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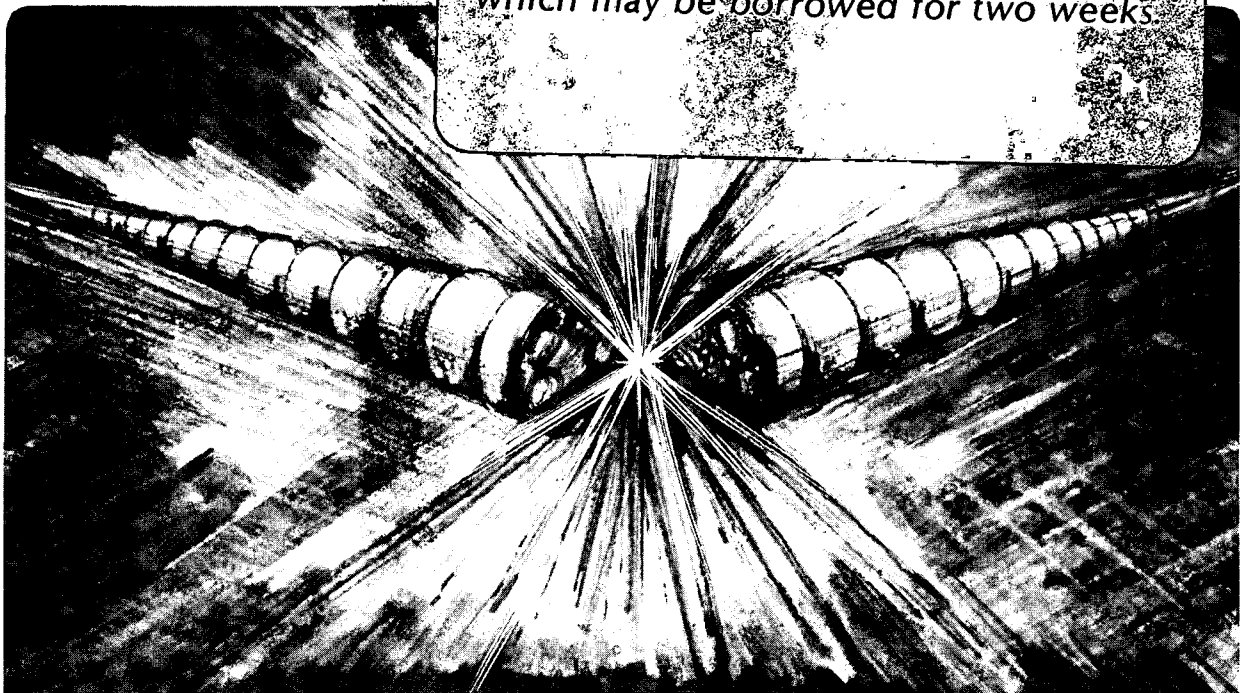
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GEOMETRY AND GUIDING CENTER MOTION

R.G. Littlejohn

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GEOMETRY AND GUIDING CENTER MOTION\*

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## Geometry and Guiding Center Motion

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### 1. Introduction

Over the years, plasma physicists have developed a number of computational techniques for solving various problems of physical interest. Often these involve perturbation methods of one sort or another, which are seldom systematized or clearly articulated. As one examines these mathematical methods of plasma physics, however, one finds that there is a surprising amount of interesting mathematics in them, especially in the area of differential geometry. A good example of this concerns guiding center motion, which is one of the most important approximation schemes in plasma physics, and which forms the subject of this paper.

To preview the main points of this paper, it turns out that guiding center

motion not only involves some interesting features of symplectic geometry, as one might expect for a mechanical system, but there is also some interesting metrical geometry. It even happens that guiding center motion is a gauge theory.

The physical principles of guiding center motion have been well understood for many years, and are reviewed in the excellent book by Northrop.<sup>1</sup> So here I will approach the subject from a mathematical standpoint, where there is much that is new. I will begin by giving a brief overview of the subject for the non-plasma physicist.

