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by

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Abstract

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In this paper we consider the long time behavior of solutions to the cubic nonlinear Schrödinger equation posed on the spatial domain $\mathbb{R}\times\mathbb{T}^d$, $1\leq d\leq 4$. We first prove the local well-posedness in $C(I;L_x^2H_y^s)\cap C(I;L_{x,y}^4)$ for solutions with initial data $u_0\in H_x^{0,1}L_y^2\cap L_x^2H_y^s$. Then, for sufficiently small, smooth, decaying data, we prove global existence and derive modified asymptotic dynamics by using the wave packet method and normal form corrections. The modified scattering behavior on $\mathbb{R}\times\mathbb{T}^d$ combines the modified scattering of the cubic NLS on real line $\mathbb{R}$ with cubic NLS dynamics on torus. We also consider the corresponding asymptotic completeness problem.

To my grandmom.

Contents

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Chapter 1

Introduction

The nonlinear Schrödinger equation (NLS) has the form

$$\left(i\partial_t + \frac{1}{2}\triangle\right)u = \lambda|u|^p u \tag{1.1}$$

where u is a complex-valued function on the spatial domain $x \in \mathbb{F}$, here $\mathbb{F}$ will be either $\mathbb{R}^n$ or $\mathbb{T}^n$ or $\mathbb{R}^n \times \mathbb{T}^m$, $t \in \mathbb{R}$, and $0 < p < \infty$. The equation is called focusing if the parameter $\lambda < 0$ and defocusing if $\lambda > 0$. It is easy to see that for suitable solutions there are the conservation law for mass

$$\int |u(t,x)|^2\, dx = \int |u(0,x)|^2\, dx, \tag{1.2}$$

and the conservation law for energy

$$E[u(t)] = \frac{1}{4}\int |\nabla u(t,x)|^2\, dx + \frac{\lambda}{2+p}\int |u(t,x)|^{2+p}\, dx \equiv E[u(0)] \tag{1.3}$$

1.1 Standard notations.

In this section we briefly collect some notation, definitions and estimates used throughout this thesis. Given two quantities A,B we will write $A \lesssim B$ if there exists some constant $C > 0$ so that $A \leq CB$, and write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. If $C = C(k)$ we will write $A \lesssim_k B$. We write $A \ll B$ if $A \lesssim B$ and the constant is sufficiently small. We denote the sets of integers, real numbers and complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. If $E \subset \mathbb{R}^n$ we denote the indicator function of the set E by $\mathbf{1}_E$. We denote the Euclidean norm by $|\cdot|$ and define the bracket $\langle\cdot\rangle = \left(1 + |\cdot|^2\right)^{\frac{1}{2}}$. If X is a normed space we denote its norm by $\|\cdot\|_X$. We denote the torus $\mathbb{R}/(2\pi\mathbb{Z})$ by $\mathbb{T}$.

In this work, we define the Fourier transform $\mathcal{F}_x f$ and $\widehat{f}$ on $\mathbb{R}$ by

$$\mathcal{F}_x f := \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi} f(x)\, dx.$$

Similarly we also have the full spatial Fourier transform

$$(\mathcal{F}f)(\xi,\mathbf{k}) := \frac{1}{(2\pi)^{(d+1)/2}}\int_{\mathbb{R}}\int_{\mathbb{T}^d} f(x,y)\, e^{-ix\xi}e^{-i\langle \mathbf{k},y\rangle}dydx.$$

Since we need to switch between $f(v,y)$ and $(\mathcal{F}_y f)(v,\mathbf{k})$ very often in this work, use the bold character for the Fourier transform in the y variable:

$$\mathbf{f}(t,v,\mathbf{k}) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{T}^d} f(t,v,y)\, e^{-i\langle \mathbf{k},y\rangle}dy.$$

We use $\partial_x f$, f_x and f' to denote a (partial) derivative in the variable x. We define the fractional derivative by using Fourier transform

$$D^\alpha f = \mathcal{F}_x^{-1}\, |\xi|^\alpha\, \widehat{f}.$$

If X is a normed space and $I \subset \mathbb{R}$ is an interval, we denote the space of continuous functions $f : I \to X$ by $C(I;X)$ equipped with the sup norm. We use the notation C^k to denote k-continuously differentiable functions; $C^\infty := \cap C^k$ to denote smooth functions; C_c^∞ to denote compact supported C^∞ functions.

For $1 < p < \infty$ we use $L_x^p(F)$ (where $F = \mathbb{R}$ or $\mathbb{C}$) to denote the space of Lebesgue-measurable functions $f : X \to F$ such that

$$\|f\|_{L^p}^p = \int |f(x)|^p\, dx < \infty,$$

with the usual modification for $p = \infty$. We will typically omit the domain and codomain when they are evident. We denote the L^2-inner product by

$$\langle u,v\rangle = \int u(x)\overline{v}(x)dx.$$

We will use Littlewood-Paley projections in x

$$(\mathcal{F}P_{\le N}f)(\xi,k) = \mathcal{X}\left(\frac{\xi}{N}\right)\mathcal{F}f(\xi,k),$$

where $\mathcal{X} \in C_c^\infty(\mathbb{R})$, $\mathcal{X}(x) = 1$ when $|x| \le 1$ and $\mathcal{X}(x) = 0$ when $|x| \ge 2$. Next, define

$$P_N = P_{\le N} - P_{\le N/2}, \quad P_{\ge N} = 1 - P_{\le N/2}.$$

When we concentrate on the frequency in y only, we will denote

$$\left(\mathcal{F}P_{\le N}^y f\right)(\xi,\mathbf{k}) = \varphi\left(\frac{|\mathbf{k}|}{N}\right)(\mathcal{F}f)(\xi,\mathbf{k}).$$

We denote the linear Schrödinger operator $e^{it\partial_x^2/2}$ on $\mathbb{R}$ by

$$e^{it\partial_x^2/2}f = \mathcal{F}_x^{-1}e^{-it|\xi|^2/2}\widehat{f}.$$

Similarly, the linear Schrödinger operator $e^{it\triangle_y/2}$ on $\mathbb{T}^d$ is given by

$$e^{it\triangle_y/2}f = \mathcal{F}_y^{-1}e^{-it|\mathbf{k}|^2/2}\mathcal{F}_y f.$$

Define the linear Schrödinger evolution on $\mathbb{R}\times\mathbb{T}^d$ as $U(t) = e^{it\triangle_y/2}e^{it\partial_x^2/2}$.

1.1.1 Norms and H^s spaces

Define the weighted Sobolev norm for $x \in \mathbb{R}$ by

$$\|f\|_{H_x^{0,1}}^2 = \|f\|_{L_x^2}^2 + \|xf\|_{L_x^2}^2 . \tag{1.4}$$

Define the Sobolev norm H_x^s for variable $x \in \mathbb{R}^n$ by

$$\|f\|_{H_x^s} = \left\| \left(1 + |\xi|^2\right)^{\frac{s}{2}} \left(\mathcal{F}_x f\right)(\xi) \right\|_{L_\xi^2} . \tag{1.5}$$

Define the Sobolev norm H_x^s for variable $y \in \mathbb{T}^n$ by

$$\|f\|_{H_y^s} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} \left(\left(1 + |\mathbf{k}|^2\right)^{\frac{s}{2}} \left|\left(\mathcal{F}_y f\right)(\mathbf{k})\right| \right)^2 \right]^{\frac{1}{2}} . \tag{1.6}$$

Define the homogeneous Sobolev norm $\dot{H}_x^s$ for variable $x \in \mathbb{R}^n$ by

$$\|f\|_{\dot{H}_x^s} = \left\| |\xi|^{\frac{s}{2}} \left(\mathcal{F}_x f\right)(\xi) \right\|_{L_\xi^2} . \tag{1.7}$$

Define the homogeneous Sobolev norm $\dot{H}_x^s$ for variable $y \in \mathbb{T}^n$ by

$$\|f\|_{\dot{H}_y^s} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} \left(|\mathbf{k}|^{\frac{s}{2}} \left|\left(\mathcal{F}_y f\right)(\mathbf{k})\right| \right)^2 \right]^{\frac{1}{2}} . \tag{1.8}$$

We now define several norms which will be used in our problem, namely the cubic NLS on $\mathbb{R} \times \mathbb{T}^d$. For local well-posedness, we use the norm $X(I)$. For energy estimates we use the norm X^+. For uniform bounds and the scattering itself, we use the norm Y. These norms are defined as follows:

When proving the local well-posedness result, we use the following norm $X(I)$:

$$\|f\|_{X(I)} := \|f\|_{L_t^\infty\left(I; L_x^2 H_y^s\right)} + \|f\|_{L_t^4\left(I, L_{x,y}^\infty\right)} , \tag{1.9}$$

where I is some finite interval with length less than or equal to 1.

The norm we will use to measure the size of the solutions will be denoted as X^+. We expect this quantity to grow as t^δ where δ is a small positive number depending on the size ϵ of the initial data u_0:

$$\|u(t)\|_{X^+}^2 := \|L_x u(t)\|_{L_{x,y}^2}^2 + \left\| D_y^s u(t) \right\|_{L_{x,y}^2}^2 . \tag{1.10}$$

While using the energy method, we also define the norm as:

$$\|f\|_Y = \|f\|_{L_x^\infty H_y^\alpha} + \|f\|_{L_{x,y}^2} , \tag{1.11}$$

We will prove that the solution tends to the modified scattering profile in the space Y.

1.1.2 Forms

Here we introduce some sets and trilinear forms associated to the cubic NLS on the torus.

We will use the following sets corresponding to momentum and resonance level sets: For a fixed $\mathbf{k}$, we define the set

$$\mathcal{M}(\mathbf{k}) = \left\{(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}) \in \mathbb{Z}^{4d} : \mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k} = 0\right\},$$

which describes the triples of frequencies which yield output $\mathbf{k}$, and the resonance level

$$\Gamma_\omega(\mathbf{k}) = \left\{(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}) \in \mathcal{M} : |\mathbf{k}_1|^2 - |\mathbf{k}_2|^2 + |\mathbf{k}_3|^2 - |\mathbf{k}|^2 = \omega\right\}.$$

In particular, for the resonance level $\omega = 0$,

$$\Gamma_0(\mathbf{k}) = \left\{(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}) \in \mathbb{Z}^{4d} : \quad \mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k} = 0, \quad |\mathbf{k}_1|^2 - |\mathbf{k}_2|^2 + |\mathbf{k}_3|^2 - |\mathbf{k}|^2 = 0\right\}.$$

Restricting the trilinear interactions in the expression $u \to |u|^2 u$ to resonant ones we obtain the resonant trilinear form

$$R[f_1, f_2, f_3](t, v, \mathbf{k}) = \sum_{\Gamma_0(\mathbf{k})} f_1(t, v, \mathbf{k}_1)\, \overline{f_2}(t, v, \mathbf{k}_2)\, f_3(t, v, \mathbf{k}_3).$$

More generally, we will also consider the non-resonant trilinear forms

$$\mathcal{E}[f_1, f_2, f_3](t, v, y) := \sum_{\Gamma_\omega(\mathbf{k}), \omega \neq 0} e^{i\omega t/2} f_1(t, v, \mathbf{k}_1)\, \overline{f_2}(t, v, \mathbf{k}_2)\, f_3(t, v, \mathbf{k}_3)\, e^{i\mathbf{k}\cdot y}.$$

The expression

$$\mathcal{D}[f_1, f_2, f_3](t, v, y) := \frac{2}{t} \sum_{\Gamma_\omega(\mathbf{k}), \omega \neq 0} \frac{e^{i\omega t/2}}{i\omega} f_1(t, v, \mathbf{k}_1)\, \overline{f_2}(t, v, \mathbf{k}_2)\, f_3(t, v, \mathbf{k}_3)\, e^{i\mathbf{k}\cdot y},$$

will be used to define an energy correction later on.

1.2 The NLS on $\mathbb{R}^n$

In this section we first consider the linear Schrödinger equation on $\mathbb{R}^n$

$$\left(i\partial_t + \frac{1}{2}\triangle\right) u = 0, \qquad u(0, x) = u_0. \tag{1.12}$$

The solution to (1.12) can be written as $u(t, x) = e^{it\triangle/2} u_0$ or a linear convolution of the initial data with the fundamental solution

$$u(t, x) = \frac{1}{\sqrt{2\pi i t}^{\,n}} \int e^{\frac{i|x-y|^2}{2t}} u_0(y)\, dy. \tag{1.13}$$

Proposition 1.2.1. *[29][12][32][25] In $\mathbb{R}^n$, we call a pair (q,r) of exponents admissible if $2 \le q, r \le \infty$, $\frac{2}{q}+\frac{n}{r}=\frac{n}{2}$, and $(q,r,n) \ne (2,\infty,2)$. For $2 \le q \le \infty$, we have the dispersion estimates in time*

$$\left\| e^{it\triangle/2} u_0 \right\|_{L_x^r} \lesssim |t|^{-\frac{n}{2}+\frac{n}{q}} \left\| u_0 \right\|_{L_x^1}. \tag{1.14}$$

Then for any admissible exponents (q,r) ,and $(\tilde{q},\tilde{r})$ we have the homogeneous Strichartz estimate

$$\left\| e^{it\triangle/2} u_0 \right\|_{L_t^q(\mathbb{R};L_x^r)} \lesssim \left\| u_0 \right\|_{L_x^2}, \tag{1.15}$$

the dual homogeneous Strichartz estimate

$$\left\| \int_{\mathbb{R}} e^{-is\triangle/2} f(s) ds \right\|_{L_x^2} \lesssim \left\| f \right\|_{L_t^{\tilde{q}'}(\mathbb{R};L_x^{\tilde{r}'})}, \tag{1.16}$$

as well as the inhomogeneous Strichartz estimate

$$\left\| \int_{s<t} e^{i(t-s)\triangle/2} f(s) ds \right\|_{L_t^q(\mathbb{R};L_x^r)} \lesssim \left\| f \right\|_{L_t^{\tilde{q}'}(\mathbb{R};L_x^{\tilde{r}'})}, \tag{1.17}$$

where $\frac{1}{q}+\frac{1}{q'}=\frac{1}{r}+\frac{1}{r'}=\frac{1}{\tilde{q}}+\frac{1}{\tilde{q}'}=\frac{1}{\tilde{r}}+\frac{1}{\tilde{r}'}=1$.

Proof. The proof of non-end point Strichartz estimate $(q,r) \ne (2, \frac{2n}{n-2})$ can be obtained by a standard TT^* argument. First by (1.13) and a direct computation, we obtain

$$\left\| e^{it\triangle/2} u_0 \right\|_{L_x^\infty} \lesssim |t|^{-\frac{n}{2}} \left\| u_0 \right\|_{L_x^1}. \tag{1.18}$$

Using the conservation law of mass, we obtain

$$\left\| e^{it\triangle/2} u_0 \right\|_{L_x^2} = \left\| u_0 \right\|_{L_x^2}. \tag{1.19}$$

Interpolating between these two inequalities, we obtain (1.14).

In order to prove (1.15), by the definition of dual space, we consider all the test functions $\varphi \in L_t^{q'} L_x^{r'}$ with $\|\varphi\|_{L_t^{q'} L_x^{r'}} \le 1$. Then we have

$$\begin{aligned} \left\| e^{it\triangle/2} u_0 \right\|_{L_t^q(\mathbb{R};L_x^r)} &= \sup_{\varphi} \int \left\langle \varphi(t), e^{it\triangle/2} u_0 \right\rangle dt = \sup_{\varphi} \left\langle \int e^{-it\triangle/2} \varphi(t) dt, u_0 \right\rangle \\ &\lesssim \sup_{\varphi} \left\| \int e^{-it\triangle/2} \varphi(t) dt \right\|_{L_x^2} \left\| u_0 \right\|_{L_x^2} \end{aligned}$$

Therefore if (1.16) is true , (1.15) is also true. Here we use standard TT^* argument.

$$\left\langle \int_{\mathbb{R}} e^{-is\triangle/2} f(s)ds, \int_{\mathbb{R}} e^{-it\triangle/2} f(t)dt \right\rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} \left\langle f(s), e^{-i(t-s)\triangle/2} f(t) \right\rangle dtds$$
$$\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \|f(s)\|_{L_x^{r'}} \left\| e^{-i(t-s)\triangle/2} f(t) \right\|_{L_x^r} dtds$$
$$\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |t-s|^{-\frac{n}{2}+\frac{n}{r}} \|f(s)\|_{L_x^{r'}} ds \|f(t)\|_{L_x^{r'}} dt$$
$$\lesssim \left\| \int_{\mathbb{R}} |t-s|^{-\frac{n}{2}+\frac{n}{r}} \|f(s)\|_{L_x^{r'}} ds \right\|_{L_t^q} \|f(t)\|_{L_t^{q'} L_x^{r'}}$$
$$\lesssim \|f\|_{L_t^{q'} L_x^{r'}}^2 .$$

The last line is obtained by the Riesz potential inequality. To prove (1.17), we combine (1.15) and (1.16) together and obtain the inequality

$$\left\| \int_{\mathbb{R}} e^{i(t-s)\triangle/2} f(s)ds \right\|_{L_t^q(\mathbb{R};L_x^r)} \lesssim \left\| \int_{\mathbb{R}} e^{-is\triangle/2} f(s)ds \right\|_{L_x^2} \lesssim \|f\|_{L_t^{\tilde{q}'}(\mathbb{R};L_x^{\tilde{r}'})} .$$

By the Christ-Kiselev lemma (see below), we prove (1.17).

The end point $(q,r) = (2, \frac{2n}{n-2})$, $(q,r,n) \neq (2,\infty,2)$ is given in [25].

□

Lemma 1.2.2 (Christ-Kiselev[6]). *Let X, Y be Banach spaces, let I be a time interval, and let $K \in C^0(I \times I \to B(X \to Y))$ be a kernel taking values in the space of bounded operators from X to Y. Suppose that $1 \leq p < q \leq \infty$ is such that*

$$\left\| \int_I K(t,s)f(s)ds \right\|_{L_t^q(I;Y)} \lesssim \|f\|_{L_t^p(I;X)}$$

for all $f \in L_t^p(I;X)$. Then we have

$$\left\| \int_{s \in I: s<t} K(t,s)f(s)ds \right\|_{L_t^q(I;Y)} \lesssim_{p,q} \|f\|_{L_t^p(I;X)} .$$

Next we consider the linear inhomogeneous Schrödinger equation

$$\left(i\partial_t + \frac{1}{2}\triangle \right) u = f, \qquad u(0,x) = u_0, \tag{1.20}$$

posed on $\mathbb{R}^n$. The solution can be presented as integral formula

$$u(t) = e^{it\triangle/2} u_0 - i \int_0^t e^{i(t-s)\triangle/2} f(s)ds. \tag{1.21}$$

The by Proposition 1.2.1 we have the following Corollary:

Corollary 1.2.3. *Suppose u is a solution to (1.20), and $I \subset \mathbb{R}$ is an interval containing 0. Then for any admissible exponents (q,r) ,$(\tilde{q},\tilde{r})$, and we have the Strichartz estimate*

$$\|u\|_{L_t^q(I;L_x^r)} \lesssim \|u_0\|_{L_x^2} + \|f\|_{L_t^{\tilde{q}'}(I;L_x^{\tilde{r}'})} . \tag{1.22}$$

Finally we consider the NLS equation (1.1) posed on the space $\mathbb{R}^n$. The energy $E[u(t)]$ in (1.3) is the Hamiltonian of the system. Recall that for a bilinear, anti-symmetric, and non-degenerate symplectic form ω, we define the symplectic gradient $\nabla_\omega H$ to be the unique function such that

$$\langle v, dH(u)\rangle = \frac{d}{d\epsilon} H\left(u+\epsilon v\right)\bigg|_{\epsilon=0} = \omega\left(\nabla_\omega H(u), v\right). \tag{1.23}$$

We can obtain the corresponding Hamiltonian flow by

$$\partial_t u(t) := \nabla_\omega H(u(t)). \tag{1.24}$$

Hence if we define the symplectic form ω as

$$\omega\left(u,v\right) = \int_{\mathbb{R}^n} \operatorname{Im}\left(u\left(x\right)\overline{v}\left(x\right)\right)dx,$$

we can verify that the Hamiltonian flow of $H(u) = E[u(t)]$ satisfies the NLS equation (1.1).

The solution u has the scaling property such that if u is a solution to (1.1) then

$$u_r(t,x) = r^{\frac{2}{p}} u\left(r^2 t, rx\right)$$

is also a solution to (1.1). We have the following equalities for u_r:

$$\|u_r\|_{L_x^2} = r^{-\frac{n}{2}+\frac{2}{p}} \|u\|_{L_x^2}, \quad \|u_r\|_{\dot{H}_x^1} = r^{-\frac{n}{2}+1+\frac{2}{p}} \|u_r\|_{\dot{H}_x^1}.$$

Letting the exponents of r to zero, we obtain the L_x^2-critical exponent $p=\frac{4}{n}$ and H_x^1-critical exponet $p=\frac{4}{n-2}$. (1.1) is locally well-posed in L_x^2 when $0<p\leq\frac{4}{n}$ and in H^1 when $0<p\leq\frac{4}{n-2}$. Also, from the scaling equation we have

$$\|u_r\|_{L_t^qL_x^r} = r^{-\frac{n}{r}-\frac{2}{q}+\frac{2}{p}} \|u\|_{L_t^qL_x^r}.$$

Therefore if we want to obtain the Strichartz estimate $\|u_r\|_{L_t^qL_x^r} \lesssim \|u_r(0)\|_{L_x^2}$ for all $r\in\mathbb{R}^+$, then we must have $\frac{n}{r}+\frac{2}{q}=\frac{n}{2}$.

We introduce the Strichartz space $S^0(I\times\mathbb{R}^n)$ for any time interval I defined as the closure of the Schwartz functions under the norm

$$\|f\|_{S^0(I)} := \sup_{(q,r)\text{ admissible}} \|f\|_{L_t^q(I;L_x^r)} . \tag{1.25}$$

Proposition 1.2.4. *[30] For $1 < p < \frac{4}{n}$, the NLS equation (1.1) is locally wellposed in L^2_x. For any $R > 0$, there exists a $T = T(p, n, R) > 0$ such that for all u_0 in the ball $B_R = \left\{ u_0 \in L^2_x, \|u_0\|_{L^2_x} \leq R \right\}$, there exists a unique strong L^2_x solution to (1.1) in the space $S^0([-T, T]) \subset C^0_t([-T, T]; L^2_x)$. Furthermore the map $u_0 \mapsto u$ from B_R to $S^0([-T, T])$ is Lipschitz continuous.*

Proof. We choose the admissible pair $(q, r) = (\frac{4}{n}\frac{p+1}{p-1}, 1 + p)$, and apply the Strichartz estimate (1.17) and Hölder inequality.

$$\begin{aligned}\left\| \int_{s<t} e^{i(t-s)\triangle/2} |u|^p u(s) - |v|^p v(s) ds \right\|_{S^0(I)} &\lesssim \left\| |u|^p u - |v|^p v \right\|_{L^{q'}_t(I; L^{r'}_x)} \\ &\lesssim |I|^\alpha \left(\|u\|^p_{L^q_t(I;L^r_x)} + \|v\|^p_{L^q_t(I;L^r_x)} \right) \|u - v\|_{L^q_t(I;L^r_x)}.\end{aligned}$$

When the interval I is small enough, by the contraction principle we obtain the existence and uniqueness of solutions in the space S^0.

If $p = \frac{4}{n}$ then the local well posedness results still holds, but the lifespan of solutions depend on the initial data profile, and do not rely on its size.

□

If the solution to (1.1) approaches a solution to the linear Schrödinger equation when $t \to \pm\infty$ then we say it is scattering. When $0 < p < \frac{2}{n}$, the solution has no scattering nor modified scattering property [28][1]. If $\frac{2}{n} < p \leq \frac{4}{n}$, the solution to the defocusing equation exhibits scattering in the functional spaces L^2_x when initial data is in the functional space $\Sigma = \left\{ f : \|f\|^2_{L^2_x} + \|xf\|^2_{L^2_x} + \|\nabla f\|^2_{L^2_x} < \infty \right\}$, and the focusing equation has scattering property when the initial data is sufficiently small in Σ. The solution has the scattering property in Σ only when $\frac{2-n+\sqrt{n^2+12n+4}}{2n} < p \leq \frac{4}{n-2}$ [5]. For the critical exponent $p = \frac{2}{n}$, when $n = 1, 2, 3$, there exists modified scattering for solutions to sufficiently small initial data [16, 24, 21]. For $n \geq 4$, the problem is still open.

Proposition 1.2.5. *[26][11][31] For $\frac{2}{n} < p < \frac{4}{n}$, consider the NLS equation (1.1) with initial data $u_0 \in \Sigma$, where*

$$\Sigma = \left\{ f : \|f\|^2_{L^2_x} + \|xf\|^2_{L^2_x} + \|\nabla f\|^2_{L^2_x} < \infty \right\}.$$

Then there exists $u_+, u_- \in L^2_x$, such that the solution u to (1.1)

$$\lim_{t\to\pm\infty} \left\| u(t) - e^{it\triangle/2} u_\pm \right\|_{L^2_x} = 0.$$

By applying the wave packet method used in [21], we can improve the result to NLS with initial data $u_0 \in H^{0,1}_x$ with error $O_{L^2}\left(t^{-\frac{np}{2}+1}\right)$.

Proof. [31] Here we consider the pseudoconformal transformation v of u

$$v(t, x) = \frac{1}{(it)^{\frac{n}{2}}} e^{i\frac{|x|^2}{2t}} \overline{u\left(\frac{1}{t}, \frac{x}{t}\right)}, \tag{1.26}$$

which satisfies the differential equation

$$\left(i\partial_t + \frac{1}{2}\triangle\right) v = t^{\frac{n}{2}p-2} |v|^p v. \tag{1.27}$$

By the regularizing technique of [10], we obtain the conservation laws and bounds for all $t \in (0, 1]$

$$t^{2-\frac{n}{2}p} \|\nabla v(t)\|_{L_x^2}^2 \lesssim 1, \qquad \|v(t)\|_{L_x^{p+1}} \lesssim 1, \qquad \|v(t)\|_{L_x^2} \lesssim 1. \tag{1.28}$$

Assuming $\varphi \in H_x^1$ and applying the inner product, we have

$$\langle v(t) - v(s), \varphi \rangle = -\frac{i}{2} \int_s^t \langle \nabla v(\tau), \nabla \varphi \rangle \, d\tau - i \int_s^t \tau^{\frac{n}{2}p-2} \langle |v|^p v(\tau), \varphi \rangle \, d\tau.$$

By the bounds (1.28), and the fact that H_x^1 is dense in L_x^2, we obtain the weak limit

$$\text{w} - \lim_{t \to +0} v(t) \equiv v(0) \tag{1.29}$$

exists in L_x^2. By choosing $\varphi = v(t)$, and letting $s \to 0^+$, we have

$$\|v(t) - v(0)\|_{L_x^2}^2 \lesssim t^{\frac{n}{2}p-1} + |\langle v(t) - v(0), v(0) \rangle| \to 0,$$

when t tends to 0^+. Since the pseudoconformal transform is invertible, we obtain

$$\lim_{t \to 0^+} \left\| u(t) - e^{it\triangle/2} \mathcal{F}_x^{-1} \overline{v}\,(0) \right\|_{L_x^2} = 0. \tag{1.30}$$

□

Proposition 1.2.6. *[16] For $p = \frac{2}{n}$ and $n = 1, 2, 3$, consider the NLS equation (1.1) with initial data $u_0 \in H_x^{\gamma,0} \cap H_x^{0,\gamma}$ for some $\frac{n}{2} < \gamma < 1 + \frac{2}{n}$, and $\|u_0\|_{H_x^{0,\gamma}} + \|u_0\|_{H_x^{\gamma,0}} \leq \epsilon$. Then there exists a function $W \in L_x^2 \cap L_x^\infty$ such that*

$$u(t, x) = \frac{1}{\sqrt{t}} e^{i\frac{|x|^2}{2t} + i\lambda \left| W\left(\frac{x}{t}\right) \right|^{\frac{2}{n}} \log t} W\left(\frac{x}{t}\right) + err_x \tag{1.31}$$

where

$$err_x \in \epsilon O_{L_x^2 \cap L_x^\infty} \left((1+t)^{-\alpha + C\epsilon^{\frac{2}{n}}} \right).$$

Here α is a positive number satisfying $\frac{n}{2} + 2\alpha < \gamma \leq 1 + \frac{2}{n}$, and $C\epsilon < \alpha < \min\{\frac{1}{2} + \frac{1}{n} - \frac{n}{4}, 1\}$.

1.3 The cubic NLS on the real line $\mathbb{R}$

The cubic NLS comes from classical field equations whose principal applications are to the propagation of light in nonlinear optical fibers and planar waveguides and to Bose-Einstein condensates. The equation also appears in the studies of small-amplitude gravity waves on the surface of deep inviscid water,

$$\left(i\partial_t + \frac{1}{2}\triangle\right) u = \lambda \left|u\right|^2 u, \qquad u\left(0, x\right) = u_0 \tag{1.32}$$

The cubic NLS on $\mathbb{R}$ is an integrable system which means it has infinity many conservation laws. Also it is the critical exponent for scattering, and has the modified scattering property.

Theorem 1.3.1. *[21] (a) Let u be a solution to (1.32) on $\mathbb{R}$ with initial data $u_0 \in H^{0,1}$ and $\|u_0\|_{H^{0,1}} \leq \epsilon$. Then there exists a function $W \in H^{1-C\epsilon^2}$ such that*

$$u(t,x) = \frac{1}{\sqrt{t}} e^{i\frac{|x|^2}{2t} + i\lambda\left|W\left(\frac{x}{t}\right)\right|^2 \log t} W\left(\frac{x}{t}\right) + err_x, \tag{1.33}$$

$$\widehat{u}(t,\xi) = e^{-\frac{it\xi^2}{2}} W(\xi) e^{i\log t|W(\xi)|^2} + err_\xi, \tag{1.34}$$

where

$$err_x \in \epsilon\left(O_{L^\infty}\left((1+t)^{-\frac{3}{4}+C\epsilon^2}\right) \cap O_{L^2_x}\left((1+t)^{-1+C\epsilon^2}\right)\right). \tag{1.35}$$

$$err_\xi \in \epsilon\left(O_{L^\infty}\left((1+t)^{-\frac{1}{4}+C\epsilon^2}\right) \cap O_{L^2_\xi}\left((1+t)^{-\frac{1}{2}+C\epsilon^2}\right)\right). \tag{1.36}$$

(b) Let C be a large universal constant. Then for each W satisfying

$$\|W\|_{H^{1+C\epsilon^2}} \ll \epsilon \ll 1$$

there exists $u_0 \in H^{0,1}$ satisfying $\|u_0\|_{H^{0,1}} \leq \epsilon$, so that the corresponding solution u has properties (1.33),(1.34),(1.35) and (1.34).

