi^{MH}(T, p; \sigma) = \int_{v^f(s^*)}^{1} p(1 - G(s^*|v))f(v)dv + \int_{\max\{v^{rb}(s^*), v^{fb}(s^*), v^f(s^*)\}} ((G(s^*|v) + \mu)v - k)f(v)dv.$$

[26] A project can be feasible but not funded, or funded but not feasible, at state v.

That is, the entrepreneur gets the crowdfunded money, $p(1-G(s^*|v))$, whenever the project is funded, but gets the retail profit $(G(s^*|v)+\mu)v-k$ only when it is positive and when the crowdfunded money is enough to fund the project.

We consider funded equilibrium in cutoff strategies with a positive cutoff, so that entrepreneur can learn the state by observing the outcome of the campaign. We call it funded equilibrium with learning.

Proposition 1.3. Given (T,p) such that $k \leq T < p < 1$, there exists a unique funded equilibrium with learning.

Proof. Given $k \leq T < p < 1$. Since $T \geq k$, whenever the project is funded it will be feasible. That is, $v^{bf} \leq v^f$.

Suppose backers use strategy σ with cutoff $s \in (0,1)$. For each state v, the entrepreneur's gain from building the project is

$$m(v,s) = (G(s|v)+\mu)v - k$$

The entrepreneur builds the project if and only if $m(v,s) \geq 0$. Let

$$v^f_{(T,p)}(s) = v^f(s) = \min\{v : p(1-G(s|v)) \geq T\}$$
$$v^{br}_{(T,p)}(s) = v^{rb}(s) = \min\{v : m(v,s) \geq 0\}$$

where $v^f(s)$ is the lowest state above which the project is funded, and by Assumption A3 $v^{rb}(s)$ is the lowest state above which the entrepreneur has an incentive to build the project. Let

$$\phi(s,s') := \int_{\max\{v^{rb}(s),v^f(s)\}}^1 v\beta(v|s')dv - \int_{v^f(s)} p\beta(v|s')dv,$$

which is the conditional expected utility to contribution for a backer with signal s' if other backers use the strategy σ with cutoff s'.

For each $s \in (0,1)$, define $\Phi : (0,1) \to [0,1]$

$$\Phi(s) = \inf_{s'} \{s' : \phi(s,s') \geq 0\},$$

which is continuous on $(0,1)$ because $g(s|v), m(s,v)$, and thus $v^f(s), v^{br}(s)$, are continuous.[27] Since $m(v,s) = (G(s|v)+\mu)v - k$ is increasing in s, $v^{br}(s)$ is decreasing on $(0,1)$ Since $v^{br}(s)$ is decreasing and $v^f(s)$ is increasing in s, $\phi(s,s') \geq 0$ implies $\phi(s+\Delta s, s') \geq 0$ for any $\Delta s > 0$. Moreover, if $v^f(s) < 1$, then $\phi(s,s')$ is strictly single crossing in s'. These imply that Φ is decreasing and the unique funded equilibrium with learning is characterized by the fixed point $s^* < 1$ of Φ.

Claim 1 There exists $0 < s^* < 1$ such that $\Phi(s^*) = 0$.

Since $p > T$, for each $v > p$ there exists s such that $p(1 - G(s|v)) = T$. On the other hand, for each $v > k/\mu$, $m(v,s) > 0$ for all s. Pick $v^* > \max\{p, k/\mu\}$, let s^* be such that $p(1 - G(s^*|v^*)) = T$. Then $m(v^*, s^*) > 0$ so $v^{rb}(s^*) < v^f(s^*) = v^*$. For any $s \in [0,1]$, the interim expected utility to contribute when other backers use cutoff s^* is then

$$\phi(s^*, s) = \int_{v^*}^{1} (v-p)\beta(v|s)dv > 0.$$

Hence $\Phi(s^*) = 0$.

Claim 2 $\lim_{s\to 0} \Phi(s) > 0$.

Let

$$s^* = \inf\left\{ s : \int_{\frac{k}{\mu}}^{1} v\beta(v|s)dv - p \geq 0 \right\},$$

which is positive by Assumption A4 and that $p \geq k$. Then, for each $0 < \epsilon < s^*$, there exists γ such that

$$\int_{\frac{k}{\mu}-x}^{1} v\beta(v|s^* - \epsilon)dv - p < 0$$

for $x < \gamma$. Note that $v^{rb}(s)$ is continuous on $(0,1)$ with

$$\lim_{s\to 0} v^b(s) = \frac{k}{\mu}.$$

Note also that since $p > T$, $v^f(s) = 0$ when s is small enough.

Choose γ' such that when $s < \gamma'$,

$$v^{rb}(s) > \frac{k}{\mu} - \gamma \text{ and } v^f(s) = 0,$$

[27] The argument repeats that of Claim 1, Proposition 3.1.

then for all $s < \gamma'$,

$$\phi(s, s^* - \epsilon) = \int_{v^{rb}(s)}^{1} v\beta(v|s^* - \epsilon)dv - \int_{v^f(s)}^{1} p\beta(v|s^* - \epsilon)dv < 0,$$

hence $\Phi(s) > s^* - \epsilon > 0$.

Claim 1 and 2 shows that $\Phi(s)$ has a fixed point in $(0, 1)$.

□

The strategy to prove Theorem 1.3 now becomes clear, and it is similar to the revenue ranking result for fixed funding versus flexible funding. Suppose the equilibrium cutoffs under some (T, p) are such that $v^f < \max\{v^{fb}, v^{rb}\}$. That is, there exists some states such that the seller will default. Then the entrepreneur should commit to take money only at a higher state $v^f + \epsilon > v^f$ by raising T to T'. This allows the entrepreneur to raise p to some p' while keeping the marginal backer s^* the same, which makes $\max\{v^{fb}, v^{rb}\}$ the same as before adjustment. This adjustment extracts more consumer surplus from buyers with high signal s, thus reducing consumer surplus. Social surplus is determined by when the project is build, hence it remains the same if ϵ is small such that $v^f + \epsilon < \max\{v^{fb}, v^{rb}\}$. So the entrepreneur obtains a higher payoff under (T', p'). Formally,

Proof of Theorem 3. We establish Theorem 3 by two claims.

Claim 1: Given $(T, p) \in C$ and a funded equilibrium σ with cutoff s^*. If

$$v^f(s^*) < \max\{v^{rb}(s^*), v^{fb}(s^*)\}$$

then there exists (T', p') with $k \leq T' < p < 1$ such that

$$\Pi^{MH}(T, p) < \Pi^{MH}(T', p').$$

Let $v^b(s^*) = \max\{v^{rb}(s^*), v^{fb}(s^*)\}$. Similar to Section 5,

$$\begin{aligned}\Pi^{MH}(T, p) = &\int_{v^f(s^*)}^{1} (1 + \mu)v - kf(v)dv \\ &- \int_{0}^{1} \int_{s^*}^{1} \left(\int_{v^b(s^*)}^{1} v\beta(\tilde{v}|s)d\tilde{v} - \int_{v^f(s^*)}^{1} p\beta(\tilde{v}|s)d\tilde{v} \right) g(s|v)dsf(v)dv\end{aligned}$$

Denote it as $\Pi(T,p) = SS - CS$.

For each $0 < \epsilon < v^b - v^f$ and each $p \leq \tilde{p} \leq p^r(1)$, define $H : [p, p^r(1)] \to \mathbb{R}$ as

$$H(\tilde{p}; \epsilon) = \int_{v^b_{(T,\tilde{p})}(s^*)}^1 v\beta(v|s^*)dv - \int_{v^f+\epsilon}^1 \tilde{p}\beta(v|s^*)dv$$

Since $(s^*, v^f(s^*), v^b(s^*))$ is an equilibrium cutoff, $H(p;\epsilon) > 0$. When $\tilde{p} \to 1$, $H(\tilde{p}) < 0$. H is also decreasing in $\tilde{p}$. Hence there exists a unique $p(\epsilon)$ such that $H(p(\epsilon);\epsilon) = 0$.

Let

$$T(\epsilon) = p(\epsilon)(1 - G(s^*|v^f(s^*) + \epsilon)).$$

Now choose any $\epsilon > 0$ such that

$$v^f(s^*) + \epsilon < v^b$$

Then by construction, under the campaign $(T(\epsilon), p(\epsilon))$, σ with cutoff s^* is still a funded equilibrium. Under such funded equilibrium, the project is funded at $v^f_{(T(\epsilon),p(\epsilon))} = v^f(s^*) + \epsilon$. Moreover, the project is feasible to build at $v^{fb}_{(T(\epsilon),p(\epsilon))} < v^{fb}(s^*)$ because the same number of backers are contributing at a higher price, but the project is rational (for the entrepreneur) to build at $v^{rb}_{(T(\epsilon),p(\epsilon))} = v^{rb}(s^*)$, because v^{rb} is determined only by s^*. Therefore, social surplus under $(T(\epsilon), p(\epsilon))$ is the same as that under (T,p).

Again we break the profit into social surplus minus consumer surplus, $\Pi^{MH}(T(\epsilon), p(\epsilon)) = SS(\epsilon) - CS(\epsilon)$. We now show that $\Pi^{MH}(T(\epsilon), p(\epsilon)) > \Pi(T,p)$ by showing that $CS(\epsilon) < CS$.

Secondly, since under $(T(\epsilon), p(\epsilon))$ the marginal backer is still the same but all backers now face with a higher price $p(\epsilon) > p$,

$$\begin{aligned}
CS &= \int_0^1 \int_{s^*}^1 \left(\int_{v^b(s^*)}^1 v\beta(\tilde{v}|s)d\tilde{v} - \int_{v^f(s^*)}^1 p\beta(\tilde{v}|s)d\tilde{v} \right) g(s|v)ds f(v)dv \\
&> \int_0^1 \int_{s^*}^1 \left(\int_{v^b(\epsilon)}^1 v\beta(\tilde{v}|s)d\tilde{v} - \int_{v^f(\epsilon)}^1 p(\epsilon)\beta(\tilde{v}|s)d\tilde{v} \right) g(s|v)ds f(v)dv \\
&= CS(\epsilon)
\end{aligned}$$

This implies

$$\Pi^{MH}(T(\epsilon), p(\epsilon)) > \Pi^{MH}(T,p).$$

Claim 2: $\arg\max_C \Pi^{MH}(T,p)$ exists, where in optimum $k = T^* < p^*$. Note that whenever $T < k$, in any funded equilibrium there is a positive probability to default. Hence by Claim 1 we can restrict the set of optimizers to $\{k \leq T \leq p \leq 1\}$. The proof steps are similar to that of Theorem 1.2.

□

An implication of Theorem 1.3 is that the entrepreneur earns less profit, and her project is less likely to be completed, when the moral hazard problem is present. This is not surprising, because if the backers take into account the possibility of default, they will be more conservative in making contributions, resulting in less overall profit. To make a comparison, note that we can add measure μ of buyers into the model without moral hazard, without affecting all the results in the previous sections. Let $\Pi^{Fix}(T,p)$ denote the profit when there is no moral hazard, and $\Pi^{MH}(T,p)$ be the profit when there is moral hazard as defined in this section.

Corollary 2.

- 1. The presence of moral hazard decreases profit. Formally,

$$\max_{C} \Pi^{MH}(T,p) \leq \max_{C_0} \Pi^{Fix}(T,p) \tag{1.9}$$

- 2. When there is moral hazard, under optimal campaign, the project is less likely to be completed than when there is no moral hazard. It is strictly less likely to be completed if and only if (1.9) is strict.[28]

Proof. By Theorem 1.3, if $(T^*, p^*) \in \arg\max_C \Pi^{MH}(T,p)$, then

$$\Pi^{MH}(T^*,p^*) = \Pi^{Fix}(T^*,p^*) \leq \max_{C_k} \Pi^{Fix}(T,p).$$

Let (T^*, p^*) be the optimal fixed funding without moral hazard with funded equilibrium cutoff v^*, s^*. Let (T^{**}, p^{**}) be the optimal fixed funding with moral hazard with funded

[28]The observation and part of the argument is due to Ichiro Obara.

equilibrium cutoff v^{**}, s^{**}. We would like to show $v^* \leq v^{**}$ and that it is strict when (1.9) is strict.

Suppose to the contrary that $v^{**} < v^*$. Then this implies $s^{**} < s^*$. To see this, note that if $s^{**} \geq s^*$, it must be $p^{**} > p^*$ (since $v^{**} < v^*$ and v^{**} is such that $p^{**}(1-G(s^{**}|v^{**})) = T^{**} = T^* = k$.). In the problem without moral hazard, the entrepreneur can then increase the price p^* to some $p^{***} > p^{**}$ such that the corresponding equilibrium cutoff is $v^{***} = v^*$ and $s^{***} > s^*$. But this means the same social welfare but smaller consumer surplus, contradicting that (T^*, p^*) is optimal.

Hence if $v^{**} < v^*$ then $s^{**} < s^*$. By Theorem 1.3,

$$v^{**}G(s^{**}|v^{**}) + \mu v^{**} \geq k.$$

This implies that

$$v^*G(s^*|v^*) + \mu v^* \geq k.$$

Hence $\Pi^{MH}(T^*, p^*) = \Pi^{Fix}(T^*, p^*) > \Pi^{MH}(T^{**}, p^{**})$, a contradiction. We have thus shown that $v^* \leq v^{**}$.

Now suppose (1.9) is strict. Suppose to the contrary that $v^{**} = v^*$.

Suppose that $v^*G(s^*|v^*)+\mu v^* \geq k$. Then (1.9) must not be strict because the equilibrium cutoff v^*, s^* can be supported by the campaign (T^*, p^*) either with or without moral hazard. But this means that the maximum profit under moral hazard is the same as that without moral hazard, contradicting that (1.9) is strict.

Suppose that $v^*G(s^*|v^*) + \mu v^* < k$. Then $v^* = v^{**}$ implies $s^{**} > s^*$. This implies $p^{**} > p^*$. But this means that with moral hazard the entrepreneur is able to maintain the same social surplus while lowering consumer surplus relative to the case without moral hazard, contradicting (1.9).

When $\max_C \Pi^{MH} = \max_{C_0} \Pi^{Fix}$, then both scenarios have the same equilibrium. □

Without clear legal consequences of default, commitment power is mostly granted by existing reputation. For example, the entrepreneur may already have a past record in business and is well-known. Corollary 2 then says that an entrepreneur has less existing reputation will be more unlikely to fund her project and also enjoys less expected profit than one with an established reputation.

1.7 Discussion

1.7.1 Why Does the Crowd Matter?

To illustrate the role of the crowd, we compare the feasibility and profitability of funding a project through a single investor with the same signal structure.

Funding by an Investor An entrepreneur with zero asset chooses to sell a share $a \in [0, 1]$ of future monopoly profit v to a potential investor in order to fund the project.[29]

Since there is only one investor, if the investor chooses not to contribute, the project will not be built. The entrepreneur asks for a price $p \geq k$.

The investor chooses to invest whenever

$$U(s) = \int_0^1 av\beta(v|s)dv - p \geq 0,$$

Lemma 1.2. If the entrepreneur chooses to fund the project through selling to an investor with share a, at price p, then in optimum $p = k$ and a necessary and sufficient condition such that the project can be funded with positive probability for some (a, p) is

$$\int_0^1 v\beta(v|1)dv > k. \tag{1.10}$$

Proof. Suppose the entrepreneur chooses to sell a share a at a price p and that (a, p) maximizes profit. An investor with signal s has payoff given by

$$U(s) = \int_0^1 av - p\beta(v|s)dv,$$

which is increasing in s. The invest will buy whenever $s \geq s^* = \min\{s : U(s) \geq \delta p\}$. Since

[29] Another interpretation is to assume that a project generates utility v to a single large partially informed investor, whose expected payoff is $\mathbb{E}[v - p|s]$. For example, Google Ventures investing in start-ups for the development of a technology that may benefit Google in the future falls in this category. This amounts to restricting $a = 1$.

the project is funded with positive probability, $s^* < 1$ Then the entrepreneur's profit is

$$\Pi^s(p,a) = \int_0^1 \left((p-k) + (1-a)v\right)(1 - G(s^*|v))f(v)dv.$$

Since $s^* < 1$, $p = \int_0^1 av\beta(v|s^*)dv$.[30] Substitute into $\Pi^s(p,a)$ and differentiate with respect to a to obtain

$$\begin{aligned}\frac{d\Pi^s}{da} &= -\int_0^1 v(1-G(s^*|v))f(v)dv + \int_0^1 \left(\int_0^1 v\beta(v|s^*)dv\right)(1-G(s^*|v))f(v)dv \\ &= -\int_0^1 \int_{s^*}^1 \left(\int_0^1 v\beta(v|s)dv\right) g(s|v)ds f(v)dv \\ &\quad + \int_0^1 \int_{s^*}^1 \left(\int_0^1 v\beta(v|s^*)dv\right) g(s|v)ds f(v)dv \\ &< 0\end{aligned}$$

Hence, being optimal, (a,p) must be that

$$p = \int_0^1 av\beta(v|s^*)dv = k. \tag{1.11}$$

If there exists (a,p) such that the project can be funded with positive probability, then $s^* < 1$ and thus (1.10) follows from (1.11). If (1.10) holds, then there exists $s^* < 1$ and (a,p) such that (11) holds. □

We show that some projects can be funded with positive probability by the crowd but not by an investor, but all projects that can be funded by an investor can be funded by the crowd.

Let

$$k_s = \sup\{k : \text{The project can be funded by an investor.}\}$$

$$k_{fix} = \sup\{k : \text{The project can be funded by fixed funding with } T \geq k\}$$

[30] If $s^* > 0$ then this holds by the definition of s^*, if $s^* = 0$ and this does not hold then the seller should increase p to increase profit.

For each $a \in [0,1]$, let $s^*(a) = \min\{s : \int_0^1 av - k\beta(v|s)dv \geq 0\}$ be the cutoff signal for the investor if he can receive a shares of the profit. Let

$$SS^{Fix}(k) = \max_{(T,p)\in C_0} \int_{v^*(T,p)}^1 v - kf(v)dv$$
$$SS^{SI}(k) = \max_{a\in[0,1]} \int_0^1 (v-k)(1-G(s^*(a)|v))f(v)dv$$

be the social welfare benchmark under fixed funding and a single investor respectively,.

We have the following theorem regarding the strength of crowdfunding versus a single large buyer.

Proposition 1.4.

$$k_{fix} \geq k_s.$$

The inequality is strict whenever $\beta(\cdot|1)$ does not put probability one on $\{v = 1\}$. For each $k \in (0,1)$,

$$SS^{Fix}(k) > SS^{SI}(k).$$

Proof. Proposition 1.1 says $k_{fix} = 1$.

By Lemma 1.2,

$$k_s = \int_0^1 v\beta(v|1)dv \leq k_{fix} = 1$$

where the inequality is strict when $\beta(\cdot|1) \neq \delta_1$.

