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Author
Marolf, Donald

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The Spectral Analysis Inner Product for Quantum Gravity

DONALD MAROLF
Department of Physics, The University of California, Santa Barbara, CA 93106, USA

ABSTRACT
This submission to the Proceedings of the Seventh Marcel-Grossman Conference is an advertisement for the use of the "spectral analysis inner product" for minisuperspace models in quantum gravity.

The following is intended to be an *advertisement* for what will be called the "spectral analysis construction" of a physical Hilbert space for cosmological models. It contains no proofs, but instead describes the problem to be addressed and the spectral analysis "solution" with references to the literature.

Our aim is to address Dirac quantization of homogeneous cosmological models or, more generally, of time reparametrization invariant systems with a finite number of degrees of freedom. When such a system is presented in canonical form, it is described by a Hamiltonian constraint $h = 0$ on some phase space. We use lower case letters to represent classical functions on this phase space while capital letters will represent quantum operators.

The Dirac quantization prescription states that the classical constraint function h should be replaced by a quantum operator H that acts on some linear space $\mathcal{H}_{aux}$ of "states," or "wavefunctions." The quantum version of the constraint is to be the selection of so-called "physical states" ψ_{phys} through the condition

$$H\psi_{phys} = 0.$$

Of course, ambiguities remain in this proposal since, for example, the structure (Hilbert space, Banach space, etc.) of the "auxiliary space" $\mathcal{H}_{aux}$ on which H acts has not been specified. Also, it is not clear that it is strictly necessary that the physical states be members of $\mathcal{H}_{aux}$ (this point will be illustrated shortly).

In addition, the so called "physical operators" should preserve the space of physical states. This is guaranteed if every physical operator A is required to commute with the constraint operator H. This is the quantum version of the statement that a classical gauge invariant (a) has vanishing Poisson bracket with the constraint, $\{a, h\} = 0$.

By analogy with quantum mechanics of unconstrained systems, we would like our physical states to form a Hilbert space. The inner product on this space is not specified by the Dirac prescription and its specification is exactly the problem that we wish to address here. Since this question has been discussed a number of times and various proposals have been made, we quickly mention a few popular ones and comment on their properties before turning to the spectral analysis approach itself.

One such proposal is known as the Klein-Gordon inner product and is motivated by the fact that, in a common presentation, the constraint $H\psi_{phys} = 0$ takes the form of a Klein-Gordon equation with a potential. Difficulties arise, however, because the potential is usually "time-dependent." While Wald[1] has shown that a form of Klein-Gordon inner product can still be defined in such cases and that it is conserved, the corresponding quantum cosmologies expand forever[1] even when the corresponding classical models reach a maximum volume and then recollapse. Such models also display additional unusual behavior at this turning point[1].

A second proposal, called the "deparametrization" proposal, asks that a canonical transformation be performed that renders the constraint linear in some momentum P_0. This form is associated with deparametrization of the model using the conjugate variable x^0 as an internal clock. If the quantum H is constructed by replacing P_0 with $-i\frac{\partial}{\partial x^0}$, then the quantum constraint takes the form of a Schrödinger equation. As in unconstrained quantum mechanics, some sort of L^2 inner product can usually then be imposed. Unfortunately, such a canonical transformation is not always apparent and has not been shown to exist for a generic model.

In addition to noting these technical difficulties, we object in-principle to the choice of a certain classical phase space function for special treatment as a "time" variable in these approaches. The philosophy followed in *this* work and in the references is that all degrees of freedom are to be treated on an equal footing [1].

The spectral analysis proposal gives a construction of an inner product for the physical Hilbert space of finite dimensional time reparametrization invariant quantum theories for which the quantum constraint is presented in the form $H\psi = 0$ where H is an operator in some Hilbert space and zero lies in it's continuous spectrum. This inner product was proposed independently in Refs. 4,5,6 and Ref. 7 (in the context of linearized quantum gravity) and was in fact used in Ref. 7 in a more general form appropriate to the presence of an arbitrary number of constraints that form a Lie algebra. We will use a presentation along the lines of Refs. 4 and 5 as it is in these works that the properties of the approach claimed below are derived.

We first introduce the spectral analysis inner product by analogy. Suppose that the spectrum of our constraint operator (H) is entirely discrete. Then, there is a natural inner product to use on the solutions of $H\psi = 0$; namely, just the inner product induced from the auxiliary space $\mathcal{H}$. However, let us introduce this inner product on $\mathcal{H}_{phys}$ in a slightly different way. We note that the spectral theorem allows us to write the auxiliary Hilbert space uniquely in the form $\mathcal{H}_{aux} = \oplus_{\lambda\in\sigma(H)}\mathcal{H}_\lambda$ where each $\mathcal{H}_\lambda$ contains the eigenvectors of H with eigenvalue λ. The physical Hilbert space may thus be defined to be $\mathcal{H}_0$. Operators that commute with H will have a natural induced action on this Hilbert space which is Hermitian if the operator is Hermitian on $\mathcal{H}_{aux}$.

The spectral analysis proposal is based on the observation that a similar fact is true in the case where H has purely continuous spectrum. That is, at least when the spectrum of H is "uniform" (see Ref. 5) near zero, the auxiliary space can be written as

$$\mathcal{H}_{aux} = \oplus \int_{\lambda\in\sigma(H)} d\lambda \; \mathcal{H}_\lambda \tag{1}$$

where, in fact, $\mathcal{H}_\lambda$ and $\mathcal{H}_{\lambda'}$ are isomorphic for λ, λ' sufficiently close to zero. Formally, at least, this decomposition is unique so that we may define $\mathcal{H}_{phys} \equiv \mathcal{H}_0$. Now, "sufficiently smooth" operators have a natural induced action on $\mathcal{H}_{phys}$ and this action is symmetric if the original operators were symmetric in $\mathcal{H}_{aux}$.

[1] On the other hand, we have no such objection to, for example, the algebraic program of Ref. 3 and the spectral analysis approach can be used to supplement such ideas.

The following is a list of claims for the spectral analysis approach that are verified in Refs. 4 and 5.

1) This method defines a positive definite inner product.

2) Spectral analysis treats all degrees of freedom on an equal footing.

3) The method can be applied to all (Hamiltonian) Bianchi models, including Bianchi IX.

4) An overcomplete set of quantum observables (perennials in the terminology of Ref. 11) can be constructed for these models that act as hermitian operators in the Hilbert space defined by the spectral analysis inner product.

5) The spectral analysis results coincide with the usual results in the case of parameterized Newtonian systems.

6) In a spectral analysis quantization of recollapsing cosmological models, the quantum cosmologies also recollapse.

This concludes our advertisement for the spectral analysis approach. It is hoped the the results of Refs. 4 and 5 will be extended in the near future.

Acknowledgements

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