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UNIVERSITY OF CALIFORNIA, SAN DIEGO

Mirror Symmetry on Toric Surfaces via Tropical Geometry

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

M. Brandon Meredith

Committee in charge:

Professor Mark Gross, Chair
Professor Ken Intriligator
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2013

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Chair

University of California, San Diego

2013

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ABSTRACT OF THE DISSERTATION

Mirror Symmetry on Toric Surfaces via Tropical Geometry

by

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Mirror symmetry is a curious duality, first noticed by physicists and then excitedly embraced by mathematicians, between certain manifolds (the A-side) and their “mirror” spaces (the B-side). On the A-side, certain counts of objects on a manifold are carried out (the purview of enumerative geometry), while on the B-side, special functions are integrated. Interestingly, the same numbers come out on both sides.

This dissertation considers mirror symmetry on toric surfaces, which are varieties with certain convenient combinatorial properties and include many well-known surfaces such as the affine and projective planes and $\mathbb{P}^1 \times \mathbb{P}^1$. These surfaces are especially suited to being exploited by tropical geometry, which is a form of algebraic geometry over the “tropical semi-ring.” Dramatically, key information about both sides of mirror symmetry on toric surfaces can be gleaned from the tropics.

The mirror symmetry of a certain subclass of toric varieties, those that are Fano, has seen much progress. In the case of toric surfaces, being Fano means that tropical geometry can “see” all the curves of the surface; this makes enumerative problems more straight-forward. This dissertation attempts a generalization to non-Fano toric surfaces, where some curves are “hidden” from the viewpoint of tropical geometry. The generalization is accomplished following the Gross-Seibert Program wherein singularities are added to the tropical picture in order to “pull” these curves into view. We verify these methods by carrying out explicit calculations on $\mathbb{F}_2$, the second Hirzebruch surface, which is a non-Fano toric variety.

Chapter 1

Introduction

This thesis covers how to carry out B-side calculations for mirror symmetry on toric varieties by dropping down into the simpler, more combinatorial, framework of tropical geometry. This dissertation can be thought of as coming in two parts. The first three chapters will introduce mirror symmetry with its component A-side and B-side, toric varieties, and tropical geometry. They will also cover what has been shown to date about mirror symmetry for $\mathbb{P}^2$. They will end with the statement of a general mirror symmetry conjecture for toric varieties, which has never been explicitly written down before but which is derived from Gross's work on the $\mathbb{P}^n$ case. The last three chapters will introduce some more recent tools of mirror symmetry and carry out explicit calculations on the second Hirzebruch surface $\mathbb{F}_2$.

1.1 Some History

Mirror symmetry is a mathematical phenomenon that was discovered by physicists in the 1990s who were studying string theory. String theory is a “theory of everything,” which attempts to reconcile Einstein's theory of general relativity with quantum mechanics. The key idea is to consider particles as tiny strings, rather than as points. For the mathematics to work out nicely, the string theorists realized that the universe had to have not just the four dimensions of space and time that we are accustomed to, but ten dimensions.

The extra six dimensions would have to be so small so as to be practically

unnoticeable, except by the even smaller “superstrings.” These strings sweep out into these extra dimensions, and it is the different ways that they vibrate that gives all the different fundamental particles (and forces). This leads to the beautiful notion of the universe being one great string symphony.

The physicists found that this six dimensional shape had to have certain properties, and it was their hope that only one such shape was possible. To their dismay, there eventually turned out to be hundreds of millions of shapes, called Calabi-Yau manifolds, that fit their requirements. These geometric objects had been studied by algebraic geometers for some time. Hence a link between string theory and algebraic geometry had been formed.

Around 1990, it was discovered that Calabi-Yau manifolds came in pairs, which they called “mirror pairs.” (The word “mirror” comes from the switching of their Hodge numbers, a technical detail for us.) These two shapes in a mirror pair can be quite different from each other geometrically, but the physicists saw that putting either of these matched shapes into their models produced the same physics in space-time. Then the physicists made a prediction that excited the mathematical community. Though lacking rigorous mathematical proof, they claimed that they had calculated the number of rational curves on a certain Calabi-Yau manifold called a quintic three-fold. A rational curve is a curve that can be parameterized by a straight line.

I will now attempt to explain what they found while also demonstrating why algebraic geometers were so excited by the results. Let’s take a step back to a simpler shape than the quintic three-fold. Let’s instead consider a cubic surface. A cubic surface is a two dimensional shape such that each point on this shape is a solution to a single polynomial equation of degree three, for example $x_1^3 + x_2^3 + x_3^3 + x_4^3 = (x_1 + x_2 + x_3 + x_4)^3$. In the 1860s, Arthur Cayley and George Salmon asked a simple question: how many straight lines lay perfectly on a cubic surface? Their answer, in a theorem since named after them, stated that this can be calculated in a more or less straightforward fashion, and the answer turns out to be 27, which is, itself, a charming result.

Similar questions can be asked about the quintic three-fold. How many lines

are in a typical quintic three-fold? 2,875. How many conic sections (degree two rational curves)? 609,250. But to demonstrate how difficult these calculations were to make, I'll note that the first number was calculated by Schubert in the nineteenth century, while the latter number was first calculated much more recently by Katz in the 1980s. [17] And while Katz was able to prove that there were only finitely many rational curves up to degree seven in the quintic three-fold, he was only able to calculate exactly how many there were up to degree two.

And so it was that in 1990 the physicists Candelas, de la Ossa, Green, and Parkes claimed that they could predict the number of rational curves on the quintic three-fold not just up to degree two as Katz had done, but up to any degree. [2] This was exciting news for the mathematical community. Once again, physics had provided fodder for intriguing mathematical discoveries. And the method that these physicists had used was mysterious and compelling.

Candelas et al. predicted that they could come up with the number of rational curves on quintic three-fold by carrying out a surprising calculation on its mirror. The new calculation, though not simple, was straightforward, manageable.

Let me not oversimplify: the counting of rational curves was very impressive, but it was the mysterious connection of two disparate parts of geometry that most excited the mathematicians. If the curve count of Candelas et al. proved true, this hinted at some hidden duality of shape that could potentially reveal much, much more about geometry and perhaps even answer some other long-standing mathematical questions.

But again, physicists made these calculations, and while they were quite sophisticated, they lacked the rigor that the mathematical community required. That is, mathematicians required proof, not just evidence. Much activity in the math world surfaced at this point, but it was not until six years had past that Givental [8] and Lian, Liu, and Yau [18] showed that Candelas et al. had in fact been correct, at least for the quintic three-fold.

Unfortunately, the proof did not reveal why the method of Candelas et al. worked, only that it did. In the same year, though, a conjecture was published that spoke of a direct geometric connection between mirror pairs. This was called

the SYZ conjecture after the authors Strominger, Yau, and Zaslow. [20] This conjecture was very promising and created a lot of interest. Due to the nature of the conjecture, it was very difficult to even come up with examples where it could be tested. In some restricted instances, though, the SYZ conjecture was shown to be true. For instance, Gross proved it in the topological category.

Gross and Siebert went on in 2001 to reformulate the SYZ conjecture in an algebro-geometric setting, a setting where it could be proved. In fact, this new setting made the entire subject of mirror symmetry much more accessible by showing that at its base many of these questions of curve counts, among others, could be formulated as combinatorial questions.

It turns out that mirror symmetry is not only a phenomenon occurring on Calabi-Yau manifolds. For instance, Fano manifolds have a mirror called a Landau-Ginzburg model which includes a mirror space along with a regular function. My work is in this more general setting.

1.2 Mirror Symmetry on Toric Varieties

This section gives a broad outline of the background material and goals of this dissertation.

The toric varieties are a special set of varieties with certain nice combinatorial properties. One might say, in fact, that toric varieties are generalizations of affine and projective space. Specifically, a toric variety is a variety that contains a complex torus $(\mathbb{C}^\times)^n$ as a dense set such that the group product of $(\mathbb{C}^\times)^n$ on itself extends naturally to the rest of the variety as a group action. Toric geometry is a rich and well-developed field and much more than can fit in this can be found in many great references including the giant and lucid tome by Cox, Little, and Schenck [4] and Fulton’s slim work [5].

For any toric variety X of dimension n , one can construct a “fan” representation in $\mathbb{R}^n$. This gives a simple description of the torus orbits of X . In the case of toric surfaces, the fan is given by a set of rays originating from the origin of the plane, where the angles of these rays contain important information about the variety.

In mirror symmetry, one refers to an A-side and a B-side. On the A-side is where the curve counts happen on our original variety, the quintic three-fold for instance. On the B-side, where our mirror variety lives, one takes a specific integral, which gives the same whole number answer as the A-side.

Tropical geometry contributes key information on both sides of mirror symmetry for toric varieties. Tropical geometry is a version of algebraic geometry built around a particular semi-ring, the tropical semiring, instead of an algebraically closed field, and it is the correct geometry for studying the combinatorial properties of toric varieties. It is an accessible and intriguing area of study in itself.

Let's consider the toric variety $\mathbb{P}^2$, the projective plane. Now, an A-model calculation for $\mathbb{P}^2$ exists already. Indeed, it was Mikhalkin [19] who gave a curve counting formula for all two-dimensional Fano toric varieties, and this formula is derived from tropical geometry.

The B-model requires calculating *period integrals*, integrals of the form $\int e^{W/\hbar} \Omega$. The function W is called a potential function, and it is a fascinating fact that this function can be read off from the tropical representation of the toric fan of $\mathbb{P}^2$.

From this integral, we obtain a function from which we may read off some curve counts. In order to get the rest of the Gromov-Witten invariants, we need to perturb the potential function by inputting information from certain important tropical objects (Maslov index two tropical disks). This gives us Gromov-Witten invariants with point corrections; in the case of curves, this is the count of curves passing through a certain subset of point.

Now this is all carefully explained by Gross in [12] for Fano toric varieties, and in particular for $\mathbb{P}^2$. The object of interest for this dissertation, though, will be $\mathbb{F}_2$, the simplest non-Fano toric variety. The gist of this dissertation is that the tools that were used for Fano toric varieties like $\mathbb{P}^2$ can be used with non-Fano toric surfaces like $\mathbb{F}_2$ but only after an important modification. Let us consider $\mathbb{F}_2$ in what follows.

In the end, the methods used for $\mathbb{P}^2$ do not quite work for $\mathbb{F}_2$. We need to calculate Gromov-Witten invariants, but the Mikhalkin curve counting formula misses some curves. In fact, tropical geometry only “sees” curves that are transverse

to the toric boundary. We have a way around this, though. The Gross-Siebert Program [9] has machinery for extracting just these extra curves. The strategy is to add a singularity to the picture, in a sense pulling the hidden curves into view and then calculating from there.

As with F_2 , we can make similar computations for other examples of toric surfaces. It is expected that certain underlying patterns will prove to underpin a proof of mirror symmetry for all toric surfaces. From here, a generalization to toric varieties of any dimension is a natural extension of my work.

Chapter 2

The A-Model

I will use this section to explain some of the “lower tech” parts of the A-model of mirror symmetry.

2.1 What is a curve count?

The point of the A-model is the age-old curve counting question: Given a surface, how many of a certain class of curve sit on it? Now, it turns out that to make the definition of “counting a curve” precise, an entire industry of mathematicians inventing different invariants on curve classes had to be built. In hindsight, the consensus view is that the so-called Gromov-Witten invariants have the right amount of invariance. That is, the requirements to be counted by a Gromov-Witten invariant are neither too stringent to exclude too many cases nor too lax to allow too many cases.

The path that we take to even get to the definition of what a curve count is may seem a little elliptical at first, but this just goes to show what rich problem this simple-sounding question presents us with. The first step on the journey is to describe what we mean by “curve.” Indeed, it turns out that if two curves are fairly similar, we want to lump them together. This creates an equivalence class of curves. So part of what we have to do is decide “how similar is similar enough?” Next, we want to discard some pathological curves. These happen to be curves that have infinite sets of automorphisms.

Once we have set up these ideas we have almost come to the full idea of stable maps of curves. Let us define those now.

Definition 2.1.1. Let X be a variety. A *stable n -pointed map* to X is a map

$$f : (C, x_1, \dots, x_n) \rightarrow X$$

such that C is a proper connected reduced nodal algebraic curve, $x_1, \dots, x_n \in C$ are distinct points not coinciding with any nodes of C , such that f has a finite automorphism group. Here an automorphism of f is an automorphism φ of $(C, x_1, \dots, x_n)$ such that $f \circ \varphi = f$.

This definition explicitly defines the objects we want to count when we say we want to “count curves passing through n points.” Now, the set of such maps may be small or large, even empty or infinite. When we consider all of them together, though, we are creating a moduli space. The theory of moduli spaces is all grown up at this point with endless intricacies and pitfalls to watch out for. The main gist is that a moduli space is a “shape” where each point is an object of interest, such as an n -pointed stable map. This moduli space has its own geometry, which may make it, for instance, a variety or a complex manifold, or, unfortunately much more likely, a stack. We will continue to chop our set of maps until we have relatively nice sets that we can mold into comely moduli spaces.

We will chop up our set by considering the homology classes of curves. This is akin to dividing up the set of curves on $\mathbb{R}^2$ based on their degree. That is, the lines, then the conics, then the degree three curves, followed by the degree four curves, and so on. If X is a surface over the complex numbers, then the homology classes of curves live in $H_2(X, \mathbb{Z})$. Let $\beta \in H_2(X, \mathbb{Z})$ be one of these curve classes. Then we say that one of our stable n -pointed maps f *represents* β if the pushforward of the fundamental class of the curve C is equal to β . That is, f represents β if $f_*([C]) = \beta$, where $[C] \in H_2(C, \mathbb{Z})$.

Furthermore, we would like to chop up our set of maps based on genus. We will only be working with genus zero curves in this work. Our moduli space is then ready to be defined.

Definition 2.1.2. The moduli space of n -pointed stable maps of genus g representing a class $\beta \in H_2(X, \mathbb{Z})$ will be denoted as $\overline{\mathcal{M}}_{g,n}(X, \beta)$.

Now we need to consider the dimension of these moduli spaces. It may be that the dimension of one of them is zero, at which point our goal would be to count the number of points in the space. In general, though, the question of dimension for these moduli spaces is complicated. The moduli space may have different components that are different dimensions, for instance. There is something in the literature, though, called the *expected*, or *virtual*, dimension of such a moduli space. We will not go into detail here, but the expected dimension of one of these moduli space is

$$n + (\dim_{\mathbb{C}} X - 3)(1 - g) + \int_{\beta} c_1(\mathcal{T}_X)$$

where c_1 is the first Chern class.

Using this virtual dimension, we can come up with a *virtual fundamental class*

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}} \in H_{2d}(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q}).$$

We will use this idea of a virtual fundamental class to define Gromov-Witten invariants. Going back to the goal of the A-model, which is to “count curves,” if “ n -pointed stable maps of genus g representing a class $\beta \in H_2(X, \mathbb{Z})$ ” is what we mean when we say “curves,” then “computing Gromov-Witten invariants” is what we mean by “count.” Basically, we are going to carry out the count by looking at the moduli space of maps we are interested in and restricting our gaze to the ones that pass through the points we want them to. That ends up looking like a cap product between the virtual fundamental class (in the homology) and a class in the cohomology. To wit:

Definition 2.1.3. *Gromov-Witten invariants.* There are natural maps

$$\text{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \rightarrow X$$

with, for $f : (C, x_1, \dots, x_n) \rightarrow X$ a stable map, $\text{ev}_i([f]) = f(x_i)$. Putting these together gives

$$\text{ev} = \text{ev}_1 \times \dots \times \text{ev}_n : \overline{\mathcal{M}}_{g,n}(X, \beta) \rightarrow X^n.$$

For classes $\alpha_1, \dots, \alpha_n \in H^*(X, \mathbb{Q})$, $\beta \in H_2(X, \mathbb{Z})$, we define the *Gromov-Witten invariant*

$$\langle \alpha_1, \dots, \alpha_n \rangle_{g, \beta} = \int_{[\bar{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}} \text{ev}^*(\alpha_1 \times \dots \times \alpha_n) \in \mathbb{Q}.$$

Here by the integral we mean we are taking a cap product of a cohomology class with a homology class.

Representing a cap product as an integral makes especially good sense in de Rham cohomology where the cohomology classes are forms and the homology classes are cycles. Then capping is equivalent to integrating a form over a cycle.

Now a consequence of this definition is that the Gromov-Witten invariant will be zero unless the degree of the cohomology class matches the degree of the homology class. This will lead to a relation on the virtual dimension that will be useful to us. We will delay giving this formula, though, until we describe the descendent invariants, so that we may state it in full generality.

Now there is a variant of the Gromov-Witten invariants called the *gravitational descendent invariants* that will be important in what follows. Intuitively we may think of it as follows: if the usual Gromov-Witten invariant allows us to count curves passing through points, then the descendent invariants allow us to count curves that are tangent to certain geometrical objects. They are defined as follows. First we construct a set of fairly natural line bundles on our moduli space $\bar{\mathcal{M}}_{g,n}(X, \beta)$. Consider a point $[(C, x_1, \dots, x_n)]$ in our moduli space. Then this point is a map, and the domain C of the map is a curve with its own geometry, independent of X or $\bar{\mathcal{M}}_{g,n}(X, \beta)$. Given $i \in 1, \dots, n$, we may consider the cotangent line $\mathfrak{m}_{x_i}/\mathfrak{m}_{x_i}^2$ of the curve C at the marked point x_i . Here $\mathfrak{m}_{x_i} \subseteq \mathcal{O}_{C, x_i}$ is the maximal ideal. We can in this way construct n line bundles $\mathcal{L}_i$ on $\bar{\mathcal{M}}_{g,n}(X, \beta)$ by considering the set of these cotangent lines all together, one line bundle for each marked point. Now let

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\bar{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q}).$$

Here, c_1 is the first Chern class. The first Chern class acts on a line bundle by mapping it to its natural place in the second cohomology group, the group of classes of codimension one shapes in our moduli space.

Now given the classes $\alpha_1, \dots, \alpha_n \in H^*(X, \mathbb{Q})$, and $\beta \in H_2(X, \mathbb{Z})$, we define *gravitational descendent Gromov-Witten invariants* as

$$\langle \psi^{p_1} \alpha_1, \dots, \psi^{p_n} \alpha_n \rangle_{g, \beta} = \int_{[\bar{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}} \psi_1^{p_1} \cup \dots \cup \psi_n^{p_n} \cup \text{ev}^*(\alpha_1 \times \dots \times \alpha_n) \in \mathbb{Q}.$$

Now we will give the promised relation on the virtual dimension. Let us say that $\deg \alpha_i = d_i$. In order for the descendent invariants to be nonzero, we would need $\sum_{i=1}^n d_i + \sum_{i=1}^n 2p_i$ to be twice the virtual dimension of $\bar{\mathcal{M}}_{g,n}(X, \beta)$. Using Riemann-Roch, this gives the relation

$$2(n + (\dim_{\mathbb{C}} X - 3)(1 - g) + \int_{\beta} c_1(\mathcal{T}_X)) = \sum_{i=1}^n d_i + \sum_{i=1}^n 2p_i. \quad (2.1)$$

Now there are a series of *Axioms* which (descendent) Gromov-Witten invariants satisfy. We will give here only the ones that will be pertinent.

Definition 2.1.4. The Fundamental Class Axiom. If $n + 2g \geq 4$ or $\beta \neq 0$ and $n \geq 1$, then

$$\langle \psi^{p_1} \alpha_1, \dots, \psi^{p_{n-1}} \alpha_{n-1}, [X] \rangle_{g, \beta} = \sum_{i=1}^{n-1} \langle \psi^{p_1} \alpha_1, \dots, \psi^{p_i-1} \alpha_i, \dots, \psi^{p_{n-1}} \alpha_{n-1} \rangle_{g, \beta}$$

where a summand on the right is taken to be zero if ψ appears with a negative power.

Definition 2.1.5. The Point Mapping Axiom. If $n \leq 3$, then

$$\langle \psi^{\nu_1} \alpha_1, \dots, \psi^{\nu_n} \alpha_n \rangle_{0,0} = 0$$

unless $n = 3$ and $\nu_1 = \dots = \nu_n = 0$. Furthermore,

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,0} = \int_X \alpha_1 \cup \alpha_2 \cup \alpha_3.$$

2.2 The Givental J -Function

We have just discussed rich and useful invariants of a manifold, the Gromov-Witten invariants. We can, though, combine these invariants into one portmanteau invariant known as the *Givental J -function*. We can think of this by analogy with how we created moduli spaces. To simplify, moduli spaces are shapes where each point represents an object of interest. That is, a moduli space is a shape of shapes. The Givental J -function is an agglomeration of invariants: with the use of extra variables, we string together into a formal sum the set of Gromov-Witten invariants of a manifold. Formal sums of this type are known as “generating functions.” Therefore, the Givental J -function for a manifold is a generating function made up of the Gromov-Witten invariants of that manifold.

The J -function is key to mirror symmetry. The B-side will produce an invariant that will match this one, thus connecting the two sides.

Here is the general form for the Givental J -function.

Definition 2.2.1. For a manifold with cohomology ring $H^*(X, \mathbb{C})$ generated as a $\mathbb{C}$ vector space by $T_0, T_1, \dots, T_m$, where by convention T_0 is the fundamental class and the second cohomology group $H^2(X, \mathbb{C})$ generated is by $T_1, T_2, \dots, T_p$, we define the Givental J -function.

$$J_X := e^{(\sum_{k=0}^p y_k T_k)/\hbar} \cup \left(T_0 + \sum_{\beta, n_k} \sum_{i=0}^m \langle T_0, T_{p+1}^{n_{p+1}}, \dots, T_m^{n_m}, \frac{T_i}{\hbar - \psi} \rangle_{0, \beta} T^i \left(\prod_{k=1}^p e^{y_k \int_{\beta} T_k} \right) \frac{y_{p+1}^{n_{p+1}} \cdots y_m^{n_m}}{n_{p+1}! \cdots n_m!} \right)$$

where the sum is over all integral homology classes of curves $\beta \in \mathbb{H}_2(X, \mathbb{Z})$ and $n_{p+1}, \dots, n_m \geq 0$. The expression $e^{(\sum_{k=0}^p y_k T_k)/\hbar}$ is interpreted in the ring $H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[y_1, \dots, y_p][[y_0, \hbar^{-1}]]$. Also, T^i is the homology element which is the Poincaré dual of the cohomology basis element T_i . Here, the expression $\frac{T_i}{\hbar - \psi}$ is interpreted as the power series expansion

$$\frac{T_i}{\hbar - \psi} = \sum_{\nu=0}^{\infty} \hbar^{-(\nu+1)} \psi^{\nu} T_i.$$

In the next section, we will show in detail how to directly modify this expression for the case of the second Hirzebruch surface. First, we will state the end result of a similar calculation for the projective plane. This calculation can be found on page 67 of [12].

The Givental J -function for $\mathbb{P}^2$ is as follows:

$$J_{\mathbb{P}^2} = e^{(y_0 T_0 + y_1 T_1)/\hbar} \cup \left(T_0 + \sum_{i=0}^2 \left(y_2 \hbar^{-1} \delta_{2,i} + \sum_{d \geq 1} \sum_{\nu \geq 0} \langle T_2^{3d+i-2-\nu}, \psi^\nu T_{2-i} \rangle_{0,d} \hbar^{-(\nu+2)} e^{dy_1} \frac{y_2^{3d+i-2-\nu}}{(3d+i-2-\nu)!} \right) T_i \right).$$

Here, d stands for the degree of a curve as that fully defines a curve's homology class for $\mathbb{P}^2$. Also, T_0 stands for the fundamental class $[\mathbb{P}^2]$, while T_1 is the cohomology class of a curve generating $H^2(\mathbb{P}^2, \mathbb{C})$, and T_2 is the cohomology class of a point on $\mathbb{P}^2$.

2.3 The $\mathbb{F}_2$ Case

In this section we will write down the Givental J -function for the second Hirzebruch surface $X = \mathbb{F}_2$.

First, we need to know about the cohomology ring for $\mathbb{F}_2$. The cohomology groups for $\mathbb{F}_2$ are $H^0(X, \mathbb{C}) = \mathbb{C}$, $H^2(X, \mathbb{C}) = \mathbb{C}^2$, and $H^4(X, \mathbb{C}) = \mathbb{C}$, while all the other cohomology groups are zero. This means that $m = 4$. As above, we let T_0 be the fundamental class of $\mathbb{F}_2$, and since $H^2(X, \mathbb{C})$ is two dimensional, we set T_1 and T_2 as the generators of this group, which makes $p = 2$. This shortens the expression to

$$e^{(\sum_{k=0}^2 y_k T_k)/\hbar} \cup \left(T_0 + \sum_{\beta, n_3} \sum_{i=0}^3 \langle T_0, T_3^{n_3}, \frac{T_i}{\hbar - \psi} \rangle_{0,\beta} T^i \left(\prod_{k=1}^2 e^{y_k \int_\beta T_k} \right) \frac{y_3^{n_3}}{n_3!} \right)$$

Using the expansion given in the definition for $\frac{T_i}{\hbar - \psi}$ we get the following.

$$e^{(\sum_{k=0}^2 y_k T_k)/\hbar} \cup \left(T_0 + \sum_{\beta, n_3} \sum_{i=0}^3 \sum_{\nu \geq 0} \langle T_0, T_3^{n_3}, \psi^\nu T_i \rangle_{0,\beta} \hbar^{-(\nu+1)} T^i \left(\prod_{k=1}^2 e^{y_k \int_\beta T_k} \right) \frac{y_3^{n_3}}{n_3!} \right)$$

Let us use the list of axioms to simplify the Gromov-Witten invariants. We can use the Fundamental Class Axiom for descendants to see that for $\beta \neq 0$

$$\langle T_0, T_3^{n_3}, \psi^\nu T_i \rangle_{0,\beta} = \langle T_3^{n_3}, \psi^{\nu-1} T_i \rangle_{0,\beta}.$$

Now, by our previous discussion, a given descendant Gromov-Witten invariant $\langle \psi^{p_1} \alpha_1, \dots, \psi^{p_n} \alpha_n \rangle_{g,\beta}$ is zero unless relation (2.1) holds. That is, if $\deg \alpha_i = d_i$, then to get a non-zero result the following must be true:

$$2(n + (\dim_{\mathbb{C}} X - 3)(1 - g) + \int_{\beta} c_1(\mathcal{T}_X)) = \sum_{i=1}^n d_i + \sum_{i=1}^n 2p_i.$$

Here $c_1(\mathcal{T}_X)$ is the first Chern class of the tangent bundle of X , and it is an element of the second integral cohomology group $H^2(X, \mathbb{Z})$. We can derive this for $\mathbb{F}_2$. Any homology class of a curve $\beta \in H_2(\mathbb{F}_2, \mathbb{Z})$ is an integral-linear combination of the curve classes $[f]$ and $[C_0]$ (where f is a fiber and C_0 is a section of $\mathbb{F}_2$ as a ruled surface). It is a well-known fact that $\int_{[f]} c_1(\mathcal{T}_{\mathbb{F}_2}) = 2$ and $\int_{[C_0]} c_1(\mathcal{T}_{\mathbb{F}_2}) = 0$; see for instance Hartshorne's section on ruled surfaces, section V.2. [13] (For general Hirzebruch surface $\mathbb{F}_n$ we have $\int_{[f]} c_1(\mathcal{T}_{\mathbb{F}_n}) = 2$ and $\int_{[C_0]} c_1(\mathcal{T}_{\mathbb{F}_n}) = -n + 2$.) Now, any curve class β can be written as $\beta = a[C_0] + b[f]$, and so $\int_{\beta} c_1(\mathcal{T}_{\mathbb{F}_2}) = \int_{a[C_0] + b[f]} c_1(\mathcal{T}_{\mathbb{F}_2}) = a \cdot 0 + b \cdot 2 = 2b$. We are only working over genus $g = 0$, and we know the dimension of our variety. So the relation reduces to

$$2(n - 1 + 2b) = \sum_{i=1}^n d_i + \sum_{i=1}^n 2p_i. \quad (2.2)$$

So in order to get a contribution from $\langle T_3^{n_3}, \psi^{\nu-1} T_i \rangle_{0,\beta}$, we must have that $2((n_3 + 1) - 1 + 2b) = 4n_3 + \deg(T_i) + 2(\nu - 1)$. That is, $n_3 = 2b - \frac{\deg(T_i)}{2} - \nu + 1$.

In the case where $\beta = 0$ we get

$$\langle T_0, T_3^{n_3}, \psi^\nu T_i \rangle_{0,0} = 0$$

unless equation (2.2) holds. That is, we only get a contribution when $2(n_3 + 2 - 1) = (0 + 4n_3 + \deg(T_i)) + (2\nu)$. That is, $n_3 = 1 - \frac{\deg(T_i)}{2} - \nu$. Now since $n_3 \geq 0$, we can immediately discount any scenario where $\nu > 1$ or $\deg(T_i) > 2$. If $\nu = 1$ then $n_3 = 0$, and by the Point Mapping Axiom, we get no contribution. Finally, if $\nu = 0$

we get three possible contributions from $i = 0, 1, 2$. But again if $i = 1$ or 2 then $n_3 = 0$, which we just showed gives no contribution. So the calculation for the only contribution when $\beta = 0$ is given by the Point Mapping Axiom:

$$\langle T_0, T_3, T_0 \rangle_{0,0} = \int_X T_0 \cup T_3 \cup T_0 = 1$$

This equals one since a point must intersect the entire space exactly once.

For brevity, define $\eta := \eta(b, i, \nu) = 2b - \frac{\deg(T_i)}{2} - \nu + 1$. Now the intersection numbers of the curves classes $[C_0]$ and $[f]$ are well known. They are $[C_0] \cdot [f] = 1$, $[C_0]^2 = -2$, and $[f]^2 = 0$. So we obtain $y_1 \int_\beta T_1 + y_2 \int_\beta T_2 = y_1 \int_{a[C_0] + b[f]} [C_0] + y_2 \int_{a[C_0] + b[f]} [f] = (-2a + b)y_1 + ay_2$. Putting this all together we get the following.

$$J_{\mathbb{F}_2} = e^{(y_0 T_0 + y_1 T_1 + y_2 T_2)/\hbar} \cup \left(T_0 + \hbar^{-1} T_3 y_3 + \sum_{i=0}^3 \sum_{\substack{\beta \neq 0 \\ \beta = a[C_0] + b[f]}} \sum_{\nu \geq 0} \langle T_3^\eta, \psi^{\nu-1} T_i \rangle_{0,\beta} \hbar^{-(\nu+1)} T^i \cdot e^{(-2a+b)y_1 + ay_2} \cdot \frac{y_3^\eta}{\eta!} \right)$$

Let us rewrite this so that it will conform to our later calculations better.

$$J_{\mathbb{F}_2} = e^{(y_0 T_0 + y_1 T_1 + y_2 T_2)/\hbar} \cup \left(T_0 + \hbar^{-1} T_3 y_3 + \sum_{i=0}^3 \sum_{\substack{\beta \neq 0 \\ \beta = a[C_0] + b[f]}} \sum_{\nu \geq 2} \langle T_3^\eta, \psi^{\nu-2} T_i \rangle_{0,\beta} \hbar^{-\nu} T^i \cdot e^{(-2a+b)y_1 + ay_2} \cdot \frac{y_3^\eta}{\eta!} \right) \quad (2.3)$$

where now $\eta := \eta(b, i, \nu) = 2b - \frac{\deg(T_i)}{2} - \nu$.