## 2. What is Guiding Center Motion?

Guiding center motion concerns the motion of a single charged particle in a given electromagnetic field. It is, therefore, represented by a Hamiltonian system of three degrees of freedom. The essence of the approximation involved in guiding center motion can be seen by examining the Lagrangian for a particle in a magnetic field,

$$L(\mathbf{x}, \mathbf{v}) = \frac{\mathbf{v}^2}{2} + \mathbf{v} \cdot \mathbf{A}(\mathbf{x}). \quad (1)$$

For simplicity I have neglected any electric field, and I have treated the particle nonrelativistically. The position of the particle is  $\mathbf{x}$ , its velocity is  $\mathbf{v}$ , and the magnetic field  $\mathbf{B}$  is represented by the vector potential  $\mathbf{A}$ , so that  $\mathbf{B} = \nabla \times \mathbf{A}$ .

I have also suppressed the physical constants  $e$ ,  $m$ , and  $c$  from the Lagrangian, in order to make the mathematics more clear. However, when the physical constants are restored and the two terms of the Lagrangian are numerically evaluated for many plasmas of physical interest, it turns out that the second term  $\mathbf{v} \cdot \mathbf{A}$ , representing the coupling of the particle to the magnetic field, greatly dominates the first term,  $v^2/2$ , which is the kinetic energy of the particle. Therefore for these physical situations one can achieve a good approximation to the particle motion simply by neglecting the kinetic energy altogether. Doing this and carrying out the variational principle, one finds the approximate equation of motion,

$$\mathbf{v} \times \mathbf{B} = 0. \tag{2}$$

The physical meaning of this is that the particle moves parallel to the direction of the magnetic field, i.e. that it is constrained to lie on the integral curves of  $\mathbf{B}$ , the magnetic field lines. This, in a nutshell, is the essence of plasma confinement by magnetic fields.

A more exact analysis of the particle motion shows that the component of the particle velocity perpendicular to the magnetic field does not really vanish, as indicated by (2), but rather undergoes high frequency oscillations. However, the average of this velocity component does, in fact, vanish to a good approximation, so that (2) is still correct in an averaged sense. The actual particle moves in tight circles around a magnetic field line, producing overall a helical trajectory as it

moves along the field line, as shown in Fig. 1.

Practically speaking, one must have a more refined picture of the particle motion than that indicated by (2), and so various schemes have been invented to analyze the motion perturbatively. These schemes effectively treat the kinetic energy as a perturbation, although they seldom explicitly acknowledge this. The physical effect of the perturbation is to cause the particle to "drift," i.e. to move slowly in a direction perpendicular to the magnetic field lines, at the same time that it moves rapidly along the field lines according to (2). The motion is pictured physically in an averaged sense, with the time average of the particle position being called the "guiding center."

### 3. The Symplectic Form in Guiding Center Theory

Now, in what sense can there be interesting symplectic geometry in guiding center motion, as I claim? After all, if you've seen one symplectic structure, you've seen them all. Darboux's theorem guarantees that local canonical coordinates always exist, so all symplectic structures look alike locally, i.e. they are all simply given by  $dp \wedge dq$ .

The answer is that the canonical coordinates whose existence is guaranteed by Darboux's theorem have different physical meanings in different contexts, so that it is the physical interpretation of the symplectic structure which provides interest and variety. To say this another way, the quantities of most immediate



physical interest do not always form canonical coordinates, so that the symplectic structure expressed in terms of physical variables is not simply  $dp \wedge dq$ . For example, in the case of guiding center motion, the symplectic 2-form has four contributions of different physical origin: one is the (kinetic) mechanical action, the second is the magnetic flux, the third is associated with a symmetry group producing rotations in the plane perpendicular to the magnetic field, and the fourth is a curvature form associated with the transport of triads of unit vectors in Euclidean space.

The decomposition of the symplectic form into physically interesting pieces can be seen in simpler form in the case of the motion of a charged particle in a magnetic field (completely apart from any guiding center approximation). For this system, the canonical momentum  $\mathbf{p}$  is often not considered a physical quantity, because it is not invariant under a change of gauge of the magnetic field. The velocity  $\mathbf{v}$ , however, is physical. In terms of the canonical momentum, the symplectic structure in phase space is just  $\omega = \sum dp_i \wedge dq_i$ . But in terms of the physically meaningful velocity, the symplectic structure is

$$\omega = \sum_i dv_i \wedge dq_i + \sum_{i < j} B_{ij} dq_i \wedge dq_j. \quad (3)$$

The first term may be called the kinetic action; it is the only term present for mechanical problems with Lagrangians of the form  $L = T - V$ , where  $T$  is the kinetic energy and  $V$  is the potential energy. Indeed, this is the only class of problems considered in older treatises on mechanics. The second term is the

magnetic flux (or rather the lift of it to phase space); it is responsible for the magnetic forces on the particle, and by our previous discussion of the Lagrangian (1), it is the dominant term when the guiding center approximation is valid. The tensor  $B_{ij}$  is expressed in terms of the usual magnetic field vector  $\mathbf{B}$  by  $B_{ij} = \epsilon_{ijk} B_k$ .