In this paper, the authors use the wave packet method, originally developed in the work of Ifrim and Tataru on the $1d$ cubic NLS[21] and $2d$ water waves [19][20]. Let $\mathcal{X} \in C_0^\infty(\mathbb{R})$ be a real-valued function localized in both space and frequency near 0 at scale ~ 1. To simplify the computations, we will normalized $\int \mathcal{X} = 1$. We define a wave packet adapted to the ray $\Upsilon_v = \{x = tv\}$ by

$$\Psi_v = e^{i\phi}\mathcal{X}\left(\frac{x-tv}{\sqrt{t}}\right), \tag{1.37}$$

where the phase function is defined by

$$\phi = \frac{x^2}{2t}.$$

We expect the approximate solution will stay coherent on the time scale $\Delta t \ll t$. By direct computations, we have

$$\left(i\partial_t + \frac{1}{2}\partial_x^2\right)\Psi_v = \frac{1}{2t}e^{i\phi}\left[t^{\frac{1}{2}}\mathcal{X}'\left(\frac{x-tv}{\sqrt{t}}\right) + i\left(x-vt\right)\mathcal{X}\left(\frac{x-tv}{\sqrt{t}}\right)\right]. \tag{1.38}$$

To measure the decay of u along the ray Υ_v we use the function

$$\gamma\left(t,v\right) = \int u\overline{\Psi}_v dx.$$

The function γ satisfies the differential equation

$$i\partial_t\gamma = \frac{\lambda}{t}\left|\gamma\right|^2\gamma + R(t,v), \tag{1.39}$$

where $R(t,v)$ decaying fast in L_v^∞ and L_v^2.

1.4 The NLS equation on the torus

In contrasts to NLS on the real line, the solutions for NLS on the torus exhibit no scattering property and even the global existence becomes difficult [4, 13, 14, 17, 18, 23]. In this case, many different long time behaviors can be sustained even on arbitrarily small open data sets around zero. The NLS on torus were first studied by Bourgain [2][3]. The Strichartz estimate for $\mathbb{R}^n$ fails in the case of $\mathbb{T}^d$.

Conjecture 1.4.1. For $q \geq 2$, and $N > 1$, we have

$$\left\|P_N e^{it\triangle/2}\phi\right\|_{L^q_{t,x}\left(\mathbb{T}\times\mathbb{T}^d\right)} \lesssim_q \left\|P_N\phi\right\|_{L^2_x} \text{ if } q < \frac{2(d+2)}{d}, \tag{1.40}$$

$$\left\|P_N e^{it\triangle/2}\phi\right\|_{L^q_{t,x}\left(\mathbb{T}\times\mathbb{T}^d\right)} \ll N^\epsilon \left\|P_N\phi\right\|_{L^2_x} \text{ if } q = \frac{2(d+2)}{d}, \tag{1.41}$$

$$\left\|P_N e^{it\triangle/2}\phi\right\|_{L^q_{t,x}\left(\mathbb{T}\times\mathbb{T}^d\right)} \lesssim_q N^{\frac{d}{2}-\frac{d+2}{q}} \left\|P_N\phi\right\|_{L^2_x} \text{ if } q > \frac{2(d+2)}{d}. \tag{1.42}$$

The conjecture is partially resolved in [2][3].

Proposition 1.4.2. *[18]There exists $\delta > 0$ such that for all dyadic $N \geq M \geq 1$ it holds*

$$\left\|\left(P_N e^{it\triangle/2}\phi_1\right)\left(P_M e^{it\triangle/2}\phi_2\right)\right\|_{L^2(\mathbb{T}\times\mathbb{T}^4)} \lesssim M\left(\frac{M}{N}+\frac{1}{N}\right)^\delta \left\|P_N\phi_1\right\|_{L^2}\left\|P_M\phi_2\right\|_{L^2}. \tag{1.43}$$

The cubic NLS with H_x^1 initial data is locally well-posed in $C\left([0,T);H_x^1\right)\cap X^1\left([0,T)\right)$ for $d=1,2,3$. For $d=4$, the equation reaches the energy critical exponent, hence we do not expect locally well-posed for $d\geq 5$. Since the NLS on torus have no dispersive property, we do not expect any decay in time. The asymptotic equation can be written as an infinite dimensional dynamical system. Notice that in the case $d=1$, due to the complete integrable property, it has infinitely many conservation laws. For any given parameters $s\in(0,\infty)\backslash\{1\}$, $K\gg 1$, and $0<\delta\ll 1$, there exists a global solution $u(t,x)$ of (1.32) and a time $T>0$ such that

$$\|u(0)\|_{H^s}\leq\delta \quad \text{and} \quad \|u(T)\|_{H^s}\geq K.$$

Therefore the orbit of any H^s-neighborhood of the origin under the nonlinear flow of the cubic NLS is not uniformly bounded in H^s.

Because of the sharp contrast in behavior between $\mathbb{R}^n$ and $\mathbb{T}^d$, considerable interest has emerged to study questions of long time behavior on the product spaces. Considering the product spaces $\mathbb{R}^n\times\mathbb{T}^d$, there is the expectation that at least if $\frac{np}{2}\geq 1$ the solutions will globally exist and decay like $t^{-\frac{np}{2}}$ for sufficiently small initial data. When $\frac{np}{2}>1$, the global solutions scatter to linear solutions. When $\frac{np}{2}=1$, the global solutions exhibit some modified scattering. This paper is a specific case of the latter scenario, $n=1, \quad p=2, \quad 1\leq d\leq 4$.

The purpose of this work is to show that we can apply the newly developed tools of wave packet testing [21, 19, 20] and normal form correction [22] to establish the same asymptotic behavior as in the work of Hani, Pausader, Tzvetkov and Visciglia [15] in a simpler manner and with lower regularity compared to the existing result. The result applies also to the focusing case.

Theorem 1.4.3. *Let $1\leq d\leq 4$. Consider the equation (2.1) with initial data u_0, which satisfies*

$$\|xu_0\|_{L^2_{x,y}}+\left\|D_y^s u_0\right\|_{L^2_{x,y}}\leq\epsilon, \tag{1.44}$$

where $s=3\alpha$, $\alpha>\frac{d}{2}$ is an arbitrary positive number, and $\epsilon<\epsilon(d)$. Then:

(a) There exists a unique global solution $u\in C(\mathbb{R};L_x^2H_y^s)$ for (2.1) with initial data u_0, and $xe^{-it\partial_x^2/2}u\in C\left(\mathbb{R};L^2_{x,y}\right)$. Moreover we have the time decay

$$\|u(t)\|_{L_x^\infty H_y^1}\lesssim\epsilon\,|t|^{-\frac{1}{2}}, \tag{1.45}$$

and the energy bound

$$\left\|xe^{-it\partial_x^2/2}u\right\|_{L^2_{x,y}},\left\|D_y^s u\right\|_{L^2_{x,y}}\lesssim\epsilon\,(1+t)^{C\epsilon^2}. \tag{1.46}$$

(b) There exists $W\left(t,v,y\right)\in C\left([1,\infty);L^2_{v,y}\cap L_v^\infty H_y^\alpha\right)$, which along rays $v=constant$ is a solution to the equation

$$i\partial_t W+\frac{1}{2}\triangle_y W=\frac{1}{t}\left|W\right|^2W, \tag{1.47}$$

such that for $t \geq 1$

$$u(t,x,y) = \frac{1}{\sqrt{t}} e^{i\frac{x^2}{2t}} W\left(t, \frac{x}{t}, y\right) + err_x \tag{1.48}$$

where

$$err_x \in \epsilon \left(O_{L^2_{v,y}}\left((1+t)^{-\frac{1}{2}+C\epsilon^2}\right) \cap O_{L^\infty_x H^\alpha_y}\left((1+t)^{-\frac{7}{12}+C\epsilon^2}\right)\right).$$

A similar statement holds as $t \to -\infty$.

Complementing the above scattering result, we also have the asymptotic completeness property.

Theorem 1.4.4. *Let* $1 \leq d \leq 4$ *and* C *be a large universal constant. There exists* $\epsilon = \epsilon(d) > 0$ *such that if* W_1 *satisfies*

$$\left\| D_v^{1+C\epsilon^2} W_1 \right\|_{L^2_{v,y}} + \left\| D_y^s D_v^{C\epsilon^2} W_1 \right\|_{L^2_{v,y}} + \left\| D_y^s W_1 \right\|_{L^2_{v,y}} \ll \epsilon \ll 1,$$

there exists $W(t)$ *solving (2.7) on* $t \in [1,\infty)$ *with initial data* $W(1) = W_1$*, and there exists a solution* u *of (2.1) with initial data* u_0 *satisfies (2.4), hence (2.8) holds for* u*.*

The work starts with the proof of Theorem 2.0.4 (a). First we prove that when ϵ is small enough, there exists $T \approx e^{\frac{C}{\epsilon^2}} \gg 1$ such that the local well-posedness and (2.5), (2.6) hold on the interval $[0,T]$. These imply that Theorem 2.0.4 (b) holds on $[0,T]$, and by a more careful analysis (2.7), (2.8) will give us better bounds for (2.5), (2.6). Hence the interval $[0,T]$ can be extended to $[0,\infty)$. For the second part we will show Theorem 5.0.8 by applying a contraction mapping argument to the resulting equation for the difference $\tilde{w} := u - u_{app}$. Here u_{app} is an approximate solution constructed from W.

Remark 1.4.5. In the original paper [15] the authors prove instead that u approaches a solution for the resonant equation (4.1). Here we use (2.7) to characterize the asymptotic profile, which is similar to cubic NLS equation on $\mathbb{T}^d$. We will prove later in Proposition 4.0.3 that these two forms actually are equivalent.

Lemma 1.4.6. *Let* $x \in \mathbb{R}$ *,* $y \in \mathbb{T}^d$*, and* $\alpha > \frac{d}{2}$*. Denote* $U(t) = e^{it\partial_x^2/2} e^{it\triangle_y/2}$*, then we have the following Strichartz inequality on product space* $\mathbb{R} \times \mathbb{T}^d$*:*

(1) $\|U(t)u_0\|_{L^\infty_{x,y}} \lesssim |t|^{-\frac{1}{2}} \|u_0\|_{L^1_x H^\alpha_y}, \quad \|U(t)u_0\|_{L^4_t(I, L^\infty_{x,y})} \lesssim \|u_0\|_{L^2_x H^\alpha_y},$

(2) $\left\| \int_I U^*(t) F(t)\, dt \right\|_{L^2_x H^{-\alpha}_y} \lesssim \|F(t)\|_{L^{4/3}_t(I, L^1_{x,y})},$

(3) $\left\| \int_I U(t-s) F(s) ds \right\|_{L^4_t(L^\infty_{x,y})} \lesssim \|F(t)\|_{L^{4/3}_t(I, L^1_x H^\alpha_y)},$

$$\left\| \int_I U(t-s) F(s) ds \right\|_{L^4_t(L^\infty_{x,y})} \lesssim \|F\|_{L^1_t(I; L^2_x H^\alpha_y)}.$$

Proof. The estimate:
$$\|U(t)u_0\|_{L^\infty_{x,y}} \lesssim |t|^{-\frac{1}{2}} \|u_0\|_{L^1_x H^\alpha_y}$$
can be obtained by direct computation.

By standard UU^* argument, consider $\varphi \in L^{4/3}_t \left(L^\infty_{x,y}\right)^* \cap L^{4/3}_t L^1_{x,y}$ with $\|\varphi\|_{L^{4/3}_t\left(I, L^1_{x,y}\right)} = 1$, we will have

$$\|U(t)u_0\|_{L^4_t\left(I, L^\infty_{x,y}\right)} \leq \sup_\varphi \int_I \langle U(t)\, u_0, \varphi\rangle_{L^2_{x,y}}\, dt \leq \sup_\varphi \|u_0\|_{L^2_x H^\alpha_y} \left\|\int_I U^*(t)\,\varphi(t)\, dt\right\|_{L^2_x H^{-\alpha}_y}.$$

Then estimate $\|\int_I U^*(t)\,\varphi(t)\,dt\|_{L^2_x H^{-\alpha}_y}$ by inner product, then for $\alpha_1, \alpha_2 > 0$ satisfying $\alpha_1 - \alpha > 0$, $\alpha_1 + \alpha_2 = 2\alpha$. By the dual space property, $H^{\alpha'} \subset H^{\frac{d}{2}+} \subset L^\infty$ hence $(L^\infty)^* \subset H^{-\frac{d}{2}-} \subset H^{-\alpha'}$ if $0 < \alpha' < \frac{d}{2}$, we will have

$$\begin{aligned}\int_I\int_I \langle U(t')\,\varphi(t'), U(t)\,\varphi(t)\rangle_{L^2_x H^{-\alpha}_y}\, dt'dt &= \int_I\int_I \langle U^*(t)\,U(t')\,\varphi(t'), \varphi(t)\rangle_{L^2_x H^{-\alpha}_y}\, dt'dt \\ &\lesssim \int_I\int_I \|U^*(t)\,U(t')\,\varphi(t')\|_{L^\infty_x H^{-\alpha_1}_y} \|\varphi(t)\|_{(L^\infty_x)^* H^{-\alpha_2}_y}\, dt'dt.\end{aligned}$$

Since

$$\|U^*(t)\,U(t')\,\varphi(t')\|_{L^\infty_x H^{-\alpha_1}_y} \lesssim |t'-t|^{-\frac{1}{2}} \|\varphi(t')\|_{L^1_x H^{-\alpha_1+\alpha}_y} \lesssim |t'-t|^{-\frac{1}{2}} \|\varphi(t')\|_{L^1_x\left(L^\infty_y\right)^*}$$

and
$$\|\varphi(t)\|_{(L^\infty_x)^* H^{-\alpha_2}_y} \lesssim \|\varphi(t)\|_{(L^\infty_x)^*\left(L^\infty_y\right)^*} \lesssim \|\varphi(t)\|_{L^1_x L^1_y},$$
we prove the second inequality,

$$\begin{aligned}\left\|\int_I U^*(t)\,\varphi(t)\,dt\right\|_{L^2_x H^{-\alpha}_y} &\lesssim \|\varphi\|_{L^{4/3}_t\left(I, \left(L^\infty_{x,y}\right)^*\right)} \left\||t|^{-\frac{1}{2}} *_t \varphi(t)\right\|_{L^4_t\left(I, L^1_x L^1_y\right)} \\ &\lesssim \|\varphi\|_{L^{4/3}_t\left(I, L^1_{x,y}\right)} \|\varphi\|_{L^{4/3}_t\left(I, L^1_{x,y}\right)}.\end{aligned}$$

The last inequality comes from

$$\begin{aligned}\left\|\int_I U(t-s)\,F(s)ds\right\|_{L^4_t\left(I; L^\infty_{x,y}\right)} &\lesssim \left\|\int_I \|U(t-s)F(s)\|_{L^\infty_{x,y}}\, ds\right\|_{L^4_t(I)} \\ &\lesssim \left\|\int_I |t-s|^{-\frac{1}{2}} \|F(s)\|_{L^1_x H^\alpha_y}\, ds\right\|_{L^4_t(I)} \\ &\lesssim \|F\|_{L^{4/3}_t\left(I; L^1_x H^\alpha_y\right)}.\end{aligned}$$

By Christ-Kiselev lemma we have

$$\left\|\int_{s<t} U(t-s)\,F(s)ds\right\|_{L^4_t\left(I; L^\infty_{x,y}\right)} \lesssim \|F\|_{L^{4/3}_t\left(I; L^1_x H^\alpha_y\right)}.$$

By direct computation we have

$$\begin{aligned}\left\|\int_I U\left(t-s\right)F(s)ds\right\|_{L_t^4\left(L^\infty_{x,y}\right)} &= \left\|U(t)\int_I U\left(-s\right)F(s)ds\right\|_{L_t^4\left(L^\infty_{x,y}\right)}\\ &\lesssim \left\|\int_I U\left(-s\right)F(s)ds\right\|_{L_x^2H_y^\alpha}\\ &\lesssim \|F\|_{L_t^1\left(I;L_x^2H_y^\alpha\right)},\end{aligned}$$

and by Christ-Kiselev lemma

$$\left\|\int_{s<t} U\left(t-s\right)F(s)ds\right\|_{L_t^4\left(I;L^\infty_{x,y}\right)} \lesssim \|F\|_{L_t^1\left(I;L_x^2H_y^\alpha\right)}.$$

□

Corollary 1.4.7. *For any function f_1, f_2 and f_3, suppose that $f_i \in L_x^2H_y^\alpha \cap L_t^4L^\infty_{x,y}$. We have the following inequality:*

$$\begin{aligned}&\left\|\int_0^t U\left(t-s\right)\left(f_1\left(s\right)f_2\left(s\right)f_3\left(s\right)\right)ds\right\|_{L_t^\infty\left(I;L_x^2H_y^\alpha\right)}\\ \lesssim& \sum_{\tau\in\mathfrak{S}(3)}|I|^{\frac{1}{2}}\left\|f_{\tau(1)}\right\|_{L_t^4\left(I,L^\infty_{x,y}\right)}\left\|f_{\tau(2)}\right\|_{L_t^4\left(I;L^\infty_{x,y}\right)}\left\|f_{\tau(3)}\right\|_{L_t^\infty\left(I;L_x^2H_y^\alpha\right)},\end{aligned}$$

$$\begin{aligned}&\left\|\int_0^t U\left(t-s\right)\left(f_1\left(s\right)f_2\left(s\right)f_3\left(s\right)\right)ds\right\|_{L_t^4\left(I,L^\infty_{x,y}\right)}\\ \lesssim& \sum_{\tau\in\mathfrak{S}(3)}|I|^{\frac{3}{4}}\left\|f_{\tau(1)}\right\|_{L_t^4\left(I,L^\infty_{x,y}\right)}\left\|f_{\tau(2)}\right\|_{L_t^\infty\left(I,L_x^2H_y^\alpha\right)}\left\|f_{\tau(3)}\right\|_{L_t^\infty\left(I,L_x^2H_y^\alpha\right)}.\end{aligned}$$

1.5 Additional estimates

The proof is given in [15] Lemma 7.1 and Lemma 7.4: see also [2, 18, 3]

Lemma 1.5.1. *Let R be defined as*

$$R\left[a^1(\mathbf{k}_1),a^2(\mathbf{k}_2),a^3(\mathbf{k}_3)\right](\mathbf{k}_4) := \sum_{\Gamma_0(\mathbf{k}_4)} a^1(\mathbf{k}_1)\overline{a^2(\mathbf{k}_2)}a^3(\mathbf{k}_3). \tag{1.49}$$

For every sequences $a^1(\mathbf{k}), a^2(\mathbf{k}), a^3(\mathbf{k})$ indexed by $\mathbb{Z}^d, d \le 4$, we have

$$\left\|R\left[a^1,a^2,a^3\right]\right\|_{l^2_{\mathbf{k}}} \lesssim_d \min_{\tau\in\mathfrak{S}(3)}\left\|a^{\tau(1)}\right\|_{l^2_{\mathbf{k}}}\left\|a^{\tau(2)}\right\|_{h^1_{\mathbf{k}}}\left\|a^{\tau(3)}\right\|_{h^1_{\mathbf{k}}}. \tag{1.50}$$

For any $s > 0$, there is the inequality

$$\left\|R\left[a^1,a^2,a^3\right]\right\|_{h^s_{\mathbf{k}}} \lesssim_d \max_{\tau\in\mathfrak{S}(3)}\left\|a^{\tau(1)}\right\|_{h^s_{\mathbf{k}}}\left\|a^{\tau(2)}\right\|_{h^1_{\mathbf{k}}}\left\|a^{\tau(3)}\right\|_{h^1_{\mathbf{k}}},$$

and if $a^i = a^i(v,\mathbf{k})$ *for* $i=1,2,3$*, there is also the inequality*

$$\left\| D_v^s R\left[a^1,a^2,a^3\right]\right\|_{L_v^2 l_{\mathbf{k}}^2} \lesssim_d \max_{\tau\in\mathfrak{S}(3)} \left\|D_v^s a^{\tau(1)}\right\|_{L_v^2 l_{\mathbf{k}}^2} \left\|a^{\tau(2)}\right\|_{L_v^\infty h_{\mathbf{k}}^1} \left\|a^{\tau(3)}\right\|_{L_v^\infty h_{\mathbf{k}}^1}.$$

Proof. One can deduce by duality, we need to prove that

$$\left|\sum_{\Gamma_0} a^0(k_0)\, a^1(k_1)\, a^2(k_2)\, a^3(k_3)\right| \lesssim \left\|a^0\right\|_{l_k^2} \min_{\tau\in\sigma(3)} \left\|a^{\tau(1)}\right\|_{l_k^2} \left\|a^{\tau(2)}\right\|_{h_k^1} \left\|a^{\tau(3)}\right\|_{h_k^1}.$$

In this section we will assume that $\alpha = \frac{d}{2}^+$, and in the setting of $d=1$, we may use the fact $|\gamma|$ is bounded and get better estimate.We will reduce to a bound on free solutions on the torus $\mathbb{T}^d$. set

$$\phi_j(y) = \sum_{k\in\mathbb{Z}^d} \tilde{a}^j(k)\, e^{ik\cdot y} : \mathbb{T}^d \to \mathbb{C}$$

$$\sum_{\Gamma_0} a^0(k_0)\, a^1(k_1)\, a^2(k_2)\, a^3(k_3) = \int_{\mathbb{T}_y^d\times\mathbb{T}_t} u_0(y,t)\overline{u_1(y,t)}u_2(y,t)\overline{u_3(y,t)}dydt,$$

where $u_j(y,t) = e^{it\triangle_{\mathbb{T}^d}}\phi_j(y)$. Therefore it follows from

$$\left|\int_{\mathbb{T}_y^d\times\mathbb{T}_t}\prod_{j=0}^3 \tilde{u}_j(y,t)dydt\right| \lesssim \|\phi_0\|_{L_y^2} \min_{\tau\in\sigma(3)} \left\|\phi_{\tau(1)}\right\|_{L_y^2} \left\|\phi_{\tau(2)}\right\|_{H_y^1} \left\|\phi_{\tau(3)}\right\|_{H_y^1},$$

$$\left\|P_N e^{it\triangle_{\mathbb{T}^d}}\phi\right\|_{L_{y,t}^4(\mathbb{T}^d)} \lesssim N^{s(d)}\|\phi\|_{L_y^2}$$

where L_y^2 and H_y^1 denote the corresponding Sobolev norms on $\mathbb{T}^d$. Estimates follow from the analysis in [2, 18, 3] as we explain below. By a slight abuse of notation we denote again by

$$\sum_{\substack{N_0\lesssim N_1\\ N_3\le N_2\le N_1}} (N_2N_3)^{-1}\left|\int_{\mathbb{T}^d} P_{N_0}u_0 P_{N_1}u_1 P_{N_2}u_2 P_{N_3}u_3 dydt\right| \lesssim \prod_{j=0}^3 \|\phi_j\|_{L_y^2} \tag{1.51}$$

we invoke the classical L^4 Strichartz estimate by Bourgain [2]

$$\left\|P_N e^{it\triangle_y}\phi\right\|_{L_{y,t}^4(\mathbb{T}^{d+1})} \lesssim N^{s(d)}\|\phi\|_{L_y^2}, \tag{1.52}$$

where $s(1)=0$, $s(d) = \frac{d-2}{4}+\epsilon$ for every $\epsilon>0$ when $d=2,3$ and $s(4)=\frac{1}{2}$.

One gets a bilinear refinement

$$\left\|\left(P_{N_1}e^{it\triangle_{\mathbb{T}^4}}\phi_1\right)\left(P_{N_2}e^{it\triangle_{\mathbb{T}^4}}\phi_2\right)\right\|_{L_{y,t}^2(\mathbb{T}^{d+1})} \lesssim N_2^{2s(d)}\|\phi_1\|_{L_y^2}\|\phi_2\|_{L_y^2}$$

where $N_2\le N_1$. For $d=4$,

$$\left\|\left(P_{N_1}e^{it\triangle_{\mathbb{T}^4}}\phi_1\right)\left(P_{N_2}e^{it\triangle_{\mathbb{T}^4}}\phi_2\right)\right\|_{L_{y,t}^2(\mathbb{T}^{d+1})} \lesssim N_2\left(\frac{N_2}{N_1}+\frac{1}{N_1}\right)^\delta \|\phi_1\|_{L_y^2}\|\phi_2\|_{L_y^2}$$

for some $\delta>0$, where again $N_2\le N_1$.

□

Lemma 1.5.2. *Consider three sequences* $\{c_1\}$, $\{c_1\}$ *and* $\{c_3\}$*. We will have the following elementary bound*

$$\left\| \sum_{\mathcal{M}(k)} c_1(k_1)\, c_2(k_2)\, c_3(k_3) \right\|_{l_k^2} \lesssim \min_{\tau \in \mathfrak{S}(3)} \left\| c_{\tau(1)} \right\|_{l_k^2} \left\| c_{\tau(2)} \right\|_{l_k^1} \left\| c_{\tau(3)} \right\|_{l_k^1}.$$

Lemma 1.5.3. *Let* $a_s(\xi, \eta, \theta) = |\eta + \theta\xi|^s$*. Define*

$$A_s^m(\theta)(f, g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix(\xi+\eta)} \frac{1}{m!} \partial_\theta^m a_s(\xi, \eta, \theta)\, \hat{f}(\xi)\, \hat{g}(\eta)\, d\xi d\eta.$$

Let $l \in \mathbb{N}$*. Let* p, p_1, p_2 *satisfy* $1 < p, p_1, p_2 < \infty 1$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$*. Let* s, s_1, s_2 *satisfy* $0 \leq s_2, s_2$ *and* $l - 1 \leq s = s_1 + s_2 \leq l$*. Then the following bilinear estimate*

$$\begin{aligned} &\left\| D^s(fg) - \sum_{k \in \mathbb{Z}} \sum_{m=0}^{l-1} A_s^m(0)(P_{\leq k-3} f, P_k g) - \sum_{k \in \mathbb{Z}} \sum_{m=0}^{l-1} A_s^m(0)(P_{\leq k-3} g, P_k f) \right\|_{L^p} \\ \leq &C \left\| D^{s_1} f \right\|_{L^{p_1}} \left\| D^{s_2} g \right\|_{L^{p_2}} \end{aligned}$$

holds for all $f, g \in \mathcal{S}$*, where* $C = C(n, p, p_1, p_2)$*.*

See [8].

Chapter 2

Small Data Scattering

In this paper, we work with the cubic defocusing nonlinear Schrödinger equation which has the form

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 + \frac{1}{2}\triangle_y\right) u = |u|^2 u, \tag{2.1}$$

where u is a complex-valued function on the spatial domain $(x, y) \in \mathbb{R} \times \mathbb{T}^d$, $1 \leq d \leq 4$. Let $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. The equation is known to be well-posed in $H^1_{x,y}$ [23]. A suitable solution will satisfy the conservation law for the mass

$$\int_{\mathbb{R}\times\mathbb{T}^d} |u(t)|^2 \, dxdy = \int_{\mathbb{R}\times\mathbb{T}^d} |u(0)|^2 \, dxdy, \tag{2.2}$$

and also for the energy

$$\int_{\mathbb{R}\times\mathbb{T}^d} |\partial_x u(t)|^2 + |\nabla_y u(t)|^2 + |u(t)|^4 \, dxdy = \int_{\mathbb{R}\times\mathbb{T}^d} |\partial_x u(0)|^2 + |\nabla_y u(0)|^2 + |u(0)|^4 \, dxdy. \tag{2.3}$$

The purpose of this work is to show that we can apply the newly developed tools of wave packet testing [21, 19, 20] and normal form correction [22] to establish the same asymptotic behavior as in the work of Hani, Pausader, Tzvetkov and Visciglia [15] in a simpler manner and with lower regularity compared to the existing result. The result applies also to the focusing case.

Theorem 2.0.4. *Let $1 \leq d \leq 4$. Consider the equation (2.1) with initial data u_0, which satisfies*

$$\|xu_0\|_{L^2_{x,y}} + \left\|D^s_y u_0\right\|_{L^2_{x,y}} \leq \epsilon, \tag{2.4}$$

where $s = 3\alpha$, $\alpha > \frac{d}{2}$ is an arbitrary positive number, and $\epsilon < \epsilon(d)$. Then:

(a) There exists a unique global solution $u \in C(\mathbb{R}; L^2_x H^s_y)$ for (2.1) with initial data u_0, and $xe^{-it\partial_x^2/2}u \in C\left(\mathbb{R}; L^2_{x,y}\right)$. Moreover we have the time decay

$$\|u(t)\|_{L^\infty_x H^1_y} \lesssim \epsilon \, |t|^{-\frac{1}{2}}, \tag{2.5}$$

and the energy bound

$$\left\|xe^{-it\partial_x^2/2}u\right\|_{L^2_{x,y}}, \left\|D_y^s u\right\|_{L^2_{x,y}} \lesssim \epsilon\,(1+t)^{C\epsilon^2}. \tag{2.6}$$

(b) There exists $W(t,v,y) \in C\left([1,\infty); L^2_{v,y} \cap L^\infty_v H^\alpha_y\right)$, which along rays $v =$ constant is a solution to the equation

$$i\partial_t W + \frac{1}{2}\triangle_y W = \frac{1}{t}\,|W|^2\,W, \tag{2.7}$$

such that for $t \geq 1$

$$u\,(t,x,y) = \frac{1}{\sqrt{t}} e^{i\frac{x^2}{2t}} W\left(t, \frac{x}{t}, y\right) + err_x \tag{2.8}$$

where

$$err_x \in \epsilon\left(O_{L^2_{v,y}}\left((1+t)^{-\frac{1}{2}+C\epsilon^2}\right) \cap O_{L^\infty_x H^\alpha_y}\left((1+t)^{-\frac{7}{12}+C\epsilon^2}\right)\right).$$

A similar statement holds as $t \to -\infty$.

Complementing the above scattering result, we also have the asymptotic completeness property.

