The Power of Negative Thinking: The Central Role of *Modus Tollens* in Human Cognition

Stellan Ohlsson
Learning Research and Development Center
University of Pittsburgh
Pittsburgh, PA 15260
stellan@vm1.cis.pitt.edu

Nina Robin
Department of Psychology
University of California at Los Angeles
Los Angeles, CA 90024
nina@cognet.ucla.edu

Abstract

Thinking is governed by abstract schemas. Verbal protocols illustrate spontaneous use, by logically unsophisticated subjects, of the schema known as modus tollens. The tollens inference schema appeared embedded within two reasoning strategies, the classical reductio ad absurdum and reasoning by elimination. The psychological reality of modus tollens is implicitly assumed by many theories in cognitive science and the hypothesis that it is a basic component of human cognition cannot be dismissed.

The *Modus Tollens* Schema

To explain the cohesion and organization of human cognition, thinkers in what we might call the structuralist tradition (Plato, Kant, Wertheimer, Piaget) have claimed that reasoning does not merely traverse the network of elements in memory, guided by nothing more than the relative strengths of associations, but applies cognitive structures that impose a form on the process. Recent authors have called such structures *thought-forms* (Keegan, 1989), *epistemic forms* (Collins & Ferguson, 1993), *abstract rules* (Smith, Langston & Nisbett, 1992), and *abstract schemas* (Ohlsson, 1993); we prefer the latter term.

Abstract cognitive structures come in many varieties (Collins & Ferguson, 1993; Ohlsson, 1993). An *argument schema* is a standard way of justifying a conclusion. The entities related by argument schemas are beliefs (propositions). The structure of argument schemas is provided by logical and semantic relations, e. g., coherence, conjunction, disjunction, equivalence, implication and negation.

The recent *Zeitgeist* in psychology has favored the hypotheses that people reason with non-propositional representations (Johnson-Laird, 1993) or through analogies with specific cases (Holyoak & Thagard, 1994). However, the debate continues, because careful scrutiny of the empiricial evidence supports the psychological reality of at least some argument schemas (Rips, 1994; Falmagne, 1990; Smith, Langston & Nisbett, 1992).

In this paper we argue for the reality and centrality of the schema known in logic as *modus tollens*. A tollens argument has the form *if P implies Q, and Q is false, then P must be false*. Although not intended as such by logicians, we use modus tollens as a condensed description of a cognitive process. In the schematic we use to exhibit arguments, modus tollens appears as follows:

1. *P* is the case.
2. It follows (through some chain of inferences) that *Q* is the case.
3. But *Q* is known to be false.
4. So *P* cannot, in fact, be the case.

The three key features of a tollens inference are (a) a cognitive process that develops the implications of a current belief (line 2), (b) the establishment of a cognitive conflict (line 3), and (c) the propagation of negation back to the initial belief (line 4), which is then rejected (or revised).

Casual observations are consistent with the hypothesis that modus tollens inferences are common in human reasoning. A standard gambit in everday arguments is to show that an opponent's claim has absurd or disastrous consequences. People take for granted that if the move succeeds, the targeted claim has indeed been disqualified. The typical defence is to try to block or invalidate the inference from the belief *P* to the purported and damaging consequence *Q*. Never have we met anyone who defends himself or herself against a tollens argument by saying "so my view has absurd or false consquences; so what?". The validity of the tollens schema is intuitively obvious.

However, casual observations are suggestive at best and systematic studies have so far failed to support the psychological reality of modus tollens. One study matched the empirical evidence regarding a variety of abstract rules against a set of eight criteria and concluded that "the consensus among students of the problem that most people do not use modus tollens is justified in terms of the criteria studied to date" (Smith, Langston & Nisbett, 1992, p. 31).

Our case for modus tollens has two parts. First, we provide empirical evidence for tollens inferences in naive reasoning. In two studies, modus tollens appeared embedded in more complex reasoning strategies. Second, we analyze the literature to show that the tollens schema is a (typically unacknowledged) keystone in several theories proposed in such diverse branches of cognitive science as philosophy, psychology and artificial intelligence. Both the empirical evidence and the analysis of the literature suggest that we cannot dismiss the possibility that modus tollens is a basic component of the human cognitive architecture.