It is useful to pursue a little further the notion of using only physically significant quantities to represent a dynamical system. Consider, for example, the equations of motion. In canonical coordinates these are just Hamilton's equations, but what do we do if the canonical coordinates are physically undesirable? The best answer, I believe, is to use the Poincaré-Cartan 1-form  $\theta$ , which is defined in terms of canonical coordinates by

$$\theta = \sum_i p_i dq_i - H dt, \quad (4)$$

where  $H$  is the Hamiltonian. This form lives on the odd-dimensional space which is phase space augmented by time, for which I will write  $P \times \mathcal{R}$ . As explained by Arnold,<sup>2</sup> the equations of motion are implicitly contained in the 1-form  $\theta$  through the construction of its "vortex lines." That is, if the 2-form  $\Omega = d\theta$  is of maximal rank, it defines a 1-dimensional distribution on  $P \times \mathcal{R}$ . A vector field  $X$  lying in this distribution satisfies  $i_X \Omega = 0$ , and any such  $X$  specifies the equations of motion through the relation

$$X = \left( \frac{dq}{ds}, \frac{dp}{ds}, \frac{dt}{ds} \right), \quad (5)$$

for some parameter  $s$  (which has no physical significance). This formulation also deals nicely with time-dependent Hamiltonians.

It is significant that the equations of motion depend only on  $\Omega = d\theta$ , so that  $\theta$  can be subjected to a transformation  $\theta \rightarrow \theta + dS$ , for any scalar  $S$ , without changing any physical results. I call this a “gauge transformation on phase space,” to distinguish it from a gauge transformation on the magnetic vector potential  $\mathbf{A}$  in physical space.

A completely equivalent formulation, and one that is easier to use in practice, is the variational principle,

$$\delta \int \theta = 0, \tag{6}$$

where the variations of the path in  $P \times \mathfrak{R}$  are required to vanish at the endpoints. In canonical coordinates this variational principle is trivially equivalent to Hamilton’s equations, and it is discussed in many mechanics books as a curiosity. However, one can easily use physically interesting variables in this variational principle, and as a result one can clearly see the physical ingredients in the symplectic structure as well as in the Hamiltonian.

Consider again, for example, the motion of a charged particle in a magnetic field. Setting  $\mathbf{p} = \mathbf{v} + \mathbf{A}$  in (4) and (6), we have

$$\delta \int \left[ (\mathbf{A} + \mathbf{v}) \cdot d\mathbf{x} - \frac{\mathbf{v}^2}{2} dt \right] = 0. \tag{7}$$

It is easy to show that this is equivalent to the usual Newton-Lorentz equations

of motion. But notice that the symplectic structure neatly breaks up into the kinetic action, represented by  $\mathbf{v} \cdot d\mathbf{x}$ , and the magnetic flux, represented by  $\mathbf{A} \cdot d\mathbf{x}$ . Furthermore, the Hamiltonian  $H = v^2/2$  is just the kinetic energy.

Now let us consider the same system in the guiding center approximation. Just as with the Lagrangian (1), the Poincaré-Cartan form  $\theta$  has a dominant term and a perturbation, for which we write  $\theta = \theta_0 + \theta_1$ . The dominant term is  $\theta_0 = \mathbf{A} \cdot d\mathbf{x}$ , and the perturbation is  $\theta_1 = \mathbf{v} \cdot d\mathbf{x} - (v^2/2)dt$ . The physical picture we drew above for the guiding center approximation is easily verified in the formulation (7).

#### 4. Geometry and Perturbation Theory

Now we are ready to get serious about a perturbation calculation. It turns out that this is not just an unhappy exercise in algebra, but that there is some interesting geometry involved. This is mainly because we require a perturbation theory which is applicable to 1-forms like  $\theta = \theta_0 + \theta_1$ , and this problem in turn causes us to think about the structure of perturbation theory in general. A more complete accounting of this analysis may be found in Refs. 3-4.