To see that $SS^{Fix}(k) > SS^{SI}(k)$, note that by Claim 3 of Theorem 1.2, there exists an $k < p < 1$ such that $v^*(k,p) = k$, which achieves the social welfare under full information. □

Hence, when the fixed cost is high, crowdfunding is the only way to fund a project, since it aggregates more information and thus enables a more efficient allocation. Moreover, through information aggregation, fixed funding can implement the socially efficient benchmark, while an investor sometimes invest in bad states or fails to invest in good states.

1.7.2 Equity Crowdfunding

Although our model focuses on reward-based crowdfunding, in which the backers derive utility directly from a unit of the product, the entrepreneur can potentially use other means to fund the project as well. For example, the entrepreneur can sell a portion of future profit to the crowd, such as in equity crowdfunding. We show how to incorporate equity crowdfunding into our framework.

Definition 1.1. An equity crowdfunding campaign is a tuple (T, p, a) where T is the threshold, p is the price per share, and a is the portion of future profit on sale.

Given an equity funding plan, an action profile, σ, with cutoff s^*, the expected utility to contribute is

$$U(s) = \int_{v^*}^{1} (av - p)\beta(v|s)dv$$

The entrepreneur's profit is

$$\Pi(T, p, a) = \int_{v^*}^{1} (p(1 - G(s^*|v)) + ((1 - a) + aG(s|v))v - k)f(v)dv.$$

Note that the analysis in previous sections still go through. The scalar a does not affect any of our arguments before.

Suppose that there is a measure μ of retail stage consumers and that $k > 1$. Then there will be no funded equilibrium in reward based funding, while in equity funding the backers will be able to enjoy the profit brought by the measure μ of retail consumers. Therefore the project may still be funded by equity funding as the entrepreneur can charge a per capita price higher than $k > 1$. Hence equity funding potentially enables more projects to be funded than reward-based crowdfunding because it can front-load the project's total value to the backers, alleviating problems of insufficient market reach.

1.7.3 When Will Flexible Funding Be Preferable?

One may wonder, why is flexible funding still adopted from time to time on some online platforms?

We first note that empirical findings in [CLS14] suggest that fixed funding has a higher rate of success and raises more money for projects about consumer products. The production function in these projects are often of the threshold type, and the entrepreneur knows that the cost is a constant k independent of v.[31]

In other types of projects, the cost may be a function of v as well. For example, consider an NPO running a campaign to contain an epidemic outbreak in a developing country, where the number of infections is uncertain and will be realized at a future date.

Formally, assume $k(v) = cv$ for $v \in [0, 1]$. This means that at each state v, the number of infections is v and the total cost to contain it is cv. Also assume $f(v) = 1$ and $g(s|v) = 2vs + 2(1-v)(1-s)$ for all $s, v \in [0, 1]$. This implies $\beta(v|s) = g(s|v)$. The NPO's problem is to maximize social surplus using crowdfunding

$$\int_{v^*}^{1} (v - cv)\, f(v)dv$$

subject to an ex-post budget balance constraint

$$X_0^\sigma(v) \geq cv$$

on the states where the project is funded.

Backers derive utility from helping out, with utility $v - p$ on the states where money is taken. That is, the bigger the problem a backer helps to alleviate, the higher his utility. Hence, a backer with a high signal assumes the problem is more serious and is thus more likely to contribute.

Consider the campaign $(T, p) = (0, 1/2)$. Then when $p = 1/2$, $\int_0^1 (v-p)f(v)dv = 0$. Since $\beta(v|1/2) = f(v)$, the indifferent backer is $s^* = 1/2$. And the money raised in each state will be

$$X_0^\sigma(v) = p\left(1 - G\left(\frac{1}{2}\,|v\right)\right) = \frac{v}{4} + \frac{1}{8}.$$

[31] [CLS14] examines data from Indiegogo and finds that entrepreneurs with different types of project or risk preference will self-select to adopt different funding mechanisms. An entrepreneur with scalable projects or with risk averse preferences are more likely to adopt flexible funding than fixed funding. While we assume the entrepreneur is risk-neutral throughout the paper, we do note that by choosing fixed funding the entrepreneur will get zero with positive probability, and it is possible to construct a risk preference under which an entrepreneur prefers flexible funding even she has a non-scalable project.

Therefore, whenever $c \leq 3/8$, the flexible campaign $(T, p) = (0, 1/2)$ maximizes social surplus with ex-post budget balance.

While we do not formally treat general production functions, we stress that it is hardly optimal to take money away while doing nothing. In any such occasion the entrepreneur can make the adjustment we proposed in Theorem 1 and receive more fund. Thus, a good flexible funding campaign often comes with a promise to build. Projects that can benefit from such a promise are those that are socially efficient to build at any states, which are more likely to involve charity projects than the development of consumption goods or investment plans.

1.8 Conclusion

This paper models crowdfunding as an entrepreneur posting a price to partially informed backers. The incentive for the backers to contribute to the campaign is driven by the expectation of paying a lower price than the retail price, when the quality of the good turns out to be good. Fixed funding reinforces this incentive by refunding the money when the quality of the good is bad, and thus it is able to achieve a higher probability of getting funded and also raises a higher amount of funds. In terms of moral hazard, there can be several reasons to explain an empirically low default rate. Instead of a reputational argument, we show how the entrepreneur could use third party online platforms as a commitment device to eliminate the possibility of defaults.

Our model thus explains the success of crowdfunding and in particular the popularity of fixed funding, both when the entrepreneur can commit or can not commit to developing her project. In spite of the lack of regulation, to a certain degree crowdfunding is not subject to serious moral hazard problems. Measures to deregulate the equity crowdfunding market, such as the JOBS act, thus seems to be a right step.

There are several interesting directions one can further investigate. For instance, one can extend the model to general production functions where the entrepreneur may need to choose an investment policy. Another direction is to consider the dynamics of crowdfunding and study how information is aggregated and how the current backers learn through observing previous backers, which relates to a vast learning/herding literature pioneered by [BHW98]. It is also of interest to more extensively study equity funding and see how it compares to methods such as going through an IPO.[32]

[32] On a practical level, crowdfunding is more accessible to IPO to small businesses. Another difference is the determination of prices. Crowdfunding is through posted price, but the commonly employed book-building method endogenously determines a market clearing price. IPO also has principal-agent problems.

CHAPTER 2

Should the Talk be Cheap in Contribution Games?

2.1 Introduction

In static contribution games with incomplete information, free riding and mis-coordination due to incomplete information often causes inefficiency. A potential solution is to allow cheap talk before contribution, that the agents are allowed to freely communicate whether they like to contribute, without cost and commitment. Intuitively, these cheap talks can be used to coordinate contributors with high private values, so the project will be built when it is worth it. However, it is subject to credibility problems. People may be just bragging about their willingness to contribute to induce others to contribute more. Another solution is making the contribution game sequential, allowing agents to take turns to contribute, with commitment. Under such setting the signals becomes much more credible and may potentially incentive people with lower valuation to contribute. It is thus not immediately clear whether a period or multiple periods of cheap talk will be more socially efficient compared to multiple periods of committed contribution.

In this paper we identify a factor that determines which setting admits a more efficient equilibrium: the number of levels of contribution the agents can choose. We consider production functions that satisfy increasing differences and that agents are only able to free ride others' efforts if they contribute(such as joining a membership). We focus on monotone equilibria, in which the contribution decision is weakly increasing in history(last period contributions) and type. This is a natural solution concept given the property of the production function.

With binary contribution choices, one of the monotone equilibria of the contribution game with cheap talk implements the ex-post efficient and ex-post individually rational allocation when the period is long enough. On the other hand no equilibrium of the contribution game with commitment can implement the same allocation.

However, with a continuum of contribution choices, free riding will be so severe that cheap talk does not convey any information in equilibrium. So the game with cheap talk has the same equilibrium as the BNE in an one-shot game. On the other hand, we construct a PBE in a dynamic contribution game with commitment that significantly increases social welfare. Hence, costless signalling and costly signalling are both useful, depending on the underlying environment. Our paper thus provides insight in designing indirect mechanisms in relevant environments, in which whether signalling is costly depends on the designer's choice. One example is crowdfunding, in which people contribute to a project over time. The popular online platform Kickstarter allows the contributors to withdraw the money anytime before a funding campaign is closed, essentially making it cheap talk, while another platform Indiegogo has a more strict policy of refund in most of its campaigns.

The contribution dynamics identified in our paper is also interesting in its own right. In a contribution game with commitment, high types contribute first and low types wait to the next round until they observe the high types' contribution. In contrast, in a game with cheap talk, the dynamics resembles that of an open ascending bid auction. Every agents with type above a certain threshold signals that they will contribute. Upon observing the number of such claims, some begin to drop out. In subsequent rounds more agents drop out, until a subset of agents such that each one of them is willing to contribute as long as others in the subset contribute, is reached. In this fashion, even with only a binary message space, if the length of cheap talk stages is long enough the efficient allocation(with a continuum of private types) will still be reached.

Our paper seems to be in contrast with the results obtained in cost sharing games, such as in Agastya et al.[AMS07], where they showed cheap talk improves efficiency in a game with a continuum of actions. However, in their paper with a cost sharing game, the two players coordinates to share costs at a pre-determined level that, once the project is decided to be built, is independent of the reports. Hence, it is essentially binary: either no one contributes or everyone contributes a pre-specified quantity. With our production function, a player always benefits from the other player's additional contribution. Free riding thus makes cheap talk non-credible.

Related Literature Cheap talk has been shown in Crawford and Sobel[CS82], among others, to improve efficiency in incomplete information games where agents interests are aligned. Palfrey and Rothensal[PR91] compare a game with cheap talk versus a one shot game. Agastya, et al.[AMS07] show that, in cost sharing games, adding a stage of cheap talk before the actual contribution improves efficiency. Using a similar production function, Palfrey et al.[PRR15] demonstrate efficiency gains of cheap talk in experiments.

Making the contribution little my little with commitment also seems to alleviate incomplete information problems. In his seminal book, Schelling[Sch60] suggests dividing contribution sequentially to overcome the credibility issue. Duffy et al.[DOV07] show using experiments that making the contribution game dynamic did increase contributions relative to its static counterpart. However, making the contribution sunk may hinder efficiency because it can make agents too conservative, as shown in the model by Admati and Perry[AP91].

Following these papers, Barberi[Bar12] simultaneously looks at dynamic contribution and cheap talk. He shows that in addition to a cheap talk stage, adding an early contribution stage prior to the cheap talk stage increases the credibility of the cheap talk, thus further improves efficiency.

Built on these set of papers, we compare the efficiency of equilibria in games with cheap talk and games with commitment, and make clear the underlying economic forces that makes cheap talk or commitment better than one another.

The rest of the paper is organized as follows. In Section 2 we set up the underlying environment. In Section 3, we look at binary contribution levels. We first characterize the incentive compatible and efficient allocation as a benchmark, and then characterize equilibria in dynamic models with cheap talk and with commitment, and show that cheap talk has an edge over dynamic games with commitment. In Section 4 we turn to a model where the level of contribution is a continuum, and show that the result reverses. Section 5 concludes.

2.2 The Environment

N agents, $i \in \{1, ..., N\}$, contribute jointly to a project. The action set for each agent is a subset $M \subset \mathbb{R}^+$ with $0 \in M$. The utility of an agent is given by

$$u(m_i, m_{-i}, \theta_i) = g(m_i, m_{-i}, \theta_i) - m_i,$$

where $\theta_i \in \Theta = [0,1]$ is private value and $m_{-i} = \sum_{j \neq i} m_j$ and $g : M \times M \times \Theta$. Assume also that $g_i(0, m_{-i}, \theta_i) = 0$ for all m_{-i}, θ_i and i. That is, this is a joint contribution to an excludable good[1]

Agents' values are distributed i.i.d with common prior F with full support on Θ. The central assumption for our production function is

Assumption 1. $g(\cdot)$ is increasing in all variables and satisfies strict increasing differences in (m_i, m_{-i}) and (m_i, θ_i).

To make the problem interesting, assume

Assumption 2. $g(1, 0, 1) > 1$ and $g(1, N-1, 0) - 1 < 0$.

This assumption amounts to the the inclusion of two extreme preferences: people that want to contribute anyways and people that will never contribute no matter what.

2.3 The Binary Action Model

In this section we assume $M = \{0, 1\}$. We will first see what the ex-post efficient allocation is, and then consider two regimes of dynamic contribution games , one with cheap talk and one without, to see whether the equilibrium allocation coincide with the efficient one.

2.3.1 Ex-post Efficient and Individually Rational Allocation

As a benchmark, we characterize the F-almost unique ex-post efficient and individually rational allocation to our environment and compare the outcomes of the models we are

[1]Co-authoring a paper, membership, etc. are examples of excludable good.

interested in against this benchmark.

An allocation is a function

$$m : \Theta^N \to \{0, 1\}^N.$$

An allocation is dominant strategy incentive compatible(dsIC) if $u_i(m(\theta), \theta_i) \geq u_i(m(\theta'_i, \theta_{-i}), \theta_i)$ for all $\theta_i, \theta'_i, \theta_{-i}$ and all i. An allocation is ex-post dominated if there exists some other allocation m' such that $u_i(m'(\theta), \theta_i) \geq u_i(m(\theta), \theta_i)$ for all i and all θ and that the inequality holds strictly for some i and θ. m is ex-post efficient if it is not ex-post dominated. m is individually rational(IR) if $u_i(m(\theta), \theta_i) \geq 0$ for all i and θ. Intuitively, ex-post efficiency and individual rationality require us to find, for each profile θ, the maximal subset of agents such that if they contribute they can all get non-negative utility. We now show formally this intuition is true. For each $n = 0, ..., N-1$, define $\tilde{\theta}(n)$ such that

$$g(1, n, \tilde{\theta}(n)) - 1 = 0,$$

which exists by A2. For each type profile $\theta = (\theta_1, ..., \theta_N)$, let $k(n)(\theta) = |\{i | g(1, n, \theta_i) - 1 \geq 0\}|$ for each n. So $k(n)(\theta)$ is the size of the set of agents who are willing to contribute as long as there are at least n other people who contribute. Let $K(\theta) = \max_n \{k(n)(\theta) | k(n)(\theta) > n\}$. Define an allocation $m : \Theta^N \to \{0, 1\}^N$ as

$$m_i(\theta) = \begin{cases} 1 & \text{if } \theta_i \geq \tilde{\theta}(K(\theta)) \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2.1. m is IR, dsIC, and ex-post efficient. Let m' be another IR and ex-post efficient allocation, then $\{\theta : m'(\theta) \neq m(\theta)\}$ is an F-measure zero set.

Proof. Individual rationality of m follows from the definition. Suppose m' ex-post dominates m. In particular, in state θ there exists i such that $u_i(m'(\theta), \theta_i) > u_i(m(\theta), \theta_i)$. We separate two cases: $K(\theta) = 0$ and $K(\theta) > 0$. In the former case, $m^*(\theta) = (0, ..., 0)$, so everyone gets zero utility under m. Hence $m'_i(\theta) = 1$. It implies there exists some j such that $u_j(m'(\theta), \theta_j) < 0$, otherwise $K(\theta) \neq 0$. But this again contradicts that m' ex-post dominates m. In the latter case, suppose first $\theta_i < \tilde{\theta}(K(\theta))$. Then $u_i(m'(\theta), \theta_i) > 0 = u_i(m(\theta), \theta_i)$. Then there must exists some j such that $u_j(m'(\theta), \theta_j) < 0$, otherwise m' contradicts the definition of $K(\theta)$. Suppose instead that $\theta_i \geq \tilde{\theta}(K(\theta))$, then it must be that the number of contributors under $m'(\theta)$ is more than $m(\theta)$. By the definition of $K(\theta)$ this again implies the existence of some agent j with $u_j(m'(\theta), \theta_j) < 0$. Either way contradicts the assumption that m' ex-post dominates m.

For the almost uniqueness assertion, since F is assumed to be continuous, it suffices to show that for all type profile $\theta = (\theta_1, ..., \theta_N)$ with $\theta_i \neq \tilde{\theta}(n)$ for all i, n, the efficient and individually rational allocation is unique.[2] To this end, let $m(\theta) \neq m'(\theta)$ be two ex-post efficient and IR allocations. Let $S = \{i : m_i(\theta) = 1\}, S' = \{i : m'_i(\theta) = 1\}$. Let $\tilde{m}(\theta)$ be defined as $\tilde{m}(\theta) = 1$ iff $i \in S \cup S'$. We claim that $\tilde{m}$ ex-post dominates m, m' and is IR. Since $|S \cup S'| \geq \max\{|S|, |S'|\}$, IR of $m(\theta), m'(\theta)$ implies

$$u_i(\tilde{m}(\theta), \theta_i) \geq \max\{u_i(m(\theta), \theta_i), u_i(m'(\theta), \theta_i)\} \geq 0$$

for all i. Since $S \neq S'$, without loss of generality assume $m_i(\theta) = 1$ for some i. We claim either i strictly prefers $\tilde{m}(\theta)$ to $m(\theta)$ or there exists some j that strictly prefers $\tilde{m}(\theta)$ to $m'(\theta)$. This contradicts that both $m(\theta), m'(\theta)$ are ex-post efficient. Suppose i is indifferent between $\tilde{\theta}$ and $m(\theta)$. Then $|S \cup S'| = |S|$, which implies $S' \subset S$. If $S' \neq \phi$, since $S \neq S'$ there exists $j \neq i$ with $m'(\theta_j) = 1$. But then $|S'| < |S|$, so $u_j(\tilde{m}(\theta), \theta_j) > u_j(m'(\theta), \theta_j)$. If $S' = \phi$, then the ex-post efficiency of m' implies $u_i(m(\theta), \theta_i) = u_i(m'(\theta), \theta_i) = 0$, which implies $\theta_i = \tilde{\theta}(|S| - 1)$, violating that $\theta_o \neq \tilde{\theta}(n)$ for all i, n.

For ex-post IC, suppose type θ_i reports θ'_i. Suppose first that $m_i(\theta_i, \theta_{-i}) = 1$. Then either $m(\theta_i, \theta_{-i}) = m(\theta'_i, \theta_{-i})$ or $m(\theta'_i, \theta_{-i}) = 0$. In the former case the utlity for θ_i is unchanged, in the latter it changes from weakly positive to zero($u_i(m(\theta), \theta_i) = 0$ when $\theta_i = \tilde{\theta}(n)$ for some appropriate n). Suppose instead that $m_i(\theta_i, \theta_{-i}) = 0$. Then either $m(\theta_i, \theta_{-i}) = m(\theta'_i, \theta_{-i})$ or $m(\theta'_i, \theta_{-i}) = 1$. In the former the utility again remains unchanged. In the latter, by definition of m every j with $\theta_j \geq \theta'_i$ will also contribute. But this means $u_i(m(\theta'_i, \theta_{-i}), \theta_i) < 0$, otherwise we will have $m_i(\theta_i, \theta_{-i}) = 1$ in the first place. □

Remark 2. The multiplicity of ex-post efficient and IR allocation comes from the existence of the cutoff types $\tilde{\theta}(n)$, that is, the types that are indifferent between contribution or not

[2] Here we abuse the word allocation slightly to mean an element in $\{0, 1\}^N$ instead of a function.

given for a given number of other contributers. Excluding such cases the ex-post efficient and IR allocation will be unique.