Chapter 3

The B-Model

The basis of mirror symmetry is that Gromov-Witten invariants of a variety can be calculated by calculating period integrals on a mirror variety. Therefore it is the goal of this chapter to describe period integrals and show how they relate back to Gromov-Witten invariants. But first let us put some of the novel aspects of the B-side work of this dissertation into historical perspective.

As described in the introduction, the mirror to a Calabi-Yau manifold is another Calabi-Yau manifold, but one can talk about mirror symmetry in a more general setting. In particular, if X is a Fano toric variety, then its mirror is called a *Landau-Ginzburg model* $(\check{X}, W)$. This model involves a mirror space $\check{X}$ and a regular function $W : \check{X} \rightarrow \mathbb{C}$. The regular function is called the *Landau-Ginzburg potential*, and it keeps track of some of the geometry of X that is lost when looking to the mirror space $\check{X}$.

Givental first described mirrors of Fano toric varieties in [7], and Hori and Vafa gave a physics derivation in [14]. Gross described the geometric connection between the A-side and B-side of Fano toric varieties using tropical geometry and carefully proved that this worked for $\mathbb{P}^2$ in [12].

It is the contention of this dissertation that mirror symmetry can be extended to non-Fano toric varieties. The expectation is that the farther a variety is from being Fano, the more one has to perturb the construction of the Landau-Ginzburg model. This will be described in detail later on. In this chapter we will focus on what is known in the Fano case, as this will serve as a base for our work on

non-Fano toric varieties.

3.1 The Period Integral

A period integral is simply an integral in the form

$$\int_{\Xi} e^{W/\hbar} \Omega.$$

In this section we will describe the main parts of this expression and also answer a few questions. In what space does the cycle Ξ live? How do we choose the cycle? What space does the form Ω act on? Whence the $\hbar$? And hopefully other questions that might come to mind will be answered.

Beginning with a toric surface X (in two complex dimensions) let us explicitly set up the mirror Landau-Ginzburg model $(\check{X}, W)$ for it. This will take us on our first big step towards understanding the period integral. Simply stated, $\check{X} = (\mathbb{C}^\times)^2$. And W , a regular function on $\check{X}$, is determined by the fan Σ of the toric surface X . Specifically, if Σ has $n + 1$ rays, then $W = x^{a_0}y^{b_0} + x^{a_1}y^{b_1} + \cdots + x^{a_n}y^{b_n}$ where (a_i, b_i) are the primitive vectors of the rays.

3.1.1 The Twisted de Rham Complex

The twisted de Rham complex provides us with natural cohomology and homology theories on our mirror space.

First, let us fix a nonsingular variety X of dimension n and a regular function $W : X \rightarrow \mathbb{C}$, a Landau-Ginzburg potential. After fixing an $\hbar \in \mathbb{C}^\times$, let us consider the *twisted de Rham complex*

$$(\Omega_X^\bullet, d + \hbar^{-1}dW \wedge).$$

This is the following complex

$$\Omega_X^0 \rightarrow \Omega_X^1 \rightarrow \Omega_X^2 \rightarrow \cdots$$

where the differential is given by $\omega \mapsto d\omega + \hbar^{-1}dW \wedge \omega$. Now we have a choice of $\hbar$ from which we obtain an entire family of twisted de Rham complexes.

The details of the cohomology theory are arcane and are unimportant for us, but the interested reader can find more details in section 2.2.1 in [12]. The homology theory is of interest. The gist is that for a fixed $\hbar$ we can construct a homology group

$$H_n(X, \operatorname{Re} W/\hbar \ll 0; \mathbb{C})$$

that is naturally dual to the top cohomology group. That is to say that a cycle in this homology group pairs with a closed n -form in the twisted de Rham complex. Given such a cycle Ξ and such a form ω we can compute the *oscillatory integral*

$$\int_{\Xi} e^{W/\hbar} \omega.$$

This expression should look familiar as the period integral is an example of an oscillatory integral.

Again, the details of $H_n(X, \operatorname{Re} W/\hbar \ll 0; \mathbb{C})$ are unimportant to us, but the idea is that this homology group represents cycles in X that are allowed to be unbounded but only in the direction in which $\operatorname{Re} W/\hbar \rightarrow -\infty$.

3.1.2 Cycles of Toric Surface Mirrors

Now that we have our mirror space and a homology theory on it, we can begin to talk about the cycles Ξ of a mirror to a toric surface. (We should emphasize that we have just transitioned from n dimensions to 2.) The answer to the question of where the cycles reside is now easy, they reside in the mirror space $\check{X} = (\mathbb{C}^\times)^2$. The cycles come in three flavors. The first and easiest cycle Ξ to deal with is the unit torus $T^2 \subseteq (\mathbb{C}^\times)^2$. The unit torus is mirror to the single point at the origin of the fan of the toric surface X . That is, it is mirror to the zero-dimensional cone in Σ . (In fact, we will see in the next couple paragraphs that the “flavors” of these cycles correspond to the different dimensions of the cones in Σ .)

The next set of cycles are those that are dual to particular unions of rays in Σ (see section 3.2 for an example). Given a ray ρ in the fan for a toric surface, we can associate to ρ a subset of $(\mathbb{C}^\times)^2$ that is topologically isomorphic (i.e., homeomorphic) to a cylinder. This subset is obtained in the following way. Define the map

$$\operatorname{Log} : (\mathbb{C}^\times)^2 \rightarrow \mathbb{R}^2$$

$$\text{Log}(x, y) = (\log|x|, \log|y|).$$

Then take the inverse image of $\rho \subseteq \mathbb{R}^2$ and restrict this set to a topological cylinder by choosing a “direction” for the cylinder to point. The key here is that one end of the cylinder will be unbounded and run off to infinity; we want the potential function W to run to negative infinity as we travel in the unbounded direction. This will solve some convergence issues for us before they crop up.

The specific method for choosing the direction of the cylinder is to choose the appropriate argument for the monomial $z^{(a,b)}$ where (a, b) is the primitive vector for ρ .

The previous two paragraphs can be summed up succinctly as follows. Let our ray be ρ with primitive vector (a, b) and the cylinder associated to ρ be called S , then

$$S := \{(x, y) \in \text{Log}^{-1}(\rho) : \arg(x^a y^b) = 0\} \quad (3.1)$$

In particular, the bounded edge of the cylinder will sit on the unit torus in $(\mathbb{C}^\times)^2$; one can read this from the definition of S . This fact is useful to us because it demonstrates that if we have a set of these cylinders $S_1, S_2, \dots, S_\nu$, where ν is just some natural number, then we may be able to choose some subset of the unit torus that, when unioned with the cylinders, will provide a connected, orientable surface (in two real dimensions). This surface will be the second “flavor” of cycle Ξ over which we will integrate.

The main question to answer here is, “Which subset of rays ρ_i will give cylinders S_i that glue with a portion of the torus to create a 2-cycle?” The answer to this is any subset of rays that meets the *balancing condition*, $\sum_i d_i(a_i, b_i) = 0$. Here we have chosen rays ρ_i with primitive vectors (a_i, b_i) and with weights $d_i \in \mathbb{N}$.

Why do we require the balancing condition? Well, that will take a few paragraphs to answer. If $\Lambda \cong \mathbb{Z}^2$ is a lattice in $\mathbb{R}^2$, we have that the first fundamental group $\pi_1(\mathbb{R}^2/\Lambda, \mathbb{Z})$ of the torus $\mathbb{R}^2/\Lambda$ is isomorphic to Λ . And then $\Lambda \cong H_1(\mathbb{R}^2/\Lambda, \mathbb{Z})$ since the first homology group is the abelianization of the fundamental group by the Hurewicz Theorem and since the fundamental group of the torus is already abelian. The identification of the fundamental group with the first homology group tells us that we can consider an element $(a, b) \in \Lambda$ as a loop on the torus. This loop,

when considered on the universal covering space $\mathbb{R}^2$ of the torus is a line segment connecting the origin to the point (a, b) .

We can be more explicit. The boundary of a cylinder S in (3.1), will be a loop on the unit torus. The loop is given by $\arg(x^a y^b) = 0$, $|x|=1$, $|y|=1$. Writing $x = e^{i\theta_1}$ and $y = e^{i\theta_2}$, we can consider this loop on the universal covering space $\mathbb{R}^2$ with the coordinates (θ_1, θ_2) . So the equation for the loop is now given by $a\theta_1 + b\theta_2 = 0 \pmod{2\pi}$. This is just the line segment connecting the origin of $\mathbb{R}^2$ to the point $(-b, a)$.

Now consider a weighted sum of cylinders $\sum d_i S_i$. The boundary of this sum resides on the unit torus and can be given by sum of loops $\sum d_i \partial S_i$, or by the reasoning just given $\sum d_i (-b_i, a_i) \in H_1(\mathbb{R}^2/\Lambda, \mathbb{Z})$. If we would like to turn the sum of cylinders into a cycle, we would need to glue on a piece of the torus, a “buffer,” to connect them all. This may not be possible. Indeed, it is possible if and only if $\sum d_i (-b_i, a_i)$ actually bounds a buffer, or 2-chain, on the torus. This follows from the definition of homology.

So to be able to turn $\sum d_i S_i$ into a cycle, there has to exist a suitable buffer. And thus we must have that $\sum d_i (-b_i, a_i) = 0$. This is equivalent to $\sum d_i (a_i, b_i) = 0$, the balancing condition.

The choice of subset of the unit torus is not unique, and in fact one can consider a double (or triple, etc.) cover of the unit torus when making this choice. For an example of this, see section A.6.

The third flavor of cycle, mirror to the fundamental class of the toric surface, is $(\mathbb{R}_{\geq 0})^2 \subseteq (\mathbb{C}^\times)^2$.

So now we see that the possible cycles Ξ that we will integrate over are all connected, orientable surfaces (in two real dimensions) sitting inside $(\mathbb{C}^\times)^2$. One of the cycles, T^2 , is bounded, the rest are unbounded.

We now turn our attention to the differential form Ω in the period integral. We define Ω to be $\frac{dx \wedge dy}{xy}$. After choosing an orientation on Ξ , this form acts with $e^{W/\hbar}$ to integrate Ξ .

Finally, the $\hbar$ term in the period integral acts as a compass, pointing our period integral in a direction that ensures convergence.

3.2 The $\mathbb{F}_2$ Case

In this work, we will focus our study of mirror symmetry on the second Hirzebruch surface F_2 , a toric surface given by the fan in figure 3.1 with four rays going in the negative x direction (ρ_1), the positive y direction (ρ_2), the positive x direction (ρ_3), and passing from the origin through the point $(-2, -1)$ (ρ_4). We will consider the mirror to F_2 , with $(\check{F}_2, W)$ where $\check{F}_2 = (\mathbb{C}^\times)^2$ and $W = x + y + x^{-1} + x^{-2}y^{-1}$.

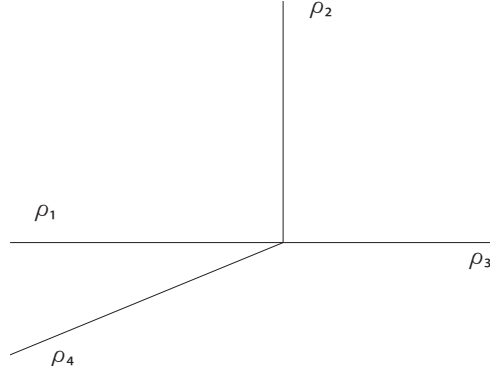


Figure 3.1: The fan for $\mathbb{F}_2$.

We will need the map

$$\begin{aligned} \text{Log} : (\mathbb{C}^\times)^2 &\rightarrow \mathbb{R}^2 \\ \text{Log}(x, y) &= (\log|x|, \log|y|) \end{aligned}$$

We will be taking an integral over several different cycles Ξ_i . The first of those cycles is the unit torus T^2 which we will call Ξ_1 . With regards to the above map, this cycle is simply $\text{Log}^{-1}((0, 0))$.

We will construct the remaining cycles by considering the following topological cylinders, each corresponding to a different leg of the fan for F_2 .

$$\begin{aligned} S_1 &:= \{(x, y) \in \text{Log}^{-1}(\rho_1) : \arg(x) = 0\} \\ S_2 &:= \{(x, y) \in \text{Log}^{-1}(\rho_2) : \arg(y) = 0\} \\ S_3 &:= \{(x, y) \in \text{Log}^{-1}(\rho_3) : \arg(x) = 0\} \\ S_4 &:= \{(x, y) \in \text{Log}^{-1}(\rho_4) : \arg(x^{-2}y^{-1}) = 0\} \end{aligned}$$

Given these sets we can build the remaining cycles. We set $\Xi_2 := S_1 \cup S_3$ simply. We then set $\Xi_3 := S_2 \cup S_3 \cup S_4 \cup \{\text{buffer}\}$, where the buffer is the part of the unit

torus needed to connect the other three components (this buffer will be covered in more detail in the appendix, section A.6). The last cycle Ξ_4 is the positive real quadrant, that is, $\Xi_4 := \mathbb{R}_{>0}^2$.

3.2.1 Givental's Mirrors of Toric Varieties

In this section we will describe through the example of the second Hirzebruch surface $\mathbb{F}_2$ the details behind the choice of Landau-Ginzburg models $(\check{X}, W)$ of a toric variety X . We will be using only basic concepts from toric geometry, which can be found in any text on the subject, see for instance [4] and [5].

Let us first set some notation. Fix the lattice $M = \mathbb{Z}^2$, the dual lattice $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Fix the fan of F_2 as Σ , which is complete in $M_{\mathbb{R}}$. Now F_2 , which is equal to X_{Σ} by definition, is a nonsingular toric variety. Let T_{Σ} be the free abelian group generated by the elements of $\Sigma^{[1]}$, the one-dimensional cones of Σ , with $\rho \in \Sigma^{[1]}$ corresponding to a generator t_{ρ} . That is, $T_{\Sigma} = \mathbb{Z}^4$ with basis t_1, t_2, t_3, t_4 corresponding to the rays $\rho_1, \rho_2, \rho_3, \rho_4$ respectively.

We have the map $r : T_{\Sigma} \rightarrow M$ given by $r(t_{\rho}) = m_{\rho}$, the primitive generator of ρ . Since F_2 is non-singular, r is surjective.

So there is a natural exact sequence of groups

$$0 \rightarrow K_{\Sigma} \xrightarrow{i} T_{\Sigma} \xrightarrow{r} M \rightarrow 0 \quad (3.2)$$

defining K_{Σ} . Let us explore the kernel K_{Σ} of the map r . K_{Σ} will be two dimensional since the domain was four dimensions and the image is two dimensions. To find out exactly how this two dimensional group sits in T_{Σ} , we need to find the nullspace of the matrix associated to r . That is, solve the following equation.

$$\begin{pmatrix} -1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

Using this equation, we see that any vector in the nullspace will have the properties

that $b = d$ and $c = 2d + a$. That is,

$$\text{Null}(r) = \left\{ \begin{pmatrix} a \\ d \\ 2d+a \\ d \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \in T_\Sigma = \mathbb{Z}^4 : a, d \in \mathbb{Z} \right\}$$

Or in other words, elements in the kernel K_Σ are precisely the vectors of the form $at_1 + dt_2 + (2d + a)t_3 + dt_4$ in T_Σ . That is, the toric divisors that satisfy the “balancing condition,” which we described in the previous section. Furthermore,

$$\text{thinking of } K_\Sigma \text{ as } \mathbb{Z}^2, \text{ we can describe the map } i \text{ as } i(a, d) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ d \end{pmatrix}.$$

One can show that the Weil divisor class group of X_Σ is given as follows. First note that if τ a cone in the fan Σ of a toric variety, then D_τ is the subvariety, or toric stratum, associated to τ . Thus D_ρ is a (Weil) divisor on X_Σ for any ray ρ . For further information on the class group, consider the transpose map ${}^t r$ is given by

$$\begin{aligned} {}^t r : N &\rightarrow \check{T}_\Sigma \\ n &\mapsto \sum_{\rho \in \Sigma[1]} \langle n, m_\rho \rangle t_\rho^* \end{aligned}$$

which is the same as

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -2 & -1 \end{pmatrix} : \mathbb{Z}^2 \rightarrow \mathbb{Z}^4$$

$$(a, b) \mapsto (-a, b, a, -2a - b).$$

This gives us a short exact sequence dual to (3.2).

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{{}^t r} & \check{T}_\Sigma & \xrightarrow{{}^t i} & \check{K}_\Sigma \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & \mathbb{Z}^2 & & \mathbb{Z}^4 & & \mathbb{Z}^2 \end{array} \quad (3.3)$$

We claim that $\text{Cl}(X_\Sigma) \cong \text{coker } {}^t r$. Indeed, an element $\psi : T_\Sigma \rightarrow \mathbb{Z}$ of $\check{T}_\Sigma$ corresponds to the Weil divisor $\sum_{\rho \in \Sigma[1]} \psi(t_\rho) D_\rho$, and ${}^t r(n)$ is the divisor of zeroes and poles of (the character) z^n , for each $n \in N$. So the class group will be the first group modulo the second: the Weil divisors mod their linear equivalences. We can think of the nonzero elements in the class group as Weil divisors as “coming from the character z^0 .” That is we want to choose the characters z^n where n is in the nullspace of the function r . In particular, for $\mathbb{F}_2$, they are characters of the form $d_1^a d_2^b$ where $a, b \in \mathbb{Z}$ and $d_1 := z^{t_1+t_3}$ and $d_2 := z^{t_2+2t_3+t_4}$. Otherwise stated: the nullspace (of r) is the orthogonal complement of the row space. Or $\ker(r) = (\text{im}({}^t r))^\perp$.

It turns out that all other (that is: non-toric) divisors are linearly equivalent to a toric divisor; this is elucidated in the proof of Theorem 4.1.3 in [4]; basically, the class group of a big torus is 0.

In sum, let D_1 be the toric divisor corresponding to t_1+t_3 , and let D_2 correspond to $t_2+2t_3+t_4$, then

$$\text{Cl}(F_2) = \{aD_1 + bD_2 | a, b \in \mathbb{Z}\}.$$

We can plug this result into the short exact sequence (3.3):

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{{}^t r} & \check{T}_\Sigma & \xrightarrow{{}^t i} & \text{Cl}(X_\Sigma) \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & \mathbb{Z}^2 & & \mathbb{Z}^4 & & \mathbb{Z}^2 \end{array}$$

If a variety is smooth, then its class group is equal to its Picard group. So this gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{{}^t r} & \text{Hom}_{\mathbb{Z}}(T_\Sigma, \mathbb{Z}) & \xrightarrow{{}^t i} & \text{Pic} X_\Sigma \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & \mathbb{Z}^2 & & \mathbb{Z}^4 & & \mathbb{Z}^2 \end{array}$$

After tensoring with $\mathbb{C}^\times$, we get another exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & N \otimes \mathbb{C}^\times & \xrightarrow{{}^t r} & \text{Hom}(T_\Sigma, \mathbb{C}^\times) & \xrightarrow{\kappa} & \text{Pic} X_\Sigma \otimes \mathbb{C}^\times \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & (\mathbb{C}^\times)^2 & & (\mathbb{C}^\times)^4 & & (\mathbb{C}^\times)^2 \end{array}$$

We should think of κ as providing the family of mirrors to X_Σ in the following sense. We define the *Kahler moduli space* of X_Σ , that is, the space where each point represents a mirror, to be

$$\mathcal{M}_\Sigma := \text{Pic} X_\Sigma \otimes \mathbb{C}^\times = \text{Spec } \mathbb{C}[K_\Sigma] = (\mathbb{C}^\times)^2.$$

So we have a morphism

$$\kappa : \text{Hom}(T_\Sigma, \mathbb{C}^\times) = \text{Spec } \mathbb{C}[T_\Sigma] = (\mathbb{C}^\times)^4 \rightarrow \mathcal{M}_\Sigma = (\mathbb{C}^\times)^2.$$

given by

$$(a, b, c, d) \mapsto {}^t i(a, b, c, d) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Note that a fiber of κ over a closed point of $\mathcal{M}_\Sigma$ is canonically isomorphic to $\text{Spec } \mathbb{C}[M] = (\mathbb{C}^\times)^2$.

Keep in mind that T_Σ is a group, and though it is generated by only four elements (the t_i) as a free abelian group, as a monoid it is generated by those elements and their inverses. That is how we obtain $\text{Spec } (\mathbb{C}[T_\Sigma]) = \text{Spec } (\mathbb{C}[z^{\pm t_1}, \dots, z^{\pm t_4}]) = (\mathbb{C}^\times)^4$. (Remember that $t_i = t_{\rho_i}$.) And the correspondence between $\text{Hom}(T_\Sigma, \mathbb{C}^\times)$ and $\text{Spec } \mathbb{C}[T_\Sigma]$ is exactly as was explained before with characters on T_Σ and Weil divisors. This correspondence of course also applies to $\text{Pic } X_\Sigma \otimes \mathbb{C}^\times$ and $\text{Spec } \mathbb{C}[K_\Sigma]$.

We then set

$$\check{\mathcal{X}}_\Sigma := \text{Hom}(T_\Sigma, \mathbb{C}^\times) = (\mathbb{C}^\times)^4.$$

We will take W_0 to be the function of $\check{\mathcal{X}}_\Sigma$ defined by

$$W_0 := \sum_{\rho \in \Sigma^{[1]}} z^{t_\rho} = z^{t_1} + z^{t_2} + z^{t_3} + z^{t_4}.$$

Now if we set $x = z^{t_3}$ and $y = z^{t_2}$, then we have two relations $d_1 = z^{t_1+t_3} = xz^{t_1}$ and $d_2 = z^{t_2+2t_3+t_4} = x^2yz^{t_4}$. This gives $z^{t_1} = d_1x^{-1}$ and $z^{t_4} = d_2x^{-2}y^{-1}$. So now we have

$$W_0 = x + y + d_1x^{-1} + d_2x^{-2}y^{-1}$$

Now W_0 makes sense as a function on $\text{Spec } \mathbb{C}[T_\Sigma]$, and hence as a function on $\check{\mathcal{X}}_\Sigma$. Think of $(\check{\mathcal{X}}_\Sigma, W_0)$ as a family of Landau-Ginzburg models which are mirror to X_Σ . This is explained in more detail in [12], section 2.2.3.

The take-away here is that if we choose $d_1 = d_2 = 1$ then we obtain $\check{X} = (\mathbb{C}^\times)^2$ and $W_0 = x + y + x^{-1} + x^{-2}y^{-1}$ just as we would expect, given the discussion at the beginning of section 3.1. Different choices for the parameters d_1 and d_2 will give different perturbations of W_0 . In later calculations, we will use the useful change of variables $y_1 = \log(d_1)$ and $y_2 = \log(d_2)$. We will see in the following section that these perturbations match, at least intuitively, with what we will define as the universal unfolding of W_0 . Furthermore, y_1 and y_2 are what we will refer to as flat coordinates.

3.3 General Conjecture

Now that we have an understanding of the period integral, we need some way to relate it back to Gromov-Witten invariants. This is the key calculation in mirror symmetry. This is generally done with a theorem, but as these theorems have been stated other places, we will do this via a conjecture. The conjecture, found at the end of this section is a generalization of Proposition 2.45 in [12] from the case of $\mathbb{P}^n$ to the case of all toric surfaces.

The point of the conjecture is that we can combine the period integrals (B-side) in such a way that we can, via a simple change of variable, obtain the J -function (A-side). Recall that the J -function encapsulates the Gromov-Witten invariants; so this conjecture nicely connects period integrals to Gromov-Witten invariants.

But this is putting the cart before the horse. First we need to develop the concept of a universal unfolding.

For complex manifolds, a *universal unfolding* of W_0 is a germ of a complex manifold $0 \in \mathcal{M}$ along with a holomorphic function

$$W : \mathcal{M} \times \check{X} \rightarrow \mathbb{C}$$

such that :

- $W|_{\{0\} \times \check{X}} = W_0$
- For any germ $0 \in Y$ of a complex space, and holomorphic function

$$W' : Y \times \check{X} \rightarrow \mathbb{C}$$

such that $W'|_{\{0\} \times \check{X}} = W_0$, there exists a commutative diagram

$$\begin{array}{ccc} Y \times \check{X} & \xrightarrow{\eta} & \mathcal{M} \times \check{X} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\varphi} & \mathcal{M} \end{array}$$

where $\varphi : (0 \in Y) \rightarrow (0 \in \mathcal{M})$ is a map of germs whose induced map on Zariski tangent spaces is unique, the vertical arrows are projections, $\eta|_{\{0\} \times \check{X}} : \{0\} \times \check{X} \rightarrow \{0\} \times \check{X}$ is the identity, and $W' = W \circ \eta$.

Now, given a toric surface X with fan Σ , we will use a specific notation with regards to the cohomology ring $H^*(X)$. We will use the convention that $T_1, T_2, \dots, T_p$ generate $H^2(X)$ while T_0 generates $H^0(X)$ and T_{p+1} is the fundamental class of X . Then Ξ_0 is the unit torus, $\Xi_1, \Xi_2, \dots, \Xi_p$ correspond to the generating set of balanced unions of rays, and $\Xi_{p+1} = (\mathbb{R}_{\geq 0})^n$.

In what follows, K_{X_Σ} signifies the canonical class of the toric variety $X = X_\Sigma$.

Conjecture 3.3.1 (Mirror Symmetry for Toric Surfaces). *Let W be the usual Givental potential. Then there exists some germ $(\mathbb{C}^{p+2}, 0)$ and a universal unfolding of W*

$$\hat{W} : Y := (\mathbb{C}^{p+2}, 0) \times (N \otimes \mathbb{C}^\times) \rightarrow \mathbb{C}.$$

There exists a form $f\Omega$ on $Y \times \{|\hbar| < \epsilon\}$ such that if we write

$$s(t, \hbar) = \sum_{i=0}^{p+1} T_i \int_{\Xi_i} e^{\hat{W}/\hbar} f\Omega,$$

then

$$s(t, \hbar) = \hbar^{K_{X_\Sigma}} \sum_{i=0}^{p+1} \varphi_i(t, \hbar^{-1}) T_i \hbar^{\deg(T_i)/2}$$

for functions φ_i satisfying

$$\varphi_i(t, \hbar^{-1}) = \delta_{0,i} + \sum_{j=1}^{\infty} \varphi_{i,j}(t) \hbar^{-j}$$

for $0 \leq i \leq n$. If we set

$$y_i(t) = \varphi_{i,1}(t), \quad 0 \leq i \leq p+1,$$

$y_0, \dots, y_n$ form a system of coordinates called flat coordinates. Second, if we write the Givental J -function as

$$J(y_0, \dots, y_n, \hbar^{-1}) = \sum_{i=0}^n J_i(y_0, \dots, y_n, \hbar^{-1}) T_i,$$

then in the $\mathbb{C}$ -vector space $\mathbb{C}[[y_0, \dots, y_n, \hbar^{-1}]]$,

$$J_i(y_0, \dots, y_{p+1}) = \varphi_i(y_0, \dots, y_{p+1}).$$

It should be noted that we consider the variables t_i to be formal in the sense that we allow ourselves to make formal power series with them. This gives us the ability to avoid certain convergence issues. For instance, in Y , above, when considering this space as a family of mirrors, there may be “too many” special fibers. Formal series do not “see” these; thus allowing us to circumvent their consideration.

3.4 Tropical Geometry and Mirror Symmetry

Now that we have a conjecture relating the A-side and B-side of mirror symmetry, let us see if we cannot improve on it with the use of tropical geometry. Before we can see how tropical geometry interrelates with mirror symmetry, we must introduce some notation and definitions.

Let $\bar{\Gamma}$ be a connected graph with no bivalent vertices. Define

$$\Gamma = \bar{\Gamma} \setminus \bar{\Gamma}_{\infty}^{[0]}$$

where $\bar{\Gamma}_{\infty}^{[0]}$ represents the set of univalent vertices of $\bar{\Gamma}$. If $\Gamma^{[0]}$ and $\Gamma^{[1]}$ represent the vertices and edges of Γ , respectively, then we think of Γ as having noncompact edges where the univalent vertices were removed. Γ will be a *weighted* graph, defined in the following way

$$w : \Gamma^{[1]} \rightarrow \mathbb{N} = \{0, 1, 2, \dots\}.$$

A *marked graph* $(\Gamma, x_1, \dots, x_k)$ is a graph where $x_1, \dots, x_k$ are used to label the non-compact edges of weight zero. We will use the convention that all non-marked edges have non-zero weight.

Recall the lattice notation that we used on page 22, for the following definition.

Definition 3.4.1. A *marked parameterized tropical curve*

$$h : (\Gamma, x_1, \dots, x_k) \rightarrow M_{\mathbb{R}}$$

is a continuous map h satisfying the following two properties:

- If $E \in \Gamma^{[1]}$ and $w(E) = 0$, then $h|_E$ is constant; otherwise, $h|_E$ is a proper embedding of E into a line of rational slope in $M_{\mathbb{R}}$.
- *The balancing condition.* Let $V \in \Gamma^{[0]}$, and let $E_1, \dots, E_l \in \Gamma^{[1]}$ be the edges adjacent to V . Let $m_i \in M$ be a primitive tangent vector to $h(E_i)$ pointing away from $h(V)$. Then

$$\sum_{i=1}^l w(E_i) m_i = 0.$$

Two marked parameterized tropical curves $h : (\Gamma, x_1, \dots, x_k) \rightarrow M_{\mathbb{R}}$ and $h' : (\Gamma', x'_1, \dots, x'_k) \rightarrow M_{\mathbb{R}}$ are *equivalent* if there is a homeomorphism $\varphi : \Gamma \rightarrow \Gamma'$ with $\varphi(E_{x_i}) = E_{x'_i}$ and $h = h' \circ \varphi$. We will define a *marked tropical curve* to be an equivalence class of parameterized marked tropical curves. $\square$

Definition 3.4.2. A curve h is a *marked tropical curve in X_{Σ}* if for each $E \in \Gamma_{\infty}^{[1]}$ which is not a marked edge, $h(E)$ is a translate of some $\rho \in \Sigma^{[1]}$.

3.4.1 Perturbations from tropical disks

We will now perturb W_0 using information from tropical disk counts taken with suitable multiplicity.

Definition 3.4.3. Let $\bar{\Gamma}$ be a weighted, connected finite graph without bivalent vertices, with the additional data of a choice of univalent vertex V_{out} , adjacent to a unique edge E_{out} . Let

$$\Gamma' := (\bar{\Gamma} \setminus \bar{\Gamma}_{\infty}^{[0]}) \cup \{V_{\text{out}}\} \subseteq \bar{\Gamma}.$$

Suppose furthermore that Γ' has first Betti number zero (i.e., Γ' is a tree with one compact external edge and a number of non-compact external edges). Then a *parameterized d -pointed tropical disk in $M_{\mathbb{R}}$ with domain Γ'* is a choice of inclusion $\{p_1, \dots, p_d\} \hookrightarrow \Gamma_{\infty}^{[1]} \setminus \{E_{\text{out}}\}$ written as $p_i \mapsto E_{p_i}$, with $w(E) = 0$ if and only if $E = E_{p_i}$ for some i , and a continuous map $h : \Gamma' \rightarrow M_{\mathbb{R}}$ satisfying the same conditions as definition 3.4.1, except that there is *no* balancing condition at V_{out} .