A great deal is known about standard forms of Hamiltonian perturbation theory, which are applicable to problems for which the Hamiltonian  $H$  consists of a dominant term  $H_0$  and a perturbation  $H_1$ . In terms of the Poincaré-Cartan for  $\theta$ , we could write  $\theta_0 = \sum p_i dq_i - H_0 dt$ , and  $\theta_1 = -H_1 dt$ . That is, for these

standard problems in Hamiltonian perturbation theory, the symplectic structure is given exactly in terms of some physically significant canonical coordinate system, and all approximations are focused on the Hamiltonian. Therefore for these problems, one typically uses a sequence of canonical transformations to transform the perturbation into something easier to solve, the virtue of canonical transformations being that they exactly preserve the canonical form of the symplectic structure.

For the guiding center problem of (7), however, the symplectic structure itself is perturbed, with the perturbation being given by  $\mathbf{v} \cdot d\mathbf{x}$ , and the entire Hamiltonian is also treated as a perturbation. Therefore we require a generalization of canonical perturbation theory which allows us to transform not only the time component of  $\theta$ , i.e. the scalar Hamiltonian, but the whole 1-form in all of its components.

It is easiest to see how to do this by examining the geometric foundations of Lie transforms, which are often used in canonical perturbation theory. The goal of canonical Lie transforms is to perform a change of coordinates in order to simplify the scalar Hamiltonian. The Lie transforms themselves are a special kind of coordinate transformation. They are given by the advance map of some vector field  $G$ , which is called the *generator* of the transformation. That is, what is often viewed as a change of coordinates can also be seen as the application of the pullback of the advance map associated with  $G$ :

$$H = T^* \bar{H}, \quad (8)$$

where

$$T = \exp(G). \quad (9)$$

For time-dependent Lie transforms, the vector field  $G$  is seen as a vector field on the space  $P \times \mathcal{R}$ . In the canonical theory  $G$  must be a Hamiltonian vector field, since one requires the map  $\exp(G)$  to be a canonical transformation. That is, we must have  $Gf = \{g, f\}$  for some  $g$  and any  $f$ . (The bracket shown is the Poisson bracket.) Typically, the scalar  $g$ , and hence the vector field  $G$  and the transformation  $T = \exp(G)$ , is chosen so as to make  $\bar{H}$  easier to solve than the original  $H$ . Usually this involves finding coordinates in which an approximate symmetry of the original problem becomes exact.

These notions are easily generalized and applied to the Poincaré-Cartan form of (7). Again we define a Lie transform as a mapping  $T = \exp(G)$  for some vector field  $G$ , and we apply the obvious generalization of (8),

$$\theta = T^* \bar{\theta} = \exp(L_G) \bar{\theta}. \quad (10)$$

Now, however,  $G$  should not be a Hamiltonian vector field, because it is precisely the form of the symplectic structure which we wish to change (so as to deal with the perturbation in it).

Thus, the practical perturbation program for guiding center motion is the following. We use (10) to obtain an explicit relation connecting  $\bar{\theta}$  and  $G$  in

terms of the given  $\theta = \theta_0 + \theta_1$ . Then we choose  $G$  to make  $\bar{\theta}$  easier to understand or solve than  $\theta$ . This comes down to a kind of averaging of  $\theta$  over the rapid oscillations; an analogous averaging procedure is often applied in canonical perturbation theory and gives the “averaged Hamiltonian”  $\bar{H}$ . Here we produce an averaged symplectic structure as well as an averaged Hamiltonian, both contained in  $\bar{\theta}$ . To go to higher order in the perturbation series, the process has to be repeated with a new  $G$ . The result is the Poincaré-Cartan form  $\bar{\theta}$  for guiding center motion.

The use of Lie transforms in this process has some unexpected benefits. If one simply takes the 1-form  $\theta$  of (7) and tries to transform it by some arbitrary change of coordinates (not necessarily a Lie transform), one quickly finds a proliferation of magnetic gauge-dependent terms, coming from the transformation of the term  $\mathbf{A} \cdot d\mathbf{x}$ . These can always be eliminated by performing a gauge transformation in phase space,  $\theta \rightarrow \theta + dS$ , but it is not always easy to find the appropriate  $S$ . Using Lie transforms, however, we are applying the Lie derivatives  $L_G$  to forms such as  $\theta$ , as shown by expanding the exponential in (10). This allows us to use a nice formula from differential geometry,

$$L_G \theta = i_G d\theta + d(i_G \theta). \quad (11)$$

The second term represents a gauge transformation in phase space, and can be dropped without any effect on the dynamics. But by doing so, one finds that all the magnetic gauge dependencies are dropped also. Thus, by using Lie

transforms, one easily obtains a perturbation expansion which is magnetic gauge invariant to all orders (except for the original magnetic flux term,  $\mathbf{A} \cdot d\mathbf{x}$ , which stays around.) The ability to guarantee gauge invariance to arbitrary order is an important benefit of the geometric approach to perturbation theory.

