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TWO-DIMENSIONAL STRESS ANALYSIS WITH INCREMENTAL CONSTRUCTION AND CREEP

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STRUCTURAL ENGINEERING LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY CALIFORNIA

STRUCTURES AND MATERIALS RESEARCH
Department of Civil Engineering

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ABSTRACT

The purpose of this report is to present a method of analysis and a computer program for the determination of stresses and displacements within two-dimensional structures involving incremental construction and creep. The finite element method utilizing linear variation of strain within each triangular element is used for the analysis of complex two-dimensional structures of arbitrary geometry composed of bimodular material. A mathematical model has been developed which represents the influence of creep, thermal and residual stresses. The procedure is illustrated by a thermal stress analysis of a section of a gravity dam constructed incrementally.

ACKNOWLEDGMENT

The method of analysis presented in this report has been developed over a several year period. Research Projects DA-45-164-CIVENG-63-263 DA-45-164-CIVENG-66-275 and DACW-68-67-C-0049 between the University of California and the U.S. Army Corps of Engineers, Walla Walla District have provided the financial sponsorship. In addition to the three authors of the report, Professor R. W. Clough and Dr. Ian P. King participated significantly in the initial phases of this work. The purpose of this report is to summarize work to date on the finite element analysis of problems involving creep and incremental construction and to present to the sponsoring office an efficient computer program which may be used for the analysis of this type of structure.

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NOTATION AND LIST OF SYMBOLS

Matrix notation is used throughout the report. Column vectors are represented by alphabetic characters or expressions enclosed in parentheses of the type $\{\cdot\}$. Wherever the elements of a vector are themselves vectors, they are underlined $\sim$. Square matrices are represented by $[\cdot]$ and diagonal matrices by $\lceil \cdot \rfloor$. In many cases, the elements of a square matrix may themselves be vectors or square matrices. No symbolism is used to distinguish these as their nature is obvious in the context. Subscripts i, j, k, m, n have been used to define summations. Subscripted zero represents the initial value. Wherever necessary, the independent variables of a function have been parenthetically associated with it e.g. $f(\underline{x},t)$ denotes the function f of vector $\underline{x}$ and scalar t. Special notations, wherever introduced, have been defined in the text. Some of the more common symbols are listed below:

u, v	=	displacements of a point in the x and y directions
u_i, v_i	=	displacements of nodal point i in the x and y directions
ϕ_i	=	the i^{th} interpolation function
ϕ_x	=	$\partial\phi/\partial x$, $\phi_y = \partial\phi/\partial y$
ϵ_x, ϵ_y, γ_{xy}	=	strain components at any point
ϵ_{xi}	=	strain in direction x at nodal point i
ϵ_{yi}	=	strain in direction y at nodal point i
γ_{xyi}	=	shearing strain at nodal point i
σ_x, σ_y, τ_{xy}	=	stress components at any point
σ_{xi}	=	stress in direction x at nodal point i
σ_{yi}	=	stress in direction y at nodal point i

τ_{xyi}	=	shearing stress at nodal point i
C_{ij}	=	elements of the strain stress transformation matrix [C]
ζ_i	=	natural or triangular coordinates
a_i, b_i	=	global dimensions of an element
A_i	=	areas of subtriangles in a triangular element
V	=	volume of the spatial domain under consideration, bounded by the surface A
A	=	surface bounding the spatial domain N. Also used to denote area of a triangular element
h	=	thickness of the two-dimensional element, assumed unity throughout
$\Delta\sigma_j^{ⓡ}$	=	increment of stress in principal direction r applied during the j^{th} time interval
T_s	=	temperature history

I. INTRODUCTION

The Finite Element Method

The finite element method of analysis as applied to continuous structures was introduced by Clough (1,2). Within the past ten years, the method has developed into a powerful tool for the analysis of various field problems in structural and continuum mechanics. Many excellent papers and reports have been published (2,3,4,5,6) and therefore, the historical development of method will not be discussed in this report.

Essentially, the finite element method involves the replacement of the actual continuum by a finite number of discrete elements or sub-regions. These elements generally have planar or rectilinear boundaries, though recent developments permit the consideration of curved boundaries. In all cases, the geometry of the elements is defined completely by their nodal points -- points on the element boundary. For the analysis of two-dimensional problems, triangular and quadrilateral shapes are most commonly used.

For the analysis of a field problem, a spatial distribution pattern for the unknown field is assumed within each element as a function of the values at the nodal points. Constitutive relationships within each element and balance laws on the complete system of elements are then applied to develop a matrix equation relating the set of known nodal point force quantities to the set of unknown nodal point field variables.

The spatial distribution patterns are generally polynomials, and their order depends on the requirements of compatibility and the accuracy desired. The development of the matrix equations may be achieved from a direct physical equilibrium approach or from a variational formulation of the problem.

Two-dimensional Stress Problem

In the analysis of the stress problem, an assumption is made regarding the displacement pattern in an element as a spatial function of the set of nodal point displacements. For the "in-plane" problem, strains are defined in terms of displacements and hence as functions of the nodal point displacements. Equilibrium of the system is satisfied by a minimization of the potential energy. In this case, the nodal displacements act as Ritz parameters for the system. Solution of the set of resulting equilibrium equations gives the nodal point displacements. From these displacements the strain field, and hence the stress field for the system is then defined.

Incremental Construction

In the case of a structure constructed incrementally, the analysis is carried out in the usual fashion for the changes in the loads during a stage of construction, and the resulting stresses and displacements are stored by the program. As each new increment is added, the new structure is analyzed for its response to the load increment. Superposition of the sequence of incremental loads on an incrementing structure gives the history of the element stresses and the nodal displacements. Therefore, the method does predict "locked in" construction stresses.

Bimodular Material Properties

Many materials -- notably concrete, rock and soil -- even when fairly isotropic, behave differently under tensile and compressive stresses. Ambartsumyan (7,8) has shown that for linear, isotropic, bimodular materials, Clapeyron's law and Green's theorem are applicable. In this report, stress-strain relationships for bimodular behavior have been developed for plane strain as well as for plane stress.

Thermal and Initial Stresses

The analysis presented incorporates the effect of thermal and initial stresses. Thermal stresses arise in the case of temperature changes under restrained geometry and in the case of steep temperature gradients. Initial stresses can be of predominant importance in application of the method to excavation problems -- this is the reverse of the incremental construction problem.

Creep Effects

Assumption of linear viscoelasticity has been made in considering creep effects. McHenry's formulation (9) has been used with minor modification. In allowing for creep, it is assumed that relaxation of stress takes place without nodal displacements over a small time increment during which the material properties do not change. This change in stress is then neutralized by releasing the constraints and treating the stress changes as residual stresses. It is assumed that during relaxation the principal stress direction does not change significantly during the time interval.

The analysis presented allows for change in elastic and creep characteristics with time or with temperature. Thus, aging and temperature-dependent materials can be considered. Also, linearity of creep formulation assumed in the analysis is not essential. King (10) has shown that certain types of non-linearity can be treated.

Selection of Element Type and Displacement Pattern

The constant strain triangle using linear displacement functions has been extensively used (3,4,11) in the solution of two-dimensional and axisymmetric problems. However, in many cases, this element fails to give satisfactory definition of stress because of lack of continuity of the strain field across element boundaries. Averaging techniques (3) have been employed along with careful selection of element shapes to obtain acceptable results. Quadrilateral shape elements of various types have been tried to improve the analysis. Wilson (4) used a quadrilateral element composed of four constant strain triangles and reduced the degrees of freedom to eight by using "condensation process" to eliminate the additional nodal point. Recent work (5) has demonstrated the use of eight and sixteen degrees of freedom elements. In this report, an element composed of two 4-nodal point linear strain triangles is used. This element has linear displacement along edges of the element insuring compatibility of displacements between adjacent elements and has a quadratic displacement form within the element.

II. THE DISPLACEMENT METHOD OF ANALYSIS

Consider a polygonal element defined by a set of n nodal points. Assuming two degrees of displacement freedom at each nodal point, the nodal point displacement vector for the element can be written as $\begin{Bmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{Bmatrix}$ where u_i and v_i are components of displacement along mutually independent axes of reference.

Assuming a displacement pattern, the displacement components for any point in the element are

$$u = \langle \phi_1 \phi_2 \phi_3 . \ . \ . \rangle \ \{\underset{\sim}{u}\} = \{\phi\}^T \ \{\underset{\sim}{u}\} \qquad \text{(II-1)}$$

and

$$v = \{\phi\}^T \ \{\underset{\sim}{v}\}$$

Here ϕ_i represents the interpolation functions.

Strain-Displacement Equations

Assuming infinitesimal strains, the strain field in a cartesian system of coordinates is given by

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \end{Bmatrix} = \begin{bmatrix} \phi_x^{\ T} & 0 \\ 0 & \phi_y^{\ T} \\ \phi_y^{\ T} & \phi_x^{\ T} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{Bmatrix} \qquad \text{(II-2)}$$

where subscript x denotes differentiation with respect to x etc. Symbolically $\{\epsilon\} = [\phi_\epsilon]^T \ \{r\}$ where $\{r\} = \begin{Bmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{Bmatrix}$.

Stress Strain Law

Assuming linear stress strain relationship for a given time or temperature, the general anisotropic relation is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{II-3}$$

or symbolically

$$\{\sigma\} = [C]\ \{\epsilon\}$$

Potential Energy of the System

The potential energy for an elastic solid is given by the formula

$$U = \frac{1}{2}\int_V \{\sigma\}^T\ \{\epsilon\}dv - \int_V \{u\}^T\ \{F\}dv - \int_A \{u\}^T\ \{P\}dA \tag{II-4}$$

where F is the body force function and P is the boundary force function. For a finite element system, composed of M arbitrary elements, the above equation can be written as a sum of integrals, each integral covering one element. Thus,

$$U = \sum_{m=1}^{M} \left[\frac{1}{2}\int_{V_m} \{\sigma^m\}^T\ \{\epsilon^m\}dV_m - \int_{V_m} \{u^m\}^T\ \{F^m\}dV_m - \int_{A_m} \{u^m\}^T\ \{P^m\}dA_m \right] \tag{II-5}$$

The surface integral exists only if the m^{th} element is on the boundary of the structure and subjected to surface tractions P.

Substituting for stresses and strains and displacements in terms of nodal point displacements, we have

$$U = \sum_{m=1}^{M} \left[\frac{1}{2}\int_{V_m} \{r\}^T[\phi_\epsilon^{\ m}][C^m][\phi_\epsilon^{\ m}]^T\{r\}dV_m - \int_{V_m} \{r\}^T \begin{bmatrix} \phi^m & 0 \\ 0 & \phi^m \end{bmatrix} \{F^m\}dV_m - \int_{A_m} \{r\}^T\{P^m\}dA_m \right] \tag{II-6}$$

If thermal and initial stresses are present, these will do work in going through the mechanical displacements. These can be included in the expression for potential energy:

$$U = \sum_{m=1}^{M} \left[\frac{1}{2} \int_{V_m} \{r\}^T [\phi_\epsilon^{\ m}][C^m][\phi_\epsilon^{\ m}]^T \{r\} dV_m - \int_{V_m} \{r\}^T \begin{bmatrix} \phi^m & 0 \\ 0 & \phi^m \end{bmatrix} \{F^m\} dV_m \right.$$

$$- \int_{V_m} \{r\}^T [\phi_\epsilon^{\ m}]\{\sigma_t^{\ m}\} dV_m + \int_{V_m} \{r\}^T [\phi_\epsilon^{\ m}]\{\sigma_o^{\ m}\} dV_m \qquad \text{(II-7)}$$

$$\left. - \int_{A_m} \{r\}^T \{P^m\} dA_m \right]$$

where $\{\sigma_t^{\ m}\}$ is the stress vector due to temperature rise under complete constraint and $\{\sigma_o^{\ m}\}$ is the initial stress vector existing in the element prior to application of the external forces and displacements. Total stress is then defined as

$$\{\sigma^m\} = [C^m]\{\epsilon^m\} - \{\sigma_t^{\ m}\} + \{\sigma_o^{\ m}\} \qquad \text{(II-8)}$$

Treating the nodal point displacements $\{r\}$ as Ritz parameters and selecting them such that the change in potential energy is an extremum, we obtain the set of equations

$$\frac{\partial U}{\partial r_i} = 0 \ , \qquad i = 1,2,\ldots, N \qquad \text{(II-9)}$$

where N is the total number of unknown nodal point displacements. These equations give the following relationship for the equilibrium of a finite element system:

$$\sum_{m=1}^{M} \int_{V_m} [\emptyset_\epsilon{}^m][C^m][\emptyset_\epsilon{}^m]^T\{r\}dV_m = \sum_{m=1}^{M} \int_{V_m} \left(\begin{bmatrix} \emptyset^m & 0 \\ 0 & \emptyset^m \end{bmatrix} \{F^m\} + [\emptyset_\epsilon{}^m]\{\sigma_t{}^m - \sigma_o{}^m\} \right) dV_m$$

$$+ \sum_{m=1}^{M} \int_{A_m} \{P^m\} dA_m \qquad \text{(II-10)}$$

As displacements $\{r\}$ are the only variable set for a given system, we have, symbolically,

$$\{Q\} = [K]\{r\} \qquad \text{(II-11)}$$

where $[K]$ is the stiffness matrix for the complete finite element system, given by

$$[K] = \sum_{m=1}^{M} [K^m] \qquad \text{(II-12)}$$

where the element stiffness $[K^m]$ is

$$[K^m] = \int_{V_m} [\emptyset_\epsilon{}^m][C^m][\emptyset_\epsilon{}^m]^T \, dV_m \qquad \text{(II-13)}$$

The load vector $\{Q\}$ is defined as

$$\{Q\} = \sum_{m=1}^{M} \{L^m\} + \sum_{m=1}^{M} \{R^m\} \qquad \text{(II-14)}$$

where the body force matrix is

$$\{L^m\} = \int_{V_m} \left(\begin{bmatrix} \emptyset^m & 0 \\ 0 & \emptyset^m \end{bmatrix} \{F^m\} + [\emptyset_\epsilon{}^m]\{\sigma_t{}^m - \sigma_o{}^m\} \right) dV_m \qquad \text{(II-15}$$

and the load vector due to surface forces is

$$\{R^m\} = \int_{A_m} \{P^m\} dA_m \qquad \text{(II-16)}$$

Solution of the equation $\{Q\} = [K]\{r\}$ gives the nodal point displacements. These are resubstituted in the equation for stress, i.e. $\{\sigma^m\} = [C^m][\emptyset_\epsilon{}^m]^T\{r\} - \{\sigma_t{}^m - \sigma_o{}^m\}$ to evaluate the stress field.

III. THE LINEAR STRAIN QUADRILATERAL

As the quadrilateral element is composed of two triangles, properties of the linear strain triangle will first be examined.

The Linear Strain Triangle

For linear variation of strain with a triangular element, the displacement pattern has to be quadratic. For a complete quadratic expression, including rigid body displacements and the state of constant strain, six generalized coordinates are needed to define each of the two displacement components.

Using nodal point displacements as the generalized coordinates, an element with six nodal points is selected. Three nodal points are the three corners of the triangular element. The additional nodal points are, for convenience, located at mid points of the three edges (Fig. 1).

To define the displacement along an edge completely in terms of displacements at corners, it is necessary to have linear variation of displacement along the boundary of the quadrilateral. Thus, for the linear strain triangle to be part of a quadrilateral element, it will be assumed that displacements at the nodal point 4 are the average of those at nodal points 1 and 2, and, similarly, the displacements at nodal point 6 are the average of those at nodal points 1 and 3. With these assumptions, the generalized coordinates associated with nodal points 4 and 6 are eliminated, and the number of degrees of freedom for the triangular element is reduced to eight.

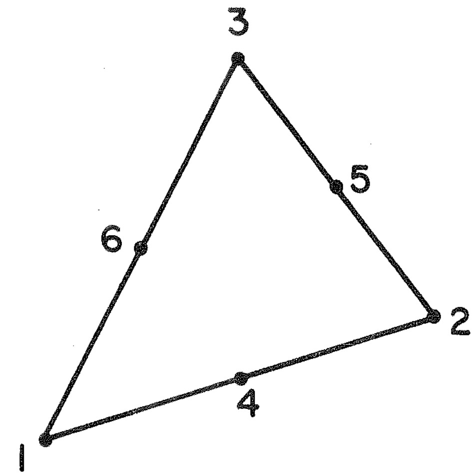


FIG. 1 LINEAR STRAIN TRIANGLE

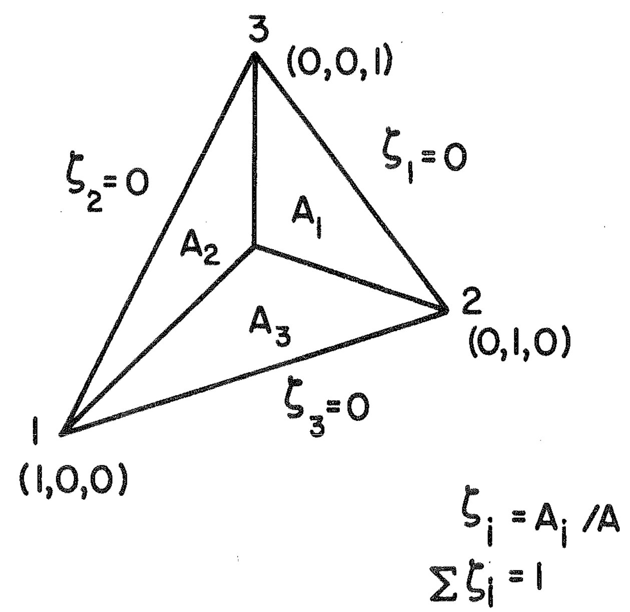


FIG. 2 TRIANGULAR (NATURAL) COORDINATES

Interpolation Functions

For a linear strain triangle, using quadratic interpolation, we have the following relationship between displacement within the element to nodal point displacements:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix} \begin{Bmatrix} \zeta_1(2\zeta_1-1) \\ \zeta_2(2\zeta_2-1) \\ \zeta_3(2\zeta_3-1) \\ 4\zeta_1\zeta_2 \\ 4\zeta_2\zeta_3 \\ 4\zeta_3\zeta_1 \end{Bmatrix} \qquad \text{(III-1)}$$

where ζ_1, ζ_2, ζ_3 are "triangular" or "natural" coordinates of any point. These coordinates are illustrated in Fig. 2 and have the following properties:

$$\begin{aligned} \zeta_i &= \frac{A_i}{A} \\ \zeta_1 + \zeta_2 + \zeta_3 &= 0 \\ \frac{\partial \zeta_i}{\partial x} &= \frac{b_i}{2A} \\ \frac{\partial \zeta_i}{\partial y} &= \frac{a_i}{2A} \end{aligned} \qquad \text{(III-2)}$$

where A_1, A_2, A_3 are the areas of three sub-triangles in Fig. 2; A is the area of the element, and a_i, b_i are the global dimensions of the element shown in Fig. 3. Using the assumptions

$$\begin{aligned} u_4 &= \frac{1}{2}(u_1 + u_2) \\ v_4 &= \frac{1}{2}(v_1 + v_2) \\ u_6 &= \frac{1}{2}(u_1 + u_3) \\ v_6 &= \frac{1}{2}(v_1 + v_3) \end{aligned} \qquad \text{(III-3)}$$

we have for the four nodal point linear strain triangle shown in Fig. 4 the following expressions for displacements

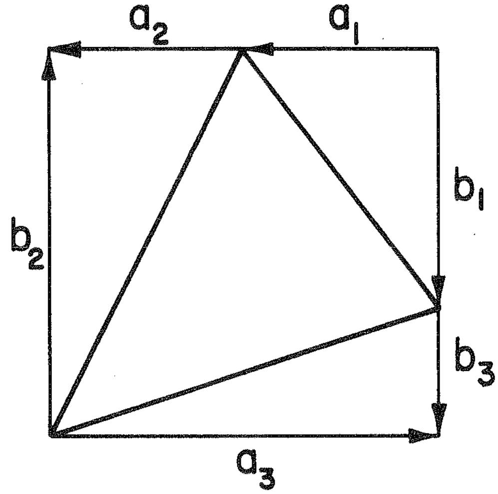


FIG. 3 GLOBAL DIMENSIONS

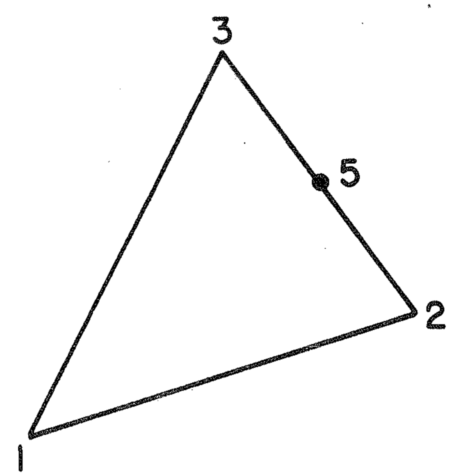


FIG. 4 FOUR-NODAL-POINT TRIANGLE

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 & u_5 \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{Bmatrix} \zeta_1 \\ \zeta_2(1-2\zeta_3) \\ \zeta_3(1-2\zeta_2) \\ 4\zeta_2\zeta_3 \end{Bmatrix} \tag{III-4}$$

Strain - Displacement Relationship

For the two-dimensional problem the infinitesimal strains are related to the displacement field by

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{III-5}$$

Substituting for displacements $\begin{Bmatrix} u \\ v \end{Bmatrix}$ in terms of nodal displacements we obtain

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \zeta_1 & \zeta_2(1-2\zeta_3) & \zeta_3(1-2\zeta_2) & 4\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_1\zeta_2(1-2\zeta_3) & \zeta_3(1-2\zeta_2) & 4\zeta_2\zeta_3 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{Bmatrix}$$

$$= \begin{bmatrix} \phi_x^T & 0 \\ 0 & \phi_y^T \\ \phi_y^T & \phi_x^T \end{bmatrix} \begin{Bmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{Bmatrix} \tag{III-6}$$

where $\{\underset{\sim}{u}\}$, $\{\underset{\sim}{v}\}$ are the nodal point displacement vectors.