Remark 3. A mechanism designer may have other goals than obtaining efficiency. For example, maximizing $\mathbb{E}[\sum_i m_i(\theta)]$ subject to IC and IR constraints. Proposition 2.1 already maximizes the total contribution subject to ex-post IR. However, it is still possible to relax ex-post IR to interim IR and obtain a higher expected total contribution.

2.3.2 The Dynamic Contribution Game

In what follows we construct indirect mechanisms where the message space is the contribution levels instead of types, and find conditions where an indirect mechanism implements the efficient allocation.

The mechanism we consider is a game in which agents make contribution decisions over T periods. There is no time discount.

In each period, each agent takes an action $a_i^t \in A_i^t \subset \{0, 1\}$. The available actions may depend on history.

A history in the beginning of date $t+1$ for player i for games in which individual contribution is unobservable is $h^t = (h_i^t, h_{-i}^t) = (\{a_i^s\}_{s=1}^t, \sum_{j \neq i} a_j^s\}_{s=1}^t)$, while in the observable setting it is $h^t = \{a_1^s, ..., a_N^s\}_{s=1}^t$. It turns out that in $T = 2$ both assumptions are the same. However, if $T > 2$ then unobservability creates complications in that a deviation may not be common knowledge among players. Hence, in this paper the result for $T > 2$ periods model will be derived under the assumption that each player's action is perfectly observable.

Let H^t denotes the set of date t histories, with $H^0 = \{1\}$. A strategy for player i, $s_i = \{m_i^t\}$, is a sequence of functions $\{m_i^t\}_{t=1}^T$ with $m_i^t : \Theta \times H^{t-1} \to \{0, 1\}$.

For a given pure strategy profile $(\{m_i^t\}_{t=1}^T)_{i=1}^N$, let $m(T) \in \{0, 1\}^N$ be the induced terminal node. Then the utility is given by

$$u_i(m(T), \theta_i) = g(m_i(T), m_{-i}(T), \theta_i) - m_i(T).$$

The contribution game has commitment if for each t and each i, $A_i^t = \{1\}$ for player i

whenever $a_i^s = 1$ for some $s < t$. That is, whenever agent i chooses to contribute, he becomes inactive and must choose $m_i^T = 1$ in the end.

The contribution game has cheap talk if $A_i^t = \{0, 1\}$ for all history.

Thus, a game with no cheap talk is a T-period contribution game where one decides the period one puts in the effort/money. A game with cheap talk comes with $T - 1$ periods of cheap talk and 1 last period of real contribution.

Call a strategy $\{m_i^1(\theta), m_i^t(\theta, h^{t-1})\}$ history-monotone if $m_i^t(\theta, h_i^{t-1}, h_{-i}^{t-1}) \geq m_i^t(\theta, h_i^{t-1}, h'^{t-1}_{-i})$ when $h_{-i}^{t-1} > h'^{t-1}_{-i}$. Call a strategy $\{m_i^1(\theta), m_i^t(\theta, h^{t-1})\}$ monotone if for every t, $m_i^t(\theta, h^t)$ is nondecreasing in each and every of the arguments θ, h_i^t, h_{-i}^t.

2.3.3 Two Period Contribution Game With Commitment

In this section we characterize the class of monotone symmetric perfect Bayesian equilibria(PBE) of the contribution game with commitment when $T = 2$. We will show that all PBEs that are history-monotone are also monotone.

Given a strategy profile, fix an agent. Let $\alpha(n_1)$ be the probability(belief of that agent) that there are n_1 contributors out of $N - 1$ agents in the first period, $\beta(k, n_1, 0)$ be the probability that there are k other contributors in the second period out of $N - 1 - n_1$ agents conditional on that there are n_1 other contributors in the first period and the agent itself did not contribute in the first period. Similarly, $\beta(k, n_1, 1)$ is the conditional belief on the number of second period other contributors if the agent contributes in the first period.

For two discrete probability density $f(t), g(t)$, we say $f(t)$ first order stochastically dominates $g(t)$(henthforth FOSD) if $\sum_q^\infty f(t) \geq \sum_q^\infty g(t)$ for all q. We say $f(t)$ strictly FOSD $g(t)$ if the inequality holds strict for some q. A standard result regarding FOSD that will be used later is that for any non-decreasing function $u(t)$, if f strictly FOSD g then $\mathbb{E}_f[u] > \mathbb{E}_g[u]$.

Lemma 2.1. Assume A1,A2. Given any symmetric PBE that is history-monotone

- 1.$\beta(k, n_1, 1)$ FOSD $\beta(k, n_1, 0)$ for all n_1.
- 2.$\beta(k, N - 2, 1)$ strictly FOSD $\beta(k, N - 2, 0)$.

Proof. We need to show that for $q = 0, 1, ..., N - n_1 - 1$,

$$\sum_{k=q}^{N-n_1-1} \beta(k, n_1, 1) \geq \sum_{k=q}^{N-n_1-1} \beta(k, n_1, 0).$$

Fix any PBE and any n_1. Let $S = m_1^{-1}(1)$, $T = \{\theta : m_2(\theta, 0, n_1) = 1\}$. Let $p = \mathbb{P}(T|S^c)$. Then

$$\beta(k, n_1, 0) = \binom{N-1-n_1}{k} p^k (1-p)^{N-1-n_1-k}.$$

Let $T' = \{\theta : m_2(\theta, 0, n_1 + 1) = 1\}$ $p' = \mathbb{P}(T'|S^c)$. Then

$$\beta(k, n_1, 1) = \binom{N-1-n_1}{k} p'^k (1-p')^{N-1-n_1-k}.$$

Since the strategies are monotone in h_{-i}, $T \subset T'$, which implies $p' \geq p$, establishing item 1.

Let $n_1 = N - 2$. Then in equilibrium the sets T, T' are given by

$$T := \left\{ \theta : \sum_{k=0}^{1} \binom{1}{k} p^k (1-p)^{N-1-n_1} g(1, N-2+k, \theta) - 1 > 0 \right\}$$
$$T' := \{\theta : g(1, N-1, \theta) - 1 > 0\}$$

Since A1 and continuity of g w.r.t. θ implies $p = \mathbb{P}(T|S^c) < 1$ in equilibrium[3], $I := T' \backslash T$ will be an interval with positive length. Suppose $\mathbb{P}(T'|S^c) = \mathbb{P}(T|S^c)$, then $\mathbb{P}(I|S^c) = 0$, so types in I all contribute in the first period. Consider the type $\inf I$, that is, the type such that $g(1, N-1, \inf I) = 1$. Since $p < 1$, he strictly prefers to contribute in the second period than contribute in the first period. By continuity of expected utility with respect to θ some types in I also strictly prefers to delay contribution, contradicting that the proposed strategy profile is a PBE. Hence $\mathbb{P}(T'|S^c) > \mathbb{P}(T|S^c)$, establishing item 2. □

In the next Lemma we show that all history-monotone PBEs are monotone(i.e. in all arguments). This follows from increasing differences and the fact that early contribution induces late contributions (Lemma 2.1).

Lemma 2.2. Assume A1,A2. For any symmetric PBE that is monotone in history, $m_1(\theta)$ is monotone in θ, $m_2(\theta, \cdot, n_1)$ is monotone in θ given n_1.

[3]There is some $\theta^* > 0$ s.t. $g(1, N-1, \theta) - 1 < 0$ for all $\theta \in [0, \theta^*]$. In equilibrium these types will not contribute in the first or second period no matter what.

Proof. First we show that m_1 is monotone in θ. The expected payoffs for type θ to contribute in the first period and delay contribution are

$$EU_1(\theta) = \sum_{n_1=0}^{N=1} \sum_{k=0}^{N-1-n_1} \alpha(n_1)\beta(k, n_1, 1)g(1, n_1 + k, \theta) - 1$$
$$EU_d(\theta) = \sum_{n_1=0}^{N=1} \alpha(n_1) \max \left\{ 0, \sum_{k=0}^{N-1-n_1} \beta(k, n_1, 0)g(1, n_1 + k, \theta) - 1 \right\}$$

Suppose $m_1(\theta) > m_1(\theta')$, then being an equilibrium implies

$$EU_1(\theta) \geq EU_d(\theta) \tag{2.1}$$
$$EU_d(\theta') \geq EU_1(\theta') \tag{2.2}$$

Adding (2.1),(2.2) and rearrange to obtain

$$\begin{aligned} &\sum_{n_1=0}^{N=1} \sum_{k=0}^{N-1-n_1} \alpha(n_1)\beta(k, n_1, 1)[g(1, n_1 + k, \theta) - g(1, n_1 + k, \theta')] \\ &\geq \sum_{n_1=0}^{N-1} \alpha(n_1) \left(\max \left\{ 0, \sum_{k=0}^{N-1-n_1} \beta(k, n_1, 0)g(1, n_1 + k, \theta) - 1 \right\} \right. \\ &\quad \left. - \max \left\{ 0, \sum_{k=0}^{N-1-n_1} \beta(k, n_1, 0)g(1, n_1 + k, \theta') - 1 \right\} \right) \end{aligned} \tag{2.3}$$

Suppose $\theta \leq \theta'$. Then increasing difference of g implies $g(1, n_1 + k, \theta) - g(1, n_1 + k, \theta')$ is decreasing in k. Applying Lemma 1 to the LHS of (2.3) yields

$$\begin{aligned} &\sum_{n_1=0}^{N-1} \sum_{k=0}^{N-1-n_1} \alpha(n_1)\beta(k, n_1, 0)[g(1, n_1 + k, \theta) - g(1, n_1 + k, \theta')] \\ &> \sum_{n_1=0}^{N-1} \alpha(n_1) \left(\max \left\{ 0, \sum_{k=0}^{N-1-n_1} \beta(k, n_1, 0)g(1, n_1 + k, \theta) - 1 \right\} \right. \\ &\quad \left. - \max \left\{ 0, \sum_{k=0}^{N-1-n_1} \beta(k, n_1, 0)g(1, n_1 + k, \theta') - 1 \right\} \right) \end{aligned} \tag{2.4}$$

Now observe that for every $0 \leq n_1 \leq N - 1$, each term in the LHS of (2.4) must be less than or equal to the corresponding terms in the RHS. This contradicts (2.4). Hence $\theta > \theta'$. This shows that m_1 is monotone in θ.

Given $m_2(\theta, \cdot, n_1) > m_2(\theta', \cdot, n_1)$. Then $m_1(\theta') = m_2(\theta', \cdot, n_1) = 0$. If $m_1(\theta) > 0$ then by monotonicity, $\theta > \theta'$. If $m_1(\theta) = 0$, both type θ, θ' postpone the contribution decision to second period. $m_2(\theta, \cdot, n_1) > m_2(\theta', \cdot, n_1)$ then implies

$$\sum_{k=0}^{N-n_1-1} \beta(k, n_1, 0)(g(1, k+n_1, \theta) - g(1, k+n_1, \theta')) > 0.$$

Since g is monotone in θ, this implies $\theta > \theta'$. Hence $m_2(\theta, \cdot, n_1)$ is monotone in θ.

□

It follows from Lemma 2.2 that in any history-monotone PBE the high types contribute first, and the middle types decides whether to contribute based on the number of first period contributors. This pattern of contribution is the same as the one found in Gradstein[Gra92], however, in his paper agents contribute first in order to enjoy it first(like early members to some club), while in our model early contributors simply want to induce follow-ups.

To show equilibrium exists it thus suffices to show the existence of the cutoff types. That is, the type θ^1 which is indifferent between contributing in the first period and delay, and the type $\theta^2(n_1)$ which is indifferent between contributing and not contributing if there are n_1 contributers in the first period. The assumption of monotonicity in h^1_{-i} requires $\theta^2(n) \leq \theta^2(m)$ whenever $n > m$.

Theorem 2.1. Assume A1,A2. Symmetric monotone PBEs of the contribution game with commitment exist and are characterized by cutoffs $\{\theta^1, \{\theta^2(n_1)\}_{n_1=0}^{N-1}\}$:

$$m_1^{nr}(\theta) = \begin{cases} 1 & \text{if } \theta > \theta^1 \\ 0 & \text{else} \end{cases}$$

$$m_2^{nr}(\theta, h^1_i, h^1_{-i}) = \begin{cases} 1 & \text{if } h^1_i = 1 \text{ or } \theta > \theta^2(h^1_{-i}) \\ 0 & \text{else} \end{cases}$$

Proof. We will use a fixed point argument to show existence of cutoff points. Define

$$S = \{(s^1, s^2_0, ..., s^2_{N-1}) \in [0,1]^{N+1} : s^2_0 \geq s^2_1 \geq ... \geq s^2_{N-1}\}.$$

Observe that S is a compact and convex set. Define a correspondence $(T^1, T^2) : S \to S$ as follows: $T^2_{n_1}(s)$ is defined to be the solution to the following equation:

$$\sum_{k=0}^{N-n_1-1} \binom{N-n_1-1}{k} P(\theta > s^2_{n_1} | \theta < s^1)^k P(\theta < s^2_{n_1} | \theta \leq s^1)^{N-n_1-1-k} g(1, n_1+k, s) - 1 = 0. \tag{2.5}$$

It is well defined since (2.5) is continuous and increasing in s, and by A1 when (2.5) evaluated at $\theta = 0$ it is negative, when evaluated at $\theta = 1$ it is positive. Furthermore, since (2.5) is continuous in all variables, T^2 is upper-hemicontinuous. To verify that T^2 satisfies $T^2_n(s) \geq T^2_m(s)$ whenever $n < m$, let $\theta = T^i_n(s)$ be the solution to (2.5) with $n_1 = n$. Then since $s^2_m \leq s^2_n$,

$$\begin{aligned}
&\sum_{k=0}^{N-m-1} \binom{N-m-1}{k} P(\theta > s^2_m | \theta < s^1)^k P(\theta < s^2_m | \theta \leq s^1)^{N-m-1-k} g(1, m+k, T^i_n(s)) - 1 \\
>&\sum_{k=0}^{N-n-1} \binom{N-n-1}{k} P(\theta > s^2_n | \theta < s^1)^k P(\theta < s^2_n | \theta \leq s^1)^{N-n-1-k} g(1, n+k, T^i_n(s)) - 1 \\
=&0.
\end{aligned}$$

Hence $T^i_n(s) > T^i_m(s)$ whenever $m > n$.

Finally, define $T^1(s)$ for $s \in S$ to be the solution to

$$\sum_{n_1=0}^{N=1} \sum_{k=0}^{N-1-n_1} \alpha(n_1)\beta(k, n_1, 1) g(1, n_1+k, \theta) - \sum_{n_1=0}^{N-1} \alpha(n_1) \max\left\{0, \sum_{k=0}^{N-1-n_1} \beta(k, n_1, 0) g(1, k+n_1, \theta) - 1\right\} = 1, \tag{2.6}$$

where $\alpha(n_1) = \binom{N-1}{n_1}(1 - F(s^1))^{n_1} F(s^1)^{N-1-n_1}$ and $\beta(k, n_1, j)$ is given by lemma 1, with $p' = P(\theta > s^2_{n_1+1} | \theta \leq s^1)$. Since $s \in S$, $\beta(k, N-2, 1)$ FOSDs(but not necessarily strict FOSD) $\beta(k, N-2, 0)$.

By A1 and Lemma 1, (2.6) when evaluated at $\theta = 0$ is negative, when evaluated at $\theta = 1$ is positive. Furthermore, the derivative of (6) with respect to θ is

$$\sum_{n_1=0}^{N-1} \sum_{k=0}^{N-1-n_1} \alpha(n_1)\beta(k, n_1, 1) \frac{\partial g(1, n_1+k, \theta)}{\partial \theta} - \sum_{n_1 \geq K(\theta)}^{N-1} \alpha(n_1) \sum_{k=0}^{N-1-n_1} \beta(k, n_1, 0) \frac{\partial g(1, n_1+k, \theta)}{\partial \theta} \geq 0$$

where $K(\theta)$ is the smallest integer n_1 such that $\sum_{k=0}^{N-1-n_1} \beta(k, n_1, 0) g(1, k+n_1, \theta) - 1 > 0$. Hence the solution to (2.6) exists and is an interval. Accordingly, $(T^1, T^2) : S \to S$ is a upper-hemicontinuous convex-valued correspondence on the compact and convex set S. The Kakutani fixed point theorem implies T has a fixed point. By Lemma 2, the fixed points of T are exactly the set of monotone symmetric PBEs. □

Remark 4. Note that if $e \in T(e)$ then $T(e)$ must be a singleton, because for any monotone PBE, by Lemma 1 $\beta(k, N-2, 1)$ strictly FOSDs $\beta(k, N-2, 0)$, which implies the derivative of (2.6) w.r.t. θ is strictly positive, so the solution to (2.6) is unique.

Corollary 3. In any monotone PBE, $\theta^2(N-1) < \theta^1$, $\theta^2(0) > \theta^1$.

Proof. The first claim is implied by Lemma 1. For the second, note that if $\theta^2(0) \leq \theta^1$, then for every n, $\beta(k, n, 1)$ strictly FOSD $\beta(k, n, 0)$(Theorem 1 shows $\theta^2(n) < \theta^2(m)$ for $n < m$. Hence when (2.6) is evaluated at θ^1 the value will be positive instead of zero, a contradiction. □

2.3.4 Two Period Contribution with Cheap Talk

In this section we characterize the set of monotone PBEs of the contribution game with cheap talk. Again let $\alpha(n_1)$ be the probability such that the number of other contributers in the first period is n_1 and let $\beta(k; 0, n^1), \beta(k; 1, n^1)$ be the conditional probabilities of second period contributions on the number of first period contributors.

The key lemma here we want to show, for the class of monotone PBEs is still that $\beta(k; 1, n^1)$ strictly FOSDs $\beta(k; 0, n^1)$, which is the main driving force of early contribution: that it induces middle types to contribute later. Intuitively, we may argue as follows: since $\theta \leq \theta^1_r$ will never contribute, observing a contribution in period 1 means $\theta_i > \theta^1_r$, hence i is more likely to contribute in the second period. So the incentive to contribute becomes strictly higher if the observed first period contribution is higher. However, since period 1 contribution is simply a cheap talk, it can also be that the agents use 1 to signal low type and 0 high type. To focus on equilibria which are more informative(if there is one), in addition to assume monotonicity of $m^2(\theta, \cdot, n)$ on n, we also need to assume $m^1(\theta)$ is monotonically increasing on θ. In particular, we assume there exists $\theta^1 \in (0, 1)$ such that $m^1(\theta) = 0$ for

$\theta < \theta^1$ and $m^1(\theta) = 1$ for $\theta > \theta^1$. So we are again characterizing the set of monotone equilibria, except that monotonicity is not implied by weaker conditions as in the no cheap talk case.