Now I shall discuss the actual results of the perturbation calculation I have just outlined. These results consist of a sequence of generating vector fields  $G$ , which I shall not display, and an averaged Poincaré-Cartan form  $\bar{\theta}$ . The generators specify the coordinate transformation (or map of  $P \times \mathcal{R}$  onto itself) which connects the original set of phase space coordinates  $(\mathbf{x}, \mathbf{v})$  with a new set of averaged coordinates. For the latter it is convenient to take the set  $(\mathbf{X}, U, \zeta, \mu)$ , in which  $\mathbf{X}$  is the guiding center position,  $U$  is the component of the guiding center velocity parallel to  $\mathbf{B}$ ,  $\mu$  is the magnetic moment, and  $\zeta$  is the "gyrophase." The magnetic moment is approximately given by  $\mu = v_{\perp}^2 / 2B$ ; its dynamical significance is that it is the generator of the  $U(1)$  symmetry group whose action consists of rotations in the plane perpendicular to the magnetic field. The gyrophase  $\zeta$  is some conventional angle in this plane, so that the Hamiltonian vector field associated with  $\mu$  is  $\partial / \partial \zeta$ . The conjugate variable pair  $(\zeta, \mu)$  describe the symmetry, and indicate that the perturbation calculation has achieved a "reduction." Thus,  $\zeta$  is an ignorable coordinate,  $\mu$  is a constant of motion, and the four remaining variables  $(\mathbf{X}, U)$ , for fixed  $\mu$ , can be taken as coordinates on the reduced phase space.



The averaged Poincaré-Cartan form is given by

$$\bar{\theta} = (\mathbf{A} + \epsilon U \mathbf{b} - \epsilon^2 \mu \mathbf{R}) \cdot d\mathbf{X} + \epsilon^2 \mu d\zeta - \epsilon H dt, \quad (12)$$

where

$$H = \mu B + \frac{1}{2} U^2 + \epsilon(\dots). \quad (13)$$

In deriving this result I have introduced a formal expansion parameter  $\epsilon$  into the original 1-form  $\theta$ , i.e. I have set  $\theta = \theta_0 + \epsilon \theta_1$  so that the final 1-form  $\bar{\theta}$  appears as a series in  $\epsilon$ . The vector  $\mathbf{R}$  will be discussed presently; additional notation appearing in (12) is defined by  $B = |\mathbf{B}|$ ,  $\mathbf{b} = \mathbf{B}/B$ . The ellipsis in (12) represents higher order terms which I have not displayed.

When this 1-form is applied to the variational principle of (6), the result is the set of drift equations familiar in plasma physics. A practical benefit of this formulation of the drift equations is that several important conservation laws (those for energy, angular momentum, and phase volume) emerge easily and naturally. The status of these conservation laws has been obscure in traditional guiding center theory.

The symplectic structure appearing in (12) consists of several contributions with different physical interpretations. The dominant term is  $\mathbf{A} \cdot d\mathbf{X}$ , which is still the magnetic flux, although it can now be interpreted as living on the reduced phase space. The next term is  $U \mathbf{b} \cdot d\mathbf{X}$ , which is in a sense the average of the kinetic action (since the perpendicular velocity components average to zero). At

next order, the term  $\mu d\zeta$  can be interpreted in terms of the symmetry group generated by  $\mu$ . The term  $-\mu \mathbf{R} \cdot d\mathbf{X}$  will of the same order; it will be discussed in the next section. Finally, the Hamiltonian (the coefficient of  $dt$ ) is still the kinetic energy, now expressed in terms of the averaged variables.

## 5. Gyrogauge Invariance: The Geometrical Picture

I turn now to the issue of defining the gyrophase, which involves some interesting metrical geometry. Because this section contains material of occasional use in plasma physics, I will express the results in terms of three-dimensional vector calculus.