2.1 Local well-posedness

Before we study the long time behavior, it is necessary to consider the local well-posedness of equation (2.1). Here we introduce the vector field $L_x := x + it\partial_x$,which is the conjugate of x with respect to the linear flow, $U(t)x = L_x U(t)$. The vector field is also the generator for the Galilean group of symmetries. The function $L_x u$ satisfies the following equation,

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 + \frac{1}{2}\triangle_y\right) L_x u = 2|u|^2 L_x u - u^2\overline{L_x u}, \tag{2.9}$$

which is the linearized equation of (2.1). The operater L_x allows us to capture the effect of the initial data localization $xu_0 \in L^2_{x,y}$ as the time increases.

Proposition 2.1.1. *The equation (2.1) is locally well-posed for initial data $u_0 \in H^{0,1}_x L^2_y \cap L^2_x H^s_y$ for any $s > \frac{d}{2}$. For such data we have a unique local solution $u \in C\left(I; L^2_x H^s_y \cap L^4_t L^\infty_{x,y}\right)$. If in addtion $xu_0 \in L^2_{x,y}$, then the solution has $L_x u \in C\left(I; L^2_{x,y}\right)$.*

In the case $d \leq 3$, global existence can be established in a much more general setting ($u_0 \in H^1_{x,y}$), see [23].

Proof. Using the normed space X defined in (1.9) and the Strichartz estimate from Lemma **??**, for arbitrary function $f, g \in L_x^2 H_y^s$, there is the inequality

$$\left\| \int_0^t U(t-s) \left(|f|^2 f - |g|^2 g \right)(s)\, ds \right\|_{X(I)} \lesssim \left(|I|^{\frac{1}{2}} + |I|^{\frac{3}{4}} \right) \left(\|f\|_{X(I)}^2 + \|g\|_{X(I)}^2 \right) \|f-g\|_{X(I)} . \tag{2.10}$$

The estimate (2.10) allows us to obtain the unique local solution through the contraction principle, if we let the interval I be small enough. Therefore equation (2.9) is locally well-posed in the space $X(I)$. Due to the lack of uniform $L_x^2 H_y^s$ estimates, for $d > 1$ we can not extend the iteration directly to global well-posedness. Later on we will prove that for solutions with suitable small initial data, we will have the bound $\|u(t)\|_{L_x^2 H_y^s} \lesssim t^{\delta}$, which implies the global well-posedness. For the linearized equation (2.9), supposing that $f, g \in L_{x,y}^2$, we use the iteration scheme and take $|I|$ to be small enough,

$$\left\| \int_0^t U(t-s) \left(|u|^2 f - |u|^2 g \right)(s)\, ds \right\|_{L_t^\infty\left(I; L_{x,y}^2\right)} \lesssim |I|^{\frac{1}{2}} \|u\|_{L_t^4\left(I, L_{x,y}^\infty\right)}^2 \|f-g\|_{L_t^\infty\left(I; L_{x,y}^2\right)} . \tag{2.11}$$

A similar bound can be applied to the nonlinear term $u^2 \overline{L_x u}$. Hence the local well-posedness for the equation (2.9) in $L_t^\infty \left(I; L_{x,y}^2\right)$ can be readily obtained once we have the well-posedness for the equation of u in $X(I)$.

□

2.2 The asymptotic equation

Here we use the wave packet testing method following the work of Ifrim and Tataru on the $1d$ cubic NLS [21]. A wave packet in the context here is an approximate solution to the linear system with $\mathcal{O}(1/t)$ errors. For each trajectory $\Upsilon_v := \{x = vt\}$ traveling with velocity v we establish decay along this ray by testing with a wave packet moving along the ray. Here we use a slightly different notation; one can verify that the function γ here is the same as in the original paper [21]. Define

$$w(t,v,y) := t^{\frac{1}{2}} e^{-i\frac{tv^2}{2}} u(t, tv, y), \tag{2.12}$$

$$\gamma(t,v,y) := P_{\le \sqrt{t}} w(t,v,y). \tag{2.13}$$

Our first result asserts that u is well approximated by a function associated with γ for as long as we have good control of the energy bounds $\|L_x u\|_{L_{x,y}^2}$ and $\left\|D_y^s u\right\|_{L_{x,y}^2}$.

Lemma 2.2.1. *The functions γ and w satisfy the following bounds for any $\alpha \ge 0$:*

$$\|\gamma\|_{L_v^\infty L_y^2\left(L_v^\infty H_y^s\right)} \lesssim \|w\|_{L_v^\infty L_y^2\left(L_v^\infty H_y^s\right)} = t^{\frac{1}{2}} \|u\|_{L_x^\infty L_y^2\left(L_x^\infty H_y^s\right)} , \tag{2.14}$$

$$\left\|D_y^s \gamma\right\|_{L_{v,y}^2} \le \left\|D_y^s w\right\|_{L_{v,y}^2} = \left\|D_y^s u\right\|_{L_{x,y}^2} , \quad \|\partial_v \gamma\|_{L_{v,y}^2} \le \|\partial_v w\|_{L_{v,y}^2} = \|L_x u\|_{L_{x,y}^2} , \tag{2.15}$$

$$\|\gamma\|_{L_v^\infty H_y^\alpha} \lesssim \|w\|_{L_v^\infty H_y^\alpha} \lesssim \|u\|_{L_{x,y}^2}^{\frac{1}{6}} \|L_x u\|_{L_{x,y}^2}^{\frac{1}{2}} \left\|D_y^{3\alpha} u\right\|_{L_{x,y}^2}^{\frac{1}{3}}. \tag{2.16}$$

We also have the physical space bounds

$$\left\|u(t,x,y) - \frac{1}{\sqrt{t}} e^{-i\frac{x^2}{2t}} \gamma\left(t, \frac{x}{t}, y\right)\right\|_{L_x^\infty H_y^\alpha} \lesssim t^{-\frac{7}{12}} \|L_x u\|_{L_{x,y}^2}^{\frac{2}{3}} \left\|D_y^{3\alpha} u\right\|_{L_{x,y}^2}^{\frac{1}{3}}, \tag{2.17}$$

$$\left\|u(t,x,y) - \frac{1}{\sqrt{t}} e^{-i\frac{x^2}{2t}} \gamma\left(t, \frac{x}{t}, y\right)\right\|_{L_{x,y}^2} \lesssim t^{-\frac{1}{2}} \|L_x u\|_{L_{x,y}^2}, \tag{2.18}$$

and the Fourier space bounds

$$\left\|\hat{u}(t,\xi,k) - e^{-it\xi^2/2} \gamma(t,\xi,\mathbf{k})\right\|_{L_\xi^2 l_{\mathbf{k}}^2} \lesssim t^{-\frac{1}{2}} \|L_x u\|_{L_{x,y}^2}. \tag{2.19}$$

Proof. By Bernstein's inequality and interpolation, we have the straightforward bounds:

$$\|\gamma\|_{L_v^\infty H_y^\alpha} \lesssim \|w\|_{L_v^\infty H_y^\alpha} \lesssim \left\|D_v^{\frac{3}{4}} w\right\|_{L_{v,y}^2}^{\frac{2}{3}} \left\|D_y^{3\alpha} w\right\|_{L_{v,y}^2}^{\frac{1}{3}} \lesssim \|u\|_{L_{x,y}^2}^{\frac{1}{6}} \|L_x u\|_{L_{x,y}^2}^{\frac{1}{2}} \left\|D_y^{3\alpha} u\right\|_{L_{x,y}^2}^{\frac{1}{3}},$$

$$\|w - \gamma\|_{L_v^\infty H_y^\alpha} \lesssim \left\|D_v^{\frac{3}{4}} P_{\geq\sqrt{t}} w\right\|_{L_{v,y}^2}^{\frac{2}{3}} \left\|D_y^{3\alpha} w\right\|_{L_{v,y}^2}^{\frac{1}{3}} \lesssim t^{-\frac{1}{12}} \|L_x u\|_{L_{x,y}^2}^{\frac{2}{3}} \left\|D_y^{3\alpha} u\right\|_{L_{x,y}^2}^{\frac{1}{3}}, \tag{2.20}$$

and

$$\|w - \gamma\|_{L_{v,y}^2} \lesssim t^{-\frac{1}{2}} \|\partial_v w\|_{L_{v,y}^2} \lesssim t^{-\frac{1}{2}} \|L_x u\|_{L_{x,y}^2}. \tag{2.21}$$

□

We will assume that $\alpha = \frac{d}{2}^+$and $s = 3\alpha$ from here on.

The next objective is to show that γ is an approximate solution to the asymptotic equation (2.7).

Lemma 2.2.2. *If u solves (2.1) then we have*

$$i\partial_t \gamma + \frac{1}{2} \triangle_y \gamma = \frac{1}{t} |\gamma|^2 \gamma + I, \tag{2.22}$$

where the remainder I satisfies

$$\|I\|_{L_{v,y}^2} \lesssim t^{-\frac{3}{2}} \left(\|u\|_{L_{x,y}^2}^{\frac{1}{3}} \|u\|_{X^+}^{\frac{5}{3}} + 1\right) \|u\|_{X^+} \tag{2.23}$$

$$\|I\|_{L_v^\infty H_y^\alpha} \lesssim t^{-\frac{13}{12}} \left(\|u\|_{L_{x,y}^2}^{\frac{1}{3}} \|u\|_{X^+}^{\frac{5}{3}} + 1\right) \|u\|_{X^+}, \tag{2.24}$$

where $s = 3\alpha$ and $\alpha = \frac{d}{2}^+$.

Proof. Let ξ be the Fourier variable in v. A direct computation yields

$$\partial_t\gamma + \frac{1}{2}\triangle_y\gamma = \mathcal{F}\left[\mathcal{X}'\left(\frac{\xi}{\sqrt{t}}\right)\cdot\frac{\xi}{2t^{\frac{3}{2}}} + \frac{|\xi|^2}{2t^2}\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\right]\hat{w} + t^{-1}P_{\leq\sqrt{t}}|w|^2 w.$$

Hence we can write an evolution equation for γ of the form

$$i\partial_t\gamma + \frac{1}{2}\triangle_y\gamma = \frac{1}{t}|\gamma|^2\gamma + I,$$

where the error term $I(t,v)$ can be written as a sum of three quantities which can be easily bounded:

$$\begin{aligned} I &:= \mathcal{F}\left[D\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\cdot\frac{\xi}{2t^{\frac{3}{2}}} + \frac{|\xi|^2}{2t^2}\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\right]\hat{w} + t^{-1}P_{\leq\sqrt{t}}|w|^2 w - t^{-1}|\gamma|^2\gamma \\ &:= \mathcal{F}\left[D\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\cdot\frac{\xi}{2t^{\frac{3}{2}}} + \frac{|\xi|^2}{2t^2}\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\right]\hat{w} + t^{-1}P_{\leq\sqrt{t}}\left(|w|^2 w - |\gamma|^2\gamma\right) + t^{-1}P_{\geq\sqrt{t}}|\gamma|^2\gamma \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

The first term I_1 can be expressed as a convolution and by Young's inequality we obtain the bound

$$\|I_1(t,v,y)\|_{L^2_{v,y}} \lesssim t^{-\frac{3}{2}}\left\|P_{\leq\sqrt{t}}\partial_v w\right\|_{L^2_{v,y}} \lesssim t^{-\frac{3}{2}}\|L_x u\|_{L^2_{x,y}} \lesssim t^{-\frac{3}{2}}\|u\|_{X^+},$$

$$\begin{aligned} \|I_1(t,v,y)\|_{L^\infty_v H^\alpha_y} &\lesssim t^{-\frac{5}{4}}\left\|P_{\leq\sqrt{t}}\partial_v w\right\|_{L^2_v H^\alpha_y} \lesssim t^{-\frac{13}{12}}\left\|P_{\leq\sqrt{t}}D_v^{\frac{2}{3}} w\right\|_{L^2_v H^\alpha_y} \\ &\lesssim t^{-\frac{13}{12}}\left\|P_{\leq\sqrt{t}}\partial_v w\right\|^{\frac{2}{3}}_{L^2_{v,y}}\left\|P_{\leq\sqrt{t}} w\right\|^{\frac{1}{3}}_{L^2_v H^{3\alpha}_y} \\ &\lesssim t^{-\frac{13}{12}}\|u\|_{X^+}. \end{aligned}$$

For the second and third term I_2, I_3, we apply the Bernstein's inequality in order to get:

$$\begin{aligned} \|I_2\|_{L^2_{v,y}} &\lesssim t^{-1}\left(\|w\|^2_{L^\infty_v H^\alpha_y} + \|\gamma\|^2_{L^\infty_v H^\alpha_y}\right)\|w-\gamma\|_{L^2_{v,y}} \lesssim t^{-\frac{3}{2}}\left(\|w\|^2_{L^\infty_v H^\alpha_y} + \|\gamma\|^2_{L^\infty_v H^\alpha_y}\right)\|L_x u\|_{L^2_{x,y}} \\ &\lesssim t^{-\frac{3}{2}}\|u\|^{\frac{1}{3}}_{L^2_{x,y}}\|L_x u\|^2_{L^2_{x,y}}\left\|D^s_y u\right\|^{\frac{2}{3}}_{L^2_{x,y}} \lesssim t^{-\frac{3}{2}}\|u\|^{\frac{1}{3}}_{L^2_{x,y}}\|u\|^{\frac{8}{3}}_{X^+}. \end{aligned}$$

$$\begin{aligned} \|I_2\|_{L^\infty_v H^\alpha_y} &\lesssim t^{-1}\left\||w|^2 w - |\gamma|^2\gamma\right\|_{L^\infty_v H^\alpha_y} \lesssim t^{-1}\left(\|w\|^2_{L^\infty_v H^\alpha_y} + \|\gamma\|^2_{L^\infty_v H^\alpha_y}\right)\|w-\gamma\|_{L^\infty_v H^\alpha_y} \\ &\lesssim t^{-\frac{13}{12}}\left(\|w\|^2_{L^\infty_v H^\alpha_y} + \|\gamma\|^2_{L^\infty_v H^\alpha_y}\right)\|L_x u\|^{\frac{2}{3}}_{L^2_{x,y}}\left\|D^s_y u\right\|^{\frac{1}{3}}_{L^2_{x,y}} \\ &\lesssim t^{-\frac{13}{12}}\|u\|^{\frac{1}{3}}_{L^2_{x,y}}\|u\|^{\frac{8}{3}}_{X^+}. \end{aligned}$$

Similarly, we have:

$$\|I_3\|_{L^2_{v,y}} \lesssim t^{-\frac{3}{2}} \left\|\partial_v |\gamma|^2 \gamma\right\|_{L^2_{v,y}} \lesssim t^{-\frac{3}{2}} \|\gamma\|^2_{L^\infty_v H^\alpha_y} \|\partial_v \gamma\|_{L^2_{v,y}} \lesssim t^{-\frac{3}{2}} \|\gamma\|^2_{L^\infty_v H^\alpha_y} \|L_x u\|_{L^2_{x,y}}$$
$$\lesssim t^{-\frac{3}{2}} \|u\|^{\frac{1}{3}}_{L^2_{x,y}} \|u\|^{\frac{8}{3}}_{X^+},$$

$$\|I_3\|_{L^\infty_v H^\alpha_y} \lesssim t^{-\frac{5}{4}} \left\|\partial_v |\gamma|^2 \gamma\right\|_{L^2_v H^\alpha_y} \lesssim t^{-\frac{13}{12}} \left\|D_v^{\frac{2}{3}} |\gamma|^2 \gamma\right\|_{L^2_v H^\alpha_y}$$
$$\lesssim t^{-\frac{13}{12}} \|\gamma\|^2_{L^\infty_v H^\alpha_y} \left\|D_v^{\frac{2}{3}} \gamma\right\|_{L^2_v H^\alpha_y} \lesssim t^{-\frac{13}{12}} \|\gamma\|^2_{L^\infty_v H^\alpha_y} \|L_x u\|^{\frac{2}{3}}_{L^2_{x,y}} \left\|D^s_y u\right\|^{\frac{1}{3}}_{L^2_{x,y}}$$
$$\lesssim t^{-\frac{13}{12}} \|u\|^{\frac{1}{3}}_{L^2_{x,y}} \|u\|^{\frac{8}{3}}_{X^+}.$$

Since

$$\|I\|_Y \leq \|I_1\|_Y + \|I_2\|_Y + \|I_3\|_Y,$$

we obtain (2.23) and (2.24).

□

2.3 The energy bound for γ.

From the equation (2.22) there is the natural guess that

$$\|\gamma(t)\|_{L^\infty_v H^1_y} \lesssim \epsilon$$

for any $t \geq 1$. Indeed, multiplying (2.22) with $\overline{\gamma}_t$, integrating over y, and taking the real part we have

$$\frac{1}{2}\mathrm{Re} \int (\triangle_y \gamma)\, \overline{\gamma_t}\, dy = \frac{1}{t}\mathrm{Re} \int |\gamma|^2 \gamma \overline{\gamma_t}\, dy + \mathrm{Re} \int I \overline{\gamma_t}\, dy,$$

which directly implies that

$$\partial_t \|\nabla_y \gamma\|^2_{L^2_y} + \frac{1}{t}\partial_t \|\gamma\|^4_{L^4_y} = -4\mathrm{Re}\, \langle \gamma_t, I\rangle_{H^{-1}_y, H^1_y}.$$

Use (2.22) again and the fact that in $\mathbb{R}^d$ for $d \leq 4$ we have $L^{\frac{4}{3}} \subset H^{-1}$, $\|f\|_{H^{-1}} \lesssim \|f\|_{L^{\frac{4}{3}}}$ and $H^{s_1} \subset H^{s_2}$ if $s_1 > s_2$,

$$\|\gamma_t\|_{H^{-1}_y} \lesssim \|\triangle_y \gamma\|_{H^{-1}_y} + t^{-1} \left\||\gamma|^2 \gamma\right\|_{H^{-1}_y} + \|I\|_{H^{-1}_y}$$
$$\lesssim \|\gamma\|_{H^1_y} + t^{-1} \|\gamma\|^3_{L^4_y} + \|I\|_{H^\alpha_y} \lesssim \|\gamma\|_{H^\alpha_y} + t^{-1} \|\gamma\|^3_{H^\alpha_y} + \|I\|_{H^\alpha_y}.$$

After integrating with respect to t, then taking the supremum over v, we get

$$\begin{aligned} &\|\nabla_y \gamma(T)\|^2_{L^\infty_v L^2_y} + \frac{1}{t}\|\gamma(T)\|^4_{L^\infty_v L^4_y} \\ \lesssim &\|\nabla_y \gamma(1)\|^2_{L^\infty_v L^2_y} + \|\gamma(1)\|^4_{L^\infty_v L^4_y} + \int_1^T t^{-2}\|u\|^{\frac{2}{3}}_{L^2_{x,y}}\|u(t)\|^{\frac{10}{3}}_{X^+}\,dt \\ &+ \int_1^T t^{-\frac{13}{12}}\|u\|^{\frac{1}{2}}_{L^2_{x,y}}\|u(t)\|^{\frac{7}{2}}_{X^+} + t^{-\frac{25}{12}}\|u\|^{\frac{5}{6}}_{L^2_{x,y}}\|u(t)\|^{\frac{31}{6}}_{X^+} + t^{-\frac{13}{6}}\|u\|^{\frac{2}{3}}_{L^2_{x,y}}\|u(t)\|^{\frac{16}{3}}_{X^+}\,dt. \end{aligned} \tag{2.25}$$

Chapter 3

The Energy Estimate

In this section, we aim to prove the energy bounds for $\|L_x u\|_{L^2_{x,y}}$ and $\left\|D^s_y u\right\|_{L^2_{x,y}}$. Here we will work with the general linearized equation of (2.1) which is given by

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 + \frac{1}{2}\triangle_y\right)\nu = 2\left|u\right|^2\nu - u^2\overline{\nu}. \tag{3.1}$$

Notice that the equation for $L_x u$, (2.9) is the same as (3.1). The function $D^s_y u$ does not directly satisfy the linearized equation, but its equation can be written in the form

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 + \frac{1}{2}\triangle_y\right)D^s_y u = 2|u|^2 D^s_y u + u^2\overline{D^s_y u} + \mathrm{cor}(t), \tag{3.2}$$

where

$$\mathrm{cor}(t) := D^s_y\left(\left|u\right|^2 u\right) - \left(2|u|^2 D^s_y u + u^2\overline{D^s_y u}\right).$$

The correction term $\mathrm{cor}(t)$ is nontrivial for $s \neq 1$ but has a commutator structure and satisfies favorable bounds. We will leave the proof of bound for $\mathrm{cor}(t)$ for the last part of the section.

To obtain $L^2_{x,y}$ estimates for the linearized equation (3.1), we make the following assumptions on u:

Hypothesis 3.0.1. *The solution u for (2.1) exists in a time interval $[0,T]$, and satisfies the bounds*

$$\left\|u\left(t\right)\right\|_{L^\infty_x H^1_y} \leq D\epsilon\left|t\right|^{-\frac{1}{2}}, \tag{3.3}$$

$$\left\|u\left(t\right)\right\|_{X^+} \leq D\epsilon\left(1+\left|t\right|\right)^{\delta}, \tag{3.4}$$

for $t \in [0,T]$. Here D is a sufficiently large positive number which does not depend on u.

Then we have the following bound for the linearized equation (3.1):

Proposition 3.0.2. *Suppose u is a solution to (2.1) satisfying Hypothesis 3.0.1 on $[0,T]$, then we will have that*

(a) The equation (3.1) in ν is $L^2_{x,y}$ well-posed.

(b) There is the bound

$$\|\nu(t)\|_{L^2_{x,y}} \lesssim \|\nu(0)\|_{L^2_{x,y}} (1+t)^{2D^3\epsilon^2} \tag{3.5}$$

for $t \in [0,T]$.

The local well-posedness property of equation (3.1) is given by Proposition 2.1.1, therefore it suffices only to prove (3.5). Denote $V(t,v,y) := e^{-\frac{itx^2}{2}}\sqrt{t}\nu(t,tx,y)$ by making a substitution $v = tx$. Instead of computing the $L^2_{x,y}$ norm of ν, it is easier to work with the $L^2_{v,y}$ norm of $V(t,v,y)$(which is the same). By direct computation, V satisfies the equation

$$\left(i\partial_t + \frac{1}{2t^2}\partial_v^2 + \frac{1}{2}\triangle_y\right) V(t,v,y) = t^{-1}\left[2|w|^2 V(t,v,y) + w^2\overline{V}(t,v,y)\right]. \tag{3.6}$$

Denote the associated linear evolution operator by

$$S(t) = e^{-it\triangle_y/2}e^{i\partial_v^2/(2t)},$$

and transform the equation (3.6) into the form

$$i\partial_t\left(S(-t)V(t,v,y)\right) = t^{-1}S(-t)\left[2|w|^2 V(t,v,y) + w^2\overline{V}(t,v,y)\right].$$

From the above equation we have the fact that

$$\partial_t \frac{1}{2}\|S(-t)V(t,v,y)\|_{L^2_{v,y}} = t^{-1}\mathrm{Im}\int_{\mathbb{R}\times\mathbb{T}^d}\left[\overline{S(-t)V}\right]\left[S(-t)w^2\overline{V}\right] dvdy,$$

and

$$\|\nu(t)\|_{L^2_{x,y}} = \|V(t)\|_{L^2_{v,y}} = \|S(-t)V(t)\|_{L^2_{v,y}} \tag{3.7}$$

for any $t \neq 0$.

Denote $\mathcal{Z} := S(-t)V$, and $W^* = e^{-it\triangle_y/2}w$. By a direct computation we write the Fourier transform of the second factor in the integrand in the form:

$$\begin{aligned}
&\mathcal{F}\left[S(-t)w^2\overline{V}\right](t,\xi,\mathbf{k}) \\
&= \sum_{\mathcal{M}(\mathbf{k})}\int\int \exp\left(i\frac{t}{2}|\mathbf{k}|^2 - i\frac{\xi^2}{2t}\right)\widehat{w}(t,\kappa,\mathbf{k}_1)\widehat{\overline{V}}(t,\xi-\eta-\kappa,\mathbf{k}_2)\widehat{w}(t,\eta,\mathbf{k}_3)\,d\kappa d\eta \\
&= \sum_{\mathcal{M}(\mathbf{k})}\int\int e^{i\Psi(t)}\widehat{W^*}(t,\kappa,\mathbf{k}_1)\widehat{\overline{\mathcal{Z}}}(t,\xi-\eta-\kappa,\mathbf{k}_2)\widehat{W^*}(t,\eta,\mathbf{k}_3)\,d\kappa d\eta.
\end{aligned} \tag{3.8}$$

Here the phase function Ψ is defined as

$$\Psi\left(t\right):=\frac{1}{2t}\left(\left(\xi-\eta-\kappa\right)^{2}+\xi^{2}\right)+\frac{t}{2}\omega, \tag{3.9}$$

where $\omega:=\left|\mathbf{k}_1\right|^2-\left|\mathbf{k}_2\right|^2+\left|\mathbf{k}_3\right|^2-\left|\mathbf{k}\right|^2$.

It is natural to separate the right hand side of the equation (3.8) into four different parts according to the v and y frequencies of the factors:

- Where the v-frequency of one of w is large, $\left\{(\kappa,\eta):|\kappa|\geq\sqrt{t}\right\}\bigcup\left\{(\kappa,\eta):|\eta|\geq\sqrt{t}\right\}$. Eliminating this case allows us to switch to the computation with w replacing by γ. The corresponding term has the expression

$$e_1\left(t,v,y\right):=t^{-1}S(-t)\left[\left(w^2-\gamma^2\right)\overline{V}\right]. \tag{3.10}$$

- Where the y frequencies are resonant,
$\left\{(\kappa,\eta):|\kappa|,|\eta|<\sqrt{t}\right\}\cap\{(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3):(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\in\mathcal{M}\left(\mathbf{k}\right),(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\in\Gamma_0\left(\mathbf{k}\right)\}$. This corresponds to

$$e_2\left(t,v,y\right):=t^{-1}S(-t)\sum_{\mathbf{k}}\left[\sum_{\Gamma_0(\mathbf{k})}\gamma\left(t,v,\mathbf{k}_1\right)\overline{V}\left(t,v,\mathbf{k}_2\right)\gamma\left(t,v,\mathbf{k}_3\right)e^{i\mathbf{k}\cdot y}\right]. \tag{3.11}$$

- Where the v and y frequencies are nonresonant,
$\left\{(\kappa,\eta):|\kappa|,|\eta|<\sqrt{t}\right\}\cap\{(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3):(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\in\mathcal{M}\left(\mathbf{k}\right)\cap\Gamma_\omega\left(\mathbf{k}\right),\omega\neq0\}\cap\left\{|\Psi'(t)|\gtrsim t^{-\frac{3}{8}}\right\}$. Therefore we choose the region Ω_t^1 as follows:

$$\Omega_t^1\left(\xi,\omega\right)=\left\{\quad\omega\neq0,\quad\left|\tfrac{(\xi-\kappa-\eta)^2}{t^2\omega}-\tfrac{1}{2}\right|\geq\tfrac{1}{2}t^{-\frac{3}{8}}\ \right\}\bigcup\left\{\omega\neq0,\left|\frac{\xi^2}{t^2\omega}-\frac{1}{2}\right|\geq2t^{-\frac{3}{8}}\right\}. \tag{3.12}$$

The corresponding term in the energy is given by

$$\begin{aligned}&e_3\left(t,v,y\right)\\:=&t^{-1}\mathcal{F}_\xi^{-1}\sum_{\mathbf{k}}\left[\sum_{\omega\neq0}\sum_{\Gamma_\omega(\mathbf{k})}\iint\mathcal{X}_1e^{i\Psi(t)}\widehat{G}\left(t,\kappa,\mathbf{k}_1\right)\widehat{\overline{\mathcal{Z}}}\left(t,\xi-\eta-\kappa,\mathbf{k}_2\right)\widehat{G}\left(t,\eta,\mathbf{k}_3\right)d\kappa d\eta\right]e^{i\mathbf{k}\cdot y}.\end{aligned} \tag{3.13}$$

Here $G:=e^{-it\triangle_y/2}\gamma$ is the linear pullback of γ, and $\mathcal{X}_1$ is a cutoff function selecting this region. Precisely, we will define the freqency cut-off function $\mathcal{X}_1$ depending on t,ξ,κ,η and ω by

$$\mathcal{X}_1:=1-\mathcal{X}_2, \tag{3.14}$$

where the function $\mathcal{X}_2$ is given in (3.17).

- Where the v and y frequencies are almost resonant,
$\left\{(\kappa,\eta):|\kappa|,|\eta|<\sqrt{t}\right\}\cap\{(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3):(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\in\mathcal{M}(\mathbf{k})\cap\Gamma_\omega(\mathbf{k}),\omega\neq 0\}\cap\left\{|\Psi'(t)|\lesssim t^{-\frac{3}{8}}\right\}$.
Therefore we choose the region Ω_t^2 as follows:

$$\Omega_t^2(\xi,\omega)=\left\{\ \omega\neq 0,\quad \left|\tfrac{(\xi-\kappa-\eta)^2}{t^2\omega}-\tfrac{1}{2}\right|<2t^{-\frac{3}{8}}\ \right\}\bigcap\left\{\ \omega\neq 0,\quad \left|\tfrac{\xi^2}{t^2\omega}-\tfrac{1}{2}\right|<2t^{-\frac{3}{8}}\ \right\}. \tag{3.15}$$

The corresponding term in the energy is given by

$$\begin{aligned}&e_4(t,v,y)\\ :=&t^{-1}\mathcal{F}_\xi^{-1}\sum_{\mathbf{k}}\left[\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\iint\mathcal{X}_2 e^{i\Psi(t)}\widehat{G}(t,\kappa,\mathbf{k}_1)\widehat{\overline{\mathcal{Z}}}(t,\xi-\eta-\kappa,\mathbf{k}_2)\widehat{G}(t,\eta,\mathbf{k}_3)\,d\kappa d\eta\right]e^{i\mathbf{k}\cdot y},\end{aligned} \tag{3.16}$$

where $\mathcal{X}_2$ is a cutoff function selecting this region. Here we define the frequency cut-off function $\mathcal{X}_2$ by

$$\mathcal{X}_2:=\mathcal{X}_{1,\omega}(\xi-\kappa-\eta)\,\mathcal{X}_{2,\omega}(\xi), \tag{3.17}$$

where

$$\mathcal{X}_{1,\omega}(\xi-\kappa-\eta):=\mathcal{X}\left(\frac{1}{2}t^{\frac{3}{8}}\left(\frac{(\xi-\kappa-\eta)^2}{t^2\omega}-\frac{1}{2}\right)\right),\quad \mathcal{X}_{2,\omega}(\xi):=\mathcal{X}\left(\frac{1}{2}t^{\frac{3}{8}}\left(\frac{\xi^2}{t^2\omega}-\frac{1}{2}\right)\right). \tag{3.18}$$

Hence we have

$$t^{-1}S(-t)\left(w^2\overline{V}\right):=e_1+e_2+e_3+e_4.$$

Assume that the initial data satisfies (2.4), by the local well-posedness we know that $\nu(t)$ and $u(t)$ exist inside the interval $[0,T]$. To advance from time 0 to time 1 we use the local well-posedness results to obtain

$$\|\nu(1)\|_{L^2_{x,y}}\lesssim\|\nu(0)\|_{L^2_{x,y}},$$

and note that by the mass conservation law (2.2) there is the inequality

$$\|u(t)\|_{L^2_{x,y}}=\|u(0)\|_{L^2_{x,y}}\leq\epsilon$$

for any $t\in[0,T]$. By (3.7) there is the inequality

$$\frac{1}{2}\|\nu(T)\|^2_{L^2_{x,y}}\leq\frac{1}{2}\|\nu(1)\|^2_{L^2_{x,y}}+\left|\int_1^T\langle S(-t)V,e_1+e_2+e_3+e_4\rangle_{L^2_{x,y}}dt\right|. \tag{3.19}$$

By (3.3), and the definition of γ, there is the property

$$\|\gamma(t)\|_{L^\infty_v H^1_y}\lesssim\sqrt{t}\,\|u(t)\|_{L^\infty_x H^1_y}\lesssim D\epsilon.$$

3.1 The high frequency estimates.

First we start with bounds for the high v-frequencies in w.