An *isomorphism* of tropical disks

$$h_1 : (\Gamma'_1, p_1, \dots, p_d) \rightarrow M_{\mathbb{R}} \text{ and } h_2 : (\Gamma'_2, p_1, \dots, p_d) \rightarrow M_{\mathbb{R}}$$

is a homeomorphism $\Phi : \Gamma'_1 \rightarrow \Gamma'_2$ respecting the marked edges and the weights with $h_1 = h_2 \circ \Phi$. A *tropical disk* is an isomorphism class of parameterized tropical disks.

Definition 3.4.4. Suppose rank $M = 2$. Let $h : \Gamma \rightarrow M_{\mathbb{R}}$ be a marked tropical disk such that $\bar{\Gamma}$ only has vertices of valency one and three. The *multiplicity* of the disk h is then

$$\text{Mult}(h) := \prod_{V \in \Gamma^{[0]}} \text{Mult}_V(h),$$

where $\text{Mult}_V(h)$ is defined as follows (note there is no contribution from V_{out}). We define for $V \in \Gamma^{[0]}$ with adjacent edges E_1 , E_2 , and E_3 ,

$$\begin{aligned} \text{Mult}_V(h) &= w_{\Gamma}(E_1)w_{\Gamma}(E_2)|m_{(V,E_1)} \wedge m_{(V,E_2)}| \\ &= w_{\Gamma}(E_2)w_{\Gamma}(E_3)|m_{(V,E_2)} \wedge m_{(V,E_3)}| \\ &= w_{\Gamma}(E_3)w_{\Gamma}(E_1)|m_{(V,E_3)} \wedge m_{(V,E_1)}| \end{aligned}$$

if none of E_1 , E_2 , E_3 are marked, and otherwise $\text{Mult}_V(h) = 1$. Here for $m_1, m_2 \in M$, we identify $\bigwedge^2 M$ with $\mathbb{Z}$ so that $|m_1 \wedge m_2|$ makes sense. The equalities follow from the balancing condition.

Definition 3.4.5. A disk h is a *tropical disk in X_{Σ}* if every $E \in \Gamma_{\infty}^{[1]} \setminus \{E_{\text{out}}\}$ has $h(E)$ either a point or a translate of some $\rho \in \Sigma^{[1]}$.

If the disk has d_{ρ} unbounded edges which are translates of $\rho \in \Sigma^{[1]}$ (counted with weight), then the *degree* of h is

$$\Delta(h) := \sum_{\rho \in \Sigma^{[1]}} d_{\rho} t_{\rho} \in T_{\Sigma}$$

and furthermore

$$|\Delta(h)| := \sum_{\rho \in \Sigma^{[1]}} d_\rho.$$

Definition 3.4.6. The *Maslov index* of a disk h with d marked points is

$$MI(h) := 2(|\Delta(h)| - d).$$

We will also need the following. Associate to the points $P_1, \dots, P_k$ the variables $u_1, \dots, u_k$ in the ring

$$R_k := \frac{\mathbb{C}[u_1, \dots, u_k]}{(u_1^2, \dots, u_k^2)}.$$

Definition 3.4.7. Let $h : (\Gamma', p_1, \dots, p_d) \rightarrow M_{\mathbb{R}}$ be a tropical disk in X_Σ with $h(V_{\text{out}} = Q, h(p_j) = P_{i_j}, 1 \leq i_1 < \dots < i_d \leq k$. (This ordering removes a $d!$ ambiguity about the labelling of the marked points.) We say h is a *tropical disk in* $(X_\Sigma, P_1, \dots, P_k)$ *with boundary* Q .

Definition 3.4.8. Given the data $P_1, \dots, P_k, Q \in M_{\mathbb{R}}$ in general position, let $h : (\Gamma', p_1, \dots, p_d) \rightarrow M_{\mathbb{R}}$ be a Maslov index two marked tropical disk with boundary Q in $(X_\Sigma, P_1, \dots, P_k)$. Then we can associate to h a monomial in $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_k[[y_0]]$,

$$\text{Mono}(h) := \text{Mult}(h) z^{\Delta(h)} u_{I(h)},$$

where $z^{\Delta(h)} \in \mathbb{C}[T_\Sigma]$ is the monomial corresponding to $\Delta(h) \in T_\Sigma$, the subset $I(h) \subseteq \{1, \dots, k\}$ is defined by

$$I(h) := \{i : h(p_j) = P_i \text{ for some } j\},$$

and

$$u_{I(h)} = \prod_{i \in I(h)} u_i.$$

Define the *k-pointed Landau-Ginzburg potential*

$$W_k(Q) := y_0 + \sum_h \text{Mono}(h) \in \mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_k[[y_0]]$$

where the sum is over all Maslov index two disks h in $(X_\Sigma, P_1, \dots, P_k)$ with boundary Q .

3.4.2 The $\mathbb{P}^2$ Case

Now a proof of conjecture 3.3.1 has been worked out in deep detail for the case of $\mathbb{P}^2$ in [12]. We will describe here then, for $\mathbb{P}^2$, the Landau-Ginzburg potential W_0 and how to perturb it correctly to obtain the k -pointed potential function W_k .

The model for each Fano toric variety, like $\mathbb{P}^2$, is its fan. The fan for $\mathbb{P}^2$ is given in figure 3.2. And for each Fano toric variety, W_0 can be read off from rays of the fan. For instance, for $\mathbb{P}^2$ we see the rays heading off through the points $(-1,-1)$, $(1,0)$, and $(0,1)$. So we label the rays with the monomials $x^{-1}y^{-1}$, x , and y , respectively, following the rules of definition 3.4.8. This gives us the following potential: $W_0 = x + y + x^{-1}y^{-1}$. (As y_0 in the definition acts as only a place holder for us, we will omit the mention of it in what follows.)

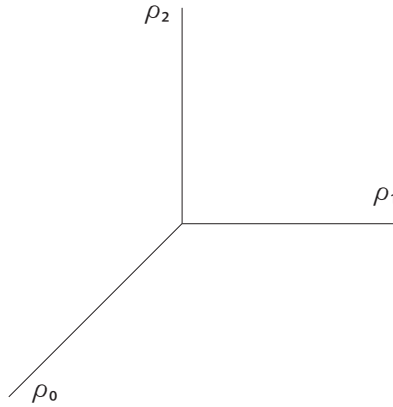


Figure 3.2: The fan for $\mathbb{P}^2$.

The reason that we are able to obtain the zero-pointed potential so readily is that, given a point Q , there is only one possible Maslov index two tropical disk with boundary Q for each ray. Splitting the three disks up, we get figure 3.3.

To get the one-pointed Landau-Ginzburg potential, we now have to choose points Q and P in general position and consider the Maslov index two marked tropical disks with boundary Q passing through P . It turns out that there is only one such disk, but that is not immediately obvious. We will describe a method in a later chapter that will allow us to keep track of the relevant disks; the method is that of scattering diagrams. The following figure describes the sole disk in question.

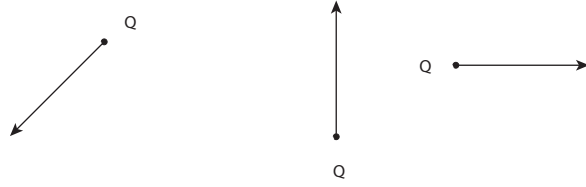


Figure 3.3: The zero-pointed Maslov index 2 tropical disks for $\mathbb{P}^2$.

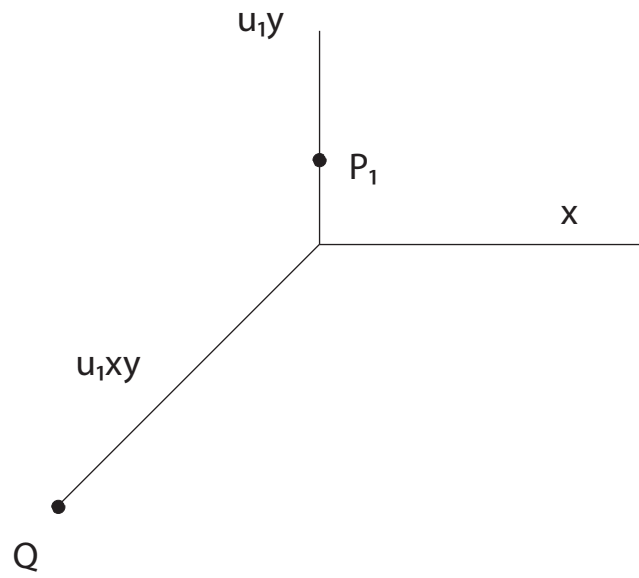


Figure 3.4: The one-pointed Maslov index 2 tropical disk for $\mathbb{P}^2$.

As we can see from figure 3.4, the monomial u_1xy is attached to the disk. Again following definition 3.4.8, we obtain a perturbed potential $W_1 = W_0 + u_1xy = x + y + x^{-1}y^{-1} + u_1xy$. Note that the original three disks contribute here, also, the take-away being that for the one-pointed potential we consider the disks passing through one *or fewer* marked points. Calculations for more marked points are carried out in much the same way.

As I mentioned at the beginning of the example, this method is expected to work for *Fano* toric varieties and has been shown to work for $\mathbb{P}^2$. This work, though, is about extending beyond the Fano case. In particular, $\mathbb{F}_2$ is a non-Fano toric manifold, and the method given above is not quite right in this case. As we will see, we can attempt to calculate W_0 as we did for $\mathbb{P}^2$, but it will introduce some errors. In following chapters we will correct the model for the non-Fano case, while still putting the methods from the Fano case to good use.

Chapter 4

Affine Manifolds with Singularities

The interesting thing about $\mathbb{F}_2$ is that it is the simplest non-Fano toric surface. Fano toric surfaces are now fairly well understood with regards to how tropical geometry interplays with mirror symmetry. The non-Fano case is more difficult, but the basic idea is that we can adjust the model for the Fano case to apply to the non-Fano case. The further the variety is away from being Fano, the more adjustments need to be made.

The adjustments are made to the tropical model. We want to introduce singularities into the tropical affine manifold. The reason that the old model does not work for non-Fano manifolds is that tropical geometry cannot “see” the curves sitting in the toric boundary. In a sense, adding singularities to the affine manifold we are working over pulls these curves into view.

We will need to consider the group $\text{Aff}(M_{\mathbb{R}}) = M_{\mathbb{R}} \rtimes \text{GL}(M_{\mathbb{R}})$ of affine linear automorphisms of $M_{\mathbb{R}}$ given by $m \mapsto Am + b$, where $A \in \text{GL}_n(\mathbb{R})$ and $b \in M_{\mathbb{R}}$.

Definition 4.0.9. A *tropical affine manifold* is a real topological manifold B (possibly with boundary) with an atlas of coordinate charts $\psi_i : U_i \rightarrow M_{\mathbb{R}}$ with transition functions $\psi_i \circ \psi_j^{-1} \in M_{\mathbb{R}} \rtimes \text{GL}(M) \subseteq \text{Aff}(M_{\mathbb{R}})$.

Definition 4.0.10. A *tropical affine manifold with singularities* is a topological manifold B along with the data

- a subset $\Delta \subseteq B$ which is a locally finite union of codimension ≥ 2 locally closed submanifolds of B ;
- a tropical affine structure on $B_0 := B \setminus \Delta$.

The set Δ is called the *singular locus* or *discriminant locus* of B .

In general we will want to restrict the behavior of Δ , but that is outside the scope of this report. Any example we look at will have a suitably “nice” singular locus.

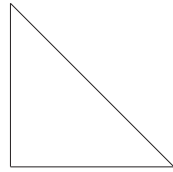


Figure 4.1: The Newton polytope for $\mathbb{P}^2$.

The Newton polytope for $\mathbb{P}^2$, shown in figure 4.1, is an example of a tropical affine manifold (with boundary). Its triangle sits snugly inside $\mathbb{R}^2$; its only transition functions is the identity, which is assuredly affine linear.

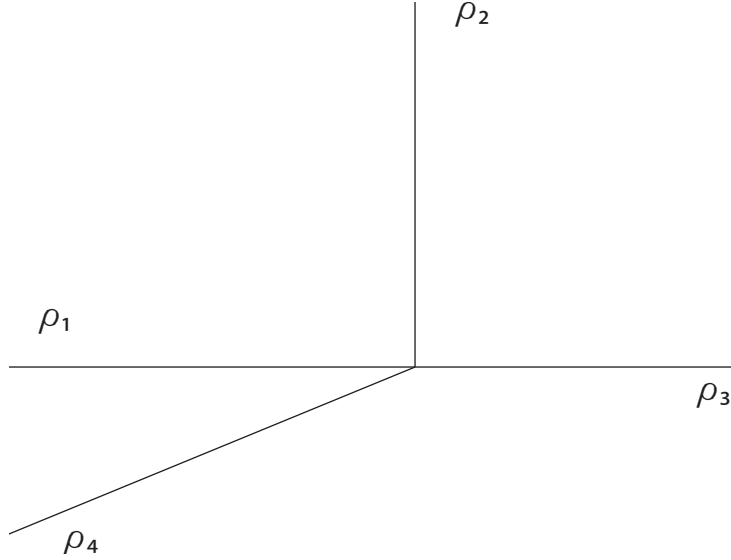
The discrete Legendre transformation of this polytope gives the fan for $\mathbb{P}^2$. And the reader may recall that, by example 3.4.2, this fan provides us with the model for constructing Maslov index two tropical disks and thereby k -pointed Landau-Ginzburg potential functions.

The second Hirzebruch surface provides us with another example of a tropical affine manifold. For $\mathbb{F}_2$, we will also construct a tropical affine manifold, this time with singularities, whose discrete Legendre transform will present us with the appropriate model for constructing potentials for this non-Fano case.

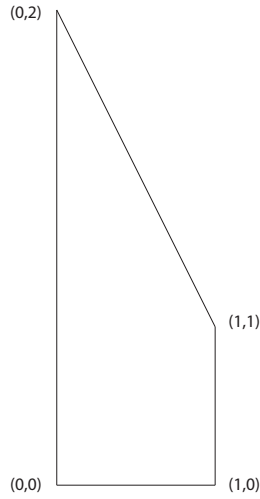
4.1 The $\mathbb{F}_2$ Case

The fan for $\mathbb{F}_2$ is given by figure 4.2.

We can readily derive the Newton polytope for $\mathbb{F}_2$ by deducing what polytope with integral vertices has inward pointing normal vectors equal to those in the

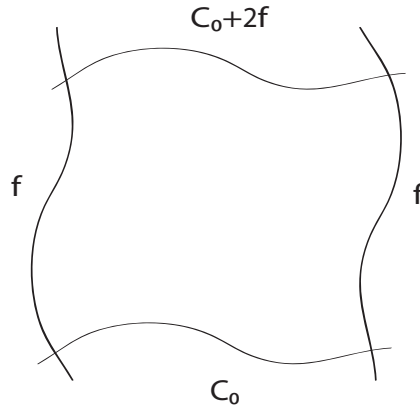
Figure 4.2: The fan for $\mathbb{F}_2$.

fan for $\mathbb{F}_2$. As in the case of $\mathbb{P}^2$, this polytope, embedded in $\mathbb{R}^2$, gives an example of a tropical affine manifold with transition functions equal to the identity. The polytope for $\mathbb{F}_2$ is given in figure 4.3.

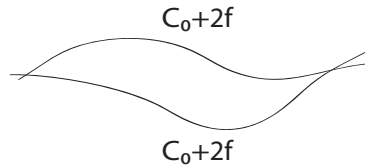
Figure 4.3: The Newton polytope for $\mathbb{F}_2$.

Now we had mentioned that $\mathbb{F}_2$ is non-Fano, which implies that there are divisors that are “hidden” in the toric boundary. The divisors of $\mathbb{F}_2$ are represented in figure 4.4.

It turns out that we can give the hidden divisor explicitly. One can imagine a

Figure 4.4: The divisors of $\mathbb{F}_2$.

holomorphic disk that just touches the divisor C_0 . While the tropics would count the holomorphic disk, it would fail to count the union of the disk and divisor C_0 . We want to *smooth* out the three divisors on the bottom part of the figure above in order to pull the divisor C_0 unioned with the holomorphic disk into view. This will give a picture like in figure 4.5.

Figure 4.5: The smoothed divisors of $\mathbb{F}_2$.

The method by which we carry out this smoothing is known as “pushing in corners.” Now, the bottom three divisors in figure 4.4 that we smoothed out correspond to the three edges on the right hand side of figure 4.3. In order to smooth the two right-hand corners of this Newton polytope for $\mathbb{F}_2$, we create a new tropical affine manifold, this time with two charts and with singularities. The charts of this manifold are given in figure 4.6. The singularities are denoted by the x’s, while the shaky lines represent where the charts have been “cut.” It can be seen in the diagram that an affine linear function can be used to send one of

the lower triangles of the charts to the other lower triangle. Similarly, the middle parallelogram can be sent to the other via an affine linear function. And this procedure can be applied to the top triangles as well.

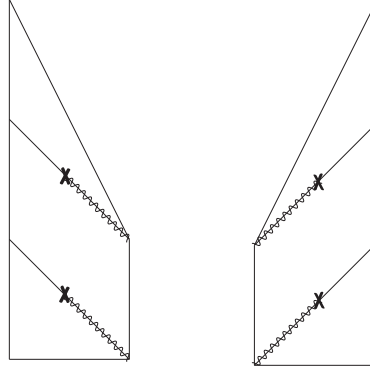


Figure 4.6: A tropical affine manifold with singularities for $\mathbb{F}_2$.

We need to make an aside, now, which will soon give us the *discrete Legendre transformation* which we will be able to use to transform our affine manifolds with singularities into fan-like pictures. Many of the more basic terms relating to polytopes and fans will be assumed as understood by the reader without further definition; for more details see chapter 1 of [12].

Let B be a topological manifold. We say that B has a *polyhedral decomposition* $\mathcal{P}$ if

$$B = \bigcup_{\sigma \in \mathcal{P}} \sigma$$

where

- $\sigma \in \mathcal{P}$ is a subset of B equipped with a homeomorphism to a (not necessarily compact) polyhedron in $M_{\mathbb{R}}$ with faces of rational slope and at least one vertex. Thus in particular any $\sigma \in \mathcal{P}$ has a set of faces: these faces are inverse images of faces of the polyhedron in $M_{\mathbb{R}}$.
- If $\sigma \in \mathcal{P}$ and $\tau \subseteq \sigma$ is a face, then $\tau \in \mathcal{P}$.
- If $\sigma_1, \sigma_2 \in \mathcal{P}$, $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a face of σ_1 and σ_2 .

for each $\sigma \in \mathcal{P}$, viewing $\sigma \subseteq M_{\mathbb{R}}$ yields a tangent space $\Lambda_{\sigma, \mathbb{R}} \subseteq M_{\mathbb{R}}$ to σ , and we can set

$$\Lambda_{\sigma} := \Lambda_{\sigma, \mathbb{R}} \cap M.$$

Let $\tau \in \mathcal{P}$. The *open star* of τ is

$$U_{\tau} := \bigcup_{\sigma \in \mathcal{P} \text{ s.t. } \tau \subseteq \sigma} \text{Int}(\sigma).$$

A *fan structure* along $\tau \in \mathcal{P}$ is a continuous map $S_{\tau} : U_{\tau} \rightarrow \mathbb{R}^k$ where $k = \dim B - \dim \tau$, satisfying

1. $S_{\tau}^{-1}(0) = \text{Int}(\tau)$.
2. If $\tau \subseteq \sigma$, then $S_{\tau}|_{\text{Int}(\sigma)}$ is an integral affine submersion onto its image, with $\dim S_{\tau}(\sigma) = \dim \sigma - \dim \tau$. By integral affine submersion we mean the following. We can think of σ as a lattice polytope in $\Lambda_{\sigma, \mathbb{R}}$. Then the map $S_{\tau}|_{\sigma}$ is induced by a surjective affine map $\Lambda_{\sigma} \rightarrow W \cap \mathbb{Z}^k$, for some vector subspace $W \subseteq \mathbb{R}^k$ of codimension equal to the codimension of σ in B .
3. For $\tau \subseteq \sigma$, define $K_{\tau, \sigma}$ to be the cone generated by $S_{\tau}(\sigma \cap U_{\tau})$. Then

$$\Sigma_{\tau} := \{K_{\tau, \sigma} | \tau \subseteq \sigma \in \mathcal{P}\}$$

is a fan with $|\Sigma_{\tau}|$ convex.

A function $f : M_{\mathbb{R}} \rightarrow \mathbb{R}$ is *affine linear* if it is of the form $m \mapsto \langle n, m \rangle + b$ for some $n \in N_{\mathbb{R}}$, $b \in \mathbb{R}$. If B is an affine manifold, an *affine linear function* $f : B \rightarrow \mathbb{R}$ is a continuous function given on a coordinate chart $\psi_i : U_i \rightarrow M_{\mathbb{R}}$ by the composition of ψ_i with an affine linear function $M_{\mathbb{R}} \rightarrow \mathbb{R}$.

In what follows, we would like any singular locus Δ or boundary of a tropical manifold to be well-behaved in a technical sense. The details of this can be found on page 21 in [12], but we will assume here that our tropical manifolds are “nice enough.” If $(B, \mathcal{P})$ is a tropical manifold with singular locus Δ , an *affine function* on an open set $U \subseteq B$ is a continuous map $\varphi : U \rightarrow \mathbb{R}$ that is affine on $U \setminus \Delta$. A *piecewise-linear* or *PL function* on U is a continuous map $\varphi : U \rightarrow \mathbb{R}$ such that if $S_{\tau} : U_{\tau} \rightarrow \mathbb{R}^k$ is the fan structure along $\tau \in \mathcal{P}$, then

$$\varphi|_{U \cap U_{\tau}} = \lambda + \varphi_{\tau} \circ S_{\tau}, \tag{4.1}$$

where

$$\varphi_\tau : |\Sigma_\tau| \rightarrow \mathbb{R}$$

is a PL function (in the usual sense) on the fan Σ_τ determined by the fan structure S_τ along τ , and λ is affine linear.

Denote by $\mathcal{Aff}(B, \mathbb{R})$ the sheaf of affine linear functions on B , and similarly by $\mathcal{PL}_\mathcal{P}(B, \mathbb{R})$ the sheaf of PL functions.

A *multi-valued PL function* is a section of $\mathcal{PL}_\mathcal{P}(B, \mathbb{R})/\mathcal{Aff}(B, \mathbb{R})$. In other words, it is a collection $\{(U_i, \varphi_i)\}$ of PL functions with $\varphi_k - \varphi_j$ affine linear on $U_i \cap U_j$.

We say a multi-valued PL function on U is *convex* if it is locally represented on $U \cap U_\tau$ by $\varphi_\tau \circ S_\tau$ with φ_τ a strictly convex function.

Given a triple $(B, \mathcal{P}, \varphi)$, where $(B, \mathcal{P})$ is a tropical manifold and φ is a multi-valued strictly convex PL function on B , we will define the discrete Legendre transform of $(B, \mathcal{P}, \varphi)$, denoted by $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$. $\check{B}$, $\check{\mathcal{P}}$, and $\check{\varphi}$ will each be defined in turn.

For any $\tau \in \mathcal{P}$, we have a representative $\varphi_\tau : |\Sigma_\tau| \rightarrow \mathbb{R}$ of φ as in (4.1), and set $\check{\tau} := \Delta_{\varphi_\tau}$, the Newton polyhedron of φ_τ . If $\tau \subseteq \sigma$ for $\tau, \sigma \in \mathcal{P}$, then Σ_σ is the quotient fan of Σ_τ by the cone in Σ_τ corresponding to σ and, up to a linear map, $\varphi_\sigma : |\Sigma_\sigma| \rightarrow \mathbb{R}$ is induced by $\varphi_\tau : |\Sigma_\tau| \rightarrow \mathbb{R}$. Hence $\check{\sigma}$ can be identified with a face of $\check{\tau}$.

Indeed, $\varphi_\sigma : |\Sigma_\sigma| \rightarrow \mathbb{R}$ is induced by $\varphi_\tau : |\Sigma_\tau| \rightarrow \mathbb{R}$ in the following way. Choose $n \in N_\mathbb{R}$ such that $\varphi_\sigma(m) = \langle n, m \rangle$ for $m \in \sigma$. Then for any ν containing σ , $\varphi|_\nu - n$ is zero on σ , hence descends to a linear function on $(\nu + \mathbb{R}\sigma)/\mathbb{R}\sigma$. These piece together to give a PL function φ_σ on Σ_σ , well-defined up to a linear function (determined by the choice of n).

This gives us the pair $(\check{B}, \check{\mathcal{P}})$, where $\check{\mathcal{P}} = \{\check{\sigma} | \sigma \in \mathcal{P}\}$, and $\check{B}$ is obtained by identifying $\check{\tau}_1$ and $\check{\tau}_2$ along the common face $\check{\sigma}$ if $\sigma \in \mathcal{P}$ is the smallest cell of $\mathcal{P}$ containing τ_1 and τ_2 .

For $\sigma \in \mathcal{P}_{\max}$, $\check{\sigma}$ is a vertex of $\check{\mathcal{P}}$, and let $\check{\Sigma}_\sigma$ denote the normal fan to σ . The cones of $\check{\Sigma}_\sigma$ are in one-to-one inclusion reversing correspondence with faces of σ , and for $\tau \subseteq \sigma$ the tangent wedge to $\check{\tau}$ at $\check{\sigma}$ is $N_\sigma(\tau)$. Thus there is a natural fan

structure

$$\check{S}_{\check{\sigma}} : U_{\check{\sigma}} \rightarrow N_{\mathbb{R}}.$$

Indeed, locally $U_{\check{\sigma}} \cap \check{\tau}$ looks like the tangent wedge to $\check{\tau}$ at $\check{\sigma}$, which is identified via $\check{S}_{\check{\sigma}}$ with the normal cone $N_{\sigma}(\tau)$. Finally, we define $\check{\varphi}$ by taking $\check{\varphi}_{\tau}$ on $\Sigma_{\check{\tau}} = \check{\Sigma}_{\tau}$ to be the function induced by τ in the following way:

$$\check{\varphi}_{\tau}(m) = -\inf\{\langle n, m \rangle | n \in \tau\}.$$

From here we can take a *discrete Legendre transformation* (DLT) of the chart on the left of figure 4.6. This transformation will leave us with the diagram in figure 4.7. The line segments in the manifold have been replaced with their inward-pointing normals. The singularities are in their appropriate respective locations, and of interest is how the singularities affect this transformed view of the manifold. In a sense, we want this new diagram to represent the DLT of both charts. Considering the chart on the right, though, we notice that the inward pointing normals of the right-hand side of this chart should be three vectors, all parallel, pointing directly left. This does not appear to be what is happening in figure 4.6, but in fact it is. The singularities allow us to cut out two sectors of the plane (the sectors emanating to the left of the singularities). Using a shearing transformation, we can take each sector and glue its edges together. This will leave us with the expected behavior of the left pointing vectors, they will all be “parallel” after the shearing action.

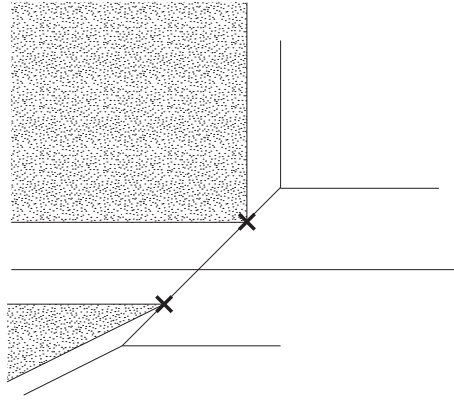


Figure 4.7: The discrete Legendre transformation of figure 4.6.

We made a choice when carrying out the DLT, which resulted in the particular

placement of the singularities. We can instead make a different choice, which would move the singularities out in opposite directions from each other, but along the line on which they both sit. We will do so; in fact, we will move these singularities all the way out to infinity. The singularities continue to affect our model even after they are moved out, and the behavior is elucidated fully in [10].

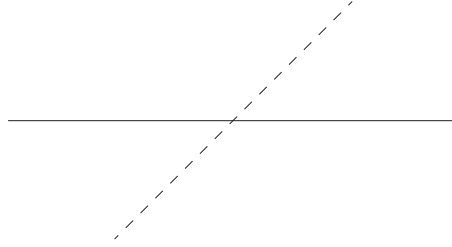


Figure 4.8: The model for $\mathbb{F}_2$ obtained after moving the singularities in figure 4.7 to infinity.

Figure 4.8 presents us with the model we will use to create and perturb the k -pointed Landau-Ginzburg potentials for $\mathbb{F}_2$. Its simplicity belies the complicated diagrams and calculations it will lead to. There are four key parts to the above diagram. There are the two horizontal rays, starting in the middle and heading to either side of the diagram. The remaining two parts are the two diagonal rays that appear as one diagonal line since they overlap one another. The diagonal rays are denoted a dashed line. One of these rays begins off at infinity in the upper-right and heads down to the lower left, and the other ray runs in the opposite direction.

Now the horizontal rays in our model for $\mathbb{F}_2$ act as the rays in the fan for $\mathbb{P}^2$ do in its model. That is, the outgoing rays for any tropical disk that we will consider will be heading off in the direction of these two rays. This is, thankfully, quite limiting. Without the action of the diagonal line, which we will describe shortly, we would be limited to only two possible Maslov index two tropical disks with zero marked points, and none for a higher number of marked points! But this cannot be right.

The diagonal line, though, gives us some added flexibility. Remember that this dashed line comes from moving the two singularities out towards infinity. Remember

also that we had cut regions out of the model and used a shearing map to glue what remained back together. This shearing allows the tropical disks to bend at this dashed line.

We will now give a breakdown of the Maslov index two tropical disks through no marked points for $\mathbb{F}_2$. This figure, figure 4.9, is analogous to figure 3.3.

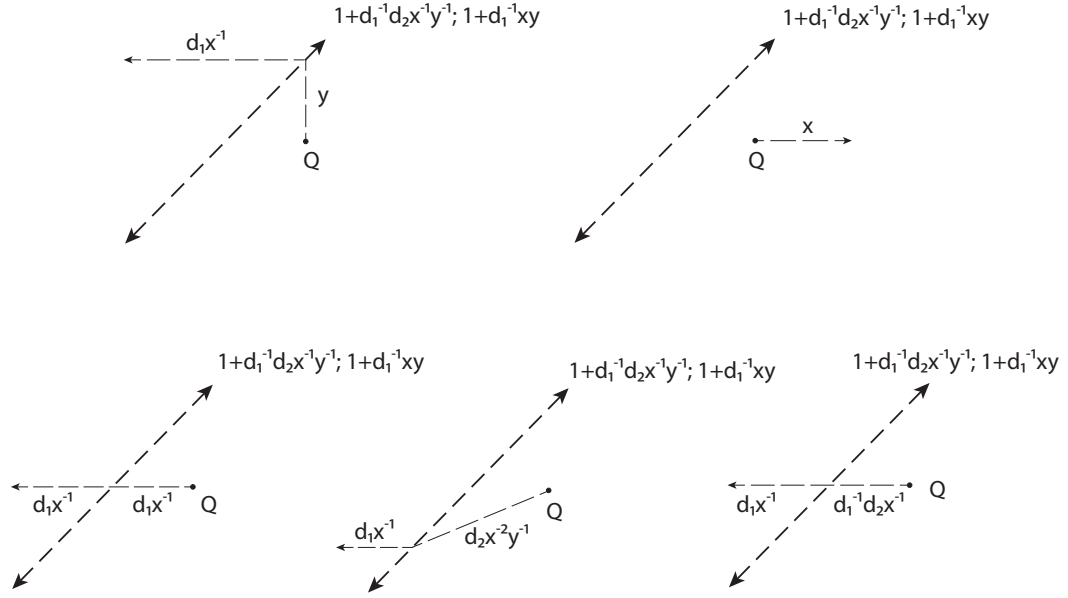


Figure 4.9: The zero-pointed Maslov index two tropical disks for $\mathbb{F}_2$.