Physically, the gyrophase is an angle which represents the rapid circular motion of the particle in the perpendicular plane. Figure 2 shows the geometrical situation; the unit vector  $\mathbf{b}$  is parallel to the magnetic field, and the unit vectors  $\mathbf{e}_1, \mathbf{e}_2$  span the plane perpendicular to  $\mathbf{b}$ . The set  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$  form an orthonormal triad. The use of such a triad of unit vectors is necessary to coordinatize the motion of the particle, but only the vector  $\mathbf{b}$  has an immediate physical and geometrical significance. The other two vectors,  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , are constrained to form an orthonormal triad with  $\mathbf{b}$ , but otherwise their particular orientation in the perpendicular plane is immaterial. Nevertheless, in a practical problem some specific choice for the vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$  must be made, in order to define the

gyrophase  $\zeta$ . For example,  $\zeta$  may be taken as the angle between  $\mathbf{e}_1$  and the perpendicular component of the particle velocity.

It is intuitively clear that no physical results should depend on the orientation of  $\mathbf{e}_1$  and  $\mathbf{e}_2$  in the perpendicular plane. It turns out that this is not a problem when the guiding center theory is carried only to lowest order, because  $\mathbf{e}_1$  and  $\mathbf{e}_2$  do not appear in the drift equations anyway. But when the equations are carried to next order, these vectors do appear, and a number of plasma physicists over the years have wondered what to do about them. Some people have suggested using some privileged choice for  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , such as the principal normal and binormal vectors of the field line. But such choices have certain esthetic drawbacks, and they do not simplify any of the calculations or results. A better answer is to let the arbitrariness in  $\mathbf{e}_1$  and  $\mathbf{e}_2$  be a free parameter of the problem, and then to study the invariance principle which results. It turns out that this invariance principle involves a kind of gauge transformation.

The arbitrariness inherent in  $\mathbf{e}_1$  and  $\mathbf{e}_2$  is that they can be rotated about  $\mathbf{b}$  by an arbitrary angle  $\psi$  with no effect on the physics. Furthermore, the amount of rotation can vary from one point of space to another, i.e.  $\psi$  is allowed to depend on  $\mathbf{x}$ . Explicitly, the transformation is

$$\begin{aligned} \mathbf{e}'_1 &= +\mathbf{e}_1 \cos \psi + \mathbf{e}_2 \sin \psi, \\ \mathbf{e}'_2 &= -\mathbf{e}_1 \sin \psi + \mathbf{e}_2 \cos \psi. \end{aligned} \tag{14}$$

I call this a "gyrogauge transformation." It is easy to see the effect of a

gyrogauge transformation on the gyrophase  $\zeta$ ; one is simply redefining the origin of gyrophase, i.e. the reference direction in the perpendicular plane which corresponds to  $\zeta = 0$ . Thus, we have

$$\zeta' = \zeta + \psi(\mathbf{x}) \quad (15)$$

under a gyrogauge transformation. We see that  $\zeta$  is not gyrogauge invariant. Naturally, we expect that any quantity which is gyrogauge invariant can be expressed purely in terms of the vector  $\mathbf{b}$  and other physical quantities.

Now, when one carries out the perturbation analysis of guiding center motion, the first quantity to appear which is not gyrogauge invariant and which requires further interpretation is a certain vector which I call  $\mathbf{R}$ :

$$\mathbf{R} = (\nabla \mathbf{e}_1) \cdot \mathbf{e}_2. \quad (16)$$

This vector is not gyrogauge invariant, because, as one easily verifies from (14),

$$\mathbf{R}' = \mathbf{R} + \nabla \psi. \quad (17)$$

One can see already that  $\mathbf{R}$  looks like a 1-form, and that (17) is a kind of gauge transformation. But to proceed from a physical point of view, let us consider the equation of motion for the gyrophase  $\zeta$ . This equation can be obtained from (12) or by other means; in any case, the result is

$$\dot{\zeta} = \frac{B}{\epsilon} + \dot{\mathbf{X}} \cdot \mathbf{R} + \text{other terms.} \quad (18)$$

The first term on the right hand side,  $B/\epsilon$ , shows the rapid evolution of the gyrophase due to the rapid orbiting of the particle around the magnetic field

line. The frequency of this motion is  $B$ , and the term appears perturbatively at order  $\epsilon^{-1}$ .