Lemma 3.1.1. *Assume that $T \geq 1$. Then the following estimates hold uniformly in t:*

$$\int_1^T \left|\langle S(-t)V, e_1(t)\rangle_{L^2_{v,y}}\right| dt \lesssim \int_1^T D^{\frac{11}{6}} \epsilon^2 t^{-\frac{13}{12}} (1+t)^{\frac{11}{6}\delta} \|\nu(t)\|^2_{L^2_{x,y}} dt \tag{3.20}$$

Proof. Proof: Since $S(-t)$ is an unitary operator, using (2.16) we have

$$\begin{aligned}\|e_1(t)\|_{L^2_{v,y}} &\lesssim t^{-1}\left(\|w\|_{L^\infty_v H^\alpha_y} + \|\gamma\|_{L^\infty_v H^\alpha_y}\right)\|w-\gamma\|_{L^\infty_v H^\alpha_y}\|V\|_{L^2_{v,y}} \\ &\lesssim t^{-\frac{13}{12}}\left(\|w\|_{L^\infty_v H^\alpha_y} + \|\gamma\|_{L^\infty_v H^\alpha_y}\right)\|L_x u\|^{\frac{2}{3}}_{L^2_{x,y}}\left\|D^s_y u\right\|^{\frac{1}{3}}_{L^2_{x,y}}\|V\|_{L^2_{v,y}} \\ &\lesssim t^{-\frac{13}{12}}\|u\|^{\frac{1}{6}}_{L^2_{x,y}}\|u\|^{\frac{11}{6}}_{X^+}\|\nu\|_{L^2_{x,y}}.\end{aligned}$$

□

3.2 The y frequencies resonant term.

The growth of the energy mainly comes from the resonant term and will be smaller then t^{-1}, hence we can apply Grownwall's inequality.

Lemma 3.2.1. *Assume that $T \geq 1$. we will have*

$$\int_1^T \left|\langle S(-t)V, e_2(t)\rangle_{L^2_{v,y}}\right| dt \lesssim \int_1^T D^2\epsilon^2 t^{-1}\|\nu(t)\|^2_{L^2_{x,y}} dt. \tag{3.21}$$

Proof. Here we use the inequality (**??**), which provides a good fixed time estimate for the y-resonant interactions. The original proof of (**??**) is given in Lemma 7.1 in [15]. By the fact that $S(-t)$ is unitary, and the inequality (**??**), the factor e_2 satisfies the following inequality:

$$\|e_2(t)\|_{L^2_{v,y}} \lesssim \frac{1}{t}\left\|\|\gamma\|^2_{H^1_y}\|V\|_{L^2_y}\right\|_{L^2_v} \lesssim \frac{1}{t}\|\gamma\|^2_{L^\infty_v H^1_y}\|V\|_{L^2_{v,y}} \lesssim \frac{1}{t}\|\gamma\|^2_{L^\infty_v H^1_y}\|\nu\|_{L^2_{x,y}}. \tag{3.22}$$

□

3.3 The fast time oscillations.

Here we use a normal form energy correction to cancel out the non-resonant frequencies in $\frac{1}{t}\gamma^2\overline{L}$, using a technique developed in the papers [22, 27, 9]. The idea is that we may apply integration by parts in time to get a better decay where the nonlinear term is non-resonant.

From the equation (3.8) we may rewrite the remaining terms with low frequency of w and non-resonat y frequencies as

$$\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\int\int e^{i\Psi(t)}\mathcal{X}_1\widehat{G}\left(t,\kappa,\mathbf{k}_1\right)\widehat{\overline{\mathcal{Z}}}\left(t,\xi-\eta-\kappa,\mathbf{k}_2\right)\widehat{G}\left(t,\eta,\mathbf{k}_3\right)d\kappa d\eta. \tag{3.23}$$

The resonance function $\Psi(t)$ is given in (3.9). One can only apply the normal form correction in the region where $\Psi'\neq 0$. By the following equation

$$\Psi'\left(t\right)=-\frac{1}{2t^2}\left(\left(\xi-\eta-\kappa\right)^2+\xi^2\right)+\frac{1}{2}\omega, \tag{3.24}$$

it is obvious that inside the area Ω_t^1 we have $|\Psi'\left(t\right)|\gtrsim t^{-\frac{3}{8}}\omega$.

Lemma 3.3.1. *Assume that $T\geq 1$. Then we have the following estimate*

$$\begin{aligned}\left|\int_1^T\langle S(-t)V,e_3\left(t\right)\rangle_{L^2_{v,y}}dt\right|\lesssim{}& D^2\epsilon^2T^{-\frac{5}{8}}\left(1+T\right)^{2\delta}\left\|\nu\left(T\right)\right\|^2_{L^2_{v,y}}+D^2\epsilon^2\left\|\nu(1)\right\|^2_{L^2_{v,y}}\\&+\int_1^T\left[D^2\epsilon^2t^{-\frac{5}{4}}\left(1+t\right)^{2\delta}+D^2\epsilon^2t^{-\frac{13}{8}}\left(1+t\right)^{2\delta}\right]\left\|\nu\left(t\right)\right\|^2_{L^2_{x,y}}dt\\&+\int_1^T\left[D^{\frac{11}{3}}\epsilon^4t^{-\frac{13}{8}}\left(1+t\right)^{\frac{11}{3}\delta}+D^{\frac{8}{3}}\epsilon^4t^{-\frac{13}{8}}\left(1+t\right)^{\frac{8}{3}\delta}\right]\left\|\nu\left(t\right)\right\|^2_{L^2_{x,y}}dt.\end{aligned} \tag{3.25}$$

Proof. First observe that

$$e^{i\Psi(t)}=\frac{1}{i\Psi'(t)}\left(\partial_te^{i\Psi(t)}\right)\text{ and }\partial_t\left(\frac{1}{i\Psi'(t)}\right)=-\frac{\Psi''\left(t\right)}{i\left(\Psi'(t)\right)^2},$$

where $\Psi''\left(t\right)=\frac{1}{t^3}\left(\left(\xi-\kappa-\eta\right)^2+\xi^2\right)$. Thus it is natural to define the trilinear form as follows

$$\begin{aligned}&\mathcal{O}_1^t\left[f_1,f_2,f_3,f_4\right]\\:={}&t^{-1}\sum_{\mathbf{k}}\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\int\iint\mathcal{X}_1\frac{e^{i\Psi(t)}}{i\Psi'}\widehat{f_1}\left(t,\kappa,\mathbf{k}_1\right)\widehat{\overline{f_2}}\left(t,\xi-\eta-\kappa,\mathbf{k}_2\right)\widehat{f_3}\left(t,\eta,\mathbf{k}_3\right)\widehat{\overline{f_4}}\left(t,-\xi,\mathbf{k}\right)d\kappa d\eta d\xi,\end{aligned}$$

$$\begin{aligned}\mathcal{O}_2^t\left[f_1,f_2,f_3,f_4\right]:={}&t^{-1}\sum_{\mathbf{k}}\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\int\iint\left[\mathcal{X}_1\frac{\Psi''}{i\left(\Psi'\right)^2}+\frac{\partial_t\mathcal{X}_1}{i\Psi'}\right]e^{i\Psi(t)}\widehat{f_1}\\&\left(t,\kappa,\mathbf{k}_1\right)\widehat{\overline{f_2}}\left(t,\xi-\eta-\kappa,\mathbf{k}_2\right)\widehat{f_3}\left(t,\eta,\mathbf{k}_3\right)\widehat{\overline{f_4}}\left(t,-\xi,\mathbf{k}\right)d\kappa d\eta d\xi.\end{aligned}$$

Then observe that

$$\begin{aligned}e_3\left(t\right)={}&\partial_t\left(\mathcal{O}_1^t\left[G,\mathcal{Z},G,\mathcal{Z}\right]\right)+\mathcal{O}_2^t\left[G,\mathcal{Z},G,\mathcal{Z}\right]+t^{-1}\mathcal{O}_1^t\left[G,\mathcal{Z},G,\mathcal{Z}\right]\\&-\mathcal{O}_1^t\left[\partial_tG,\mathcal{Z},G,\mathcal{Z}\right]-\mathcal{O}_1^t\left[G,\partial_t\mathcal{Z},G,\mathcal{Z}\right]-\mathcal{O}_1^t\left[G,\mathcal{Z},\partial_tG,\mathcal{Z}\right]-\mathcal{O}_1^t\left[G,\mathcal{Z},G,\partial_t\mathcal{Z}\right].\end{aligned}$$

We start with the estimates associated with $\mathcal{O}_1^t$:

Lemma 3.3.2. *Assume that $\mathcal{O}_1^t$ defined as above. Then for any $t \geq 1$*

$$\left|\mathcal{O}_1^t\left[G, \mathcal{Z}, G, \mathcal{Z}\right]\right| \lesssim \epsilon^2 D^2 t^{-\frac{5}{8}}\left(1+t\right)^{2\delta}\left\|\nu\left(t\right)\right\|_{L^2_{x,y}}^2, \tag{3.26}$$

$$\left|\mathcal{O}_1^t\left[\partial_t G, \mathcal{Z}, G, \mathcal{Z}\right]\right|, \left|\mathcal{O}_1^t\left[G, \mathcal{Z}, \partial_t G, \mathcal{Z}\right]\right| \lesssim \epsilon^4 D^{\frac{11}{3}} t^{-\frac{13}{8}}\left(1+t\right)^{\frac{11}{3}\delta}\left\|\nu\right\|_{L^2_{x,y}}^2, \tag{3.27}$$

and

$$\left|\mathcal{O}_1^t\left[G, \partial_t\mathcal{Z}, G, \mathcal{Z}\right]\right|, \left|\mathcal{O}_1^t\left[G, \mathcal{Z}, G, \partial_t\mathcal{Z}\right]\right| \lesssim \epsilon^4 D^{\frac{8}{3}} t^{-\frac{13}{8}}\left(1+t\right)^{\frac{8}{3}\delta}\left\|\nu\right\|_{L^2_{x,y}}^2. \tag{3.28}$$

Proof. By lemma (**??**), (2.15), (2.2) and Minkowski's integral inequality

$$\begin{aligned}\left|\mathcal{O}_1^t\left[G, \mathcal{Z}, G, \mathcal{Z}\right]\right| &\lesssim t^{-\frac{5}{8}}\left\|\widehat{G}\left(t,\xi,\mathbf{k}\right)\right\|_{L^1_\xi h^\alpha_{\mathbf{k}}}^2\left\|\widehat{\mathcal{Z}}\left(t,\xi,\mathbf{k}\right)\right\|_{L^2_\xi l^2_{\mathbf{k}}}^2 \lesssim t^{-\frac{5}{8}}\left\|\langle\xi\rangle^{\frac{2}{3}}\widehat{G}\left(t,\xi,\mathbf{k}\right)\right\|_{L^2_\xi h^\alpha_{\mathbf{k}}}^2\left\|\mathcal{Z}\right\|_{L^2_{v,y}}^2\\ &\lesssim t^{-\frac{5}{8}}\left\|G\right\|_{L^2_\xi H^{3\alpha}_y}^{\frac{2}{3}}\left\|\partial_v G\right\|_{L^2_{v,y}}^{\frac{4}{3}}\left\|\mathcal{Z}\right\|_{L^2_{v,y}}^2 \lesssim t^{-\frac{5}{8}}\left\|D^s_y u\right\|_{L^2_{x,y}}^{\frac{2}{3}}\left\|L_x u\right\|_{L^2_{x,y}}^{\frac{4}{3}}\left\|\nu\right\|_{L^2_{x,y}}^2.\end{aligned}$$

From (2.16) and Bernstein's inequality, for $t > 1$ there is the bound

$$\left\|\partial_t G\right\|_{\dot{H}^1_v L^2_y\left(L^2_v H^s_y\right)} \lesssim t^{-1}\left\|w\right\|_{L^\infty_v H^\alpha_y}^2\left\|w\right\|_{\dot{H}^1_v L^2_y\left(L^2_v H^s_y\right)} + t^{-1}\left\|w\right\|_{\dot{H}^1_v L^2_y\left(L^2_v H^s_y\right)} \lesssim \epsilon^3 D^{\frac{8}{3}} t^{-1}\left(1+t\right)^{\frac{8}{3}\delta}.$$

Hence using the same procedure, there is the estimate

$$\begin{aligned}&\left|\mathcal{O}_1^t\left[\partial_t G, \mathcal{Z}, G, \mathcal{Z}\right]\right|\\ \lesssim& t^{-\frac{5}{8}}\left\|\partial_t D^s_y G\right\|_{L^2_{v,y}}^{\frac{1}{3}}\left\|\partial_t\partial_v G\right\|_{L^2_{v,y}}^{\frac{2}{3}}\left\|D^s_y G\right\|_{L^2_{v,y}}^{\frac{1}{3}}\left\|\partial_v G\right\|_{L^2_{v,y}}^{\frac{2}{3}}\left\|\mathcal{Z}\right\|_{L^2_{v,y}}^2\\ \lesssim& D^{\frac{11}{3}}\epsilon^4 t^{-\frac{13}{8}}\left(1+t\right)^{\frac{11}{3}\delta}\left\|\nu\right\|_{L^2_{x,y}}^2.\end{aligned}$$

Also there is the bound

$$\left\|\partial_t\mathcal{Z}\right\|_{L^2_{v,y}} \lesssim t^{-1}\left\|w^2 V\right\|_{L^2_{v,y}} \lesssim t^{-1}\left\|w\right\|_{L^\infty_v H^\alpha_y}^2\left\|V\right\|_{L^2_{v,y}} \lesssim D^{\frac{5}{3}}\epsilon^2 t^{-1}\left(1+t\right)^{\frac{5}{3}\delta}\left\|\nu\right\|_{L^2_{x,y}},$$

and therefore

$$\left|\mathcal{O}_1^t\left[G, \partial_t\mathcal{Z}, G, \mathcal{Z}\right]\right| \lesssim t^{-\frac{5}{8}}\left\|D^s_y u\right\|_{L^2_{x,y}}^{\frac{2}{3}}\left\|L_x u\right\|_{L^2_{x,y}}^{\frac{1}{3}}\left\|\partial_t\mathcal{Z}\right\|_{L^2_{v,y}}\left\|\mathcal{Z}\right\|_{L^2_{v,y}} \lesssim D^{\frac{8}{3}}\epsilon^4 t^{-\frac{13}{8}}\left(1+t\right)^{\frac{8}{3}\delta}\left\|\nu\right\|_{L^2_{x,y}}^2.$$

□

It remains to establish the bound of $\mathcal{O}_2^t$. Note that inside Ω_t^1 we have the bounds

$$\left|\frac{\Psi''}{\left(\Psi'\right)^2}\right| \lesssim t^{-\frac{1}{4}}\left|\omega\right|^{-1} \text{ for } \omega \neq 0.$$

By a direct computation we have that

$$\partial_t \mathcal{X}_1 = -\mathcal{X}'_{1,\omega}\mathcal{X}_{2,\omega}\left[\frac{3}{16}t^{-\frac{5}{8}}\left(\frac{(\xi-\eta-\kappa)^2}{t^2\omega}-\frac{1}{2}\right)-t^{-\frac{5}{8}}\left(\frac{(\xi-\eta-\kappa)^2}{t^2\omega}\right)\right]$$
$$-\mathcal{X}_{1,\omega}\mathcal{X}'_{2,\omega}\left[\frac{3}{16}t^{-\frac{5}{8}}\left(\frac{\xi^2}{t^2\omega}-\frac{1}{2}\right)-t^{-\frac{5}{8}}\left(\frac{\xi^2}{t^2\omega}\right)\right].$$

Hence we have $|\partial_t \mathcal{X}_1| \lesssim t^{-\frac{5}{8}}$, and

$$\left|\frac{\partial_t \mathcal{X}_1}{\Psi'}\right| \lesssim t^{-\frac{1}{4}} |\omega|^{-1}.$$

Lemma 3.3.3. *Assuming $t \geq 1$, we have*

$$\left|\mathcal{O}_2^t [G, \mathcal{Z}, G, \mathcal{Z}]\right| \lesssim D^2\epsilon^2 t^{-\frac{5}{4}} (1+|t|)^{2\delta} \|\nu(t)\|^2_{L^2_{x,y}}. \tag{3.29}$$

Applying the same estimate as Lemma (3.3.2) one can obtain the bound. Hence we have

$$\left|\int_1^T \langle S(-t)V, e_3(t)\rangle_{L^2_{v,y}} dt\right|$$
$$\lesssim \left|\mathcal{O}_1^t [G, \mathcal{Z}, G, \mathcal{Z}]\right|\Big|_{t=1}^T + \int_1^T \left|\mathcal{O}_2^t [G, \mathcal{Z}, G, \mathcal{Z}]\right| dt + \left|t^{-1}\mathcal{O}_1^t [G, \mathcal{Z}, G, \mathcal{Z}]\right| dt$$
$$+ \int_1^T \left|\mathcal{O}_1^t [G_t, \mathcal{Z}, G, \mathcal{Z}]\right| + \left|\mathcal{O}_1^t [G, \mathcal{Z}_t, G, \mathcal{Z}]\right| + \left|\mathcal{O}_1^t [G, \mathcal{Z}, G_t, \mathcal{Z}]\right| + \left|\mathcal{O}_1^t [G, \mathcal{Z}, G, \mathcal{Z}_t]\right| dt.$$

Using the estimates from $\mathcal{O}_1^t$ and $\mathcal{O}_2^t$, the proof of Lemma 3.3.3 is finished.

□

3.4 Almost resonant interactions.

The remaining case corresponds to frequency interactions localized in the region Ω_t^2. Inside this region we will have $|\Psi'(t)| \lesssim t^{-\frac{3}{8}}$, which correspond to almost is resonance both in y and v frequency. It is crucial to have the t^{-1} decay of $\left|\langle S(-t)V, e_4\rangle_{L^2_{x,y}}\right|$. Since we are not able to use a normal form correction to gain extra decay, one needs to use the bilinear Strichartz estimate on the torus. Since the bilinear Strichartz estimate only works on unit time scale, we divide $[1, T]$ into nonoverlapping unit time intervals. Inside each interval, the phase function is almost constant on unit time scale, hence we can replace the phase function by a constant with small errors. Then we do a frequency localization in v. After the frequency localization, the quantity e_4 can be described as the output of interaction of linear waves, with very small errors.

The idea here is to separate $\langle S(-t)V, e_4\rangle_{L^2_{x,y}}$ into two parts:

$$\langle S(-t)V, e_4\rangle_{L^2_{x,y}} := \langle S(-t)V, M(t)\rangle_{L^2_{x,y}} + \langle S(-t)V, \mathrm{err}\,(t)\rangle_{L^2_{x,y}}$$

where $M(t)$ is a product of linear flow when $t \in [T_n, T_{n+1})$ where $T_1 = 1$, $T_{n+1} = T_n + 2\pi$. For $M(t)$ we have the bound

$$\|M\,(t)\|_{L^2_t(T_n,T_{n+1};L^2_{v,y})} \lesssim t^{-1}\,\|G(T_n, v, y)\|^2_{L^\infty_v H^1_y}\,\|V\,(T_n, v, y)\|_{L^2_{v,y}}\,. \tag{3.30}$$

Using the bootstrap assumptions (3.3), (3.4) to bound the errors, we will obtain the estimate

$$\int_1^T \|\mathrm{err}\,(t)\|_{L^2_{v,y}}\,\|\nu\|_{L^2_{x,y}}\,dt \lesssim \epsilon.$$

Lemma 3.4.1. *For $T \geq 1$, we will have*

$$\begin{aligned}\int_1^T \left|\langle S(-t)V, e_4\,(t)\rangle_{L^2_{v,y}}\right| dt \lesssim & \int_1^T D^2\epsilon^2 t^{-1}\,\|\nu\,(t)\|^2_{L^2_{x,y}}\,dt \\ &+ \int_1^T \left[D^{\frac{11}{6}}\epsilon^2 t^{-\frac{65}{64}}\,(1+t)^{\frac{11}{6}\delta} + D^2\epsilon^2 t^{-\frac{25}{24}}\,(1+t)^{2\delta}\right]\|\nu(t)\|^2_{L^2_{x,y}}\,dt \\ &+ \int_1^T \left[D^{\frac{11}{3}}\epsilon^4 t^{-2}\,(1+t)^{\frac{5}{3}\delta} + D^4\epsilon^4 t^{-2}\,(1+t)^{4\delta}\right]\|\nu\,(t)\|^2_{L^2_{x,y}}\,dt.\end{aligned} \tag{3.31}$$

Recall that P^y denotes the frequency projection of y. We may restrict the case to where $\max\{|k_1|, |k_3|\} \leq t^{\frac{1}{16}}$, and since $s > \frac{3d}{2} \geq \frac{3}{2}$, we peel off the high y-freqency in γ:

Lemma 3.4.2. *Define*

$$\begin{aligned} err_1\,(t) := {}& e_4\,(t, v, y) \\ &- t^{-1}\mathcal{F}^{-1}_\xi \sum_{\mathbf{k}} \left[\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})} \iint \mathcal{X}_2 e^{i\Psi(t)}\hat{\tilde{G}}\,(t,\kappa,\mathbf{k}_1)\,\hat{\tilde{\mathcal{Z}}}\,(t,\xi-\eta-\kappa,\mathbf{k}_2)\,\hat{\tilde{G}}\,(t,\eta,\mathbf{k}_3)\,d\kappa d\eta\right] e^{i\mathbf{k}\cdot y},\end{aligned}$$

where

$$\tilde{G} = P^y_{\leq t^{\frac{1}{16}}} G.$$

Then we have

$$\|err_1\,(t)\|_{L^2_{v,y}} \lesssim D^{\frac{11}{6}}\epsilon^2 t^{-\frac{65}{64}}\,(1+|t|)^{\frac{11}{6}\delta}\,\|\nu\|_{L^2_{x,y}}\,. \tag{3.32}$$

Proof.

$$\begin{aligned}\|\mathrm{err}_1\,(t)\|_{L^2_{v,y}} =& \frac{1}{t}\left\|P^y_{\geq t^{\frac{1}{16}}}\gamma\right\|_{L^\infty_v H^\alpha_y}\|\gamma\|_{L^\infty_v H^\alpha_y}\|V\|_{L^2_{v,y}} \\ \lesssim& \frac{1}{t}\left\|P^y_{\geq t^{\frac{1}{16}}}\gamma\right\|^{\frac{1}{6}}_{L^2_{v,y}}\|\gamma\|^{\frac{1}{6}}_{L^2_{v,y}}\|\partial_v\gamma\|_{L^2_{v,y}}\left\|D^{3\alpha}_y\gamma\right\|^{\frac{2}{3}}_{L^2_{v,y}}\|V\|_{L^2_{v,y}} \\ \lesssim& t^{-\frac{65}{64}}\,\|\gamma\|^{\frac{1}{6}}_{L^2_{v,y}}\left\|D^s_y\gamma\right\|^{\frac{5}{6}}_{L^2_{v,y}}\|\partial_v w\|_{L^2_{v,y}}\|V\|_{L^2_{v,y}}\,.\end{aligned}$$

Thus we can reduce the problem to the case $0 \leq \frac{1}{2}\omega \leq t^{\frac{1}{8}}$ since display

$$\frac{1}{2}\omega = \frac{1}{2}\left(|\mathbf{k}_1|^2 + |\mathbf{k}_3|^2 - |\mathbf{k}_2|^2 - |\mathbf{k}_4|^2\right) \leq t^{\frac{1}{8}}.$$

□

Next we consider unit time intervals, and show that we can freeze $\widehat{\widetilde{G}}$ and $\widehat{\mathcal{Z}}$ at end points. First we show that on unit time intervals there are uniform bounds for linearized equation.

Lemma 3.4.3. *For any $s, t \in [T_n, T_{n+1})$, and $s \leq t$, there are the bounds*

$$\|\mathcal{Z}(t) - \mathcal{Z}(s)\|_{L^2_{v,y}} \lesssim D^{\frac{5}{3}}\epsilon^2 t^{-1}(1+t)^{\frac{5}{3}\delta}\|\mathcal{Z}(t)\|_{L^2_{v,y}}, \tag{3.33}$$

$$\int_s^t \|\mathcal{Z}(\sigma)\|_{L^2_{v,y}} d\sigma \lesssim \|\mathcal{Z}(t)\|_{L^2_{v,y}}. \tag{3.34}$$

Proof. Since

$$\|\mathcal{Z}(t) - \mathcal{Z}(s)\|_{L^2_{v,y}} \lesssim \int_s^t \|\partial_t \mathcal{Z}(\sigma)\| d\sigma \lesssim \int_s^t \sigma^{-1}\|w(\sigma)\|^2_{L^\infty_v H^\alpha_y}\|\mathcal{Z}(\sigma)\|_{L^2_{v,y}} d\sigma,$$

applying (3.3), (3.4), there is the bound (3.33). By applying (3.33)

$$\begin{aligned}\int_s^t \|\mathcal{Z}(\sigma)\|_{L^2_{v,y}} d\sigma &\lesssim \|\mathcal{Z}(t)\|_{L^2_{x,y}} + \int_s^t \|\mathcal{Z}(\sigma) - \mathcal{Z}(t)\|_{L^2_{v,y}} d\sigma \\ &\lesssim \|\mathcal{Z}(t)\|_{L^2_{x,y}} + D^{\frac{5}{3}}\epsilon^2 t^{-1}(1+t)^{\frac{5}{3}\delta}\int_s^t \|\mathcal{Z}(\sigma)\|_{L^2_{v,y}} d\sigma.\end{aligned}$$

By recursion, when T_n is large hence t is large, the inequality (3.34) holds.