As displayed in the diagram, the rays coming in from the left side start with the monomial d_1x^{-1} attached to them. If they are affected by one of the diagonal rays or both of them, they will “bend.” The monomials will change via multiplication with that of the affecting dashed ray, and the new monomial gives the angle of the “bend.” In the last case in the above figure, the disk was bent by one ray and then bent back by the other, leaving the disk heading in the same direction as before, but with a new monomial.

In chapter 5 we keep track of these tropical disks with something known as a “broken line.” It so happens that in the zero marked points case, the broken lines coincide exactly with the Maslov index two disks. So one can see all of the disks

overlaid onto each other in figure 5.2.

From this list of monomials comes something exciting. We get the correct potential function for $\mathbb{F}_2$. That is, we can apply definition 3.4.8 with little change to our model to construct the k -pointed Landau-Ginzburg potential for $\mathbb{F}_2$. Let us start with $k = 0$. Then $W_0 = x + y + d_2 x^{-2} y^{-1} + d_1 x^{-1} + d_2 d_1^{-1} x^{-1}$. It turns out that this is the correct potential, and we verify this in section 6.3 by showing that the period integral over this potential (B-side) matches the appropriate Gromov-Witten invariants (A-side).

Notice that this correct W_0 only differs by one monomial from what we would have expected if we had just used the fan for $\mathbb{F}_2$ as our model. (We covered this briefly on page 34.) That is, we have the one extra monomial $d_2 d_1^{-1} x^{-1}$. In fact, this uncorrected potential was known by Givental in the 1990s and then by Hori and Vafa in 2000. The corrected version, which coincides with the W_0 just calculated, was not known until 2011 by Chan, et al.

We could at this point continue lashing about for Maslov index two tropical disks in an ad hoc fashion, like we did for $\mathbb{P}^2$ in example 3.4.2. But the following chapter will provide us with systematic tools for keeping track of these disks. So we will withhold any further search until we have these tools.

Chapter 5

Scattering Diagrams

In this section we introduce *scattering diagrams* which are a diagrammatical method for keeping track of the perturbations of the Landau-Ginzburg potential that we will need. We fix $M \cong \mathbb{Z}^2$ and set Σ to be a complete fan in $M_{\mathbb{R}}$ defining a non-singular toric surface.

Associate to the points $P_1, \dots, P_k$ the variables $u_1, \dots, u_k$ in the ring

$$R_k := \frac{\mathbb{C}[u_1, \dots, u_k]}{(u_1^2, \dots, u_k^2)}.$$

Definition 5.0.1. Fix $k \geq 0$.

1. A *ray* or line is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ such that

- $\mathfrak{d} \subseteq M_{\mathbb{R}}$ is given by

$$\mathfrak{d} = m'_0 - \mathbb{R}_{\geq 0} r(m_0)$$

if $\mathfrak{d}$ is a ray and

$$\mathfrak{d} = m'_0 - \mathbb{R} r(m_0)$$

if $\mathfrak{d}$ is a line, for some $m'_0 \in M_{\mathbb{R}}$ and $m_0 \in T_{\Sigma}$ with $r(m_0) \neq 0$. The set $\mathfrak{d}$ is called the *support* of the line or ray. If $\mathfrak{d}$ is a ray, m'_0 is called the *initial point* of the ray, written as $\text{Init}(\mathfrak{d})$.

- $f_{\mathfrak{d}} \in \mathbb{C}[z^{m_0}] \otimes_{\mathbb{C}} R_k \subseteq \mathbb{C}[T_{\Sigma}] \otimes_{\mathbb{C}} R_k[[y_0]]$.
- If $\mathfrak{d}$ is a ray, then $f_{\mathfrak{d}} \equiv 1 \pmod{(u_1, \dots, u_k)z^{m_0}}$.

2. A *scattering diagram* $\mathfrak{D}$ is a finite collection of rays.

If $\mathfrak{D}$ is a scattering diagram, we write

$$\text{Supp}(\mathfrak{D}) := \bigcup_{\mathfrak{d} \in \mathfrak{D}} \sigma \subseteq M_{\mathbb{R}}$$

and

$$\text{Sing}(\mathfrak{D}) := \bigcup_{\mathfrak{d} \in \mathfrak{D}} \partial \mathfrak{d} \cup \bigcup_{\substack{\mathfrak{d}_1, \mathfrak{d}_2 \\ \dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = 0}} \mathfrak{d}_1 \cap \mathfrak{d}_2$$

Definition 5.0.2. Given a smooth immersion $\gamma : [0, 1] \rightarrow M_{\mathbb{R}} \setminus \text{Sing}(\mathfrak{D})$ with endpoints not in $\text{Supp}(\mathfrak{D})$, such that γ intersects elements of $\mathfrak{D}$ transversally, we can define a ring automorphism $\theta_{\gamma, \mathfrak{D}}$, the γ -ordered product of $\mathfrak{D}$, on the ring $\mathbb{C}[T_{\Sigma}] \otimes_{\mathbb{C}} R_k[[y_0]]$. Explicitly, we can find numbers

$$0 < t_1 \leq t_2 \leq \dots \leq t_s < 1$$

and elements $\mathfrak{d}_i \in \mathfrak{D}$ such that $\gamma(t_i) \in \mathfrak{d}_i$ and $\mathfrak{d}_i \neq \mathfrak{d}_j$ if $t_i = t_j$, $i \neq j$, and s taken as large as possible. Then for each i , define $\theta_{\gamma, \mathfrak{d}_i}$, an automorphism on the same ring, to be

$$\begin{aligned} \theta_{\gamma, \mathfrak{d}_i}(z^m) &= z^m f^{\langle n_0, r(m) \rangle} \\ \theta_{\gamma, \mathfrak{d}_i}(a) &= a \end{aligned}$$

for $m \in T_{\Sigma}$, $a \in R_k[[y_0]]$, where $n_0 \in N$ is chosen to be primitive, annihilates the tangent space to $\mathfrak{d}_i$, and is finally completely determined by the sign convention that

$$\langle n_0, \gamma'(t_i) \rangle < 0$$

We then define

$$\theta_{\gamma, \mathfrak{D}} = \theta_{\gamma, \mathfrak{d}_s} \circ \dots \circ \theta_{\gamma, \mathfrak{d}_1}$$

Definition 5.0.3. A *tropical tree* in $(X_{\Sigma}, P_1, \dots, P_k)$ is a d -pointed tropical curve $h : (\Gamma, p_1, \dots, p_d) \rightarrow M_{\mathbb{R}}$ with $h(p_j) = P_{i_j}$ for $1 \leq i_1 < \dots < i_d \leq k$, along with the additional data of a choice of an unmarked unbounded edge $E_{\text{out}} \in \Gamma_{\infty}^{[1]}$ such that for any $E \in \Gamma_{\infty}^{[1]} \setminus \{E_{\text{out}}\}$, $h(E)$ is a point or is a translate of some $\rho \in \Sigma^{[1]}$. The

degree of h , $\Delta(h)$, is defined *without* counting the edge $h(E_{\text{out}})$, which need not be a translate of any $\rho \in \Sigma^{[1]}$.

The *Maslov index* of h is

$$MI(h) := 2(|\Delta(h)| - d).$$

Before moving on to the next definition, we should note that a standard dimension counting argument tells us that the set of Maslov index zero trees is finite.

Definition 5.0.4. Let Σ be the fan for a toric variety. We define $\mathfrak{D}(\Sigma, P_1, \dots, P_k)$ to be the scattering diagram which contains one ray for each Maslov index zero tree h passing through some subset of the points $P_1, \dots, P_k$. The ray corresponding to h is of the form $(\mathfrak{d}, f_{\mathfrak{d}})$, where

- $\mathfrak{d} = h(E_{\text{out}})$.
- $f_{\mathfrak{d}} = 1 + w_{\Gamma}(E_{\text{out}})\text{Mult}(h)z^{\Delta(h)}u_{I(h)}$, where $u_{I(h)} = \prod_{i \in I(h)} u_i$ and $I(h) \subseteq \{1, \dots, k\}$ is defined by

$$I(h) := \{i \mid h(p_j) = P_i \text{ for some } j\}.$$

Now scattering diagrams of this type purposefully encode information about Maslov index zero disks. In fact, scattering diagrams break up our space into chambers using this information. This is useful because we are better able to keep track of Maslov index two disks. These disks change suddenly when their end points are dragged across one of the rays of a scattering diagram into a new chamber. In fact, this gives us a method for listing all the Maslov index two disks readily. That method is the method of *broken lines*.

Definition 5.0.5. A *broken line* is a continuous proper piecewise linear map

$$\beta : (-\infty, 0] \rightarrow M_{\mathbb{R}}$$

with endpoint $Q = \beta(0)$, along with some additional data described as follows. Let

$$-\infty = t_0 < t_1 < \dots < t_n = 0$$

be the smallest set of real numbers such that $\beta|_{(t_{i-1}, t_i)}$ is linear. Then for each $1 \leq i \leq n$, we are given the additional data of a monomial $c_i z^{m_i^\beta} \in \mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_k[[y_0]]$ with $m_i^\beta \in T_\Sigma \setminus K_\Sigma$ and $0 \neq c_i \in R_k$. Furthermore, this data should satisfy the following properties:

- For each i , $r(m_i^\beta) = -\beta'(t)$ for $t \in (t_{i-1}, t_i)$.
- $m_1^\beta = t_\rho$ for some $\rho \in \Sigma^{[1]}$ and $c_1 = 1$.
- $\beta(t_i) \in \text{Supp}(\mathfrak{D}(\Sigma, P_1, \dots, P_k)) \setminus \text{Sing}(\mathfrak{D}(\Sigma, P_1, \dots, P_k))$ for $1 \leq i \leq n$.
- If $\beta(t_i) \in \mathfrak{d}_1 \cap \dots \cap \mathfrak{d}_s$ (necessarily this intersection is one-dimensional), then $c_{i+1} z^{m_{i+1}^\beta}$ is a term in

$$(\theta_{\beta, \mathfrak{d}_1} \circ \dots \circ \theta_{\beta, \mathfrak{d}_s})(c_i z^{m_i^\beta}).$$

By this, we mean the following. Suppose $f_{\mathfrak{d}_j} = 1 + c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}}$, $1 \leq j \leq s$, with $c_{\mathfrak{d}_j}^2 = 0$, and $n \in N$ is primitive, orthogonal to all the $\mathfrak{d}_j$'s, chosen so that

$$\begin{aligned} (\theta_{\beta, \mathfrak{d}_1} \circ \dots \circ \theta_{\beta, \mathfrak{d}_s})(c_i z^{m_i^\beta}) &= c_i z^{m_i^\beta} \prod_{j=1}^s (1 + c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}})^{\langle n, r(m_i^\beta) \rangle} \\ &= c_i z^{m_i^\beta} \prod_{j=1}^s (1 + \langle n, r(m_i^\beta) \rangle c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}}). \end{aligned}$$

Then we must have

$$c_{i+1} z^{m_{i+1}^\beta} = c_i z^{m_i^\beta} \prod_{j=1}^s \langle n, r(m_i^\beta) \rangle c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}}$$

for some index set $J \subseteq \{1, \dots, s\}$. We think of this as β being bent at time t_i by the rays $\{\mathfrak{d}_j \mid j \in J\}$.

5.1 The $\mathbb{F}_2$ Case

Let us apply these scattering diagrams to the case of the second Hirzebruch surface. Counter-intuitively, we must actually begin with the scattering diagram for zero points. Now there are no Maslov index zero disks through zero marked

points; so what do we mean by this? Well, we have to use a model wherein we consider singularities created by “pushing in corners.” We have already discussed this in section 4.1, which culminated in the model in figure 4.8. This leaves us with a line, as given in definition 5.0.1, or in our case two lines, that a broken line may bend at. The unadorned scattering diagram for $\mathbb{F}_2$ with no marked points is deceptively simple, as we see in figure 5.1.

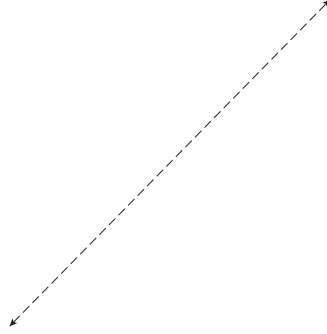


Figure 5.1: The scattering diagram $\mathfrak{D}(\Sigma)$.

This simplicity is belied by figure 5.2 where we can see all the broken lines ending at a point Q that pass through the two lines of this scattering diagram. The broken lines that originate from the left hand side of the figure all begin with the monomial d_1x^{-1} . When they pass through the dashed lines, their monomials may change and their directions may also. There is one broken line that is labeled with the number two; this represents two broken lines laying on top of each other. As is represented in the figure, one of the overlapping broken lines changes its monomial while the other does not. There is only one broken line originating from the right, which is to be expected as there is nowhere for that line to “break.”

The scattering diagram becomes more interesting when we allow one marked point. We see in figure 5.3 what this looks like. The double arrows represent two overlapping rays. There is one subtle point to the diagram: there is one ray originating from P_1 and pointing left, but it is joined by a second ray pointing left when it reaches the dashed line. The function $1 + xu_1$ attached to the first ray is given just to the left of P_1 , and then again at the end of the ray along with the function $1 + d_1^{-2}d_2xu_1$ of the other ray.

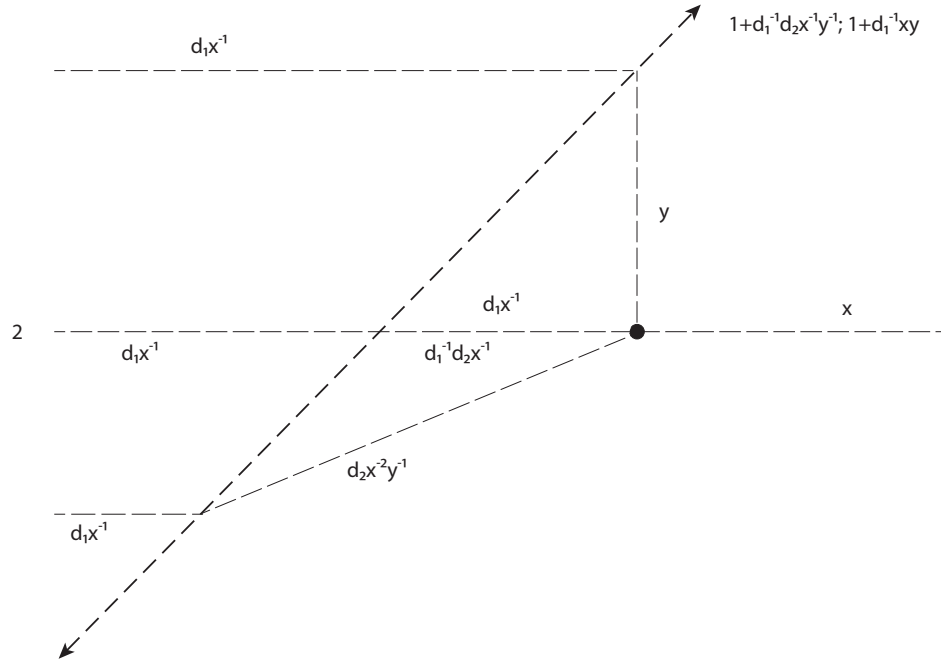


Figure 5.2: The scattering diagram $\mathfrak{D}(\Sigma)$ with broken lines. Scattering here comes from “pushing in the corners” of our affine manifold.

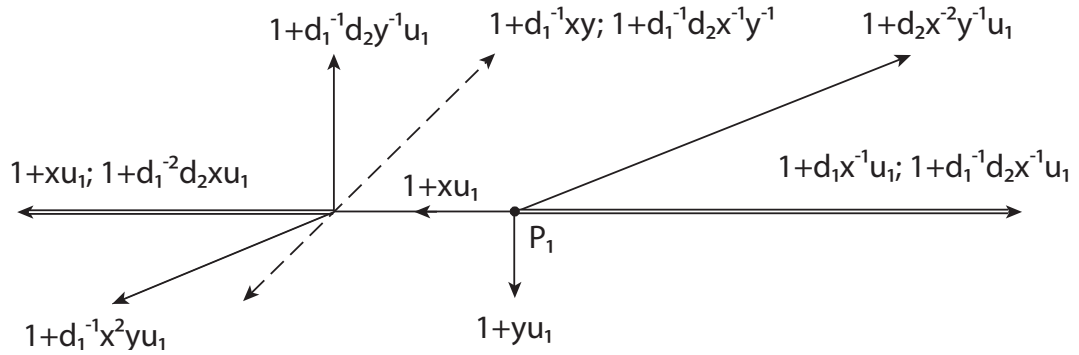


Figure 5.3: The scattering diagram $\mathfrak{D}(\Sigma, P_1)$.

Next, in figure 5.4 we zoom out from the diagram. This simplifies the procedure of creating broken lines, and there is no cost to it: we maintain the same level of accuracy. The solid rays represent the scattering diagram, one ray for every direction of a ray given in our original one-pointed diagram. The dashed lines ending at the point Q are the broken lines. We can see there is really only one new broken line added compared to figure 5.2, the broken line on the lower right. The monomial attached to this broken line starts out as x as it originates from the right hand side of the figure, and when it bends towards the point Q it takes the new monomial $d_2x^{-1}y^{-1}u_1$.

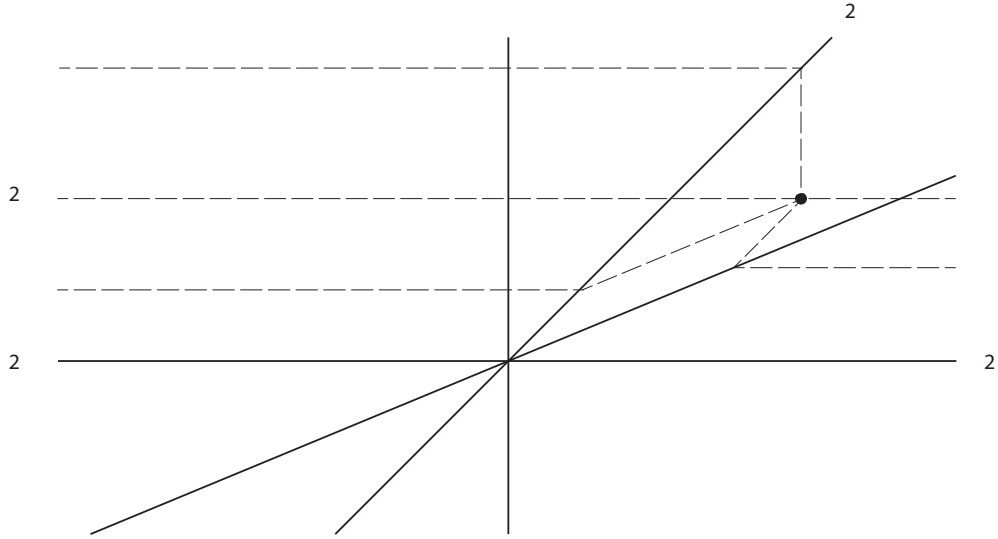
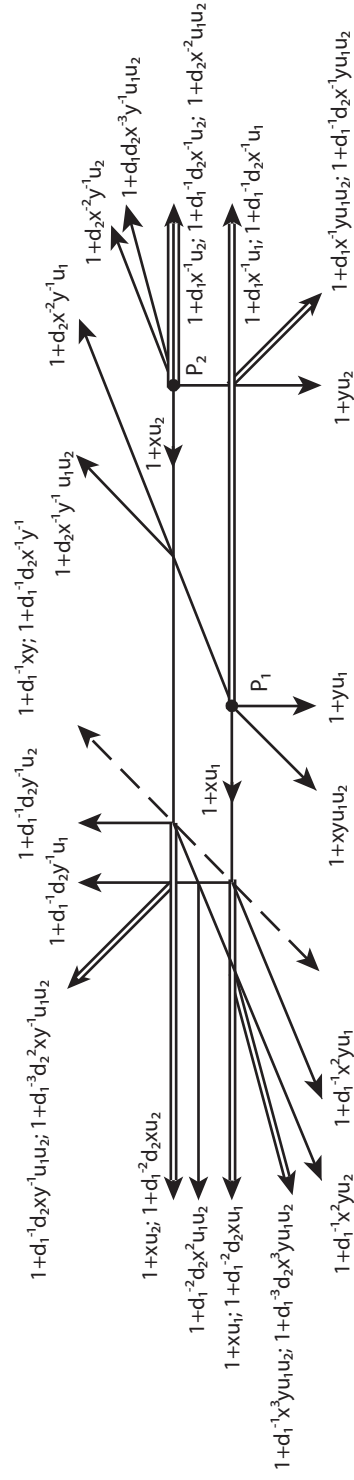


Figure 5.4: The scattering diagram $\mathfrak{D}(\Sigma, P_1)$ zoomed out with broken lines.

Example 5.1.1. Figure 5.5 is the two-pointed scattering diagram $\mathfrak{D}(\Sigma, P_1, P_2)$ for $\mathbb{F}_2$. Here Σ is the fan for $\mathbb{F}_2$. The function attached to each ray is given. There is some subtlety with this diagram; the following will hopefully help with its interpretation. Some rays coincide, that is they overlap each other; in this case, this is represented by a double arrow which is then labeled with the two appropriate functions. There is one case with a triple arrow emanating from P_1 and pointing directly to the right; it is labeled with three functions. There is one arrow originating at each of the points P_1 and P_2 that is directed exactly left and continues all the way across the diagram; in each case, a second arrow joins this

one beginning at the dashed line.

Figure 5.5: The scattering diagram $\mathfrak{D}(\Sigma, P_1, P_2)$

Once the appropriate scattering diagram is set up, we may look at the broken lines through this diagram that end at a point Q . Gross carefully showed in chapter 5 of [12] that the location of Q does not matter as long as it does not lie on a ray. So we may zoom out from this diagram far enough so that only the angles of the various rays matter. This simplifies the picture a great deal, and we can choose a location for Q and find the broken lines that lead to it. The sum of the monomials associated with these broken lines gives us the correct perturbation for the Landau-Ginzburg potential. Integrating over this potential will give us the B-side calculations which we desire, and these calculations will coincide with the A-side two-pointed Gromov-Witten invariants.

Example 5.1.2. Continuing example 5.1.1, we can zoom out and introduce the broken lines. This is displayed in figure 5.6. Some solid lines have numbers near them; this means that there are that many rays from the scattering diagram pointing in the direction. Similarly, some of the broken lines are overlapping, and are marked appropriately.

The monomials associated to these broken lines are as follows, starting from the upper-right (the broken line pointing in the positive x direction) and going counter-clockwise: x , y , d_1x^{-1} , $d_2d_1^{-1}x^{-1}$, $d_2x^{-1}u_1u_2$, $d_2x^{-2}y^{-1}$, $d_1d_2x^{-2}y^{-1}u_1u_2$, $d_2^2d_1^{-1}x^{-2}y^{-1}u_1u_2$, $d_2^2x^{-3}y^{-2}u_1u_2$, $d_2x^{-1}y^{-1}u_1$, and the final one being $d_2x^{-1}y^{-1}u_2$. These have been placed into table 5.1 for ease of reading.

Table 5.1: Monomials attached to the broken lines in figure 5.6. (Read top to bottom and then right.)

x	$d_2 x^{-1} u_1 u_2$	$d_2^2 x^{-3} y^{-2} u_1 u_2$
y	$d_2 x^{-2} y^{-1}$	$d_2 x^{-1} y^{-1} u_1$
$d_1 x^{-1}$	$d_1 d_2 x^{-2} y^{-1} u_1 u_2$	$d_2 x^{-1} y^{-1} u_2$
$d_2 d_1^{-1} x^{-1}$	$d_2^2 d_1^{-1} x^{-2} y^{-1} u_1 u_2$	

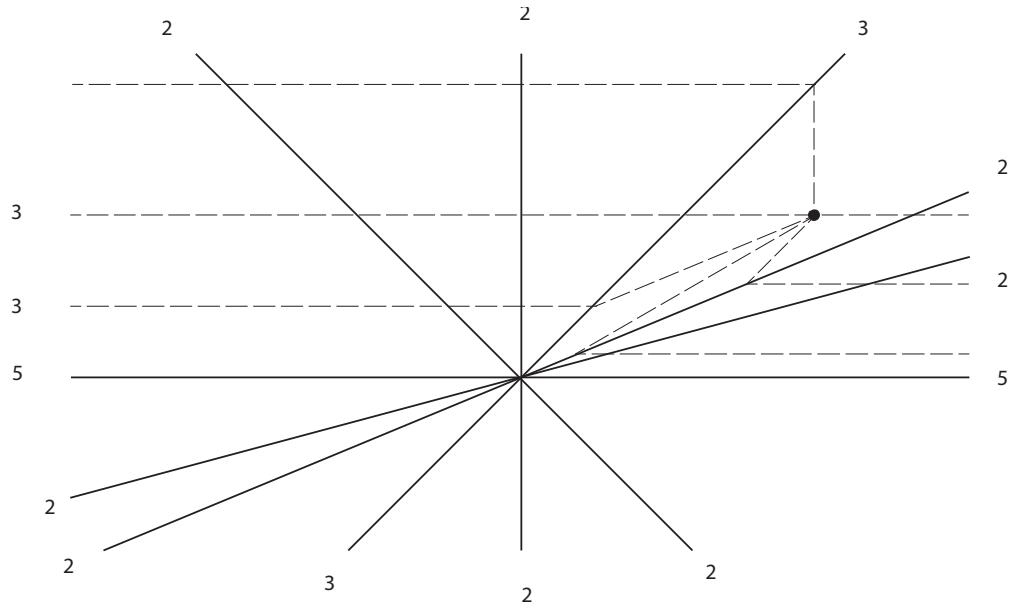


Figure 5.6: The scattering diagram $\mathfrak{D}(\Sigma, P_1, P_2)$ zoomed out with broken lines

Chapter 6

Main Calculation

The bulk of work that went into this dissertation was calculating the B-side integrals for the second Hirzebruch surface $\mathbb{F}_2$. That is, we identified the appropriate cycles $\Xi'_1, \dots, \Xi'_4$ to integrate over and then carried out the period integral calculations over the naive potential function $W_0 = x + y + d_1 x^{-1} + d_2 x^{-2} y^{-1}$. All of this we have placed in the appendix. We give here the end result of these calculations. First, we define ξ'_i as the end result of the period integral once a certain coefficient has been removed.

$$\int_{\Xi'_i} e^{W_0/\hbar} \Omega = d_1^{C_0} d_2^f \hbar^{-2C_0-4f} \xi'_i$$

Then we give all four ξ'_i explicitly.

$$\begin{aligned}
\xi'_1 &= \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1+2k_2)!} \\
\xi'_2 &= - \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1+2k_2)!} (H_{k_1} + H_{k_1+2k_2}) \\
&\quad - \sum_{\substack{k_3, k_4 \geq 0 \\ k_3-2k_4+1 \leq 0}} \frac{d_1^{k_3-2k_4} d_2^{k_4} (-k_3+2k_4-1)!}{k_3! (k_4!)^2 \hbar^{2k_3}} (-1)^{-k_3+2k_4} \\
\xi'_3 &= - \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1+2k_2)!} (2H_{k_2} + 2H_{k_1+2k_2}) \\
\xi'_4 &= - \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-4k_2}}{k_1! (k_2!)^2 (k_1+2k_2)!} \left(-2H_{k_1}H_{k_2} + H_{k_1}^2 \right. \\
&\quad \left. - H_{k_1+2k_2}^2 - 2H_{k_1+2k_2}H_{k_2} - H_{k_1+2k_2}^{(2)} + H_{k_1}^{(2)} \right) \\
&\quad - 2 \sum_{k_4=1}^{\infty} \sum_{k_3=0}^{2k_4-1} \frac{d_1^{k_3-2k_4} d_2^{k_4} \hbar^{-2k_3} (-k_3+2k_4-1)! (-1)^{-k_3+2k_4}}{k_3! (k_4!)^2} \\
&\quad \cdot (H_{-k_3+2k_4-1} - H_{k_4})
\end{aligned}$$

6.1 Corrected Zero-Point Calculations

As was discussed earlier, (for instance in the last paragraph of page 34) since the above calculations are given from the uncorrected potential function, they are not quite right. That is, they are not quite the right B-side calculations, they will not match up with the expected Gromov-Witten invariants. In order to get the correct calculations, we will have needed to calculate the period integrals over the *corrected* potential function. The uncorrected potential function was $W_0 = x + y + d_1 x^{-1} + d_2 x^{-2} y^{-1}$, while the corrected potential function has one more monomial $d_1^{-1} d_2 x^{-1}$. Luckily, we do not have to reinvent the wheel and carry out the integrals again. Instead, we can do a change of variable in what we calculated for the ξ' . That is, in order to get

$$\int_{\Xi'_i} e^{(x+y+d_1 x^{-1}+d_2 x^{-2} y^{-1}+d_1^{-1} d_2 x^{-1})/\hbar} \Omega$$

we just need the following change of variable:

$$d_1 \mapsto d_1 + d_1^{-1}d_2 \tag{6.1}$$

For the rest of this section, we will carry out this substitution and compare the results to the low-order Gromov-Witten invariants we are able to calculate using the computer program GROWI [6].

6.1.1 The Corrected Integral over the Unit Torus

We saw that our integral over the unit torus gave us the following result.

$$\xi'_1 = \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!}$$

In order to get the corrected period integral over the torus, we need to apply the substitution (6.1) to this. We get

$$\sum_{k_1, k_2 \geq 0} \frac{(d_1 + d_1^{-1} d_2)^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!}$$

We would like to simplify this. Let us do a substitution taking d_2 to $d_1 d_2$.

$$\sum_{k_1, k_2 \geq 0} \frac{(d_1 + d_2)^{k_1} (d_1 d_2)^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!}$$

Let us expand $(d_1 + d_2)^{k_1}$ via the binomial expansion.

$$\sum_{k_1, k_2 \geq 0} \frac{\sum_{i=0}^{k_1} \binom{k_1}{i} d_1^{k_1-i+k_2} d_2^{i+k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!}$$

At this point we want to rejigger the triple summation that we are working with. This will involve a change of variables. Let $\alpha = k_1 - i + k_2$ and $\beta = i + k_2$. If we fix α and β , then the previous two equations give us two planes in (k_1, k_2, i) -space. The normal vector for the first plane is $(1, 1, -1)$, and the normal vector for the second plane is $(0, 1, 1)$. The cross product of these give the direction vector of the line which is the intersection of the two planes: $(0, 1, 1) \times (1, 1, -1) = (-2, 1, -1)$. A point that these two planes have in common is $(\alpha + \beta, 0, \beta)$. So the parametric equation for this line sitting at the intersection of the two planes is $\vec{r}(t) = (\alpha + \beta, 0, \beta) + t(-2, 1, -1)$.