Lemma 2.3. Assume A1,A2. In the game with cheap talk, in any monotone PBE, $\beta(k; 1, n^1)$ FOSDs $\beta(k; 0, n^1)$. sFOSD holds for $n^1 = 1, ..., N-2$.

Remark 5. The proof strategy is as follows: First we show if an agent contributes in the first period, then the posterior probability that he is going to contribute in the second period is higher than that if he does not contribute in the first period. Then we show that given any first period action of an agent, he is more likely to contribute in the second period if he sees more contributors in the first period. Finally we show that by contributing in the first period, one induces the other agents to contribute in the second period.

Proof. Fix a PBE $\{m_i^1(\theta), m_i^2(\theta, h_i^t, h_{-i}^t)\}$. Let θ^1 be such that $\{\theta : m^1(\theta) = 1\} = [\theta^1, 1]$. Monotonicity of m^2 with respect to θ implies $\{\theta : m^2(\theta, 1, n) = 1\}$ and $\{\theta : m^2(\theta, 0, n) = 1\}$ are two connected intervals with right endpoint 1. Denote them by $[a(n), 1]$ and $[b(n), 1]$ respectively, where $a(n) \le b(n)$.(What the marginal type a, b do are not essential, they could as well be half open intervals.) Let $p(1, n) = \mathbb{P}(m^2(\theta, 1, n) = 1 | m^1(\theta) = 1)$, $p(0, n) = \mathbb{P}(m^2(\theta, 0, n) = 1 | m^1(\theta) = 0)$.

We then show that first period contribution indicates a higher chance to contribution in the second period:

Claim 1 $p(1, n) > p(0, n)$

By the definition of conditional probability, this writes as

$$\frac{1 - \max\{a(n), \theta^1\}}{1 - \theta^1} > \frac{\theta^1 - \min\{b(n), \theta^1\}}{\theta^1}, \tag{2.7}$$

which is verified by observing that when $a(n) > \theta^1$ the right hand side is zero, and when $a(n) \le \theta^1$ the left hand side is one, and by assumption A1, $a(n) \neq 1$ and $b(n) \neq 0$ for all n.

Claim 2 $a(n+1) < a(n)$, $b(n+1) < b(n)$.

To see this, fix a type θ. Consider his continuation payoff for contributing in $t = 2$ under history $h^1 = (0, n+1)$ and under $h^1 = (0, n)$ respectively:

$$E[u(1, m^2, \theta)|h^1 = (0, n+1)] = E[g(1, X_1 + Y_1, \theta)] - 1, \tag{2.8}$$

$$E[u(1, m^2, \theta)|h^1 = (0, n)] = E[g(1, X_2 + Y_2, \theta)] - 1, \tag{2.9}$$

where $X_1 \sim \text{Bin}(n+1, p(1, n))$, $Y_1 \sim \text{Bin}(N-2-n, p(0, n+1))$, and $X_2 \sim \text{Bin}(n, p(1, n-1))$, $Y_2 \sim \text{Bin}(N - n - 1, p(0, n))$. It then follows from Claim 1 that $X_1 + Y_1$ strictly FOSDs $X_2 + Y_2$. Since $\{\theta : m^2(\theta, 0, n+1) = 1\} = [b(n+1), 1]$ is the set of θ on which (2.8) is positive, and $\{\theta, m^2(\theta, \cdot, n) = 1\} = [b(n), 1]$ is the set of θ on which (2.9) is positive, one part of claim is established. The same argument applies to the other part.

A direct implication by the expression in (2.7) is that

$$p(1, n) \geq p(1, n-1) \text{ and } p(0, n+1) \geq p(0, n). \tag{2.10}$$

Now observe that $\beta(k, n, 1)$ is the probability of $X_1 + Y_1 = k$, where $X_1 \sim \text{Bin}(n, p(1, n))$, $Y_1 \sim \text{Bin}(N - n - 1, p(0, n+1))$, and $\beta(k|h^1 = (0, n))$ is the probability of $X_2 + Y_2 = k$, where $X_2 \sim \text{Bin}(n, p(1, n-1))$, $Y_2 \sim \text{Bin}(N - n - 1, p(0, n))$. Hence (2.10) shows FOSD for all n. To show sFOSD for $n = 1, ...N - 2$, it suffices to show that either one of the inequalities in (11) holds strictly. To this end, note that if $a(n) \geq \theta^1$, then by Claim 2 we have $p(1, n) > p(1, n-1)$. If $a(n) < \theta^1$, then by Claim 1, $1 = p(1, n) > p(0, n)$. This will imply $b(n) > 0$, and Claim 2 will thus imply $p(0, n+1) > p(0, n)$.

□

The argument does not apply to the case $n = N - 1$ as Y_1, Y_2 will be zero.

Lemma 2.4. Assume A1,A2. Given a profile of symmetric monotone equilibrium. Suppose

$$\beta(k, 1, N-1)\text{sFOSDs}\beta(k, 0, N-1)$$

then a type θ who contribute with positive probability on the equilibrium path contribute in the first period. Suppose sFOSD fails for $n_1 = N - 1$, then any type above $\tilde{\theta}(N-2)$ contribute in the first period. In this case, $a(N-1) = \tilde{\theta}(N-1), a(N-2) = \theta^1 < \tilde{\theta}(N-2)$.

Proof. Suppose sFOSD holds for all n_1. For any type θ, the expected payoffs for contributing in the first period(EU_1) and delay contribution(EU_2) are

$$EU_1(\theta) = \sum_{n_1=0}^{N-1} \alpha(n_1) \max\left\{0, \sum_{k=0}^{N-1} \beta(k,1,n_1) g(1,k,\theta) - 1\right\}$$
$$EU_2(\theta) = \sum_{n_1=0}^{N-1} \alpha(n_1) \max\left\{0, \sum_{k=0}^{N-1} \beta(k,0,n_1) g(1,k,\theta) - 1\right\}$$

That type θ contribute with positive probability implies there exists some n_1 such that

$$\sum_{k=0}^{N-1} \beta(k,n_1,1) g(1,k,\theta) - 1 > 0.$$

Since sFOSD holds for n_1, $EU_1(\theta) > EU_2(\theta)$ and $EU_1(\theta) > 0$. Hence it is optimal for the type θ to contribute in the first period.

Suppose sFOSD fails for $n_1 = N-1$. The expression in (2.7) then implies $p(1, N-1) = p(1, N-2) = 1$, which implies $a(N-1) < a(N-2) \leq \theta^1$. If $a(N-2) < \theta^1$, then any type $\theta \in (a(N-2), \theta^1)$ will be better off contributing in period 1 since sFOSD holds for $N-2$ and $\sum_k \beta(k, 1, N-2) g(1, k, \theta) - 1 > 0$. Hence we must have $a(N-2) = \theta^1$. Furthermore, $p(1, N-1) = 1$ implies $a(N-1) = \tilde{\theta}(N-1) < \theta^1$. Since $p(1, N-2) = 1$ and $p(0, N-1) = P(\tilde{\theta}(N-1) < \theta | \theta < \theta^1) > 0$, in a history $h = (1, N-2)$ there will be at least $N-2$ final contributors, and $N-1$ with a positive probability, hence $a(N-2) < \tilde{\theta}(N-2)$. □

The converse of Lemma 2.4 does not hold: It is not necessarily true that in any monotone PBE where sFOSD hold for all n, all agents who contribute in the first period will contribute with positive probability.[4] Lemma 2.3 and 2.4 imply the monotone PBEs of the cheap talk game fall into three categories

- a. $\theta^1 = a(N-1) < a(N-2) < ... < a(0)$
- b. $\theta^1 < a(N-1) < a(N-2) < ... < a(0)$
- c. $a(N-1) < a(N-2) = \theta^1 < ... < a(0)$

[4]Because contribution is cheap talk, low type agents contributing in period 1 can be supported as an equilibrium even though their utility is zero with probability one.

Category a,b are cases where strict FOSD holds for all n. In category a, $\theta^1 = a(N-1)$ implies $p(N-1,1) = 1$, which implies all $\theta > \tilde{\theta}(N-1)$ will contribute in the second period, so $\theta^1 = \tilde{\theta}(N-1)$. In category b, some players who contribute in the first period will never contribute in the second period. In category c, all types $a(N-1) < a(N-2)$ are indifferent between contributing in the first period or not, because they will contribute in the second period only if $n^1 = N-1$, and sFOSD does not hold in this case.

In order to characterize the monotone PBE of the cheap talk game, by lemma 2.3 and 2.4 what is left is to show existence the PBE that belongs to the above three categories. Similar to the game with commitment, it suffices to show the existence of cutoff types who are indifferent between contribution or not in period 1 and the types $\theta^2(n_1)$ who contributes in period 1 and is indifferent between keeping the contribution or withdrawing it in period 2.

Theorem 2.2. Assume A1,A2. The set of monotone symmetric PBEs of the contribution game with cheap talk is non-empty and is characterized by cutoffs $(\theta^1, \{a(n_1), b(n_1)\}_{n_1=0}^{N-1})$.

$$m_1^r(\theta) = \begin{cases} 1 & \text{if } \theta > \theta^1 \\ 0 & \text{else} \end{cases}$$

$$m_2^r(\theta, h_i^1, h_{-i}^1) = \begin{cases} 1 & \text{if } h_i^1 = 1, \theta_i > a(h_{-i}^1) \\ & \text{or } h_i^1 = 0, \theta_i > b(h_{-i}^1) \\ 0 & \text{else} \end{cases}$$

For each category (a),(b),(c) mentioned above, there exist cutoffs that satisfy that category,

Proof.

- (Category a,b) The first period cutoff is $\tilde{\theta}$. We only need to show the existence of cutoff types $a(n_1) \in (\theta^1, 1)$ for $n_1 = 0, ..., N-1$. Given that there are n_1 other contributers in the first period, $\theta^2(n_1)$ solves

$$\sum_{k=0}^{n_1} \binom{n_1}{k} \left(\frac{1-F(\theta)}{1-F(\theta^1)}\right)^k \left(\frac{F(\theta)-F(\theta^1)}{1-F(\theta^1)}\right)^{n_1-k} g(1,k,\theta) - 1 = 0. \tag{2.11}$$

When (11) is evaluated at $\theta = 1$ we get $g(1,0,1) - 1 > 0$, when evaluated at $\theta = \theta^1$ we get $g(1, n_1, \theta^1) - 1 \leq 0$ for $n_1 < N-1$ by the definition of θ^1. Hence there exists at least one solution to (2.11) in the interval $[\theta^1, 1)$. In particular, when $n_1 = N-1$, a solution $a(N-1) \in [\theta^1, 1)$ exists.

To constitute a monotone equilibrium, we need $\{\theta^2(n_1)\}$ to be decreasing in n_1. For any $n_1 > 0$, let $a(n_1)$ be a solution to (2.11). Then (2.11) evaluated at $a(n_1)$ when $n_1 - 1$ is in place of n_1 is negative.

$$\sum_{k=0}^{n_1-1} \binom{n_1-1}{k} \left(\frac{1-F(a(n_1))}{1-F(\theta^1)}\right)^k \left(\frac{F(a(n_1))-F(\theta^1)}{1-F(\theta^1)}\right)^{n_1-1-k} g(1,k,a(n_1)) - 1 < 0$$

Hence a solution $a(n_1-1) \in (a(n_1), 1)$ exists to the above equation. Finally, $\tilde{\theta}(N-1)$ is a solution to (2.11) when $n_1 = N-1$ and $\theta^1 = \tilde{\theta}(N-1)$. This deals with the strategies on the equilibrium path. The cutoffs $\{b(n_1)\}$'s are found in a similar manner.

To check incentives to stay on the equilibrium path, note that since $\theta^1 < \tilde{\theta}(N-1)$, type $\theta_i < \theta^1$ agents have no incentive to deviate since they will not contribute anyways. For type $\theta > \theta^1$, they have no incentive to deviate at period 1 either since sFOSD holds and they can always switch to $m_i = 0$ in the second period. In the second period, the way we select $\{a(n_1)\}$ already guarantees that the second period action is optimal.

- (Category c) By Lemma 2.4, in this case $a(N-1) = \tilde{\theta}(N-1) < a(N-2) = \theta^1 < \tilde{\theta}(N-2)$. Hence a solution to (11) in $(\theta_1, 1)$ when $n_1 = N-2$ exists. The arguments for finding other cutoffs are the same as before. As for the incentives whether to deviate, type $\theta \leq \tilde{\theta}(N-1)$ surely won't. Types $\theta \in (a(N-1), \theta^1)$ will not either. This is because they will contribute in period 2 only if $n_1 = N-1$, and since $p(1, N-2) = 1$, whether this agent contribute will not affect the final decision of the $N-1$ first period contributors. The second period actions are optimal by the definition of $a(n_1)$'s.

□

Example 1. Consider $N = 2$. The cutoff given by $\{\theta^1, \theta^2(1), \theta^2(0)\} = \{\tilde{\theta}(1), \tilde{\theta}(1), \tilde{\theta}(0)\}$ determines an equilibrium of the game with cheap talk that is ex-post efficient and ex-post individually rational. This amounts to that, in the first period, every agent who is willing to contribute as along as the other contributes say Yes. In the second period, both agents contribute if both says Yes, and one withdraws if one is the only one saying so and his type is unwilling to contribute along.

The cutoff θ^1 of the game with commitment is determined by

$$(1-F(\theta^2(1)))g(1,1,\theta)+F(\theta^2(1))g(1,0,\theta)-1 = (1-F(\theta))\max\{0, g(1,1,\theta)-1\}+F(\theta)\max\{0, g(1,0,\theta)-1\}$$

The equation when evaluated at $\theta \le \theta^2(1)$ is negative, when evaluated at $\theta \ge \theta^2(0)$ is positive, hence $\theta \in (\theta^2(1), \theta^2(0))$. Moreover, the derivative with respect to θ is

$$(1-F(\theta^2(1)))\frac{\partial g(1,1,\theta))}{\partial \theta}+F(\theta^2(1))\frac{\partial g(1,0,\theta)}{\partial \theta}+f(\theta)(g(1,1,\theta)-1)-(1-F(\theta))\frac{\partial g(1,1,\theta)}{\partial \theta} > 0,$$

hence the solution is unique. The equilibrium outcomes is neither ex-post efficient nor ex-post individually rational. For example, if both $\theta_i \in (\theta^2(1), \theta^1)$ then no one will contribute in the end, and if $\theta_1 < \theta^2(1)$ while $\theta_2 \in (\theta^1, \theta^2(0))$ then in the end only agent 2 contributes but he will get negative utility.

The game with cheap talk can approximately achieve efficiency when the number of agents goes to infinity more easily than the game with commitment.

Theorem 2.3. If $\lim_{N\to\infty} NF(\tilde{\theta}(N-1)) = 0$, then the equilibrium with $\theta_r^2(N-1) = \tilde{\theta}(N-1)$(type a) is asymptotically ex-post efficient, in the sense that

$$\lim_{N\to\infty} \mathbb{P}(\{\theta : m(\theta) \text{ is efficient at } \theta\}) = 1$$

Proof. Note that for any such equilibrium, $m(\theta)$ is efficient whenever $\theta_i > \tilde{\theta}(N-1)$ for all i. The probability that there exists at least one agent with $\theta_i \le \tilde{\theta}(N-1)$ is given by $1-(1-F(\tilde{\theta}(N-1)))^N$, which is approximately $NF(\tilde{\theta}(N-1))$[5] and converges to zero as $N \to \infty$. □

Such condition does not guarantee efficiency for games with commitment by Corollary 3.

Our result differs from Gradstein's[Gra92] in that in our mode, players' incentives to contribute are purely driven by the ability to induce follow-ups rather than to enjoy the product earlier. Also, in our model the inefficiency could come from both kinds of mis-coordination: either too few people contribute or too many people contribute.

[5]This follows from $\lim_{x\to 0} \log(1-x)/x = 1$ and $F(\tilde{\theta}(N-1))$ converges to 0.

2.3.5 Multiple Period Cheap Talk

Instead of completely characterizing the equilibrium outcomes like we did to $T = 2$ models, we show that the PBEs of the game with commitment are inefficient and construct an efficient equilibrium for the game with cheap talk.

The equilibrium we construct for the cheap talk game implements the efficient allocation in Proposition 2.1. We simply make everyone who can possibly contribute for some realization of θ(in the ex-post sense) signal $a^1 = 1$. And then we gradually let agents drop out, until a maximal set of agents who are willing to contribute remains.

Theorem 2.4. Assume A1,A2. Assume individual players' contributions are observable.

1 For any T, all PBEs outcome of the game with commitment are not ex-post efficient.

2 For $T \geq N$, there exists a monotone PBE of the game with cheap talk whose outcome is ex-post efficient.

Proof. Let $\sigma = \{m_i^t\}$ be an equilibrium strategy profile of the game with commitment. Consider a profile $\theta \in \Theta^N$ such that $\theta_i \in (\tilde{\theta}(N-1), \tilde{\theta}(N-2))$ for all i. Then ex-post efficiency requires under σ everyone in this interval contributes. However, let i be the earliest contributor on the equilibrium path of σ. Say $m_i^t(\theta_i, h^{t-1}) = 1$. In this period, his continuation expected payoff is positive only if he believes in the future everyone is going to contribute with probability one. But this is impossible since by definition the history h^{t-1} is that no one ever contributes before, and the posterior probability that there are some agents whose type is below $\tilde{\theta}(N-1)$ given this history is thus still positive. Hence player i's action at t given by σ is not sequentially rational, contradicting that σ is an equilibrium.

Now we proceed to prove the claim about game with cheap talk. A history up to date t is $h^t = (h_1^t, ..., h_N^t)$, where $h_i^t = (a_i^1, ..., a_i^t)$. h_i^t is called consistent if $a_i^t \geq a_i^{t'}$ whenever $t > t'$. Hence, the beliefs in any consistent history is the same as that on the equilibrium path, and the beliefs in any inconsistent history can be defined in an arbitrary way. We are going to construct a strategy and a belief such that the strategy is to contribute and refund in the most conservative way, while the belief is such that any deviator is assumed to be the low types who will never contribute in the end. To define it formally, fix a history h^t and a player i, let

$$n_t = \sum_{j \neq i, h_j^t \text{ is consistent}} a_j^t.$$

The dependence of n_t on i will be omitted in the following to keep notations cleaner, since we are describing a symmetric strategy profile. Note that n_t is decreasing in t. The strategy for player i(and hence all other players) is defined as

$$m^1(\theta) = \begin{cases} 1 & \text{if } \theta \geq \tilde{\theta}(N-1) \\ 0 & \text{else} \end{cases}$$

$$m^t(\theta, h^{t-1}) = \begin{cases} 1 & \text{if } \theta \geq \tilde{\theta}(n_{t-1}) \\ 0 & \text{else} \end{cases}$$

for each $t = 2, ..., T$, where n_{t-1} is the number of other agents who has not deviated from the induced path before and who contributes in period $t-1$.