□

Lemma 3.4.4. *For $t \in [T_n, T_{n+1})$, define*

$\tilde{M}(t, v, y) :=$

$$t^{-1}\mathcal{F}_\xi^{-1}\sum_{\mathbf{k}}\left[\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\iint \mathcal{X}_2(T_n)\, e^{i\Psi(T_n)}\widehat{\widetilde{G}}(T_n, \kappa, \mathbf{k}_1)\,\overline{\widehat{\widetilde{\mathcal{Z}}}}(T_n, \xi-\eta-\kappa, \mathbf{k}_2)\,\widehat{\widetilde{G}}(T_n, \eta, \mathbf{k}_3)\, d\kappa d\eta\right] e^{i\mathbf{k}\cdot y},$$

and

$$err_2(t) = e_4(t, v, y) - err_1(t) - \tilde{M}(t, v, y).$$

Then there is the bound

$$\|err_2(t)\|_{L^2_{v,y}} \lesssim \left[D^2\epsilon^2 t^{-\frac{5}{4}}(1+t)^{2\delta} + D^4\epsilon^4 t^{-2}(1+t)^{4\delta}\right]\|\nu\|_{L^2_{x,y}}. \tag{3.35}$$

Proof. Define

$$\tilde{\mathcal{R}}^t\left[f_1,f_2,f_3\right]:=\mathcal{F}_\xi^{-1}\sum_{\mathbf{k}}\left[\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\iint\mathcal{X}_2\left(t\right)e^{i\Psi(t)}\widehat{f_1}\left(\kappa,\mathbf{k}_1\right)\widehat{\overline{f_2}}\left(\xi-\eta-\kappa,\mathbf{k}_2\right)\widehat{f_3}\left(\eta,\mathbf{k}_3\right)d\kappa d\eta\right]e^{i\mathbf{k}\cdot y}.$$

Then we have

$$\mathrm{err}_2(t)=t^{-1}\int_{T_n}^t\left(\left(\partial_\sigma\tilde{\mathcal{R}}^\sigma\right)\left[\tilde{G},\mathcal{Z},\tilde{G}\right]+\tilde{\mathcal{R}}^\sigma\left[\tilde{G}_t,\mathcal{Z},\tilde{G}\right]+\tilde{\mathcal{R}}^\sigma\left[\tilde{G},\mathcal{Z}_t,\tilde{G}\right]+\tilde{\mathcal{R}}^\sigma\left[\tilde{G},\mathcal{Z},\tilde{G}_t\right]\right)d\sigma.$$

By a direct computation,

$$\left(\partial_t\tilde{\mathcal{R}}^t\right)\left[G,\mathcal{Z},G\right]:=$$

$$\mathcal{F}_\xi^{-1}\sum_{\mathbf{k}}\left[\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\iint e^{i\Psi(t)}\left[i\Psi'\mathcal{X}_2+\partial_t\mathcal{X}_2\right](t)\widehat{\tilde{G}}\left(t,\kappa,\mathbf{k}_1\right)\widehat{\overline{\mathcal{Z}}}\left(t,\xi-\eta-\kappa,\mathbf{k}_2\right)\widehat{\tilde{G}}\left(t,\eta,\mathbf{k}_3\right)d\kappa d\eta\right]e^{i\mathbf{k}\cdot y}.$$

Using the bound on ω, we have

$$|\Psi'(t)\mathcal{X}_2|\lesssim\omega t^{-\frac{3}{8}}\lesssim t^{-\frac{1}{4}},$$

and

$$|\partial_t\mathcal{X}_2|=|-\partial_t\mathcal{X}_1|\lesssim t^{-\frac{5}{8}}.$$

Therefore by Young's inequality and Parseval's indentity we have

$$\begin{aligned}&\left\|\left(\partial_t\tilde{\mathcal{R}}^t\right)\left[G,\mathcal{Z},G\right]\right\|_{L^2_{v,y}}\\ \lesssim&\left\|\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\iint\left|i\Psi'\mathcal{X}_2+\partial_t\mathcal{X}_2\right|\left|\widehat{\tilde{G}}\left(t,\kappa,\mathbf{k}_1\right)\right|\left|\widehat{\overline{\mathcal{Z}}}\left(t,\xi-\eta-\kappa,\mathbf{k}_2\right)\right|\left|\widehat{\tilde{G}}\left(t,\eta,\mathbf{k}_3\right)\right|d\kappa d\eta\right\|_{L^2_\xi l^2_{\mathbf{k}}}\\ \lesssim&\left\|i\Psi'\mathcal{X}_2+\partial_t\mathcal{X}_2\right\|_{L^\infty_\xi l^\infty_{\mathbf{k}}}\left\|\widehat{\tilde{G}}(t)\right\|^2_{L^1_\xi l^1_{\mathbf{k}}}\left\|\widehat{\overline{\mathcal{Z}}}\left(t\right)\right\|_{L^2_\xi l^2_{\mathbf{k}}}.\end{aligned}$$

Hence

$$\begin{aligned}\left\|\left(\partial_t\tilde{\mathcal{R}}^t\right)\left[\tilde{G},\mathcal{Z},\tilde{G}\right]\right\|_{L^2_{v,y}}&\lesssim t^{-\frac{1}{4}}\left\|\widehat{G}\right\|^2_{L^1_\xi h^\alpha_{\mathbf{k}}}\left\|\widehat{\overline{\mathcal{Z}}}\right\|_{L^2_\xi h^2_{\mathbf{k}}}\lesssim t^{-\frac{1}{4}}\left\|\widehat{G}\left(t,\xi,\mathbf{k}\right)\right\|^{\frac{2}{3}}_{h^s_{\mathbf{k}}L^2_\xi}\left\|\xi\widehat{G}\left(t,\xi,\mathbf{k}\right)\right\|^{\frac{4}{3}}_{l^2_{\mathbf{k}}L^2_\xi}\left\|\mathcal{Z}\right\|_{L^2_{v,y}}\\ &\lesssim t^{-\frac{1}{4}}D^2\epsilon^2\left(1+t\right)^{2\delta}\left\|\nu\right\|_{L^2_{x,y}}.\end{aligned}$$

Similarly, we have

$$\left\|\tilde{\mathcal{R}}^t\left[\tilde{G}_t,\mathcal{Z},\tilde{G}\right]\right\|_{L^2_{v,y}}\lesssim\left\|\mathcal{X}_2\right\|_{L^\infty_\xi l^\infty_{\mathbf{k}}}\left\|\widehat{\tilde{G}_t}(t)\right\|_{L^1_\xi l^1_{\mathbf{k}}}\left\|\widehat{\tilde{G}}(t)\right\|_{L^1_\xi l^1_{\mathbf{k}}}\left\|\widehat{\overline{\mathcal{Z}}}\left(t\right)\right\|_{L^2_\xi l^2_{\mathbf{k}}}.$$

Hence by a similar estimate as Lemma 3.3.2,

$$\left\|\tilde{\mathcal{R}}^t\left[\tilde{G}_t,\mathcal{Z},\tilde{G}\right]\right\|_{L^2_{v,y}} \lesssim t^{-1}\|\widehat{w}\|^4_{L^1_\xi h^\alpha_{\mathbf{k}}}\left\|\widehat{\mathcal{Z}}\right\|_{L^2_\xi h^2_{\mathbf{k}}} \lesssim t^{-1}\|\widehat{w}(t,\xi,\mathbf{k})\|^{\frac{4}{3}}_{h^s_{\mathbf{k}}L^2_\xi}\|\xi\widehat{w}(t,\xi,\mathbf{k})\|^{\frac{8}{3}}_{l^2_{\mathbf{k}}L^2_\xi}\|\mathcal{Z}\|_{L^2_{v,y}}$$
$$\lesssim t^{-1}D^4\epsilon^4(1+t)^{4\delta}\|\nu\|_{L^2_{x,y}},$$

The same estimate yields

$$\left\|\tilde{\mathcal{R}}^t\left[\tilde{G},\mathcal{Z}_t,\tilde{G}\right]\right\|_{L^2_{v,y}} \lesssim t^{-1}D^4\epsilon^4(1+t)^{4\delta}\|\nu\|_{L^2_{x,y}}.$$

Therefore we have

$$\|\mathrm{err}_2(t)\|_{L^2_{v,y}} \lesssim \int_{T_n}^t\left[D^2\epsilon^2\sigma^{-\frac{5}{4}}(1+\sigma)^{2\delta}+D^4\epsilon^4\sigma^{-2}(1+\sigma)^{4\delta}\right]\|\nu(\sigma)\|_{L^2_{x,y}}\,d\sigma. \tag{3.36}$$

In order to switch this quantity into a form where Gronwall's inequality can be applied, we use Lemma 3.4.3.

Using that $\|\mathcal{Z}(t)\|_{L^2_{v,y}} = \|\nu(\sigma)\|_{L^2_{x,y}}$ and Lemma 3.4.3, we can switch the bound (3.36) to

$$\|\mathrm{err}_2(t)\|_{L^2_{v,y}} \lesssim \left[D^2\epsilon^2t^{-\frac{5}{4}}(1+t)^{2\delta}+D^4\epsilon^4t^{-2}(1+t)^{4\delta}\right]\|\nu(t)\|_{L^2_{x,y}},$$

therefore proving the lemma.

□

Since from (3.17), we can define the interval $A(\omega)$ corresponding to the cutoff function $\mathcal{X}_{1,\omega}(T_n)$ by

$$A(\omega) := \left\{\xi : \left|\frac{(\xi-\kappa-\eta)^2}{T_n^2\omega}-\frac{1}{2}\right| < 2T_n^{-\frac{3}{8}}\right\}, \tag{3.37}$$

to be the $v-$frequencies contributing to the Ω_t^2 region for a fixed ω. Then we have the following equality for fixed $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$:

$$\begin{aligned}&\iint \mathcal{X}_2(T_n)\,e^{i\Psi(T_n)}\hat{\tilde{G}}(T_n,\kappa,\mathbf{k}_1)\,\hat{\tilde{\mathcal{Z}}}(T_n,\xi-\eta-\kappa,\mathbf{k}_2)\,\hat{\tilde{G}}(T_n,\eta,\mathbf{k}_3)\,d\kappa d\eta\\ =&\mathcal{X}_{2,\omega}(T_n,\xi)\iint e^{i\Psi(T_n)}\hat{\tilde{G}}(T_n,\kappa,\mathbf{k}_1)\,\widehat{\mathcal{Z}_{A(\omega)}}(T_n,\xi-\eta-\kappa,\mathbf{k}_2)\,\hat{\tilde{G}}(T_n,\eta,\mathbf{k}_3)\,d\kappa d\eta.\end{aligned} \tag{3.38}$$

Since the $\mathcal{X}_{2,\omega}(T_n,\xi)$ factor does not affect the computation of $L^2_{v,y}$ norm, we can omit it. Inside this interval $A(\omega)$,

$$(\xi-(\eta+\kappa))^2 = \frac{T_n^2}{2}\omega+\mathcal{O}\left(\omega T_n^{\frac{13}{8}}\right),$$

hence

$$\xi = \pm T_n\sqrt{\frac{\omega}{2}}\sqrt{1+\mathcal{O}\left(T_n^{-\frac{3}{8}}\right)}+(\eta+\kappa).$$

By Taylor expansion $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ for x very small and $\sqrt{\frac{\omega}{2}} \leq T_n^{\frac{1}{16}}$,

$$\left|\xi \mp T_n\sqrt{\frac{\omega}{2}}\right| \lesssim |\eta+\kappa| + \mathcal{O}\left(T_n^{\frac{11}{16}}\right).$$

For each $A(k)$ we compute the distance of sets by the distance of their "center":

$$\left|T_n\sqrt{\frac{k}{2}} - T_n\sqrt{\frac{k+1}{2}}\right| \approx \frac{T_n}{\sqrt{k}\sqrt{k+1}}.$$

Since $\omega < T_n^{\frac{1}{16}}$, the centers are separated by distance at least $T_n^{\frac{15}{16}}$, and the width of each set is at most $T_n^{\frac{11}{16}}$. Hence the sets are disjoint when T_n is sufficiently large.

Since we are not able to directly localize to this interval, we instead divide $A(\omega)$ into several disjoint intervals, where each interval has length $\sqrt{T_n}$. Denote these intervals by $A(\omega, k)$, where $k \in \mathbb{Z}$,

$$A(\omega, k) := \left\{\xi : \left|\xi \mp T_n\sqrt{\frac{\omega}{2}} - 2k\sqrt{T_n}\right| \leq \sqrt{T_n}\right\}.$$

Each $A(\omega)$ has length $T_n^{\frac{11}{16}}$ and each $A(\omega, k)$ has length $\sqrt{T_n}$, therefore $|k| \lesssim T_n^{\frac{11}{16}}/T_n^{\frac{1}{2}} \lesssim T_n^{\frac{3}{16}}$. By (3.38), the quantity $\tilde{M}$ can be written as

$$\tilde{M}(t,v,y) := t^{-1}\mathcal{F}_\xi^{-1}\sum_{\mathbf{k}}\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_{k} \left[\iint e^{i\Psi(T_n)}\hat{\tilde{G}}(T_n,\kappa,\mathbf{k}_1)\,\widehat{\overline{\mathcal{Z}_{A(\omega,k)}}}(T_n,\xi-\eta-\kappa,\mathbf{k}_2)\,\hat{\tilde{G}}(T_n,\eta,\mathbf{k}_3)\,d\kappa d\eta\right]e^{i\mathbf{k}\cdot y}.$$

After the frequency localization, we can transform the effect of phase function into translation of the v variable with small errors. In the following computation, let ξ_0 be either $T_n\sqrt{\frac{\omega}{2}} + 2k\sqrt{T_n}$ or $-T_n\sqrt{\frac{\omega}{2}} + 2k\sqrt{T_n}$,

$$\begin{aligned}&\frac{1}{2T_n}\left((\xi-\eta-\kappa)^2 + \xi^2\right)\\ =&\frac{\xi^2}{T_n} - \frac{\xi_0(\eta+\kappa)}{T_n} - \frac{(\xi-\xi_0)(\eta+\kappa)}{T_n} + \frac{(\eta+\kappa)^2}{2T_n}.\end{aligned}$$

The last factor $\frac{(\eta+\kappa)^2}{2T_n} \lesssim 1$, hence it is negligible. By a direct computation of Fourier transformation, we have

$$\begin{aligned}&\mathcal{F}_\xi^{-1}\iint \exp\left(i\left(\frac{\xi^2}{T_n} - \frac{\xi_0(\eta+\kappa)}{T_n} - \frac{(\xi-\xi_0)(\eta+\kappa)}{T_n}\right)\right)\widehat{f_1}(\kappa)\,\widehat{\overline{f_2}}(\xi-\eta-\kappa)\,\widehat{f_3}(\eta)\,d\kappa d\eta\\ =&e^{-i\frac{\partial_v^2}{T_n}}\left[f_1\left(v-\frac{\xi_0}{T_n}\right)f_2(v)f_3\left(v-\frac{\xi_0}{T_n}\right)\right].\end{aligned}$$

Hence we can split $\tilde{M}$ into three parts:

$$\tilde{M}(t,v,y) := M_+(t,v,y) + M_-(t,v,y) + \mathrm{err}_3(t,v,y).$$

Where M_+ and M_- correspond to positive freqency localization $A^+(\omega,k) := A(\omega,k)\cap\{\xi > 0\}$ and negative freqency localization $A^-(\omega,k) := A(\omega,k)\cap\{\xi<0\}$.

$$\begin{aligned}
&M_+(t,v,y)\\
&:= t^{-1}\mathcal{F}_\xi^{-1}\sum_{\mathbf{k}}\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_{k}\\
&\qquad e^{-i\frac{\partial_v^2}{T_n}+i\frac{\omega}{2}T_n}\sum_k \tilde{G}\left(T_n, v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_1\right)\overline{\mathcal{Z}_{A^+(\omega,k)}}(T_n,v,\mathbf{k}_2)\,\tilde{G}\left(T_n, v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_3\right),
\end{aligned}$$

$$\begin{aligned}
&M_-(t,v,y)\\
&:= t^{-1}\mathcal{F}_\xi^{-1}\sum_{\mathbf{k}}\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_{k}\\
&\qquad e^{-i\frac{\partial_v^2}{T_n}+i\frac{\omega}{2}T_n}\sum_k \tilde{G}\left(T_n, v+\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_1\right)\overline{\mathcal{Z}_{A^-(\omega,k)}}(T_n,v,\mathbf{k}_2)\,\tilde{G}\left(T_n, v+\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_3\right).
\end{aligned}$$

The error from transforming the effect of phase function into spatial translation is given by

$$\mathrm{err}_3(t,v,y) := \tilde{M}(t,v,y) - M_+(t,v,y) - M_-(t,v,y).$$

Lemma 3.4.5. *For $t\geq 1$, the error from approximating the phase function by spatial translation of v has the bound*

$$\left\|err_3(t,v,y)\right\|_{L^2_{v,y}} \lesssim D^2\epsilon^2 t^{-\frac{25}{24}}(1+t)^{2\delta}\left\|\nu(t)\right\|_{L^2_{x,y}}. \tag{3.39}$$

Proof. From the inequality

$$\left|-\frac{(\xi-\xi_0)(\eta+\kappa)}{T_n}+\frac{(\eta+\kappa)^2}{2T_n}\right| \lesssim \frac{|\eta+\kappa|}{\sqrt{T_n}}+\frac{(\eta+\kappa)^2}{2T_n},$$

thus we know that

$$\begin{aligned}
&\left\|\iint|\eta+\kappa|^{\frac{1}{12}}\left|\widehat{\tilde{G}}(T_n,\kappa,\mathbf{k}_1)\,\widehat{\overline{\mathcal{Z}_{A(\omega)}}}(T_n,\xi-\eta-\kappa,\mathbf{k}_2)\,\widehat{\tilde{G}}(T_n,\eta,\mathbf{k}_3)\right|d\kappa d\eta\right\|_{L^2_\xi}\\
\lesssim& T_n^{-\frac{25}{24}}\left\||\xi|^{\frac{1}{12}}\widehat{\tilde{G}}(T_n,\xi,\mathbf{k}_1)\right\|_{L^1_\xi}\left\|\widehat{\overline{\mathcal{Z}_{A(\omega)}}}(T_n,\xi,\mathbf{k}_2)\right\|_{L^2_\xi}\left\|\widehat{\tilde{G}}(T_n,\xi,\mathbf{k}_3)\right\|_{L^1_\xi}\\
&+T_n^{-\frac{25}{24}}\left\|\widehat{\tilde{G}}(t,\xi,\mathbf{k}_1)\right\|_{L^1_\xi}\left\|\widehat{\overline{\mathcal{Z}_{A(\omega)}}}(T_n,\xi,\mathbf{k}_2)\right\|_{L^2_\xi}\left\||\xi|^{\frac{1}{12}}\widehat{\tilde{G}}(T_n,\xi,\mathbf{k}_3)\right\|_{L^1_\xi}.
\end{aligned}$$

Therefore when summing over $\mathcal{M}(\mathbf{k})$ by using Lemma ??, and the fact that the sets $A(\omega)$ are disjoint for different ω, we have

$$\|\mathrm{err}_3(t,v,y)\|_{L^2_{v,y}} \lesssim t^{-\frac{25}{24}} \left\|\langle\xi\rangle^{\frac{2}{3}} \widehat{\tilde{G}}(T_n,\xi,\mathbf{k})\right\|^2_{l^1_{\mathbf{k}}L^2_\xi} \left(\sum_j \left\|\widehat{\mathcal{Z}_{A(j)}}(T_n,\xi,\mathbf{k})\right\|^2_{l^2_{\mathbf{k}}L^2_\xi}\right)^{\frac{1}{2}}$$
$$\lesssim t^{-\frac{25}{24}} \left\|\widehat{\tilde{G}}(T_n,\xi,\mathbf{k})\right\|^{\frac{2}{3}}_{h^s_{\mathbf{k}}L^2_\xi} \left\|\xi\widehat{\tilde{G}}(T_n,\xi,\mathbf{k})\right\|^{\frac{4}{3}}_{l^2_{\mathbf{k}}L^2_\xi} \|\mathcal{Z}(T_n)\|_{L^2_{v,y}} \lesssim D^2\epsilon^2 t^{-\frac{25}{24}}(1+|t|)^2\|\nu(t)\|_{L^2_{x,y}}.$$

□

To simplify the next computations, we consider the case where $\xi>0$; the case $\xi<0$ is similar by changing the sign. First notice that while doing summation over all the possible triples, due to the fact that the $A(\omega)$'s are disjoint, the $e^{i\frac{\omega}{2}T_n}$does not affect the computation of $L^2_{v,y}$ norm

$$\Bigg\|\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_k e^{-i\frac{\partial_v^2}{T_n}+i\frac{\omega}{2}T_n}\tilde{G}\left(T_n, v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_1\right)$$
$$\overline{\mathcal{Z}_{A^+(\omega,k)}}(T_n,v,\mathbf{k}_2)\,\tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_3\right)\Bigg\|_{L^2_v}$$
$$=\left\|\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_k \tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_1\right)\overline{\mathcal{Z}_{A^+(\omega,k)}}(T_n,v,\mathbf{k}_2)\,\tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_3\right)\right\|_{L^2_v}.$$

Hence it suffices only to consider the L^2 norm of the following function

$$M(t,v,y)=t^{-1}\sum_{\mathbf{k}}\sum_{\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_k$$
$$\Bigg[\tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_1\right)\overline{\mathcal{Z}_{A^+(\omega,k)}}(T_n,v,\mathbf{k}_2)\,\tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_3\right)$$
$$+\tilde{G}\left(T_n,v+\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_1\right)\overline{\mathcal{Z}_{A^-(\omega,k)}}(T_n,v,\mathbf{k}_2)\,\tilde{G}\left(T_n,v+\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_3\right)\Bigg]e^{i\mathbf{k}\cdot y}.$$

When computing the L^2_v norm, we can apply a shift to v and obtain the same value,

$$\left\|\tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_1\right)\overline{\mathcal{Z}_{A^+(\omega,k)}}(T_n,v,\mathbf{k}_2)\,\tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_3\right)\right\|_{L^2_v}$$
$$=\left\|\tilde{G}(T_n,v,\mathbf{k}_1)\,\overline{\mathcal{Z}_{A^+(\omega,k)}}\left(t,v+\sqrt{\frac{\omega}{2}}+\frac{2k}{\sqrt{T_n}},\mathbf{k}_2\right)\tilde{G}(T_n,v,\mathbf{k}_3)\right\|_{L^2_v}.$$

Since dist $\left(A^{+}\left(\omega,k\right),A^{+}\left(\omega,k+5\right)\right)\geq 8\sqrt{t}$, if we sort the k into five groups,and the frequency for different groups are disjoint, there is the equality

$$\begin{aligned}&\left\|\sum_{\omega\neq0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_{m}\tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2(5m+i)}{\sqrt{T_n}},\mathbf{k}_1\right)\right.\\&\qquad\left.\overline{\mathcal{Z}_{A^+(\omega,5m+i)}}\left(T_n,v,\mathbf{k}_2\right)\tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2(5m+i)}{\sqrt{T_n}},\mathbf{k}_3\right)\right\|_{L^2_v}\\&=\left\|\sum_{\omega\neq0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_{m}\tilde{G}\left(T_n,v,\mathbf{k}_1\right)\overline{\mathcal{Z}_{A^+(\omega,5m+i)}}\left(T_n,v+\sqrt{\frac{\omega}{2}}+\frac{2(5m+i)}{\sqrt{T_n}},\mathbf{k}_2\right)\tilde{G}\left(T_n,v,\mathbf{k}_3\right)\right\|_{L^2_v},\end{aligned}$$

where $i=0,1,2,3,4$. Hence we have the following bound

$$\begin{aligned}&\left\|\sum_{\omega\neq0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_{k}\tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_1\right)\overline{\mathcal{Z}_{A^+(\omega,k)}}\left(T_n,v,\mathbf{k}_2\right)\tilde{G}\left(T_n,v-\sqrt{\frac{\omega}{2}}-\frac{2k}{\sqrt{T_n}},\mathbf{k}_3\right)\right\|_{L^2_v}\\&\lesssim\left\|\sum_{\omega\neq0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_{k}\tilde{G}\left(T_n,v,\mathbf{k}_1\right)\overline{\mathcal{Z}_{A^+(\omega,k)}}\left(T_n,v+\sqrt{\frac{\omega}{2}}+\frac{2k}{\sqrt{T_n}},\mathbf{k}_2\right)\tilde{G}\left(T_n,v,\mathbf{k}_3\right)\right\|_{L^2_v}.\end{aligned}$$

Moreover we define the function

$$\mathcal{Z}_+\left(\sigma,v,\mathbf{k}_2\right)(T):=\sum_{1\leq j\leq T_n^{\frac{1}{16}}}\sum_{k}e^{ij\sigma/2}\mathcal{Z}_{A^+(j,k)}\left(T_n,v+\sqrt{\frac{j}{2}}+\frac{2k}{\sqrt{T_n}},\mathbf{k}_2\right),$$

$$\mathcal{Z}_-\left(\sigma,v,\mathbf{k}_2\right)(T):=\sum_{1\leq j\leq T_n^{\frac{1}{16}}}\sum_{k}e^{ij\sigma/2}\mathcal{Z}_{A^-(j,k)}\left(T_n,v-\sqrt{\frac{j}{2}}+\frac{2k}{\sqrt{T_n}},\mathbf{k}_2\right),$$

where $\sigma\in[0,2\pi)$ which is akin to $\sigma=t-T_n$. Notice that when T_n is sufficiently large we have each $A^+(j)$ are disjoint hence $\|\mathcal{Z}_+\|_{L^2_{v,y}}\leq\|V\|_{L^2_{v,y}}$. Using that the distance between $A^+(j)$ and $A^+(j+1)$ is greater than $\sqrt{t}$ again, it follows that $\mathcal{F}_vG\overline{\mathcal{Z}_{A_j}}$ also have disjoint support for different j.

Lemma 3.4.6. *For $t\geq1$, there is the bound*

$$\|M(t)\|_{L^2_{v,y}}\lesssim t^{-1}\left\|e^{-i\sigma\triangle_y/2}\left[\left(e^{i\sigma\triangle_y/2}\tilde{G}\left(T_n,v,y\right)\right)^2\overline{e^{i\sigma\triangle_y/2}\left[\mathcal{Z}_++\mathcal{Z}_-\right](\sigma,v,y)}\right]\right\|_{L^2_\sigma\left(\mathbb{T};L^2_{v,y}\right)}.$$

Proof. Given any $\varphi\in L^2_{v,y}$,

$$
\begin{aligned}
&\left\langle \varphi\left(v,y\right), e^{-i\sigma\triangle_y/2}\left[\left(e^{i\sigma\triangle_y/2}\tilde{G}\right)^2\left(T_n,v,y\right)\overline{e^{i\sigma\triangle_y/2}\mathcal{Z}_+}\left(\sigma,v,y\right)\right]\right\rangle_{L^2_{v,y}}\\
=&\left\langle e^{i\sigma\triangle_y/2}\varphi\left(v,\mathbf{k}\right), \left(e^{i\sigma\triangle_y/2}\tilde{G}\right)^2\left(T,v,y\right)\overline{e^{i\sigma\triangle_y/2}\mathcal{Z}_+}\left(\sigma,v,y\right)\right\rangle_{L^2_v l^2_k}\\
=&\sum_{\omega\neq0}\sum_{\Gamma_\omega}\sum_{1\leq j\leq T_n^{\frac{1}{16}}} e^{i(\omega-j)\sigma/2}\Big\langle\varphi\left(v,\mathbf{k}\right),\tilde{G}\left(T_n,v,\mathbf{k}_1\right),\\
&\qquad\sum_k\overline{\mathcal{Z}_{A^+(j,k)}}\left(T,v+\sqrt{\frac{j}{2}}+\frac{2k}{\sqrt{T_n}},\mathbf{k}_4\right)\tilde{G}\left(T,v,\mathbf{k}_3\right)\Big\rangle_{L^2_\xi L^2_y}
\end{aligned}
$$

Only when $j=\omega=|k_1|^2-|k_2|^2+|k_3|^2-|k_4|^2$ can the integral in time be nonzero. In other cases there will be $e^{it(j-\omega)/2}$ in front the inner product and integrating in t with unit time interval will be 0. Hence

$$
\begin{aligned}
&t^{-1}\int_0^{2\pi}\left\langle \varphi\left(v,y\right), e^{-i\sigma\triangle_y/2}\left[\left(e^{i\sigma\triangle_y/2}\tilde{G}\right)^2\left(T_n,v,y\right)\overline{e^{i\sigma\triangle_y/2}\mathcal{Z}_+}\left(\sigma,v,y\right)\right]\right\rangle_{L^2_{v,y}}d\sigma\\
=&\left\langle\varphi(v,y),t^{-1}\sum_{\omega\neq0}\sum_{\Gamma_\omega(\mathbf{k})}\sum_k\tilde{G}\left(T_n,v,\mathbf{k}_1\right)\overline{\mathcal{Z}_{A^+(\omega,k)}}\left(t,v+\sqrt{\frac{\omega}{2}}+\frac{2k}{\sqrt{T_n}},\mathbf{k}_2\right)\tilde{G}\left(T_n,v,\mathbf{k}_3\right)\right\rangle_{L^2_{v,y}}.
\end{aligned}
$$

Since φ is an arbitrary function, by the definition of L^2-norm , we know that the lemma holds.