Evidence for *Modus Tollens* Inferences

In two separate studies we identified two reasoning strategies that include a tollens inference as a key step. One is the classical *reductio ad absurdum* and the other we call *reasoning by elimination.* In contrast to most psychological studies of reasoning, we did not ask subjects to complete or evaluate pre-formulated arguments or proofs but engaged them in tasks that seemingly had nothing to do with logic. Also unlike many other studies of reasoning, we recorded verbal protocols rather than proportion correct answers or solution times.

Reductio ad Absurdum

To reason with the reductio ad absurdum schema, one decides which conclusion *C* one wants to prove and assumes its negation *notC*. The argument proceeds by deriving a conclusion *Q* that is known to be false. From the contradiction *Q & notQ* one then infers that *notC* must be false. But if *notC* is false, then *C* is true. Schematically:

1. *C* must be the case.
2. For suppose that *notC* were the case.
3. Then it follows (through some chain of inferences) that *Q* is the case.
4. But *Q* is known to be false (so we have the contradiction *Q & notQ*).
5. Hence, *notC* cannot be the case.
6. Therefore, *C* is the case.

Line 2 sets the stage for the reductio by introducing the assumption *notC*. Lines 3-5 specify a tollens inference, with *notC* playing the role of *P*. Line 3 spells out the implications of the assumption. Because the conclusion *Q* is false (line 4), the assumption *notC* cannot be true (line 5). The reductio is completed by asserting that the negation of the initial assumption, i. e., the desired conclusion *C*, is true (line 6). The tollens inference is embedded within the reductio: It is preceeded by the act of assuming the opposite of what one wants to conclude and succeeded by the act of concluding that because the assumption turned out false, the desired conclusion is true.

Historical instances. One of the first recorded uses of the reductio strategy is Euclid's proof that there is no largest prime number[1]:

1. There is no largest prime.
2. For suppose some number M was the largest prime.
3. Then we could form the successor to the product of all primes, N = (3*5* ... *M) +1. By construction, N is larger than M and it is not divisible by any of the prime numbers up to and including M, always leaving the remainder 1. Hence, N is either itself prime or else it must be divisible by some other prime not included in the sequence 3, 5, ... , M.
4. So either N itself or its divisor is a prime number larger than M.
5. Hence, it is not the case that M is the largest prime.
6. That is, there is no largest prime.

Another fameous reductio argument was put forward by Galileo Galilei to disprove the Aristotelian principle that heavier objects fall faster than lighter objects:

> "If then we take two bodies whose natural speeds are different, it is clear that on uniting the two, the more rapid one will be partly retarded by the slower, and the slower will be somewhat hastened by the swifter. ... But if this is true, and if a large stone moves with a speed of, say, eight while a smaller moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to [the initial] supposition ... ." (Galilei, 1954/1638, p. 63)

The historical examples document that the reductio argument schema was not invented by logicians but first appeared in the intellectual practice of scholars. Also, both of these arguments were highly convincing to their authors' contemporaries. However, Euclid and Galilei hardly qualify as average thinkers, so it could still be true that "the reductio strategy of assuming the opposite of what one wants to prove is not an obvious move for subjects who haven't had extensive mathematics training." (Rips, 1994, p. 157). We next present evidence for spontaneous reductio arguments in precisely such subjects.

Contemporary instances. We interviewed ten logically and mathematically unschooled psychology students about mechanical motion (Robin & Ohlsson, 1989). The tape recorded interviews were transcribed verbatim. Because verbal protocols are necessarily incomplete, we would not expect a subject who is thinking aloud to produce explicit verbalizations of all six steps in the reductio schema. Instead, we would expect to see an abreviated discourse which emphasizes the conclusions, e. g., "*notC*, therefore *Q*; but *notQ*, so *C*." In the following excerpts, "S" means "subject" and "I" means "interviewer". The crucial step of assuming the opposite of what is to be concluded is underlined:

<u>Case 1. Subject S1</u>:

I: Does gravity act on all objects at all times?