Carrying out the differentiation and evaluating ϕ_x^T, ϕ_y^T for the nodal points 1, 2, 3 having natural coordinates (1,0,0); (0,1,0); (0,0,1), the nodal point strains are:

$$\begin{Bmatrix} \underset{\sim}{\varepsilon}_x \\ \underset{\sim}{\varepsilon}_y \\ \underset{\sim}{\varepsilon}_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{x1} \\ \epsilon_{x2} \\ \epsilon_{x3} \\ \epsilon_{y1} \\ \epsilon_{y2} \\ \epsilon_{y3} \\ \epsilon_{xy1} \\ \epsilon_{xy2} \\ \epsilon_{xy3} \end{Bmatrix} = \begin{bmatrix} U & 0 \\ 0 & V \\ V & U \end{bmatrix} \begin{Bmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{Bmatrix} \qquad \text{(III-7)}$$

where,

$$[U] = \frac{1}{2A} \begin{bmatrix} b_1 & b_2 & b_3 & 0 \\ b_1 & b_2-2b_3 & -b_3 & 4b_3 \\ b_1 & -b_2 & b_3-2b_2 & 4b_2 \end{bmatrix}$$

$$[V] = \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ a_1 & a_2-2a_3 & -a_3 & 4a_3 \\ a_1 & -a_2 & a_3-2a_2 & 4a_3 \end{bmatrix}$$

Linear variation of strain within the element implies the following relationship

$$\epsilon_x = \zeta_1 \epsilon_{x1} + \zeta_2 \epsilon_{x2} + \zeta_2 \epsilon_{x3}$$

or in matrix notation

$$\epsilon_x = \langle \zeta_1 \zeta_2 \zeta_3 \rangle \begin{Bmatrix} \epsilon_{x1} \\ \epsilon_{x2} \\ \epsilon_{x3} \end{Bmatrix} = \{\zeta\}^T \{\epsilon_x\} \qquad \text{(III-8)}$$

and similar relationships for ϵ_y and γ_{xy}. Thus, the strain at any point $(\zeta_1, \zeta_2, \zeta_3)$ is given by

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \zeta^T & 0 & 0 \\ 0 & \zeta^T & 0 \\ 0 & 0 & \zeta^T \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\epsilon}_x \\ \underset{\sim}{\epsilon}_y \\ \underset{\sim}{\epsilon}_{xy} \end{Bmatrix}$$

$$= \begin{bmatrix} \zeta^T & 0 & 0 \\ 0 & \zeta^T & 0 \\ 0 & 0 & \zeta^T \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \\ V & U \end{bmatrix} \begin{Bmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{Bmatrix} = [\phi_\epsilon]^T \{r\} \tag{III-9}$$

where $\{r\} = \begin{Bmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{Bmatrix}$.

<u>Stress-Strain Relationship</u>

For the two-dimensional problem, the linear stress-strain relationship is adequately represented by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{22} & C_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{III-10}$$

or, symbolically

$$\{\sigma\} = [C]\{\epsilon\} = [C][\phi_\epsilon]^T \{r\}$$

Here $[C]$ is a symmetric matrix. For the case of linear viscoelasticity, C_{ij} will be functions of time.

It is possible to assume a variation of material properties with each element. In that case, a suitable interpolation formula can be used to express C_{ij} for any point $(\zeta_1, \zeta_2, \zeta_3)$ in terms of the constitutive relationships for the nodal points. In this report the material property matrix [C] is assumed to be constant within each element.

Calculation of Stiffness of Triangular Element

We have already shown (Eq. II-13) that element stiffness is given by

$$[K^m] = \int_{V_m} [\phi^m_\epsilon]\,[C^m]\,[\phi^m_\epsilon]^T\, dVm$$

Substituting from Eq. (III-9) for $[\phi^m_\epsilon]^T$ we obtain

$$[K^m] = \int_{V_m} \begin{bmatrix} {U^m}^T & 0 & {V^m}^T \\ 0 & {V^m}^T & {U^m}^T \end{bmatrix} \begin{bmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{bmatrix} \begin{bmatrix} C^m_{11} & C^m_{12} & C^m_{13} \\ C^m_{21} & C^m_{22} & C^m_{23} \\ C^m_{31} & C^m_{32} & C^m_{33} \end{bmatrix} \begin{bmatrix} \zeta^T & 0 & 0 \\ 0 & \zeta^T & 0 \\ 0 & 0 & \zeta^T \end{bmatrix} \begin{bmatrix} U^m & 0 \\ 0 & V^m \\ V^m & U^m \end{bmatrix} dVm \qquad \text{(III-11)}$$

In this equation $[U^m]$ and $[V^m]$ are functions of the element dimensions only and are not a function of space; thus, the integration is confined to

$$[J] = \int_{V_m} \begin{bmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{bmatrix} \begin{bmatrix} C^m_{11} & C^m_{12} & C^m_{13} \\ C^m_{21} & C^m_{22} & C^m_{23} \\ C^m_{31} & C^m_{32} & C^m_{33} \end{bmatrix} \begin{bmatrix} \zeta^T & 0 & 0 \\ 0 & \zeta^T & 0 \\ 0 & 0 & \zeta^T \end{bmatrix} dVm$$

For homogeneous material properties over the element, the above integral is

$$[J] = \begin{bmatrix} C^m_{11}Q & C^m_{12}Q & C^m_{13}Q \\ C^m_{21}Q & C^m_{22}Q & C^m_{23}Q \\ C^m_{31}Q & C^m_{32}Q & C^m_{33}Q \end{bmatrix}$$

where

$$[Q] = \int_{V_m} \begin{bmatrix} \zeta_1^2 & \zeta_1\zeta_2 & \zeta_1\zeta_3 \\ \zeta_2\zeta_1 & \zeta_2 & \zeta_2\zeta_3 \\ \zeta_3\zeta_1 & \zeta_3\zeta_2 & \zeta_3^2 \end{bmatrix} dVm$$

Volume integration yields (5,12)

$$Q = \frac{A^m h^m}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \text{(III-12)}$$

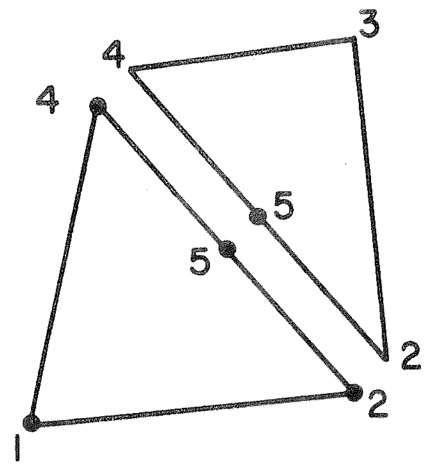
where h is the thickness of the element and is assumed to be uniform. Thus, the element stiffness, Eq. (III-11), reduces to

$$[K^m] = \frac{A^m h^m}{12} \begin{bmatrix} U^{m^T} & 0 & V^{m^T} \\ 0 & V^{m^T} & U^{m^T} \end{bmatrix} \begin{bmatrix} C^m_{11}Q & C^m_{12}Q & C^m_{13}Q \\ C^m_{21}Q & C^m_{22}Q & C^m_{23}Q \\ C_{31}Q & C_{32}Q & C_{33}Q \end{bmatrix} \begin{bmatrix} U^m & 0 \\ 0 & V^m \\ V^m & U^m \end{bmatrix} \qquad \text{(III-13)}$$

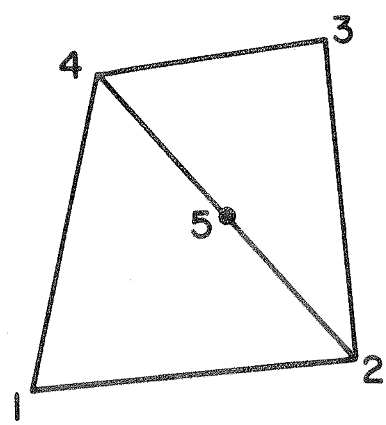
<u>Stiffness of the Quadrilateral Element</u>

The stiffness matrix for the linear strain triangle relates the nodal forces at the four nodal points to the corresponding displacements. In the quadrilateral element (Fig. 5), the element has five nodal points. To add the stiffnesses of the two component triangles, the triangle stiffness is expanded to accommodate the additional nodal point.

Thus, for triangle ①



a. TRIANGULAR SUBELEMENTS



b. ASSEMBLED ELEMENT

FIG. 5 QUADRILATERAL ELEMENT

$$[K^{\textcircled{1}}]\{r^{\textcircled{1}}\} = \begin{bmatrix} K_{11} & K_{12} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{24} & K_{25} \\ K_{41} & K_{42} & K_{34} & K_{45} \\ K_{51} & K_{42} & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_4 \\ r_5 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 & K_{14} & K_{15} \\ K_{21} & K_{22} & 0 & K_{24} & K_{25} \\ 0 & 0 & 0 & 0 & 0 \\ K_{41} & K_{42} & 0 & K_{44} & K_{45} \\ K_{51} & K_{52} & 0 & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

$$= [K_1]\{r\}$$

where $\{r_n\} = \begin{Bmatrix} u_n \\ v_n \end{Bmatrix}$, and each K_{ij} is a 2 x 2 matrix corresponding to the two degrees of freedom at each nodal point.

Also for triangle 2

$$[K^{\textcircled{2}}]\{r^{\textcircled{2}}\} = \begin{bmatrix} K_{33} & K_{34} & K_{32} & K_{35} \\ K_{43} & K_{44} & K_{42} & K_{45} \\ K_{23} & K_{24} & K_{22} & K_{25} \\ K_{53} & K_{54} & K_{52} & K_{55} \end{bmatrix} \begin{Bmatrix} r_3 \\ r_4 \\ r_2 \\ r_5 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & K_{22} & K_{23} & K_{24} & K_{25} \\ 0 & K_{32} & K_{33} & K_{34} & K_{35} \\ 0 & K_{42} & K_{43} & K_{44} & K45 \\ 0 & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

$$= [K_2]\{r\}$$

These stiffness matrices $[K_1]$ and $[K_2]$ can be directly added to give the total stiffness matrix for the quadrilateral element.

Elimination of Center Point

The central nodal point (No. 5) can be eliminated by standard "condensation" procedure (4) to obtain the 8 x 8 stiffness matrix.

Partitioning the stiffness and the load matrix, we can write

$$\begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix} \begin{Bmatrix} r_a \\ r_b \end{Bmatrix} = \begin{Bmatrix} Q_a \\ Q_b \end{Bmatrix} \qquad \text{(III-14)}$$

From the above equations,

$$\{r_b\} = [K_{bb}]^{-1} \left[\{Q\} - [K_{ba}]\{r_a\}\right]$$

Substituting in the other equation, we have

$$[K^*]\{r_a\} = \{Q_a^*\} \qquad \text{(III-15)}$$

where,

$$[K^*] = [K_{aa}] - [K_{ab}][K_{bb}]^{-1} [K_{ba}] \qquad \text{(III-16)}$$

and,

$$\{Q_a^*\} = \{Q_a\} - [K_{ab}][K_{bb}]^{-1} \{Q_b\} \qquad \text{(III-17)}$$

If desired, the stiffness calculation can be repeated with the other diagonal of the quadrilateral as the common diagonal between subelement triangles, and stiffness averaged. However, in most stress problems this would be unnecessarily expensive in computer effort.

Calculation of Load Vector

The body forces are assumed to be uniform within each element and lumped at the four nodal points of the quadrilateral. One-fourth of the total body force on the element volume is assigned to each nodal point. Actually, by carrying out the evaluation of the integral

$$\int_{V_m} \begin{bmatrix} \phi^m & 0 \\ 0 & \phi^m \end{bmatrix} \{F^m\} \, dVm$$

in Eq. (II-10), we obtain equal distribution of body loads over the four nodal points I, J, K, and L of a quadrilateral, and twice that load at the center point, for the case of two triangular subelements having equal area. Thus, it is often reasonable to assume equal distribution of the body forces for all the four nodal points.

The boundary loads are replaced by equivalent nodal point forces having the same resultant magnitude, direction and position as the boundary loads.

The initial stresses $\{\sigma_o^m\}$ and thermal stress $\{\sigma_t^m\}$ contribute to the load matrix the term

$$\left\{L_\sigma^m\right\} = \int_{V_m} \left[\phi_\epsilon^m\right] \left\{\sigma_t^m - \sigma_o^m\right\} \, dVm \qquad \text{(III-18)}$$

The stress values specified are assumed to apply to the mid-point of the common diagonal. The coordinates of this point are $(0, \frac{1}{2}, \frac{1}{2})$ in the natural system. Thus, all the quantities under the integral are constants, and we obtain

$$\left\{L_\sigma^m\right\} = A^m h^m \begin{bmatrix} {U^m}^T & 0 & {V^m}^T \\ 0 & {V^m}^T & {U^m}^T \end{bmatrix} \begin{bmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{bmatrix} \begin{Bmatrix} \sigma_t^m - \sigma_{xo}^m \\ \sigma_t^m - \sigma_{yo}^m \\ - \tau_{xyo}^m \end{Bmatrix}$$

Substituting for $\zeta = (0, \frac{1}{2}, \frac{1}{2})$,

$$\left\{L_\sigma^m\right\} = A^m h^m \begin{bmatrix} {U^m}^T \zeta & 0 & {V^m}^T \zeta \\ 0 & {V^m}^T \zeta & {U^m}^T \zeta \end{bmatrix} \begin{Bmatrix} \sigma_t^m - \sigma_{xo}^m \\ \sigma_t^m - \sigma_{yo}^m \\ - \tau_{xyo}^m \end{Bmatrix}$$

$$= A^m h^m \left[\phi_{\epsilon 0}^m\right] \left\{\underset{\sim}{\sigma}_t^m - \underset{\sim}{\sigma}_0^m\right\}$$

where

$$[\phi^m_{\epsilon 0}] = \begin{bmatrix} U^{m^T} \zeta & 0 & V^{m^T} \zeta \\ 0 & V^{m^T} \zeta & U^{m^T} \zeta \end{bmatrix}$$

and

$$\{U^{m^T} \zeta\} = \frac{1}{2A^m} \begin{Bmatrix} b^m_1 \\ -b^m_3 \\ -b^m_2 \\ -2b^m_1 \end{Bmatrix}$$

$$\{V^{m^T} \zeta\} = \frac{1}{2A^m} \begin{Bmatrix} a^m_1 \\ -a^m_3 \\ -a^m_2 \\ -2a^m_1 \end{Bmatrix}$$

Thus

$$\{L^m{}_\sigma\} = \begin{bmatrix} b^m_1 & 0 & a^m_1 \\ -b^m_3 & 0 & -a^m_3 \\ -b^m_2 & 0 & -a^m_2 \\ -2b^m_1 & 0 & -2a^m_1 \\ 0 & a^m_1 & b^m_1 \\ 0 & -a^m_3 & -b^m_3 \\ 0 & -a^m_2 & -b^m_2 \\ 0 & -2a^m_1 & -2b^m_1 \end{bmatrix} \begin{Bmatrix} \sigma^m_t - \sigma^m_{xo} \\ \sigma^m_t - \sigma^m_{yo} \\ -\tau^m_{xyo} \end{Bmatrix} \qquad \text{(III-19)}$$

for each triangular subelement. For the quadrilateral, the load vector is expanded to dimension 10 x 1 and then the vectors for the two subelements can be added directly.

IV. CREEP ANALYSIS

One-Dimensional Creep

If a material that exhibits creep behavior is subjected to a constant stress level σ, the strain response $\varepsilon(t)$ is a function of time. Under uniaxial loading, the creep law can be written as

$$\varepsilon(t) = \sigma J(t) \tag{IV-1}$$

where $J(t)$ is defined as the uniaxial creep compliance. The basic assumption made in the field of linear viscoelasticity is that the compliance $J(t)$ is independent of the stress level σ. This assumption allows the principle of superposition to be used (Boltzmann, Volterra). Therefore, if the stress is applied incrementally over a period of time, the strain at some later time "t" may be calculated by summing the effects of each increment of stress. Mathematically, this can be stated as

$$\varepsilon(t) = \int d\varepsilon(t) \tag{IV-2}$$

From Eq. (IV-1), the incremental strain is defined as

$$\begin{aligned} d\varepsilon(t) &= J(\bar{t})\ d\sigma(\tau) \\ &= J(\bar{t})\ \frac{\partial\sigma(\tau)}{\partial\tau}\ d\tau \end{aligned} \tag{IV-3}$$

where τ and $\bar{t}$ are defined in Fig. 6.