□

Lemma 3.4.7. *For any $t\in[T_n,T_{n+1})$, there is the bound*

$$
\|M\left(t\right)\|_{L^2_{v,y}}\lesssim t^{-1}\left\|G(T_n,v,y)\right\|^2_{L^\infty_v H^1_y}\left\|V\left(T_n,v,y\right)\right\|_{L^2_{v,y}}. \tag{3.40}
$$

Proof. From the previous lemma, for any $\varphi\in L^2_{v,y}$ there is the inequality

$$
\begin{aligned}
&t^{-1}\left\langle \varphi\left(v,y\right), e^{-i\sigma\triangle_y/2}\left[\left(e^{i\sigma\triangle_y/2}\tilde{G}\right)^2\left(T_n,v,y\right)\overline{e^{i\sigma\triangle_y/2}\mathcal{Z}_+}\left(\sigma,v,y\right)\right]\right\rangle_{L^2_{\sigma,v,y}}\\
\lesssim& t^{-1}\left\|\overline{e^{i\sigma\triangle_y/2}\varphi}e^{i\sigma\triangle_y/2}\tilde{G}\left(T_n,v,y\right)\right\|_{L^2_{\sigma,v,y}}\left\|e^{i\sigma\triangle_y/2}\tilde{G}\left(T_n,v,y\right)\overline{e^{i\sigma\triangle_y/2}\mathcal{Z}_+}\left(\sigma,v,y\right)\right\|_{L^2_{\sigma,v,y}}.
\end{aligned}
$$

Thus by Fourier transformation and disjointness of domains for different j, applying the bilinear Strichartz estimate we will have the L^2 bound:

$$\begin{aligned}
&\left\| \left[e^{i\sigma\triangle_y/2}\tilde{G}\left(T_n,v,y\right)\right] \overline{e^{i\sigma\triangle_y/2}\mathcal{Z}_+}\left(\sigma,v,y\right)\right\|_{L^2_\sigma\left(\mathbb{T};L^2_{v,y}\right)} \\
&= \sum_{1\le j\le T^{\frac{1}{16}}} \left\| e^{ij\sigma/4}\mathcal{F}_v\left(e^{i\sigma\triangle_y/2}\tilde{G}\left(T_n,v,y\right)e^{-i\sigma\triangle_y/2}\sum_k \overline{\mathcal{Z}_{A^+(j,k)}}\left(T_n, v+\sqrt{\frac{j}{2}+\frac{2k}{\sqrt{T_n}}},y\right)\right)\right\|_{L^2_\sigma\left(\mathbb{T};L^2_{\xi,y}\right)} \\
&= \left\| e^{i\sigma\triangle_y/2}\tilde{G}\left(T_n,v,y\right)e^{-i\sigma\triangle_y/2}\left[\sum_{1\le j\le T^{\frac{1}{16}}}\sum_k \overline{\mathcal{Z}_{A^+(j,k)}}\left(T_n, v+\sqrt{\frac{j}{2}+\frac{2k}{\sqrt{T_n}}},y\right)\right]\right\|_{L^2_\sigma\left(\mathbb{T};L^2_{v,y}\right)} \\
&\lesssim \left\|\tilde{G}(T,v,y)\right\|_{L^\infty_v H^1_y}\left\|\sum_{1\le j\le T^{\frac{1}{16}}}\sum_k \overline{\mathcal{Z}_{A^+(j,k)}}\left(T_n, v+\sqrt{\frac{j}{2}+\frac{2k}{\sqrt{T_n}}},y\right)\right\|_{L^2_{v,y}} \\
&\lesssim \left\|G(T_n,v,y)\right\|_{L^\infty_v H^1_y}\left\|V\left(T_n,v,y\right)\right\|_{L^2_{v,y}}.
\end{aligned}$$

The proof of bilinear Strichartz estimate on $L^2_{\sigma,y}$ is given in [18]. Similarly,

$$\left\|\overline{e^{i\sigma\triangle_y/2}\varphi}e^{i\sigma\triangle_y/2}\tilde{G}\left(T_n,v,y\right)\right\|_{L^2_\sigma\left(\mathbb{T};L^2_{v,y}\right)} \lesssim \left\|\tilde{G}(T_n,v,y)\right\|_{L^\infty_v H^1_y}\|\varphi\|_{L^2_{v,y}} \lesssim \|G(T_n,v,y)\|_{L^\infty_v H^1_y}\|\varphi\|_{L^2_{v,y}}.$$

Since φ is an arbitrary $L^2_{v,y}$ function, by the property of L^2-norm we know that for $t\in[T_n,T_{n+1})$,

$$\|M\left(t\right)\|_{L^2_{v,y}} \lesssim t^{-1}\left\|\gamma(T_n,v,y)\right\|^2_{L^\infty_v H^1_y}\left\|\nu\left(T_n,v,y\right)\right\|_{L^2_{x,y}}.$$

□

In order to switch back to estimate of integration in time, by (3.33) for any $t\in[T_n,T_{n+1})$ we have

$$\begin{aligned}
\|M\left(t\right)\|_{L^2_{v,y}} &\lesssim D^2\epsilon^2 t^{-1}\left\|\mathcal{Z}\left(T_n,v,y\right)\right\|_{L^2_{v,y}} \\
&\lesssim D^2\epsilon^2 t^{-1}\|\nu\left(t\right)\|_{L^2_{x,y}} + D^{\frac{11}{3}}\epsilon^4 t^{-2}\left(1+t\right)^{\frac{5}{3}\delta}\|\nu\left(t\right)\|_{L^2_{x,y}}.
\end{aligned}$$

Since

$$e_4(t,v,y) = M_+(t,v,y)+M_-(t,v,y)+\mathrm{err}_1(t,v,y)+\mathrm{err}_2(t,v,y)+\mathrm{err}_3(t,v,y),$$

combining all the estimates (3.32), (3.35), (3.39) together, the estimate (3.31) is proved.

$$\begin{aligned}
\|e_4\left(t\right)\|_{L^2_{v,y}} &\lesssim D^2\epsilon^2 t^{-1}\|\nu\left(t\right)\|_{L^2_{x,y}} + D^{\frac{11}{3}}\epsilon^4 t^{-2}\left(1+t\right)^{\frac{5}{3}\delta}\|\nu\left(t\right)\|_{L^2_{x,y}} \\
&\quad + \|\mathrm{err}_1\left(t\right)\|_{L^2_{v,y}} + \|\mathrm{err}_2\left(t\right)\|_{L^2_{v,y}} + \|\mathrm{err}_3\left(t\right)\|_{L^2_{v,y}}.
\end{aligned}$$

3.5 Proof of Proposition 3.0.2

By (3.19), we have the bound

$$\|\nu(T)\|_{L^2_{x,y}}^2 \le \|\nu(1)\|_{L^2_{x,y}}^2 + 2\int_1^T \left(\|e_1\|_{L^2_{v,y}} + \|e_2\|_{L^2_{v,y}} + \|e_4\|_{L^2_{v,y}}\right)\|\nu(t)\|_{L_{x,y}}\, dt$$
$$+ \left|\int_1^T \langle \nu(t), e_3\rangle_{L^2_{x,y}}\, dt\right|.$$

Combining (3.20),(3.21),(3.25) and (3.31) together, we assume that D is a sufficiently large positive constant which only depends on d and s. Noticing that the choice of D does not depend on ϵ and T. Therefore the following inequality holds:

$$\left[1 - D^3\epsilon^2 T^{-\frac{5}{8}}(1+T)^{2\delta}\right]\|\nu(T)\|_{L^2_{x,y}}^2$$
$$\le \|\nu(1)\|_{L^2_{x,y}}^2 + D^3\epsilon^2\|\nu(1)\|_{L^2_{x,y}}^2 + \int_1^T D^3\epsilon^2 t^{-1}\|\nu(t)\|_{L^2_{x,y}}^2\, dt$$
$$+ D\int_1^T \left(D^{\frac{11}{6}}\epsilon^3 + D^{\frac{11}{3}}\epsilon^4 + D^{\frac{8}{3}}\epsilon^4 + D^2\epsilon^2 + D^4\epsilon^4\right)(1+t)^{-\frac{65}{64}+\frac{11}{6}\delta}\|\nu(t)\|_{L^2_{x,y}}^2\, dt.$$

By Gronwall's inequality for any $t \in [1, T]$ there is the inequality

$$\|\nu(t)\|_{L^2_{x,y}}^2 \le \frac{(1 + D^3\epsilon^2)}{1 - D^3\epsilon^2}\|\nu(1)\|_{L^2_{x,y}}^2 \exp\left(\frac{D^3\epsilon^2}{1 - D^3\epsilon^2}\log(1+t)\right)$$
$$\exp\left(\frac{D}{1 - D^3\epsilon^2}\left(D^{\frac{11}{6}}\epsilon^3 + D^{\frac{11}{3}}\epsilon^4 + D^{\frac{8}{3}}\epsilon^4 + D^2\epsilon^2 + D^4\epsilon^4\right)\right).$$

Thus if we take $\epsilon \ll D^{-\frac{3}{2}}$ such that $0 < D^3\epsilon^2 < \delta < \frac{1}{2}$, we have the desired inequality

$$\|\nu(t)\|_{L^2_{x,y}}^2 \lesssim \|\nu(1)\|_{L^2_{x,y}}^2 (1+t)^{2D^3\epsilon^2} \lesssim \|\nu(0)\|_{L^2_{x,y}}^2 (1+t)^{2D^3\epsilon^2}. \tag{3.41}$$

Remark 3.5.1. In the special case $d = 1$, since we have the fact $|\gamma|$ is bounded, we can get a much simpler estimate.

3.5.1 Correction to the Leibnitz derivative rule

In order to finish proof of the estimate for $D_y^s u$, we need to prove the following lemma. Then the same estimate of $\|\nu\|_{L^2_{x,y}}$ for the solution ν to linearized equation can be directly applied to $\left\|D_y^s u\right\|_{L^2_{x,y}}$.

Lemma 3.5.2. *Assume that u satisfies Hypothesis 3.0.1. Then there is the bound*

$$\begin{aligned}
&\left|\int_1^T \left\langle D_y^s u, cor(t)\right\rangle_{L^2_{x,y}} dt\right| \\
\lesssim & D^{\frac{5}{3}}\epsilon^2 \left\|D_y^s u(1)\right\|^2_{L^2_{x,y}} + D^{\frac{5}{3}}\epsilon^2 T^{-1}(1+T)^{\frac{5}{3}\delta}\left\|D_y^s u(T)\right\|^2_{L^2_{x,y}} \\
&+\int_1^T t^{-1}D^2\epsilon^2\left\|D_y^s u\right\|^2_{L^2_{x,y}} dt + \int_1^T D^{\frac{d}{s}+\tilde{\beta}}\epsilon^2 t^{-1-\tilde{\beta}}(1+t)^{\left(\frac{d}{s}+\tilde{\beta}\right)\delta}\left\|D_y^s u(t)\right\|^2_{L^2_{x,y}} dt \\
&+\int_1^T \left[D^{\frac{5}{3}}\epsilon^2 t^{-2}(1+t)^{\frac{5}{3}\delta} + D^{\frac{10}{3}}\epsilon^4 t^{-2}(1+t)^{\frac{10}{3}\delta}\right]\left\|D_y^s u(t)\right\|^2_{L^2_{x,y}} dt,
\end{aligned} \tag{3.42}$$

for some positive number $\tilde{\beta}$.

Proof. We first split the nonlinear term into two parts: the low v-frequency part and high frequency part,

$$t^{-1}|w|^2 w = t^{-1}|\gamma|^2\gamma + t^{-1}\left(|w|^2 w - |\gamma|^2\gamma\right).$$

The first factor $t^{-1}|\gamma|^2\gamma$ is easy to estimate. The inner product can be written as

$$\left\langle D_y^s w, D_y^s\left(t^{-1}|\gamma|^2\gamma\right)\right\rangle_{L^2_{v,y}} = \left\langle P_{\leq 5\sqrt{t}} D_y^s w, D_y^s\left(t^{-1}|\gamma|^2\gamma\right)\right\rangle_{L^2_{v,y}}.$$

Then separate the inner product into a y resonant factor and a y nonresonant factor. The resonant factor can be estimated by (**??**), and the elementary inequality

$$\|D^s(fg)\|_{L^2} \lesssim \|(D^s f)g\|_{L^2} + \|f(D^s g)\|_{L^2},$$

hence we have

$$t^{-1}\|R[\gamma,\gamma,\gamma]\|_{L^2_v H^s_y} \lesssim t^{-1}\|\gamma\|^2_{L^\infty_v H^1_y}\left\|D_y^s w\right\|_{L^2_{v,y}}.$$

The nonresonant factor can be estimated by integration by parts in time, which is almost the same as the computations in the proof of Proposition 4.0.3 (b), hence we omit it. Therefore we have

$$\begin{aligned}
\left|\int_1^T \left\langle D_y^s u, t^{-1}|\gamma|^2\gamma\right\rangle_{L^2_{x,y}} dt\right| \lesssim & D^{\frac{5}{3}}\epsilon^2\left\|D_y^s u(1)\right\|^2_{L^2_{x,y}} + D^{\frac{5}{3}}\epsilon^2 T^{-1}(1+T)^{\frac{5}{3}\delta}\left\|D_y^s u(T)\right\|^2_{L^2_{x,y}} \\
&+\int_1^T\left[D^{\frac{5}{3}}\epsilon^2 t^{-2}(1+t)^{\frac{5}{3}\delta} + D^{\frac{10}{3}}\epsilon^4 t^{-2}(1+t)^{\frac{10}{3}\delta}\right]\left\|D_y^s u(t)\right\|^2_{L^2_{x,y}} dt.
\end{aligned} \tag{3.43}$$

In order to estimate the second term $t^{-1}\left(|w|^2 w - |\gamma|^2\gamma\right)$, we separate it into three parts

$$t^{-1}\left(P_{\geq\sqrt{t}}w|w|^2 + \gamma\overline{P_{\geq\sqrt{t}}w}w + |\gamma|^2 P_{\geq\sqrt{t}}w\right).$$

We move the bound for the first term, the others follow in the same manner.

By Lemma **??**, the Leibnitz rule correction form, it suffices to work with the estimate for

$$t^{-1}\left(D_y^{s_1}P_{\geq\sqrt{t}}w\right)\left(\overline{D_y^{s_2}w}\right)\left(D_y^{s_3}w\right),$$

where

$$s_1, s_2, s_3 \geq 0, \quad s_1+s_2+s_3 = s, \quad , \max\{s_1, s_2, s_3\} \leq \max\{s-1, 1\}.$$

Let

$$p_i = \frac{s}{s_i}, \quad \text{for } i = 1, 2, 3.$$

If $\alpha_i = 0$, we let $p_i = \infty$.

$$\begin{aligned}\left\|\left(D_y^{s_1}P_{\geq\sqrt{t}}w\right)\left(\overline{D_y^{s_2}w}\right)\left(D_y^{s_3}w\right)\right\|_{L_y^2} &\lesssim \left\|D_y^{s_1}P_{\geq\sqrt{t}}w\right\|_{L_y^{2p_1}}\left\|D_y^{s_2}w\right\|_{L_y^{2p_2}}\left\|D_y^{s_3}w\right\|_{L_y^{2p_3}} \\ &\lesssim \left\|D_y^{s_1^*}P_{\geq\sqrt{t}}w\right\|_{L_y^2}\left\|D_y^{s_2^*}w\right\|_{L_y^2}\left\|D_y^{s_3^*}w\right\|_{L_y^2},\end{aligned}$$

where

$$s_i^* = s_i + \frac{d}{2}\left(1 - \frac{s_i}{s}\right), \quad \text{for } s_i > 0, \quad s_i^* = \frac{d}{2} + \delta^*, \quad \text{for } s_i = 0.$$

Here δ^* is some sufficiently small positive number. Notice that $s_i^* < s$ if $s_i < s$.

There exist $q_i > 1$, satisfying

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1,$$

and

$$\frac{1}{2}\left[1 - \frac{1}{q_i}\right] + \frac{s_i^*}{s} < 1.$$

Define $\beta_i := \frac{1}{2}\left(1 - \frac{1}{q_i}\right)$. Then we well have

$$\begin{aligned}\left\|\left(D_y^{s_1}P_{\geq\sqrt{t}}w\right)\left(\overline{D_y^{s_2}w}\right)\left(D_y^{s_3}w\right)\right\|_{L_{v,y}^2} &\lesssim \left\|D_y^{s_1^*}P_{\geq\sqrt{t}}w\right\|_{L_v^{2q_1}L_y^2}\left\|D_y^{s_2^*}w\right\|_{L_v^{2q_2}L_y^2}\left\|D_y^{s_3^*}w\right\|_{L_v^{2q_3}L_y^2} \\ &\lesssim \left\|D_v^{\beta_1}D_y^{s_1^*}P_{\geq\sqrt{t}}w\right\|_{L_{v,y}^2}\left\|D_v^{\beta_2}D_y^{s_2^*}w\right\|_{L_{v,y}^2}\left\|D_v^{\beta_3}D_y^{s_3^*}w\right\|_{L_{v,y}^2}.\end{aligned}$$

Since by interpolation

$$\left\|D_v^{\beta_i}D_y^{s_i^*}w\right\|_{L_{v,y}^2} \lesssim \|w\|_{L_{v,y}^2}^{1-\beta_i-s_i^*/s}\|\partial_v w\|_{L_{v,y}^2}^{\beta_i}\left\|D_y^s w\right\|_{L_{v,y}^2}^{s_i^*/s},$$

for $D_y^{s_i}P_{\geq\sqrt{t}}w$, we will have

$$\left\|D_v^{\beta_1}D_y^{s_1^*}P_{\geq\sqrt{t}}w\right\|_{L_{v,y}^2} \lesssim t^{-\frac{1}{2}\left(1-\beta_1-s_1^*/s\right)}\|\partial_v w\|_{L_{v,y}^2}^{1-s_1^*/s}\left\|D_y^s w\right\|_{L_{v,y}^2}^{s_1^*/s},$$

which allows extra decay in time. Defining

$$\tilde{\beta} := \min_i \left\{ \frac{1}{2} \left(1 - \beta_i - s_i^* / s\right) \right\}$$

for all possible values of i, we will have

$$\left\| D_y^s \left(|w|^2 w - |\gamma|^2 \gamma \right) \right\|_{L^2_{v,y}} \lesssim D^{\frac{d}{s} + \tilde{\beta}} \epsilon^2 t^{-\tilde{\beta}} \left(1 + t\right)^{\frac{d}{s} + \tilde{\beta}} \left\| D_y^s u \right\|_{L^2_{x,y}}. \tag{3.44}$$

Combining (3.43) and (3.44), the lemma is proved.

□

By (3.42) and the same estimate for ν can be apply to $2\left|u\right|^2 D_y^s u + u^2 \overline{D_y^s u}$, therefore we have

$$\begin{aligned}
&\left[1 - D^3 \epsilon^2 T^{-\frac{5}{8}} \left(1 + T\right)^{2\delta} - D^{\frac{8}{3}} \epsilon^2 T^{-1} \left(1 + T\right)^{\frac{5}{3}\delta}\right] \left\| D_y^s u\left(T\right) \right\|^2_{L^2_{x,y}} \\
\le &\left(1 + D^3 \epsilon^2 + D^{\frac{8}{3}} \epsilon^2\right) \left\| D_y^s u\left(1\right) \right\|^2_{L^2_{x,y}} + \int_1^T D^3 \epsilon^2 t^{-1} \left\| D_y^s(t) \right\|^2_{L^2_{x,y}} dt \\
&+ \int_1^T D^{\frac{d}{s} + \tilde{\beta}} \epsilon^2 t^{-1-\tilde{\beta}} \left(1 + t\right)^{\left(\frac{d}{s} + \tilde{\beta}\right)\delta} \left\| D_y^s u(t) \right\|^2_{L^2_{x,y}} dt \\
&+ D \int_1^T \left(D^{\frac{11}{6}} \epsilon^3 + D^{\frac{11}{3}} \epsilon^4 + D^{\frac{8}{3}} \epsilon^4 + D^2 \epsilon^2 + D^4 \epsilon^4 + D^{\frac{5}{3}} \epsilon^2 + D^{\frac{10}{3}} \epsilon^4 \right) \\
&\qquad \left(1 + t\right)^{-\frac{65}{64} + \frac{11}{6}\delta} \left\| D_y^s u(t) \right\|^2_{L^2_{x,y}} dt.
\end{aligned}$$

By Gronwall's inequality and the assumption $0 < D^3 \epsilon^2 < \delta < \frac{1}{2}$, there is the bound on $[1, T]$ that

$$\left\| D_y^s u\left(t\right) \right\|^2_{L^2_{x,y}} \lesssim \left\| D_y^s u\left(1\right) \right\|^2_{L^2_{x,y}} \left(1 + t\right)^{2D^2 \epsilon^3} \lesssim \left\| D_y^s u\left(0\right) \right\|^2_{L^2_{x,y}} \left(1 + t\right)^{2D^2 \epsilon^3}. \tag{3.45}$$

3.6 Proof of Theorem 2.0.4

We want to show that under the bootstrap assumptions (3.4) and (3.3) in time interval $[0, T]$ with T arbitrary, u satisfies a better energy bound and a better decay bound after a more careful computation. By (3.41) and (3.45), we know that when u_0 satisfies (2.4), the solution u satisfies

$$\left\| u\left(t\right) \right\|_{X^+} \lesssim \epsilon \left(1 + t\right)^{D^3 \epsilon^2}. \tag{3.46}$$

The inequality holds with an implicit constant which does not depend on D and T.

Hence we turn to the global decay of u. From (2.17), (3.46) and $s = 3\alpha > 1$

$$\left\| u(t) - \frac{1}{\sqrt{t}} e^{-i\frac{x^2}{2t}} \gamma\left(t, \frac{x}{t}, y\right) \right\|_{L^\infty_x H^1_y} \lesssim t^{-\frac{7}{12}} \left\| u \right\|_{X^+} \lesssim \epsilon t^{-\frac{7}{12} + D^3 \epsilon^2}.$$

There is also the Sobolev embedding $H^1_y \subset L^4_y$. Assuming that u satisfies (2.4), by local well-posedness we have

$$\|\nabla_y \gamma(1)\|_{L^\infty_v L^2_y} \lesssim \|\nabla_y u(1)\|_{L^\infty_x L^2_y} \le \epsilon, \quad \|\gamma(1)\|_{L^\infty_v H^\alpha_y} \lesssim \|u(1)\|_{L^\infty_x H^\alpha_y} \le \epsilon.$$

By (2.25) we have that

$$\begin{aligned}
&\|\nabla_y \gamma(T)\|^2_{L^\infty_v L^2_y} + \frac{1}{t}\|\gamma(T)\|^4_{L^\infty_v L^4_y} \\
\le& \epsilon^2 + \epsilon^4 + \int_1^T D\left[D^{\frac{10}{3}}\epsilon^4(1+t)^{-2+\frac{10}{3}D^3\epsilon^2} + D^{\frac{7}{2}}\epsilon^4(1+t)^{-\frac{13}{12}+\frac{7}{2}D^3\epsilon^2}\right. \\
&\left. + D^{\frac{31}{6}}\epsilon^6(1+t)^{-\frac{25}{12}+\frac{31}{6}D^3\epsilon^2} + D^{\frac{16}{3}}\epsilon^6(1+t)^{-\frac{13}{6}+\frac{16}{3}D^3\epsilon^2}\right] dt \\
\le& \epsilon^2 + \epsilon^4 + D\left[D^{\frac{10}{3}}\epsilon^4 + D^{\frac{7}{2}}\epsilon^4 + D^{\frac{31}{6}}\epsilon^6 + D^{\frac{16}{3}}\epsilon^6\right] \lesssim \epsilon^2.
\end{aligned}$$

If we have $\epsilon \ll D^{-\frac{3}{2}}$ as before, we will have

$$\|u(t)\|_{L^\infty_x H^1_y} \le t^{-\frac{1}{2}}\|\gamma(t)\|_{L^\infty_v H^1_y} + \epsilon t^{-\frac{7}{12}+D^3\epsilon^2} \lesssim \epsilon t^{-\frac{1}{2}}. \tag{3.47}$$

The bounds (3.47), (3.46) are better then the original bootstrap assumptions (3.3), (3.4). Thus we have closed the bootstrap argument, and the time interval of local well-posedness $t \in [1, T]$ can be extended to $t \in [1, \infty)$.

We go back to the estimates of asymptotic profile of u. By (2.17), (2.18), and (3.41),

$$\left\| u(t,x,y) - \frac{1}{\sqrt{t}} e^{-i\frac{x^2}{2t}} \gamma\left(t, \frac{x}{t}, y\right)\right\|_{L^2_{x,y}} \lesssim \epsilon t^{-\frac{1}{2}+D^3\epsilon^2},$$

$$\left\| u(t,x,y) - \frac{1}{\sqrt{t}} e^{-i\frac{x^2}{2t}} \gamma\left(t, \frac{x}{t}, y\right)\right\|_{L^\infty_x H^\alpha_y} \lesssim \epsilon t^{-\frac{7}{12}+D^3\epsilon^2}.$$

Combining Lemma 2.2.2 and (3.41), we obtain

$$\|I\|_{L^2_{v,y}} \lesssim \epsilon t^{-\frac{3}{2}+D^3\epsilon^2}, \quad \|I\|_{L^\infty_v H^\alpha_y} \lesssim \epsilon t^{-\frac{13}{12}+D^3\epsilon^2}. \tag{3.48}$$

The local well-posedness property of the equation

$$\left(i\partial_t + \frac{1}{2}\triangle_y\right)\gamma(t,v,y) = \frac{1}{t}|\gamma|^2\gamma(t,v,y) + I(t,v,y) \tag{3.49}$$

can be obtained by treating the $I(t,v,y)$ term as a perturbation.

Lemma 3.6.1. *If γ satisfies the equation (3.49) with the bounds (3.48), there exists exists a solution $W(t)$ to (2.7) with*

$$\gamma(t,v,y) = W(t,v,y) + \mathcal{O}_{L^2_{v,y}}\left(\epsilon t^{-\frac{1}{2}+2D^3\epsilon^2}\right),$$

and

$$\gamma(t,v,y) = W(t,v,y) + \mathcal{O}_{L^\infty_v H^\alpha_y}\left(\epsilon t^{-\frac{1}{12}+2D^3\epsilon^2}\right).$$

Proof. Let W_n be a solution to the homogeneous equation (2.7) with initial data $W_n(2^n, v, y) = \gamma(2^n, v, y)$, where $n \in \mathbb{N}$, $n \geq 1$. By (2.2), we have the conservation law $\|W_n(t)\|_{L^2_{v,y}} = \|\gamma(2^n)\|_{L^2_{v,y}} \lesssim \epsilon$. By Lemma 4.0.7, there is the conservation law $\|W_n(t)\|_{L^\infty_v H^1_y} \lesssim \|W_n(2^n)\|_{L^\infty_v H^1_y} = \|\gamma(2^n)\|_{L^\infty_v H^1_y} \lesssim \epsilon$, and the bound $\|W_n(2^n)\|_{L^\infty_v H^\alpha_y} \lesssim \epsilon 2^{\frac{5}{6}nD^3\epsilon^2}$. The local and global well-posedness of the homogeneous equation (2.7) is given in Proposition 4.0.3, hence W_n exists on $[1, \infty)$ for each n. By (2.7) and (3.49), we have

$$\begin{aligned}\left(i\partial_t + \frac{1}{2}\triangle_y\right)(\gamma - W_n) &= \frac{1}{t}\left(|\gamma|^2\gamma - |W_n|^2 W_n\right) + I \\ &= \frac{1}{t}\left(R[\gamma,\gamma,\gamma] - R[W_n,W_n,W_n]\right) + \frac{1}{t}\left(\mathcal{E}[\gamma,\gamma,\gamma] - \mathcal{E}[W_n,W_n,W_n]\right).\end{aligned}$$

Therefore we have the bounds

$$\begin{aligned}&\|(\gamma - W_n)(t)\|_{L^\infty_v H^\alpha_y} \\ &\lesssim \int_{2^n}^t \frac{1}{\sigma}\left(\|\gamma\|^2_{L^\infty_v H^1_y} + \|W_n\|^2_{L^\infty_v H^1_y}\right)\|(\gamma - W_n)(\sigma)\|_{L^\infty_v H^\alpha_y}\, d\sigma + \int_{2^n}^t \|I(\sigma)\|_{L^\infty_v H^\alpha_y}\, d\sigma \\ &\quad + \left\|\int_{2^n}^t \frac{1}{\sigma}\mathcal{E}[\gamma,\gamma,\gamma]\, d\sigma\right\|_{L^\infty_v H^\alpha_y} + \left\|\int_{2^n}^t \frac{1}{\sigma}\mathcal{E}[W_n,W_n,W_n]\, d\sigma\right\|_{L^\infty_v H^\alpha_y}.\end{aligned}$$

By (3.48), (4.5), and (4.6) , we have the bounds

$$\int_{2^n}^{2^{n+1}} \|I(\sigma)\|_{L^\infty_v H^\alpha_y}\, d\sigma \lesssim \epsilon 2^{n\left(-\frac{1}{12}+D^3\epsilon^2\right)},$$

$$\sup_{2^n \leq t \leq 2^{n+1}} \left\|\int_{2^n}^t \frac{1}{\sigma}\mathcal{E}[\gamma,\gamma,\gamma]\, d\sigma\right\|_{L^\infty_v H^\alpha_y} \lesssim \epsilon^3 2^{-n+\frac{5}{2}D^3\epsilon^2},$$

$$\sup_{2^n \leq t \leq 2^{n+1}} \left\|\int_{2^n}^t \frac{1}{\sigma}\mathcal{E}[W_n,W_n,W_n]\, d\sigma\right\|_{L^\infty_v H^\alpha_y} \lesssim \epsilon^3 2^{-n+\frac{5}{2}D^3\epsilon^2}.$$

Therefore by Gronwall's inequality, at $t = 2^{n+1}$ we obtain

$$\left\|(\gamma - W_n)(2^{n+1})\right\|_{L^\infty_v H^\alpha_y} \lesssim \left[\epsilon 2^{n\left(-\frac{1}{12}+D^3\epsilon^2\right)} + \epsilon^3 2^{-n+\frac{5}{2}D^3\epsilon^2}\right] 2^{nD\epsilon^2},$$

$$\begin{aligned}&\left\|W_n(2^{n+1}) - W_{n+1}(2^{n+1})\right\|_{L^\infty_v H^\alpha_y} \\ &\lesssim \left\|(\gamma - W_n)(2^{n+1})\right\|_{L^\infty_v H^\alpha_y} + \left\|(\gamma - W_{n+1})(2^{n+1})\right\|_{L^\infty_v H^\alpha_y} \lesssim \epsilon 2^{n\left(-\frac{1}{12}+D\epsilon^2+D^3\epsilon^2\right)}.\end{aligned}$$

Solving the equation for the difference $W_n - W_{n+1}$ backward from $t = 2^{n+1}$,

$$\begin{aligned}\left(i\partial_t + \frac{1}{2}\triangle_y\right)(W_n - W_{n+1}) &= \frac{1}{t}\left(R[W_n,W_n,W_n] - R[W_{n+1},W_{n+1},W_{n+1}]\right) \\ &\quad + \frac{1}{t}\left(\mathcal{E}[W_n,W_n,W_n] - \mathcal{E}[W_{n+1},W_{n+1},W_{n+1}]\right).\end{aligned}$$

By Gronwall's inequality we obtain

$$\|W_n(1) - W_{n+1}(1)\|_{L_v^\infty H_y^\alpha} \lesssim \epsilon 2^{n\left(-\frac{1}{12}+D\epsilon^2+D^3\epsilon^2\right)} 2^{(n+1)D\epsilon^2} = \epsilon 2^{n\left(-\frac{1}{12}+2D\epsilon^2+D^3\epsilon^2\right)+D\epsilon^2}.$$

Here we assume that $\epsilon \ll D^{-\frac{3}{2}}$, hence $D\epsilon^2 < \frac{1}{2}D^3\epsilon^2$. Therefore there exists $W_\infty(1)$ such that

$$\lim_{n\to\infty} \|W_n(1) - W_\infty(1)\|_{L_v^\infty H_y^\alpha} \lesssim \lim_{n\to\infty} \epsilon 2^{n\left(-\frac{1}{12}+\frac{3}{2}D^3\epsilon^2\right)}.$$

We obtain a solution $W_\infty(t)$ to the equation (2.7) with initial data $W_\infty(1)$ at $t = 1$. Then W_∞has the desired property. By Lemma 4.0.7,

$$\|W_n(t) - W_\infty(t)\|_{L_v^\infty H_y^\alpha} \lesssim t^{D\epsilon^2} \|W_n(1) - W_\infty(t)\|_{L_v^\infty H_y^\alpha} \lesssim \epsilon 2^{n\left(-\frac{1}{12}+2D^3\epsilon^2\right)} t^{D\epsilon^2}.$$

For $t \in (2^n, 2^{n+1})$, we have

$$\begin{aligned}\|\gamma(t) - W_\infty(t)\|_{L_v^\infty H_y^\alpha} &\leq \|W_n(t) - W_\infty(t)\|_{L_v^\infty H_y^\alpha} + \|\gamma(t) - W_n(t)\|_{L_v^\infty H_y^\alpha} \\ &\lesssim \epsilon 2^{n\left(-\frac{1}{12}+\frac{3}{2}D^3\epsilon^2\right)} t^{D\epsilon^2} + \epsilon 2^{n\left(-\frac{1}{12}+D^3\epsilon^2\right)} \left(t-2^n\right)^{D\epsilon^2} \lesssim \epsilon t^{\left(-\frac{1}{12}+2D^3\epsilon^2\right)}.\end{aligned}$$

The computations for the $L^2_{v,y}$ norm follows a similar steps as the $L_v^\infty H_y^\alpha$ norm, we obtain

$$\|\gamma(t) - W_\infty(t)\|_{L^2_{v,y}} \lesssim \epsilon 2^{n\left(-\frac{1}{2}+2D^3\epsilon^2\right)}.$$

□

Together with (3.47), the proof of Theorem 2.0.4 is complete.

Chapter 4

The cubic NLS on torus

The main results of this section are to develop the same resonant equation as in [15], and prove some important properties for the solution of (2.7), such as local (global) well-posedness, asymptotic dynamics, asymptotic completeness and energy bounds.