S: Yeah, or else <u>if it didn't</u>, you'd see objects floating away or something.

[1] Book IX, Proposition 20. The proof appears on pp. 412-413 in Volume 3 of the second Dover edition of Euclid's *Elements* (Heath, 1956). The streamlined version presented here is due to Hardy (1967/1940, pp. 92-93).

Reconstruction:
1. It must be the case that gravity acts on all objects at all times.
2. For suppose that gravity did not act on all objects at all times.
3. Then objects would float away.
4. But, in fact, objects don't float away.
5. Hence, it is not the case that gravity does not act on all objects all times.
6. Therefore, gravity does act on all objects at all times.

Case 2. Subject S3:

S: ... in space there are no forces. ...

I: But how do you know that?

S: Just because that's, they've sent people out into space. ... You don't put a man on the moon without realizing there's no gravity in space. Um, if there was gravity, if the earth's gravity extended all the way to the moon, then it would be almost impossible to get to the moon.

Reconstruction:
1. It must be the case that there is no gravity in space.
2. For suppose gravity extended all the way to the moon.
3. Then it would be impossible to get to the moon.
4. But people have, in fact, travelled to the moon.
5. Hence, gravity does not extend all the way to the moon.
6. Therefore, there is no gravity in space.

Case 3. Subject S4:

I: Does gravity act on all objects at all times?

S: Hmm, well, I guess it does because although, for example, birds, although they can, you know, lift off the ground and fly, if there, if it wasn't acting on them, once they lifted off the ground and [fly], they would just disappear into space, and they don't.

Reconstruction:
1. It must be the case that gravity acts on birds.
2. For suppose gravity was not acting on birds.
3. Then birds would disappear into space.
4. But birds do not, in fact, disappear into space.
5. Hence, it is not the case that gravity is not acting on birds.
6. Therefore, gravity does act on birds.

Case 4. Subject S5:

I: And is gravity acting on [a projectile] as it moves, up, down, and at the top?

S: Yeah, it has to act on it as it moves up because if it's not, then how will anything slow down. That's what causes it to slow down.

Reconstruction:
1. It must be the case that gravity acts on projectiles travelling upwards.
2. For suppose gravity was not acting on a projectile travelling upwards.
3. Then a projectile travelling upwards would not slow down.
4. But projectiles travelling upwards do, in fact, slow down.
5. Hence, gravity cannot not be acting on projectiles travelling upwards.
6. Therefore, gravity is acting on projectiles travelling upwards.

In each excerpt, the subject begins by indicating his/her belief ("yeah", "there are no forces", "I guess it does", "it has to act") and continues by negating it ("if it didn't", "if there was gravity","if it wasn't acting", "if it's not"). He or she then carries out a modus tollens argument by deriving a conclusion that he or she knows is wrong (that objects float away, that it is impossible to get to the moon, that birds disappear into space, that projectiles travelling upwards do not slow down). The reductio is then completed by inferring the opposite of the initial assumption, i. e., by affirming the subject's own belief. In short, these four spontaneous arguments instantiate the reductio schema, with its embedded tollens inference, quite closely.

Reasoning by Elimination

Consider the following verbal reasoning puzzle, which we will refer to as the Bench Problem:

> *Some boys are sitting on a bench. Jonas is further right than Ingvar. Olaf is further left than Ingvar. David is immediately to the left of Jonas. Who is immediately to the right of Ingvar?*

People often solve such spatial arrangement problems by constructing a mental model of the arrangement and reading off the desired answer by inspecting the model (Mani & Johnson-Laird, 1982; Ohlsson, 1980, 1984, 1990). However, some subjects solve this type of problem by successively eliminating those objects that cannot be the answer until only a single object remains; that object is then inferred to be the answer to the problem. For example, the following is the complete think aloud protocol of one subject called SII8 (Ohlsson, 1980) on the Bench Problem (the elimination inferences are underlined and information read from the problem statement appears in quotes):

1. "Who is sitting immediately to the right of Ingvar?"
2. "Olof is sitting further left than Ingvar."
3. Therefore not Olof.
4. "Jonas is sitting further right than Ingvar."
5. Therefore it could be Jonas.
6. "David is sitting to the left of Jonas."
7. Therefore it cannot be David.
8. Therefore it must be ...
9. Jonas is the correct answer.