We have a couple constraints on this line given by the constraints on the triple sums: $k_1, k_2 \geq 0$, $0 \leq i \leq k_1$. This implies that $0 \leq t \leq \min(\alpha, \beta)$.

So we get the following:

$$\sum_{\alpha, \beta \geq 0} \sum_{t=0}^{\min(\alpha, \beta)} \binom{\alpha + \beta - 2t}{\beta - t} \frac{d_1^\alpha d_2^\beta \hbar^{-2(\alpha+\beta)}}{(\alpha + \beta - 2t)! (t!)^2 (\alpha + \beta)!}$$

Now we make the claim that we can dramatically simplify the coefficients of $\frac{d_1^\alpha d_2^\beta \hbar^{-2(\alpha+\beta)}}{(\alpha+\beta)!}$. That is, by manipulating all the terms with the t variable, we are going to get rid of the t variable.

$$\begin{aligned}
\text{coefficient} &= \sum_{t=0}^{\min(\alpha,\beta)} \binom{\alpha+\beta-2t}{\beta-t} \frac{1}{(\alpha+\beta-2t)! (t!)^2} \\
&= \sum_{t=0}^{\min(\alpha,\beta)} \frac{1}{(\beta-t)! (\alpha-t)! (t!)^2} \\
&= \frac{1}{\alpha! \beta!} \sum_{t=0}^{\alpha} \binom{\alpha}{t} \binom{\beta}{t} \quad \text{WLOG assume } \alpha \leq \beta \\
&= \frac{1}{\alpha! \beta!} \binom{\alpha+\beta}{\alpha} \quad \text{by Vandermonde's convolution} \\
&= \frac{(\alpha+\beta)!}{(\alpha!)^2 (\beta!)^2}
\end{aligned}$$

Plugging this coefficient back into our expression, we get the following.

$$\sum_{\alpha,\beta \geq 0} \frac{d_1^\alpha d_2^\beta \hbar^{-2(\alpha+\beta)}}{(\alpha!)^2 (\beta!)^2}$$

Now let us undo the substitution that gave us this nice simplification. That is, we will send d_2 back to d_2/d_1 . This gives the following.

$$\sum_{\alpha,\beta \geq 0} \frac{d_1^{\alpha-\beta} d_2^\beta \hbar^{-2(\alpha+\beta)}}{(\alpha!)^2 (\beta!)^2}.$$

One can easily extract some low order terms of this expression. Indeed, we get

$$1 + \hbar^{-2}(d_1 + d_1^{-1}d_2) + \hbar^{-4}(d_1^2/4 + d_2 + d_1^{-2}d_2^2/4) + \text{H.O.T.}$$

We are now able to match these terms with calculations from the computer program GROWI [6] for certain Gromov-Witten invariants. That is, our B-side calculations match the A-side calculation of $\langle [pt]^\eta, \psi^{\nu-2}[pt] \rangle_{0,a[C_0]+b[f]}$, at least for low order terms. Remember that the medium for comparing A-side and B-side calculations is the J -function from section 2.2. (The exact comparison is given by conjecture 3.3.1.) Therefore, the notation of the Gromov-Witten invariant just given is the same as in equation (2.3). And it should be noted that η will be negative in this

specific case so that we are actually calculating $\langle \psi^{\nu-2}[pt] \rangle_{0,a[C_0]+b[f]}$. We also have the special case of the coefficient of $T_0 h^0$ in the J -function, which is one and which we will treat as the case where $\nu = 0$ and $(a, b) = (0, 0)$. We have put the low order coefficients into table 6.1 coinciding with their appropriate Gromov-Witten invariants. Note that the value of ν matches with the negative of the power of $\hbar$ for each term. This is expected, because by conjecture 3.3.1 these should differ by $\deg(T_0)/2 = 0$. (That is, the powers of $\hbar$ here are the same as the powers of $\hbar$ in the J -function.)

Table 6.1: Gromov-Witten invariant calculations for $\langle \psi^{\nu-2}[pt] \rangle_{0,a[C_0]+b[f]}$.

	$\nu = 0$	$\nu = 2$		$\nu = 4$		
(a,b)	(0,0)	(0,1)	(1,1)	(0,2)	(1,2)	(2,2)
GW	1	1	1	1/4	1	1/4

6.1.2 The Corrected Integral over Ξ'_2

Calculating the low order terms for the corrected period integral over Ξ'_2 is a more complicated business. We have to do more than just change the variables in ξ'_2 . We have to take into account the term $d_1^{C_0} d_2^f$ in the solution given by expression (A.8). When we make the substitution where d_1 goes to $d_1 + d_1^{-1} d_2$, we get that $d_1^{C_0} d_2^f$ goes to $(d_1 + d_2/d_1)^{C_0} (d_2)^f = (1 + d_2/d_1^2)^{C_0} d_1^{C_0+f} (d_2/d_1)^f$. We will leave $d_1^{C_0+f} (d_2/d_1)^f$ out front, but we will need to multiply the rest of expression (A.8) through by $(1 + d_2/d_1^2)^{C_0}$. This term, though, has a finite Taylor expansion since $C_0^3 = 0$. We get

$$\begin{aligned}
(1 + d_2/d_1^2)^{C_0} &= 1 + C_0 \ln(1 + d_2/d_1^2) + C_0^2 \ln(1 + d_2/d_1^2)^2/2 \\
&= 1 + C_0 \ln(1 + d_2/d_1^2) - [pt] \ln(1 + d_2/d_1^2)^2
\end{aligned}$$

Let us plug this into the pertinent part of expression (A.8). Keep in mind for the following calculation that $\xi'_i = \xi'_i(d_1, d_2)$ is a function, and in this calculation we

are considering it as the function after our substitutions, that is $\xi'_i(d_1 + d_2/d_1, d_2)$.

$$\begin{aligned}
& (1 + d_2/d_1^2)^{C_0} (\xi'_1 + \xi'_2 C_0 + \xi'_3 f + \xi'_4 [pt]) \\
&= \left(1 + C_0 \ln(1 + d_2/d_1^2) - [pt] \ln(1 + d_2/d_1^2)^2 \right) (\xi'_1 + \xi'_2 C_0 + \xi'_3 f + \xi'_4 [pt]) \\
&= \xi'_1 + [\xi'_2 + \ln(1 + d_2/d_1^2) \xi'_1] C_0 + \xi'_3 f \\
&\quad + \left[\xi'_4 + \ln(1 + d_2/d_1^2) (-2\xi'_2 + \xi'_3) - \ln(1 + d_2/d_1^2)^2 \xi'_1 \right] [pt] \tag{6.2}
\end{aligned}$$

So in order to obtain the low order terms for the coefficient of C_0 , we need to expand out $\ln(1 + d_2/d_1^2)$ and then multiply these through. (Notice that the coefficient of f – section 6.1.4 – remains unchanged by the change in variable, as does the first term ξ'_1 – section 6.1.1.) The computer program Maple can carry these calculations out with the following result.

$$-2 \frac{d_1}{\hbar^2} - 3/4 \frac{d_1^2}{\hbar^4} - 2 \frac{d_2}{\hbar^4} - 1/2 \frac{d_2^2 d_1^{-1}}{\hbar^6} - \frac{11}{108} \frac{d_1^3}{\hbar^6} - 3/4 \frac{d_1 d_2}{\hbar^6}$$

And these terms will match the Gromov-Witten invariants of the form $\langle [pt]^\eta, \psi^{\nu-2}[f] \rangle_{0,a[C_0]+b[f]}$. Again η is not positive. Note that the value of $\nu - 1$ matches with the negative of the power of $\hbar$ for each term. This is expected, because by conjecture 3.3.1 these should differ by $\deg(T_1)/2 = 1$. (That is, the powers of $\hbar$ here differ by one with the powers of $\hbar$ in the J -function.)

Table 6.2: Gromov-Witten invariant calculations for $\langle \psi^{\nu-2}[f] \rangle_{0,a[C_0]+b[f]}$.

	$\nu = 3$	$\nu = 5$		$\nu = 7$		
(a,b)	(0,1)	(0,2)	(1,2)	(0,3)	(1,3)	(2,3)
GW	-2	-3/4	-2	-11/108	-3/4	-1/2

6.1.3 The Corrected Integral over Ξ'_3

Here, we are back to the situation of section 6.1.1 where all we have to do to get the corrected period integral over Ξ'_3 is to apply substitution (6.1) to ξ'_3 , simplify it, and view the low order terms. Recall that ξ'_3 is as follows.

$$- \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!} (2H_{k_2} + 2H_{k_1+2k_2})$$

The calculations follow those of section 6.1.1 except for the following change. We need this calculation:

$$\begin{aligned} & \sum_{t=0}^{\min(\alpha, \beta)} \binom{\alpha + \beta - 2t}{\beta - t} \frac{1}{(\alpha + \beta - 2t)! (t!)^2} H_t \\ &= \sum_{t=0}^{\min(\alpha, \beta)} \frac{1}{(\beta - t)! (\alpha - t)! (t!)^2} H_t \\ &= \frac{1}{\alpha! \beta!} \sum_{t=0}^{\alpha} \binom{\alpha}{t} \binom{\beta}{t} H_t \quad \text{WLOG assume } \alpha \leq \beta \\ &= \frac{1}{\alpha! \beta!} \binom{\alpha + \beta}{\alpha} (H_\alpha + H_\beta - H_{\alpha+\beta}) \quad \text{by identity (7) given in [21]} \\ &= \frac{(\alpha + \beta)!}{(\alpha!)^2 (\beta!)^2} (H_\alpha + H_\beta - H_{\alpha+\beta}) \end{aligned}$$

Using this fact, we get:

$$\begin{aligned} & - \sum_{\alpha, \beta \geq 0} \frac{d_1^\alpha d_2^\beta \hbar^{-2(\alpha+\beta)}}{(\alpha!)^2 (\beta!)^2} [2(H_\alpha + H_\beta - H_{\alpha+\beta}) + 2H_{\alpha+\beta}] \\ &= -2 \sum_{\alpha, \beta \geq 0} \frac{d_1^\alpha d_2^\beta \hbar^{-2(\alpha+\beta)}}{(\alpha!)^2 (\beta!)^2} (H_\alpha + H_\beta) \end{aligned}$$

Just as in section 6.1.1, we need to send d_2 back to d_2/d_1 so that we get

$$-2 \sum_{\alpha, \beta \geq 0} \frac{d_1^{\alpha-\beta} d_2^\beta \hbar^{-2(\alpha+\beta)}}{(\alpha!)^2 (\beta!)^2} (H_\alpha + H_\beta).$$

Again, we can easily extract the low order terms of this expression:

$$-2 \left(\hbar^{-2} (d_1 + d_1^{-1} d_2) + \hbar^{-4} ((H_2) d_1^2 / 4 + 2d_2 + (H_2) d_1^{-2} d_2^2 / 4) + \text{H.O.T.} \right)$$

And again these terms match with the calculations we get from the computer program GROWI [6] for Gromov-Witten invariants of the form $\langle [pt]^\eta, \psi^{\nu-2}[C_0 + 2f] \rangle_{0, a[C_0] + b[f]}$. Again in this case, η will be negative so that $[pt]^\eta$ drops off. Note that the value of $\nu - 1$ matches with the negative of the power of $\hbar$ for each term. This is expected, because by conjecture 3.3.1 these should differ by $\deg(T_2)/2 = 1$. (That is, the powers of $\hbar$ here differ by one with the powers of $\hbar$ in the J -function.)

Table 6.3: Gromov-Witten invariants calculations for $\langle \psi^{\nu-2}[C_0 + 2f] \rangle_{0, a[C_0] + b[f]}$.

	$\nu = 3$		$\nu = 5$		
(a,b)	(0,1)	(1,1)	(0,2)	(1,2)	(2,2)
GW	-2	-2	-3/4	-4	-3/4

6.1.4 The Corrected Integral over Ξ'_4

In this section we will calculate the low order terms of the coefficient of $[pt]$ in the equation (6.2). This will give the solution to the corrected period integral over the cycle Ξ'_4 . The calculations here follow section 6.1.2 very closely. It is only slightly more complicated to obtain the low order terms for the coefficient of $[pt]$ than for C_0 . We need to expand out $\ln(1 + d_2/d_1^2)$ and $\ln(1 + d_2/d_1^2)^2$ and then multiply these through. The computer program Maple can carry these calculations out with the following result.

$$4 \frac{d_2}{\hbar^4} + 3/2 \frac{d_1 d_2}{\hbar^6} + 3/2 \frac{d_1^{-1} d_2^2}{\hbar^6} + \frac{9}{16} \frac{d_2^2}{\hbar^8} + \frac{11}{54} \frac{d_1^{-2} d_2^3}{\hbar^8} + \frac{11}{54} \frac{d_1^2 d_2}{\hbar^8}$$

And these terms match what we expect to obtain by using the computer program GROWI [6] to calculate the Gromov-Witten invariants of the form $\langle [pt]^\eta, \psi^{\nu-2} \rangle_{0, a[C_0] + b[f]}$. Again η is not positive. Note that the value of $\nu - 2$ matches with the negative of the power of $\hbar$ for each term. This is expected, because by conjecture 3.3.1 these should differ by $\deg(T_3)/2 = 2$. (That is, the powers of $\hbar$ here differ by two with the powers of $\hbar$ in the J -function.)

Table 6.4: Gromov-Witten invariant calculations for $\langle \psi^{\nu-2} \rangle_{0, a[C_0] + b[f]}$.

	$\nu = 6$	$\nu = 8$		$\nu = 10$		
(a,b)	(1,2)	(1,3)	(2,3)	(1,4)	(2,4)	(3,4)
GW	4	3/2	3/2	11/54	9/16	11/54

6.2 Integral Formula

In this section, we lay out a straightforward method for calculating the period integrals after a point correction to the potential function. Using the methods laid out in this section, we will not have to recalculate the period integrals from scratch but rather use what we have already done to our advantage.

Remember that the period integral we originally computed was

$$I_i(d_1, d_2) = \int_{\Xi_i} e^{(x+y+d_2x^{-2}y^{-1}+d_1x^{-1})/\hbar} \Omega. \quad (6.3)$$

We want to use these calculations to calculate integrals of the form

$$\int_{\Xi_i} x^a y^b e^{(x+y+d_2x^{-2}y^{-1}+d_1x^{-1})/\hbar} \Omega$$

for any integers a and b .

As a first pass, we can see that if we differentiate the integrand by d_1 a copy of $x^{-1}/\hbar$ comes down, and if we differentiate by d_2 a copy of $x^{-2}y^{-1}/\hbar$ comes down. If we continue differentiating in this way we obtain for any nonnegative integers a and b :

$$\boxed{\hbar^{a+b} \frac{\partial^{a+b}}{\partial^a d_1 \partial^b d_2} (I_i(d_1, d_2)) = \int_{\Xi_i} (x^{-1})^a (x^{-2}y^{-1})^b e^{(x+y+d_2x^{-2}y^{-1}+d_1x^{-1})/\hbar} \Omega.} \quad (6.4)$$

But this is not enough, as we will want to be able to integrate with any monomial $x^a y^b$ out front for any integers a and b , not just the monomials of the form $(x^{-1})^a (x^{-2}y^{-1})^b$ for nonnegative integers a and b . In a sense, we have only found the monomials from the cone between the first and fourth legs of the fan for $\mathbb{F}_2$.

To get monomials from other cones, we can use substitution. Make a change of variables by sending x to $d_1 x$; now this substitution will not change the integral (6.3) since $\Omega = \frac{\partial x \wedge \partial y}{xy}$. We will also need to send d_2 to $d_2 d_1^2$. Thus we get

$$I(d_1, d_2 d_1^2) = \int e^{(d_1 x + y + d_2 x^{-2} y^{-1} + x^{-1})/\hbar} \Omega. \quad (6.5)$$

Differentiating as before gives powers of x and $x^{-2}y^{-1}$ as the monomial out front.

$$\begin{aligned}
\hbar^{a+b} \frac{\partial^{a+b}}{\partial^a d_1 \partial^b d_2} (I(d_1, d_2 d_1^2)) &= \int x^a (x^{-2}y^{-1})^b e^{(d_1 x + y + d_2 x^{-2}y^{-1} + x^{-1})/\hbar} \Omega \\
&= \int \left(\frac{x}{d_1}\right)^a \left(\left(\frac{x}{d_1}\right)^{-2} y^{-1}\right)^b e^{(x+y+d_2 d_1^2 x^{-2}y^{-1} + d_1 x^{-1})/\hbar} \Omega \\
&= d_1^{-a+2b} \int x^a (x^{-2}y^{-1})^b e^{(x+y+d_2 d_1^2 x^{-2}y^{-1} + d_1 x^{-1})/\hbar} \Omega.
\end{aligned}$$

In the second step above, we reversed one of the substitutions from above. That is, we sent x to x/d_1 ; again, this does not change the integral. In the last step, we pulled out an offending constant. The following equation is the obvious conclusion of these calculations, and it is also how we will calculate the integral on the right hand side.

$$\boxed{d_1^{a-2b} \hbar^{a+b} \frac{\partial^{a+b}}{\partial^a d_1 \partial^b d_2} (I(d_1, d_2 d_1^2)) = \int x^a (x^{-2}y^{-1})^b e^{(x+y+d_2 d_1^2 x^{-2}y^{-1} + d_1 x^{-1})/\hbar} \Omega.} \quad (6.6)$$

That is to say, we will calculate the expression on the left hand side, and then make the substitution where d_2 goes to d_2/d_1^2 in order to obtain for nonnegative integers a and b :

$$\int x^a (x^{-2}y^{-1})^b e^{(x+y+d_2 x^{-2}y^{-1} + d_1 x^{-1})/\hbar} \Omega.$$

Now, for a third case, starting with (6.5), if we now send y to $d_2 y$ then we will get the monomials generated by x and y . But this substitution does not change the integral, and so

$$I(d_1, d_2 d_1^2) = \int e^{(d_1 x + d_2 y + x^{-2}y^{-1} + x^{-1})/\hbar} \Omega.$$

and

$$\begin{aligned}
\hbar^{a+b} \frac{\partial^{a+b}}{\partial^a d_1 \partial^b d_2} (I(d_1, d_2 d_1^2)) &= \int x^a y^b e^{(d_1 x + d_2 y + x^{-2}y^{-1} + x^{-1})/\hbar} \Omega \\
&= \int x^a \left(\frac{y}{d_2}\right)^b e^{(d_1 x + y + d_2 x^{-2}y^{-1} + x^{-1})/\hbar} \Omega \quad (6.7)
\end{aligned}$$

$$\begin{aligned}
&= \int \left(\frac{x}{d_1}\right)^a \left(\frac{y}{d_2}\right)^b e^{(x+y+d_2 d_1^2 x^{-2}y^{-1} + d_1 x^{-1})/\hbar} \Omega \quad (6.8) \\
&= d_1^{-a} d_2^{-b} \int x^a y^b e^{(x+y+d_2 d_1^2 x^{-2}y^{-1} + d_1 x^{-1})/\hbar} \Omega.
\end{aligned}$$

Again, in the above calculations we reversed the substitutions we had used. In line (6.7), we sent y to y/d_2 . In line (6.8) we again sent x to x/d_1 . This gives

$$\boxed{d_1^a d_2^b \hbar^{a+b} \frac{\partial^{a+b}}{\partial^a d_1 \partial^b d_2} (I(d_1, d_2 d_1^2)) = \int x^a y^b e^{(x+y+d_2 d_1^2 x^{-2} y^{-1} + d_1 x^{-1})/\hbar} \Omega.} \quad (6.9)$$

That is, we have to make the calculation on the left hand side of the above equation and then send d_2 to d_2/d_1^2 in order to obtain for nonnegative integers a and b :

$$\int x^a y^b e^{(x+y+d_2 x^{-2} y^{-1} + d_1 x^{-1})/\hbar} \Omega.$$

And for the fourth and final case, we would need to do a substitution of the original period integral (6.3) by sending y to $d_2 y$ in order to get the monomials generated by y and x^{-1} . Again, the integral is not changed by this substitution.

$$I(d_1, d_2) = \int e^{(x+d_2 y+x^{-2} y^{-1} + d_1 x^{-1})/\hbar} \Omega.$$

And so we get

$$\begin{aligned} \hbar^{a+b} \frac{\partial^{a+b}}{\partial^a d_1 \partial^b d_2} I(d_1, d_2) &= \int (x^{-1})^a y^b e^{(x+d_2 y+x^{-2} y^{-1} + d_1 x^{-1})/\hbar} \Omega \\ &= \int (x^{-1})^a \left(\frac{y}{d_2}\right)^b e^{(x+y+d_2 x^{-2} y^{-1} + d_1 x^{-1})/\hbar} \Omega \end{aligned}$$

The last line was gotten by sending y to y/d_2 . And so

$$\boxed{d_2^b \hbar^{a+b} \frac{\partial^{a+b}}{\partial^a d_1 \partial^b d_2} (I(d_1, d_2)) = \int (x^{-1})^a y^b e^{(x+y+d_2 x^{-2} y^{-1} + d_1 x^{-1})/\hbar} \Omega} \quad (6.10)$$

6.3 One Point Correction

As we saw on page 52, when we add one marked point to the situation we get one more Maslov index two disk, which we represented by a broken line with the monomial $d_2x^{-1}y^{-1}u_1$ associated to it. Using the formulas from section 6.2, we can come up with the B-side calculations for one marked point.

Before, in the case with zero marked points, we computed the B-side period integrals of the form

$$I_i(d_1, d_2) = \int_{\Xi_i} e^{(x+y+d_2x^{-2}y^{-1}+d_1x^{-1})/\hbar} \Omega.$$

Recall that the potential function was this part of the above expression: $x + y + d_2x^{-2}y^{-1} + d_1x^{-1}$. Before we can move on to the period integrals for one marked point, we recall that we needed to perturb the potential function according to our model wherein we “pushed in the corners” of the underlying affine manifold to get $x + y + d_2x^{-2}y^{-1} + d_1x^{-1} + d_1^{-1}d_2x^{-1}$. Now, we point out that this new integral can be gotten simply by doing a change of variables, sending d_1 to $d_1 + d_1^{-1}d_2$. That is,

$$I_i(d_1 + d_1^{-1}d_2, d_2) = \int_{\Xi_i} e^{(x+y+d_2x^{-2}y^{-1}+(d_1+d_1^{-1}d_2)x^{-1})/\hbar} \Omega.$$

To get this to align with the boxed items in section 6.2, we need only make the above substitution into the final calculation.

In order to obtain the period integrals for one marked point, we need to perturb the potential function further by adding the monomial coming from the scattering diagram and broken lines for one marked point, which we said at the top of the section was $d_2x^{-1}y^{-1}u_1$. This gives us the new perturbed potential function $x + y + d_2x^{-2}y^{-1} + (d_1 + d_2/d_1)x^{-1} + d_2x^{-1}y^{-1}u_1$. Now let us plug this back into a

period integral.

$$\begin{aligned}
& \int_{\Xi_i} e^{(x+y+d_2x^{-2}y^{-1}+(d_1+d_2/d_1)x^{-1}+d_2x^{-1}y^{-1}u_1)/\hbar} \Omega \\
&= \int_{\Xi_i} e^{(d_2x^{-1}y^{-1}u_1)/\hbar} e^{(x+y+d_2x^{-2}y^{-1}+(d_1+d_2/d_1)x^{-1})/\hbar} \Omega \\
&= \int_{\Xi_i} \sum_{n=0}^{\infty} \frac{(d_2x^{-1}y^{-1}u_1)^n}{\hbar^n n!} e^{(x+y+d_2x^{-2}y^{-1}+(d_1+d_2/d_1)x^{-1})/\hbar} \Omega \\
&= \sum_{n=0}^{\infty} \frac{(d_2u_1)^n}{\hbar^n n!} \int_{\Xi_i} (x^{-1}y^{-1})^n e^{(x+y+d_2x^{-2}y^{-1}+(d_1+d_2/d_1)x^{-1})/\hbar} \Omega \\
&= \int_{\Xi_i} e^{(x+y+d_2x^{-2}y^{-1}+(d_1+d_2/d_1)x^{-1})/\hbar} \Omega \\
&\quad + \frac{d_2u_1}{\hbar} \int_{\Xi_i} (x^{-1}y^{-1}) e^{(x+y+d_2x^{-2}y^{-1}+(d_1+d_2/d_1)x^{-1})/\hbar} \Omega \quad \text{since } u_1^2 = 0
\end{aligned}$$

Now this expression following the last equal sign is important. It comes in two parts, and the first part is just the zero-point calculation that we have already made. This will give us the Gromov-Witten invariants for zero points. The second part will give us the Gromov-Witten invariants with one marked point.

For each of the cycles Ξ_i we can calculate

$$\frac{d_2u_1}{\hbar} \int_{\Xi_i} (x^{-1}y^{-1}) e^{(x+y+d_2x^{-2}y^{-1}+(d_1+d_2/d_1)x^{-1})/\hbar} \Omega$$

using formula (6.6) with $(a, b) = (1, 1)$. We used the computer program Maple to carry out the calculations and obtained some low order terms. These low order terms were then matched against Gromov-Witten invariant calculations carried out in the computer program GROWI ??.

- For the torus Ξ_1 , the coefficient of $\hbar^{-\nu} d_1^{-2a+b} d_2^a$ coincides with the Gromov-Witten invariant $\langle [pt], \psi^{\nu-2}[pt] \rangle_{0,a[C_0]+b[f]}$ where :

$$\hbar^{-3}d_2 + \hbar^{-5}((1/2)d_1^{-1}d_2^2 + (1/2)d_1d_2) + \hbar^{-7}((1/12)d_1^2d_2 + (1/12)d_1^{-2}d_2^3 + (1/4)d_2^2) + \text{H.O.T.}$$

- For Ξ_2 , the coefficient of $\hbar^{-\nu} d_1^{-2a+b} d_2^a$ coincides with the Gromov-Witten invariant $\langle [pt], \psi^{\nu-1}[f] \rangle_{0,a[C_0]+b[f]}$. These numbers were calculated in a less direct fashion against the results from the computer program GROWI ??.

$$\hbar^{-1} \left(d_1^{-1} d_2 \right) + \hbar^{-3} \left((1/2) d_1^{-2} d_2^2 - d_2 \right) + \hbar^{-5} \left((1/12) d_1^{-3} d_2 - (1/2) d_1^{-1} d_2^2 - (5/4) d_1 d_2 \right) + \text{H.O.T.}$$

- For Ξ_3 , the coefficient of $\hbar^{-\nu} d_1^{-2a+b} d_2^a$ coincides with the Gromov-Witten invariant $\langle [pt], \psi^{\nu-1} [C_0 + 2f] \rangle_{0, a[C_0] + b[f]}$:

$$\hbar^{-1} (d_1 + d_1^{-1} d_2) + \hbar^{-3} ((1/2) d_1^{-2} d_2^2 - 2d_2 + (1/2) d_1^2) + \hbar^{-5} ((1/12) d_1^3 - (7/4) d_1 d_2 + (1/12) d_1^{-3} d_2^3 - (7/4) d_1^{-1} d_2^2) + \text{H.O.T.}$$

- For Ξ_4 , the coefficient of $\hbar^{-\nu} d_1^{-2a+b} d_2^a$ coincides with the Gromov-Witten invariant $\langle [pt], \psi^\nu \rangle_{0, a[C_0] + b[f]}$:

$$\hbar^{-1} (-d_1^{-1} d_2 - d_1) + \hbar^{-3} ((-5/4) d_1^{-2} d_2^2 + d_2 - (5/4) d_1^2) + \hbar^{-5} ((-5/18) d_1^{-3} d_2^3 + (5/4) d_1^{-1} d_2^2 + (5/4) d_1 d_2 - (5/18) d_1^3) + \text{H.O.T.}$$

6.4 Two Point Correction

The calculation for finding the Gromov-Witten invariants for two marked points follows similarly. First we need to perturb the potential function using the broken line calculations from page 55. The perturbed potential function is $x+y+d_2x^{-2}y^{-1}+(d_1+d_2d_1^{-1})x^{-1}+d_2x^{-1}u_1u_2+d_1d_2x^{-2}y^{-1}u_1u_2+d_2^2d_1^{-1}x^{-2}y^{-1}u_1u_2+d_2^2x^{-3}y^{-2}u_1u_2+d_2x^{-1}y^{-1}u_1+d_2x^{-1}y^{-1}u_2$. For brevity's sake, we will refer to the corrected zero point potential function as W'_0 where appropriate, that is $W'_0 = x + y + d_2x^{-2}y^{-1} + (d_1 + d_2d_1^{-1})x^{-1}$. Plugging this into a period integral, we get.

$$\begin{aligned}
& \int_{\Xi_i} \exp[(1/h)(W'_0 + u_1u_2((d_1d_2 + d_2^2d_1^{-1})x^{-2}y^{-1} \\
& \quad + d_2x^{-1} + d_2^2x^{-3}y^{-2}) + d_2x^{-1}y^{-1}(u_1 + u_2))]\Omega \\
&= \int_{\Xi_i} e^{u_1u_2((d_1d_2+d_2^2d_1^{-1})x^{-2}y^{-1}+d_2x^{-1}+d_2^2x^{-3}y^{-2})/h} \cdot e^{(u_1+u_2)d_2x^{-1}y^{-1}/h} \cdot e^{W'_0/h}\Omega \\
&= \int_{\Xi_i} e^{W'_0/h}\Omega \\
& \quad + \frac{d_2(u_1 + u_2)}{\hbar} \int_{\Xi_i} (x^{-1}y^{-1})e^{W'_0/h}\Omega \\
& \quad + \frac{d_2^2u_1u_2}{\hbar^2} \int_{\Xi_i} (x^{-2}y^{-2})e^{W'_0/h}\Omega \\
& \quad + \frac{u_1u_2}{\hbar} \int_{\Xi_i} ((d_1d_2 + d_2^2d_1^{-1})x^{-2}y^{-1} + d_2x^{-1} + d_2^2x^{-3}y^{-2})e^{W'_0/h}\Omega \tag{6.11}
\end{aligned}$$

What follows the last equal sign in the equation just above is a breakdown of the calculations of the Gromov-Witten invariants for no marked points (the first line), for the first marked point or the second marked point (the second line), and for both of the marked points (the last two lines). This is made clear in the expression by the presence of the u_i which tell us which points are being passed through.

It remains for us to calculate the last two lines of expression (6.11). We can calculate

$$\frac{d_2^2u_1u_2}{\hbar^2} \int_{\Xi_i} (x^{-2}y^{-2})e^{W'_0/h}\Omega$$

for each of the cycles Ξ_i by applying formula (6.6). Now,

$$\frac{u_1u_2}{\hbar} \int_{\Xi_i} ((d_1d_2 + d_2^2d_1^{-1})x^{-2}y^{-1} + d_2x^{-1} + d_2^2x^{-3}y^{-2})e^{W'_0/h}\Omega$$

breaks up into three parts

$$\frac{u_1 u_2}{\hbar} \int_{\Xi_i} ((d_1 d_2 + d_2^2 d_1^{-1}) x^{-2} y^{-1}) e^{W'_0/\hbar \Omega} \quad (6.12)$$

$$+ \frac{u_1 u_2}{\hbar} \int_{\Xi_i} (d_2 x^{-1}) e^{W'_0/\hbar \Omega} \quad (6.13)$$

$$+ \frac{u_1 u_2}{\hbar} \int_{\Xi_i} (d_2^2 x^{-3} y^{-2}) e^{W'_0/\hbar \Omega} \quad (6.14)$$

Line (6.14) can again be calculated using formula (6.6). And lines (6.12) and (6.13) will need formula (6.4). The sixteen cases involved required the use of the computer package Maple as well as somewhat robust computer. These low order terms were then matched against Gromov-Witten invariant calculations carried out in the computer program GROWI ??. The low order terms for each cycle are as follows.