The asymptotic dynamics of small solutions to (2.1) is related to solutions of the resonant system:

$$i\partial_t \mathcal{G}\left(t,v,y\right) = \frac{1}{t} R\left[\mathcal{G}\left(t,v,y\right), \mathcal{G}\left(t,v,y\right), \mathcal{G}\left(t,v,y\right)\right]. \tag{4.1}$$

Note that this system is none other that the resonant system for the cubic NLS equation on $\mathbb{T}^d$ up to a change of timescale. Here we introduce the global well-posedness of (4.1), which will be used in the prove of asymptotic dynamics and asymptotic completeness for (2.7).

Proposition 4.0.2. *(Properties of $\mathcal{G}$) (a) Let $d = 1, 2, 3, 4$. For any $\mathcal{G}(1) \in L_v^\infty H_y^\alpha \cap L^2_{v,y}$, there exists a unique global solution $\mathcal{G}(t) \in C^1\left([1,\infty), L_v^\infty H_y^\alpha \cap L^2_{v,y}\right)$ for any $\alpha \geq 1$.*

(b) There are the conservation laws $\|\mathcal{G}(t)\|_{L^2_{v,y}} \equiv \|\mathcal{G}(1)\|_{L^2_{v,y}}$, $\quad \|\mathcal{G}(t)\|_{L_v^\infty \dot{H}_y^1} \equiv \|\mathcal{G}(1)\|_{L_v^\infty \dot{H}_y^1}$.

(c) (Infinite cascade) If $d \geq 2$ and $\alpha > 1$, there exist global solutions $\mathcal{G}(t) \in C^1\left([1,\infty), L_v^\infty H_y^\alpha \cap L^2_{v,y}\right)$ such that

$$\sup_{t>1} \|\mathcal{G}(t)\|_{L_v^\infty H_y^\alpha} = \infty.$$

The proof is given in [15], Section 4. See [7, 14, 13] for the energy cascade property, this is stated here for completeness, but not used in any way.

Proposition 4.0.3. *(a) The equation (2.7) is locally well-posed for initial data $W_1 \in Y$ ($\alpha > \frac{d}{2}$, $d = 1, 2, 3, 4$). Precisely, for each such data there is a unique solution $W \in C([1,b];Y)$ for some $b > 1$ with $W(1) = W_1$, and Lipschitz continuous dependence on initial data locally on time.*

(b) There exists $\epsilon = \epsilon(d) > 0$ such that if in addition

$$\|W_1\|_Y \leq \epsilon, \tag{4.2}$$

then the equation (2.7) is globally well-posed in $[1,\infty)$*, and there exists* $\mathcal{G} \in Y$ *a solution to (4.1) such that*

$$\left\| W(t,v,y) - e^{it\triangle_y/2}\mathcal{G}(t,v,y) \right\|_Y \to 0 \quad \text{as } t \to \infty. \tag{4.3}$$

(c) If $\mathcal{G}_1$ *satisfies* $\|\mathcal{G}_1\|_Y \ll \epsilon < 1$*, then there exists a global solution* $\mathcal{G} \in C([1,\infty);Y)$ *of (4.1) with initial data* $\mathcal{G}(1) = \mathcal{G}_1$ *and* $\mathcal{W} \in C([1,\infty);Y)$ *which satisfies (2.7) with* $\|\mathcal{W}(1)\|_Y \le \epsilon$ *such that (4.3) holds.*

Remark 4.0.4. Here the variable v is merely a parameter. We can prove the well-posedness and asymptotic pofile in the space $Y = H_y^s$ for the function $W(t,y)$ satisfying (2.7). The proof is following in a similar manner to the proof of Proposition 4.0.3.

Proof of (a): The equation can also be solved in critical functional spaces, see [17, 18, 23]. Consider a short interval $I = [a,b] \subset \mathbb{R}^+$, and let $f, g \in Y$. There is the following iteration scheme:

$$\begin{aligned}
&\left\| \int_0^t s^{-1} e^{i(t-s)\triangle_y/2} \left(|f|^2 f - |g|^2 g \right)(s)\, ds \right\|_{L_t^\infty\left(I; L_v^\infty H_y^\alpha\right)} \\
\lesssim &\left\| t^{-1} \left(\|f\|^2_{L_v^\infty H_y^\alpha} + \|g\|^2_{L_v^\infty H_y^\alpha} \right) \|f-g\|_{L_v^\infty H_y^\alpha} \right\|_{L_t^\infty(I)} \\
\lesssim &\log\left(\frac{b}{a}\right) \left(\|f\|^2_{L_t^\infty\left(I; L_v^\infty H_y^\alpha\right)} + \|g\|^2_{L_t^\infty\left(I; L_v^\infty H_y^\alpha\right)} \right) \|f-g\|_{L_t^\infty\left(I; L_v^\infty H_y^\alpha\right)},
\end{aligned}$$

$$\begin{aligned}
&\left\| \int_0^t s^{-1} e^{i(t-s)\triangle_y/2} \left(|f|^2 f - |g|^2 g \right)(s)\, ds \right\|_{L_t^\infty\left(I; L^2_{v,y}\right)} \\
\lesssim &\left\| t^{-1} \left(\|f\|^2_{L_v^\infty H_y^\alpha} + \|g\|^2_{L_v^\infty H_y^\alpha} \right) \|f-g\|_{L^2_{v,y}} \right\|_{L_t^\infty(I)} \\
\lesssim &\log\left(\frac{b}{a}\right) \left(\|f\|^2_{L_t^\infty\left(I; L_v^\infty H_y^\alpha\right)} + \|g\|^2_{L_t^\infty\left(I; L_v^\infty H_y^\alpha\right)} \right) \|f-g\|_{L_t^\infty\left(I; L^2_{v,y}\right)}.
\end{aligned}$$

If we take $|I|$ to be small enough, by the contraction principle the local well-posedness is obtained in Y.

□

Proof of (b): The solution of (2.7) with initial condition (4.2) will have the conservation laws and inequality for $t \ge 1$:

$$\|W(t)\|_{L^2_{v,y}} = \|W_1\|_{L^2_{v,y}}, \quad \|W(t)\|^2_{L_v^\infty \dot{H}_y^1} + \frac{1}{t}\|W(t)\|^4_{L_v^\infty L_y^4} \le \|W(1)\|^2_{L_v^\infty \dot{H}_y^1} + \|W(1)\|^4_{L_v^\infty L_y^4}.$$

From local well-posedness, assume that the solution $W(t)$ exists on the time interval $[1,T]$. Here we make an extra bootstrap assumption

$$\|W(t)\|_{L_v^\infty H_y^\alpha} \lesssim D\epsilon\,(1+t)^\delta \tag{4.4}$$

where δ is some small positive number. Denote $\mathcal{W} := e^{-it\triangle_y/2}W$, we have

$$\|\mathcal{W}(1)\|_{L_v^\infty H_y^\alpha} = \|W(1)\|_{L_v^\infty H_y^\alpha} \leq \epsilon$$

and

$$\|\mathcal{W}(t)\|_{L_v^\infty H_y^1} = \|W(t)\|_{L_v^\infty H_y^1} \leq \epsilon$$

for any $t \in [1, T]$. The equation of $\mathcal{W}$ is

$$i\partial_t \mathcal{W}(t, v, y) = \sum_{\omega} \sum_{\Gamma_\omega(\mathbf{k})} e^{i\omega t/2} \mathcal{W}(t, v, \mathbf{k}_1) \overline{\mathcal{W}}(t, v, \mathbf{k}_2) \mathcal{W}(t, v, \mathbf{k}_3) e^{i\mathbf{k}\cdot y}.$$

To obtain the resonant equation, separate the nonlinear term of $\partial_t \mathcal{W}$ into the resonant term and nonresonant term,

$$i\partial_t \mathcal{W}(t, v, y) = \frac{1}{t} R[\mathcal{W}, \mathcal{W}, \mathcal{W}](t, v, y) + \frac{1}{t}\mathcal{E}\left[\mathcal{W}, \overline{\mathcal{W}}, \mathcal{W}\right](t, v, y).$$

Using the definition of the non-resonant trilinear form $\mathcal{D}$ and we will have

$$\begin{aligned}
&\frac{1}{t}\mathcal{E}\left[\mathcal{W}, \overline{\mathcal{W}}, \mathcal{W}\right](t, v, y) \\
=&\partial_t \mathcal{D}[\mathcal{W}, \mathcal{W}, \mathcal{W}] + t^{-1}\mathcal{D}[\mathcal{W}, \mathcal{W}, \mathcal{W}] - \mathcal{D}[\mathcal{W}_t, \mathcal{W}, \mathcal{W}] \\
&- \mathcal{D}[\mathcal{W}, \mathcal{W}_t, \mathcal{W}] - \mathcal{D}[\mathcal{W}, \mathcal{W}, \mathcal{W}_t].
\end{aligned} \tag{4.5}$$

It is easy to verify that

$$\begin{aligned}
\|\mathcal{D}[\mathcal{W}, \mathcal{W}, \mathcal{W}](t)\|_{L_v^\infty H_y^\alpha} &\lesssim t^{-1}\|\mathcal{W}(t)\|^3_{L_v^\infty H_y^\alpha}, \\
\|\partial_t \mathcal{W}\|_{L_v^\infty H_y^\alpha} &\lesssim t^{-1}\|\mathcal{W}(t)\|^3_{L_v^\infty H_y^\alpha}.
\end{aligned} \tag{4.6}$$

By (4.6), (4.4), we have the bound

$$\begin{aligned}
&\left\|\int_1^T t^{-1}\mathcal{E}\left[\mathcal{W}, \overline{\mathcal{W}}, \mathcal{W}\right](t, v, y)dt\right\|_{L_v^\infty H_y^\alpha} \\
\lesssim& T^{-1}\|\mathcal{W}(T)\|^3_{L_v^\infty H_y^\alpha} + \|\mathcal{W}(1)\|^3_{L_v^\infty H_y^\alpha} + \int_1^T t^{-2}\left(\|\mathcal{W}(t)\|^3_{L_v^\infty H_y^\alpha} + \|\mathcal{W}(t)\|^6_{L_v^\infty H_y^\alpha}\right)dt \\
\lesssim& D^3\epsilon^3 T^{-1}(1+T)^{3\delta} + \epsilon^3 + \int_1^T t^{-2}\left(D^3\epsilon^3(1+t)^{3\delta} + D^6\epsilon^6(1+t)^{6\delta}\right)dt.
\end{aligned}$$

By (**??**), we also have

$$\|\mathcal{R}[\mathcal{W}, \mathcal{W}, \mathcal{W}](t)\|_{L_v^\infty H_y^\alpha} \lesssim \|\mathcal{W}(t)\|^2_{L_v^\infty H_y^1}\|\mathcal{W}(t)\|_{L_v^\infty H_y^\alpha} \lesssim \epsilon^2\|\mathcal{W}(t)\|_{L_v^\infty H_y^\alpha}.$$

Combining the above estimates together,

$$\|\mathcal{W}(T)\|_{L_v^\infty H_y^\alpha} \leq \epsilon + D^4\epsilon^3 T^{-1}(1+T)^{3\delta} + D\epsilon^3 + \int_1^T D^2\epsilon^3 t^{-1} + t^{-2}D\left(D^3\epsilon^3(1+t)^{3\delta} + D^6\epsilon^6(1+t)^{6\delta}\right)dt.$$

Hence if $\epsilon \ll D^{-\frac{3}{2}}$and $D^2\epsilon^3 < \delta$, we have

$$\|W(t)\|_{L_v^\infty H_y^\alpha} = \|\mathcal{W}(t)\|_{L_v^\infty H_y^\alpha} \lesssim \left(\epsilon + D\epsilon^3 + D^4\epsilon^3 + D^7\epsilon^6\right)t^{D^2\epsilon^3} \lesssim \epsilon t^{D^2\epsilon^3}, \tag{4.7}$$

which is a better bound. Due to the boundness of $\|W(t)\|_{L_v^\infty H_y^\alpha}$, we can extend the local well-posedness result to well-posedness on $[1,\infty)$.

To obtain the asymptotic equation, define the function

$$F(t,v,y) := -i\int_t^\infty \sum_{\omega,\omega\neq 0}\sum_{\Gamma_\omega(\mathbf{k})} e^{i\omega\sigma/2}\mathcal{W}(\sigma,v,\mathbf{k}_1)\overline{\mathcal{W}}(\sigma,v,\mathbf{k}_2)\mathcal{W}(\sigma,v,\mathbf{k}_3)e^{i\mathbf{k}\cdot y}d\sigma.$$

For this function we have the bounds:

Lemma 4.0.5. *For $t \geq 1$, we have*

$$\|F(t,v,y)\|_{L_v^\infty H_y^\alpha} \lesssim \epsilon^6 t^{-1+6D^2\epsilon^3}, \quad \|F(t,v,y)\|_{L_{v,y}^2} \lesssim \epsilon^7 t^{-1+5D^2\epsilon^3}.$$

Proof. By (4.7), (4.6) and integration by parts in time t,

$$\begin{aligned}\|F(t,v,y)\|_{L_v^\infty H_y^\alpha} &\lesssim t^{-1}\|\mathcal{W}(t)\|^3_{L_v^\infty H_y^\alpha} + \int_t^\infty \sigma^{-2}\left(\|\mathcal{W}(\sigma)\|^3_{L_v^\infty H_y^\alpha} + \|\mathcal{W}(\sigma)\|^6_{L_v^\infty H_y^\alpha}\right)d\sigma \\ &\lesssim \epsilon^3 t^{-1+3D^2\epsilon^3} + \epsilon^6 t^{-1+6D^2\epsilon^3} \lesssim \epsilon^6 t^{-1+6D^2\epsilon^3},\end{aligned}$$

$$\begin{aligned}\|F(t,v,y)\|_{L_{v,y}^2} &\lesssim t^{-1}\|\mathcal{W}(t)\|^2_{L_v^\infty H_y^\alpha}\|\mathcal{W}(t)\|_{L_{v,y}^2} \\ &\quad + \int_t^\infty \sigma^{-2}\left(\|\mathcal{W}(\sigma)\|^2_{L_v^\infty H_y^\alpha} + \|\mathcal{W}(\sigma)\|^5_{L_v^\infty H_y^\alpha}\right)\|\mathcal{W}(\sigma)\|_{L_{v,y}^2}d\sigma \\ &\lesssim \epsilon^4 t^{-1+2D^2\epsilon^3} + \epsilon^7 t^{-1+5D^2\epsilon^3} \lesssim \epsilon^7 t^{-1+5D^2\epsilon^3}.\end{aligned}$$

□

Then we have the property

$$i\partial_t(\mathcal{W}(t,v,y) + F(t,v,y)) = \frac{1}{t}R[\mathcal{W}(t,v,y),\mathcal{W}(t,v,y),\mathcal{W}(t,v,y)].$$

In order to rewrite this equation into the form of (4.1), define a modified function of $\mathcal{W}$ to be

$$\tilde{\mathcal{W}}(t,v,y) := \mathcal{W}(t,v,y) + F(t,v,y).$$

Thus the equation becomes

$$\begin{aligned} i\partial_t\tilde{\mathcal{W}}(t,v,y) &= \frac{1}{t}R\left[\tilde{\mathcal{W}}-F,\tilde{\mathcal{W}}-F,\tilde{\mathcal{W}}-F\right] \\ &= \frac{1}{t}R\left[\tilde{\mathcal{W}},\tilde{\mathcal{W}},\tilde{\mathcal{W}}\right]-\frac{1}{t}R\left[F,\mathcal{W},\mathcal{W}\right]-\frac{1}{t}R\left[\tilde{\mathcal{W}},F,\mathcal{W}\right]-\frac{1}{t}R\left[\tilde{\mathcal{W}},\tilde{\mathcal{W}},F\right]. \end{aligned}$$

Lemma 4.0.6. *For $t>1$,*

$$i\partial_t\tilde{\mathcal{W}}(t,v,y) = \frac{1}{t}R\left[\tilde{\mathcal{W}},\tilde{\mathcal{W}},\tilde{\mathcal{W}}\right]+\mathcal{O}_{L^2_{v,y}}\left(\epsilon^9 t^{-2+5D^2\epsilon^3}\right)\cap\mathcal{O}_{L^\infty_v H^\alpha_y}\left(\epsilon^8 t^{-2+8D^2\epsilon^3}\right).$$

Proof. Since $\left\|\tilde{\mathcal{W}}\right\|_{L^2_{v,y}}\lesssim\|\mathcal{W}\|_{L^2_{v,y}}$ and $\left\|\tilde{\mathcal{W}}\right\|_{L^\infty_v H^\alpha_y}\lesssim\|\mathcal{W}\|_{L^\infty_v H^\alpha_y}$, and the estimate

$$\begin{aligned} \left\|t^{-1}R\left[F,\mathcal{W},\mathcal{W}\right]\right\|_{L^2_{v,y}} &\lesssim t^{-1}\|\mathcal{W}\|^2_{L^\infty_v H^1_y}\|F\|_{L^2_{v,y}}\lesssim\epsilon^9 t^{-2+5D^2\epsilon^3}, \\ \left\|t^{-1}R\left[F,\mathcal{W},\mathcal{W}\right]\right\|_{L^\infty_v H^\alpha_y} &\lesssim t^{-1}\|\mathcal{W}\|^2_{L^\infty_v H^\alpha_y}\|F\|_{L^\infty_v H^\alpha_y}\lesssim\epsilon^8 t^{-2+8D^2\epsilon^3}, \end{aligned}$$

the lemma is proved.

□

Therefore $\tilde{\mathcal{W}}$ solves the resonant equation (4.1) with a perturbative error. By Proposition 4.0.2, there is a solution to the homogeneous equation (4.1) for any initial data in Y. Hence go through the similar argument as in Lemma 3.6.1, there exists a solution $\mathcal{G}$ to equation (4.1) satisfying

$$\tilde{\mathcal{W}}(t,v,y)=\mathcal{G}\left(t,v,y\right)+\mathcal{O}_{L^2_{v,y}}\left(\epsilon^9 t^{-1+5D^2\epsilon^3}\right)\cap\mathcal{O}_{L^\infty_v H^\alpha_y}\left(\epsilon^8 t^{-1+8D^2\epsilon^3}\right).$$

Using the estimate of $F=\tilde{\mathcal{W}}-W$, we obtain

$$W(t,v,y)=e^{it\triangle_y/2}\mathcal{G}\left(t,v,y\right)+\mathcal{O}_{L^2_{v,y}}\left(\epsilon^7 t^{-1+5D^2\epsilon^3}\right)\cap\mathcal{O}_{L^\infty_v H^\alpha_y}\left(\epsilon^8 t^{-1+8D^2\epsilon^3}\right).$$

□

Proof of (c): Let $\mathcal{G}$ be a solution to (4.1), but where we put extra assumptions on the initial data:

$$\|\mathcal{G}\left(1\right)\|_{L^2_{v,y}}+\|\mathcal{G}\left(1\right)\|_{L^\infty_v H^\alpha_y}\le M,$$

where $0<M\ll\delta$. By the conservation law of (4.1) and (**??**) we have the following inequalities

$$\|\mathcal{G}\left(t\right)\|_{L^2_{v,y}},\|\mathcal{G}\left(t\right)\|_{L^\infty_v H^1_y}\le M,\quad \|\mathcal{G}\|_{L^\infty_v H^\alpha_y}\lesssim Mt^\delta.$$

If there exists a solution W to (2.7) tending to $e^{it\triangle_y/2}\mathcal{G}$ at $t=\infty$, then the difference $\mathcal{V}:=W-e^{it\triangle_y/2}\mathcal{G}$ satisfies the following equation:

$$\begin{cases} \left(i\partial_t+\frac{1}{2}\triangle_y\right)\mathcal{V}:=h_1+h_2, \\ \mathcal{V}(\infty)=0. \end{cases} \tag{4.8}$$

The functions h_1, h_2 are given by

$$h_1 := \frac{1}{t} e^{it\triangle_y/2}\left[\mathcal{R}\left[e^{-it\triangle_y/2}\mathcal{V}+\mathcal{G}, e^{-it\triangle_y/2}\mathcal{V}+\mathcal{G}, e^{-it\triangle_y/2}\mathcal{V}+\mathcal{G}\right]-\mathcal{R}\left[\mathcal{G},\mathcal{G},\mathcal{G}\right]\right],$$

and denoting $\mathcal{W} := e^{-it\triangle_y/2}\mathcal{V}+\mathcal{G}$,

$$h_2 := \frac{e^{i\frac{\omega}{2}t+it\triangle_y/2}}{t}\sum_{\omega}\sum_{\Gamma_\omega(\mathbf{k})}\mathcal{W}\left(t,v,\mathbf{k}_1\right)\overline{\mathcal{W}}\left(t,v,\mathbf{k}_2\right)\mathcal{W}\left(t,v,\mathbf{k}_3\right)e^{i\mathbf{k}\cdot y}.$$

Here our goal is to solve the $\mathcal{V}$ equation from $t=\infty$. The solution $\mathcal{V}$ will satisfy the equation

$$\mathcal{V}(t) = i\int_t^\infty e^{i(t-s)\triangle_y/2}\left(h_1+h_2\right)(s)ds. \tag{4.9}$$

Hence we define a function space with time decay, and solve $\mathcal{V}$ in the space

$$\|f\|_{\mathscr{X}} := \sup_{T>1} T^\delta\left(\|f\|_{L_t^\infty\left(T,2T;L_v^\infty H_y^\alpha\right)}+\|f\|_{L_t^\infty\left(T,2T;L_{v,y}^2\right)}\right).$$

Since $\mathcal{R}$ is a trilinear form,

$$\begin{aligned}
&\|h_1\|_{L_v^\infty H_y^\alpha\left(L_{v,y}^2\right)}\\
\lesssim & t^{-1}\Bigg(\left\|R\left[\mathcal{G},\mathcal{G},e^{-it\triangle_y/2}\mathcal{V}\right]\right\|_{L_v^\infty H_y^\alpha\left(L_{v,y}^2\right)}+\left\|R\left[\mathcal{G},e^{-it\triangle_y/2}\mathcal{V},e^{-it\triangle_y/2}\mathcal{V}\right]\right\|_{L_v^\infty H_y^\alpha\left(L_{v,y}^2\right)}\\
&+\left\|R\left[e^{-it\triangle_y/2}\mathcal{V},e^{-it\triangle_y/2}\mathcal{V},e^{-it\triangle_y/2}\mathcal{V}\right]\right\|_{L_v^\infty H_y^\alpha\left(L_{v,y}^2\right)}\Bigg).
\end{aligned}$$

By (**??**) there are the following bounds:

$$\left\|t^{-1}R\left[\mathcal{G},\mathcal{G},e^{-it\triangle_y/2}\mathcal{V}\right]\right\|_{L_t^1\left(T,2T;L_v^\infty H_y^\alpha\right)+L_t^1\left(T,2T;L_{v,y}^2\right)}\lesssim M^2T^{-\delta}\|\mathcal{V}\|_{\mathscr{X}},$$

$$\left\|t^{-1}R\left[\mathcal{G},e^{-it\triangle_y/2}\mathcal{V},e^{-it\triangle_y/2}\mathcal{V}\right]\right\|_{L_t^1\left(T,2T;L_v^\infty H_y^\alpha\right)+L_t^1\left(T,2T;L_{v,y}^2\right)}\lesssim MT^{-2\delta}\|\mathcal{V}\|_{\mathscr{X}}^2,$$

$$\left\|t^{-1}R\left[e^{-it\triangle_y/2}\mathcal{V},e^{-it\triangle_y/2}\mathcal{V},e^{-it\triangle_y/2}\mathcal{V}\right]\right\|_{L_t^1\left(T,2T;L_v^\infty H_y^\alpha\right)+L_t^1\left(T,2T;L_{v,y}^2\right)}\lesssim T^{-3\delta}\|\mathcal{V}\|_{\mathscr{X}}^3.$$

We obtain the $\mathscr{X}$ bound for h_1,

$$\left\|\int_t^\infty e^{-i(t-s)\triangle_y/2}h_1(s)ds\right\|_{\mathscr{X}}\lesssim M^2\|\mathcal{V}\|_{\mathscr{X}}+M\|\mathcal{V}\|_{\mathscr{X}}^2+\|\mathcal{V}\|_{\mathscr{X}}^3. \tag{4.10}$$

For the h_2 part, use the integration by parts in time and break the time interval into dyadic subintervals and estimate $e^{-i(t-s)\triangle_y/2}h_1(s)$ in each interval and sum up:

$$\begin{aligned}
\int_T^{2T} e^{-is\triangle_y/2}h_2(s)ds = &\,\mathcal{D}\left[\mathcal{W},\mathcal{W},\mathcal{W}\right]\Big|_T^{2T}+\int_T^{2T}s^{-1}\mathcal{D}\left[\mathcal{W},\mathcal{W},\mathcal{W}\right]ds\\
&+\int_T^{2T}\mathcal{D}\left[\partial_t\mathcal{W},\mathcal{W},\mathcal{W}\right]+\mathcal{D}\left[\mathcal{W},\partial_t\mathcal{W},\mathcal{W}\right]+\mathcal{D}\left[\mathcal{W},\mathcal{W},\partial_t\mathcal{W}\right]ds.
\end{aligned}$$

By the formula for $\mathcal{G}$ and $\mathcal{V}$, we have

$$i\partial_t \mathcal{W} = \frac{e^{-it\triangle_y/2}}{t} |\mathcal{W}|^2 \mathcal{W},$$

and

$$\|\mathcal{W}\|_{L_v^\infty H_y^\alpha} \lesssim T^{-\delta} \|\mathcal{V}\|_{\mathscr{X}} + MT^\delta, \quad \|\mathcal{W}\|_{L_{v,y}^2} \lesssim T^{-\delta} \|\mathcal{V}\|_{\mathscr{X}} + M.$$

Hence we have

$$\left\| \int_T^{2T} e^{-is\triangle_y/2} h_2(s) ds \right\|_{L_v^\infty H_y^\alpha} \lesssim T^{-1} \left(T^{-\delta} \|\mathcal{V}\|_{\mathscr{X}} + MT^\delta \right)^3 + T^{-1} \left(T^{-\delta} \|\mathcal{V}\|_{\mathscr{X}} + MT^\delta \right)^5,$$

$$\left\| \int_T^{2T} e^{-is\triangle_y/2} h_2(s) ds \right\|_{L_{v,y}^2} \lesssim T^{-1} \left[\left(T^{-\delta} \|\mathcal{V}\|_{\mathscr{X}} + MT^\delta \right)^2 + \left(T^{-\delta} \|\mathcal{V}\|_{\mathscr{X}} + MT^\delta \right)^4 \right] \left(T^{-\delta} \|\mathcal{V}\|_{\mathscr{X}} + M \right).$$

Therefore we have the bound

$$\left\| \int_t^\infty e^{-i(t-s)\triangle_y/2} h_2(s) ds \right\|_{\mathscr{X}} \lesssim \left(\|\mathcal{V}\|_{\mathscr{X}} + M \right)^3 + \left(\|\mathcal{V}\|_{\mathscr{X}} + M \right)^5. \tag{4.11}$$

By (4.10), (4.11), and the assumption that M is a small positive number, we obtain the bound

$$\|\mathcal{V}\|_{\mathscr{X}} \lesssim M^3 + M^2 \|\mathcal{V}\|_{\mathscr{X}} + M \|\mathcal{V}\|_{\mathscr{X}}^2 + \|\mathcal{V}\|_{\mathscr{X}}^3 + \|\mathcal{V}\|_{\mathscr{X}}^4 + \|\mathcal{V}\|_{\mathscr{X}}^5. \tag{4.12}$$

In a similar manner, for $\mathcal{V}_1$ and $\mathcal{V}_2$ both satisfying the equation, we have the Lipschitz bounds

$$\|\mathcal{V}_1 - \mathcal{V}_2\|_{\mathscr{X}} \lesssim \left(M^2 + \|\mathcal{V}_1\|_{\mathscr{X}}^4 + \|\mathcal{V}_2\|_{\mathscr{X}}^4 \right) \|\mathcal{V}_1 - \mathcal{V}_2\|_{\mathscr{X}}. \tag{4.13}$$

By (4.12), we have the bound $\|\mathcal{V}\|_{\mathscr{X}} \lesssim M^3$ if M is small enough. Hence by (4.13), we can solve the equation (4.8) for γ by the contraction principle in the function space $\mathscr{X}$.

□

In order to prove the asymptotic completeness of (2.1), we introduce the following lemma:

Lemma 4.0.7. *Suppose W is a solution to the equation (2.7), let $\alpha > \frac{d}{2}$, and assume that $\|W(1)\|_{L_v^\infty H_y^1}^2 \lesssim \delta$. By Lemma* **??**, *there are the growth bounds:*

(a) For any $s \geq 1$, if at $t = 1$, we have $\|W(1)\|_{L_v^2 H_y^s + L_v^\infty H_y^\alpha} < \infty$, then for any $t \geq 1$, there is the bound $\|W(t)\|_{L_v^2 H_y^s + L_v^\infty H_y^\alpha} \lesssim t^\delta \|W(1)\|_{L_v^2 H_y^s + L_v^\infty H_y^\alpha}$.

(b) For any $s > 0$, if at $t = 1$, we have $\|W(1)\|_{H_v^s L_y^2 + L_v^\infty H_y^\alpha} < \infty$, then for any $t \geq 1$, there is the bound $\|W(t)\|_{H_v^s L_y^2 + L_v^\infty H_y^\alpha} \lesssim t^\delta \|W(1)\|_{H_v^s L_y^2 + L_v^\infty H_y^\alpha}$.

The lemma can be proved by following the same steps in Proposition 4.0.3.

Chapter 5

Asymptotic Completeness

Theorem 5.0.8. *Let $1 \leq d \leq 4$ and C be a large universal constant. There exists $\epsilon = \epsilon(d) > 0$ such that if W_1 satisfies*

$$\left\| D_v^{1+C\epsilon^2} W_1 \right\|_{L^2_{v,y}} + \left\| D_y^s D_v^{C\epsilon^2} W_1 \right\|_{L^2_{v,y}} + \left\| D_y^s W_1 \right\|_{L^2_{v,y}} \ll \epsilon \ll 1,$$

there exists $W(t)$ solving (2.7) on $t \in [1, \infty)$ with initial data $W(1) = W_1$, and there exists a solution u of (2.1) with initial data u_0 satisfies (2.4), hence (2.8) holds for u.