This subject was quite consistent in using the elimination method, as can be seen in the initial segment of his protocol on a more complicated spatial arrangment problem (the elimination steps are underlined and information read from the problem statement appears in quotes):

1. "Some boys are standing in line at an ice-cream stand."

2. "Who was immediately behind Erik?"
3. We take the same method again.
4. Looking at which lines Erik occurs in.
5. "Rolf is further towards the front than Erik."
6. Therefore we eliminate Rolf.
7. Where do we have Erik on other lines?
8. "Erik is immediately behind Hans."
9. Therefore Hans is immediately in front of Erik
10. and consequently not immediately behind Erik.
11. Therefore we eliminate Rolf and Hans.

...

20. And "Erik is immediately behind Hans."
21. Therefore everybody who is in front of Hans is eliminated

How should the method of elimination be described and what is the relation between this method and modus tollens? Let P be the goal predicate (e. g., "immediately behind Erik"), and let $x_1, x_2, ..., x_n$ stand for the objects mentioned in the premises. The structure of the elimination strategy can then be described as follows:

1. $P(x_1)$ ... $P(x_{i-1})$, $P(x_i)$, $P(x_{i+1})$ ... v $P(x_n)$ are the possible hypotheses.
2. $P(x_i)$ implies Q.
3. But the premises claim or imply *notQ*.
4. Hence, $P(x_i)$ is not the case.
5. So $P(x_1)$, ... $P(x_{i-1})$, $P(x_{i+1})$ ... $P(x_n)$ are now the possible hypotheses.
6. If $n = 1$, then conclude that the only remaining hypothesis is the answer; otherwise, go to step 1.

Lines 2-4 correspond to the tollens schema: The hypothesis that the object x_i is the answer implies some conclusion which in turn contradicts one of the givens of the problem; hence, x_i is not the answer. The method of elimination can be conceptualized as an iterative application of modus tollens to each candidate in turn in a situation where there is a small and well specified set of candidates. The iterations continue until only a single hypothesis is left; that hypothesis is then asserted.

Although we only show data from this one subject here, we have observed spontaneous use of this reasoning method in several subjects (Ohlsson, 1980). We have also shown that people who do not spontaneously reason by elimination on spatial arrangement problems can be induced to do so by giving them problems which are not solvable by other methods (Ohlsson, 1984).

Summary

We observed logically unschooled subjects spontaneously employ two reasoning strategies, the classical reductio ad absurdum and reasoning by elimination. Both methods contain modus tollens as an essential component. In the reductio, the tollens inference is flanked by the acts of deliberately assuming the opposite of what one wants to conclude and of affirming the negation of that assumption. In the method of elimination, the tollens step is the 'inner loop' in an iterative procedure. Neither strategy can work without the tollens inference. Because subjects spontaneously use both strategies, they must be capable of carrying out modus tollens. Of course, these data constitute an existence proof only; they do not say anything about the frequency or prevalence of tollens inferences in everyday reasoning.

The Ubiquitous *Modus Tollens*

The idea that people revise or reject their knowledge structures when those structures are revealed to have negative consequences has been proposed again and again in various branches of the cognitive sciences. The purpose of this section is to point out similarities between several seemingly unrelated theories.