- For the torus Ξ_1 , the coefficient of $\hbar^{-\nu} d_1^{-2a+b} d_2^a$ coincides with the Gromov-Witten invariant $\langle [pt]^2, \psi^{\nu-2}[pt] \rangle_{0,a[C_0]+b[f]}$:

$$\hbar^{-2}(d_2) + \hbar^{-4}(d_1^{-1} d_2^2 + d_1 d_2) + \hbar^{-6}\left((1/4)d_1^{-2} d_2^3 + d_2^2 + (1/4)d_1^2 d_2\right) + \text{H.O.T.}$$

- For Ξ_2 , the coefficient of $\hbar^{-\nu} d_1^{-2a+b} d_2^a$ coincides with the Gromov-Witten invariant $\langle [pt]^2, \psi^{\nu-1}[f] \rangle_{0,a[C_0]+b[f]}$. These numbers were calculated in a less direct fashion against the results from the computer program GROWI ??.

$$(-1) \left(\hbar^{-4}(2d_1 d_2) + \hbar^6 \left((1/36)d_1^{-4} d_2^4 + 2d_2 + (3/4)d_1^2 d_2 \right) \right) + \text{H.O.T.}$$

- For Ξ_3 , the coefficient of $\hbar^{-\nu} d_1^{-2a+b} d_2^a$ coincides with the Gromov-Witten invariant $\langle [pt]^2, \psi^{\nu-1}[C_0 + 2f] \rangle_{0,a[C_0]+b[f]}$:

$$(-1) \left(\hbar^{-4}(2d_1^{-1} d_2^2 + 2d_1 d_2) + \left((3/4)d_1^{-2} d_2^3 + 4d_2^2 + (3/4)d_1^2 d_2 \right) \right) + \text{H.O.T.}$$

- For Ξ_4 , the coefficient of $\hbar^{-\nu} d_1^{-2a+b} d_2^a$ coincides with the Gromov-Witten invariant $\langle [pt]^2, \psi^{\nu} \rangle_{0,a[C_0]+b[f]}$:

$$\hbar^{-6}(4d_2) + \hbar^{-8}\left((3/2)d_1^{-1} d_2^3 + (3/2)d_1 d_2^2\right) + \left((11/54)d_1^{-2} d_2^4 + (9/16)d_2^3 + (11/54)d_1^2 d_2^2\right) + \text{H.O.T.}$$

Appendix A

Final Notes

This appendix contains the bulk of the calculations necessary for deriving the period integrals for the second Hirzebruch surface, $\mathbb{F}_2$. In section 3.1, we give a method for choosing cycles Ξ_i to integrate over. Then at the start of section 3.2 we give a more explicit description for $\mathbb{F}_2$.

In the following sections we will first calculate the period integral over the unit torus. Then we will do the same for the various cylinders, which we will glue together to form two different cycles, in sections A.6 and A.7. Furthermore, in A.7 we will simplify these calculations somewhat. In section A.9 we develop a different method for calculating the period integral over the final cycle. This will allow us to obtain a new set of cycles Ξ'_i whose period integrals are rational and single-valued.

A.1 The Unit Torus

We begin by taking the period integral over $\Xi_1 = T^2$, the unit torus.

$$\begin{aligned}
I &= \int_{T^2} e^{(x+y+d_1x^{-1}+d_2x^{-2}y^{-1})/\hbar} \frac{dx \wedge dy}{xy} \\
&= \int_{T^2} \sum_{n=0}^{\infty} \frac{(x+y+d_1x^{-1}+d_2x^{-2}y^{-1})^n}{n! \hbar^n} \frac{dx \wedge dy}{xy} \\
&= \sum_{n=0}^{\infty} \int_{T^2} \sum_{\substack{0 \leq k_1, k_2, k_3, k_4 \leq n \\ k_1+k_2+k_3+k_4=n}} \frac{\binom{n}{k_1, k_2, k_3, k_4}}{n! \hbar^n} x^{k_3} y^{k_2} d_1^{k_1} x^{-k_1} d_2^{k_4} x^{-2k_4} y^{-k_4} \frac{dx \wedge dy}{xy} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1+k_2+k_3+k_4=n}} \frac{\binom{n}{k_1, k_2, k_3, k_4}}{n! \hbar^n} d_1^{k_1} d_2^{k_4} \int_{T^2} x^{k_3-k_1-2k_4-1} y^{k_2-k_4-1} dx \wedge dy
\end{aligned}$$

Notice above that the k_i were chosen to go along with the ordering of the rays of the fan for $\mathbb{F}_2$ given in figure A.1 .

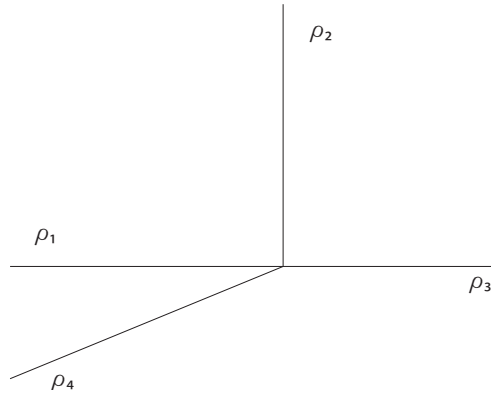


Figure A.1: The fan for $\mathbb{F}_2$.

Just for the moment, let $\oint$ represent integration over the unit circle in the complex plane. Then we know that $\oint z^a dz = 2\pi i$ if $a = -1$ and the integral is zero otherwise. So the above integral becomes:

$$\begin{aligned}
I &= \sum_{n=0}^{\infty} \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1+k_2+k_3+k_4=n}} \frac{\binom{n}{k_1, k_2, k_3, k_4}}{n! \hbar^n} d_1^{k_1} d_2^{k_4} \oint x^{k_3-k_1-2k_4-1} dx \oint y^{k_2-k_4-1} dy \\
&= -4\pi^2 \sum_{n=0}^{\infty} \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1+k_2+k_3+k_4=n \\ k_3-k_1-2k_4-1=-1 \\ k_2-k_4-1=-1}} \frac{\binom{n}{k_1, k_2, k_3, k_4}}{n! \hbar^n} d_1^{k_1} d_2^{k_4} \\
&= -4\pi^2 \sum_{n=0}^{\infty} \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1+k_2+k_3+k_4=n \\ k_3-k_1-2k_4=0 \\ k_2-k_4=0}} \frac{d_1^{k_1} d_2^{k_4}}{k_1! k_2! k_3! k_4! \hbar^n} \\
&= -4\pi^2 \sum_{n=0}^{\infty} \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1+k_2+k_3+k_4=n \\ k_3-k_1-2k_4=0 \\ k_2-k_4=0}} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1+2k_2)!} \\
&= \boxed{-4\pi^2 \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1+2k_2)!}}
\end{aligned}$$

Notice the vast simplification of the double sum in the last step.

A.2 The First Leg

Here we will integrate over the cylinder $S_1 = \{(x, y) \in \text{Log}^{-1}(\rho_1) : \arg(x) = 0\}$ where ρ_1 is as in figure A.1.

$$\begin{aligned} I &= \int_{S_1} e^{(x+y+d_1x^{-1}+d_2x^{-2}y^{-1})/\hbar} \Omega \\ &= \int_{S_1} e^{(d_1x^{-1})/\hbar} e^{(x+y+d_2x^{-2}y^{-1})/\hbar} \frac{dx \wedge dy}{xy} \end{aligned}$$

We pull out the factor above to preclude convergence issues later.

$$= \int_{S_1} e^{(d_1x^{-1})/\hbar} \sum_{n=0}^{\infty} \frac{(x+y+d_2x^{-2}y^{-1})^n}{n! \hbar^n} \frac{dx \wedge dy}{xy}$$

The above comes from the Taylor expansion of e^x .

$$= \sum_{n=0}^{\infty} \int_{S_1} e^{(d_1x^{-1})/\hbar} \sum_{\substack{k_2, k_3, k_4 \\ k_2+k_3+k_4=n}} \frac{\binom{n}{k_2, k_3, k_4}}{n! \hbar^n} x^{k_3} y^{k_2} (d_2x^{-2}y^{-1})^{k_4} \frac{dx \wedge dy}{xy}$$

The above coming from the multinomial theorem where we choose the k_i depending on which leg the contribution is coming from.

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \sum_{\substack{k_2, k_3, k_4 \\ k_2+k_3+k_4=n}} d_2^{k_4} \frac{\binom{n}{k_2, k_3, k_4}}{n! \hbar^n} \int_{S_1} e^{(d_1x^{-1})/\hbar} x^{k_3-2k_4} y^{k_2-k_4} \frac{dx \wedge dy}{xy} \\ &= \sum_{k_2, k_3, k_4 \geq 0} \frac{d_2^{k_4} \hbar^{-k_2-k_3-k_4}}{k_2! k_3! k_4!} \int_{S_1} e^{(d_1x^{-1})/\hbar} x^{k_3-2k_4} y^{k_2-k_4} \frac{dx \wedge dy}{xy} \end{aligned}$$

Let us now focus on the double integral we need to compute. We will call it L to make it easier to reference. Remember that we are integrating over a cylinder of radius x from $x = 0$ to $x = 1$. So here we can substitute $e^{i\theta}$ for y which will give:

$$\begin{aligned}
L &= \int_{S_1} e^{(d_1 x^{-1})/\hbar} x^{k_3-2k_4} y^{k_2-k_4} \frac{dx \wedge dy}{xy} \\
&= \int_0^1 \int_0^{2\pi} e^{d_1(x^{-1})/\hbar} x^{k_3-2k_4} e^{i\theta(k_2-k_4)} \frac{dx d(e^{i\theta})}{x e^{i\theta}} \\
&= \int_0^1 \int_0^{2\pi} e^{d_1(x^{-1})/\hbar} x^{k_3-2k_4-1} e^{i\theta(k_2-k_4)} i dx d\theta \\
&= i \int_0^1 e^{(d_1 x^{-1})/\hbar} x^{k_3-2k_4-1} dx \int_0^{2\pi} e^{i\theta(k_2-k_4)} d\theta
\end{aligned}$$

Let us put this integral into more manageable form by substituting r^{-1} for x .

$$\begin{aligned}
L &= i \int_{-\infty}^1 e^{d_1 r/\hbar} r^{-(k_3-2k_4-1)} d(r^{-1}) \int_0^{2\pi} e^{i\theta(k_2-k_4)} d\theta \\
&= i \int_{-\infty}^1 \frac{e^{d_1 r/\hbar}}{r^{k_3-2k_4-1}} (-r^{-2}) dr \int_0^{2\pi} e^{i\theta(k_2-k_4)} d\theta \\
&= i \int_1^{\infty} \frac{e^{d_1 r/\hbar}}{r^{k_3-2k_4+1}} dr \int_0^{2\pi} e^{i\theta(k_2-k_4)} d\theta
\end{aligned}$$

Notice that in the last line we cancelled out the negative sign by swapping the interval we were integrating over.

Now the integral over θ is equal to 0 except when $k_2 - k_4 = 0$. In this case, we get 2π .

$$L = \begin{cases} 2\pi i \int_1^{\infty} \frac{e^{d_1 r/\hbar}}{r^{k_3-2k_4+1}} dr & \text{when } k_2 - k_4 = 0 \\ 0 & \text{else} \end{cases}$$

So plugging this back into our original integral gives:

$$\begin{aligned}
I &= 2\pi i \sum_{\substack{k_2, k_3, k_4 \geq 0 \\ k_2 - k_4 = 0}} \frac{d_2^{k_4} \hbar^{-k_2-k_3-k_4}}{k_2! k_3! k_4!} \int_1^{\infty} \frac{e^{d_1 r/\hbar}}{r^{k_3-2k_4+1}} dr \\
&= 2\pi i \sum_{k_3, k_4 \geq 0} \frac{d_2^{k_4} \hbar^{-k_3-2k_4}}{k_3! (k_4!)^2} \int_1^{\infty} \frac{e^{d_1 r/\hbar}}{r^{k_3-2k_4+1}} dr
\end{aligned}$$

The coefficient above will reappear a lot; so let us call it

$$a_{k_3, k_4} := \frac{d_2^{k_4} \hbar^{-k_3-2k_4}}{k_3! (k_4!)^2}.$$

Then restating our integral we have

$$I = 2\pi i \sum_{k_3, k_4 \geq 0} a_{k_3, k_4} \int_1^\infty \frac{e^{d_1 r / \hbar}}{r^{k_3 - 2k_4 + 1}} dr$$

Now the interior integral

$$\int_1^\infty \frac{e^{d_1 r / \hbar}}{r^{k_3 - 2k_4 + 1}} dr$$

must be split up into two cases based on the sign of $k_3 - 2k_4 + 1$. These are special cases of a pair of integral functions $E_n(z)$ and $\alpha_n(z)$ found in [1] in the section on exponential integrals, page 228. We will give them here for easy reference.

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n} dt \quad (n = 0, 1, 2, \dots; \operatorname{Re}(z) > 0) \quad (\text{A.1})$$

$$\alpha_n(z) = \int_1^\infty t^n e^{-zt} dt \quad (n = 0, 1, 2, \dots; \operatorname{Re}(z) > 0) \quad (\text{A.2})$$

So

$$\int_1^\infty \frac{e^{d_1 r / \hbar}}{r^{k_3 - 2k_4 + 1}} dr = \begin{cases} E_{k_3 - 2k_4 + 1} \left(-\frac{d_1}{\hbar} \right) & \text{when } k_3 - 2k_4 + 1 \geq 1 \\ \alpha_{-k_3 + 2k_4 - 1} \left(-\frac{d_1}{\hbar} \right) & \text{when } k_3 - 2k_4 + 1 \leq 0 \end{cases}$$

So our original integral becomes a sum of two expressions J and K :

$$I = 2\pi i \underbrace{\sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \geq 1}} a_{k_3, k_4} E_{k_3 - 2k_4 + 1} \left(-\frac{d_1}{\hbar} \right)}_{=: J} + 2\pi i \underbrace{\sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} a_{k_3, k_4} \alpha_{-k_3 + 2k_4 - 1} \left(-\frac{d_1}{\hbar} \right)}_{=: K}$$

A.2.1 The Expansion

Focusing now on J , one can find a series expansion of $E_n(z)$ on page 229 of [1].

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!} [-\ln z + \Psi(n)] - \sum_{\substack{m=0 \\ m \neq n-1}}^{\infty} \frac{(-z)^m}{(m-n+1)m!} \quad (|\arg z| < \pi) \quad (\text{A.3})$$

where

$$\begin{aligned} \Psi(1) &= -\gamma \\ \Psi(n) &= -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \quad (n > 1) \\ \gamma &= .57721\ 56649\dots \quad \text{is Euler's constant.} \end{aligned}$$

So if we choose $|\arg(-\frac{d_1}{\hbar})| < \pi$, we can plug this into our integral.

$$\begin{aligned} E_{k_3-2k_4+1} \left(-\frac{d_1}{\hbar} \right) &= \\ &= \frac{(-(-\frac{d_1}{\hbar}))^{(k_3-2k_4+1)-1}}{((k_3-2k_4+1)-1)!} \left[-\ln \left(-\frac{d_1}{\hbar} \right) + \Psi(k_3-2k_4+1) \right] \\ &\quad - \sum_{\substack{m=0 \\ m \neq (k_3-2k_4+1)-1}}^{\infty} \frac{(-(-\frac{d_1}{\hbar}))^m}{(m-(k_3-2k_4+1)+1)m!} \\ &= \frac{\hbar^{-k_3+2k_4}}{(k_3-2k_4)! d_1^{-k_3+2k_4}} \left[\ln \left(-\frac{\hbar}{d_1} \right) + \Psi(k_3-2k_4+1) \right] \\ &\quad - \sum_{\substack{m=0 \\ m \neq k_3-2k_4}}^{\infty} \frac{d_1^m \hbar^{-m}}{(m-k_3+2k_4)m!} \end{aligned}$$

This gives the following expansion of J (remember that $I = J + K$).

$$J = \boxed{2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \geq 1}} a_{k_3, k_4} \left(\frac{\hbar^{-k_3 + 2k_4}}{(k_3 - 2k_4)! d_1^{-k_3 + 2k_4}} \right) \left[\ln \left(-\frac{\hbar}{d_1} \right) + \Psi(k_3 - 2k_4 + 1) \right]}$$

$$\boxed{-2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \geq 1}} a_{k_3, k_4} \sum_{\substack{m=0 \\ m \neq k_3 - 2k_4}}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!}}$$

Moving our attention to K , [1] gives on page 228 an explicit expression for $\alpha_n(z)$:

$$\alpha_n(z) = n! z^{-n-1} e^{-z} \left(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} \right) \quad (\text{A.4})$$

But this expression does not jibe well with our previous calculations. So we will come up with our own.

Proposition A.2.1.

$$\boxed{\alpha_n(z) = n! z^{-n-1} - \sum_{m=0}^{\infty} \frac{(-z)^m}{(m+n+1)m!}}$$

Proof. (First notice that in the equation given in the proposition, we have a problem if $m+n+1=0$, but since $\alpha_n(z)$ was originally only defined for nonnegative n , we have no problem.)

We will complete this proof using induction on n by comparing the expansion that we have come up with to the expansion in equation A.4.

Let us call our derived expression:

$$G_n(z) = n! z^{-n-1} - \sum_{m=0}^{\infty} \frac{(-z)^m}{(m+n+1)m!}$$

Now when $n=0$,

$$G_0(z) = z^{-1} - \sum_{m=0}^{\infty} \frac{(-z)^m}{(m+1)m!}$$

and

$$\begin{aligned}
\alpha_0(z) &= z^{-1}e^{-z} \\
&= z^{-1} \sum_{m=0}^{\infty} \frac{(-z)^m}{m!} \\
&= - \sum_{m=0}^{\infty} \frac{(-z)^{m-1}}{m!} \\
&= z^{-1} - \sum_{m=1}^{\infty} \frac{(-z)^{m-1}}{m!} \\
&= z^{-1} - \sum_{m=0}^{\infty} \frac{(-z)^m}{(m+1)!}
\end{aligned}$$

And we immediately see that $G_0(z) = \alpha_0(z)$.

Let us assume that it is true that $G_n(z) = \alpha_n(z)$ for some positive integer n , then

$$\begin{aligned}
\alpha_{n+1} &= (n+1)! z^{-n-2} e^{-z} (1 + z + \cdots + \frac{z^{n+1}}{(n+1)!}) \\
&= (n+1)z^{-1} [n! z^{-n-1} e^{-z} (1 + z + \cdots + \frac{z^n}{n!}) + n! z^{-n-1} e^{-z} \frac{z^{n+1}}{(n+1)!}] \\
&= (n+1)z^{-1} [\alpha_n(z) + n! z^{-n-1} e^{-z} \frac{z^{n+1}}{(n+1)!}] \\
&= (n+1)z^{-1} [G_n(z) + n! z^{-n-1} e^{-z} \frac{z^{n+1}}{(n+1)!}] \quad \text{by the induction hypothesis} \\
&= (n+1)z^{-1} [G_n(z) + \frac{e^{-z}}{(n+1)}] \\
&= (n+1)z^{-1} [G_n(z) + \sum_{m=0}^{\infty} \frac{(-z)^m}{(n+1)m!}] \\
&= (n+1)z^{-1} [n! z^{-n-1} - \sum_{m=0}^{\infty} \frac{(-z)^m}{(m+n+1)m!} + \sum_{m=0}^{\infty} \frac{(-z)^m}{(n+1)m!}]
\end{aligned}$$

But now we can combine the summations by considering the coefficients of the $-z^m$ terms. That is, for $m \geq 1$ we have

$$\begin{aligned}
-\frac{1}{(m+n+1)m!} + \frac{1}{(n+1)m!} &= \frac{1}{m!} \left(\frac{-(n+1) + m + n + 1}{(m+n+1)(n+1)} \right) \\
&= \frac{1}{(m-1)!(m+n+1)(n+1)}
\end{aligned}$$

While for $m = 0$ we get

$$-\frac{1}{n+1} + \frac{1}{n+1} = 0$$

And so

$$\begin{aligned}
\alpha_{n+1} &= (n+1)z^{-1} \left[n! z^{-n-1} + \sum_{m=1}^{\infty} \frac{(-z)^m}{(m+n+1)(n+1)(m-1)!} \right] \\
&= (n+1)! z^{-n-2} + \sum_{m=1}^{\infty} \frac{(-z)^{m-1}}{(m+n+1)(m-1)!} \\
&= (n+1)! z^{-n-2} - \sum_{m=0}^{\infty} \frac{(-z)^m}{(m+n)(m)!} \\
&= G_{n+1}(z)
\end{aligned}$$

□

Plugging the expansion of $\alpha_n(z)$ into K :

Recall that

$$K = 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} a_{k_3, k_4} \alpha_{-k_3 + 2k_4 - 1} \left(-\frac{d_1}{\hbar} \right)$$

But now after recalling that we chose $|\arg(-\frac{d_1}{\hbar})| < \pi$ earlier, we can say

$$\begin{aligned}
\alpha_{-k_3 + 2k_4 - 1} \left(-\frac{d_1}{\hbar} \right) &= \\
&= (-k_3 + 2k_4 - 1)! \left(-\frac{d_1}{\hbar} \right)^{-(-k_3 + 2k_4 - 1) - 1} - \sum_{m=0}^{\infty} \frac{\left(-\left(-\frac{d_1}{\hbar} \right) \right)^m}{(m + (-k_3 + 2k_4 - 1) + 1)m!} \\
&= (-k_3 + 2k_4 - 1)! \left(-\frac{d_1}{\hbar} \right)^{k_3 - 2k_4} - \sum_{m=0}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!}
\end{aligned}$$

Plugging this into K we get

$$K = \boxed{2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} a_{k_3, k_4} (-k_3 + 2k_4 - 1)! \left(-\frac{d_1}{\hbar}\right)^{k_3 - 2k_4}}$$

$$\boxed{-2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} a_{k_3, k_4} \sum_{m=0}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!}}$$

Combining J and K :

So we have:

$$\begin{aligned} I &= J + K \\ &= 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \geq 1}} a_{k_3, k_4} \left(\frac{\hbar^{-k_3 + 2k_4}}{(k_3 - 2k_4)! d_1^{-k_3 + 2k_4}} \right) \left[\ln \left(-\frac{\hbar}{d_1} \right) + \Psi(k_3 - 2k_4 + 1) \right] \\ &\quad - 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \geq 1}} a_{k_3, k_4} \sum_{\substack{m=0 \\ m \neq k_3 - 2k_4}}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!} \\ &\quad + 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} a_{k_3, k_4} (-k_3 + 2k_4 - 1)! \left(-\frac{d_1}{\hbar}\right)^{k_3 - 2k_4} \\ &\quad - 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} a_{k_3, k_4} \sum_{m=0}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!} \end{aligned}$$

Notice in the last expression that the second and fourth lines combine very nicely. So we will get

$$\begin{aligned}
I &= 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \geq 1}} a_{k_3, k_4} \left(\frac{\hbar^{-k_3+2k_4}}{(k_3 - 2k_4)! d_1^{-k_3+2k_4}} \right) \left[\ln \left(-\frac{\hbar}{d_1} \right) + \Psi(k_3 - 2k_4 + 1) \right] \\
&\quad + 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} a_{k_3, k_4} (-k_3 + 2k_4 - 1)! \left(-\frac{d_1}{\hbar} \right)^{k_3 - 2k_4} \\
&\quad - 2\pi i \sum_{k_3, k_4 \geq 0} a_{k_3, k_4} \sum_{\substack{m=0 \\ m \neq k_3 - 2k_4}}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!}
\end{aligned}$$

Now we need to unpack the coefficients $a_{k_3, k_4} := \frac{d_2^{k_4} \hbar^{-k_3-2k_4}}{k_3! (k_4!)^2}$.

$$\begin{aligned}
I &= 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \geq 1}} \frac{d_2^{k_4} \hbar^{-k_3-2k_4}}{k_3! (k_4!)^2} \left(\frac{d_1^{k_3-2k_4} \hbar^{-k_3+2k_4}}{(k_3 - 2k_4)!} \right) \left[\ln \left(-\frac{\hbar}{d_1} \right) + \Psi(k_3 - 2k_4 + 1) \right] \\
&\quad + 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} \frac{d_2^{k_4} \hbar^{-k_3-2k_4}}{k_3! (k_4!)^2} (-k_3 + 2k_4 - 1)! \left(-\frac{d_1}{\hbar} \right)^{k_3 - 2k_4} \\
&\quad - 2\pi i \sum_{k_3, k_4 \geq 0} \frac{d_2^{k_4} \hbar^{-k_3-2k_4}}{k_3! (k_4!)^2} \sum_{\substack{m=0 \\ m \neq k_3 - 2k_4}}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!} \\
&= 2\pi i \underbrace{\sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \geq 1}} \frac{d_1^{k_3-2k_4} d_2^{k_4} \hbar^{-2k_3}}{k_3! (k_4!)^2 (k_3 - 2k_4)!} \left[\ln \left(-\frac{\hbar}{d_1} \right) + \Psi(k_3 - 2k_4 + 1) \right]}_{=:R} \\
&\quad + 2\pi i \underbrace{\sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} \frac{d_1^{k_3-2k_4} d_2^{k_4} (-k_3 + 2k_4 - 1)!}{k_3! (k_4!)^2 \hbar^{2k_3}} (-1)^{-k_3+2k_4}}_{=:S} \\
&\quad - 2\pi i \underbrace{\sum_{k_3, k_4 \geq 0} \frac{d_2^{k_4} \hbar^{-k_3-2k_4}}{k_3! (k_4!)^2} \sum_{\substack{m=0 \\ m \neq k_3 - 2k_4}}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!}}_{=:T}
\end{aligned}$$

Notice that we have labeled the three parts of this integral as R , S , and T .

A.3 The Second Leg

In this section we will be taking the period integral over $S_2 = \{(x, y) \in \text{Log}^{-1}(\rho_2) : \arg(y) = 0\}$, where ρ_2 is the ray in $\mathbb{R}^2$ given in figure A.1.

$$\begin{aligned}
I &= \int_{S_2} e^{(x+y+d_1x^{-1}+d_2x^{-2}y^{-1})/\hbar} \Omega \\
&= \int_0^{2\pi} \int_1^\infty e^{(x+y+d_1x^{-1}+d_2x^{-2}y^{-1})/\hbar} \frac{i}{y} dy d\theta \\
&\quad \text{Here we substituted } x = e^{i\theta} \text{ into } \frac{dx}{x} \text{ giving } i d\theta. \\
&= \int_0^{2\pi} \int_1^\infty e^{y/\hbar} e^{(x+d_1x^{-1}+d_2x^{-2}y^{-1})/\hbar} \frac{i}{y} dy d\theta \\
&\quad \text{We pulled out the factor above to avoid convergence issues later.} \\
&= \int_0^{2\pi} \int_1^\infty e^{y/\hbar} \sum_{n=0}^\infty \frac{(x + d_1x^{-1} + d_2x^{-2}y^{-1})^n}{n! \hbar^n} \frac{i}{y} dy d\theta
\end{aligned}$$

Here we will expand the multinomial and index the k_i in manner that will make our calculations easier later.

$$\begin{aligned}
I &= \int_0^{2\pi} \int_1^\infty e^{y/\hbar} \sum_{n=0}^\infty \sum_{\substack{k_3, k_1, k_2 \\ k_3+k_1+k_2=n}} \frac{\binom{n}{k_3, k_1, k_2}}{n! \hbar^n} x^{k_3} (d_1x^{-1})^{k_1} (d_2x^{-2}y^{-1})^{k_2} \frac{i}{y} dy d\theta \\
&= \int_0^{2\pi} \int_1^\infty e^{y/\hbar} \sum_{k_1, k_3, k_2 \geq 0} \frac{1}{k_1! k_3! k_2! \hbar^{k_1+k_3+k_2}} x^{k_3} (d_1x^{-1})^{k_1} (d_2x^{-2}y^{-1})^{k_2} \frac{i}{y} dy d\theta \\
&= i \sum_{k_1, k_3, k_2 \geq 0} \frac{1}{k_1! k_3! k_2! \hbar^{k_1+k_3+k_2}} d_1^{k_1} d_2^{k_2} \int_0^{2\pi} \int_1^\infty e^{y/\hbar} x^{k_3-k_1-2k_2} y^{-k_2-1} dy d\theta \\
&= i \sum_{k_1, k_3, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1-k_3-k_2}}{k_1! k_3! k_2!} \int_0^{2\pi} x^{k_3-k_1-2k_2} d\theta \int_1^\infty e^{y/\hbar} y^{-k_2-1} dy \\
&= i \sum_{k_1, k_3, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1-k_3-k_2}}{k_1! k_3! k_2!} \int_0^{2\pi} e^{i\theta(k_3-k_1-2k_2)} d\theta \int_1^\infty e^{y/\hbar} y^{-k_2-1} dy
\end{aligned}$$

Now, the integral over θ is 0 except when $k_3 - k_1 - 2k_2 = 0$ for which we get 2π .

$$\begin{aligned} I &= 2\pi i \sum_{\substack{k_1, k_3, k_2 \geq 0 \\ k_3 - k_1 - 2k_2 = 0}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1 - k_3 - k_2}}{k_1! k_3! k_2!} \int_1^\infty e^{y/\hbar} y^{-k_2 - 1} dy \\ &= 2\pi i \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1 - 3k_2}}{k_1! (k_1 + 2k_2)! k_2!} \int_1^\infty e^{y/\hbar} y^{-k_2 - 1} dy \end{aligned}$$

Now by applying equation A.1, we obtain:

$$E_{k_2+1} \left(-\frac{1}{\hbar} \right) = \int_1^\infty \frac{e^{-(\frac{1}{\hbar})y}}{y^{k_2+1}} dy \quad \left(\operatorname{Re} \left(-\frac{1}{\hbar} \right) > 0 \right)$$

So we get

$$I = 2\pi i \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1 - 3k_2}}{k_1! (k_1 + 2k_2)! k_2!} E_{k_2+1} \left(-\frac{1}{\hbar} \right)$$

If we choose $(|\arg(-\frac{1}{\hbar})| < \pi)$, then equation A.1 will give us the following expansion of our integral:

$$2\pi i \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1 - 3k_2}}{k_1! (k_1 + 2k_2)! k_2!} \left(\frac{\hbar^{-k_2}}{k_2!} [\ln(-\hbar) + \Psi(k_2 + 1)] - \sum_{\substack{m=0 \\ m \neq k_2}}^\infty \frac{\hbar^{-m}}{(m - k_2)m!} \right)$$

A.4 The Third Leg

Here we will compute the period integral over the cylinder $S_3 = \{(x, y) \in \text{Log}^{-1}(\rho_3) : \arg(x) = 0\}$ where ρ_3 is the ray in figure A.1. This integral will behave similarly to that of the second leg.