The second term is more subtle. As the guiding center moves along with velocity  $\dot{\mathbf{X}}$ , the local  $\mathbf{e}_1$  and  $\mathbf{e}_2$  vectors, to which the definition of  $\zeta$  is tied, change from point to point. In a certain sense, this change is due to two causes. One is the fact that  $\mathbf{b}$  itself changes from point to point, and  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are constrained to be perpendicular to  $\mathbf{b}$ . A second is that the orientation of  $\mathbf{e}_1$  and  $\mathbf{e}_2$  in the perpendicular plane can also change, i.e. these vectors may rotate in their plane of definition as one moves about. I will show momentarily that it is the latter effect which is represented by the term  $\dot{\mathbf{X}} \cdot \mathbf{R}$ . Accepting this for a moment, we can now interpret  $\mathbf{R}$  as a 1-form by writing

$$\rho = \mathbf{R} \cdot d\mathbf{X}, \quad (19)$$

and we see that the integral of  $\rho$  along some path is the net angle of rotation which  $\mathbf{e}_1$  and  $\mathbf{e}_2$  undergo in their plane of definition along the given path.

But what sense does it make to talk about an angle of rotation, when the plane in which  $\mathbf{e}_1$  and  $\mathbf{e}_2$  lie is changing from point to point? Clearly we need some concept of transport, so that  $\mathbf{e}_1$  and  $\mathbf{e}_2$  lying in the perpendicular plane at one point can be compared to their neighbors a short distance away, lying in another plane. Nor can this be the usual parallel transport of Riemannian geometry, because physical space is flat.

Instead, we find the following geometrical picture. Consider two neighboring

points  $\mathbf{x}$  and  $\mathbf{x} + \Delta\mathbf{x}$ , and the corresponding vectors  $\mathbf{e}_1(\mathbf{x})$ ,  $\mathbf{e}_2(\mathbf{x})$  and  $\mathbf{e}_1(\mathbf{x} + \Delta\mathbf{x})$ ,  $\mathbf{e}_2(\mathbf{x} + \Delta\mathbf{x})$ . Take  $\mathbf{e}_1(\mathbf{x})$  and  $\mathbf{e}_2(\mathbf{x})$  and move them parallel to themselves over to the point  $\mathbf{x} + \Delta\mathbf{x}$ . Then project these vectors onto the perpendicular plane at the new point, to create new vectors,  $\mathbf{f}_1$ ,  $\mathbf{f}_2$ . The angle  $\Delta\alpha$  between  $\mathbf{f}_1$ ,  $\mathbf{f}_2$  and  $\mathbf{e}_1(\mathbf{x} + \Delta\mathbf{x})$ ,  $\mathbf{e}_2(\mathbf{x} + \Delta\mathbf{x})$  is interpreted as the rotation of  $\mathbf{e}_1$  and  $\mathbf{e}_2$  on passing between the two points. Indeed, a simple calculation shows that

$$\Delta\alpha = \mathbf{R} \cdot \Delta\mathbf{x}, \quad (20)$$

which confirms the interpretation of  $\mathbf{R}$  given above.

At this point one might be tempted to say that  $\mathbf{e}_1$  and  $\mathbf{e}_2$  rotate as one moves about in space only because they were poorly defined. If we were somehow able to define a set of unit vector fields  $\mathbf{e}_1$  and  $\mathbf{e}_2$  which were "rotationless;" then the troublesome terms involving the vector  $\mathbf{R}$  would vanish. Nevertheless, it turns out that it is impossible, in general, to define such rotationless vector fields. One way to see this is to take the curl of (17):

$$\nabla \times \mathbf{R}' = \nabla \times \mathbf{R}. \quad (21)$$

Thus, although  $\mathbf{R}$  itself is gyrogauge dependent, its curl is gyrogauge invariant. If this curl is nonzero, as it sometimes is, then clearly no choice of  $\mathbf{e}_1$  and  $\mathbf{e}_2$  can make  $\mathbf{R}$  vanish.