Philosopy

In *The Logic of Scientific Discovery* Karl Popper proposed that scientists revise their theories when they find that predictions derived from those theories are falsified by observations:

> "According to the view that will be put forward here, the method of critically testing theories ... always proceeds on the following lines. ... With the help of other statements, previously accepted, certain singular statements--which we may call 'predictions'--are deduced from the theory ... Next we seek a decision as regards these (and other) derived statements by comparing them with the results of practical applications and experiments. ... if the decision is negative, or in other words, if the conclusions have been *falsified*, then their falsification also falsifies the theory from which they were logically deduced."
> (Popper, 1972/1934, pp. 32-33; italics in original)

As this passage shows, Popper is quite explicit about the three key components of modus tollens: (a) the derivation of the consequences of current knowledge, (b) the establishment of conflict, and (c) the backward propagation of negation from the consequences to the knowledge. Later developments have shown that these processes are more complicated than Popper depicted them, but nobody has suggested that scientists are unaffected by conflicts between their theories and their data.

Reaction to conflict is also taken as fundamental in epistemology and logic. The "characteristic occasion" for belief revision is "... when a new belief, up for adoption, conflicts somewhat with the present body of beliefs as a body. Now when a set of beliefs is inconsistent, at least one of the beliefs must be rejected as false..." (Quine & Ullian, 1978, pp. 16-17; see Harman, 1986, for similar a view). This dictum is, of course, only true for agents who operate with modus tollens.

Psychology

Attitude change. In the period 1945-1970 several theories proposed within social psychology were based on the idea that people strive towards consistency among their attitudes and beliefs and that belief revision operates to restore consistency when the latter is disturbed by new information. The most well-known of these theories is the cognitive dissonance theory formulated by Leon Festinger and co-workers (Festinger, 1957).

> "The core notion of the theory is extremely simple: Dissonance is a negative drive state which occurs whenever an individual simultaneously holds two cognitions (ideas, beliefs, opinions) which are psychologically inconsistent. Stated differently, two cognitions are dissonant if, considering these two cognitions alone, the opposite of one follows from the other. Since the occurrence of dissonance is presumed to be unpleasant, individuals strive to reduce it by adding 'consonant' cognitions or by changing one or both cognitions to make them 'fit together' better; i. e., so that they become more consonant with each other."
> (Aronson, 1978, pp. 182-183).

Alternative formulations of the same theme were proposed by Abelson and Rosenberg (1958), Cartwright and Harari (1956) and Osgood and Tannenbaum (1955). Long dormant, the cognitive consistency tradition is currently undergoing a modest revival (see, e. g., Shultz & Lepper, 1992).

The language in which cognitive consistency theories are formulated is far removed from the austere formalism of logic. However, claims to the effect that 'dissonance' and similar concepts cannot be reduced to logical contradiction are unconvincing. Kruglanski (1989) has recently argued that every example of dissonance discussed in the cognitive consistency literature does, in fact, reduce to logical contradiction. Hence, the core cognitive process postulated in this tradition is the process described by modus tollens.

Learning, development, and education. Schank (1986) has proposed that people learn new cognitive schemas by revising existing schemas that do not quite fit observed events. The current stock of explanation patterns generates expectations about events. When those expectations are violated, learning occurs. The explanation pattern that gave rise to the violated expectation is 'tweaked' (i. e., revised) in such a way as to fit the observed event better; the result is a new explanation pattern.

The language of Schank's theory differs from the language of classical logic: Knowledge structures are called explanation patterns instead of beliefs, propositions, or theories; derived consequences are called expectations rather than implications or predictions; cognitive conflicts are called expectation violations instead of contradictions; and so on. As with the case of the cognitive dissonance theory, close scrutiny shows that these differences are terminological. Schank's explanation patterns are in fact (complex) propositions, the process of deriving an expectation is deductive, and expectation violations consist of one or more logical contradictions. The central assumption that underlies the computational machinery Schank describes is the one captured in modus tollens.

The central role of cognitive conflict in learning and development has also been emphasized in the Piagetian tradition (Piaget, 1985; Zimmerman & Blom, 1983). Similarly, science educators have proposed that cognitive conflict is required for students to overcome their misconceptions about science topics (e. g., Posner et al., 1982). Finally, Berkson and Wettersten (1984) have reformulated Popper's philosophy of science as a theory of learning from error. In each case, it is taken as given that the negation of the consequences of current knowledge structures causes alterations in those structures.