The posterior probability density for θ_j held by player i at each history h^t is defined, on the regions where the density is non-negative, as

$$p(\theta_j | h^t) = \begin{cases} \frac{f(\theta_j)}{1-F(\tilde{\theta}(n_{t-1})} & \text{if } a_j^{t-1} = 1, h_j^t \text{ is consistent} \\ \frac{f(\theta_j)}{F(\tilde{\theta}(n_{k-1}) - F(\tilde{\theta}(n_{k-2})))} & \text{if } a_j^k = 1, a_j^{k+1} = 0 \text{ for some } k \leq t-2, h_j^t \text{ is consistent} \\ \frac{f(\theta)}{F(\tilde{\theta}(N-1))} & \text{if } h_j^t \text{ is inconsistent.} \end{cases}$$

Now we verify that the strategy belief pair $\{m^t, p(\cdot|h^t)\}$ defined above forms a monotone PBE in the game where $T \geq N$ whose equilibrium outcome equals the allocation defined in Section 2.

First note that, on the equilibrium path, if $n_t = n_{t+1}$ for some t then the equality will continue to hold until $t = T$. Since n_t is monotonely decreasing and $T \geq N$, the equilibrium outcome is indeed $m(\theta)$ except for profiles with some agent whose type equal the cutoff types $\tilde{\theta}(n)$.

Second, we check there is no incentive to deviate for any given history. Suppose h^t is a history where player i hasn't deviated before. There are two kinds of deviations, one is to deviate in a way that the player himself's history is consistent. This kind of deviation is not profitable because $m(\theta)$ is dominant strategy incentive compatible. Another kind of deviation is to deviate in a way such that h_i^t is inconsistent-jumping from 0 to 1 at some period. In player i's belief, for players who haven't deviated yet, this drives down their contribution since they now treat i the same as the lowest type. For players who have deviated, player i already believes them be the lowest type. Hence overall the continuation expected payoff also decreases. Suppose h^t is a history where player i has deviated before. Suppose h_i^t is consistent. Then player i has either delayed refund or refunded earlier than the prescribed strategy. In the first case since n_t is not larger than his cutoff number, he should follow the prescribed strategy and quite in the next period. In the second case, the remaining players already believed that he will not contribute, he should again follow the prescribed strategy. Suppose h_i^t is inconsistent, then again his future actions will not affect others' actions, and he should again follow the prescribed strategy, that is, to contribute whenever $\theta_i > \tilde{\theta}(n_t)$ as if there are n_t players who's going to contribute. □

Remark 6. To show the PBE just constructed is sequential, consider the sequence of completely mixed strategies indexed by $\epsilon > 0$:

$$m^t(\theta, h^{t-1}) = \begin{cases} 1 \quad \text{with prob } 1-\epsilon, & \theta \geq \tilde{\theta}(n_{t-1}) \\ 1 \quad \text{with prob } \epsilon^2, & \tilde{\theta}(N-1) \leq \theta < \tilde{\theta}(n_{t-1}) \\ 1 \quad \text{with prob } \epsilon, & \theta < \tilde{\theta}(N-1) \end{cases}$$

2.4 The Continuum Choice Model

In binary models, cheap talk is useful as it avoids mis-coordination and the free riding problem is assumed away, because when one contributes 0 one gets nothing.

With multiple contribution levels, free riding problem kicks in and this turns out to undermine the usefulness of cheap talk entirely.

Intuitively, if the strict FOSD lemma holds, agents have an incentive to induce others to contribute more by making a high contribution in the first period, and switching back to a lower level m^1 while free riding others' high contributions. This argument is vividly illustrated in a two period binary type model.

2.4.1 The Environment

We illustrate the phenomena within a simplified setting. 2 agents contribute jointly to a project over 2 periods. Contribution decision is continuous for each agent: $m_i \in M = [0, \overline{m}]$ for some $\overline{m} < \infty$. The utility function and production function are the same as before.

The set of uncertainty is $\theta_i \in \{\theta_L, \theta_H\}$ for $i = 1, 2$ with prior $f(\theta_L) = p$. Since there are two types, we modify A2 to

Assumption 3. For each $\theta \in \{\theta_H, \theta_L\}$ and each $m_{-i} > 0$, there exists $m_i > 0$ such that

$$g(m_i, m_{-i}, \theta) - m_i > 0$$

In addition, we also assume strict concavity of g with respect to m_i:

Assumption 4. $g(m_i, m_{-i}, \theta)$ is strictly concave in m_i for all $m_{-i} > 0$ and all θ.

2.4.2 Contribution Game With Cheap Talk

We now show that free riding completely eliminates the advantage of cheap talk. Consider a two stage model with cheap talk in which $A_i = [0, \overline{m}]$[6]. It is equivalent to adopt a binary message space, since there are only two types.

Lemma 2.5. Assume A1,A3,A4. In any symmetric PBE where $m_i^2(h_i, h_{-i}, \theta_H) > m_i^2(h_i, h_{-i}, \theta_L)$ for all history h on the equilibrium path, then for all i, the posterior $\mu_i(\theta_j | m_j^1)$ must be the same as the prior.

Proof. Suppose to the contrary that there exists such a symmetric PBE in which $m_i^1(H) \neq m_i^1(L)$. Suppose player i is of type L and $m_i^1(\theta_L) < m_i^1(\theta_H)$. Then conditional on $m_i^1 = m_i^1(\theta_L)$ player j in the beginning of the second period believes player i is of low type with

[6] It is completely fine to use a binary set for cheap talk since there are only two types. The currently presentation is more consistent with the comparison to games with commitment.

probability one. Then in equilibrium $m_j^2(m_i, \theta_j)$ satisfies

$$m_j^2(m_i^1(\theta_L), \theta_j) \in \arg\max_{m_j} g(m_j, m_i^2(m_j^1(\theta_j), \theta_L), \theta_j) - m_j.$$

Let $m_j(H) = m_j^2(m_i^1(\theta_L), \theta_H)$, $m_j(L) = m_j^2(m_i^1(\theta_L), \theta_L)$. Then player i's expected utility given by the proposed equilibrium is

$$p \max_{m_i}(g(m_i, m_j(H), \theta_L) - m_i) + (1-p) \max_{m_i}(g(m_i, m_j(L), \theta_L) - m_i). \tag{2.12}$$

Consider a deviation of type θ_L player i to $m_i^1(\theta_H)$. Then j's belief in the beginning of period 2 will be that player i is of high type with probability one. Thus $m_j^2(m_i, \theta_j)$ satisfies

$$m_j^2(m_i^1(\theta_H), \theta_j) \in \arg\max_{m_j} g(m_j, m_i^2(m_j^1(\theta_j), \theta_H), \theta_j) - m_j.$$

Let $m_j(H)' = m_j^2(m_i^1(\theta_H), \theta_H)$, $m_j^2(L)' = m_j^2(m_i^1(\theta_H), \theta_L)$. Since by assumption second period action is monotone in type, $m_i^2(m_j, \theta_H) > m_i^2(m_j, \theta_L)$. Since player i are now expected to take a higher action after his deviation to $m_i(H)$, it follows from A1, A3 and A4 that $m_j(H)' > m_j(H)$ and $m_j(L)' > m_j(L)$.

But then player i's expected payoff in the beginning of period 1 is

$$p \max_{m_i}(g_i(m_i, m_j(H)', \theta_L) - m_i) + (1-p) \max_{m_i}(g_i(m_i, m_j(L)', \theta_L) - m_i), \tag{2.13}$$

By A3, (2.13) is larger than (2.12).

This shows that the low type deviating to $m_i^1(H)$ is profitable, contradicting that the proposed strategy profile is an equilibrium. $\square$

The intuition behind this lemma is that the player always have incentive to pretend to be the high type, so that he can free ride other player's increased effort even if he is of low type.

Theorem 2.5. Suppose A1,A3 A4. Then all symmetric PBEs of the cheap talk game have the same equilibrium allocation as some BNE of the static contribution game.

Proof. By Lemma 2.5 every PBE $\{m^1, m^2\}$ in which the on path period 2 contribution is different is optimal when the belief is the prior, hence $(m_i^2(m^1, \theta_H), m_i^2(m^1, \theta_L))_{i \in \{1,2\}}$ constitutes a BNE.

Suppose there exists a PBE such that both types contribute the same amount m in period 2 on the equilibrium path. If $m \in (0, \overline{m})$, then since g satisfies strict increasing difference in (m_i, θ), together with A4 it implies m can not be a best response to θ_L, θ_H simultaneously, a contradiction. Hence it is only possible that $m \in \{0, \overline{m}\}$. But then

$$g(m, m, \theta) - m \geq g(m', m, \theta) - m'$$

for all $m' \in [0, \overline{m}]$, implying that $(m_H, m_L) = (m, m)$ for $i = 1, 2$ also constitutes a BNE for the static game. □

2.4.3 Contribution Game With Commitment

Previous section shows that cheap talk does not convey any information when the action set is a continuum. On the other hand, if signalling is costly, high type can then credibly signal that he is high type by commit to contribute, and this induces low type to contribute in corresponding periods. We now give such an example in which efficiency improves over the equilibrium in the cheap talk game, which is the BNE in the one shot game without communication.

Take $M = [0, 9]$, $\Theta = \{0.5, 8\}$ and $g(m_1, m_2, \theta) = \theta m_1^{0.5} m_2^{0.5}$. Probability of low type is $p = 0.5$. One can verify that the setting satisfies Theorem 2.5.

The BNEs of the static game includes $(m_L, m_H) = (0, 0)$, and another equilibrium with positive contribution:

$$m_L = \left(\frac{p\theta_L}{2} + \frac{1-p}{2} (\theta_L \theta_H)^{1/2} \right)^2$$
$$m_H = \left(\frac{p}{2} (\theta_L \theta_H)^{1/2} + \frac{(1-p)\theta_H}{2} \right)^2$$

Figure 1 gives the non-zero PBE allocation and its social welfare of the game with cheap talk , which is simply that of the static game by Theorem 2.5. The ex-ante social surplus is roughly

$$SW = 1/4(-25/64 + 425/64 + 425/64 + 350/4) \sim 25$$

In what follows we construct a PBE in the contribution game without cheap talk that

	θ_L	θ_H
θ_L	$(5/8)^2, (5/8)^2$	$(5/8)^2, (5/2)^2$
θ_H	$(5/2)^2, (5/8)^2$	$(5/2)^2, (5/2)^2$

BNE

	θ_L	θ_H
θ_L	$-25/64$	$425/64$
θ_H	$425/64$	$350/4$

Social Welfare

Figure 2.1: Cheap Talk

ex-ante dominates the BNE of the one-shot game. Before we start, note that since the action set is a continuum, unlike the binary case where the agent becomes inactive once he decides to contribute, an agent may split his contribution over two periods. To have a parallel comparison to Section 2.3.2 we assume once a player chooses to contribute $m_i^1 > 0$ in the first period he becomes inactive. We can allow players to splic contributions over multiple periods and obtain a similar construction, with the sacrifice of becoming more notationally involved.

Consider the following strategy profile.

$$m_i^1(\theta_L) = 0$$

$$m_i^2(\theta_L, m_i^1, m_{-i}^1) = \begin{cases} \frac{\theta_L^2}{4} m_{-i} & m_i^1 = 0 \\ 0 & m_i^1 > 0 \end{cases}$$

$$m_i^1(\theta_H) = 9$$

$$m_i^2(\theta_H, m_i^1, m_{-i}^1) = \begin{cases} 0 & \text{if } m_i^1 > 0 \text{ or } m_{-i}^1 = 0 \\ \max\{9, \frac{\theta_H^2}{4} m_{-i}\} & \text{else} \end{cases}$$

The belief is given by the the posterior p such that

$$p(\theta_L; m_{-i}^1 \neq 9) = 1.$$

That is, whenever a player does not commit to contribute 9 initially, he is believed to be the low type. The belief does not matter since a player who already contributes is inactive by the assumption of the model.

This is a profile in which the high type commits to contribute in advance. Low type follows to contribute whenever he observes a commitment to contribute.

To check it is PBE, first note that the low type's equilibrium payoff is

$$U(\theta_L) = 0.5 \times 0 + 0.5(0.5 \times 3 \times \frac{3}{4} - \frac{9}{16}) = \frac{9}{32}.$$

If the low type deviates to $m_i > 0$, he gets

$$U(m_i; \theta_L) = 0.5(0.5 m_i^{1/2} \times \frac{m_i^{1/2}}{4}) + 0.5(0.5 \times m_i^{1/2} \times 3) - m_i$$

Solving FOC, we see the optimal deviation is $m_i^* = (2/5)^2$ which yields a payoff of

$$U(m_i^*, \theta_L) = 0.15 < U(\theta_L).$$

Hence low type will not deviate in the first stage. The second stage actions are already best responses for the given history so he will not deviate either.

Secondly, note that the high type's equilibrium payoff is

$$U(\theta_H) = 0.5 \times 8 \times 3 \times \frac{1}{4} + 0.5 \times 8 \times 3 \times 3 - 9 = 30$$

If he deviates to m_i, he gets

$$U(m_i; \theta_H) = 0.5(8 \times m^{1/2} \times \frac{m^{1/2}}{4}) + 0.5(8 \times m^{1/2} \times 3) - m = 12m^{1/2},$$

hence choosing $m = 9$ is best response.

The allocation on the equilibrium path and social welfare is then The ex-ante social

	θ_L	θ_H
θ_L	$0, 0$	$9/16, 9$
θ_H	$9, 16/9$	$9, 9$

BNE

	θ_L	θ_H
θ_L	0	18
θ_H	18	126

Social Welfare

Figure 2.2: No Cheap Talk

surplus is roughly

$$SW = 1/4(0 + 18 + 18 + 126) = 40.5$$

There is a significant increase in social welfare if the signalling is costly.

2.5 Conclusion

This paper shows that for production functions with increasing differences, when contribution choices are binary it may be more efficient to have multiple stages of cheap talk rather than commited contribution. However, if contribution choices are not binary, incentives to free ride can become so severe that equilibria in games with cheap talk do not convey information. In this situation, multiple rounds of committed contribution helps to improve efficiency.

A large amount of dynamic contribution games we encounter in daily life are in the form of crowdfunding projects, and many of which allow people to withdraw or change contribution levels over time. On a normative side, allowing contributors to withdraw is argued as necessary for consumer protection, but our paper identifies situations that, in a social welfare perspective, such regulation may not be optimal. However, different projects have different production functions and different value structure(private, common, interdependent), it is of interest to further explore the differences of the two regimes and its welfare implications in those environments.

CHAPTER 3

Belief-Preserving Morphism and Bayes Nash Equilibrium

[1]

3.1 Introduction

Ever since Harsanyi's type space model become the foundation of incomplete information games, it has been asked what kind of type space is rich enough to capture certain solution concepts. For correlated rationalizable actions, it has been shown in Dekel,Fudenberg, and Morris[DFM06, DFM07](DFM 2006,2007) that two types have the same θ-hierarchy if and only if they have the same interim correlated rationalizable actions across all games, so the Mertens-Zamir universal type space is the universal space for correlated rationalizability. For independent rationalizable actions, it has been shown in Ely and Peski[EP06](EP2006) that two types have the same Δ-hierarchy if and only if they have the same interim independent rationalizable actions across all games, so the universal space over conditional beliefs is the universal space for independent rationalizability, which is larger than the space of infinite hierarchy of beliefs.

Our paper provides a characterization to the above type of questions when the solution concept is Bayes Nash equilibrium. Sadzik[Sad11] provided such a characterization for Polish type spaces, which states that two types have the same BNE across all games if and only if they have the same X-hierarchies. Our characterization, for countable type spaces, is in terms of a well-known concept: a type t_i in some type space T has the same equilibrium

[1] This chapter is co-authored with Ichiro Obara.

actions as some other type t_i' in some other type space T' across all games if and only if T and T' are identical. More precisely, there exists an *injective* belief preserving map from some subset of T containing t_i, to T' that maps t_i to t_i'. This condition has a direct game theoretical appeal: same BNE behavior for any two types from different type spaces is guaranteed if and only the type spaces containing the two types are identical, not just that they have the same θ hierarchy or Δ-hierarchy. So for Bayes Nash equilibrium, the interpretation is that all the information carried by a type in a type space is indispensable, in the sense that once the type space is mapped into a smaller one there will exists a game such that the equilibrium prediction is different.

We provide a direct proof using scoring rules and relatively elementary mathematical tools. It is a full characterization, which includes the case 1.the set of BNE actions of one type is contained in that of another(which is covered in Sadzik's paper), and the case 2.when the two types have exactly the same set of BNE actions across all games. Finally, we apply our characterization to show the non existence of universal type spaces for BNE, we then weaken the conditions for universality and construct a universal space for BNE under these weaker requirements(compared to parallel requirements for rationalizability).

Related Literature Type space characterization of other solution concepts than BNE is treated in DFM(2006,2007)[DFM06, DFM07], EP2006[EP06]. A characterization for BNE is in Sadzik[Sad11]. Bergemann and Morris[BM15] uses scoring rule to demonstrate a similar result to ours for correlated BNE. What they achieved is a ranking of finite information structures, according to when the set of correlated BNE of one information structure is contained in that of another, for all finite games. This constitutes one direction of our Theorem 3.1. Our result holds for countable spaces and provides a more complete ordering: when two type spaces have the same BNE across all finite games, they are identical.

3.2 Example

In this section we represent type spaces graphically in terms of Aumann style information structures, and illustrate the implications of these "different" type spaces to solution concepts of interest. In sum, we construct four type spaces, in which types have the same θ-hierarchy, and types from three out of the four spaces have the same Δ-hierarchy, which is the hierarchies on conditional beliefs(conditioning on knowing the opponent's type) about Θ. These type spaces are shown to be "different" in the sense that for any two type spaces there always exists some game such that the BNE on the two type spaces are different. However, it is also shown that the set of BNE actions of one of the type space is a superset of another type space for all games. The structural dissimilarity or similarity that makes the examples so is a consequence of Theorem 1.

Consider players 1 and 2. Fix the set of payoff uncertainty to be $\Theta = \{-1, 1\}$ throughout. Player 1's partitions are drawn using solid lines and Player 2's are drawn with dashed lines.