Therefore, the total strain at time "t" due to the initial stress σ_o and the incremental changes in stress is

$$\varepsilon(t) = \sigma_o\ J(t) + \int_o^t J(t-\tau)\ \frac{\partial\sigma(\tau)}{\partial\tau}\ d\tau \tag{IV-4}$$

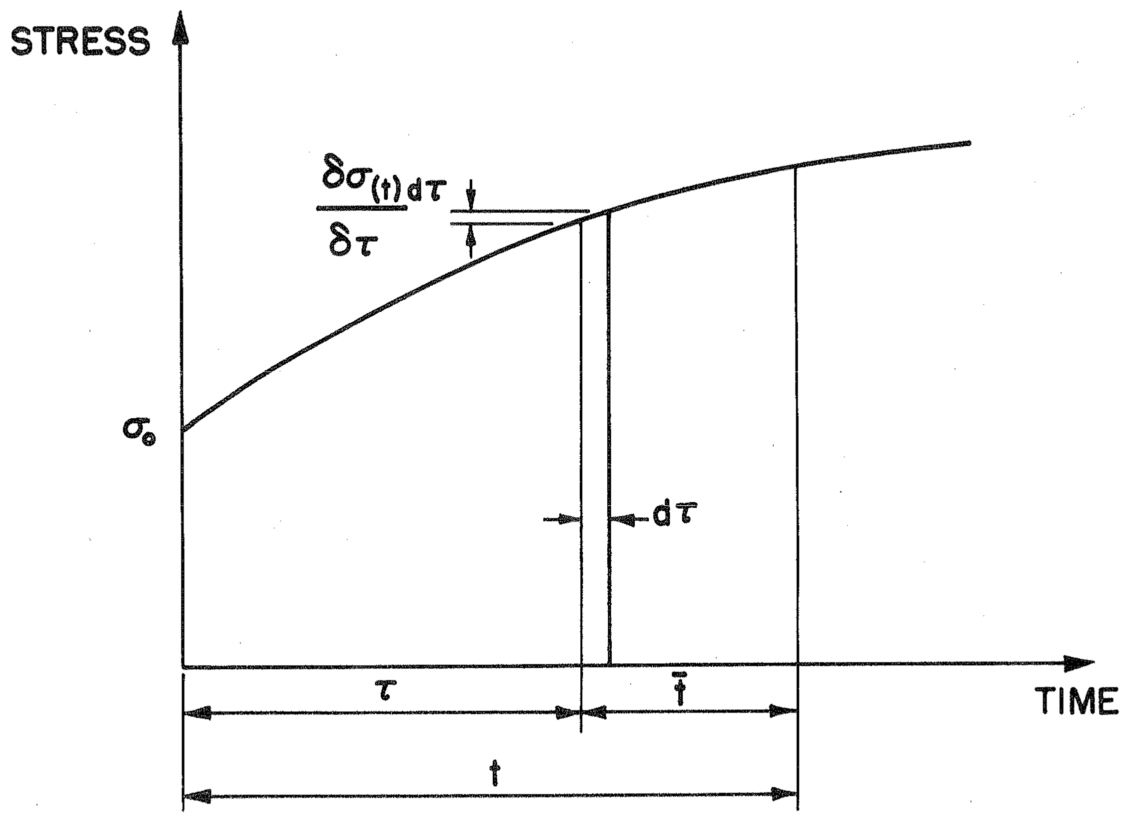


FIG. 6 TYPICAL STRESS-TIME CURVE

It is of interest to note that for statically determinate structures, where the stress must be independent of time, the integral vanishes.

One possible representation for the compliance $J(\bar{t})$ is the following exponential form:

$$J(\bar{t}) = J(o) + \sum_{n=1}^{\infty} J_n e^{-m_n \bar{t}}$$

For a material like concrete, which has properties that change with age, the above equation is generalized to include the age dependence. Then

$$J(\bar{t},\tau) = J(o,\tau) + \sum_{n=1}^{\infty} J_n(\tau) \left(1-e^{-m_n(\tau)\bar{t}}\right) \qquad \text{(IV-5)}$$

The above equation represents McHenry's creep laws according to which the compliance function

$$f(t,\tau) = f_o(t) + \sum_{n=1}^{N} f_n(\tau) \left(1-e^{-m_n(\tau)(t-\tau)}\right) \qquad \text{(IV-6)}$$

where N is made sufficiently large to adequately represent the material.

For simplicity, including the initially applied stress under the integral, for aging materials,

$$\epsilon(t) = \int_{t_o}^{t} f(t,\tau) \frac{\partial\sigma(\tau)}{\partial\tau}$$

$$= \int_{\sigma_o}^{\sigma_t} f(t,\tau) d\sigma_\tau \qquad \text{(IV-7)}$$

Or, expressing the integral as a rectangular sum

$$\epsilon(t_m) = \sum_{j=1}^{m} f(t_m,t_j) \Delta\sigma_j \qquad \text{(IV-8)}$$

where

$$f(t_m,t_j) = f_o(t_m) + \sum_{n=1}^{N} f_n(t_j) \left(1-e^{-m_n(t_j)[t_m-t_j]}\right) \qquad \text{(IV-9)}$$

Three-Dimensional Creep

For the case of three-dimensional stress, the above formulation may be generalized to include the influence of the other stress components. If Poisson's ratio is assumed to be independent of stress level and time, and if the principal stress directions do not change significantly during the time interval, the relationship between principal strains and stresses is

$$\epsilon_1 = \sum_{m,1}^{m} - \nu_2 \sum_{m,2}^{m} - \nu_3 \sum_{m,3}^{m}$$

where

$$\sum_{n,r}^{m} = \sum_{j=1}^{m} f^{(r)}(t_n, t_j)\ \Delta\sigma_j^{(r)} \qquad (r = 1,\ 2,\ 3)$$

ϵ_2 and ϵ_3 are given by similar equations. The superscripts (r) denote the principal stress direction and m is the number of time steps at which stress increments are applied.

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{Bmatrix} = \begin{bmatrix} 1- & \nu_2 & -\nu_3 \\ -\nu_1 & 1 & -\nu_3 \\ -\nu_1 & -\nu_2 & 1 \end{bmatrix} \begin{Bmatrix} \sum_{m,1}^{m} \\ \sum_{m,2}^{m} \\ \sum_{m,3}^{m} \end{Bmatrix} \qquad \text{(IV-10)}$$

or symbolically

$$\{\epsilon\} = [M] \left\{ \sum_{m}^{m} \right\} \qquad \text{(IV-10a)}$$

If the strain is kept constant and the stress allowed to relax,

$$\{\epsilon\} = [M] \sum_{m+1}^{m+1} = [M] \sum_{m}^{m}$$

Repeated application of this equality yields

$$\{\epsilon\} = [M] \left\{ \sum_{m+1}^{m+1} \right\} = \, . \, . \, . \, . \, . \, . = [M] \sum_{1}^{1} = \{\epsilon_{(o)}\} \qquad \text{(IV-11)}$$

where $\{\epsilon_{(o)}\}$ is the matrix of initial strains. Noting that

$$\sum_{m+1,r}^{m+1} = \sum_{m+1,r}^{m} + f^{(r)}(t_{m+1}, t_{m+1}) \, \Delta\sigma_{m+1}^{(r)} \quad ,$$

we can write the above equation as

$$[M] \left\{ f(t_{m+1}, t_{m+1}) \, \Delta\sigma_{m+1} \right\} = [M] \left\{ f(t_1, t_1) \, \Delta\sigma_1 - \sum_{m+1}^{m} \right\} \qquad \text{(IV-12)}$$

These equations give the stress changes $\Delta\sigma_{m+1}$ for the $(m+1)^{th}$ time interval as a function of the stress changes for all the preceding intervals.

Using the formulation in exponential form for $f(t_m, t_j)$, we have

$$\left\{ \sum_{m+1}^{m} \right\} = \left\{ \sum_{j=1}^{m} \left[f_o(t_{m+1}) + \sum_{n=1}^{N} f_n(t_j) \left(1 - e^{-m_n(t_j)[t_{m+1} - t_j]} \right) \right] \Delta\sigma_j \right\}$$

Changing the order of summation, we obtain, for time invariant m_n

$$\left\{ \sum_{m+1}^{m} \right\} = \left\{ \sum_{n=1}^{N} \sum_{j=1}^{m} \left(f_n(t_j) - f_n(t_j) \, e^{-m_n(t_j)[t_{m+1} - t_j]} \right) \Delta\sigma_j + \sum_{j=1}^{m} f_o[t_{m+1}] \Delta\sigma_j \right\}$$

$$= \left\{ \sum_{n=1}^{N} \left[\sum_{j=1}^{m} f_n(t_j) \, \Delta\sigma_j - e^{-m_n t_{m+1}} \sum_{j=1}^{m} f_n(t_j) \, e^{m_n t_j} \, \Delta\sigma_j \right] + f_o(t_{m+1}) \sigma_m \right\} \qquad \text{(IV-13)}$$

Let

$$b_{nm} = \sum_{j=1}^{m} f_n(t_j)\ \Delta\sigma_j$$

and (IV-14)

$$C_{nm} = \sum_{j=1}^{m} f_n(t_j)\ e^{m_n t_j}\ \Delta\sigma_j$$

then

$$b_{nm} = b_{nm-1} + f_n(t_m)\ \Delta\sigma_m$$

and (IV-15)

$$C_{nm} = C_{nm-1} + f_n(t_m)\ e^{m_n t_m}\ \Delta\sigma_m$$

Using the above symbolism, Eq. (IV-13) can be written as,

$$\left\{\sum_{m+1}^{m}\right\} = \left\{\sum_{n=1}^{N} b_{nm} - \sum_{n=1}^{N} e^{-m_n t_{m+1}}\ C_{nm} + f_o(t_{m+1})\ \sigma_m\right\} \qquad \text{(IV-16)}$$

Substituting (IV-16) in (IV-12), we have

$$[M]\{f(t_{m+1},t_{m+1})\ \Delta\sigma_{m+1}\} = [M]\left\{f_o(t_1)\ \Delta\sigma_1 - \sum_{n=1}^{N} b_{nm} + \sum_{n=1}^{N} e^{-m_n t_{m+1}}\ C_{nm} - f_o(t_{m+1})\ \sigma_m\right\} \qquad \text{(IV-17)}$$

Also, at any stage

$$\frac{\partial\epsilon}{\partial\sigma(t_j)} = f(t_j,t_j)$$

Hence,

$$[M]\{f(t_{m+1},t_{m+1})\ \Delta\sigma_{m+1}\} = [M]\{f_o(t_{m+1})\ \Delta\sigma_{m+1}\} \qquad \text{(IV-18)}$$

Substitution of (IV-18) in (IV-17) and transposition of the terms $f_o(t_{m+1})\sigma_m$ yields, on restricting summations to $N = 2$,

$$[M]\ \{f_o(t_{m+1})\sigma_{m+1}\} = [M]\ \left\{f_o(t_1)\Delta\sigma_1 - b_{1m} - b_{2m} + e^{-m_1 t_{m+1}} C_{1m} + e^{-m_2 t_{m+1}} C_{2m}\right\}$$

(IV-19)

or, as $\Delta\sigma_1 = \sigma_1$, we can write symbolically

$$[M]\ \lceil F_{m+1} \rfloor\ \{\sigma_{m+1}\} = [M]\ \lceil F_1 \rfloor\ \{\sigma_1\} - [M]\ \{L_m\} \qquad \text{(IV-20)}$$

where,

$\lceil F_{m+1} \rfloor$ = strain-stress or flexibility relationship at time stage t_{m+1},

$\lceil F_1 \rfloor$ = strain-stress or flexibility relationship at time stage t_1,

$\{\sigma_{m+1}\}, \{\sigma_1\}$ are the stress state vectors at time stages t_{m+1}, t_1 respectively, and

$$\{L_m\} = \left\{b_{1m} + b_{2m} - e^{-m_1 t_{m+1}} C_{1m} - e^{-m_2 t_{m+1}} C_{2m}\right\}$$

If $\{\sigma_m\}$ is the stress vector at time stage t_m, then

$$[M]\ \lceil F_m \rfloor\ \{\sigma_m\} = [M]\ \lceil F_1 \rfloor\ \{\sigma_1\} - [M]\ \{L_{m-1}\} \qquad \text{(IV-21)}$$

From (IV-20) and (IV-21),

$$[M]\ \ \lceil F_{m+1} \rfloor\ \{\sigma_{m+1}\} - \lceil F_m \rfloor\ \{\sigma_m\} = -\ [M]\{L_m - L_{m-1}\} \qquad \text{(IV-22)}$$

For use in computer program, a stage for stress relaxation is chosen small enough so that the stress-strain law and the creep functions can be assumed constant during this small interval. Thus, for $m + 1 = 2$, we have, assuming m_1, m_2 for any direction to be time independent,

$$[M] \; \lceil F_2 \rfloor \{\sigma_2\} - \lceil F_1 \rfloor \{\sigma_1\} = - [M] \{L_1\} \qquad \text{(IV-23)}$$

For $[F_2] = [F_1]$ and $\lceil K_1 \rfloor = \lceil F_1 \rfloor^{-1}$, we have

$$\{\Delta\sigma_2\} = \{\sigma_2 - \sigma_1\} = - \lceil K_1 \rfloor \{L_1\} \qquad \text{(IV-24)}$$

Also using

$$e^{-m_1(t_2-t_1)} = e^{-m_1 \Delta t} \approx (1 - m_1 \Delta t)$$

and

$$e^{-m_2(t_2-t_1)} \approx (1 - m_2 \Delta t)$$

$$\{L_1\} = \left\{ \Big((f_1(t_1) + f_2(t_1) - (1-m_1\Delta t)\, f_1(t_1) - (1-m_2\Delta t)\, f_2(t_1) \Big) \sigma_1 \right\}$$

$$= \left\{ (f_1(t_1)\, m_1 + f_2(t_1) m_2) \sigma_1 \right\} \qquad \text{(IV-25)}$$

Thus, $\{\sigma_2 - \sigma_1\}$ is completely defined in terms of $\{\Delta\sigma_1\}$. In general, for $\lceil F_m \rfloor$ practically equal to $\lceil F_{m+1} \rfloor$, we have

$$[M] \; \lceil F_m \rfloor \{\Delta\sigma_{m+1}\} = - [M] \{L_m - L_{m-1}\}$$

and therefore, because $[M]$ is a non-singular square matrix

$$\{\Delta\sigma_{m+1}\} = - \lceil K_m \rfloor \{L_m - L_{m-1}\} \qquad \text{(IV-26)}$$

For $m = 2$, a typical term in the column vector $\{L_m - L_{m-1}\}$ is

$$L_2^{(1)} - L_1^{(1)} = [m_1 f_1(t_2) + m_2 f_2(t_2)]\, \Delta\sigma_2 \Delta t + \Big(m_1 f_1(t_1)[1-m_1\Delta t] + m_2 f_2(t_1)[1-m_2\Delta t] \Big) \Delta\sigma_1 \Delta t$$

and so on, in general, for equal intervals of time Δt,

$$\{L_m - L_{m-1}\} = \Big\{ \big[m_1 f_1(t_m) + m_2 f_2(t_m) \big] \Delta\sigma_m \Delta t + \big[m_1 f_1(t_{m-1})[1-m_1\Delta t] + m_2 f_2(t_{m-1})[1-m_2\Delta t] \big] \Delta\sigma_{m-1} \Delta t + \ldots + \big[m_1 f_1(t_1)[1-m_1\Delta t]^{m-1} + m_2 f_2(t_1)[1-m_2\Delta t]^{m-1} \big] \Delta\sigma_1 \sigma t \Big\} \qquad \text{(IV-27)}$$

It is to be noticed that the three terms in the $\{L_m - L_{m-1}\}$ matrix are mutually independent and can follow different laws.

Specialization for Plane Stress and Plane Strain

The relaxation of the three stress components is uncoupled because of $[\![K]\!]$ being a diagonal matrix. Thus, reduction to cases of plane stress and plane strain is direct and consists only in including the effects of relaxation of only two stresses.

Noting that $[\![K_m]\!]$ is a diagonal matrix of elastic moduli in the three principal directions at time t_m, we have both for plane stress and plane strain the equality

$$\begin{Bmatrix} \Delta\sigma^{(1)}_{m+1} \\ \Delta\sigma^{(2)}_{m+1} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E^{(1)}(t_m)} & 0 \\ 0 & \frac{1}{E^{(2)}(t_m)} \end{bmatrix} \begin{Bmatrix} L^{(1)}_m - L^{(1)}_{m-1} \\ L^{(2)}_m - L^{(2)}_{m-1} \end{Bmatrix} \qquad \text{(IV-28)}$$

Thermal Creep

In case the coefficient functions in Eq. (IV-9) depend upon temperature and are not time-dependent, the system can be analyzed as follows:

$$\epsilon(t) = \int_0^t J(t-\tau, T_s) \frac{\partial\sigma_\tau}{\partial\tau} d\tau + \underset{\tau=0}{\overset{t}{g}} [T(\tau)] \qquad \text{(IV-29)}$$

where

$$\underset{\tau=0}{\overset{t}{g}} [T(\tau)] = g(T_s)$$

represents the thermal strain at zero stress and equals free thermal expansion. Also, the initial compliance is included in the integral.

In the above equation J is a function of $(t-\tau)$ and is a functional of temperature history T_s. For constant temperature and exponential variation of strain with time, it would be reasonable to assume analogously to Eq. (IV-5), taking m_n to be time and temperature independent and J_n to be temperature dependent

$$J(\bar{t},T) = J(o,T) + \sum_{n=1}^{\infty} J_n(T)\, e^{-m_n \bar{t}} \qquad \text{(IV-30)}$$

or, in a form analogous to McHenry's equation for aging materials (IV-6)

$$f(\bar{t},T) = f_o(T) + \sum_{n=1}^{N} f_n(T) \left(1-e^{-m_n(t-\tau)}\right) \qquad \text{(IV-31)}$$

Thus, using a rectangular summation to replace the integral in (IV-29)

$$\epsilon(t_m) = \sum_{j=1}^{m} f(t_m-t_j,T)\,\Delta\sigma_j + g(T_s)$$

$$= \sum_{j=1}^{m} \left[f_o(T) + \sum_{n=1}^{N} f_n(T)\left(1-e^{-m_n(t_m-t_j)}\right)\right] \Delta\sigma_j + g(T_s)$$

Changing the order of summation, we have

$$\epsilon(t_m) = \left[f_o(T) + \sum_{n=1}^{N} f_n(T)\right] \sigma_m + g(T_s) - \sum_{n=1}^{N} f_n(T) \sum_{j=1}^{m} e^{-m_n(t_m-t_j)}\Delta\sigma_j$$

If the strain is kept constant and stress allowed to relax,

$$\epsilon_1 = \sum_{j=1}^{m+1} f(t_{m+1}-t_j,T)\,\Delta\sigma_j + g(T_s)$$

$$= \sum_{j=1}^{m} f(t_m-t_j,T)\,\Delta\sigma_j + g(T_s) \qquad \text{(IV-32)}$$

and eventually

$$\epsilon_1 = f_o(T)\ \Delta\sigma_1 + g(T_s)$$

the initial strain due to first load application.