$$\begin{aligned}
I &= \int_{S_3} e^{(x+y+d_1x^{-1}+d_2x^{-2}y^{-1})/\hbar} \frac{dx \wedge dy}{xy} \\
&= \int_{S_3} e^{x/\hbar} e^{(y+d_1x^{-1}+d_2x^{-2}y^{-1})/\hbar} \frac{dx \wedge dy}{xy} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{k_2, k_1, k_4 \\ k_2+k_1+k_4=n}} \frac{\binom{n}{k_2, k_1, k_4}}{n! \hbar^n} \int_{S_3} e^{x/\hbar} y^{k_2} d_1^{k_1} x^{-k_1} d_2^{k_4} (x^{-2}y^{-1})^{k_4} \frac{dx \wedge dy}{xy} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{k_2, k_1, k_4 \\ k_2+k_1+k_4=n}} d_2^{k_4} d_1^{k_1} \frac{\binom{n}{k_2, k_1, k_4}}{n! \hbar^n} \int_{S_3} e^{x/\hbar} y^{k_2-k_4} x^{-k_1-2k_4} \frac{dx \wedge dy}{xy} \\
&= i \sum_{n=0}^{\infty} \sum_{\substack{k_2, k_1, k_4 \\ k_2+k_1+k_4=n}} d_2^{k_4} d_1^{k_1} \frac{\binom{n}{k_2, k_1, k_4}}{n! \hbar^n} \int_0^{2\pi} e^{i\theta(k_2-k_4)} d\theta \int_1^{\infty} e^{x/\hbar} x^{-k_1-2k_4-1} dx
\end{aligned}$$

The integral over θ is 0 except when $k_2 - k_4 = 0$. In this case we get 2π .

$$I = 2\pi i \sum_{n=0}^{\infty} \sum_{\substack{k_2, k_1, k_4 \\ k_2+k_1+k_4=n \\ k_2-k_4=0}} d_2^{k_4} d_1^{k_1} \frac{\binom{n}{k_2, k_1, k_4}}{n! \hbar^n} \int_1^{\infty} e^{x/\hbar} x^{-k_1-2k_4-1} dx$$

Using the integral function in equation A.1 we get:

$$I = 2\pi i \sum_{n=0}^{\infty} \sum_{\substack{k_2, k_1, k_4 \\ k_2+k_1+k_4=n \\ k_2-k_4=0}} d_2^{k_4} d_1^{k_1} \frac{\binom{n}{k_2, k_1, k_4}}{n! \hbar^n} E_{k_1+2k_4+1} \left(-\frac{1}{\hbar} \right) \quad \left(\text{Re} \left(-\frac{1}{\hbar} \right) > 0 \right)$$

Expanding via equation A.3 and choosing $(|\arg(-\frac{1}{\hbar})| < \pi)$:

$$\begin{aligned}
I &= 2\pi i \sum_{n=0}^{\infty} \sum_{\substack{k_2, k_1, k_4 \\ k_2+k_1+k_4=n \\ k_2-k_4=0}} d_2^{k_4} d_1^{k_1} \frac{\binom{n}{k_2, k_1, k_4}}{n! \hbar^n} \\
&\quad \cdot \left(\frac{\hbar^{-k_1-2k_4}}{(k_1+2k_4)!} [\ln(-\hbar) + \Psi(k_1+2k_4+1)] - \sum_{\substack{m=0 \\ m \neq k_1+k_4}}^{\infty} \frac{\hbar^{-m}}{(m-k_1-2k_4)m!} \right)
\end{aligned}$$

We can now condense the summation signs.

$$\begin{aligned}
I &= 2\pi i \sum_{k_1, k_4 \geq 0} d_2^{k_4} d_1^{k_1} \frac{1}{k_1! (k_4!)^2 \hbar^{k_1+2k_4}} \cdot \left(\frac{\hbar^{-k_1-2k_4}}{(k_1+2k_4)!} [\ln(-\hbar) + \Psi(k_1+2k_4+1)] - \sum_{\substack{m=0 \\ m \neq k_1+2k_4}}^{\infty} \frac{\hbar^{-m}}{(m-k_1-2k_4)m!} \right) \\
&= 2\pi i \sum_{k_1, k_4 \geq 0} d_2^{k_4} d_1^{k_1} \frac{1}{k_1! (k_4!)^2 (k_1+2k_4)! \hbar^{2k_1+4k_4}} [\ln(-\hbar) + \Psi(k_1+2k_4+1)] \\
&\quad - 2\pi i \sum_{k_1, k_4 \geq 0} d_2^{k_4} d_1^{k_1} \frac{1}{k_1! (k_4!)^2 \hbar^{k_1+2k_4}} \sum_{\substack{m=0 \\ m \neq k_1+2k_4}}^{\infty} \frac{\hbar^{-m}}{(m-k_1-2k_4)m!}
\end{aligned}$$

For later convenience, let us replace the index k_4 with k_2 , but only in the first summation.

$$\begin{aligned}
I &= \boxed{2\pi i \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-4k_2}}{k_1! (k_2!)^2 (k_1+2k_2)!} [\ln(-\hbar) + \Psi(k_1+2k_2+1)]} \\
&\quad \boxed{-2\pi i \sum_{k_1, k_4 \geq 0} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-k_1-2k_4}}{k_1! (k_4!)^2} \sum_{\substack{m=0 \\ m \neq k_1+2k_4}}^{\infty} \frac{\hbar^{-m}}{(m-k_1-2k_4)m!}}
\end{aligned}$$

A.5 The Fourth Leg

In this section we are integrating over $S_4 = \{(x, y) \in \text{Log}^{-1}(\rho_4) : \arg(x^{-2}y^{-1}) = 0\}$.

$$\begin{aligned}
 I &= \int_{S_4} e^{(x+y+d_1x^{-1}+d_2x^{-2}y^{-1})/\hbar} \Omega \\
 &= \int_{S_4} e^{(d_2x^{-2}y^{-1})/\hbar} e^{(x+y+d_1x^{-1})/\hbar} \frac{dx \wedge dy}{xy} \\
 &= \int_{S_4} e^{(d_2x^{-2}y^{-1})/\hbar} \sum_{n=0}^{\infty} \sum_{\substack{k_1, k_2, k_3 \\ k_1+k_2+k_3=n}} \frac{\binom{n}{k_1, k_2, k_3}}{n! \hbar^n} d_1^{k_3} x^{k_1-k_3} y^{k_2} \frac{dx \wedge dy}{xy}
 \end{aligned}$$

We got this exactly as before with Taylor and multinomial expansions.

$$\begin{aligned}
 &= \int_{S_4} e^{(d_2x^{-2}y^{-1})/\hbar} \sum_{k_1, k_2, k_3 \geq 0} \frac{d_1^{k_3} \hbar^{-k_1-k_2-k_3}}{k_1! k_2! k_3!} x^{k_1-k_3} y^{k_2} \frac{dx \wedge dy}{xy} \\
 &= \sum_{k_1, k_2, k_3 \geq 0} \frac{d_1^{k_3} \hbar^{-k_1-k_2-k_3}}{k_1! k_2! k_3!} \int_{S_4} e^{(d_2x^{-2}y^{-1})/\hbar} x^{k_1-k_3} y^{k_2} \frac{dx \wedge dy}{xy}
 \end{aligned}$$

Let us examine S_4 a little more closely. It, too, should be topologically equivalent to a cylinder (or a “leg”) since we get a loop for each point $(a, b) \in \rho_4$. Its actual form is a little more complicated, though. First let us look at the preimage $\text{Log}^{-1}(a, b)$ of a point (a, b) in ρ_4 (remember that $\text{Log}(x, y) = (\log|x|, \log|y|)$). The preimage is a set that is the product of two circles in $\mathbb{C}^2$ centered at the origin with radii e^a and e^b (and the more negative a and b are, the smaller the radii become). That is, $\text{Log}^{-1}(a, b)$ is a torus. Now, restricting on this torus to the points (x, y) where $\arg(x^{-2}y^{-1}) = 0$ actually gives us a curve that loops around the torus twice. What we mean by this is that the argument of y increases at twice the rate of the argument of x on this torus.

Now, keeping this visual in mind, we need to make a substitution of variables. What we would like to do is integrate over the argument and then the magnitude of y (i.e. over $b = \arg(|y|)$). That is, if $x = r_1 e^{\theta_1}$ and $y = r_2 e^{\theta_2}$, we want to get x and y in terms of r_2 and θ_2 . Conveniently, we get some relations that aid us in this. In particular, since $\arg(x^{-2}y^{-1}) = 0$ in S_4 , then $\boxed{\theta_1 = -\theta_2/2}$. Indeed, $0 = \arg(x^{-2}y^{-1}) = \arg((r_1 e^{\theta_1})^{-2} (r_2 e^{\theta_2})^{-1}) = \arg(r_1^{-2} r_2^{-1} e^{-2\theta_1 - \theta_2}) \Rightarrow -2\theta_1 - \theta_2 =$

$0 \Rightarrow \theta_1 = -\theta_2/2$. And since we are wrapping around the torus twice in the y variable, we need θ_2 to run from 0 to 4π .

Also, since each (x, y) in S_4 is in the preimage $\text{Log}^{-1}(a, b)$ of some point $(a, b) \in \rho_4$, and since every such point (a, b) has the relation $a = 2b$, we get that $\boxed{r_1 = r_2^2}$. Indeed, $a = 2b \Rightarrow \log|r_1 e^{i\theta_1}| = 2 \log|r_2 e^{i\theta_2}| \Rightarrow r_1 = r_2^2$. Notice that r_2 will run from 1 to 0 as b heads off towards negative infinity. That is, $b = 0 \Rightarrow \log|r_2 e^{i\theta_2}| = 0 \Rightarrow r_2 = 1$, and $b \rightarrow -\infty \Rightarrow \log|r_2 e^{i\theta_2}| \rightarrow -\infty \Rightarrow r_2 \rightarrow 0$.

Finally, $\arg(x^{-2}y^{-1}) = 0$ tells us that $x^{-2}y^{-1}$ is a positive real number, and hence $x^{-2}y^{-1} = r_1^{-2}r_2^{-1} = (r_2^2)^{-2}r_2^{-1} = r_2^{-5}$. That is, $\boxed{x^{-2}y^{-1} = r_2^{-5}}$.

That is, $\boxed{y = r_2 e^{i\theta_2}}$ and $\boxed{x = r_2^2 e^{i(-\theta_2/2)} = r_2^2 e^{-i\theta_2/2}}$. Since there will be no confusion, let us lose the index on r_2 and θ_2 so that we are only working with the two variables r and θ .

Now consider the integral

$$J = \int_{S_4} e^{(d_2 x^{-2} y^{-1}/\hbar)} x^a y^b \frac{dx \wedge dy}{xy}$$

Let us separate out the form for special attention for clarity in calculation.

$$\begin{aligned} dx \wedge dy &= \\ &= d(r^2 e^{-i\theta/2}) \wedge d(r e^{i\theta}) \\ &= (2r e^{-i\theta/2} dr + (-i/2) r^2 e^{-i\theta/2} d\theta) \wedge (e^{i\theta} dr + i r e^{i\theta} d\theta) \\ &= 2i r^2 e^{i\theta/2} dr \wedge d\theta + (-i/2) r^2 e^{i\theta/2} d\theta \wedge dr \\ &= 2i r^2 e^{i\theta/2} dr \wedge d\theta + (i/2) r^2 e^{i\theta/2} dr \wedge d\theta \\ &= (5/2) i r^2 e^{i\theta/2} dr \wedge d\theta \end{aligned}$$

Dividing this by xy gives:

$$\begin{aligned} \frac{dx \wedge dy}{xy} &= \frac{(5/2) i r^2 e^{i\theta/2} dr \wedge d\theta}{r^2 e^{-i\theta/2} r e^{i\theta}} \\ &= -(5/2) i r^{-1} d\theta \wedge dr \end{aligned}$$

Plugging all this into J gives

$$\begin{aligned}
J &= \int_{S_4} e^{(d_2 x^{-2} y^{-1}/\hbar)} x^a y^b \frac{dx \wedge dy}{xy} \\
&= \int_1^0 \int_0^{4\pi} e^{d_2 r^{-5}/\hbar} (r^2 e^{-i\theta/2})^a (r e^{i\theta})^b (-5/2) i r^{-1} d\theta \wedge dr \\
&= - \int_1^0 \int_0^{4\pi} e^{(d_2 r^{-5}/\hbar)} r^{(2a+b)} e^{i\theta(-a/2+b)} (5/2) i r^{-1} d\theta \wedge dr
\end{aligned}$$

Then

$$\begin{aligned}
J &= -(5/2) i \int_1^0 \int_0^{4\pi} e^{(d_2 r^{-5}/\hbar)} r^{(2a+b-1)} e^{i\theta(-a/2+b)} d\theta \wedge dr \\
&= -(5/2) i \int_1^0 e^{(d_2 r^{-5}/\hbar)} r^{(2a+b-1)} dr \int_0^{4\pi} e^{i\theta(-a/2+b)} d\theta
\end{aligned}$$

In what follows, we will use that $\int_0^{4\pi} e^{ai\theta} d\theta = 2 \int_0^{2\pi} e^{ai2\theta} d\theta$. Now, plugging in for $a = k_1 - k_3$ and $b = k_2$ gives

$$\begin{aligned}
J &= -(5/2) i \int_1^0 e^{(d_2 r^{-5}/\hbar)} r^{(2(k_1-k_3)+(k_2)-1)} dr \int_0^{4\pi} e^{i\theta(-(k_1-k_3)/2+(k_2))} d\theta \\
&= -(5/2) i \int_1^0 e^{(d_2 r^{-5}/\hbar)} r^{2k_1+k_2-2k_3-1} dr \int_0^{4\pi} e^{i\theta(-\frac{1}{2}k_1+k_2+\frac{1}{2}k_3)} d\theta \\
&= -5i \int_1^0 e^{(d_2 r^{-5}/\hbar)} r^{2k_1+k_2-2k_3-1} dr \int_0^{2\pi} e^{i\theta(-k_1+2k_2+k_3)} d\theta
\end{aligned}$$

Again, the integral $\int_0^{2\pi} e^{ai\theta} d\theta = 2\pi$ when $a = 0$, and the integral is zero otherwise. The integral over r will come from the exponential function given by equation A.1. Let us plug our results, so far, into the original integral.

$$\begin{aligned}
I &= - \sum_{k_1, k_2, k_3 \geq 0} \frac{d_1^{k_3} \hbar^{-k_1-k_2-k_3}}{k_1! k_2! k_3!} (5i) \int_1^0 e^{(d_2 r^{-5}/\hbar)} r^{2k_1+k_2-2k_3-1} dr \int_0^{2\pi} e^{i\theta(-k_1+2k_2+k_3)} d\theta \\
&= - \sum_{\substack{k_1, k_2, k_3 \geq 0 \\ -k_1+2k_2+k_3=0}} \frac{d_1^{k_3} \hbar^{-k_1-k_2-k_3}}{k_1! k_2! k_3!} (5i) \int_1^0 e^{(d_2 r^{-5}/\hbar)} r^{2k_1+k_2-2k_3-1} dr (2\pi)
\end{aligned}$$

And here we use the new relation under the summation sign to obtain a very convenient substitution. In particular, we are going to set $k_1 = 2k_2 + k_3$.

$$I = -10\pi i \sum_{k_2, k_3 \geq 0} \frac{d_1^{k_3} \hbar^{-3k_2-2k_3}}{(2k_2 + k_3)! k_2! k_3!} \int_1^0 e^{(d_2 r^{-5}/\hbar)} r^{5k_2-1} dr$$

The very convenient substitution will be $t = r^{-5}$.

$$\begin{aligned} I &= -10\pi i \sum_{k_2, k_3 \geq 0} \frac{d_1^{k_3} \hbar^{-3k_2-2k_3}}{(2k_2 + k_3)! k_2! k_3!} \int_1^\infty e^{(d_2 t/\hbar)} t^{(-1/5)(5k_2-1)} d(t^{(-1/5)}) \\ &= -10\pi i \sum_{k_2, k_3 \geq 0} \frac{d_1^{k_3} \hbar^{-3k_2-2k_3}}{(2k_2 + k_3)! k_2! k_3!} \int_1^\infty e^{(d_2 t/\hbar)} t^{(-1/5)(5k_2-1)} \left(-\frac{1}{5} t^{-6/5} dt\right) \\ &= 2\pi i \sum_{k_2, k_3 \geq 0} \frac{d_1^{k_3} \hbar^{-3k_2-2k_3}}{(2k_2 + k_3)! k_2! k_3!} \int_1^\infty e^{(d_2 t/\hbar)} t^{-k_2+1/5-6/5} dt \\ &= 2\pi i \sum_{k_2, k_3 \geq 0} \frac{d_1^{k_3} \hbar^{-3k_2-2k_3}}{(2k_2 + k_3)! k_2! k_3!} \int_1^\infty e^{(d_2 t/\hbar)} t^{-k_2-1} dt \\ &= 2\pi i \sum_{k_2, k_3 \geq 0} \frac{d_1^{k_3} \hbar^{-3k_2-2k_3}}{(2k_2 + k_3)! k_2! k_3!} E_{k_2+1} \left(\frac{-d_2}{\hbar} \right) \end{aligned}$$

Let us expand this out. According to equation A.3, for $|\arg \frac{-d_2}{\hbar}| < \pi$:

$$E_{k_2+1} \left(\frac{-d_2}{\hbar} \right) = \frac{(\frac{d_2}{\hbar})^{k_2}}{k_2!} [\ln \hbar - \ln(-d_2) + \Psi(k_2 + 1)] - \sum_{\substack{m=0 \\ m \neq k_2}}^\infty \frac{(\frac{d_2}{\hbar})^m}{(m - k_2)m!}$$

Plugging this back into our integral we get:

$$\begin{aligned} I &= 2\pi i \sum_{k_2, k_3 \geq 0} \frac{d_1^{k_3} \hbar^{-3k_2-2k_3}}{(2k_2 + k_3)! k_2! k_3!} \\ &\quad \cdot \left(\frac{(\frac{d_2}{\hbar})^{k_2}}{k_2!} [\ln \hbar - \ln(-d_2) + \Psi(k_2 + 1)] - \sum_{\substack{m=0 \\ m \neq k_2}}^\infty \frac{(\frac{d_2}{\hbar})^m}{(m - k_2)m!} \right) \end{aligned}$$

For convenience later, let us replace the index k_3 with k_1 .

$$I = 2\pi i \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} \hbar^{-2k_1-3k_2}}{(k_1 + 2k_2)! k_2! k_1!}.$$

$$\cdot \left(\frac{(\frac{d_2}{\hbar})^{k_2}}{k_2!} \left[\ln \left(\frac{-\hbar}{d_2} \right) + \Psi(k_2 + 1) \right] - \sum_{\substack{m=0 \\ m \neq k_2}}^\infty \frac{(\frac{d_2}{\hbar})^m}{(m - k_2)m!} \right)$$

A.6 The Buffer

There are three cylinders that are contained in the cycle Ξ_3 . They are

$$S_2 = \{(x, y) \in \text{Log}^{-1}(\rho_2) : \arg(y) = 0\}$$

$$S_3 = \{(x, y) \in \text{Log}^{-1}(\rho_3) : \arg(x) = 0\}$$

$$S_4 = \{(x, y) \in \text{Log}^{-1}(\rho_4) : \arg(x^{-2}y^{-1}) = 0\}$$

But that is not all of Ξ_3 . We need to connect these three cylinders somehow, and we do so by considering the boundaries of the cylinders, which all lie on a torus. The boundaries are

$$\partial S_2 = \{(x, y) \in \text{Log}^{-1}(0) : \arg(y) = 0\}$$

$$\partial S_3 = \{(x, y) \in \text{Log}^{-1}(0) : \arg(x) = 0\}$$

$$\partial S_4 = \{(x, y) \in \text{Log}^{-1}(0) : \arg(x^{-2}y^{-1}) = 0\}$$

Each of these boundaries is a circle sitting on the unit torus. In particular, when we flatten this unit torus out, we obtain the $[-\pi, \pi]^2$ region in figure A.2 (each point in this region is given by the coordinate $(\arg(x), \arg(y))$ for some $(x, y) \in \text{Log}^{-1}(0)$).

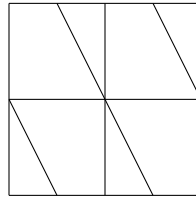


Figure A.2: The unit torus with cylinder boundaries for Ξ_3 .

The following is not necessarily obvious, but these three circles are situated in a way that allows us to wrap a (filled-in) triangle around the unit torus (twice) so that the edges of the triangle correspond to the circles. We will call this wrapped-up triangle S_5 . The union of the three legs along with this wrapped-up triangle forms the entire cycle Ξ_3 , a “manifold with corners”. That is, S_5 fills in the gap (on the torus) between the different legs.

Unwrapping S_5 gives the region in figure A.3, which is twice as tall (i.e. has a height of 4π) as the flat torus in figure A.2. We define this triangle as $\Delta := \{(\theta_1, \theta_2) \in [0, 2\pi] \times [0, 4\pi] : 0 \leq \theta_2 \leq -2\theta_1\}$, but it is just S_5 after flattening out.

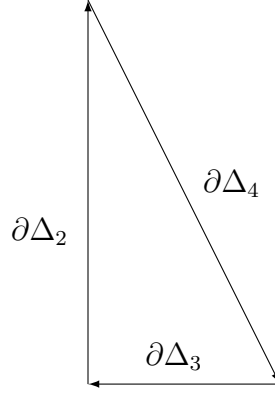


Figure A.3: The triangle Δ which wraps twice around the unit torus to give S_5 , the buffer for Ξ_3 . The orientation of Δ is given by the arrow.

The edges $\partial\Delta_i$ of the triangle correspond to the boundaries ∂S_i after flattening.

Now let us compute the integral over S_5 :

$$\begin{aligned} I &= \int_{S_5} e^{(x+y+d_1x^{-1}+d_2x^{-2}y^{-1})/\hbar} \Omega \\ &= \int_{S_5} \sum_{n=0}^{\infty} \frac{(x+y+d_1x^{-1}+d_2x^{-2}y^{-1})^n}{n! \hbar^n} \Omega \\ &= \sum_{n=0}^{\infty} \int_{S_5} \frac{(x+y+d_1x^{-1}+d_2x^{-2}y^{-1})^n}{n! \hbar^n} \frac{dx \wedge dy}{xy} \end{aligned}$$

And from earlier calculations we get:

$$\begin{aligned} &= \sum_{n=0}^{\infty} \int_{S_5} \sum_{\substack{k_3, k_2, k_1, k_4 \\ k_3+k_2+k_1+k_4=n}} \frac{\binom{n}{k_3, k_2, k_1, k_4}}{n! \hbar^n} d_1^{k_1} d_2^{k_2} x^{k_3-k_1-2k_4} y^{k_2-k_4} \frac{dx \wedge dy}{xy} \\ &= \sum_{n=0}^{\infty} \sum_{\substack{k_3, k_2, k_1, k_4 \\ k_3+k_2+k_1+k_4=n}} \frac{\binom{n}{k_3, k_2, k_1, k_4}}{n! \hbar^n} d_1^{k_1} d_2^{k_2} \int_{\Delta} e^{i\theta_1(k_3-k_1-2k_4)} e^{i\theta_2(k_2-k_4)} (-1) d\theta_1 \wedge d\theta_2 \\ &= \sum_{k_3, k_2, k_1, k_4 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1-k_2-k_3-k_4}}{k_1! k_2! k_3! k_4!} \int_{\Delta} e^{i\theta_1(k_3-k_1-2k_4)} e^{i\theta_2(k_2-k_4)} (-1) d\theta_1 \wedge d\theta_2 \end{aligned}$$

To compute this we will need to integrate

$$\int_{\Delta} e^{ia\theta_1} e^{ib\theta_2} d\theta_1 \wedge d\theta_2.$$

This will require the use of Stoke's Theorem and several cases.

When $a = b = 0$ the integral is just the area of the triangle: $I = \frac{2\pi \cdot 4\pi}{2} = 4\pi^2$.

When $b \neq 0$:

$$\begin{aligned} I &= \int_{\Delta} e^{ia\theta_1} e^{ib\theta_2} d\theta_1 \wedge d\theta_2 \\ &= - \int_{\Delta} d \left(\frac{1}{ib} e^{ia\theta_1} e^{ib\theta_2} d\theta_1 \right) \\ &= \int_{\partial\Delta} \frac{1}{ib} e^{ia\theta_1} e^{ib\theta_2} d\theta_1 \end{aligned}$$

This last integral is taken in the clockwise direction, which is why the negative sign disappeared.

Now integrating along the vertical line $\partial\Delta_2$ we get:

$$\begin{aligned} I_1 &= \int_{\partial\Delta_2} \frac{1}{ib} e^{ia\theta_1} e^{ib\theta_2} d\theta_1 \\ &= 0 \end{aligned}$$

because integrating the form $\partial\theta_1$ on a segment where θ_1 is constant will always give zero.

Integrating along the diagonal line $\partial\Delta_4$ we get

$$\begin{aligned} I_2 &= \int_{\partial\Delta_4} \frac{1}{ib} e^{ia\theta_1} e^{ib\theta_2} d\theta_1 \\ &= \frac{1}{ib} \int_0^{2\pi} e^{ia\theta_1} e^{ib(-2\theta_1)} d\theta_1 \end{aligned}$$

Here we substituted $\theta_2 = -2\theta_1$.

$$\begin{aligned} &= \frac{1}{ib} \int_0^{2\pi} e^{i\theta_1(a-2b)} d\theta_1 \\ &= -\frac{1}{b} \left(i \int_0^{2\pi} e^{i\theta_1(a-2b)} d\theta_1 \right) \\ &= -\frac{1}{b} \oint z^{a-2b-1} dz \\ &= \begin{cases} -2\pi i \frac{1}{b} & \text{if } a - 2b = 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$

Here $\oint$ means integrating around the unit circle in the complex plane.

Integrating along the horizontal line $\partial\Delta_3$ (remember clockwise) gives:

$$\begin{aligned}
 I_3 &= \int_{\partial\Delta_3} \frac{1}{ib} e^{ia\theta_1} e^{ib\theta_2} d\theta_1 \\
 &= \int_{2\pi}^0 \frac{1}{ib} e^{ia\theta_1} d\theta_1 \\
 &= \frac{1}{ib} \int_{2\pi}^0 e^{ia\theta_1} d\theta_1 \\
 &= \begin{cases} 2\pi i \frac{1}{b} & \text{if } a = 0 \\ 0 & \text{else} \end{cases}
 \end{aligned}$$

If $b = 0$ and $a \neq 0$ then

$$\begin{aligned}
 I &= \int_{\Delta} e^{ia\theta_1} d\theta_1 \wedge d\theta_2 \\
 &= \int_{\Delta} d \left(\frac{1}{ia} e^{ia\theta_1} d\theta_2 \right) \\
 &= - \int_{\partial\Delta} \frac{1}{ia} e^{ia\theta_1} d\theta_2
 \end{aligned}$$

Notice in the last line that we are integrating in the clockwise direction, which gives a negative sign.

Integrating along the vertical line $\partial\Delta_2$ will give:

$$\begin{aligned}
 I_1 &= - \int_{\partial\Delta_2} \frac{1}{ia} e^{ia\theta_1} d\theta_2 \\
 &= - \frac{1}{ia} \int_0^{4\pi} d\theta_2
 \end{aligned}$$

Notice we substituted in 0 for θ_1 .

$$\begin{aligned}
 &= - \frac{1}{ia} 4\pi \\
 &= 4\pi i \frac{1}{a}
 \end{aligned}$$

Integrating along the diagonal line $\partial\Delta_4$

$$\begin{aligned}
 I_2 &= - \int_{\partial\Delta_4} \frac{1}{ia} e^{ia\theta_1} d\theta_2 \\
 &= \int_0^{4\pi} \frac{1}{ia} e^{ia(-\frac{1}{2}\theta_2)} d\theta_2 = 0 \quad \text{since } a \neq 0.
 \end{aligned}$$

Notice that by integrating in the positive x direction, we had to change signs again. Also, here we substituted $\theta_1 = -\frac{1}{2}\theta_2$.

Integrating on the horizontal path over the form $\partial\theta_2$ will of course give 0:

$$\begin{aligned} I_3 &= - \int_{\partial\Delta_3} \frac{1}{ia} e^{ia\theta_1} d\theta_2 \\ &= 0 \end{aligned}$$

To summarize, we have four cases that give us nonzero integrals:

- Case 1: If $b \neq 0$ and $a = 2b$ (hence $a \neq 0$) then $I = -2\pi i \frac{1}{b}$
- Case 2: If $b = 0$ and $a \neq 0$ then $I = 4\pi i \frac{1}{a}$.
- Case 3: If $b \neq 0$ and $a = 0$ then $I = 2\pi i \frac{1}{b}$.
- Case 4: If $a = b = 0$ then $I = 4\pi^2$.

So we will have to split our sum from our original integral I into four parts (possibly being careful not to count terms twice).

Recall that we had shown that

$$I = \sum_{k_3, k_2, k_1, k_4 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1-k_2-k_3-k_4}}{k_1! k_2! k_3! k_4!} \int_{\Delta} e^{i\theta_1(k_3-k_1-2k_4)} e^{i\theta_2(k_2-k_4)} (-1) d\theta_1 d\theta_2$$

We will now define J , K , L , and M so that

$$I = (-1)(J + K + L + M)$$

$$\begin{aligned}
J &= \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_2 \neq k_4 \\ k_3 - k_1 - 2k_4 = 2(k_2 - k_4)}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1 - k_2 - k_3 - k_4}}{k_1! k_2! k_3! k_4!} \int_{\Delta} e^{i\theta_1(k_3 - k_1 - 2k_4)} e^{i\theta_2(k_2 - k_4)} d\theta_1 d\theta_2 \\
&= \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_2 \neq k_4 \\ k_3 - k_1 - 2k_4 = 2(k_2 - k_4)}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1 - k_2 - k_3 - k_4}}{k_1! k_2! k_3! k_4!} \frac{1}{(k_2 - k_4)} (-2\pi i) \\
&= \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_2 \neq k_4 \\ k_3 = k_1 + 2k_2}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1 - k_2 - k_3 - k_4}}{k_1! k_2! k_3! k_4!} \frac{1}{(k_2 - k_4)} (-2\pi i) \\
&= -2\pi i \sum_{\substack{k_1, k_2, k_4 \geq 0 \\ k_2 \neq k_4}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1 - 3k_2 - k_4}}{k_1! k_2! (k_1 + 2k_2)! k_4!} \frac{1}{(k_2 - k_4)}
\end{aligned}$$

$$\begin{aligned}
K &= \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_2 = k_4 \\ k_3 - k_1 - 2k_4 \neq 0}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1 - k_2 - k_3 - k_4}}{k_1! k_2! k_3! k_4!} \int_{\Delta} e^{i\theta_1(k_3 - k_1 - 2k_4)} e^{i\theta_2(k_2 - k_4)} d\theta_1 d\theta_2 \\
&= \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_2 = k_4 \\ k_3 - k_1 - 2k_4 \neq 0}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1 - k_2 - k_3 - k_4}}{k_1! k_2! k_3! k_4!} \left(\frac{1}{(k_3 - k_1 - 2k_4)} 4\pi i \right) \\
&= 4\pi i \sum_{\substack{k_1, k_3, k_4 \geq 0 \\ k_3 - k_1 - 2k_4 \neq 0}} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-k_1 - k_3 - 2k_4}}{k_1! k_3! (k_4!)^2} \left(\frac{1}{(k_3 - k_1 - 2k_4)} \right)
\end{aligned}$$

$$\begin{aligned}
L &= \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_2 \neq k_4 \\ k_3 - k_1 - 2k_4 = 0}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1 - k_2 - k_3 - k_4}}{k_1! k_2! k_3! k_4!} \int_{\Delta} e^{i\theta_1(k_3 - k_1 - 2k_4)} e^{i\theta_2(k_2 - k_4)} d\theta_1 d\theta_2 \\
&= \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_2 \neq k_4 \\ k_3 - k_1 - 2k_4 = 0}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1 - k_2 - k_3 - k_4}}{k_1! k_2! k_3! k_4!} \left(\frac{1}{(k_2 - k_4)} 2\pi i \right) \\
&= 2\pi i \sum_{\substack{k_1, k_2, k_4 \geq 0 \\ k_2 \neq k_4}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1 - k_2 - 3k_4}}{k_1! k_2! (k_1 + 2k_4)! k_4!} \left(\frac{1}{(k_2 - k_4)} \right)
\end{aligned}$$

$$\begin{aligned}
M &= \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_2 = k_4 \\ k_3 - k_1 - 2k_4 = 0}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1 - k_2 - k_3 - k_4}}{k_1! k_2! k_3! k_4!} \int_{\Delta} e^{i\theta_1(k_3 - k_1 - 2k_4)} e^{i\theta_2(k_2 - k_4)} d\theta_1 d\theta_2 \\
&= \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_2 = k_4 \\ k_3 - k_1 - 2k_4 = 0}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-k_1 - k_2 - k_3 - k_4}}{k_1! k_2! k_3! k_4!} (4\pi^2) \\
&= 4\pi^2 \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1 - 4k_2}}{k_1! (k_2!)^2 (k_1 + 2k_2)!}
\end{aligned}$$

Notice that M is just the negative of the integral over the unit torus.