Artificial Intelligence

The process of reasoning from falsified consequences is studied in many areas of Artificial Intelligence. For example, non-monotonic logic (Gardenfors, 1988) focuses on the problem of consistency in large knowledge bases. Given the falsification of one of its consequences, what are all the rational revisions of a knowledge base? This problem was not addressed in classical logic, at least partly because logicians were not faced with large knowledge bases before the advent of the computer. From our point of view, non-monotonic logic asks how far back to propagate the negation in the modus tollens inference when P is not an isolated proposition but a component of a belief system. Obviously, this problem does not arise unless modus tollens is adopted in the first place.

Research in machine learning has produced a number of systems that operate with modus tollens as their central process. For example, Rose and Langley (1986) describe a system that uses reductio ad absurdum to deduce the chemical composition of substances, given initial hypotheses about the composition of some other substances plus knowledge of chemical reactions such as reduction. When a derivation produces a so-called unbalanced null reaction, i. e., the absurd conclusion that a particular substance has *no* components, the system revises its initial hypotheses. This and many other machine learning systems are computational implementations of modus tollens in special-purpose knowledge representations.

Summary

Cognitive theories that emphasize the propagation of negation from falsified conclusions back to the knowledge structures from which those conclusions were derived are remarkably diverse and range across cognitive systems (scientific communities, individual minds, computers) and task domains. The different theories use different terminologies to refer to knowledge structures, derivation processes, cognitive conflicts, and knowledge revisions, but the cognitive mechanisms described with those terminologies are structurally isomorpic to each other; modus tollens epitomizes the shared structure. Multiple appearances of the same idea in the minds of a large number of insightful scholars do not prove that idea, but neither do they encourage its dismissal. In addition, the empirical support for the various theories--which in some cases is quite extensive--also indirectly supports the central processes that these theories share: Deriving consequences, establishing cognitive conflicts, and revising the relevant knowledge structures.

Final Words

Two arguments for the psychological reality and importance of the modus tollens argument schema are advanced in this paper. First, scholars who lived before the formalization of logic and contemporary students who have no logic training spontaneously use reductio ad absurdum and reasoning by elimination, two reasoning strategies which include a tollens inference. Second, if we look past the differences in terminology, we see that many theories proposed in such diverse disciplines as philosophy, psychology and artificial intelligence (implicitly) postulate the modus tollens schema as a basic component of intelligence. These arguments are not conclusive, but they should make us pause before we conclude that modus tollens is not a basic component of the human cognitive architecture.

Acknowledgement

Preparation of this manuscript was supported by Grant No. N00014-93-I-1013 from the Cognitive Science Program of the Office of Naval Research. The opinions expressed are not necessarily those of the sponsoring agency and no endorsement should be inferred.

References

Abelson, R. P., & Rosenberg, M. J. (1958). Symbolic psycho-logic: A model of attitudinal cognition. *Behavioral Science, 3*, 1-13.

Aronson, E. (1978). The theory of cognitive dissonance: A current perspective. In L. Berkowitz, (Ed.), *Cognitive Theories in Social Psychology* (pp. 181-214). New York: Academic Press.

Berkson, W., & Wettersten, J. (1984). *Learning from Error*. La Salle, IL: Open Court.

Cartwright, D., & Harari, F. (1956). Structural balance: A generalization of Heider's theory. *Psychological Review, 63*, 277-293.

Collins, A., & Ferguson, W. (1993). Epistemic forms and epistemic games: Structures and strategies to guide inquiry. *Educational Psychologist, 28*(1), 25-42.

Falmagne, R. J. (1990). Language and the acquisition of logical knowledge. In W. F. Overton, (Ed.), *Reasoning, Necessity, and Logic* (pp. 111-131). Hillsdale, NJ: Erlbaum.

Festinger, L. (1957). *A Theory of Cognitive Dissonance*. Stanford, CA: Stanford University Press.