Writing

$$\sum_{j=1}^{m+1} f(t_{m+1}-t_j,T)\ \Delta\sigma_j = \sum_{j=1}^{m} f(t_{m+1}-t_j,T)\ \Delta\sigma_j + f(t_{m+1}-t_{m+1},T)\ \Delta\sigma_{m+1}$$

we have, for constant strain, from (IV-32)

$$f(t_{m+1}-t_{m+1},T)\ \Delta\sigma_{m+1} = \epsilon_1 - \sum_{j=1}^{m} f(t_{m+1}-t_j,T)\ \Delta\sigma_j - g(T_s)$$

$$= f_o(T)\ \Delta\sigma_1 - \sum_{j=1}^{m} f(t_{m+1}-t_j,T)\ \Delta\sigma_j \qquad \text{(IV-33)}$$

Now

$$\sum_{j=1}^{m} f(t_{m+1}-t_j,T)\ \Delta\sigma_j = \sum_{j=1}^{m}\left[f_o(T) + \sum_{n=1}^{N} f_n(T)\left(1-e^{-m_n(t_{m+1}-t_j)}\right)\right]\Delta\sigma_j$$

$$= f_o(T)\ \sigma_m + \sum_{n=1}^{N} f_n(T)\ \sigma_m - \sum_{n=1}^{N} f_n(T)\ e^{-m_n t_{m+1}} \sum_{j=1}^{m} e^{m_n t_j}\ \Delta\sigma_j$$

$$= \left[f_o(T) + \sum_{n=1}^{N} f_n(T)\right]\sigma_m - \sum_{n=1}^{N} f_n(T)\ e^{-m_n t_{m+1}}\ C_{nm} \qquad \text{(IV-34)}$$

where

$$C_{nm} = \sum_{j=1}^{m} e^{m_n t_j}\ \Delta\sigma_j$$

and

$$C_{nm} = C_{nm-1} + e^{m_n t_j} \Delta\sigma_m \qquad \text{(IV-35)}$$

Also

$$C_{n1} = e^{m_1 t_1} \Delta\sigma_1$$

We also have

$$f(t_{m+1}-t_{m+1}, T) = f_o(T) \qquad \text{(IV-36)}$$

Hence (IV-33) becomes

$$f_o(T)\Delta\sigma_{m+1} = f_o(T)\Delta\sigma_1 - \left(f_o(T) + \sum_{n=1}^{N} f_n(T)\right) \sigma_m + \sum_{n=1}^{N} f_n(T)\ e^{-m_n t_{m+1}} C_{nm} \qquad \text{(IV-37)}$$

Using $N = 2$,

$$f_o(T)\Delta\sigma_{m+1} = f_o(T)\Delta\sigma_1 - \left(f_o(T) + f_1(T) + f_2(T)\right) \sigma_m + f_1(T)\ e^{-m_1 t_{m+1}} C_{1m} + f_2(T)\ e^{-m_2 t_{m+1}} C_{2m}$$

or transposing $f_o(T)\sigma_m$,

$$f_o(T)\sigma_{m+1} = f_o(T)\Delta\sigma_1 - \left(f_1(T) + f_2(T)\right) \sigma_m + f_1(T)\ e^{-m_1 t_{m+1}} C_{1m} + f_2(T)\ e^{-m_2 t_{m+1}} C_{2m} \qquad \text{(IV-38)}$$

For the previous time interval, the above equation will give

$$f_o(T)\sigma_m = f_o(T)\Delta\sigma_1 - \left(f_1(T) + f_2(T)\right) \sigma_{m-1} + f_1(T) e^{-m_1 t_m} C_{1m-1} + f_2(T)\ e^{-m_2 t_m} C_{2m-1} \qquad \text{(IV-39)}$$

Subtracting (IV-39) from (IV-38) yields for $t_{m+1}-t_m = \Delta t$

$$f_o(T)\Delta\sigma_{m+1} = -\left(f_1(T) + f_2(T)\right)\Delta\sigma_m + f_1(T)\, e^{-m_1 t_m}\left(C_{1m}\, e^{-m_1\Delta t} - C_{1m-1}\right)$$
$$+ f_2(T)\, e^{-m_1 t_m}\left(C_{2m}\, e^{-m_2\Delta t} - C_{2m-1}\right)$$

Using (IV-35), the above equation gives, on setting $e^{-m_n\Delta t} \simeq (1-m_n\Delta t)$ for sufficiently small value of Δt,

$$f_o(T)\Delta\sigma_{m+1} \simeq -\left[\left(m_1 f_1(T) + m_2 f_2(T)\right)\Delta\sigma_m + \left(m_1 f_1(T)\, e^{-m_1 t_m} C_{1m-1}\right.\right.$$
$$\left.\left. + m_2 f_2(T)\, e^{-m_2 t_m} C_{2m-1}\right)\right]\Delta t$$

Repeated use of (IV-35) gives eventually

$$f_o(T)\Delta\sigma_{m+1} = - m_1 f_1(T)\left((1-m_1\Delta t)^{m-1}\Delta\sigma_1 + (1-m_1\Delta t)^{m-2}\Delta\sigma_2 + \ldots\right.$$
$$\left. + (1-m_1\Delta t)\Delta\sigma_{m-1} + \Delta\sigma_m\right)\Delta t - m_2 f_2(T)\left((1-m_2\Delta t)^{m-2}\Delta\sigma_1\right.$$
$$\left. + (1-m_2\Delta t)^{m-2}\Delta\sigma_2 + \ldots + (1-m_2\Delta t)\Delta\sigma_{m-1} + \Delta\sigma_m\right)\Delta t$$

(IV-40)

This expresses the stress variation in the $(m + 1)^{th}$ time interval as a function of the previous variations. The above analysis for uniaxial thermo-viscoelasticity can be generalized exactly as in the case of time-dependent creep, and, for a constant temperature, the formulation is exactly the same. Thus, to allow for thermal creep over a small interval of time, the temperature will be assumed to be constant. For different stages of analysis of an incremental structure, if temperature changes, the creep coefficients will change. However, a significant difference from the time-dependent analysis is that, as the structure is analyzed at each stage of construction for a stress increment, in the case of time-dependence, strain rate quantities associated with the stored stresses from previous analyses are preserved and added to the influence of the stress change $\Delta\sigma_1$, applied as the increment in stress. For thermal creep, the total stress stored, as well as the stress increment, is applied as the initial stress change $\Delta\sigma_1$, for calculation of relaxation of stress over a number of small time intervals. This is because thermal creep is age-independent and can be translated along the time axis.

V. BIMODULAR MATERIAL PROPERTIES

In the two-dimensional stress problem, the relationship between strain and stress can be written in terms of principal stresses and strain as

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_2}{E_2} & -\frac{\nu_3}{E_3} \\ -\frac{\nu_1}{E_1} & \frac{1}{E_2} & -\frac{\nu_3}{E_3} \\ -\frac{\nu_1}{E_1} & -\frac{\nu_2}{E_2} & \frac{1}{E_3} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} \qquad \text{(V-1)}$$

where 1, 2, 3 are the three principal directions or, symbolically,

$$\{\epsilon\} = [F]\{\sigma\} \qquad \text{(V-2)}$$

For [F] to be symmetrical,

$$\frac{\nu_1}{E_1} = \frac{\nu_2}{E_2} = \frac{\nu_3}{E_3} \qquad \text{(V-3)}$$

For plane stress where $\sigma_3 = 0$, we have

$$\begin{Bmatrix}\epsilon_1\\ \epsilon_2\end{Bmatrix} \begin{bmatrix}\frac{1}{E_1} & -\frac{\nu_2}{E_2}\\ -\frac{\nu_1}{E_1} & \frac{1}{E_2}\end{bmatrix} \begin{Bmatrix}\sigma_1\\ \sigma_2\end{Bmatrix}$$

$$= \frac{1}{E_2}\begin{bmatrix}\frac{E_2}{E_1} & -\nu_2\\ -\nu_2 & 1\end{bmatrix} \begin{Bmatrix}\sigma_1\\ \sigma_2\end{Bmatrix} \tag{V-4}$$

For a bimodular material exhibiting isotropy, but having different elastic properties in compression and tension, the following three cases arise:

1. $\sigma_1 \geq 0$, $\sigma_2 \geq 0$
2. $\sigma_1 \geq 0$, $\sigma_2 < 0$
3. $\sigma_1 < 0$, $\sigma_2 < 0$

If E_t, ν_t represent the elastic constants for tension and E_c, ν_c the constants for compression, then for each of the three cases above, we have respectively

Case	E_1	E_2	ν_1	ν_2	[F]
1	E_t	E_t	ν_t	ν_t	$\frac{1}{E_t}\begin{bmatrix}1 & -\nu_t\\ -\nu_t & 1\end{bmatrix}$
2	E_t	E_c	ν_t	ν_c	$\frac{1}{E_c}\begin{bmatrix}E_c/E_t & -\nu_c\\ -\nu_c & 1\end{bmatrix}$
3	E_c	E_c	ν_c	ν_c	$\frac{1}{E_c}\begin{bmatrix}1 & -\nu_c\\ -\nu_c & 1\end{bmatrix}$

For the case of plane strain, $\epsilon_3 = -(\nu_1/E_1)\,\sigma_1 - (\nu_2/E_2)\sigma_2 + (1/E_3)\sigma_3 = 0$

Hence

$$\sigma_3 = E_3\left[\frac{\nu_1\sigma_1}{E_1} + \frac{\nu_2\sigma_2}{E_2}\right] = \nu_3\ (\sigma_1 + \sigma_2)$$

This gives

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \end{Bmatrix} \begin{bmatrix} \frac{1}{E_1}(1-\nu_1\nu_3) & -\frac{\nu_2}{E_2}(1+\nu_3) \\ -\frac{\nu_2}{E_2}(1+\nu_3) & \frac{1}{E_2}(1-\nu_2\nu_3) \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} \qquad \text{(V-5)}$$

In this case, for the bimodular isotropic material, the following four cases arise:

1. $\sigma_1 \geq 0$, $\sigma_2 \geq 0$
2. $\sigma_1 \geq 0$, $\sigma_2 < 0$ but $\sigma_1 + \sigma_2 \geq 0$
3. $\sigma_1 \geq 0$, $\sigma_2 < 0$ but $\sigma_1 + \sigma_2 < 0$
4. $\sigma_1 < 0$, $\sigma_2 < 0$

The following tabulation for [F] results:

Case	E_1	E_2	ν_1	ν_2	ν_3	F
1.	E_t	E_t	ν_t	ν_t	ν_t	$\begin{bmatrix} \frac{1}{E_t}(1-\nu_t^2) & -\frac{\nu_t}{E_t}(1+\nu_t) \\ -\frac{\nu_t}{E_t}(1+\nu_t) & \frac{1}{E_t}(1-\nu_t^2) \end{bmatrix}$
2.	E_t	E_c	ν_t	ν_c	ν_t	$\begin{bmatrix} \frac{1}{E_t}(1-\nu_t^2) & -\frac{\nu_c}{E_c}(1+\nu_t) \\ -\frac{\nu_c}{E_c}(1+\nu_t) & \frac{1}{E_c}(1-\nu_c\nu_t) \end{bmatrix}$
3.	E_t	E_c	ν_t	ν_c	ν_c	$\begin{bmatrix} \frac{1}{E_t}(1-\nu_c\nu_t) & -\frac{\nu_c}{E_c}(1+\nu_c) \\ -\frac{\nu_c}{E_c}(1+\nu_c) & \frac{1}{E_c}(1-\nu_c^2) \end{bmatrix}$
4.	E_c	E_c	ν_c	ν_c	ν_c	$\begin{bmatrix} \frac{1}{E_c}(1-\nu_c^2) & -\frac{\nu_c}{E_c}(1+\nu_c) \\ -\frac{\nu_c}{E_c}(1+\nu_c) & \frac{1}{E_c}(1-\nu_c^2) \end{bmatrix}$

Writing $\nu_c = \nu$, $\nu_t = \nu\, E_t/E_c$, we obtain the following inverse relationship

$$\{\sigma\} = [C]\,\{\epsilon\}$$

where

$$[C] = \frac{E_c}{(XX-\nu R)(YY-\nu R)-\nu^2(1+R)^2}\begin{bmatrix} XX-\nu R & \nu(1+R) \\ \nu(1+R) & YY-\nu R \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \qquad \text{(V-6)}$$

such that XX, YY, R have the following values for the four cases:

Case	XX	YY	R
1.	$\frac{E_c}{E_t}$	$\frac{E_c}{E_t}$	$\nu\,\frac{E_t}{E_c}$
2.	$\frac{E_c}{E_t}$	1	$\nu\,\frac{E_t}{E_c}$
3.	$\frac{E_c}{E_t}$	1	ν
4.	1	1	ν

For the shearing stress strain relationship, the shearing stress can be replaced by a pair of equal and opposite stresses at ± 45° angle to the shearing stress direction. There is no difference between the plane stress and plane strain case because shearing stresses in 1-2 plane do not cause any dilatation. This analysis yields the relationship

$$\tau_{12} = \frac{1}{\frac{1}{E_c} + \frac{1}{E_t} + 2\nu/E_c}\,\gamma_{12} = C_{33}\gamma_{12} \qquad \text{(V-7)}$$

The complete stiffness relationship for principal directions is then

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{V-8}$$

This relationship is transformed to the global coordinates using the usual scheme for transformation of fourth rank tensors. In the present case, if $\{\sigma_p\}$, $\{\epsilon_p\}$, $[C_p]$ refer to the principal stress directions, and $\{\sigma\}$, $\{\epsilon\}$, $[C]$ refer to global coordinates, then

$$\{\sigma_p\} = [C_p]\ \{\epsilon_p\} \tag{V-9}$$

$$\{\sigma\} = [C]\ \{\epsilon\} \tag{V-10}$$

Rules for transformation of stress and strain are

$$\{\sigma_p\} = [J]\ \{\sigma\} \text{ or } \{\sigma\} = [J]^{-1}\ \{\sigma_p\} \tag{V-11}$$

$$\{\epsilon_p\} = [T]\ \{\epsilon\} \tag{V-12}$$

Then

$$\{\sigma\} = [J]^{-1}\ \{\sigma_p\} = [J]^{-1}\ [C_p]\{\epsilon_p\} = [J]^{-1}\ [C_p][T]\{\epsilon\} \tag{V-13}$$

Hence

$$[C] = [J]^{-1}\ [C_p][T] = [T]^T\ [C_p][T] \tag{V-14}$$

where

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \tag{V-15}$$

represents the strain transformation from global to principal strains, θ being the angle of major principal strain to the X-coordinate in the global system.

VI. EXAMPLE

The development presented in the foregoing chapters was applied to the analysis of stresses in a concrete gravity dam constructed lift by lift and subject to temperature rise due to hydration and to surface exposure to the atmosphere. A typical cross-section of the dam is shown in Fig. 7 and the typical cooling coil arrangement in Fig. 8. Due to considerations of symmetry only a 30 inch wide slice was analyzed as shown in Fig. 9. Typical creep test data are shown in Fig. 10. The data were analyzed to obtain the coefficients in McHenry's equation which are shown in Fig. 11. The air temperature and placement schedule of lifts in dam are shown in Fig. 12. The temperature history of the dam was obtained using Wilson's (13) heat conduction analysis procedure and constituted the data input for this example.

The computer program given in Appendix C was used to solve the two-dimensional problem of incremental construction with creep. Fig. 13 shows a history of distribution of average horizontal stress on vertical sections. The analysis successfully obtained the effect of cold new concrete being placed on relatively warm old lift resulting in sudden development of high tensile stresses and the gradual subsequent dissipation of stresses on account of creep and temperature rise due to hydration of cement.