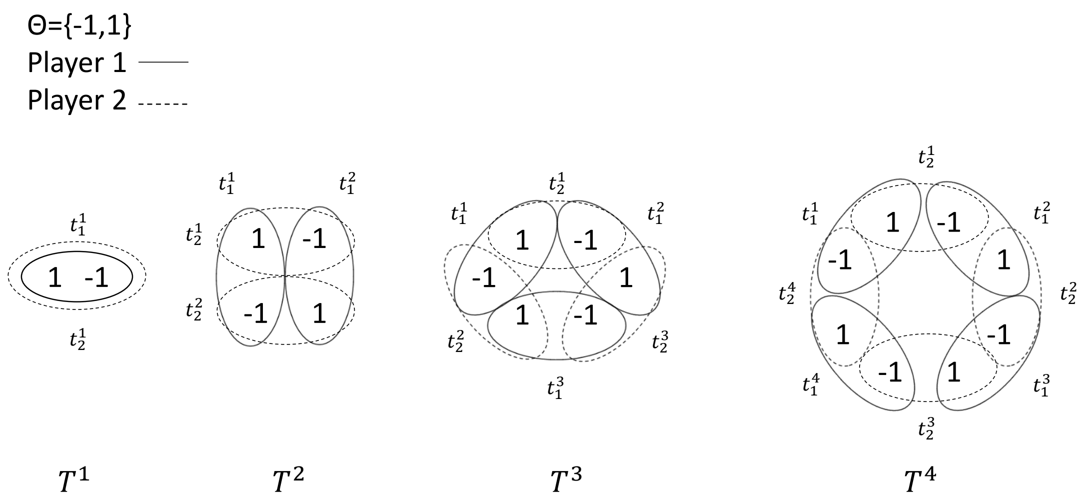

Figure 3.1: Type Spaces

Each of the information partition, when equipped with uniform prior, represents a type space. Each cell is a type. For example, T^1 is the following type space:

$$T_i = \{t_i^1\}, h_i(t_i)(1, t_{-i}) = h_i(t_i)(-1, t_{-i}) = \frac{1}{2},$$

and T^2 is the following type space:

$$T_i = \{t_i^1, t_i^2\},$$
$$h_i(t_i^1)(1, t_{-i}^1) = h_i(t_i^1)(-1, t_{-i}^2) = \frac{1}{2}$$
$$h_i(t_i^2)(1, t_{-i}^2) = h_i(t_i^2)(-1, t_{-i}^1) = \frac{1}{2}.$$

T^1, T^2 are simply the example discussed in the literature, while T^3, T^4 are extensions.

All types in the four type spaces have the same θ-hierarchy: all types believe that probability that $\theta = 1$ is 0.5, believe that the opponent believe the probability that $\theta = 1$ is 0.5, and so forth. Furthermore, the types in T^2 through T^4 induces the same Δ-hierarchy as defined in Ely and Peski(2006). However, all of the type spaces differ in the sense that no two of them have the same Nash equilibrium prediction across all games. Consider the following two player game of incomplete information, denoted by G_1.

	a_2	b_2	c_2
a_1	$1, 1$	$-10, -10$	$-10, 0$
b_1	$-10, -10$	$1, 1$	$-10, 0$
c_1	$0, -10$	$0, -10$	$0, 0$

$\theta = 1$

	a_2	b_2	c_2
a_1	$-10, -10$	$1, 1$	$-10, 0$
b_1	$1, 1$	$-10, -10$	$-10, 0$
c_1	$0, -10$	$0, -10$	$0, 0$

$\theta = -1$

Figure 3.2: G_1

3.2.1 Equilibrium Analysis

- The game (G_1, T^1)

  The only Nash equilibrium strategy profile is (c_1, c_2).

  To see this, note that when player 2 chooses a_2 with probability q, the maximum expected utility over q for player 1 to choose a_1 or b_1 is less than zero. So (c_1, c_2) is the only mutual best response.

- The game (G_1, T^2)

While the θ-hierarchy implied by T^2 over Θ is the same as that of T^1, the set of Bayes Nash equilibria actions now includes a_i, b_i in addition to c_i.

To see this, consider the strategy that t_1^1 plays a_1, t_1^2 plays b_1 , and that t_2^1 plays a_2, t_2^2 plays b_2. The expected utility for t_1^1 to play a_1 is given by $0.5(1) + 0.5(1) = 1 > 0$. This is because t_1^1 believes with probability 1/2 that $\theta = 1$ and player 2 is type t_2^1(thus is going to play a_2), with probability 1/2 that $\theta = -1$ and player 2 is type t_2^2(thus is going to play b_2). Similarly the expected utility for t_1^2 to play b_1 is 1. Hence the strategy defines a mutual best response for each type.

- The game (G_1, T^3)

  While a_i, b_i remain independently rationalizable as in (G_1, T^2), the only Nash equilibrium strategy profile is (c_1, c_2).

  To see this, suppose there exists a Nash equilibrium in which t_1^1 plays a_1 with positive probability. Then the expected utility a_1 yields must be higher than that of c_1, which is zero. Let t_2^1 play a_2 with probability q_1, and t_2^2 play b_2 with probability q_2. For a_1 to be a best response for t_1^1 requires

$$0.5(q_1 - 10(1 - q_1)) + 0.5(q_2 - 10(1 - q_2)) \geq 0,$$

  which implies $q_1 + q_2 \geq 20/11$, hence $q_1 > 0$ and $q_2 > 0$.

  Let t_1^1 plays a_1 with probability p_1, t_1^2 plays b_1 with probability p_2, t_1^3 plays b_1 with probability p_3. Reason as above, for a_2 to be a best response for t_2^1, we need $p_1 + p_2 \geq 20/11$. For b_2 to be a best response for t_2^2, we need $p_1 + p_3 \geq 20/11$.

  However, this implies that t_2^3 will strictly prefer to play c_2, since with 1/2 probability he will get at least $-90/11$ if he plays either a_2 or b_2. This contradicts $q_1 > 0$. This will result to that only (c_1, c_2) is played in equilibrium.

- The game (G_1, T^4)

  The set of Bayes Nash equilibria actions now again includes a_i, b_i in addition to c_i. To see this, simply replicate what each players do in T^2 in their opposing partitions.

The argument used for (G_1, T^4) applies to arbitrary games. That is, across games, T^4 possesses all the equilibrium actions that T^2 possesses. Does this mean that T^2 and T^4 are identical in the sense that they yield the same prediction in Nash equilibrium across different games? The answer is negative. Consider the game G_2 in Figure 3.3.

	a_2	b_2	c_2	d_2	e_2
a_1	$-10, -10$	$1, 1$	$-10, -10$	$-10, -10$	$-10, 0$
b_1	$-10, -10$	$-10, -10$	$1, 1$	$-10, -10$	$-10, 0$
c_1	$-10, -10$	$-10, -10$	$-10, -10$	$1, 1$	$-10, 0$
d_1	$1, 1$	$-10, -10$	$-10, -10$	$-10, -10$	$-10, 0$
e_1	$0, -10$	$0, -10$	$0, -10$	$0, -10$	$0, 0$

$\theta = 1$

	a_2	b_2	c_2	d_2	e_2
a_1	$1, 1$	$-10, -10$	$-10, -10$	$-10, -10$	$-10, 0$
b_1	$-10, -10$	$1, 1$	$-10, -10$	$-10, -10$	$-10, 0$
c_1	$-10, -10$	$-10, -10$	$1, 1$	$-10, -10$	$-10, 0$
d_1	$-10, -10$	$-10, -10$	$-10, -10$	$1, 1$	$-10, 0$
e_1	$0, -10$	$0, -10$	$0, -10$	$0, -10$	$0, 0$

$\theta = -1$

Figure 3.3: G_2

- The game (G_2, T_4)

  Every can be supported as Nash equilibrium.

  To see this, note that the strategy that player 2 plays b_2, c_2, d_2, a_2 for types $t_2^1, t_2^4, t_2^3, t_2^2$ respectively and player 1 plays a_1, b_1, c_1, d_1 for types $t_1^1, t_1^2, t_1^3, t_1^4$ is a NE in T^4.

- The game (G_2, T_2)

  The only Nash equilibrium action of G_2 in T^2 is (e_1, e_2).

	G_1		G_2	
Type Space	R_T^G	BNE_T^G	R_T^G	BNE_T^G
T^1	c	c	-	-
T^2	a,b,c	a,b,c	a,b,c,d,e	e
T^3	a,b,c	c	-	-
T^4	a,b,c	a,b,c	a,b,c,d,e	a,b,c,d,e

Table 3.1: Summary of example

To see this, suppose that type t_1^1 plays a_1 with probability $p_1 > 0$. Suppose t_2^1 plays b_2 with probability q_1 and t_2^2 plays a_2 with probability q_2. That a_1 is a best response for t_1^1 implies $q_1 + q_2 \geq 20/11$, so $q_1, q_2 > 0$. Suppose t_1^2 plays b_1 with probability p_2 and a_1 with probability p_3. For b_2, a_2 to be best responses for t_2^1, t_2^2 respectively, we need $p_1+p_2 \geq 20/11$ and $p_1+p_3 \geq 20/11$, which leads to $p_2+p_3 > 1$, an impossibility. Hence there exists no Bayes Nash equilibrium in which t_1^1 plays a_1 with positive probability. Other cases are argued symmetrically.

The above discussions are summarized in Table 3.1. The notation R_T^G denotes the set of rationalizable actions in a given type space in the game G, BNE_T^G similarly defined. The player index to the actions are omitted for notational simplicity.

3.3 The Model

A type space over a finite set Θ is a tuple (T_1, T_2, h_1, h_2) where each T_i is measurable, h_i a Borel measurable function from T_i to $\Delta(\Theta \times T_{-i})$, where $\Delta(A)$ denotes the set of Borel measures on the set A. A type space is Polish if T is also a Polish space. We often simply denote by T the type space and leave the associating h_i's implicit.

Given a type space T, say $S_i \subset T_i$, $i = 1, 2$, is a belief closed subspace of T_i if for all i, S_i is Borel measurable and $s_i \in S_i$,

$$h_i(s_i)(\Theta \times S_{-i}) = 1.$$

Given two type spaces T, T' and a measurable map $\phi = (\phi_1, \phi_2) : T_1 \times T_2 \to T'_1 \times T'_2$. ϕ is belief-preserving from T to T' if

$$h'_i(\phi(t_i)) = h_i(t_i) \circ \hat{\phi}_{-i}^{-1} \tag{3.1}$$

on all Borel sets $A \subset \Delta(\Theta \times T'_{-i})$, for $i \in \{1, 2\}$, where $\hat{\phi}_i = (id_\Theta, \phi_i) : \Theta \times T_i \to \Theta \times T'_i$.

Let $t_i \in T_i$, denote by $\hat{T}^{t_i}$ the smallest belief closed subspace of T that contains t_i.

A game form is a tuple $G = (A_1, A_2, u_1(a_1, a_2, \theta), u_2(a_1, a_2, \theta))$ with $u_i : A_1 \times A_2 \times \Theta \to \mathbb{R}$. G is called finite if each A_i is finite.

A tuple (G, T) is called a game with incomplete information. A pure strategy is a measurable function $s_i : T_i \to A_i$. Given $T \in \mathcal{T}(\Theta)$, a pure strategy profile (s_1, s_2) is a Nash equilibrium if for all i, all t_i, all $a_i \in A_i$,

$$\int u_i(s_i(t_i), s_{-i}(t_{-i}), \theta) dh(t_i) \geq \int u_i(a_i, s_{-i}(t_i), \theta) dh(t_i).$$

Let

$$BNE_T^G(t_i) = \{a_i \in A_i : \text{there exists a pure NE } \{s_i(t_i)\} \text{ with } a_i = s_i(t_i)\}$$

3.4 Characterization of Bayes Nash Equilibrium In Terms of Type Space Structure

In this section we prove the characterization result for Bayes Nash Equilibrium.[2]

First we summarize some of the useful properties of type spaces that follow from the definition.

Lemma 3.1. Given two type spaces T, T'. Suppose that $\phi : T \to T'$ is a bimeasurable and injective belief preserving map, and that $\phi_i(t_i) = t'_i$ for some i, t_i, t'_i. Then $\phi(T)$ is a belief closed subspace of T' that contains t'_i, and ϕ^{-1} is a belief-preserving map from $\phi(T)$ to T.

[2] If we include mixed strategies in the definition of $BNE_T^G(t)$, our results hold with the additional assumption that in each type space (T, h), h is injective. See Remark 1.

Proof. First we show that $\{\phi(T)\}$ constitutes a belief closed subspace of T' containing t'_i. For player $j \in \{1,2\}$, Let $t'_j \in \phi_j(T_j)$. Let $t_j \in T_j$ be such that $\phi_j(t_j) = t'_j$. Since ϕ is bimeasurable, $\phi_{-j}(T_{-j})$ is Borel measurable. Then

$$\begin{aligned} h'_j(t'_j)(\Theta \times \phi_{-i}(T_{-j})) &= h'_j(\phi_j(t_j))(\Theta \times \phi_{-i}(T_{-j})) \\ &= h_j(t_j)(\Theta \times \phi^{-1}_{-j}(\phi_{-j}(T_{-j})) \\ &\geq h_j(t_j)(\Theta \times T_{-j}) = 1 \end{aligned}$$

The second equality follows from that ϕ is belief preserving. Hence $\phi_j(\hat{T}^{t_i}_j)$ is belief-closed. Also note that $t'_i \in \phi_i(T_i)$.

To show ϕ^{-1} constitutes a belief preserving map from $\phi(T)$ to T, note that for all $t'_j \in \phi(T_j)$ with $\phi^{-1}_j(t'_j) = t_j$,

$$\begin{aligned} h'_j(t'_j) \circ \hat{\phi}_j &= h'_j(\phi(t_j)) \circ \hat{\phi}_j \\ &= h_j(t_j) \circ \hat{\phi}^{-1}_j \circ \hat{\phi}_j \\ &= h_j(t_j) = h_j(\phi^{-1}_j(t'_j)), \end{aligned}$$

where the second equality follows by ϕ is belief preserving, and the third follows by injectivity of ϕ, which implies that $\hat{\phi}^{-1}_j \circ \hat{\phi}_j = id$. □ □

We summarize our first main result in the following theorem.

Theorem 3.1. Given two countable type spaces T, T'. Let $t \in T^t \subset T, t' \in T'^{t'} \subset T'$, where T^t is the smallest belief closed subspace of T containing t and similar for $T'^{t'}$. Then

$$BNE^G_T(t) = BNE^G_{T'}(t') \quad \forall \text{ finite } G$$

if and only if

there exists a bijective belief-preserving map $\phi : T^t \to T'^{t'}$ with $\phi(t) = t'$.

We break down the proof into several statements, from Lemma 3.2 to Lemma 3.4. The if part of Theorem 3.1 is given in Lemma 3.2, which is proved via a standard and straightforward argument and is stronger than what Theorem 3.1 states.

Lemma 3.2 (Sufficiency)**.** Given two type spaces T, T' and $t_i \in T_i, t'_i \in T'_i$. Suppose that there exists a belief preserving map ϕ from T to T' with $\phi(t_i) = t'_i$, then

$$BNE^G_{T'}(t'_i) \subset BNE^G_T(t_i). \tag{3.2}$$

for all G. Furthermore, suppose ϕ is bimeasurable and injective, then

$$BNE^G_{\phi(T)}(t'_i) = BNE^G_T(t_i) \tag{3.3}$$

for all game G.

Proof. The implication (3.2) is Friedenberg and Meier[Ama12]'s Proposition 4.1.[3]

To show (3.3), observe that

$$BNE^G_{\phi(T)}(t'_i) \subset BNE^G_T(t_i) \subset BNE^G_{\phi(T)}(t'_i)$$

where the first inclusion follows from (3.2), the second from applying (3.2) to $\phi(T)$ and T, since by injectivity of ϕ Lemma 1 implies ϕ^{-1} is a belief preserving map from $\phi(T)$ to T with $\phi^{-1}(t'_i) = t_i$. □

Intuitively, if there exists a belief preserving map ϕ from type space T to T', then a type $\tilde{t} \in T$ can mimic the equilibrium behavior of type in T' by "pretending" that their types are $\phi(\tilde{t}) \in T'$. It might be that the space T is richer than T', in the sense that for some games types in T captures more equilibrium actions than types in T', just like IS_2 and IS_4 in the example shows, but the existence of belief preserving morphism means types in T can always discard some information to make itself look like T'.

The universal type space over Θ, U, has the property that for every Θ-based type space T, there exists a belief preserving mapping from T to U. According to Theorem 3.1, types in T can capture the equilibrium actions played by their corresponding types in U, but types in T may have more equilibrium action for some game G. The usage of universal type space as the type space for incomplete information game to predict the equilibrium action of some type from some type space therefore implies possibly strictly less predictions.

[3]The central question investigated in Friedenberg and Meier[Ama12] is the characterization of the conditions under which (3.3) will hold. (3.3) also follows from Lemma 1 and their Corollary 6.1.

Lemma 3.3 (Necessity,Part 1)**.** Suppose $T, T' \in \mathcal{T}(\Theta)$ are countable. Let $t_i \in T_i$ and $t_i' \in T_i'$. Suppose

$$BNE_{T'}^G(t_i') \subset BNE_T^G(t_i) \tag{3.4}$$

for all finite game form G, then there exists a belief preserving map $\phi : T^{t_i} \to T'$ with $\phi_i(t_i) = t_i'$.

Proof. Without loss of generality, assume $T = T^{t_i}$.

First we define a sequence of collections of finite subsets of $\Theta, T_j', j = i, -i$ as follows.

- For each $n = 1, 2, 3, ...,$ define $\widehat{T}_j^{\prime,k}(n) \subset T_j'$, $k = 1, ..., n$, $j = i, -i$ that satisfy the following properties.
  - $\widehat{T}_i^{\prime,1}(n) = \{t_i'\}$.
  - For any $k = 1, ..., n$, (1) $h_i'(t_i')\left(\widehat{T}_{-i}^{\prime,k}\right) > \frac{n-1}{n}$ for every $t_i' \in \widehat{T}_i^{\prime,k}(n)$ and (2) for any $t_{-i}' \in \widehat{T}_{-i}^{\prime,k}$, there exists $t_i' \in \widehat{T}_i^{\prime,k}(n)$ such that $h_i'(t_i')\left(t_{-i}'\right) > 0$.
  - For any $k = 2, ..., n$, (1) $h_{-i}'(t_{-i}')\left(\widehat{T}_i^{\prime,k}\right) > \frac{n-1}{n}$ for every $t_{-i}' \in \widehat{T}_{-i}^{\prime,k-1}(n)$ and (2) for any $t_i' \in \widehat{T}_i^{\prime,k}$, there exists $t_{-i}' \in \widehat{T}_{-i}^{\prime,k-1}(n)$ such that $h_{-i}'(t_{-i}')\left(t_i'\right) > 0$.
  - $\widehat{T}_j^{\prime,k}(n) \subset \widehat{T}_j^{\prime,k}(n+1)$ for any $n \geq k$ for any $j = i, -i$ and k.
- $\widehat{\Theta}(n) \subset \widehat{\Theta}(n+1)$ for any $n \geq 1$ and $\bigcup_n \widehat{\Theta}(n) = \Theta$.