Galilei, G. (1954/1638). *Dialogues Concerning Two New Sciences*. New York: Dover.

Gardenfors, P. (1988). *Knowledge in Flux*. Cambridge, MA: MIT Press.

Hardy, G. H. (1967/1940). *A Mathematician's Apology*. Cambridge, UK: Cambridge University Press.

Harman, G. (1986). *Change in View*. Cambridge, MA: MIT Press.

Heath, T. L. (1956). *The Thirteen Books of Euclid's Elements* (2nd ed.). New York: Dover Publications.

Holyoak, K., & Thagard, P. (1994). *Mental Leaps*. Cambridge, MA: MIT Press. In press.

Johnson-Laird, P. (1993). *Human and Machine Thinking*. Hillsdale, NJ: Erlbaum.

Keegan (1989). How Charles Darwin became a psychologist. In D. B. Wallace & H. E. Gruber, (Eds.), *Creative People at Work* (pp. 107-125). New York, NY: Oxford University Press.

Kruglanski, A. W. (1989). *Lay Epistemics and Human Knowledge*. New York: Plenum Press.

Mani, K., & Johnson-Laird, P. N. (1982). The mental representation of spatial descriptions. *Memory and Cognition, 10*, 181-187.

Ohlsson, S. (1980, January). *Competence and Strategy in Reasoning with Common Spatial Concepts* (Technical Report No. 6). Stockholm, Sweden: Department of Psychology, University of Stockholm.

Ohlsson, S. (1984). Induced strategy shifts in spatial reasoning. *Acta Psychologica, 57*, 47-67.

Ohlsson, S. (1990). Trace analysis and spatial reasoning: An example of intensive cognitive diagnosis and its implications for testing. In N. Frederiksen, R. Glaser, A. Lesgold, and M. G. Shafto, (Eds.), *Diagnostic Monitoring of Skill and Knowledge Acquisition* (pp. 251-296). Norwood, NJ: Ablex.

Ohlsson, S. (1993). Abstract schemas. *Educational Psychologist, 28*(1), 51-66.

Osgood, C. E., & Tannenbaum, P. H. (1955). The principle of congruity in the prediction of attitude change. *Psychological Review, 62*, 42-55.

Piaget, J. (1985). *The Equilibration of Cognitive Structures*. Chicago, IL: University of Chicago Press.

Popper, K. (1972/1934). *The Logic of Scientific Discovery*. London, UK: Hutchinson.

Posner, G., Strike, K., Hewson, P., & Gertzog, W. (1982). Accomodation of a scientific conception: Toward a theory of conceptual change. *Science Education, 66*, 211-127.

Quine, W. V., & Ullian, J. S. (1978). *The Web of Belief* (2nd ed.). New York: Random House.

Rips, L. J. (1994). *The Psychology of Proof*. Cambridge, MA: MIT Press.

Robin, N., & Ohlsson, S. (1989). Impetus then and now: A detailed comparison between Jean Buridan and a single contemporary subject. In D. E. Hergot, (Ed.), *The History and Philosophy in Science Teaching: Proceedings of the First International Conference* (pp. 292-305). Tallahassee, FL: Florida State University.

Rose, D., & Langley, P. (1986). Chemical discovery as belief revision. *Machine Learning, 1*, 423-451.

Schank, R. (1986). *Explanation Patterns*. Hillsdale, NJ: Erlbaum.

Shultz, T. R., & Lepper, M. R. (1992). A constraint satisfaction model of cognitive dissonance phenomena. In *Proceedings of the Fourteenth Annual Conference of the Cognitive Science Society* (pp. 462-467), July 29 to August 1, Indiana University, Bloomington, Indiana.

Smith, E. E., Langston, C., & Nisbett, R. (1992). The case for rules in reasoning. *Cognitive Science, 16*, 1-40.

Zimmerman, J., & Blom, D. (1983). Toward an empirical test of the role of cognitive conflict in learning. *Developmental Review, 3*, 18-38.