5.1 Asymptotic functions

In this section we prove Theorem 5.0.8. Suppose that W is the solution to the equation (2.7) for $t \in [1, \infty)$, it suffices to prove existence of a solution $u(t)$ on $[1, \infty)$ satisfying

$$\|u(1)\|_{L^2_x H^s_y} + \|L_x u(1)\|_{L^2_{x,y}} \lesssim \epsilon, \tag{5.1}$$

so that u is close to W at $t = \infty$.

Since we need extra regularity in v in order to finish the asymptotic completeness proof, to start with the proof, from Lemma 4.0.7, we make extra assumptions that the initial data of $W(t)$ satisfies

$$\left\| D_v^{1+8\delta} W(1) \right\|_{L^2_{v,y}}, \left\| D_y^s D_v^{8\delta} W(1) \right\|_{L^2_{v,y}}, \left\| D_y^s W(1) \right\|_{L^2_{v,y}} \lesssim M.$$

Therefore we will have

$$\|W(1)\|^2_{L^\infty_v H^1_y} \lesssim M^2 \lesssim \delta,$$

and the growth rate for any $t \geq 1$

$$\left\| D_v^{1+8\delta} W(t) \right\|_{L^2_{v,y}}, \left\| D_y^s D_v^{8\delta} W(t) \right\|_{L^2_{v,y}}, \left\| D_y^s W(t) \right\|_{L^2_{v,y}} \leq M t^\delta, \tag{5.2}$$

where $M, \delta > 0$ and $\delta \gg M$. For regularity reasons, instead of working with the asymptotic profile

$$u_{asy} = \frac{1}{\sqrt{t}} e^{i\frac{x^2}{2t}} W\left(t, \frac{x}{t}, y\right),$$

we work with the regularized approximate solution

$$u_{app} = \frac{1}{\sqrt{t}} e^{i\frac{x^2}{2t}} \left(P_{\leq\sqrt{t}} W\right)\left(t, \frac{x}{t}, y\right).$$

Then by Bernstein's inequality we have the bounds

$$\|u_{asy} - u_{app}\|_{L^2_{v,y}} \lesssim M t^{-\frac{1}{2}-4\delta}, \quad \|u_{asy} - u_{app}\|_{L^2_v H^s_y} \lesssim M t^{-4\delta}, \quad \|L_x u_{asy} - L_x u_{app}\|_{L^2_{v,y}} \lesssim M t^{-4\delta}. \tag{5.3}$$

Here we use direct computations showing that

$$L_x u_{app} = \frac{i}{\sqrt{t}} e^{i\frac{x^2}{2t}} \left(\partial_v W\right)\left(t, \frac{x}{t}, y\right), \quad D^s_y u_{app} = \frac{1}{\sqrt{t}} e^{i\frac{x^2}{2t}} \left(P_{\leq\sqrt{t}} D^s_y W\right)\left(t, \frac{x}{t}, y\right), \tag{5.4}$$

as well as the inequalities

$$\|u_{app}\|_{L^\infty_x H^\alpha_y} \lesssim M t^{-\frac{1}{2}+\alpha}, \quad \|u_{app}\|_{L^\infty_x H^1_y} \lesssim M.$$

Thus we have that

$$\begin{aligned}
&\left(i\partial_t + \frac{1}{2}\partial_x^2 + \frac{1}{2}\triangle_y\right) u_{app} \\
&= \left|u_{app}\right|^2 u_{app} + t^{-\frac{3}{2}} e^{i\frac{x^2}{2t}} \mathcal{F}^{-1}_\xi \left[-\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\frac{\xi^2}{t} - \mathcal{X}'\left(\frac{\xi}{\sqrt{t}}\right)\frac{\xi}{2\sqrt{t}}\right] * W\left(t, \frac{x}{t}, y\right) \\
&\quad + t^{-\frac{3}{2}} e^{i\frac{x^2}{2t}} P_{\leq\sqrt{t}} \left(|W|^2 W - \left|P_{\leq\sqrt{t}} W\right|^2 P_{\leq\sqrt{t}} W\right)\left(t, \frac{x}{t}, y\right) \\
&\quad + t^{-\frac{3}{2}} e^{i\frac{x^2}{2t}} P_{\geq\sqrt{t}} \left(\left|P_{\leq\sqrt{t}} W\right|^2 P_{\leq\sqrt{t}} W\right)\left(t, \frac{x}{t}, y\right) \\
&:= \left|u_{app}\right|^2 u_{app} + I'_1 + I'_2 + I'_3.
\end{aligned}$$

To find u solving the cubic NLS equation (2.1) and matching u_{app}, let $\tilde{w} = u - u_{app}$ solve the following equation for $\tilde{w}$:

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 + \frac{1}{2}\triangle_y\right)\tilde{w} = \left|u_{app} + \tilde{w}\right|^2 \left(u_{app} + \tilde{w}\right) - \left|u_{app}\right|^2 u_{app} - I'_1 - I'_2 - I'_3.$$

By direct computation

$$\left|u_{app} + \tilde{w}\right|^2 \left(u_{app} + \tilde{w}\right) - \left|u_{app}\right|^2 u_{app} = 2\tilde{w}\left|u_{app}\right|^2 + \overline{\tilde{w}} u_{app}^2 + \tilde{w}^2 \overline{u_{app}} + 2\left|\tilde{w}\right|^2 u_{app} + \left|\tilde{w}\right|^2 \tilde{w}.$$

Let

$$L_{lin}\left(u_{app}, \tilde{w}\right) := 2\left|u_{app}\right|^2 \tilde{w} + u_{app}^2 \overline{\tilde{w}},$$

$$Q_1(u_{app},\tilde{w}) := \tilde{w}^2\overline{u_{app}} + 2|\tilde{w}|^2 u_{app} + |\tilde{w}|^2\tilde{w}.$$

Hence we need to solve the equation from infinity:

$$\begin{gathered}\left(i\partial_t + \frac{1}{2}\partial_x^2 + \frac{1}{2}\triangle_y\right)\tilde{w} = L_{lin}(u_{app},\tilde{w}) + Q_1(u_{app},\tilde{w}) - I_1' - I_2' - I_3',\\ \tilde{w}(\infty) = 0.\end{gathered} \tag{5.5}$$

The solution operator for the inhomogeneous Schrödinger equation with zero Cauchy data at infinity is given by

$$\Phi f = i\int_t^\infty U(t-s)\,f(s)ds.$$

Then the equation for $\tilde{w}$ can be written as

$$\tilde{w} = i\int_t^\infty U(t-s)\left(L_{lin}(u_{app},\tilde{w}) + Q_1(u_{app},w) - I_1' - I_2' - I_3'\right)(s)ds. \tag{5.6}$$

We also need the backward solvability for the linearized equation to solve (5.5). Hence consider the linearized equation, where $\tilde{\nu}$ can be either $L_x\tilde{w}$ or $D_y^s\tilde{w}$ with Leibnitz rule correction terms (The correction terms can be estimated by the same computation in Lemma 3.5.2.) By direct computations, we obtain the equation

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 + \frac{1}{2}\triangle_y\right)\tilde{\nu} := L_{lin}(u_{app},\tilde{\nu}) + Q_2(u_{app},\tilde{\nu}) + g(u_{app},\tilde{w}) - \tilde{I}_1' - \tilde{I}_2' - \tilde{I}_3',$$

where

$$L_{lin}(u_{app},\tilde{\nu}) := 2|u_{app}|^2\tilde{\nu} - u_{app}^2\overline{L\tilde{\nu}},$$

$$Q_2(u_{app},\tilde{\nu}) := 2|\tilde{w}|^2\tilde{\nu} - \tilde{w}^2\overline{\tilde{\nu}} + 2\overline{u_{app}}\tilde{w}\tilde{\nu} + 2\overline{\tilde{w}}u_{app}\tilde{\nu} - 2\tilde{w}u_{app}\overline{\tilde{\nu}}.$$

For $\tilde{\nu} = L_x\tilde{w}$, defining the following functions:

$$g(u_{app},\tilde{w}) := 2w\,(L_x u_{app})\,\overline{u_{app}} - 2wu_{app}\overline{L_x u_{app}} - w^2\overline{L_x u_{app}} + 2\overline{w}u_{app}L_x u_{app} + 2|w|^2 L_x u_{app},$$

and

$$\tilde{I}_1' = L_x I_1', \quad \tilde{I}_2' = L_x\tilde{I}_2', \quad \tilde{I}_3' = L_x\tilde{I}_3'.$$

For $\tilde{\nu} = D_y^s\tilde{w}$, we define $g(u_{app},\tilde{w})$ and $\tilde{I}_1', \tilde{I}_2', \tilde{I}_3'$ in the same manner.

The equation for $\tilde{\nu}$ can be written as

$$\tilde{\nu} = i\int_t^\infty U(t-s)\left[L_{lin}(u_{app},\tilde{\nu}) + Q_2(u_{app},\tilde{\nu}) + g(u_{app},\tilde{w}) - \tilde{I}_1' - \tilde{I}_2' - \tilde{I}_3'\right](s)ds. \tag{5.7}$$

5.2 Proof of the theorem 5.0.8

The solution (5.6),(5.7) will be solved together through contraction principle, using the Strichartz bound

$$\|\Phi f\|_{L_t^\infty(T,\infty;L^2_{x,y})} \lesssim \|f\|_{L_t^1(T,\infty;L^2_{x,y})}. \tag{5.8}$$

In order to bound $\|f\|_{L_t^1(T,\infty;L^2_{x,y})}$ we divide $[T,\infty)$ into dyadic subintervals, estimate $\|f\|_{L_t^\infty L^2_{x,y}}$ in each interval, and then sum up. Hence we define a function space with appropriate time decay, and let $\tilde{w}$ be solved in the space

$$\|f\|_Z = \sup_{T>1} T^{\frac{1}{2}+\delta}\left[\|f\|_{L_t^\infty(T,2T;L^2_{x,y})}\right],$$

and the larger space for $L\tilde{w}$ is

$$\|f\|_{\tilde{Z}} = \sup_{T>1} T^{\delta}\left[\|f\|_{L_t^\infty(T,2T;L^2_{x,y})}\right].$$

Since we are unable to solve $\tilde{w}$ and $\tilde{\nu}$ separately, here also define a norm Z^+ to be

$$\|\tilde{w}\|^2_{Z^+} := \|\tilde{w}\|^2_Z + \|\tilde{\nu}\|^2_{\tilde{Z}}.$$

In order to solve the equations for $\tilde{w}$ and $\tilde{\nu}$ simultaneously in Z^+ using the contraction principle we need to show that

(1) The map

$$(\tilde{w},\tilde{\nu}) \to \left(\Phi\left[L_{lin}\left(u_{app},\tilde{w}\right)+Q_1\left(u_{app},\tilde{w}\right)\right],\Phi\left[L_{lin}\left(u_{app},\tilde{\nu}\right)+Q_2\left(u_{app},\tilde{\nu}\right)+g\left(u_{app},\tilde{w}\right)\right]\right)$$

maps Z^+ into $Z^+$with a small Lipschitz constant for $(\tilde{w},\tilde{\nu})$ in a ball of radius CM where $1 \ll C \ll M$.

(2) The nonlinear term $I_1+I_2+I_3$ satisfies

$$\|\Phi\left(I_1+I_2+I_3\right)\|_{Z^+} \lesssim M.$$

Then there is a solution $(\tilde{w},\tilde{\nu})$ satisfying

$$\|\tilde{w}\|_{Z^+} \lesssim M.$$

Lemma 5.2.1. *There are bounds associated with $\tilde{w}$:*

$$\|\Phi L_{lin}\left(u_{app},\tilde{w}\right)\|_Z \lesssim M^2\|\tilde{w}\|_Z, \tag{5.9}$$

$$\|\Phi Q_1\left(u_{app},\tilde{w}\right)\|_Z \lesssim \delta^{-1}M\|\tilde{w}\|_Z^{\frac{7}{6}}\|\tilde{\nu}\|_{\tilde{Z}}^{\frac{5}{6}}+\|\tilde{w}\|_Z^{\frac{4}{3}}\|\tilde{\nu}\|_{\tilde{Z}}^{\frac{5}{3}}. \tag{5.10}$$

Proof. For (5.9), notice that $L_{lin}(u_{app}, \tilde{w})$ is nothing but the nonlinear term of the linearized equation (2.9), if one replaces u by u_{app} and $L_x u$ by $i\tilde{w}$. Using the same computations as in the energy estimate section, we have

$$\left\| \int_T^{2T} U(-s) L_{lin}(u_{app}, \tilde{w})(s) ds \right\|_{L_t^\infty(T,2T;L^2_{x,y})}$$
$$\lesssim \left\| t^{-1} \|u_{app}\|^2_{L_x^\infty H_y^1} \|\tilde{w}\|_{L^2_{x,y}} \right\|_{L_t^1(T,2T)} \lesssim M^2 \|\tilde{w}\|_{L_t^\infty(T,2T;L^2_{x,y})}.$$

We start the proof of (5.10) by the estimate used in (2.16), and get

$$\|\tilde{w}(t,x,y)\|_{L_x^\infty H_y^\alpha} \lesssim t^{-\frac{1}{2}} \|\tilde{w}\|^{\frac{1}{6}}_{L^2_{x,y}} \|L_x \tilde{w}\|^{\frac{1}{2}}_{L^2_{x,y}} \left\| D_y^s \tilde{w} \right\|^{\frac{1}{6}}_{L^2_{x,y}}.$$

The estimate for $Q_1(u_{app}, \tilde{w})$ is straightforward by applying above inequality. Here we let $\tilde{\nu}$ be either L_x or D_y^s,

$$\left\| |\tilde{w}|^2 u_{app} \right\|_{L_t^1(T,2T;L^2_{x,y})}$$
$$\lesssim \left\| \|\tilde{w}\|_{L^2_{x,y}} \|\tilde{w}\|_{L_x^\infty H_y^\alpha} \|u_{app}\|_{L_x^\infty H_y^\alpha} \right\|_{L_t^1(T,2T)} \lesssim \left\| M t^{-1+\delta} \|\tilde{w}\|^{\frac{7}{6}}_{L^2_{x,y}} \|\tilde{\nu}\|^{\frac{5}{6}}_{L^2_{x,y}} \right\|_{L_t^1(T,2T)}$$
$$\lesssim \delta^{-1} M T^\delta \|\tilde{w}\|^{\frac{7}{6}}_{L_t^\infty(T,2T;L^2_{x,y})} \|\tilde{\nu}\|^{\frac{5}{6}}_{L_t^\infty(T,2T;L^2_{x,y})} \lesssim \delta^{-1} M T^{-\frac{7}{12}-2\delta} \|\tilde{w}\|^{\frac{7}{6}}_{Z} \|\tilde{\nu}\|^{\frac{5}{6}}_{\tilde{Z}}.$$

$$\left\| |\tilde{w}|^2 \tilde{w} \right\|_{L_t^1(T,2T;L^2_{x,y})} \lesssim \left\| \|\tilde{w}\|_{L^2_{x,y}} \|\tilde{w}\|^2_{L_x^\infty H_y^\alpha} \right\|_{L_t^1(T,2T)} \lesssim \left\| t^{-1} \|\tilde{w}\|^{\frac{4}{3}}_{L^2_{x,y}} \|\tilde{\nu}\|^{\frac{5}{3}}_{L_x^\infty} \right\|_{L_t^1(T,2T)}$$
$$\lesssim \|\tilde{w}\|^{\frac{4}{3}}_{L_t^\infty(T,2T;L^2_{x,y})} \|\tilde{\nu}\|^{\frac{5}{3}}_{L_t^\infty(T,2T;L^2_{x,y})} \lesssim T^{-\frac{2}{3}-3\delta} \|\tilde{w}\|^{\frac{4}{3}}_{Z} \|\tilde{\nu}\|^{\frac{5}{3}}_{\tilde{Z}}.$$

□

Lemma 5.2.2. *For $\delta < \frac{1}{24}$, there are the bounds associated with $\tilde{\nu}$:*

$$\|\Phi L_{lin}(u_{app}, \tilde{\nu})\|_{\tilde{Z}} \lesssim M^2 \|\tilde{\nu}\|_{\tilde{Z}}, \tag{5.11}$$

$$\|\Phi Q_2(u_{app}, \tilde{\nu})\|_{\tilde{Z}} \lesssim \|\tilde{w}\|^{\frac{1}{3}}_{Z} \|\tilde{\nu}\|^{\frac{8}{3}}_{\tilde{Z}} + \delta^{-1} M \|\tilde{w}\|^{\frac{1}{6}}_{Z} \|\tilde{\nu}\|^{\frac{11}{6}}_{\tilde{Z}}, \tag{5.12}$$

$$\|\Phi g(u_{app}, \tilde{w})\|_{\tilde{Z}} \lesssim \delta^{-1} M^2 \|\tilde{w}\|^{\frac{1}{6}}_{Z} \|\tilde{\nu}\|^{\frac{5}{6}}_{\tilde{Z}} + \delta^{-1} M \|\tilde{w}\|^{\frac{1}{3}}_{Z} \|\tilde{\nu}\|^{\frac{5}{3}}_{\tilde{Z}}. \tag{5.13}$$

Proof. By a similar process as the estimates for the Z norm, we have

$$\left\| |\tilde{w}|^2 \tilde{\nu} \right\|_{L_t^1(T,2T;L^2_{x,y})} \lesssim \left\| \|\tilde{w}\|^2_{L_x^\infty H_y^\alpha} \|\tilde{\nu}\|_{L^2_{x,y}} \right\|_{L_t^1(T,2T)} \lesssim T^{-\frac{1}{6}-3\delta} \|\tilde{w}\|^{\frac{1}{3}}_{Z} \|\tilde{\nu}\|^{\frac{8}{3}}_{\tilde{Z}},$$

$$\|\overline{u_{app}}\tilde{w}\tilde{\nu}\|_{L_t^1\left(T,2T;L_{x,y}^2\right)} \lesssim \left\| \|\tilde{w}\|_{L_x^\infty H_y^\alpha} \|u_{app}\|_{L_x^\infty H_y^\alpha} \|\tilde{\nu}\|_{L_{x,y}^2} \right\|_{L_t^1(T,2T)} \lesssim \delta^{-1} M T^{-\frac{1}{12}-\delta} \|\tilde{w}\|_Z^{\frac{1}{6}} \|\tilde{\nu}\|_{\tilde{Z}}^{\frac{11}{6}}.$$

Since the operator can either by D_y^s or L_x here, we compute the $L_x$case, and the D_y^s case follows the same manner. There are the bounds

$$\begin{aligned}
&\|w\left(L_x u_{app}\right)\overline{u_{app}}\|_{L_t^1\left(T,2T;L_{x,y}^2\right)} \lesssim \left\| \|\tilde{w}\|_{L_x^\infty H_y^\alpha} \|u_{app}\|_{L_x^\infty H_y^\alpha} \|L_x u_{app}\|_{L_{x,y}^2} \right\|_{L_t^1(T,2T)} \\
\lesssim &\delta^{-1} M^2 T^{2\delta} \|\tilde{w}\|_{L_t^\infty\left(T,2T;L_{x,y}^2\right)}^{\frac{1}{6}} \|\tilde{\nu}\|_{L_t^\infty\left(T,2T;L_{x,y}^2\right)}^{\frac{5}{6}} \lesssim \delta^{-1} M^2 T^{-\frac{1}{12}+\delta} \|\tilde{w}\|_Z^{\frac{1}{6}} \|\tilde{\nu}\|_{\tilde{Z}}^{\frac{5}{6}},
\end{aligned}$$

$$\begin{aligned}
&\left\|w^2\overline{L_x u_{app}}\right\|_{L_t^1\left(T,2T;L_{x,y}^2\right)} \lesssim \left\| \|\tilde{w}\|_{L_x^\infty H_y^\alpha}^2 \|L_x u_{app}\|_{L_{x,y}^2} \right\|_{L_t^1(T,2T)} \\
\lesssim &\delta^{-1} M T^{\delta} \|\tilde{w}\|_{L_t^\infty\left(T,2T;L_{x,y}^2\right)}^{\frac{1}{3}} \|\tilde{\nu}\|_{L_t^\infty\left(T,2T;L_{x,y}^2\right)}^{\frac{5}{3}} \lesssim \delta^{-1} M T^{-\frac{1}{6}-\delta} \|\tilde{w}\|_Z^{\frac{1}{3}} \|\tilde{\nu}\|_{\tilde{Z}}^{\frac{5}{3}}.
\end{aligned}$$

□

By (5.9),(5.10),(5.11),(5.12), (5.13), and $M \ll \delta$ we obtain the Lipschitz dependence:

$$\begin{aligned}
&\|\Phi\left[L_{lin}\left(u_{app},\tilde{w}_1\right)+Q_1\left(u_{app},\tilde{w}_1\right)-L_{lin}\left(u_{app},\tilde{w}_2\right)-Q_1\left(u_{app},\tilde{w}_2\right)\right]\|_Z \\
\lesssim &\left(M^2+\|\tilde{w}_1\|_{Z^+}^2+\|\tilde{w}_2\|_{Z^+}^2\right)\|\tilde{w}_1-\tilde{w}_2\|_{Z^+},
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
&\|\Phi\left[L_{lin}\left(u_{app},\tilde{\nu}_1\right)+Q_2\left(u_{app},\tilde{w}_1\right)+g\left(u_{app},\tilde{w}_1\right)-L_{lin}\left(u_{app},\tilde{\nu}_2\right)-Q_2\left(u_{app},\tilde{w}_2\right)-g\left(u_{app},\tilde{w}_2\right)\right]\|_{\tilde{Z}} \\
\lesssim &\left(M^2+\|\tilde{w}_1\|_{Z^+}^2+\|\tilde{w}_2\|_{Z^+}^2\right)\|\tilde{w}_1-\tilde{w}_2\|_{Z^+}.
\end{aligned} \tag{5.15}$$

Lemma 5.2.3. *We have estimates for I_1', I_2' ,I_3':*

$$\|\Phi I_1'\|_Z \lesssim M, \quad \|\Phi I_2'\|_Z, \|\Phi I_3'\|_Z \lesssim M^3. \tag{5.16}$$

$$\|\Phi L_x I_1'\|_{\tilde{Z}} \lesssim M, \quad \|\Phi L_x I_2'\|_{\tilde{Z}}, \|\Phi L_x I_3'\|_{\tilde{Z}} \lesssim M^3. \tag{5.17}$$

$$\left\|\Phi D_y^s I_1'\right\|_{\tilde{Z}} \lesssim M, \quad \left\|\Phi D_y^s I_2'\right\|_{\tilde{Z}}, \left\|\Phi D_y^s I_3'\right\|_{\tilde{Z}} \lesssim M^3. \tag{5.18}$$

Proof. By Bernstein's inequality, we have the following inequality for any $t \geq 1$,

$$\|I_1'(t)\|_{L_{x,y}^2} \lesssim t^{-\frac{3}{2}-4\delta} \left\|D_v^{1+8\delta} W\right\|_{L_{v,y}^2} \lesssim M t^{-\frac{3}{2}-3\delta},$$

$$\|I_2'(t)\|_{L_{x,y}^2} \lesssim t^{-1} \left\| |W|^2 W - \left|P_{\leq\sqrt{t}} W\right|^2 P_{\leq\sqrt{t}} W \right\|_{L_{v,y}^2} \lesssim t^{-1} \|W\|_{L_v^\infty H_y^\alpha}^2 \left\|P_{\geq\sqrt{t}} W\right\|_{L_{v,y}^2} \lesssim M^3 t^{-\frac{3}{2}-\delta},$$

$$\|I_3'(t)\|_{L^2_{x,y}} \lesssim t^{-1}\left\|P_{\geq\sqrt{t}}\left(\left|P_{\leq\sqrt{t}}W\right|^2 P_{\leq\sqrt{t}}W\right)\right\|_{L^2_{v,y}} \lesssim t^{-\frac{3}{2}-4\delta}\|W\|^2_{L^\infty_v H^\alpha_y}\left\|D_v^{1+8\delta}W\right\|_{L^2_{v,y}} \lesssim M^3 t^{-\frac{3}{2}-\delta}.$$

Then integrating the above inequalities with respect to t in $[T,2T]$, (5.16) is straightforward.

By (5.4) and letting the projection operator $\tilde{P} = \mathcal{F}_\xi^{-1}\left[-\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\frac{\xi^2}{t} - \mathcal{X}'\left(\frac{\xi}{\sqrt{t}}\right)\frac{\xi}{2\sqrt{t}}\right]$ we separate $L_x I_1'$ into two parts:

$$L_x I_1' = it^{-\frac{3}{2}}e^{i\frac{x^2}{2t}}\tilde{P}\left(\partial_v P_{\ll\sqrt{t}}W\right)\left(t,\frac{x}{t},y\right) + it^{-\frac{3}{2}}e^{i\frac{x^2}{2t}}\tilde{P}\left(\partial_v P_{\approx\sqrt{t}}W\right)\left(t,\frac{x}{t},y\right),$$

The first part is very small due to the fact that $\tilde{P}P_{\ll\sqrt{t}} \approx 0$, hence it suffices only to do the estimate for the second part. The second part we integrate with respect to t first and get

$$\begin{aligned}\left\|t^{-\frac{3}{2}}e^{i\frac{x^2}{2t}}\tilde{P}\left(\partial_v P_{\approx\sqrt{t}}W\right)\left(t,\frac{x}{t},y\right)\right\|_{L^1_t\left(T,2T;L^2_{x,y}\right)} &\lesssim \left\|\tilde{P}\partial_v P_{\approx\sqrt{t}}W\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)}\\ &\lesssim T^{-4\delta}\left\|D_v^{1+8\delta}W\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)} \lesssim MT^{-3\delta}.\end{aligned}$$

Similar estimates apply for $D_y^s I_1'$.

By Bernstein's inequality,

$$\begin{aligned}\|L_x I_2'(t)\|_{L^1_t\left(T,2T;L^2_{x,y}\right)} &\lesssim \left\|\partial_v P_{\leq\sqrt{t}}\left(|W|^2W - \left|P_{\leq\sqrt{t}}W\right|^2 P_{\leq\sqrt{t}}W\right)\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)}\\ &\lesssim T^{\frac{1}{2}}\|W\|^2_{L^\infty_t\left(T,2T;L^\infty_v H^\alpha_y\right)}\left\|P_{\geq\sqrt{t}}W\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)} \lesssim M^3T^{-\delta},\end{aligned}$$

$$\begin{aligned}\|L_x I_3'(t)\|_{L^1_t\left(T,2T;L^2_{x,y}\right)} &\lesssim \left\|P_{\geq\sqrt{t}}\partial_v\left(\left|P_{\leq\sqrt{t}}W\right|^2 P_{\leq\sqrt{t}}W\right)\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)}\\ &\lesssim T^{-4\delta}\left\|D_v^{1+8\delta}\left(\left|P_{\leq\sqrt{t}}W\right|^2 P_{\leq\sqrt{t}}W\right)\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)} \lesssim M^3T^{-\delta}.\end{aligned}$$

Applying the inequality

$$\begin{aligned}&\left\|P_{\geq\sqrt{t}}W\right\|_{L^\infty_t\left(T,2T;L^\infty_v H^\alpha_y\right)}\\ &\lesssim \left\|P_{\geq\sqrt{t}}W\right\|^{\frac{1}{6}}_{L^\infty_t\left(T,2T;L^2_{v,y}\right)}\left\|D_y^s P_{\geq\sqrt{t}}W\right\|^{\frac{1}{3}}_{L^\infty_t\left(T,2T;L^2_{v,y}\right)}\left\|\partial_v P_{\geq\sqrt{t}}W\right\|^{\frac{1}{2}}_{L^\infty_t\left(T,2T;L^2_{v,y}\right)} \lesssim M^3T^{-\frac{1}{12}-\delta},\end{aligned}$$

we will have

$$\begin{aligned}\left\|D_y^s I_2'(t)\right\|_{L^1_t\left(T,2T;L^2_{x,y}\right)} &\lesssim \left\|D_y^s\left(|W|^2W - \left|P_{\leq\sqrt{t}}W\right|^2 P_{\leq\sqrt{t}}W\right)\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)}\\ &\lesssim \left\|\left(D_y^s W\right)W\left(P_{\geq\sqrt{t}}W\right)\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)} + \left\|W^2\left(D_y^s P_{\geq\sqrt{t}}W\right)\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)}\\ &\lesssim \|W\|_{L^\infty_t\left(T,2T;L^\infty_v H^\alpha_y\right)}\left\|P_{\geq\sqrt{t}}W\right\|_{L^\infty_t\left(T,2T;L^\infty_v H^\alpha_y\right)}\left\|D_y^s W\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)}\\ &\quad + \|W\|^2_{L^\infty_t\left(T,2T;L^\infty_v H^\alpha_y\right)}\left\|D_y^s P_{\geq\sqrt{t}}W\right\|_{L^\infty_t\left(T,2T;L^2_{v,y}\right)} \lesssim M^3T^{-\delta}.\end{aligned}$$

$$\begin{aligned}\left\| D_y^s I_3'(t) \right\|_{L_t^1\left(T,2T;L_{x,y}^2\right)} &\lesssim \left\| P_{\geq\sqrt{t}} D_y^s \left(\left| P_{\leq\sqrt{t}} W \right|^2 P_{\leq\sqrt{t}} W \right) \right\|_{L_t^\infty\left(T,2T;L_{v,y}^2\right)} \\ &\lesssim T^{-4\delta} \left\| D_y^s D_v^{8\delta} \left(\left| P_{\leq\sqrt{t}} W \right|^2 P_{\leq\sqrt{t}} W \right) \right\|_{L_t^\infty\left(T,2T;L_{v,y}^2\right)} \lesssim M^3 T^{-\delta}.\end{aligned}$$

□

Combining Lemma 5.2.3 with (5.14), (5.15), we have

$$\|\tilde{w}\|_{Z^+} \lesssim M + M^3 + \left(M^2 + \|\tilde{w}\|_{Z^+}^2 \right) \|\tilde{w}\|_{Z^+},$$

hence for M small enough there is the desired property $\|\tilde{w}\|_{Z^+} \lesssim M$, and $u = u_{app} + \tilde{w}$ is a solution to (2.1). Recalling (5.3), u_{asy} tends to u in the following sense that

$$\|u_{asy} - u\|_{L_{v,y}^2} \lesssim M t^{-\frac{1}{2}-\delta}, \quad \|u_{asy} - u\|_{L_v^2 H_y^s} \lesssim M t^{-\delta}, \quad \|L_x u_{asy} - L_x u\|_{L_{v,y}^2} \lesssim M t^{-\delta}.$$

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