Intuitively $\left\{\widehat{T}_j^{\prime,k}(n), k = 1, ..., n, j = i, -i\right\}$ is a finite approximation of the set of types that are relevant to type t_i''s $2n-1$th order beliefs. We note that, if t_i' is in $\widehat{T}_i^{\prime,k}(n)$ and $h_i'(t_i')(t_{-i}') \geq \frac{1}{n}$, then t_{-i}' must be in $\widehat{T}_{-i}^{\prime,k}(n)$ (a similar condition holds for player $-i$). Let $\widehat{T}_j'(n) := \bigcup_{k=1}^n \widehat{T}_j^{\prime,k}(n)$ for $j = i, -i$, which is still a finite set.

For each n, define a function σ^n on T' as follows.

- $\sigma_j^n\left(t_j'\right) = t_j'$ for $t_j' \in \widehat{T}_j'(n)$, $j = i, -i$.
- $\sigma_j^n\left(t_j'\right) = e_j$ for any $t_j' \in T_j'/\widehat{T}_j'(n)$, $j = i, -i$.

Similarly define a function γ^n on Θ as follows.

- $\gamma^n(\theta) = \theta$ for $\theta \in \widehat{\Theta}(n)$
- $\gamma^n(\theta) = e$ for any $\theta \in \Theta / \widehat{\Theta}(n)$.

For each n, let $T'_j(n) := \widehat{T}'_j(n) \cup \{e_j\}$ for $j = i, -i$ and $\Theta(n) := \widehat{\Theta}(n) \cup \{e\}$. For each $t'_j \in \widehat{T}'_j$ let $\widehat{h}^n_j(t'_j) := h'_j(t'_j) \circ \left(\gamma^n, \sigma^n_{-j}\right)^{-1}$, which is type t'_j's belief on finite set $\left(\Theta(n) \times T'_{-j}(n)\right)$, which is generated from $\left(h'_j, \gamma^n, \sigma^n_{-j}\right)$.

Now consider the following finite game $G(n, \varepsilon)$ for each n and $\epsilon > 0$.

- **Actions**: $A_j(n, \epsilon) = T'_j(n) \times B_j(n, \epsilon)$ for $j = i, -i$, where $B_j(n, \epsilon) \in \triangle\left(\Theta(n) \times T'_{-j}(n)\right)$ is a large finite set of beliefs such that (1) it includes $\widehat{h}^n_j\left(t'_j\right)$ for every $t'_j \in \widehat{T}'_j(n)$ and (2) it is an ϵ net of $\triangle\left(\Theta(n) \times T'_{-j}(n)\right)$ for some $\epsilon > 0$. Thus player j announces his type (or e_j) and his belief on $\Theta(n) \times T'_{-j}(n)$. Player j's pure strategy is $\phi_j : T_j \to A_j(n)$ for $j = i, -i$.
- **Payoffs:**
  - If player j announces $t'_j \in \widehat{T}'_j(n)$, but did not announce $\widehat{h}^n_j\left(t'_j\right)$, then his payoff is $-\infty$.
  - If player j either announces $t'_j \in \widehat{T}_j$ and $\widehat{h}^n_j\left(t'_j\right)$ or e_j and any $p_j \in B_j(n, \epsilon)$, then player j's payoff is computed according to a strictly proper scoring rule $v_j\left(p_j, (\theta, t'_{-j})\right)$, where $p_j \in \triangle\left(\Theta(n) \times T'_{-j}(n)\right)$ is the belief announced by j and θ is a realized state and t'_{-j} is a type announced by player $-j$.[4]

    Note also that without loss, we assume the expected utility $\mathbb{E}_b[v_j(p_j, (\theta, t'_{-j}))]$, where h is the probability with respect to which the expected value is computed, is constructed to be continuous in (b, p_j). This can be achieved by the quadratic scoring rule following Definition 1.

[4] Note that a type $\widetilde{t}'_j$ with belief $\widehat{h}^n(t'_j)$ for some $t'_j\ (\neq \widetilde{t}'_j) \in \mathbf{\widehat{T}'_j(n)}$ is indifferent between $(t'_j, \widehat{h}^n(t'_j))$ and $(e_j, \widehat{h}^n(t'_j))$. So we can extract correct types in some equilibrium even if they share the same belief.

The argument will proceed by showing the following : For each $n, \epsilon > 0$,

Claim 1 There exists a pure strategy BNE s' in the incomplete information game $(G(n,\epsilon), T')$ in which $s'_j(t'_j) = (t'_j, \widehat{h}^n(t'_j))$ whenever $t'_j \in \widehat{T}_j(n)$, and $s'_j(t'_j) = (e_j, p_j)$ for some $p_j \in B(n,\epsilon)$ if $t'_j \notin \widehat{T}_j(n)$.

Proof: Consider any player j and any type t'_j. Suppose the other $-j$ adopts the above given strategy that "truthfully" reports their types. Suppose $t'_j \in \widehat{T}'_j(n)$. Then the equilibrium belief t'_j has on the distribution of strategies played by his opponent and the realization of θ then satisfies

$$h'_j(t'_j) \circ \left(\gamma^n, \sigma^n_{-j}\right)^{-1} = \widehat{h}_j(t'_j) \tag{3.5}$$

Since v_j is a strictly proper scoring rule, reporting $(t'_j, \widehat{h}_j(t'_j))$ is the (unique)best response of t_j. Suppose $t'_j \in T'_j \backslash \widehat{T}'_j(n)$. His equilibrium belief is again given by (7). Then his best response is (e_j, p_j) where

$$p_j \in \arg\max_{p \in B(n,\epsilon)} \mathbb{E}_b[v_j(p, (\theta, t'_{-j}))].$$

Claim 2 There exists a pure strategy BNE $\phi(n,\epsilon)$ of the incomplete information game $(G(n,\epsilon), T)$ such that $\phi_i(n,\epsilon)(t_i) = (t'_i, \widehat{h}^n(t'_i))$.

Proof: This follows from Claim 1 and the assumption that $BNE^G_{T'}(t'_i) \subset BNE^G_T(t_i)$ for any finite game G.

Claim 3 Let $\phi^1(n,\epsilon) : T \to T'(n)$ be the first component of $\phi(n,\epsilon)$. For any $\epsilon^k \to 0$, $\{\phi^1(n,\epsilon^k)\}$ has a point-wise convergent subsequence, with limit denoted by $\phi^{1*}(n)$. Furthermore, for every $t_j \in T_j$ such that $\phi^{1*}(n)(t_j) \in \hat{T}_j(n)$,

$$\widehat{h}^n_j(\phi^{1*}(n)(t_j)) = h_j(t_j) \circ \left(\gamma^n, \phi^{*1}_{-j}(n)\right)^{-1}.$$

Proof: Since $T'(n)$ is finite and T is countable, a diagonal argument shows the existence of point-wise convergent subsequence. Without loss of generality let $\phi^{1*}(n) = \lim_{k\to\infty} \phi^1(n,\epsilon^k)$.

Let $\delta > 0$ be given and that $\phi^{1*}(n)(t_j) = t'_j \in \hat{T}_j(n)$. Since $T'_j(n)$ is finite, there exists K_1 such that for all $k > K_1$, $\phi^1(n, \epsilon^k)(t_j) = t'_j$. By the construction of payoffs and that $\phi(n, \epsilon^k)$ is best response $\phi(n, \epsilon^k)(t_j) = (t'_j, \hat{h}^n(t'_j))$ for all $k > K_1$.

Suppose there exists a subsequence ϵ^{k_m} such that

$$d(\hat{h}^n(t'_j), h(t_j) \circ \left(\gamma^n, \phi^1(n, \epsilon^{k_m})\right)^{-1}) > \delta,$$

where $d(\cdot, \cdot)$ is an appropriate metric on the space of probability measures over some countable set. Since $\epsilon^{k_m} \to 0$ and $B(n, \epsilon^{k_m})$ is an ϵ^{k_m}-net, for each m there exists $p^m \in B(n, \epsilon^{k_m})$ such that

$$\lim_{m \to \infty} d(p^m, h(t_j) \circ \left(\gamma^n, \phi^1(n, \epsilon^{k_m})\right)^{-1}) = 0.$$

Let $v^*(m)$ be the expected utility of t_j when he reports (e_j, p^m), let $v(m)$ be the expected utility of t_j when he reports $(t'_j, \hat{h}^n(t'_j))$. Since the expected utility $E_b[v_j(p_j, (\theta, t'_{-j}))]$ is continuous in (b, p_j), and v_j is a strictly proper scoring rule, there exists $\delta' > 0$ and some $M > 0$ such that for all $m > M$, $v^*(m) - \delta' > v(m)$. This contradicts that the equilibrium action is $\phi(n, \epsilon^k)(t_j) = (t'_j, \hat{h}^n(t'_j))$ for all $k > K_1$. Hence there exists $K > K_1$ such that whenever $k > K$,

$$d(\hat{h}^n(t'_j), h(t_j) \circ \left(\gamma^n, \phi^1(n, \epsilon^k)\right)^{-1}) < \delta.$$

Since δ is arbitrary,

$$\hat{h}^n(t'_j) = h(t_j) \circ \left(\gamma^n, \phi^{1*}(n)\right)^{-1}.$$

Claim 4 There exists a convergent subsequence of $\{\phi^{*1}(n)\}$ with limit $\phi^{*1} : T \to T'^{t'_i}$.

Proof: Take any $t_j \in T_j^{t_i}$. Then there exists n' and a sequence $t_i, t^2_{-i}, t^3_i, \ldots, t^{n'}_j$ such that $h_i(t_i)(t^2_{-i}) > 0, h_{-i}(t^2_{-i})(t^3_i) > 0, \ldots, h_{-j}(t^{n'-1}_{-j})(t^{n'}_j) > 0$. Let $\eta > 0$ be the smallest probability along this sequence. Pick N such that $\frac{1}{N} < \eta$. For every $n \geq N$, we have

$$\widehat{h}^n_i(t'_i)(\phi^{*1}_{-i}(n)(t^2_{-i})) = h_i(t_i)\left(\left(\phi^{*1}_{-i}(n)\right)^{-1}\left(\phi^{*1}_{-i}(n)(t^2_{-i})\right)\right) \geq h_i(t_i)(t^2_{-i}) > \frac{1}{N}.$$

By construction of $\hat{T}(n)$, $\widehat{h}^n_i(t'_i)(t'_{-i}) > \frac{1}{N}$ implies $t'_{-i} \in \widehat{T}'^{,1}_{-i}(n)$ for $n > N$. Hence $\phi^{*1}_{-i}(n)(t^2_{-i})$ must be included in $\widehat{T}'^{,1}_{-i}(N)$ for any $n \geq N$. Similarly, by induction, we can show that $\phi^{*1}_i(n)(t^3_i) \in \widehat{T}'^{,2}_i(N), \ldots$, and $\phi^{*1}_j(n)(t_j) \in \widehat{T}'^{,k}_j(n)$ for some k for every $n \geq N$. Since $\widehat{T}'_j(N)$

is a finite set, there exists a convergent subsequence of $\phi_j^{*1}(n)(t_j)$ that converges to some point in $\widehat{T}_j'(N)$. This implies that, since T^{t_i} is countable, the standard diagonal argument guarantees a convergent subsequence of $\phi^{*1}(n)$ with some well-defined limit $\phi^{*1}: T^{t_i} \to T'^{t_i'}$.

Claim 5 $\phi^{*1}: T \to T'^{t_i'}$ is a belief-preserving morphism between (T, h) and $(T'^{t_i'}, h')$ that maps t_i to t_i'. That is, for all $t_j \in T_j$,

$$h_j'(\phi_j^{*1}(t_j)) = h_j(t_j) \circ \left(id, \phi_{-j}^{*1}\right)^{-1}. \tag{3.6}$$

Proof: By Claim 4, with out of loss, let $\phi^{*1} = \lim_{n\to} \phi^{*1}(n)$. Let $t_j \in T_j^{t_i}$. By construction of $\Theta(n)$ and γ^n, $\lim_{n\to} \gamma^n = id$, where $id: \Theta \to \Theta$ is the identity mapping. Hence

$$\lim_{n\to\infty} d(h_j(t_j) \circ \left(\gamma^n, \phi_{-j}^{*1}(n)\right)^{-1}, h_j(t_j)(id, \phi_{-j}^{*}\right)^{-1}) = 0$$ [5]

Fix any $t_j \in T_j$. By the proof of Claim 4, there exists N such for all $n > N$, $\phi^{1*}(n)(t_j) \in \hat{T}_j'(n)$, so that the condition in Claim 3 is satisfied. By Claim 3, it then suffices to show that

$$\lim_{n\to\infty} d(\widehat{h}_j^n(\phi_j^{*1}(n)(t_j)), h_j'(\phi_j^{*1}(t_j)) = 0. \tag{3.7}$$

By definition of $\widehat{h}_j^n$, for $n > N$,

$$\widehat{h}_j^n(\phi_j^{*1}(n)(t_j)) = h_j'(\phi_j^{*1}(n)(t_j)) \circ \left(\gamma^n, \sigma_{-j}^n\right)^{-1}$$

Since γ^n tends to id and σ_{-j}^n tends to $id^{T'}: T' \to T'$ by construction of $\hat{T}'(n)$, (9) thus holds.

Since t_j is arbitrary, (8) is proved, as well this lemma. □

The idea is to construct a game using strict scoring rules(defined in the Appendix) as the utility. The scoring rule asks each player to report a type in T_i' and a belief over $\Delta(\Theta \times T_{-i}')$. When this game is played with respect to the type space T' truthful reporting is a Bayes Nash equilibrium. By assumption there exists an equilibrium of the same game played with respect to the type space T such that t also reports truthfully. A contagious argument shows that the reason why t reports truthfully is that he thinks correctly that the $-i$ types within his belief's support will also report truthfully, and so on. Truncating the strategy we get a

[5] The metric d is on the space $\Delta(\Theta \times (T' \cup \{e\}))$.

map from T to T'. We show that this map is the desired belief-preserving map. The general proof of Theorem 2 is built on this idea and an approximation argument because we only use finite games.

Note that the existence of belief preserving map from T to T' is not guaranteed. For example T may be a union of two belief closed spaces $T^1 \cup T^2$, with $t_i \in T^1$, and T^2 totally different from T'(so no belief preserving maps from T^2 to T' exists). Since the notion $BNE_T^G(t_i)$ is local, what this T^2 looks like does not affect equilibrium actions of types outside it.

Remark 7. The restriction to pure strategies is essential. Consider the following example:

Define two type spaces $(T, h^t), (S, h^s)$ where

$$T_1 = \{t_1^1, t_1^2\}, T_2 = \{t_2^1\}, h^t(t_2^1)(t_1^1) = 1/3$$
$$S_1 = \{s_1^1, s_1^2\}, S_2 = \{s_2^1\}, h^s(s_2^1)(s_1^1) = 1/2.$$

First observe that there does not exist a belief preserving morphism $\phi : S \to T$ such that $\phi(s_2^1) = t_2^1$. Consider any game G with action sets A_1, A_2 and any Nash equilbrium $((f_1, f_2), g)$ with respect to the type space T, where $f_1, f_2 \in \Delta(A_1)$ are strategies used by t_1^1, t_1^2 respectively, and $g \in \Delta(A_2)$ is the strategy used by t_2^1.(f_1, f_2, g can be either pure or mixed.) Consider the following strategy profile used by (s_1^1, s_1^2) and s_2^1

$$((\frac{2}{3}f_1 + \frac{1}{3}f_2, f_2), g)$$

Since type t_1^2 and t_1^1 has the same preference on A_1 given a fixed g, $2/3f_1 + 1/3f_2$ is still a best response to g. On the other hand, the distribution of actions type s_2^1 faces is

$$\frac{1}{2}(\frac{2}{3}f_1 + \frac{1}{3}f_2) + \frac{1}{2}f_2 = \frac{1}{3}f_1 + \frac{2}{3}f_2,$$

which is the same as the distribution of actions faced by t_2^1 in type space T. Hence g is a best response for s_2^1 when type s_2^1 uses the *possibly mixed* strategy $2/3f_1 + 1/3f_2$. This shows that $BNE_T^G(t_2^1) \subset BNE_S^G(s_2^1)$ if we allow a strategy profile to contain mixed strategies in the definition of $BNE_T^G(t)$.

What's so special about the above simple counter example is that the type mapping $h_1 : T_1 \to \Delta(\Theta \times T_2)$ is not injective. Assuming the type mappings of S, T are injective, then Theorem 2 again holds when the definition of BNE_T^G accommodates mixed strategy profiles. This assumption guarantees that when the utility function is a strictly proper scoring rule, since the maximizer is unique, a type t will not be indifferent to some $t'_1 \neq t'_2$ since $h'(t'_1) \neq h'(t'_2)$ hence we can guarantee the equilibrium strategy with $s_i(t_i) = (t'_i, h_i(t'_i))$ is pure, so that $s : T \to T'$ will be a BPM.

Our next step is to show that the converse of Lemma 2 holds: for countable space, there exists an injective belief preserving map between the two types, which means that the type spaces surrounding the two types looks exactly the same.

Lemma 3.4 (Necessity Part 2)**.** Suppose T is countable and $BNE_T^G(t_i) = BNE_{T'}^G(t'_i)$ for every finite game G, then there exists a bijective belief-preserving morphism between T^{t_i} and $T'^{t'_i}$.

Proof. By Lemma 3, there exists belief preserving morphism $\phi : T^{t_i} \to T'$ and $\psi : T'^{t'_i} \to T$. We will show that ϕ is bijective. Lemma 1 together with onto-ness implies $T'^{t'_i} = \phi(T^{t_i})$.

The proof is broken down to two steps.

Step 1 For each $j \in \{i, -i\}$ and $k \geq 1$, ϕ is one-to-one on T_j^k and $\phi(T_j^k) = T_j'^k$.

We will use induction to show that this holds. First consider $k = 1$. When $j = i, T_i^1 = \{t_i\}$ is a singleton so it's trivial. For $j = -i$, suppose not. Say

$$\exists \tilde{t}_{-i}^1 \in T_{-i}'^1 \text{ such that } \phi^{-1}(\tilde{t}_{-i}^1) = \{t_{-i}^1, ..., t_{-i}^n\} \subset T_{-i}^1, n \geq 2. \tag{3.8}$$

Since ϕ is belief preserving, this implies

$$h'_i(t'_i)(\tilde{t}_{-i}^1) = h_i(t_i)(\{t_{-i}^1, ..., t_{-i}^n\}) > \max_k \{h_i(t_i)(t_{-i}^k)\} \tag{3.9}$$

The inequality is strict because $n \geq 2, t_{-i}^k \in T_{-i}^1$ and elements in the support of a discrete measure has positive probability.

Claim 1. There does not exist a composition map of arbitrary length $\gamma = \psi \circ \phi \circ ... \circ \psi$ such that $\gamma(\tilde{t}^1_{-i}) \in \{t^1_{-i}, ..., t^n_{-i}\}$.