A.7 Conclusion

So now we are able to talk about three particular cycles. The first is the unit torus, Ξ_1 . The second, Ξ_2 is the first and third legs glued together at their boundaries; this forms a cylinder. The third cycle Ξ_3 is the second, third, and fourth legs together with the buffer, where the third leg is taken twice. With some imagination, one could envision Ξ_3 as a pair of three-legged pants. Again, Xi_2 corresponds to $\rho_1 \cup \rho_3$ and Xi_3 corresponds to the union of ρ_2 , ρ_4 , and two copies of ρ_3 , where the ρ_i come from the fan in figure A.1.

As a reminder, the integral over the unit torus was:

$$\int_{T^2} e^{W/\hbar} \Omega = -4\pi^2 \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!}$$

Since the following expression will appear many times, let us shorten it:

$$c_{k_1, k_2} := \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!}$$

A.7.1 The Integral over Ξ_2

The integral over the cylinder Ξ_2 is the sum of the two integrals we calculated for the first and third legs.

$$\begin{aligned}
& \int_{\Xi_2} e^{W/\hbar} \Omega = \\
& = 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \geq 1}} \frac{d_1^{k_3 - 2k_4} d_2^{k_4} \hbar^{-2k_3}}{k_3! (k_4!)^2 (k_3 - 2k_4)!} \left[\ln \left(-\frac{\hbar}{d_1} \right) + \Psi(k_3 - 2k_4 + 1) \right] \Bigg\} a \\
& + 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} \frac{d_1^{k_3 - 2k_4} d_2^{k_4} (-k_3 + 2k_4 - 1)!}{k_3! (k_4!)^2 \hbar^{2k_3}} (-1)^{-k_3 + 2k_4} \Bigg\} b \\
& - 2\pi i \sum_{k_3, k_4 \geq 0} \frac{d_2^{k_4} \hbar^{-k_3 - 2k_4}}{k_3! (k_4!)^2} \sum_{\substack{m=0 \\ m \neq k_3 - 2k_4}}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!} \Bigg\} c \\
& + 2\pi i \sum_{k_1, k_2 \geq 0} d_2^{k_2} d_1^{k_1} \frac{1}{k_1! (k_2!)^2 (k_1 + 2k_2)! \hbar^{2k_1 + 4k_2}} [\ln(-\hbar) + \Psi(k_1 + 2k_2 + 1)] \Bigg\} d \\
& - 2\pi i \sum_{k_1, k_4 \geq 0} d_2^{k_4} d_1^{k_1} \frac{1}{k_1! (k_4!)^2 \hbar^{k_1 + 2k_4}} \sum_{\substack{m=0 \\ m \neq k_1 + 2k_4}}^{\infty} \frac{\hbar^{-m}}{(m - k_1 - 2k_4)m!} \Bigg\} e
\end{aligned}$$

But now, we can simplify. Let us first use the shorthand we developed just above. Part *a* can also, after a change of variables, be shortened: send k_4 to k_2 and

send k_3 to $k_1 + 2k_2$.

$$\begin{aligned}
\int_{\Xi_2} e^{W/\hbar} \Omega &= 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\left[\ln \left(-\frac{\hbar}{d_1} \right) + \Psi(k_1 + 1) \right] \right) \Bigg\} a \\
&\quad + 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} \frac{d_1^{k_3 - 2k_4} d_2^{k_4} (-k_3 + 2k_4 - 1)!}{k_3! (k_4!)^2 \hbar^{2k_3}} (-1)^{-k_3 + 2k_4} \Bigg\} b \\
&\quad - 2\pi i \sum_{k_3, k_4 \geq 0} \frac{d_2^{k_4} \hbar^{-k_3 - 2k_4}}{k_3! (k_4!)^2} \sum_{\substack{m=0 \\ m \neq k_3 - 2k_4}}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!} \Bigg\} c \\
&\quad + 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} (\ln(-\hbar) + \Psi(k_1 + 2k_2 + 1)) \Bigg\} d \\
&\quad - 2\pi i \sum_{k_1, k_4 \geq 0} d_2^{k_4} d_1^{k_1} \frac{1}{k_1! (k_4!)^2 \hbar^{k_1 + 2k_4}} \sum_{\substack{m=0 \\ m \neq k_1 + 2k_4}}^{\infty} \frac{\hbar^{-m}}{(m - k_1 - 2k_4)m!} \Bigg\} e \\
&= 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\ln \left(\frac{\hbar^2}{d_1} \right) + \Psi(k_1 + 1) + \Psi(k_1 + 2k_2 + 1) \right) \Bigg\} a + d \\
&\quad + 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} \frac{d_1^{k_3 - 2k_4} d_2^{k_4} (-k_3 + 2k_4 - 1)!}{k_3! (k_4!)^2 \hbar^{2k_3}} (-1)^{-k_3 + 2k_4} \Bigg\} b \\
&\quad - 2\pi i \sum_{k_3, k_4 \geq 0} \frac{d_2^{k_4} \hbar^{-k_3 - 2k_4}}{k_3! (k_4!)^2} \sum_{\substack{m=0 \\ m \neq k_3 - 2k_4}}^{\infty} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!} \Bigg\} c \\
&\quad - 2\pi i \sum_{k_1, k_4 \geq 0} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-k_1 - 2k_4}}{k_1! (k_4!)^2} \sum_{\substack{m=0 \\ m \neq k_1 + 2k_4}}^{\infty} \frac{\hbar^{-m}}{(m - k_1 - 2k_4)m!} \Bigg\} e
\end{aligned}$$

But now we claim that c and e cancel each other out. Indeed:

$$\begin{aligned}
\frac{c}{-2\pi i} &= \sum_{\substack{k_3, k_4, m \geq 0 \\ m \neq k_3 - 2k_4}} \frac{d_2^{k_4} \hbar^{-k_3 - 2k_4}}{k_3! (k_4!)^2} \frac{d_1^m \hbar^{-m}}{(m - k_3 + 2k_4)m!} \\
&= \sum_{\substack{k_1, k_3, k_4 \geq 0 \\ k_1 \neq k_3 - 2k_4}} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-k_1 - k_3 - 2k_4}}{k_1! k_3! (k_4!)^2 (k_1 - k_3 + 2k_4)} \quad \text{Replace } m \text{ with } k_1.
\end{aligned}$$

and

$$\begin{aligned}
\frac{e}{-2\pi i} &= \sum_{\substack{k_1, k_4, m \geq 0 \\ m \neq k_1 + 2k_4}} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-k_1 - 2k_4}}{k_1! (k_4!)^2} \frac{\hbar^{-m}}{(m - k_1 - 2k_4)m!} \\
&= \sum'_{\substack{k_1, k_3, k_4 \geq 0 \\ k_3 \neq k_1 + 2k_4}} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-k_1 - k_3 - 2k_4}}{k_1! k_3! (k_4!)^2 (-k_1 + k_3 - 2k_4)} \quad \text{Replace } m \text{ with } k_3.
\end{aligned}$$

One easily sees from this that $e = -c$.

So we have the final form of the integral over the cylinder:

$$\begin{aligned}
\int_{\Xi_2} e^{W/\hbar} \Omega &= 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\ln \left(\frac{\hbar^2}{d_1} \right) + \Psi(k_1 + 1) + \Psi(k_1 + 2k_2 + 1) \right) \Bigg\} a + d \\
&\quad + 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} \frac{d_1^{k_3 - 2k_4} d_2^{k_4} (-k_3 + 2k_4 - 1)!}{k_3! (k_4!)^2 \hbar^{2k_3}} (-1)^{-k_3 + 2k_4} \Bigg\} b
\end{aligned}$$

A.7.2 The Integral over Ξ_3

Moving our focus to the more complicated cycle, the integral over Ξ_3 is the sum of the integral that we calculated for the second leg (labelled f below), plus two times that for the third leg (g and h , which are related to d and e from the previous section), plus the fourth leg (i), minus the integral over the buffer $-(J + K + L + M)$.

Here we need to have a quick discursion on why we had to just *subtract* the integral over the buffer. (That is, we must *add* $J + K + L + M$.) We had chosen an orientation on the buffer S_5 , which was given by the arrows in figure A.3. It turns out we had chosen it at random, and with a fifty-fifty chance of getting it right, we got it wrong. In retrospect, a simple analysis of the orientations of the legs in Ξ_3 , determined by the form $\partial x \wedge \partial y$, will give the orientation of the boundaries of the legs. These boundaries form the triangle in figure A.3 but with the opposite orientation.

Hence the appropriate period integral over the entire cycle Ξ_3 is as follows.

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{\Xi_3} e^{W/\hbar} \Omega = \\
& = \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-3k_2}}{k_1! (k_1+2k_2)! k_2!} \left(\frac{\hbar^{-k_2}}{k_2!} [\ln(-\hbar) + \Psi(k_2+1)] - \sum'_{\substack{m=0 \\ m \neq k_2}} \frac{\hbar^{-m}}{(m-k_2)m!} \right) \Bigg\} f \\
& + 2 \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-4k_2}}{k_1! (k_2!)^2 (k_1+2k_2)!} [\ln(-\hbar) + \Psi(k_1+2k_2+1)] \Bigg\} g = 2d \\
& - 2 \sum_{k_1, k_4 \geq 0} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-k_1-2k_4}}{k_1! (k_4!)^2} \sum'_{\substack{m=0 \\ m \neq k_1+2k_4}} \frac{\hbar^{-m}}{(m-k_1-2k_4)m!} \Bigg\} h = 2e \\
& + \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} \hbar^{-2k_1-3k_2}}{(k_1+2k_2)! k_2! k_1!} \left(\frac{(\frac{d_2}{\hbar})^{k_2}}{k_2!} \left[\ln \frac{-\hbar}{d_2} + \Psi(k_2+1) \right] - \sum'_{\substack{m=0 \\ m \neq k_2}} \frac{(\frac{d_2}{\hbar})^m}{(m-k_2)m!} \right) \Bigg\} i \\
& + \left(\frac{1}{2\pi i} \right) (J + K + L + M)
\end{aligned}$$

where

$$\begin{aligned}
J &= -2\pi i \sum_{\substack{k_1, k_2, k_4 \geq 0 \\ k_2 \neq k_4}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-3k_2-k_4}}{k_1! k_2! (k_1+2k_2)! k_4!} \frac{1}{(k_2-k_4)} \\
K &= 4\pi i \sum_{\substack{k_1, k_3, k_4 \geq 0 \\ k_3-k_1-2k_4 \neq 0}} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-k_1-k_3-2k_4}}{k_1! k_3! (k_4!)^2} \left(\frac{1}{(k_3-k_1-2k_4)} \right) \\
L &= 2\pi i \sum_{\substack{k_1, k_2, k_4 \geq 0 \\ k_2 \neq k_4}} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-k_2-3k_4}}{k_1! k_2! (k_1+2k_4)! k_4!} \left(\frac{1}{(k_2-k_4)} \right) \\
M &= 4\pi^2 \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-4k_2}}{k_1! (k_2!)^2 (k_1+2k_2)!}
\end{aligned}$$

Now, let's do the same sort of simplification as we did for Ξ_2 . Below we have

further subdivided the parts of our calculation with $f_1 + f_2 = f$ and $i_1 + i_2 = i$.

$$\begin{aligned}
\int_{\Xi_3} e^{W/\hbar} \Omega = & \\
& = 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} (\ln(-\hbar) + \Psi(k_2 + 1)) \Bigg\} f_1 \\
& - 2\pi i \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-3k_2}}{k_1! (k_1 + 2k_2)! k_2!} \left(\sum_{\substack{m=0 \\ m \neq k_2}}^{\infty} \frac{\hbar^{-m}}{(m - k_2)m!} \right) \Bigg\} f_2 \\
& + 4\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} (\ln(-\hbar) + \Psi(k_1 + 2k_2 + 1)) \Bigg\} g = 2d \\
& - 4\pi i \sum_{k_1, k_4 \geq 0} d_2^{k_4} d_1^{k_1} \frac{1}{k_1! (k_4!)^2 \hbar^{k_1+2k_4}} \sum_{\substack{m=0 \\ m \neq k_1+2k_4}}^{\infty} \frac{\hbar^{-m}}{(m - k_1 - 2k_4)m!} \Bigg\} h = 2e \\
& + 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} (\ln \hbar - \ln(-d_2) + \Psi(k_2 + 1)) \Bigg\} i_1 \\
& - 2\pi i \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} \hbar^{-2k_1-3k_2}}{(k_1 + 2k_2)! k_2! k_1!} \left(\sum_{\substack{m=0 \\ m \neq k_2}}^{\infty} \frac{(\frac{d_2}{\hbar})^m}{(m - k_2)m!} \right) \Bigg\} i_2 \\
& + (J + K + L + M)
\end{aligned}$$

And now we will see some very nice simplifications.

$$\begin{aligned}
f_2 &= -2\pi i \sum_{k_1, k_4 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-3k_2}}{k_1! (k_1 + 2k_2)! k_2!} \left(\sum_{\substack{m=0 \\ m \neq k_2}}^{\infty} \frac{\hbar^{-m}}{(m - k_2)m!} \right) \\
&= -2\pi i \sum_{\substack{k_1, k_4, k_2 \geq 0 \\ k_4 \neq k_2}}' \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-k_4-3k_2}}{k_1! k_4! k_2! (k_1 + 2k_2)! (k_4 - k_2)} \quad \text{Here } m \text{ goes to } k_4 \\
&= -J
\end{aligned}$$

$$h = 2e$$

$$\begin{aligned}
&= -4\pi i \sum_{k_1, k_4 \geq 0} \frac{d_1^{k_1} d_2^{k_4} \hbar^{-k_1-2k_4}}{k_1! (k_4!)^2} \sum_{\substack{m=0 \\ m \neq k_1+2k_4}}^{\infty} \frac{\hbar^{-m}}{(m-k_1-2k_4)m!} \\
&= -4\pi i \sum_{\substack{k_1, k_3, k_4 \geq 0 \\ k_3 \neq k_1+2k_4}}' \frac{d_1^{k_1} d_2^{k_4} \hbar^{-k_1-k_3-2k_4}}{k_1! k_3! (k_4!)^2 (-k_1+k_3-2k_4)} \quad \text{from calculation of } e \text{ before} \\
&= -K
\end{aligned}$$

$$\begin{aligned}
i_2 &= -2\pi i \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} \hbar^{-2k_1-3k_2}}{(k_1+2k_2)! k_2! k_1!} \left(\sum_{\substack{m=0 \\ m \neq k_2}}^{\infty} \frac{(\frac{d_2}{\hbar})^m}{(m-k_2)m!} \right) \\
&= -2\pi i \sum_{\substack{k_1, k_2, m \geq 0 \\ m \neq k_2}}' \frac{d_1^{k_1} \hbar^{-2k_1-3k_2}}{(k_1+2k_2)! k_2! k_1!} \left(\frac{(\frac{d_2}{\hbar})^m}{(m-k_2)m!} \right) \\
&= -2\pi i \sum_{\substack{k_1, k_2, k_4 \geq 0 \\ k_2 \neq k_4}}' \frac{d_1^{k_1} d_2^{k_4} \hbar^{-2k_1-3k_4-k_2}}{k_1! k_4! k_2! (k_1+2k_4)! (k_2-k_4)} \\
&= -L
\end{aligned}$$

And also notice that the following holds.

$$\begin{aligned}
&f_1 + g + i_1 = \\
&= 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} (\ln(-\hbar) + \Psi(k_2 + 1)) \Bigg\} f_1 \\
&\quad + 4\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} (\ln(-\hbar) + \Psi(k_1 + 2k_2 + 1)) \Bigg\} g = 2d \\
&\quad + 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} (\ln \hbar - \ln(-d_2) + \Psi(k_2 + 1)) \Bigg\} i_1 \\
&= 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\ln \left(\frac{(-\hbar)(-\hbar)^2(\hbar)}{(-d_2)} \right) + 2\Psi(k_2 + 1) + 2\Psi(k_1 + 2k_2 + 1) \right) \\
&= 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\ln \left(\frac{\hbar^4}{d_2} \right) + 2\Psi(k_2 + 1) + 2\Psi(k_1 + 2k_2 + 1) \right)
\end{aligned}$$

Putting this all together, we get the following:

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{\Xi_3} e^{W/\hbar \Omega} \\
&= \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\ln \left(\frac{\hbar^4}{d_2} \right) + 2\Psi(k_2 + 1) + 2\Psi(k_1 + 2k_2 + 1) \right) \Bigg\} f_1 + g + i_1 \\
&\quad - J - K - L \qquad \qquad \qquad \Bigg\} f_2 + h + i_2 \\
&\quad + (J + K + L + M) \\
&= \boxed{\sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\ln \left(\frac{\hbar^4}{d_2} \right) + 2\Psi(k_2 + 1) + 2\Psi(k_1 + 2k_2 + 1) \right) - 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2}}
\end{aligned}$$

A.8 The Cycles

Recall that we are using the following notation for brevity.

$$c_{k_1, k_2} := \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!}$$

The integral over the unit torus:

$$\int_{\Xi_1} e^{W/\hbar} \Omega = -4\pi^2 \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!}$$

The integral over the cylinder:

$$\begin{aligned} \int_{\Xi_2} e^{W/\hbar} \Omega &= 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\ln \left(\frac{\hbar^2}{d_1} \right) + \Psi(k_1 + 1) + \Psi(k_1 + 2k_2 + 1) \right) \\ &\quad + 2\pi i \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} \frac{d_1^{k_3-2k_4} d_2^{k_4} (-k_3 + 2k_4 - 1)!}{k_3! (k_4!)^2 \hbar^{2k_3}} (-1)^{-k_3+2k_4} \end{aligned}$$

The integral over the union of three legs and a buffer;

$$\begin{aligned} \int_{\Xi_3} e^{W/\hbar} \Omega &= \\ &= 2\pi i \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\ln \left(\frac{\hbar^4}{d_2} \right) + 2\Psi(k_2 + 1) + 2\Psi(k_1 + 2k_2 + 1) \right) + 4\pi^2 \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \end{aligned}$$

A.9 Iritani's Solution

In theorem 4.19 of [15], Iritani gives an expression for the period integral of $\Xi_4 = (\mathbb{R}_{\geq 0})^2$. That solution is:

$$\frac{2\pi}{\hbar} \int_X \sum_{n_1, n_2 \geq 0} \frac{Todd(X) d_1^{C_0/2\pi i} d_2^{f/2\pi i} \hbar^{(-2C_0-4f)/2\pi i} d_1^{n_1-2n_2} d_2^{n_2} \hbar^{-2n_1}}{\Gamma(1+n_1-2n_2+\frac{C_0}{2\pi i}) \Gamma(1+n_2+\frac{f}{2\pi i})^2 \Gamma(1+n_1+\frac{C_0+2f}{2\pi i})} \quad (\text{A.5})$$

where $Todd(X) = 1 + (C_0 + 2f) + [pt] \in H^*(X, \mathbb{R})$.

It should be stated that $X = \mathbb{F}_2$, the surface of interest in this work. Then the cohomology ring $H^*(X, \mathbb{R})$ is generated as an $\mathbb{R}$ -vector space by $[X]$ in degree 0, C_0 and f in degree 2, and $[pt]$ in degree 4, with cup product given by $C_0^2 = -2[pt]$, $C_0 \cdot f = [pt]$, $f^2 = 0$ as usual.

We can imagine the generators C_0 , f are variables, and use Maple to expand this expression as a Taylor series in C_0 , f up to order 2. We then use the above cup products to interpret quadratic monomials in C_0 and f as multiples of $[pt]$. The integral over X just reads off the coefficient of $[pt]$. Also the expressions like $d_1^{(C_0+2f)}$ have to be interpreted as $e^{(C_0+2f)\log d_1}$, and again this is expanded in a Taylor series up to degree 2.

Let us strip this down to a simpler expression that we can then manipulate (the constant $4\pi^2$ simplifies the output).

$$\frac{4\pi^2}{\Gamma(1+n_1-2n_2+C_0/2\pi i)\Gamma(1+n_2+f/2\pi i)^2\Gamma(1+n_1+(C_0+2f)/2\pi i)} \quad (\text{A.6})$$

For now, let us assume that $n_1 - 2n_2 \geq 0$. We can plug this into Maple and extract the appropriate coefficients from the Taylor expansion in terms of C_0 and f of this expression. The constant term is

$$\frac{4\pi^2}{(n_1 - 2n_2)! (n_2!)^2 n_1!}.$$

The coefficient of C_0 is

$$\frac{2\pi i (\Psi(1+n_1-2n_2) + \Psi(1+n_1))}{(n_1 - 2n_2)! (n_2!)^2 n_1!}.$$

The coefficient of f is

$$\frac{4\pi i (\Psi(1+n_2) + \Psi(1+n_1))}{(n_1 - 2n_2)! (n_2!)^2 n_1!}.$$

The coefficient of $[pt]$ is slightly more complicated to obtain since we must take the coefficient of $C_0 \cdot f = [pt]$ and then subtract two times the coefficient of the $C_0^2 = -2[pt]$ term. It is as follows.

$$\frac{1}{(n_1 - 2n_2)! (n_2!)^2 n_1!} \left(-2\Psi(1 + n_1 - 2n_2)\Psi(1 + n_2) - 2\Psi(1 + n_1)\Psi(1 + n_2) \right. \\ \left. - \Psi(1 + n_1)^2 + \Psi(1 + n_1 - 2n_2)^2 + \Psi^{(1)}(1 + n_1) - \Psi^{(1)}(1 + n_1 - 2n_2) \right).$$

Now let us assume that $n_1 - 2n_2$ is negative. Then we have to be careful of the term in the denominator $\Gamma(1 + n_1 - 2n_2 + C_0/2\pi i)$. We will need to expand out its inverse in order to get the correct expansion. Now

$$\frac{1}{\Gamma(z)} = (-1)^n n! (z + n) - (-1)^n n! \Psi(1 + n) (z + n)^2 + (O(z + n))^3 \\ = (-1)^n n! \left((z + n) - \Psi(1 + n)(z + n)^2 \right) + (O(z + n))^3.$$

We will plug in $-(1 + n_1 - 2n_2)$ for n in this expression, and we will plug in $1 + n_1 - n_2 + \frac{C_0}{2\pi i}$ for z .

So we get

$$\frac{(-1)^{-1-n_1} (-1 - n_1 + 2n_2)! \left(-2\pi i C_0 - 2\Psi(-n_1 + 2n_2)[pt] \right)}{\Gamma(1 + n_2 + f/2\pi i)^2 \Gamma(1 + n_1 + (C_0 + 2f)/2\pi i)}.$$

Plugging this into Maple in a similar manner as above we get that the constant term is 0, as is the coefficient of f . The coefficient of C_0 is

$$\frac{2\pi i (-1)^{-n_1} (-n_1 + 2n_2 - 1)!}{n_1 (n_2!)^2}.$$

And the coefficient of $[pt]$ is

$$\frac{2(-1)^{-n_1} (-n_1 + 2n_2 - 1)! (\Psi(-n_1 + 2n_2) - \Psi(1 + n_2))}{n_1 (n_2!)^2}.$$

With a couple simple changes of labels, we can get a relationship between these solutions and the integrals we carefully calculated over the last several sections. For the solutions with $n_1 - 2n_2 \geq 0$, we need to send n_1 to $k_1 + 2k_2$ and n_2 to k_2 . In the case where $n_1 - 2n_2$ is negative, we need to send n_1 to k_3 and n_2 to k_4 .

Let us add back a little more information to the expression (A.6) from the expression (A.5); we will actually be comparing our old basis to this expression.

$$\sum_{n_1, n_2 \geq 0} \frac{4\pi^2 d_1^{n_1-2n_2} d_2^{n_2} \hbar^{-2n_1}}{\Gamma(1+n_1-2n_2+C_0/2\pi i) \Gamma(1+n_2+f/2\pi i)^2 \Gamma(1+n_1+(C_0+2f)/2\pi i)} \quad (\text{A.7})$$

Then we get that the coefficient of $[pt]$ for the expression (A.7) is as follows

$$\begin{aligned} \xi'_4 := & - \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-4k_2}}{k_1! (k_2!)^2 (k_1+2k_2)!} \left(-2H_{k_1}H_{k_2} + H_{k_1}^2 \right. \\ & \left. - H_{k_1+2k_2}^2 - 2H_{k_1+2k_2}H_{k_2} - H_{k_1+2k_2}^{(2)} + H_{k_1}^{(2)} \right) \\ & - 2 \sum_{k_4=1}^{\infty} \sum_{k_3=0}^{2k_4-1} \frac{d_1^{k_3-2k_4} d_2^{k_4} \hbar^{-2k_3} (-k_3+2k_4-1)! (-1)^{-k_3+2k_4}}{k_3! (k_4!)^2} \\ & \cdot (H_{-k_3+2k_4-1} - H_{k_4}) \end{aligned}$$

Notice here that we made the substitutions $\Psi(n) = H_{n+1} - \gamma$ where H_n is the n^{th} harmonic number and γ is the Euler-Mascheroni constant. We also used the relation $\Psi^{(1)}(n) = \frac{\pi^2}{6} - H_{n-1}^{(2)}$ where $H_n^{(2)} = \sum_{k=1}^n \frac{1}{k^2}$ is the generalized harmonic number.

This means expression (A.7) can be written as

$$-\xi_1 + \xi_2 C_0 + (\xi_3 + \xi_1)f + \xi'_4[pt]$$

where $\xi_i = \int_{\Xi_i} e^{W/\hbar} \Omega$ comes from our integrals over the original basis $\Xi_1, \dots, \Xi_4$. We will be using ξ'_4 to set up a new basis $\Xi'_1, \dots, \Xi'_4$ where $\Xi'_4 = (\mathbb{R}_{\geq 0})^2$, and Ξ'_1, Ξ'_2 and Ξ'_3 are linear combinations of elements in the old basis.

A.10 Summary for Period Integrals in New Basis

Here we will give the full solution to equation A.7.

$$\begin{aligned}
& \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!} \\
& - C_0 \left(\sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\ln \left(\frac{\hbar^2}{d_1} \right) + H_{k_1} + H_{k_1+2k_2} \right) \right. \\
& \quad \left. + \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} \frac{d_1^{k_3-2k_4} d_2^{k_4} (-k_3 + 2k_4 - 1)!}{k_3! (k_4!)^2 \hbar^{2k_3}} (-1)^{-k_3+2k_4} \right) \\
& - f \left(\sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \left(\ln \left(\frac{\hbar^4}{d_2} \right) + 2H_{k_2} + 2H_{k_1+2k_2} \right) + \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} \right) \\
& + [pt] \left[- \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-4k_2}}{k_1! (k_2!)^2 (k_1 + 2k_2)!} \right. \\
& \quad \cdot \left(-2H_{k_1} H_{k_2} + H_{k_1}^2 - H_{k_1+2k_2}^2 - 2H_{k_1+2k_2} H_{k_2} - H_{k_1+2k_2}^{(2)} + H_{k_1}^{(2)} \right) \\
& \quad - 2 \sum_{k_4=1}^{\infty} \sum_{k_3=0}^{2k_4-1} \frac{1}{k_3! (k_4!)^2} d_1^{k_3-2k_4} d_2^{k_4} \hbar^{-2k_3} \cdot \\
& \quad \cdot \left(-k_3 + 2k_4 - 1 \right)! (-1)^{-k_3+2k_4} \left(H_{-k_3+2k_4-1} - H_{k_4} \right) \left. \right]
\end{aligned}$$

Now we can set up a new and simpler set of period integral calculations (over our new basis elements Ξ'_i) based on the four that we already have (over our old basis Ξ_i). For bookkeeping purposes we will give all four of the solutions including the first, which basically does not change, and the fourth, which we gave in the previous section.

$$\begin{aligned}
\xi'_1 = \xi_1 &= \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!} \\
\xi'_2 &= - \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!} (H_{k_1} + H_{k_1+2k_2}) \\
&\quad - \sum_{\substack{k_3, k_4 \geq 0 \\ k_3 - 2k_4 + 1 \leq 0}} \frac{d_1^{k_3-2k_4} d_2^{k_4} (-k_3 + 2k_4 - 1)!}{k_3! (k_4!)^2 \hbar^{2k_3}} (-1)^{-k_3+2k_4} \\
\xi'_3 &= - \sum_{k_1, k_2 \geq 0} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-(2k_1+4k_2)}}{k_1! (k_2!)^2 (k_1 + 2k_2)!} (2H_{k_2} + 2H_{k_1+2k_2}) \\
\xi'_4 &= - \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{d_1^{k_1} d_2^{k_2} \hbar^{-2k_1-4k_2}}{k_1! (k_2!)^2 (k_1 + 2k_2)!} \left(-2H_{k_1} H_{k_2} + H_{k_1}^2 \right. \\
&\quad \left. - H_{k_1+2k_2}^2 - 2H_{k_1+2k_2} H_{k_2} - H_{k_1+2k_2}^{(2)} + H_{k_1}^{(2)} \right) \\
&\quad - 2 \sum_{k_4=1}^{\infty} \sum_{k_3=0}^{2k_4-1} \frac{d_1^{k_3-2k_4} d_2^{k_4} \hbar^{-2k_3} (-k_3 + 2k_4 - 1)! (-1)^{-k_3+2k_4}}{k_3! (k_4!)^2} \\
&\quad \cdot (H_{-k_3+2k_4-1} - H_{k_4})
\end{aligned}$$

The solution to expression (A.5) is then

$$d_1^{C_0} d_2^f \hbar^{-2C_0-4f} (\xi'_1 + \xi'_2 C_0 + \xi'_3 f + \xi'_4 [pt]) \quad (\text{A.8})$$

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