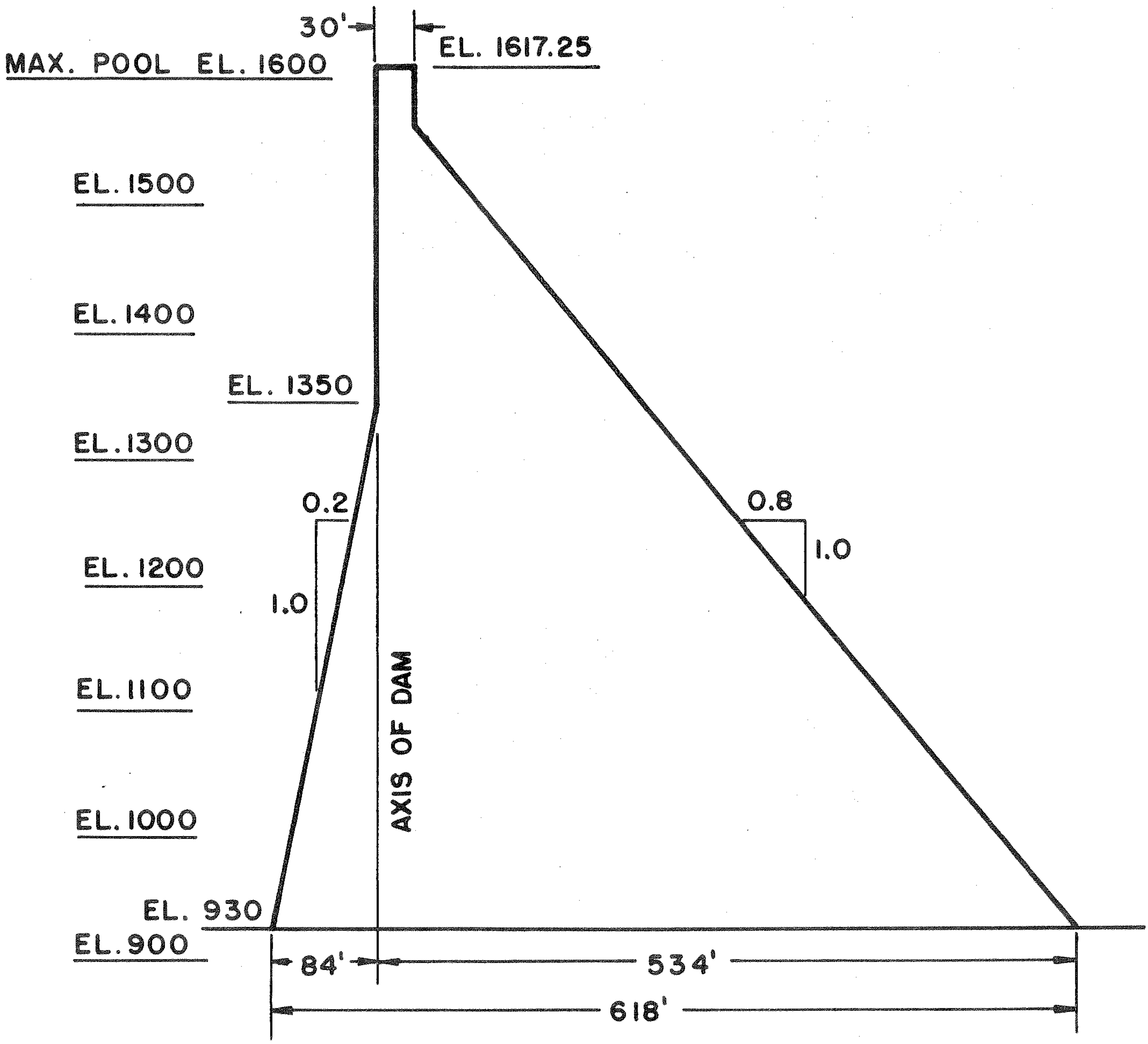


FIG. 7 CROSS SECTION OF THE DAM

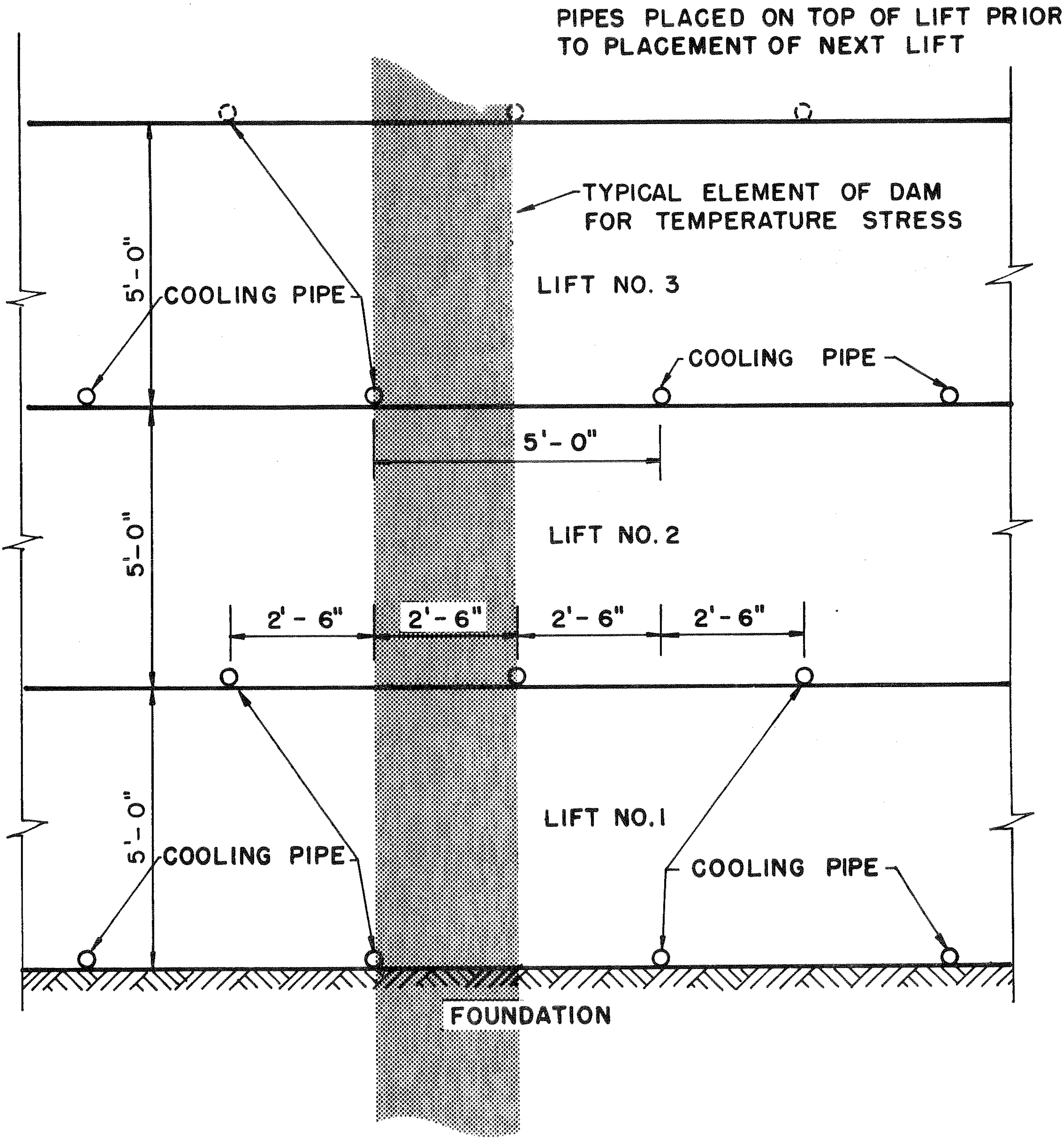


FIG. 8 TYPICAL ARRANGEMENT OF COOLING PIPES

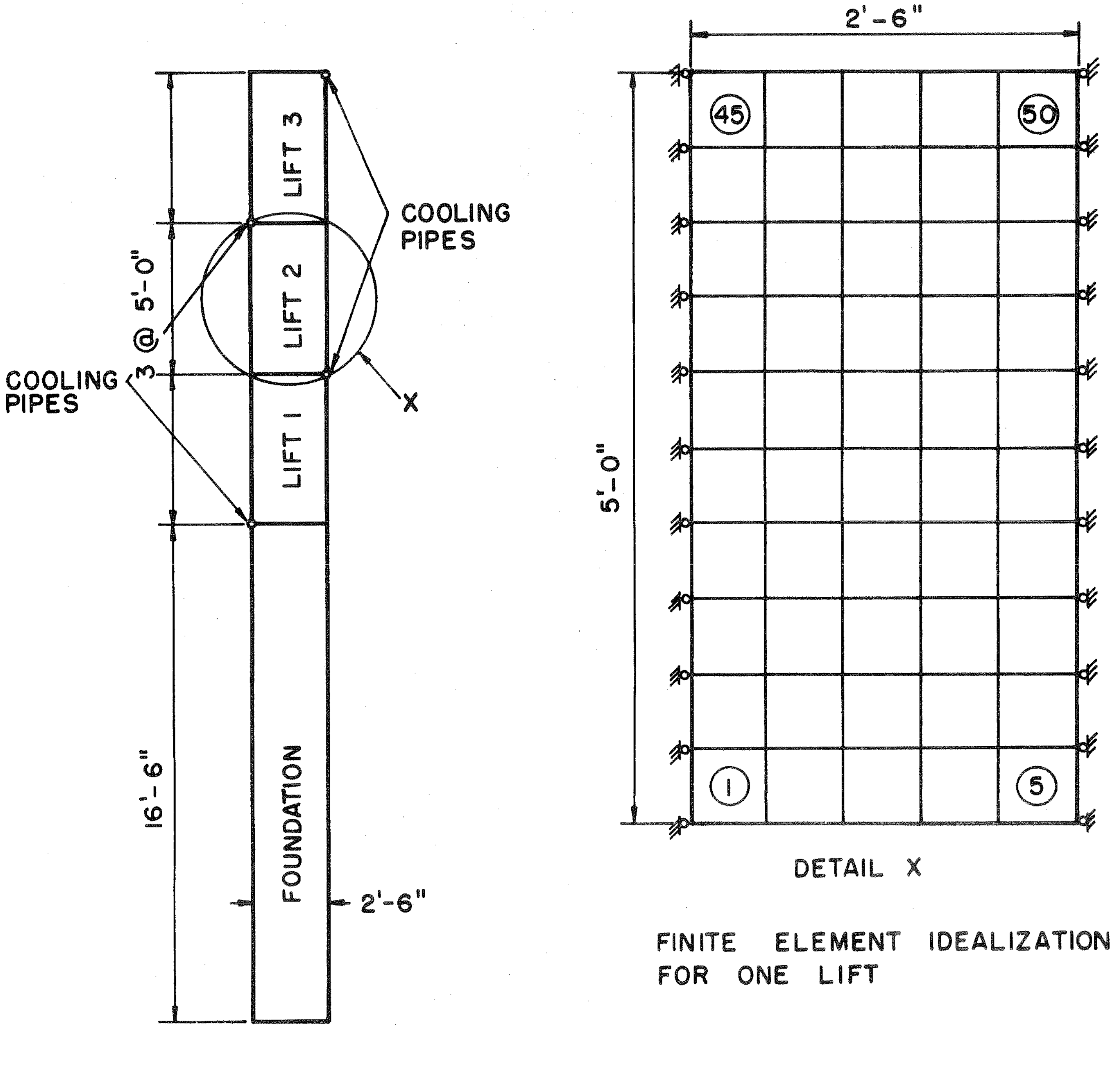


FIG. 9 STRUCTURE ANALYZED

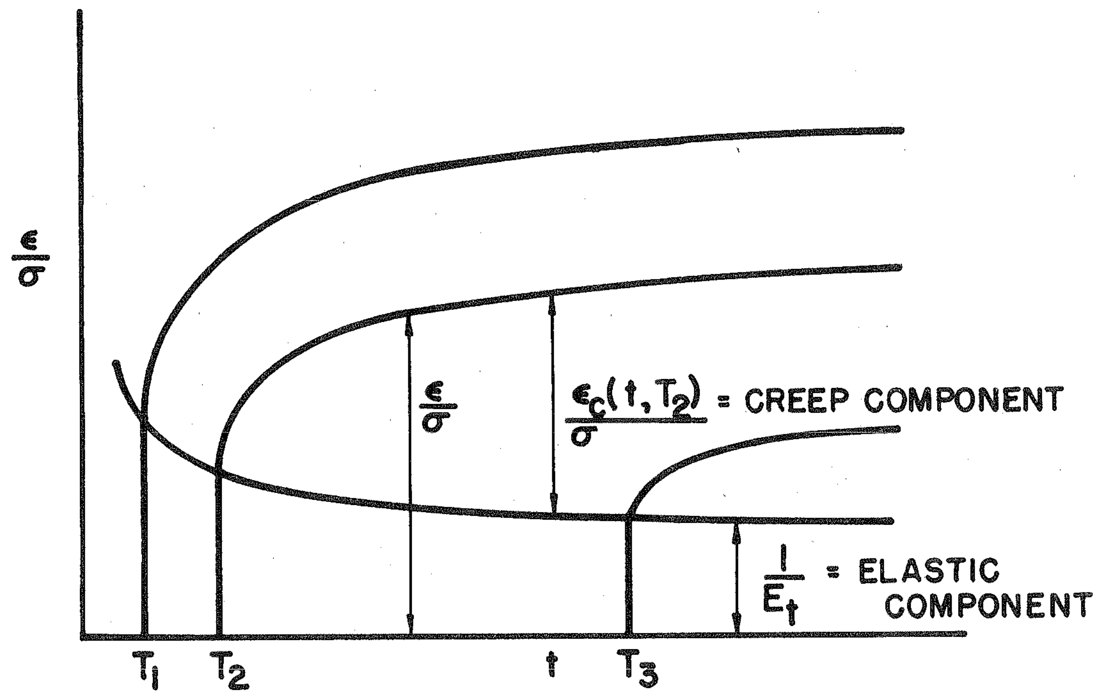


FIG. 10 TYPICAL CREEP TEST RESULTS

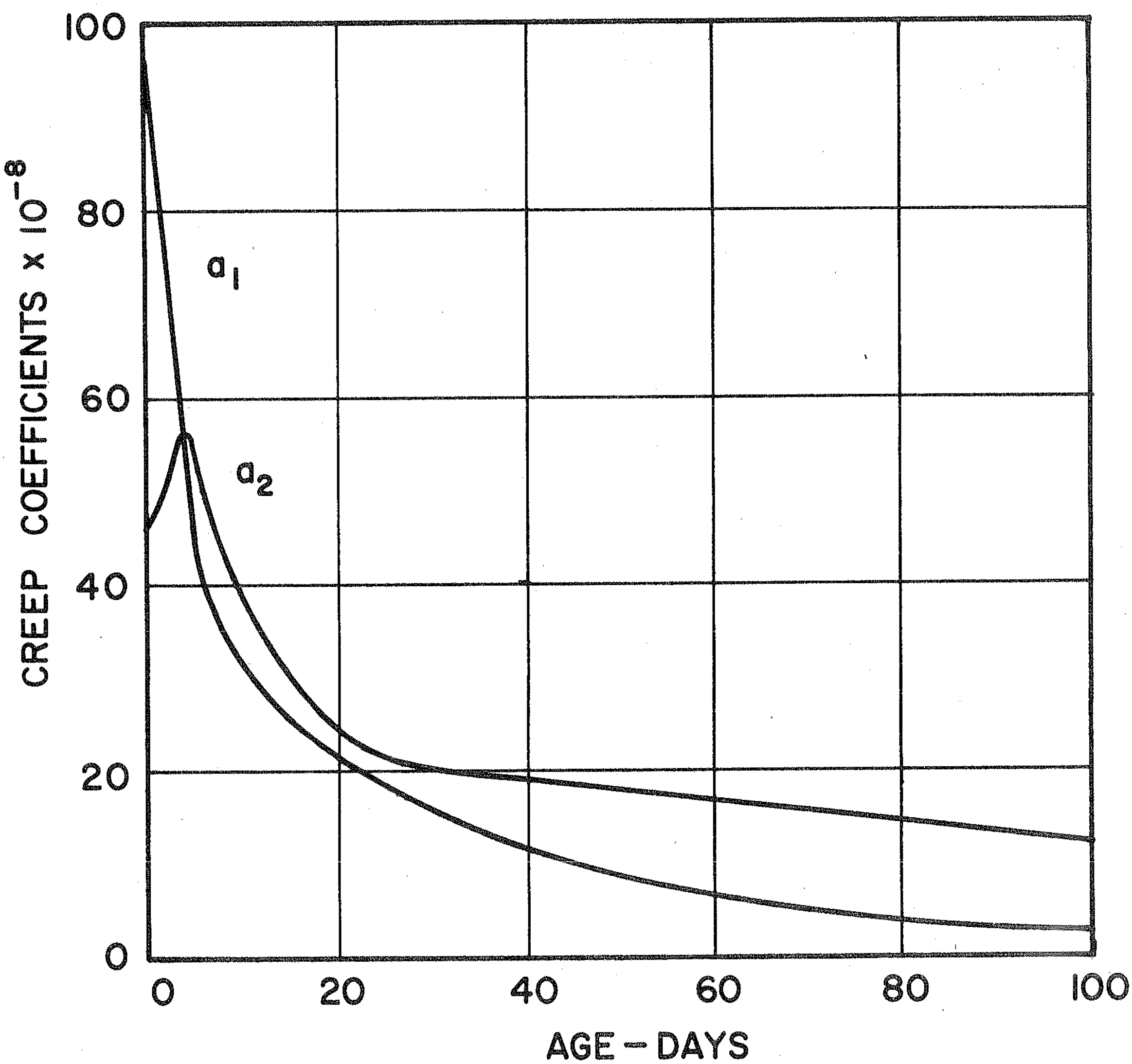


FIG. 11 VARIATION OF CREEP COEFFICIENTS WITH AGE.

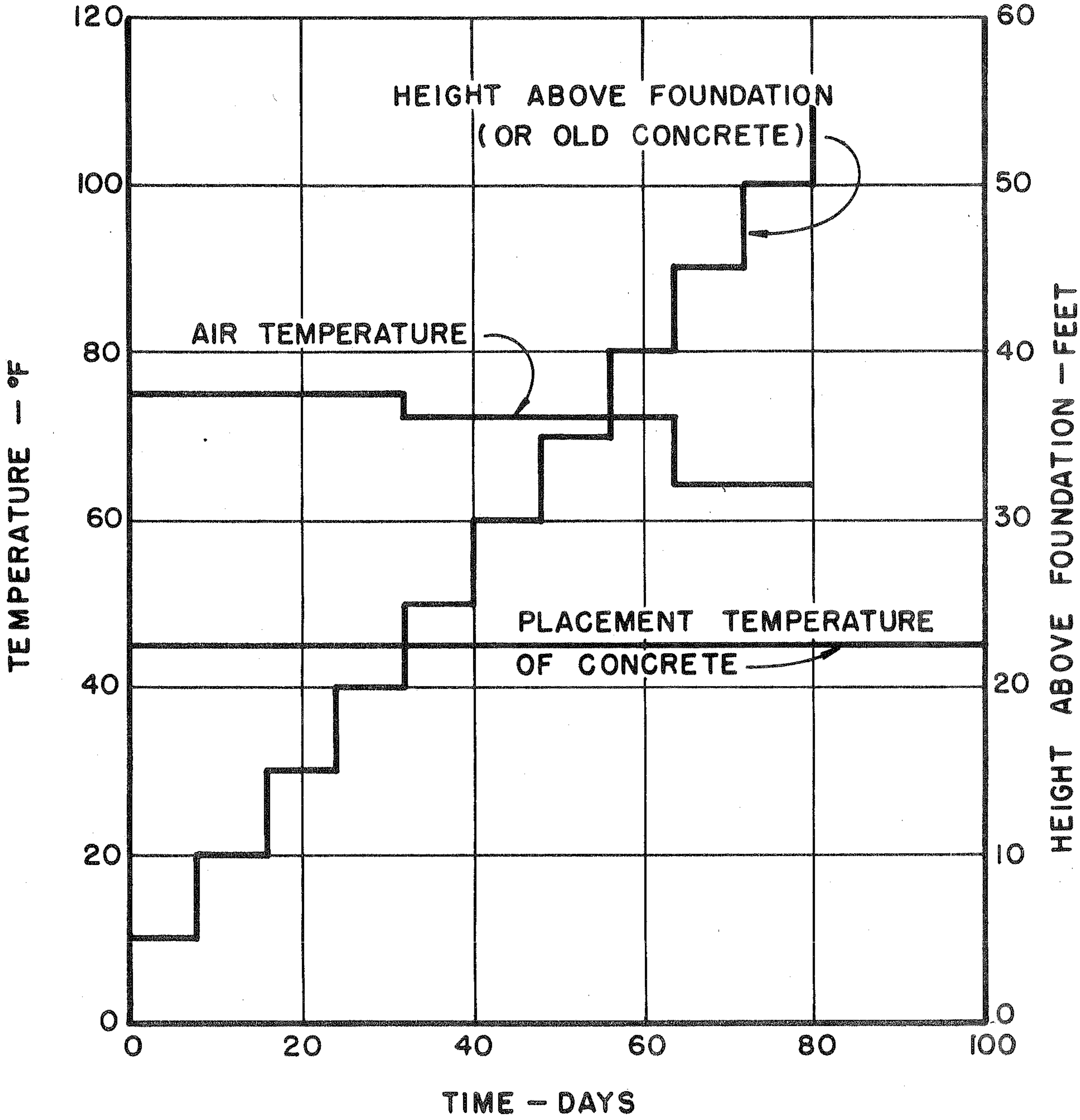


FIG. 12 AIR TEMPERATURES AND PLACEMENT SCHEDULE

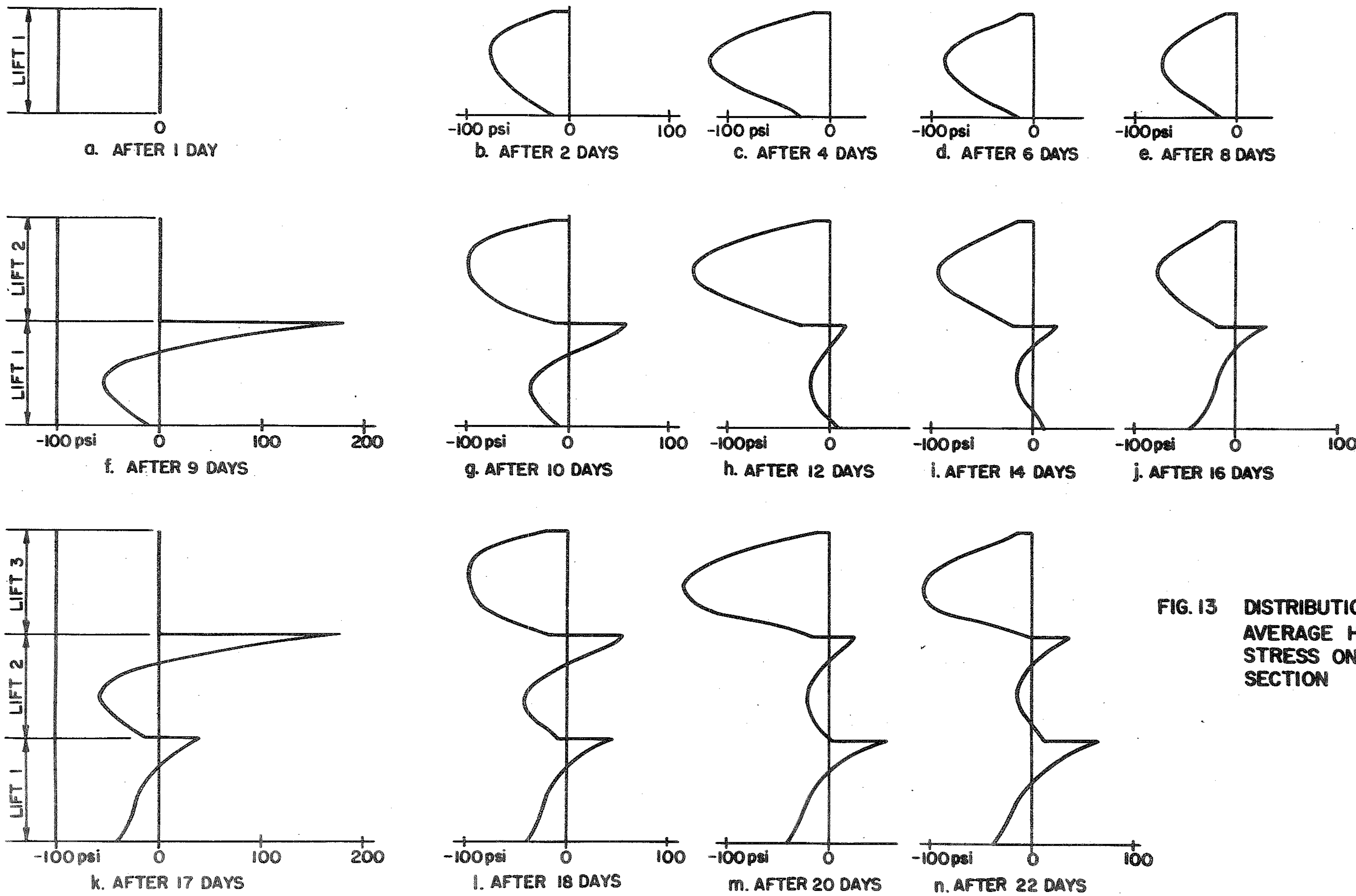


FIG. 13 DISTRIBUTION OF AVERAGE HORIZONTAL STRESS ON VERTICAL SECTION

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APPENDIX A

ORGANIZATION OF COMPUTER PROGRAM

APPENDIX A. ORGANIZATION OF COMPUTER PROGRAM

The computer program described in this Appendix is based on the theory presented in the previous sections of this report. The program is written in Fortran IV language and may be used directly on computers with 32 K or greater storage.

The program is intended to furnish a stress and displacement history for a two-dimensional structure constructed in increments, making allowance for the boundary conditions, residual stresses, stresses due to temperature change, varying pressure boundaries, bimodular behavior of the material, time- or temperature-dependent elastic properties and creep which may have time- or temperature-dependent characteristics. Small deformations and linear material laws are assumed. The structure may consist of several different materials. Creep properties may also be bimodular. Thus, the program is quite general and applicable to a wide class of problems. Examples of application would be thermal stress analysis of a concrete gravity dam constructed lift by lift allowing for creep and including effects of temperature changes, varying reservoir elevations and gravity loads. The program can be used for changing displacement boundary conditions and also for problems of incremental loading -- e.g., a beam subjected to a series of load increments, with very small modification. For large scale specialized use, some of the options in the program can easily be eliminated.