Proof When $\gamma = \psi$, having $\psi(\tilde{t}^1_{-i}) = t^k_{-i}$ for some k would imply, since ψ is belief-preserving, that

$$h_i(t_i)(\gamma(\tilde{t}^1_{-i})) = h'_i(t'_i) \circ \psi^{-1}(t^k_{-i}) \geq h'_i(t'_i)(\tilde{t}^1_{-i}), \tag{3.10}$$

contradicting (11). Suppose $\gamma = \psi \circ \phi \circ \psi(\tilde{t}^1_{-i}) = t^k_{-i}$ for some $k = 1, ..., n$, then

$$h_i(t_i)(t^k_{-i}) \geq h'_i(t'_i)(\phi \circ \psi(\tilde{t}^1_{-i})) = h_i(t_i) \circ \phi^{-1}(\phi \circ \psi(\tilde{t}^1_{-i})) = h_i(t_i)(\psi(\tilde{t})^1_{-i}) \geq h'_i(t'_i)(\tilde{t}^{-1}_{-i}), \tag{3.11}$$

again contradicting (11), where we alternatingly use the fact that ϕ, ψ are belief preserving maps. This argument can be seen to apply to γ of arbitrary length, proving Claim 1.

According to Claim 1, $\psi(\tilde{t}^1_{-i}) = t^{n+1}_{-i} \in T^1_{-i}$ and $t^{n+1}_{-i} \neq t^k_{-i}$ for $k = 1, ..., n$. Moreover, (10) implies $\phi(t^{n+1}_{-i}) = \tilde{t}^2_{-i} \neq \tilde{t}^1_{-i}$. Claim 1 implies $\psi(\tilde{t}^2_{-i}) = t^{n+2}_{-i} \neq t^k_{-i}$ for $k = 1, ..., n+1$.[6] Continue this process, we obtain an infinite sequence of distinct elements $\{t^k_{-i}\} \subset T^1_{-i}$. Moreover, for each t^m_{-i} with $m > n$, by construction there exists $m - n$ ψ's such that $t^k_{-i} = \psi \circ \phi \circ ... \circ \psi(\tilde{t}^1_{-i})$. Thus the same argument of (3) leads to

$$h_i(t_i)(t^m_{-i}) \geq h'_i(t'_i)(\tilde{t}^1_{-i}) > \max_k \{h_i(t_i)(t^k_{-i})\}.$$

But then $h_i(t_i)(T^1_{-i}) \geq h_i(t_i)(\{t^k_{-i}\}^\infty_{k=1}) = \infty$, a contradiction. Hence ϕ is injective when restricted to T^1_{-i}.

This shows that $\phi : T^1_{-i} \to T'^1_{-i}$ is one-to-one. To show onto, pick any $t\tilde{}_{-i} \in T'^1_{-i}$. Belief-preserving implies

$$h_i(t_i)(\phi^{-1}(t\tilde{}_{-i})) = h'_i(t'_i)(t\tilde{}_{-i}) > 0,$$

So $\phi^{-1}(t\tilde{}_{-i}) \cap T^1_{-i} \neq \phi$.

To prove the general case, first the type mapping $h_i : T_i \to \Delta(\Theta \times T_{-i})$ is generalized and then an inductive argument can be applied. For k even, define

$$h^k_i : T_i \to \Delta(\Pi^{k/2}_{m=1}(\Theta \times T_{-i} \times \Theta \times T_i)^m)$$

[6]Claim is true from both side of the type spaces T, T'. When $t^{n+2}_{-i} = t^{n+1}_{-i}$, use Claim 1 when $\gamma = \phi$ to get a contradiction, when $t^{n+2}_{-i} = t^k_{-i}, k = 1, ..., n$, use the claim wiht $\gamma = \psi \circ \phi \circ \psi$ to get a contradiction.

as follows:

$$h_i^k(t_i)(\theta_1, t_{-i}^1, ..., \theta_{k/2}, t_i^{k/2}) = h_i(t_i)(\theta_1, t_{-i^1}) \times h_{-i}(t_{-i}^1)(\theta_2, t_i^2) \times ... \times h_{-i}(t_{-i}^{k/2})(\theta_k, t_i^{k/2})$$

For k odd, define $h_i^k : T_i \to \Delta(\Theta \times T_{-i} \times \Pi_{m=1}^{(k-1)/2}(\Theta \times T_i \times \Theta \times T_{-i})^m)$ similarly. Also, for k even, define $\hat{\phi}^k = (\hat{\phi_{-i}}, \hat{\phi}_i)^k$[7], for k odd define $\hat{\phi}^k = (\hat{\phi_{-i}}, (\hat{\phi}_i, \hat{\phi_{-i}})^{k-1/2})$. The above definitions also apply to the space T_i' and ψ. Observe that the support of $h_i^k(t_i)$ is a subset of $\Theta \times T_{-i}^1 \times ... \times \Theta \times T_i^{k/2+1}$ for k even and $T_{-i}^1 \times ... \times \Theta \times T_{-i}^{(k+1)/2}$ for k odd.

Assume as the inductive hypothesis that ϕ is bijective when restricted to $T_i^1, T_{-i}^1, ..., T_{-i}^n$. We aim to show ϕ is bijective when restricted to T_i^{n+1}

The first task is to show that for these generalized type mappings, the belief preserving property holds. Then an argument similar to proving injectivity on T_{-i}^1 can be applied.

Claim 2. $h_i'^{2n}(t_i') = h_i^{2n}(t_i) \circ (\hat{\phi^{2n}})^{-1}$. $h_i^{2n}(t_i) = h_i'^{2n}(t_i') \circ (\hat{\psi^{2n}})^{-1}$.

Proof: Take any $(\theta_1, \tilde{t_{-i}}^1, ..., \tilde{t_{-i}}^n, \theta_{2n}, \tilde{t}_i^{n+1}) \in \operatorname{supp} h_i'^{2n}(t_i')$. By inductive hypothesis, let $t_j^k = \phi^{-1}(\tilde{t_j}^k, j \in \{i, -i\}, k = 2, ..., n$ be the unique element in T_j^k such that $\phi(t_j^k) = \tilde{t_j}^k$. Then

$$\begin{aligned}
&h_i'^{2n}(\phi(t_i))(\theta_1, \tilde{t_{-i}}^1, ..., \theta_k, \tilde{t}_i^{n+1}) \\
=&h_i'(\phi(t_i))(\theta_1, \tilde{t_{-i}}^1) \times h_{-i}'(\tilde{t_{-i}}^1)(\theta_2, \tilde{t}_i^1) \times ... \times h_{-i}'(\tilde{t_{-i}}^n)(\theta_{2n}, \tilde{t}_i^{n+1}) \\
=&h_i(t_i)(\theta_1, t_{-i}^1) \times h_{-i}(t_{-i}^1)(\theta_2, \phi^{-1}(\theta, t_i^2) ... \times h_{-i}(t_{-i}^n)(\theta_{2n}, t_i^{n+1}) \\
=&h_i^{2n}(t_i) \circ (\hat{\phi}^k)^{-1}(\theta_1, \tilde{t_{-i}}^1, ..., \theta_k, \tilde{t}_i^{k/2}).
\end{aligned}$$

The first and third equality is definition, the second one follows from ϕ is belief preserving. A symmetric argument shows this applies to T' as well. This proves Claim 2.

With a slight abuse of notation, let $h_i^{2n}(t_i)$ denote the marginal measure on T_i^{n+1}. Note that $\operatorname{supp} h_i^{2n}(t_i) = T_i^{n+1}$. Suppose ϕ is not one-to-one when restricted to T_i^{n+1}. Then there exists $\tilde{t_i} \in T_i'^{n+1}$ such that $\phi^{-1}(\tilde{t_i}) = \{t_i^1, ..., t_i^k\}$. Now the same argument in the first section

[7]Let $f : X \to Y$ be a function, then $f^k : X^k \to Y^k$ is defined to be $(x_1, ..., x_k) \mapsto (f(x_1), ..., f(x_k))$, where $(x_1, ..., x_k) \in \Pi_{i=1}^k X := X^k$.

of the proof applies and will produce a contradiction. Onto-ness is also proved in the same way.[8]

Step 2 $\phi : \cup_{k=1}^{\infty} T^k \to \cup_{k=1}^{\infty} T'^k$ is one-to-one and onto.

Ontoness follows from Step 1.

Suppose ϕ is not injective on T^{t_i}. Say there exists some $\tilde{t}_i \in T'^{t'_i}$ and $m \neq n$ such that $\phi^{-1}(\tilde{t}_i^{\,l}) = \{t_i^m, t_i^n\}$, where $t_i^m \in T_i^m, t_i^n \in T_i^n$ and $t_i^m \neq t_i^n$. Since $\phi(T_i^k) = T_i'^k$ for $k = m, n$, we have $\tilde{t}_i \in T'^m \cap T'^n$. Hence $h_i'^{2(m-1)}(t'_i)(\tilde{t}_i) > 0, h_i'^{2(n-1)}(t'_i)(\tilde{t}_i) > 0$. Note that $t_i^m \notin T_i^n$ and $t_i^n \notin T_i^m$ otherwise we will immediately reach a contradiction that ϕ is one-to-one when restricted to T_i^n and T_i^m. Hence $h_i^{2(m-1)}(t_i)(t_i^n) = 0$ and $h_i^{2(n-1)}(t_i)(t_i^m) = 0$. Suppose $\psi(\tilde{t}_i) = t_i^m$, then

$$0 = h_i^{2(n-1)}(t_i)(t_i^m) = h_i'^{2(n-1)}(t'_i)(\psi^{-1}(t_i^m)) \geq h_i^{2(n-1)}(t'_i)(\tilde{t}) > 0,$$

which is a contradiction. Similarly, if $\psi(\tilde{t}_i) = t_i^n$ there will also be a contradiction. Hence there exists $t_i^{k_1} \notin \{t_i^m, t_i^n\}$ such that $\psi(\tilde{t}_i) = t_i^{k_1}$. Since $\psi(T_i'^m) = T_i^m$ and $\psi(T_i'^n) = T_i^n$, $t_i^{k_1} \in T_i^m \cap T_i^n$. Continue this process and argue as before, we obtain an infinite sequence of distinct elements $\{t_i^{k_l}\} \subset T_i^m \cap T_i^n$.

Now $h_i^{2(m-1)}(t_i)(t_i^{k_1}) = h_i'^{2(m-1)}(t'_i) \circ \psi^{-1}(t_i^{k_1}) \geq h_i'^{2(m-1)}(t'_i)(\tilde{t}) > 0$, and the same argument shows that $h_i^{2(m-1)}(t_i)(t_i^{k_l}) \geq h_i'^{2(m-1)}(t'_i)(\tilde{t}) > 0$. But then $h_i^{2(m-1)}(t_i)(T_i^m) = \infty$, a contradiction.

Since $T^{t_i} = \cup_{k=1}^{\infty} T^k$ and $T'^{t'_i} = \cup_{k=1}^{\infty} T'^k$, this concludes the proof. □

If we assume T is finite, the lemma follows from a straightforward argument:

By Theorem 2, there exists belief preserving maps $\phi : T^{t_i} \to T'$ and $\psi : T'^{t'_i} \to T$. First we claim that $\phi(T^{t_i}) = T'^{t'_i}$.

By Lemma 1, $\phi(T^{t_i})$ is belief closed, hence, being the smallest belief closed space, $T'^{t'_i} \subset \phi(T^{t_i})$. For the other side of inclusion, pick $\tilde{t}' \in \phi^{T^{t_i}}$. To show $\tilde{t}'$ is in $T'^{t'_i}$, we need to exhibit a sequence of types $\tilde{t}^n \in T'$ such that $\tilde{t}^1 = t'_i$, $\tilde{t}^n = \tilde{t}'$, $\tilde{t}^k, \tilde{t}^{k+1}$ belongs to different players,

[8] We can as well start induction on T_{-i}^1 and postpone the argument to the general case, but presenting the main idea earlier improves clarity.

and that $h(\tilde{t}^k)(\Theta \times \{\tilde{t}^{k+1}\}) > 0$, since this is how we construct the smallest belief closed set containing t_i'.

Let $\phi(t) = \tilde{t}'$ where $t \in T^{t_i}$. Then there exists a sequence of types $t^n \in T^{t_i}$ with the properties described above. Define $\tilde{t}^k = \phi(t^k)$. Since ϕ is belief-preserving, the sequence $\{\phi(t^k)\}$ satisfies the required property, hence $\tilde{t}' \in T'^{t'}$.

The same argument applies to ψ, hence $\psi(T'^{t_i'}) = T^{t_i}$.

Since both ϕ and ψ are surjective, T^t and $T'^{t'}$ has the same cardinality. Theorem 3 now follows from the fact that surjective functions between finite sets with the same cardinality must be injective. □

Lemma 4 says that two types have the same Nash equilibria actions across all games if and only if smallest type spaces containing them are isomorphic, which means the information contained in the implicit representation $h_i : T_i \to \Delta(\Theta \times T_{-i})$ can not be collapsed, say into θ-hierarchies(for correlated rationalizability) or Δ-hierarchies(for independent rationalizability). In a sense, there is no redundant types with respect to Nash equilibrium.

3.4.1 Example Revisited

The reason that for any game G, if $\mu_i \in A_i$ is played in some BNE in the type space T^2 then the same strategy is also played in T^4 by some BNE, follows from Theorem 1. The type space for T^4 is given by $S_i = \{s_i^1, s_i^{1'}, s_i^2, s_i^{2'}\}$, where

$$\begin{aligned} h_i(s_i^1)(1, s_{-i}^{2'}) = h_i(s_i^1)(-1, s_{-i}^1) &= \frac{1}{2} \\ h_i(s_i^{1'})(1, s_{-i}^2) = h_i(s_i^{1'})(-1, s_{-i}^{1'}) &= \frac{1}{2} \\ h_i(s_i^2)(1, s_{-i}^1) = h_i(s_i^2)(-1, s_{-i}^2) &= \frac{1}{2} \\ h_i(s_i^{2'})(1, s_{-i}^{1'}) = h_i(s_i^{2'})(-1, s_{-i}^{2'}) &= \frac{1}{2} \end{aligned}$$

The map such that $\phi_i(s_i^1) = \phi_i(s_i^{1'}) = t_i^1$ and $\phi_i(s_i^2) = \phi_i(s_i^{2'}) = t_i^2$ can be shown to be a belief-preserving map from T^4 to T^2. However, it is not injective. And indeed we find the game G_2 such that $a \in BNE_{T^4}^{G_2}(s_i)$ but $a \notin BNE_{T^2}^{G_2}(t_i)$ for all $s_i \in S_i$ and $t_i \in T_i$.

Let $Q_i^3 = \{q_i^1, q_i^2, q_i^3\}$ be the type space T^3, with the h_i' defined in the obvious way as we did for T^2 and T^4. By Theorem 2 we shall also observe that there exists no belief-preserving map from T^2 to T^3 or from T^3 to T^2. The non-existence can be proved directly: Suppose a belief-preserving map ϕ from T^2 to T^3 exists. Since T^3 is already the smallest belief closed subspace containing any of its types, by the first part of Lemma 1 ϕ must be onto, which is impossible since they are both finite and T^3 is strictly bigger. On the other hand, suppose a belief-preserving map ϕ from T^3 to T^2 exists, the same reason concludes ϕ is onto. Hence we can without loss assume $\phi_2^{-1}(t_2^1) = q_2^1$. Belief-preservation implies

$$h_1'(q_1)(\theta, q_2^1) = h_1(t_1)(\theta, t_2^1)$$

where $t_1 = \phi_1(q_1)$. Observe that for all $t_1 \in T_1$, there exists θ such that the RHS is 1/2. However, there exists $q_1 \in Q_1$ such that the LHS is zero for all θ. This is a contradiction.

Lastly, note that the type mappings in all the type spaces considered in Figure 1 is injective, hence Theorem 1 continues to hold even if mixed strategies are allowed(See Remark 1). Indeed, in the analysis in Section 2, we considered all Nash equilibrium actions, pure or mixed.

3.5 Universal Type Space for Bayes Nash Equilibrium

A universal space U for a solution concept S with respect to a class of type spaces $\mathcal{T}$ satisfies the following properties:

(1). $U \in \mathcal{T}$

(2). For all $T \in \mathcal{T}$, there exists a belief preserving map $\rho : T \to U$ such that $\rho(t)$ and t has the same S-strategy across all games.

(3). For $u, u' \in U$, if $u \neq u'$ then there exists a game such that u, u' have different S-strategies.

For example, if S is correlated interim rationalizability(DFM2007), then the universal space U with respect to the class of measurable type space is a space of hierarchy of beliefs on θ. It satisfies (1), by construction. It satisfies (2), by mapping each type to its induced belief hierarchy. It satisfies (3), by results in DFM(2006). When S is independent interim rationalizability[EP06], a similar space U can be constructed. However, when S is Bayes Nash equilibria, so far no such space is constructed yet.

However, when $S = BNE$, there is a trade-off between (2) and (3). In particular, if $\mathcal{T}$ is large enough, then even a relaxed version of (3) will fail. Specifically, when $\mathcal{T}$ contains all countable type spaces, then every countable space must be injectively embedded into U. (3) will then lose its bite because for each type t in any countable type space T, there will be infinitely many $u \in U$ that has the same BNE as t. This is because we can take unions of disjoint copies of identical type spaces and maps it injectively into U. This observation is proved with an application of Theorem 1 and is documented below, where proof is relegated to the appendix.

Proposition 3.1. Let $\mathcal{T}$ be the class of all countable type spaces. Suppose there exists a type space U that satisfies (2) for BNE with respect to $\mathcal{T}$, then every countable type space can be injectively embedded into U.

Proof. Let (T, h) be an arbitrary countable type space. Define a countable type space (T', h') such that $T'_i = T_i \cup \{t'_i\}$, and define $h'_i : T'_i \to \Delta(\Theta \times T'_{-i})$ such that $h'_i(t_i) = h_i(t_i)$ for all $t_i \in T_i$, and that $h'_i(t'_i)(\Theta \times t_{-i}) > 0$ for all $t_{-i} \in T_{-i}$. By (2), there exists $\rho : T' \to U$ that preserves BNE. Let $\rho(t'_i) = u'_i$. Note that $\rho(T')$ is a type space, moreover, since T' is the smallest belief-closed space containing t'_i, $\rho(T') \subset U$ will also be the smallest belief closed space containing u'_i. By Theorem 1, ρ is injective. Restricting ρ to T, we obtain an injective belief-preserving map from T to U. □

This proposition shows that there will always be an infinite number of types in a "universal type space" having the same set of BNE actions across all games. For example, we can take the union of arbitrarily many disjoint copies of T_2 and map it to the universal space injectively.

3.6 Conclusion

In this paper, we show that Bayes Nash equilibrium actions are determined by the type space structure in its entirety. This is in contrast to rationalizability where one can collapse the information carried by a type to a hierarchy of beliefs. Thus, the goal to find the most parsimonious universal space for Nash equilibrium is not attainable.

One future direction of research is to explicitly characterize the trade-offs of non-redundancy and the number of type spaces with respect to which a space is universal. For example, if we restrict to only a subclass of countable type spaces, non-redundancy may be salvaged.

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