One-dimensional elements are permitted to accommodate planes of low shearing resistance such as joints, and also to permit sloping boundaries.

The principal program called MAIN controls all the data input and control information. It does the basic system initialization, and prints out the control data and material and geometrical properties of the structure. Stiffness formulation, equation solving, and creep analysis is carried out by subroutines called by MAIN.

a. Formation of Stiffness and Load Matrix

Stiffness matrix for each analysis is computed in blocks by the subroutine STIFF. For element stiffness, it calls additional subroutines -- ONED for one-dimensional elements and QUAD for the quadrilateral elements. The element stiffness is added to the total stiffness using the direct stiffness technique. Concentrated forces at nodal points are added for the newly input nodal points and pressure boundary changes included in the load matrix. Equations are modified for displacement boundary conditions by calling the subroutine MODIFY. The QUAD subroutine interpolates the elastic properties for the material of the element, calculates the principal stress-strain relationship, and then transforms it to global coordinates. With the constitutive law thus defined, subroutine EDLST is called twice to obtain the stiffness contribution of each of the two linear strain triangles and also to recover the strain displacement transformation. Unbalanced forces due to residual stresses and temperature changes are calculated using the material constitutive law and the force-stress relationship obtained from EDLST subroutine. Shear stiffness, if any, of the foundation is added and the center point eliminated from the stiffness and the load matrices, by condensation. Loads, due to gravity, are added for the elements in the newly-placed increment. All this load-stiffness information computed for the element is added to the total load and stiffness in the subroutine STIFF.

b. Calculation of Displacements

After the stiffness and load matrices for a stage are computed, the resulting equations are solved by calling the subroutine BANSOL. This subroutine uses Gaussian elimination technique developed for banded equations by Wilson (4). The displacements calculated are for the load increment only and are added in the MAIN program to the total displacements to obtain cumulative displacement history for all nodal points. These total displacements are printed out.

c. Calculation of Stresses and Creep Effects

With the displacements known, the next step is to calculate element stresses. This is done by CREEP subroutine. It calls STRESS subroutine for each element for evaluation of stresses which are printed out. The STRESS subroutine calls QUAD to obtain the strain displacement law and the stiffness for the element. This is used to obtain the element strains and thence the stress due to displacements. To this stress are added the previously stored element stresses and the unbalanced residual stresses to obtain resultant stresses in the global system. Determination of principal stresses is carried out in the usual manner. The principal stress values are used in CREEP to define the stress state for the bimodular material. If the material shows creep, the stresses are then modified for relaxation without any strain for the interval of time up to the next analysis. The creep parameters, if time-dependent, are stored and the change in stress stored as residual stress to be included in the next analysis.

APPENDIX B

COMPUTER PROGRAM USAGE

APPENDIX B. COMPUTER PROGRAM USAGE

Input Data

The first step in the structural analysis of a two-dimensional plane strain structure is to select a finite element representation of the cross-section of the body. Elements and nodal points are then numbered in two sequences each starting with one. The following group of punched cards numerically define the two-dimensional structure to be analyzed.

a. Identification Card - (72H)

Columns 1 to 72 of this card contain information to be printed with results.

b. System Control Card (4I5, 4F10.2)

Columns	1-5	Total number of nodal points in the structure (450 max.)
	6-10	Total number of elements in the structure (400 max.)
	11-15	Number of different materials in the structure (8 max.)
	16-20	Total number of lifts in the incremental structure (20 max.)
	21-30	Reference temperature (stress free temperature)
	31-40	Time of first analysis
	41-50	Time interval for creep analysis
	51-60	= 0 implies temperature-dependent material properties ≠ 0 implies time-dependent material properties

d. Material Property Information

The following group of cards must be supplied for each different material:

First Card - (3I5,F10.0)

Columns 1-5 Material identification - any number from 1 to 8

6-10 Number of different times/temperatures for which elastic properties are given - 30 maximum

11-15 Number of different times/temperatures for which creep properties are given - 15 maximum. Zero in this column shows that the material shows no creep

16-25 Unit weight of the material

Following Cards: These are in the following three groups:

i. Elastic Properties Cards (6F10.0). One card for each time/temperature.

Columns 1-10 Time or Temperature

11-20 Modulus of elasticity in compression

21-30 Poisson's ratio in compression

31-40 Modulus of elasticity in tension

41-50 Shear foundation factor G/H^2 or the area of a bar element

51-60 Coefficient of thermal expansion α

ii. Creep Cards (F10.0, 4E10.3) - One card for each time/temperature.

Columns 1-10 Time or Temperature

11-20, 21-30 } A_1, A_2 for creep in compression

31-40, 41-50 } A_3, A_4 for creep in tension

iii. Creep Cards (6F10.0) - One card for each material.

Columns 1-10, 11-20 } m_1, m_2 for creep in compression

21-30, 31-40 } m_3, m_4 for creep in tension

For creep, McHenry's equation is used taking only the first two terms of the sequence and considering indexes of e to be time/temperature invariant. Thus, the equation is for compression

$$\frac{\varepsilon_c(t)}{\sigma} = \frac{1}{E_c(t)} + A_1(\tau)(1-e^{-m_1(t-\tau)}) + A_2(\tau)(1-e^{-m_2(t-\tau)})$$

for tension

$$\frac{\varepsilon_t(t)}{\sigma} = \frac{1}{E_t(t)} + A_3(\tau)(1-e^{-m_3(t-\tau)}) + A_4(\tau)(1-e^{-m_4(t-\tau)})$$

d. Nodal Point Cards (I5,F5.0,4F10.0)

One card for each nodal point with the following information:

Columns 1-5 Nodal point number

6-10 Number which indicates if displacements or forces are to be specified

11-20 X-ordinate

21-30 Y-ordinate

31-40 XR

41-50 XZ

If the number in Columns 5-10 is

0. XR is the specified X-load and

X is the specified Y-load.

1. XR is the specified X-displacement and

X is the specified Y-load.

2. XR is the specified X-load and

X is the specified Y-displacement.

3. XR is the specified X-displacement and

X is the specified Y-displacement.

All loads are considered to be total forces acting on an element of unit thickness. Nodal point cards must be in numerical sequence. If cards are omitted, the omitted nodal points are generated at equal intervals along a straight line between the defined nodal points. The boundary code (Columns 6-10), XR and XZ are set equal to zero.

e. Element Cards (6I5,3F10.0)

One card for each element.

Columns		
1-5	Element	
6-10	Nodal point I	1. For a right-handed coordinate system, order nodal points counter-clockwise around.
11-15	Nodal point J	
16-20	Nodal point K	
21-25	Nodal point L	2. Maximum difference between nodal points must be less than 27.
26-30	Material identification	
31-40	Major initial stress	
41-50	Minor initial stress	
51-60	Angle of major initial stress with X-direction	

Element cards must be in element number sequence. If element cards are omitted, the program automatically generates the omitted information by incrementing by one the preceding I, J, K, and L, and by linearly interpolating the initial stresses. The material identification code for the generated cards is set equal to the value on the last card. The last element card must always be supplied.

Triangular elements are identified by repeating the last nodal point number (i.e., I, J, K, K). One-dimensional bar elements are identified by a nodal point numbering sequence of the form I, J, J, I.

In lift by lift construction, elements must be numbered in liftwise sequence.

f. Control Card for Stage Analysis (6I5,3F10.0)

This card is required to mark the change in size of the structure on placement of additional lift.

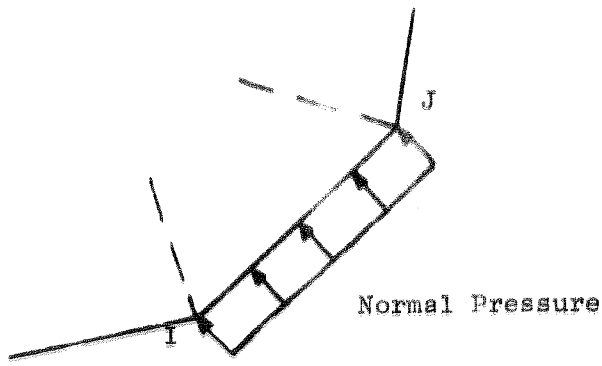
Columns	1-5	Number of lifts in the analyses
	6-10	Number of nodal points in the analyses
	11-15	Number of elements in the analyses
	16-20	Number of pressure boundary cards (100 max.)
	21-25	Number of approximations for bimodular analysis at each time stage
	26-30	Number of analyses at this stage of construction
	31-40	Time interval between successive analyses.
	41-50	Time of placement of last lift
	51-60	Time of placement of next lift

This card is followed by sets of temperature and pressure boundary cards for each analysis. (Number of sets = Number in columns 26-30 of Card F.)

i. Temperature loads (12F6.0). Temperatures of all nodal points in the analysis are input in the above format.

ii. Pressure changes (2I5,F10.0). One card for each boundary element subject to a normal pressure. (Number of cards = Number in Columns 16-20 of Card F.)

Columns	1-5	Nodal point I
	6-10	Nodal point J
	11-20	Change in normal pressure from previous input value



As shown above, the boundary element must be on the left as one progresses from I to J. Surface tensile force is defined as a negative pressure.

Output Information

The following information is developed and printed by the program:

a. Reprint of input data

b. Nodal point displacements

c. Stresses at center of each element

APPENDIX C

FORTRAN LISTING OF COMPUTER PROGRAM

```
$IBFTC MAIN    DECK
C       ARBITRARY TWO-DIMENSIONAL STRESS STRUCTURE INCLUDING INCREMENTAL
C       CONSTRUCTION, MCHENRY CREEP, RESIDUAL STRESSES, THERMAL STRESSES,
C       VARYING PRESSURE BOUNDARY CONDITIONS, AND BIMODULAR MATERIAL
C       PROPERTIES.
C
      COMMON NUMNP,NUMEL,NUMPC,N,VOL,TEMP,MTYPE,Q,NLAY,LAY,NUMN,NANAL,NP
     1,NDT,NCOUNT,TI,DT,DTT,T1,TL,XC,YC,ST(3,10),TIMLA(20),NUME(20),TIM,
     2TTT(15),NUMOL,TIME
       COMMON /MATARG/ E(30,6,8),RO(8),EE(5),HED(12),CIC(15,4,8),
     1CC(4,8),NCREEP(8)
       COMMON /ELEARG/ IX(400,5),MTAG(400),SIG(400,5),TOLD(400),
     1DE11(400),DE12(400),DE21(400),DE22(400),DSIG(6),CCO(4),CCC(4)
       COMMON /PRSARG/ IBC(100),JBC(100),PR(100)
       COMMON /ORDARG/ R(450),Z(450),UR(450),UZ(450),CODE(450),T(450)
       COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
       COMMON /LS4ARG/ I,J,K,S(10,10),C(3,3),D(3,3),H(3,3),P(10),LM(4),
     1 F(3,3)
C
      DIMENSION FF(900)
C
C***************************************************************************
C     READ AND PRINT OF CONTROL INFORMATION AND MATERIAL PROPERTIES
C***************************************************************************
   50 READ (5,1000) HED,NUMNP,NUMEL,NUMMAT,NLAY,Q,TI,DTT,T1
      WRITE (6,2000) HED,NUMNP,NUMEL,NUMMAT,NLAY,Q,TI,DTT
C
   56 DO 59 M=1,NUMMAT
      READ (5,1001) MTYPE,NUMTC,NCREEP(MTYPE),RO(MTYPE)
      WRITE (6,2011) MTYPE,NUMTC,NCREEP(MTYPE),RO(MTYPE)
      READ (5,1005) ((E(I,J,MTYPE),J=1,6),I=1,NUMTC)
      WRITE (6,2010) ((E(I,J,MTYPE),J=1,6),I=1,NUMTC)
      DO 58 I=NUMTC,30
      DO 58 J=1,6
   58 E(I,J,MTYPE)=E(NUMTC,J,MTYPE)
      IF (NCREEP(MTYPE)) 54,59,54
   54 NCR=NCREEP(MTYPE)
      READ (5,1003)  (TTT(I),(CIC(I,J,MTYPE),J=1,4),I=1,NCR)
      WRITE (6,2013) MTYPE, (TTT(I),(CIC(I,J,MTYPE),J=1,4),I=1,NCR)
      READ (5,1005) (CC(I,MTYPE),I=1,4)
      WRITE (6,2014) (CC(I,MTYPE),I=1,4)
   59 CONTINUE
C
C***************************************************************************
C     READ AND PRINT OF NODAL POINT DATA
C***************************************************************************
      WRITE (6,2004)
      L=0
   60 READ (5,1002) N,CODE(N),R(N),Z(N),UR(N),UZ(N)
      NL=L+1
      ZX=N-L
      DR=(R(N)-R(L))/ZX
      DZ=(Z(N)-Z(L))/ZX
   70 L=L+1
      IF(N-L) 100,90,80
```

```
 80 CODE(L)=0.0
    R(L)=R(L-1)+DR
    Z(L)=Z(L-1)+DZ
    UR(L)=0.0
    UZ(L)=0.0
    GO TO 70
 90 WRITE (6,2002) (K,CODE(K),R(K),Z(K),UR(K),UZ(K).K=NL,N)
    IF(NUMNP-N) 100,110,60
100 WRITE (6,2012) N
    CALL EXIT
110 CONTINUE
C*************************************************************************
C      READ AND PRINT OF ELEMENT PROPERTIES
C*************************************************************************
    WRITE (6,2001)
    N=0
130 READ (5,1006) M,(IX(M,I),I=1,5),(SIG(M,I),I=1,3)
140 N=N+1
    IF (M-N) 170,170,150
150 IX(N,1)=IX(N-1,1)+1
    IX(N,2)=IX(N-1,2)+1
    IX(N,3)=IX(N-1,3)+1
    IX(N,4)=IX(N-1,4)+1
    IX(N,5)=IX(N-1,5)
    ZX=M-N+1
    DR=(SIG(M,1)-SIG(N-1,1))/ZX
    DZ=(SIG(M,2)-SIG(N-1,2))/ZX
    DA=(SIG(M,3)-SIG(N-1,3))/ZX
    SIG(N,1)=SIG(N-1,1)+DR
    SIG(N,2)=SIG(N-1,2)+DZ
    SIG(N,3)=SIG(N-1,3)+DA
170 WRITE (6,2003) N,(IX(N,I),I=1,5),(SIG(N,I),I=1,3)
    IF (M-N) 180,180,140
180 IF (NUMEL-N) 190,190,130
190 CONTINUE
C*************************************************************************
C      DETERMINE BAND WIDTH
C*************************************************************************
    J=0
    DO 340 N=1,NUMEL
    DO 340 I=1,4
    DO 325 L=1,4
    KK=IABS(IX(N,I)-IX(N,L))
    IF (KK-J) 325,325,320
320 J=KK
325 CONTINUE
340 CONTINUE
    MBAND=2*J+2
C*************************************************************************
.      SOLVE INCREMENTAL STRUCTURE BY LAYERS
C*************************************************************************
    NUMOL=0
    NANAL=0
    TIME=TI
    TIM=TI
```

```
C*****************************************************************************
C     DETERMINE TYPE OF STRESS STATE IN ELEMENTS
C*****************************************************************************
      DO 450 N=1,NUMEL
      TOLD(N)=Q
      SIG(N,4)=0.
      SIG(N,5)=0.
      MTAG(N)=1
      IF (SIG(N,1)) 445,445,440
  440 IF (SIG(N,1)+SIG(N,2)) 441,441,442
  441 MTAG(N)=2
      GO TO 450
  442 MTAG(N)=3
  445 IF (SIG(N,2)) 450,450,448
  448 MTAG(N)=4
  450 CONTINUE
C
C     INITIALIZE DISPLACEMENTS
C
      DO 460 N=1,NUMNP
      FF(2*N-1)=0.
  460 FF(2*N)=0.
      DO 600 LLL=1,NLAY
C*****************************************************************************
C     INPUT OF LAYER INFORMATION
C*****************************************************************************
  550 READ (5,1006) LAY,NUMN,NUME(LAY),NUMPC,NP,NDT,DT,TIMLA(LAY),TIMNL
      WRITE (6,2008) LAY,NUMN,NUME(LAY),NUMPC,NP,NDT,DT,TIMLA(LAY),TIMNL
      NNAL=0
C*****************************************************************************
C     READ AND PRINT OF DATA FOR EACH LOADING STAGE
C*****************************************************************************
  400 NNAL=NNAL+1
      IF(NNAL-NDT) 410,410,590
  410 NANAL=NANAL+1
      READ (5,1007) (T(KK),KK=1,NUMN)
      WRITE (6,2009) NANAL,LAY,(KK,T(KK),KK=1,NUMN)
C*****************************************************************************
C     READ AND PRINT OF PRESSURE BOUNDARY CONDITIONS
C*****************************************************************************
      IF (NUMPC) 290,310,290
  290 WRITE (6,2005)
      DO 300 L=1,NUMPC
      READ (5,1004) IBC(L),JBC(L),PR(L)
  300 WRITE (6,2007) IBC(L),JBC(L),PR(L)
  310 CONTINUE
      NUMNL=NUME(LAY)
      IF (NP-1) 435,435,500
  435 DO 350 N=1,NUMNL
  350 MTAG(N)=1
  500 CONTINUE
C*****************************************************************************
C     SOLVE BIMODULAR STRUCTURE BY SUCCESSIVE APPROXIMATION
C*****************************************************************************
      NCOUNT=0
```

```
      DO 570 NNN=1,NP
  425 NCOUNT=NCOUNT+1
C
C     FORM STIFFNESS MATRIX
C
      CALL STIFF
C
C     SOLVE FOR DISPLACEMENTS
C
      CALL BANSOL
C
      IF (NCOUNT-NP) 525,510,510
  510 DO 520 N=1,NUMN
      FF(2*N-1)=FF(2*N-1)+B(2*N-1)
  520 FF(2*N)=FF(2*N)+B(2*N)
      WRITE (6,2006) (N,FF(2*N-1),FF(2*N),N=1,NUMN)
C
C     COMPUTE STRESSES ALLOWING FOR CREEP
C
  525 CALL CREEP
C
  570 CONTINUE
      NUMOL=NUME(LAY)
      TIM=TIME+DT/2.
      TIME=TIME+DT
      GO TO 400
  590 IF(TIME-TIMNL) 550,600,600
  600 CONTINUE
C***********************************************************************
      GO TO 50
C***********************************************************************
 1000 FORMAT (12A6/4I5,4F10.2)
 1001 FORMAT (3I5,F10.0)
 1002 FORMAT (I5,F5.0,4F10.0)
 1003 FORMAT (F10.0,4E10.3)
 1004 FORMAT (2I5,F10.0)
 1005 FORMAT (6F10.0)
 1006 FORMAT (6I5,3F10.0)
 1007 FORMAT(18X,6F6.1)
 2000 FORMAT (1H1 12A6/
     1 40H0 NUMBER OF NODAL POINTS--------------- I3/
     2 40H0 NUMBER OF ELEMENTS------------------- I3/
     3 40H0 NUMBER OF DIFFERENT MATERIALS--------- I3/
     4 40H0 NUMBER OF LAYERS IN THE STRUCTURE----- I3/
     5 40H0 REFERENCE TEMPERATURE----------------- F10.4/
     6 40H0 TIME OF FIRST ANALYSIS---------------- F10.4/
     7 40H0 TIME INTERVAL FOR CREEP ANALYSIS------ F10.4)
 2001 FORMAT (92H1ELEMENT NO.      I     J     K     L    MATERIAL    SI
     1G1-RESIDUAL    SIG2-RESIDUAL    ANGLE  )
 2002 FORMAT (I12,F12.2,2F12.3,2E24.7)
 2003 FORMAT (1I13,4I6,1I12,2F17.3,F9.3)
 2004 FORMAT  (97H1NODAL POINT        TYPE  X-ORDINATE  Y-ORDINATE  X LO
     1AD OR DISPLACEMENT  Y LOAD OR DISPLACEMENT )
 2005 FORMAT (29H0PRESSURE BOUNDARY CONDITIONS/ 24H     I     J    PRESS
     1URE )
```

```
2006 FORMAT (12H1N.P. NUMBER 18X 2HUX 18X 2HUY / (1I12,2E20.7))
2007 FORMAT (2I6,F12.3)
2008 FORMAT (50H1 NUMBER OF LAYERS IN THE ANALYSIS---------------=I5/
    1 50H0 NUMBER OF NODAL POINTS IN THE ANALYSIS----------=I5/
    2 50H0 NUMBER OF ELEMENTS IN THE ANALYSIS--------------=I5/
    3 50H0 NUMBER OF PRESSURE CARDS FOR THE ANALYSIS-------=I5/
    4 50H0 NUMBER OF APPROXIMATIONS FOR STRESS CALCULATION=I5/
    5 50H0 NUMBER OF TIME INTERVALS FOR ANALYSIS-----------=I5/
    6 50H0 TIME INTERVAL BETWEEN SUCCESSIVE ANALYSES-------=F10.3/
    7 50H0 TIME OF LAYING THE TOP LIFT---------------------=F10.3/
    8 50H0 TIME OF LAYING THE NEXT LIFT--------------------=F10.3)
2009 FORMAT (42H1 NODAL  TEMPERATURES FOR ANALYSIS NUMBER I5,
    1 21H STRUCTURE UPTO LIFT I5//
    2 120H           NP.    TEMP.           NP.    TEMP.           NP.    TEMP.
    3           NP.    TEMP.           NP.    TEMP.           NP.    TEMP.//
    4 (I12,F8.3,I12,F8.3,I12,F8.3,I12,F8.3,I12,F8.3,I12,F8.3))
2010 FORMAT (15H0   TEMP./TIME 10X 5HE(C)  9X 6HNU     11X 4HE(T)
    1 10X 5HG/H2  10X 5HALPHA/
    2 (F15.3,4F15.5,E15.5))
2011 FORMAT (17H0MATERIAL NUMBER= I3, 30H, NUMBER OF TEMP./TIME CARDS =
    1I3,24H, NUMBER OF CREEP CARDS=I3, 15H, MASS DENSITY= E12.4)
2012 FORMAT (26H0NODAL POINT CARD ERROR N= I5)
2013 FORMAT (17H0 MATERIAL NUMBER I5//
    1111H COEFFICIENT FUNCTIONS A(T) IN MCHENRYS EQUATION  STRAIN(T) =
    2STRAIN(0)+A1(T)(1-EXP(-M1*T))+A2(T)(1-EXP(-M2*T))//
    310X, 10HTEMP./TIME 11X, 24HA1,A2 FOR COMPRESS.CREEP 12X, 23HA3,A4
    4FOR TENSILE CREEP// 38X, 2HA1,13X,2HA2,18X,2HA3,13X,2HA4//
    5(10X,F10.3,10X,E10.3,5X,E10.3,10X,E10.3,5X,E10.3))
2014 FORMAT (30H0 INDEXES IN MCHENRYS EQUATION//
    130H   FOR COMPRESSIVE CREEP  M1 =E10.3,6X,4HM2 = E10.3/
    230H   FOR TENSILE CREEP      M3 = E10.3,6X,4HM4 = E10.3)
C
      END
```

```
$IBFTC STIF    DECK
      SUBROUTINE STIFF
C
      COMMON NUMNP,NUMEL,NUMPC,N,VOL,TEMP,MTYPE,Q,NLAY,LAY,NUMN,NANAL,NP
     1,NDT,NCOUNT,TI,DT,DTT,T1,TL,XC,YC,ST(3,10),TIMLA(20),NUME(20),TIM,
     2TTT(15),NUMOL,TIME
       COMMON /MATARG/ E(30,6,8),RO(8),EE(5),HED(12),CIC(15,4,8),
     1CC(4,8),NCREEP(8)
       COMMON /ELEARG/ IX(400,5),MTAG(400),SIG(400,5),TOLD(400),
     1DE11(400),DE12(400),DE21(400),DE22(400),DSIG(6),CCO(4),CCC(4)
       COMMON /PRSARG/ IBC(100),JBC(100),PR(100)
       COMMON /ORDARG/ R(450),Z(450),UR(450),UZ(450),CODE(450),T(450)
       COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
       COMMON /LS4ARG/ I,J,K,S(10,10),C(3,3),D(3,3),H(3,3),P(10),LM(4),
     1 F(3,3)
C
C*********************************************************************
C     INITIALIZATION
C*********************************************************************
      REWIND 2
      NB=27
      ND=2*NB
      ND2=2*ND
      STOP=0.0
      NUMBLK=0
C
      DO 50 N=1,ND2
      B(N)=0.0
      DO 50 M=1,ND
   50 A(N,M)=0.0
C*********************************************************************
C     FORM STIFFNESS MATRIX IN BLOCKS
C*********************************************************************
   60 NUMBLK=NUMBLK+1
      NH=NB*(NUMBLK+1)
      NM=NH-NB
      NL=NM-NB+1
      KSHIFT=2*NL-2
C
      N1=1
      DO 220 M=1,LAY
      TL=TIM-TIMLA(M)
      N2=NUME(M)
      DO 210 N=N1,N2
C
      IF (IX(N,5)) 210,210,65
   65 DO 80 I=1,4
      IF (IX(N,I)-NL) 80,70,70
   70 IF (IX(N,I)-NM) 90,90,80
   80 CONTINUE
      GO TO 210
C
   90 IF (IX(N,3)-IX(N,2)) 92,91,92
   91 CALL ONED
```

```
      GO TO 165
   92 CALL QUAD
      IF (VOL) 164,164,165
  164 WRITE (6,2003) N
C
C            ADD ELEMENT STIFFNESS TO TOTAL STIFFNESS
C
  165 IX(N,5)=-IX(N,5)
      DO 166 I=1,4
  166 LM(I)=2*IX(N,I)-2
C
      DO 200 I=1,4
      DO 200 K=1,2
      II=LM(I)+K-KSHIFT
      KK=2*I-2+K
      B(II)=B(II)+P(KK)
      DO 200 J=1,4
      DO 200 L=1,2
      JJ=LM(J)+L-II+1-KSHIFT
      LL=2*J-2+L
      IF(JJ) 200,200,175
  175 IF(ND-JJ) 180,195,195
  180 WRITE (6,2004) N
      STOP=1.0
      GO TO 210
  195 A(II,JJ)=A(II,JJ)+S(KK,LL)
  200 CONTINUE
  210 CONTINUE
      N1=N2+1
      IF(N1-NUMEL) 220,220,225
  220 CONTINUE
C
C     ADD CONCENTRATED FORCES WITHIN BLOCK
C
  225 DO 255 N=NL,NM
      K=2*N-KSHIFT
      B(K)=B(K)+UZ(N)
      B(K-1)=B(K-1)+UR(N)
      IF (NCOUNT-NP) 255,250,250
  250 IF (N-NUMN) 252,252,255
  252 UZ(N)=0.
      UR(N)=0.
  255 CONTINUE
C
C     BOUNDARY CONDITIONS
C
C       1. PRESSURE B.C.
C
      IF (NUMPC) 260,310,260
  260 DO 300 L=1,NUMPC
      I=IBC(L)
      J=JBC(L)
      PP=PR(L)/2.
      DZ=(Z(I)-Z(J))*PP
      DR=(R(J)-R(I))*PP
```

```
  264 II=2*I-KSHIFT
      JJ=2*J-KSHIFT
      IF (II) 280,280,265
  265 IF (II-ND) 270,270,280
  270 SINA=0.0
      COSA=1.0
      IF (CODE(I)) 271,272,272
  271 SINA=SIN(CODE(I)/57.3)
      COSA=COS(CODE(I)/57.3)
  272 B(II-1)=B(II-1)+(COSA*DZ+SINA*DR)
      B(II)=B(II)-(SINA*DZ-COSA*DR)
  280 IF (JJ) 300,300,285
  285 IF (JJ-ND) 290,290,300
  290 SINA=0.0
      COSA=1.0
      IF (CODE(J)) 291,292,292
  291 SINA=SIN(CODE(J)/57.3)
      COSA=COS(CODE(J)/57.3)
  292 B(JJ-1)=B(JJ-1)+(COSA*DZ+SINA*DR)
      B(JJ)=B(JJ)-(SINA*DZ-COSA*DR)
  300 CONTINUE
C
C
C        2. DISPLACEMENT B.C.
  310 DO 400 M=NL,NH
      IF (M-NUMN)  315,315,400
  315 U=UR(M)
      N=2*M-1-KSHIFT
      IF (CODE(M)) 390,400,316
  316 IF (CODE(M)-1.) 317,370,317
  317 IF (CODE(M)-2.) 318,390,318
  318 IF (CODE(M)-3.) 390,380,390
  370 CALL MODIFY(A,B,ND2,MBAND,N,U)
      GO TO 400
  380 CALL MODIFY(A,B,ND2,MBAND,N,U)
  390 U=UZ(M)
      N=N+1
      CALL MODIFY(A,B,ND2,MBAND,N,U)
  400 CONTINUE
C
C     WRITE BLOCK OF EQUATIONS ON TAPE AND SHIFT UP LOWER BLOCK
C
      WRITE (2) (B(N),(A(N,M),M=1,MBAND),N=1,ND)
C
      DO 420 N=1,ND
      K=N+ND
      B(N)=B(K)
      B(K)=0.0
      DO 420 M=1,ND
      A(N,M)=A(K,M)
  420 A(K,M)=0.0
C
C     CHECK FOR LAST BLOCK
C
      IF (NM-NUMN)  60,480,480
```

```
  480 CONTINUE
C*************************************************************************
      IF(STOP) 490,500,490
  490 CALL EXIT
  500 RETURN
C
 2003 FORMAT (26HONEGATIVE AREA ELEMENT NO. I4)
 2004 FORMAT (29HOBAND WIDTH EXCEEDS ALLOWABLE I4)
      END
```

```
$IBFTC QUDF    DECK
      SUBROUTINE QUAD
C
      COMMON NUMNP,NUMEL,NUMPC,N,VOL,TEMP,MTYPE,Q,NLAY,LAY,NUMN,NANAL,NP
     1,NDT,NCOUNT,TI,DT,DTT,T1,TL,XC,YC,ST(3,10),TIMLA(20),NUME(20),TIM,
     2TTT(15),NUMOL,TIME
       COMMON /MATARG/ E(30,6,8),RO(8),EE(5),HED(12),CIC(15,4,8),
     1CC(4,8),NCREEP(8)
       COMMON /ELEARG/ IX(400,5),MTAG(400),SIG(400,5),TOLD(400),
     1DE11(400),DE12(400),DE21(400),DE22(400),DSIG(6),CCO(4),CCC(4)
       COMMON /PRSARG/ IBC(100),JBC(100),PR(100)
       COMMON /ORDARG/ R(450),Z(450),UR(450),UZ(450),CODE(450),T(450)
       COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
       COMMON /LS4ARG/ I,J,K,S(10,10),C(3,3),D(3,3),H(3,3),P(10),LM(4),
     1 F(3,3)
C
      I=IX(N,1)
      J=IX(N,2)
      K=IX(N,3)
      L=IX(N,4)
      IX(N,5)=IABS(IX(N,5))
      MTYPE=IX(N,5)
C
C     FORM STRESS-STRAIN RELATIONSHIP INCLUDING TIME OR TEMPERATURE
C     DEPENDENCE OF ELASTIC CONSTANTS
C
      TEMP=(T(I)+T(J)+T(K)+T(L))/4.0
      TEM=(TEMP+TOLD(N))/2.
      IF (T1) 50,40,50
   40 DO 103 M=2,30
      IF (E(M,1,MTYPE)-TEM)  103,104,104
  103 CONTINUE
  104 RATIO=0.0
      DEN=E(M,1,MTYPE)-E(M-1,1,MTYPE)
      IF (DEN) 70,71,70
   70 RATIO= (TEM-E(M-1,1,MTYPE))/DEN
      GO TO 71
   50 DO 55 M=2,30
9     IF (E(M,1,MTYPE)-TL) 55,60,60
   55 CONTINUE
   60 RATIO=0.
      DEN=E(M,1,MTYPE)-E(M-1,1,MTYPE)
      IF (DEN) 64,71,64
   64 RATIO=(TL-E(M-1,1,MTYPE))/DEN
   71 DO 105 KK=1,5
  105 EE(KK)=E(M-1,KK+1,MTYPE)+RATIO*(E(M,KK+1,MTYPE)-E(M-1,KK+1,MTYPE))
C
   76 IF (MTAG(N)-2) 80,80,81
   80 RATIO=EE(2)
      GO TO 82
   81 RATIO=EE(2)*EE(3)/EE(1)
   82 XX=EE(1)/EE(3)
      YY=1.
      IF (MTAG(N)-1) 83,83,84
```

```
   83 XX=YY
   84 IF (MTAG(N)-3) 86,86,85
   85 YY=XX
   86 CONTINUE
      UU=YY-EE(2)*RATIO
      VV=XX-EE(2)*RATIO
      UV=EE(2)*(1.+RATIO)
      COMM=EE(1)/(VV*UU-UV**2)
      C(1,1)=UU*COMM
      C(1,2)=UV*COMM
      C(1,3)=0.
      C(2,1)=C(1,2)
      C(2,2)=VV*COMM
      C(2,3)=0.
      C(3,1)=0.
      C(3,2)=0.
      C(3,3)=EE(1)/(EE(1)/EE(3)+1.+2.*EE(2))
      THETA=SIG(N,3)/57.296
      SS=SIN(THETA)
      CO=COS(THETA)
      S2=SS*SS
      C2=CO*CO
      SC=SS*CO
C
      DO 87 II=1,3
      DO 87 JJ=1,3
   87 F(II,JJ)=C(II,JJ)
C
      D(1,1)=C2
      D(1,2)=S2
      D(1,3)=SC
      D(2,1)=S2
      D(2,2)=C2
      D(2,3)=-SC
      D(3,1)=-2.*SC
      D(3,2)=-D(3,1)
      D(3,3)=C2-S2
C
      DO 88 II=1,3
      DO 88 JJ=1,3
      H(II,JJ)=0.0
      DO 88 KK=1,3
   88 H(II,JJ)=H(II,JJ) +C(II,KK)*D(KK,JJ)
C
      DO 89 II=1,3
      DO 89 JJ=1,3
      C(II,JJ)=0.0
      DO 89 KK=1,3
   89 C(II,JJ)=C(II,JJ)+D(KK,II)*H(KK,JJ)
C
C
C           FORM QUADRILATERAL STIFFNESS MATRIX
C
      DO 100 II=1,10
      P(II)=0.0
```

```
      DO 100 JJ=1,10
  100 S(II,JJ)=0.0
C
      DO 150 II=1,3
      DO 150 JJ=1,10
  150 ST(II,JJ)=0.0
C
      VOL=0.0
      I=IX(N,1)
      J=IX(N,2)
      K=IX(N,4)
      CALL EDLST(1,3,7)
      I=IX(N,3)
      J=IX(N,4)
      K=IX(N,2)
      XC=(R(J)+R(K))/2.
      YC=(Z(J)+Z(K))/2.
      CALL EDLST(5,7,3)
C
C     CALCULATE UNBALANCED LOADS DUE TO TEMPERATURE CHANGE AND STRESS
C     RELAXATION
      TEMP=(TEMP-TOLD(N))*EE(5)
      IF(TIM) 170,160,170
  160 TEMP=0.
  170 CONTINUE
      DSIG(1)=SIG(N,1)*C2+SIG(N,2)*S2
      DSIG(2)=SIG(N,1)*S2+SIG(N,2)*C2
      DSIG(3)=(SIG(N,1)-SIG(N,2))*SC
C
      DO 190 JJ=1,3
  190 DSIG(JJ)=-DSIG(JJ)+(C(JJ,1)+C(JJ,2))*TEMP
      DO 200 II=1,10
      DO 200 JJ=1,3
  200 P(II)=P(II)+DSIG(JJ)*ST(JJ,II)*VOL
C
C     ADD SHEAR STIFFNESS OF FOUNDATION
C
      COMM=VOL*EE(4)
      S(9,9)=S(9,9)+COMM
      S(10,10)=S(10,10)+COMM
C
C     ELIMINATE CENTER POINT
C
      DO 500 K=1,2
      IH=10-K
      ID=IH+1
      DO 500 I=1,IH
      S(ID,I)=S(ID,I)/S(ID,ID)
      P(I)=P(I)-P(ID)*S(I,ID)/S(ID,ID)
      DO 500 J=1,IH
  500 S(J,I)=S(J,I)-S(J,ID)*S(ID,I)
C
C     CALCULATE LOADS DUE TO GRAVITY
C
      IF(N-NUMOL) 580,580,540
```

```
  540 IF(NNAL-1) 550,550,580
  550 DO 560 I=1,4
  560 P(2*I)=P(2*I)-RO(MTYPE)*VOL/4.
  580 CONTINUE
C
  130 RETURN
C
      END
```

```
$IBFTC ONE      DECK
      SUBROUTINE ONED
C
      COMMON NUMNP,NUMEL,NUMPC,N,VOL,TEMP,MTYPE,Q,NLAY,LAY,NUMN,NANAL,NP
     1,NDT,NCOUNT,TI,DT,DTT,T1,TL,XC,YC,ST(3,10),TIMLA(20),NUME(20),TIM,
     2TTT(15),NUMOL,TIME
       COMMON /MATARG/ E(30,6,8),RO(8),EE(5),HED(12),CIC(15,4,8),
     1CC(4,8),NCREEP(8)
       COMMON /ELEARG/ IX(400,5),MTAG(400),SIG(400,5),TOLD(400),
     1DE11(400),DE12(400),DE21(400),DE22(400),DSIG(6),CCO(4),CCC(4)
       COMMON /PRSARG/ IBC(100),JBC(100),PR(100)
       COMMON /ORDARG/ R(450),Z(450),UR(450),UZ(450),CODE(450),T(450)
       COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
       COMMON /LS4ARG/ I,J,K,S(10,10),C(3,3),D(3,3),H(3,3),P(10),LM(4),
     1 F(3,3)
C
      DO 100 I=1,8
      P(I)=0.0
      DO 100 J=1,8
  100 S(I,J)=0.0
C
      MTYPE=IX(N,5)
      I=IX(N,1)
      J=IX(N,2)
      DX=R(J)-R(I)
      DY=Z(J)-Z(I)
      XL=SQRT(DX**2+DY**2)
      COSA=DX/XL
      SINA=DY/XL
      COMM=E(1,2,MTYPE)*E(1,5,MTYPE)/XL
C
      S(1,1)=COSA*COSA*COMM
      S(1,2)=COSA*SINA*COMM
      S(1,3)=-S(1,1)
      S(1,4)=-S(1,2)
      S(2,1)=S(1,2)
      S(2,2)=SINA*SINA*COMM
      S(2,3)=-S(1,2)
      S(2,4)=-S(2,2)
      S(3,1)=S(1,3)
      S(3,2)=S(2,3)
      S(3,3)=S(1,1)
      S(3,4)=S(1,2)
      S(4,1)=S(1,4)
      S(4,2)=S(2,4)
      S(4,3)=S(3,4)
      S(4,4)=S(2,2)
C
      EP=SIG(N,1)/E(1,2,MTYPE)
      DX=DX*EP
      DY=DY*EP
      P(1)=S(1,1)*DX+S(1,2)*DY
      P(2)=S(2,1)*DX+S(2,2)*DY
      P(3)=-P(1)
```

```
      P(4)=-P(2)
C
      RETURN
C
      END
```

```
$IBFTC EDLS    DECK
      SUBROUTINE EDLST(N1,N2,N3)
C
      COMMON NUMNP,NUMEL,NUMPC,N,VOL,TEMP,MTYPE,Q,NLAY,LAY,NUMN,NANAL,NP
     1,NDT,NCOUNT,TI,DT,DTT,T1,TL,XC,YC,ST(3,10),TIMLA(20),NUME(20),TIM,
     2TTT(15),NUMOL,TIME
       COMMON /MATARG/ E(30,6,8),RO(8),EE(5),HED(12),CIC(15,4,8),
     1CC(4,8),NCREEP(8)
       COMMON /ELEARG/ IX(400,5),MTAG(400),SIG(400,5),TOLD(400),
     1DE11(400),DE12(400),DE21(400),DE22(400),DSIG(6),CCO(4),CCC(4)
       COMMON /PRSARG/ IBC(100),JBC(100),PR(100)
       COMMON /ORDARG/ X(450),Y(450),UR(450),UZ(450),CODE(450),T(450)
       COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
       COMMON /LS4ARG/ I,J,K,S(10,10),C(3,3),D(3,3),H(3,3),P(10),LM(4),
     1 F(3,3)
      DIMENSION BA(3,2),U(3,4),V(3,4),UV(3,4,2)
      EQUIVALENCE (UV,U),(UV(13),V)
C
      TH=1.0
      BA(1,1)=Y(J)-Y(K)
      BA(2,1)=Y(K)-Y(I)
      BA(3,1)=Y(I)-Y(J)
      BA(1,2)=X(K)-X(J)
      BA(2,2)=X(I)-X(K)
      BA(3,2)=X(J)-X(I)
      AREA=(X(J)*BA(2,1)+X(I)*BA(1,1)+X(K)*BA(3,1))/2.
      IF (AREA) 400,400,100
  100 VOL=VOL+AREA
      COMM=TH/(48.*AREA)
      C11=C(1,1)*COMM
      C12=C(1,2)*COMM
      C13=C(1,3)*COMM
      C22=C(2,2)*COMM
      C23=C(2,3)*COMM
      C33=C(3,3)*COMM
C
C
      DO 150 M=1,2
      D1=BA(1,M)
      D2=BA(2,M)
      D3=BA(3,M)
      UV(1,1,M)=D1
      UV(2,1,M)=D1
      UV(3,1,M)=D1
      UV(1,2,M)=D2
      UV(2,2,M)=D2-2.*D3
      UV(3,2,M)=-D2
      UV(1,3,M)=D3
      UV(2,3,M)=-D3
      UV(3,3,M)=D3-2.*D2
      UV(1,4,M)=0.
      UV(2,4,M)=4.*D3
  150 UV(3,4,M)=4.*D2
C
```

```
      LM(1)=N1
      LM(2)=N2
      LM(3)=N3
      LM(4)=9
C
      COMM=8.*AREA
      DO 300 I=1,4
      II=LM(I)
C
      UU=(U(2,I)+U(3,I))/COMM
      VV=(V(2,I)+V(3,I))/COMM
      ST(1,II)=ST(1,II)+UU
      ST(2,II+1)=ST(2,II+1)+VV
      ST(3,II)=ST(3,II)+VV
      ST(3,II+1)=ST(3,II+1)+UU
C
      SUM=U(1,I)+U(2,I)+U(3,I)
      SUM1=SUM+U(1,I)
      SUM2=SUM+U(2,I)
      SUM3=SUM+U(3,I)
      SUM=V(1,I)+V(2,I)+V(3,I)
      SVM1=SUM+V(1,I)
      SVM2=SUM+V(2,I)
      SVM3=SUM+V(3,I)
      DO 300 J=1,4
      JJ=LM(J)
      UQU=U(1,J)*SUM1+U(2,J)*SUM2+U(3,J)*SUM3
      VQU=V(1,J)*SUM1+V(2,J)*SUM2+V(3,J)*SUM3
      VQV=V(1,J)*SVM1+V(2,J)*SVM2+V(3,J)*SVM3
      UQV=U(1,J)*SVM1+U(2,J)*SVM2+U(3,J)*SVM3
      S(II,JJ)=S(II,JJ)+ C11*UQU+C13*(VQU+UQV)+C33*VQV
      S(II+1,JJ+1)=S(II+1,JJ+1)+ C22*VQV+C23*(VQU+UQV)+C33*UQU
      S(II,JJ+1)=S(II,JJ+1)+ C23*VQV+C13*UQU+VQU*C12+C33*UQV
  300 S(JJ+1,II)=S(II,JJ+1)
C
  400 RETURN
C
      END
```

```
$IBFTC MODI     DECK
      SUBROUTINE MODIFY(A,B,NEQ,MBAND,N,U)
C
      DIMENSION A(108,54),B(108)
C
      DO 250 M=2,MBAND
      K=N-M+1
      IF(K) 235,235,230
  230 B(K)=B(K)-A(K,M)*U
      A(K,M)=0.0
  235 K=N+M-1
      IF(NEQ-K) 250,240,240
  240 B(K)=B(K)-A(N,M)*U
      A(N,M)=0.0
  250 CONTINUE
      A(N,1)=1.0
      B(N)=U
      RETURN
C
      END
```

```
$IBFTC BAND    DECK
      SUBROUTINE BANSOL
C
      COMMON /BANARG/ MM,NUMBLK,B(108),A(54,108)
C
      DIMENSION NAB(34)
C
      NN=54
      CALL TIME (Z,NNN)
      NCOUNT = NN*NN
      JUMPA = NCOUNT/460   +   1
      JUMPB = NN/460 + 1
      NTRACK=1
      NL=NN+1
      NH=NN+NN
      REWIND 2
      NB=0
      GO TO 150
C*************************************************************************
C     REDUCE EQUATIONS BY BLOCKS
C*************************************************************************
C
C     1. SHIFT BLOCK OF EQUATIONS
C
  100 NB=NB+1
      DO 125 N=1,NN
      NM=NN+N
      B(N)=B(NM)
      B(NM)=0.0
      DO 125 M=1,MM
      A(M,N) = A(M,NM)
  125 A(M,NM) = 0.0
C
C     2. READ NEXT BLOCK OF EQUATIONS INTO CORE
C
      IF (NUMBLK-NB) 150,200,150
  150 READ (2) (B(N),(A(M   ,N), M = 1, MM), N = NL, NH )
      IF (NB) 200,100,200
C
C     3. REDUCE BLOCK OF EQUATIONS
C
  200 DO 300 N=1,NN
      IF ( A(1,N) )                     225,  300,  225
  225 B(N) = B(N) / A(1,N)
      DO 275 L=2,MM
      IF ( A(L,N) )                     230,  275,  230
  230 C = A(L,N) / A(1,N)
      I=N+L-1
      J=0
      DO 250 K=L,MM
      J=J+1
  250 A(J,I) = A(J,I) - C * A(K,N)
      B(I) = B(I) - A(L,N) * B(N)
      A(L,N) = C
```

```
  275 CONTINUE
  300 CONTINUE
C
C     4. WRITE BLOCK OF REDUCED EQUATIONS ON TAPE 2
C
      IF (NUMBLK-NB) 375,400,375
  375 IF(NCOUNT+NN.GT.(39-MOD(NTRACK,40))*460) NTRACK=(NTRACK/40)*40+40
      NAB(NB) = NTRACK
      CALL WRDISK ( NTRACK, A, NCOUNT )
      NTRACK = NTRACK + JUMPA
      CALL WRDISK ( NTRACK, B, NN )
      NTRACK = NTRACK + JUMPB
      GO TO 100
C***********************************************************************
C     BACK-SUBSTITUTION
C***********************************************************************
  400 DO 450 M=1,NN
      N=NN+1-M
      DO 425 K=2,MM
      L=N+K-1
  425 B(N) = B(N) - A(K,N) * B(L)
      NM=N+NN
      B(NM)=B(N)
  450 A(NB,NM) = B(N)
      NB=NB-1
      IF (NB) 475,500,475
  475 NTRACK = NAB(NB)
      CALL RDDISK ( NTRACK, A, NCOUNT )
      NTRACK = NTRACK + JUMPA
      CALL RDDISK ( NTRACK, B, NN )
      GO TO 400
C***********************************************************************
C***********************************************************************
C     ORDER UNKNOWNS IN B ARRAY
  500 K=0
      DO 600 NB=1,NUMBLK
      DO 600 N=1,NN
      NM=N+NN
      K=K+1
  600 B(K) = A(NB, NM)
C
      CALL TIME (Z,MMM)
      N = MMM - NNN
      WRITE (6,1) N
C
      RETURN
C
    1 FORMAT (17H4TIME IN BANSOL = I8, 13H MILLESECONDS///)
C
      END
```

```
$IBFTC CREE    DECK
      SUBROUTINE CREEP
C
      COMMON NUMNP,NUMEL,NUMPC,N,VOL,TEMP,MTYPE,Q,NLAY,LAY,NUMN,NANAL,NP
     1,NDT,NCOUNT,TI,DT,DTT,T1,TL,XC,YC,ST(3,10),TIMLA(20),NUME(20),TIM,
     2TTT(15),NUMOL,TIME
       COMMON /MATARG/ E(30,6,8),RO(8),EE(5),HED(12),CIC(15,4,8),
     1CC(4,8),NCREEP(8)
       COMMON /ELEARG/ IX(400,5),MTAG(400),STG(400,5),TOLD(400),
     1DE11(400),DE12(400),DE21(400),DE22(400),DSIG(6),CCO(4),CCC(4)
       COMMON /PRSARG/ IBC(100),JBC(100),PR(100)
       COMMON /ORDARG/ R(450),Z(450),UR(450),UZ(450),CODE(450),T(450)
       COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
       COMMON /LS4ARG/ I,J,K,S(10,10),C(3,3),D(3,3),H(3,3),P(10),LM(4),
     1F(3,3)
C
      MPRINT=0
      N1=1
      DO 600 M=1,LAY
      N2=NUME(M)
      DO 550 MM=N1,N2
      N=MM
      TL=TIM-TIMLA(M)
C
C     EVALUATE ELEMENT STRESSES
C
      CALL STRESS
C
      IF (IX(N,2)-IX(N,3)) 255,104,255
  255 MTAG(N)=1
      IF (DSIG(4)) 104,104,259
  259 IF (DSIG(4)+DSIG(5)) 260,260,261
  260 MTAG(N)=2
      GO TO 104
  261 MTAG(N)=3
  265 IF (DSIG(5)) 104,104,266
  266 MTAG(N)=4
C
  104 IF (MPRINT) 106,105,106
  105 WRITE(6,2000) LAY,NANAL,TIME,NCOUNT
      MPRINT=50
  106 MPRINT=MPRINT-1
C
  305 WRITE (6,2001) N,XC,YC,(DSIG(I),I=1,6)
C
C     MODIFY STRESSES FOR CREEP IF APPLICABLE
C
      IF (NCOUNT-NP) 550,50,50
   50 IF (IX(N,2)-IX(N,3)) 55,550,55
C
C     INTERPOLATION OF CREEP COEFFICIENTS
C
   55 I=IX(N,1)
      J=IX(N,2)
```

```
      K=IX(N,3)
      L=IX(N,4)
      IX(N,5)=IABS(IX(N,5))
      MTYPE=IX(N,5)
      TEMP=(T(I)+T(J)+T(K)+T(L))/4.
      TOLD(N)=TEMP
      IF (NCREEP(MTYPE)) 250,250,60
   60 NCR=NCREEP(MTYPE)
      IF(T1) 120,110,120
  110 TL=TEMP
  120 DO 140 NN=2,NCR
      IF (TL-TTT(NN)) 125,150,140
  125 TM=TTT(NN)-TTT(NN-1)
      DIFF= TL-TTT(NN-1)
      DO 130 KK=1,4
  130 CCO(KK)=CIC(NN-1,KK,MTYPE)+DIFF*(CIC(NN,KK,MTYPE)-CIC(NN-1,KK,MTYP
     1E))/TM
      GO TO 160
  140 CONTINUE
  150 DO 155 KK=1,4
  155 CCO(KK)=CIC(NN,KK,MTYPE)
  160 DO 165 KK=1,4
  165 CCC(KK)=CC(KK,MTYPE)
C
C     SELECT APPROPRIATE CONSTANTS
C
      IF (DSIG(4)) 170,170,175
  170 KK=1
      GO TO 180
  175 KK=3
  180 CCO1=CCO(KK)
      CCO2=CCO(KK+1)
      CCO3=CCO1
      CCO4=CCO2
      CCC1=CCC(KK)
      CCC2=CCC(KK+1)
      CCC3=CCC1
      CCC4=CCC2
      IF (DSIG(5)) 185,185,190
  185 CCO3=CCO(1)
      CCO4=CCO(2)
      CCC3=CCC(1)
      CCC4=CCC(2)
  190 CONTINUE
C
C     MODIFICATION OF STRESSES TO ALLOW FOR CREEP. RELAXATION OF STRESS
C     AT CONSTANT STRAIN ON THE APPLICATION OF A TIME INCREMENT
C
      THETA=(DSIG(6)-SIG(N,3))/57.296
      CO=COS(THETA)
      SS=SIN(THETA)
      C2=CO*CO
      S2=SS*SS
      SIG1=C2*(SIG(N,4)+SIG(N,1))+S2*(SIG(N,5)+SIG(N,2))
      SIG2=S2*(SIG(N,4)+SIG(N,1))+C2*(SIG(N,5)+SIG(N,2))
```

```
      IF (T1) 192,191,192
  191 DE11(N)=DSIG(4)*CCC2*CCO2
      DE12(N)=DSIG(4)*CCC1*CCO1
      DE21(N)=DSIG(5)*CCC4*CCO4
      DE22(N)=DSIG(5)*CCC3*CCO3
      GO TO 195
  192 DET11= C2*DE11(N)+S2*DE21(N)
      DET12= C2*DE12(N)+S2*DE22(N)
      DET21= S2*DE11(N)+C2*DE21(N)
      DET22= S2*DE12(N)+C2*DE22(N)
C
      DE11(N)= DET11+(DSIG(4)-SIG1)*CCC2*CCO2
      DE12(N)= DET12+(DSIG(4)-SIG1)*CCC1*CCO1
      DE21(N)= DET21+(DSIG(5)-SIG2)*CCC4*CCO4
      DE22(N)= DET22+(DSIG(5)-SIG2)*CCC3*CCO3
  195 CONTINUE
      SIG(N,1)=0.
      SIG(N,2)=0.
C
      TR=0.
  220 TR=TR+DTT
      DELTA1=(DE11(N)+DE12(N))*DTT
      DELTA2=(DE21(N)+DE22(N))*DTT
      SIG1=DELTA1*F(1,1)
      SIG2=DELTA2*F(2,2)
      SIG(N,1)=SIG(N,1)-SIG1
      SIG(N,2)=SIG(N,2)-SIG2
C
C        CALCULATION OF CREEP RATES FOR THE NEXT TIME INTERVAL
C
      DE11(N)= DE11(N)*(1.-DTT*CCC2)
      DE12(N)= DE12(N)*(1.-DTT*CCC1)
      DE21(N)= DE21(N)*(1.-DTT*CCC4)
      DE22(N)=DE22(N)*(1.-DTT*CCC3)
C
      DE11(N)=DE11(N)-SIG1*CCC2*CCO2
      DE12(N)=DE12(N)-SIG1*CCC1*CCO1
      DE21(N)=DE21(N)-SIG2*CCC4*CCO4
      DE22(N)=DE22(N)-SIG2*CCC3*CCO3
      IF (TR-DT) 220,250,250
  250 SIG(N,3)=DSIG(6)
      SIG(N,4)=DSIG(4)
      SIG(N,5)=DSIG(5)
  550 CONTINUE
      N1=N2+1
      IF (N1-NUMEL) 600,600,650
  600 CONTINUE
  650 CONTINUE
C
      RETURN
C
 2000 FORMAT (19H1 NUMBER OF LIFTS = I5,18H ANALYSIS NUMBER =I5/
     116H STRESSES AFTER F10.3,24HTIME, APPROXIMATION NO. I5/
     17H EL.NO. 7X 1HX 7X 1HY 4X 8HX-STRESS 4X 8HY-STRESS 3X 9HXY-STRESS
     2 2X 10HMAX-STRESS 2X 10HMIN-STRESS 7H  ANGLE)
```

```
 2001 FORMAT (I7,2F8.2,1P5E12.4,0P1F7.2)
C
      END
```

```
$IBFTC STRES   DECK
      SUBROUTINE STRESS
C
      COMMON NUMNP,NUMEL,NUMPC,N,VOL,TEMP,MTYPE,Q,NLAY,LAY,NUMN,NANAL,NP
     1,NDT,NCOUNT,TI,DT,DTT,T1,TL,XC,YC,ST(3,10),TIMLA(20),NUME(20),TIM,
     2TTT(15),NUMOL,TIMF
       COMMON /MATARG/ E(30,6,8),RO(8),EE(5),HED(12),CIC(15,4,8),
     1CC(4,8),NCREEP(8)
       COMMON /ELEARG/ IX(400,5),MTAG(400),SIG(400,5),TOLD(400),
     1DE11(400),DE12(400),DE21(400),DE22(400),DSIG(6),CCO(4),CCC(4)
       COMMON /PRSARG/ IBC(100),JBC(100),PR(100)
       COMMON /ORDARG/ R(450),Z(450),UR(450),UZ(450),CODE(450),T(450)
       COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
       COMMON /LS4ARG/ I,J,K,S(10,10),C(3,3),D(3,3),H(3,3),P(10),LM(4),
     1 F(3,3)
C*********************************************************************
C     COMPUTE ELEMENT STRESSES
C*********************************************************************
C
      DO 50 I=1,6
   50 DSIG(I)=0.0
C
      IF (IX(N,3)-IX(N,2)) 90,80,90
C***************************************************- ONE-D ELEMENT ***
   80 I=IX(N,1)
      J=IX(N,2)
      DX=R(J)-R(I)
      DY=Z(J)-Z(I)
      XL=SQRT(DX**2+DY**2)
      DU=B(2*J-1)-B(2*I-1)
      DV=B(2*J)-B(2*I)
      DL=DV*DY/XL +DU*DX/XL
      DSIG(1)=DL*E(1,2,MTYPE)/XL+SIG(N,1)+SIG(N,4)
      IF (NCOUNT-NP) 85,84,84
   84 SIG(N,4)=DSIG(1)
      SIG(N,1)=0.
   85 XC=0.0
      YC=0.0
      GO TO 320
C***************************************************- ONE-D ELEMENT ***
C
   90 CALL QUAD
C
      DO 120 I=1,4
      II=2*I
      JJ=2*IX(N,I)
      P(II-1)=B(JJ-1)
  120 P(II)=B(JJ)
C
      DO 150 I=9,10
      P(I)=0.0
      KK=I-1
      DO 150 K=1,KK
  150 P(I)=P(I)-S(I,K)*P(K)
```

```
C
      D(1,1)=0.
      D(2,1)=0.
      D(3,1)=0.0
      DO 170 I=1,3
      DO 170 K=1,10
  170 D(I,1)=D(I,1)+ST(I,K)*P(K)
C
      THETA= SIG(N,3)/57.296
      CO=COS(THETA)
      SS=SIN(THETA)
      C2=CO*CO
      S2=SS*SS
      SC=SS*CO
      DSIG(1)= SIG(N,4)*C2+SIG(N,5)*S2-DSIG(1)
      DSIG(2)= SIG(N,4)*S2+SIG(N,5)*C2-DSIG(2)
      DSIG(3)=(SIG(N,4)-SIG(N,5))*SC-DSIG(3)
      DO 180 I=1,3
      DO 180 K=1,3
  180 DSIG(I)= DSIG(I)+ C(I,K)*D(K,1)
C
C************************************************************************
C     OUTPUT STRESSES
C************************************************************************
C
C     CALCULATE PRINCIPAL STRESSES
C
      AA=(DSIG(1)+DSIG(2))/2.
      BB= (DSIG(1)-DSIG(2))/2.
      CR= SQRT(BB**2+DSIG(3)**2)
      DSIG(4)=AA+CR
      DSIG(5)=AA-CR
      IF ((BB.EQ.0.0).AND.(DSIG(3).EQ.0.0)) GO TO 320
      DSIG(6)=ATAN2(DSIG(3),BB)/2.
      DSIG(6)=57.296*DSIG(6)
C
  320 RETURN
C
      END
```