Another way to see the same thing is to attempt a geometrical construction of a rotationless set  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and see what happens. We begin by choosing  $\mathbf{e}_1$

and  $\mathbf{e}_2$  at a single point  $\mathbf{x}_0$ . Next, we take some curve passing through  $\mathbf{x}_0$  and extend the definition of  $\mathbf{e}_1$  and  $\mathbf{e}_2$  to points on this curve, in such a way that the vectors are rotationless along the curve. This is done geometrically by first moving  $\mathbf{e}_1$  and  $\mathbf{e}_2$  a small distance along the curve parallel to themselves, next projecting them down onto the perpendicular plane at the new point, and then by repeating the first two steps for the next and successive small increments of distance. This process can be described in terms of a transport differential equation. If we let  $\mathbf{V}$  be a vector we wish to transport along the curve (it might be  $\mathbf{e}_1$  or  $\mathbf{e}_2$  or something else, but it should satisfy  $\mathbf{b} \cdot \mathbf{V} = 0$  at  $\mathbf{x}_0$ ) and we let  $s$  be the arc length, then the desired transport equation is

$$\frac{d\mathbf{V}}{ds} = -\mathbf{b} \left( \frac{\partial \mathbf{b}}{\partial s} \cdot \mathbf{V} \right). \quad (22)$$

This equation follows from a simple analysis of the geometrical picture of the transport process which I have just given.

This transport equation has several notable features. First, if  $\mathbf{b} \cdot \mathbf{V} = 0$  at  $\mathbf{x}_0$ , as we require, then  $\mathbf{b} \cdot \mathbf{V} = 0$  everywhere on the curve. Thus,  $\mathbf{e}_1$  and  $\mathbf{e}_2$  transported by (22) are actually in the perpendicular plane at each point. Next, if  $\mathbf{V}_1$  and  $\mathbf{V}_2$  are two perpendicular vectors created by the transport process, then the scalar product  $\mathbf{V}_1 \cdot \mathbf{V}_2$  is constant along the curve. Thus,  $\mathbf{e}_1$  and  $\mathbf{e}_2$  created by (22) remain orthogonal to each other along the curve. Finally, by taking  $\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{e}_1$  or  $\mathbf{e}_2$ , we see that the length of  $\mathbf{e}_1$  and  $\mathbf{e}_2$  is preserved by the transport, so these vectors remain unit vectors. Thus we obtain an orthonormal

triad at each point on the curve, which is rotationless along the curve.

Equation (22) has the form of a parallel transport equation in Riemannian geometry, if we define Christoffel symbols by

$$\Gamma_{jk}^i = b_i b_{k,j}. \quad (23)$$

Unlike the usual Christoffel symbols, however, these are not symmetric, i.e.  $\Gamma_{jk}^i \neq \Gamma_{kj}^i$ , unless  $\nabla \times \mathbf{b} = 0$ . But in that case it happens that there exist surfaces which are everywhere perpendicular to  $\mathbf{b}$  (actually  $\mathbf{b} \cdot \nabla \times \mathbf{b} = 0$  is sufficient), and  $\Gamma_{jk}^i$  can be given its usual interpretation in terms of parallel transport along curves which lie in those surfaces. Altogether, we see that the transport and connection given by (22)-(23) is a kind of generalization of parallel transport on two-dimensional surfaces in Euclidean  $\mathbb{R}^3$ .

The transport process yields a rotationless triad along a given curve, but not a field of triads. One could fill up a finite volume of space with triads by drawing many curves radiating from  $\mathbf{x}_0$ , but these triads would be rotationless only along the given curves. Along some other curve, such as the path of the guiding center, they might not be rotationless. Thus, the way to see if a rotationless field of triads can be set up is to consider the transport of a triad along a closed curve (and not to worry that actual guiding centers might never follow a closed curve), in order to see if the property of being rotationless is path dependent.

Let us transport a perpendicular vector  $\mathbf{V}$  around a small parallelogram defined by two small displacements,  $\Delta \mathbf{x}_1$  and  $\Delta \mathbf{x}_2$ . The area of the parallelogram



is represented by the vector  $\Delta\mathbf{S} = \Delta\mathbf{x}_1 \times \Delta\mathbf{x}_2$ . In taking  $\mathbf{V}$  around this small parallelogram, one must carry (22) out to second order in the displacements, because the first order terms cancel. We know that when we bring  $\mathbf{V}$  back to its starting point it must have the same length as when it started, and it must still be perpendicular to  $\mathbf{b}$ . Therefore at worst it has rotated in the perpendicular plane by a certain angle  $\Delta\alpha$ . This angle is given by  $\Delta\alpha = \mathbf{N} \cdot \Delta\mathbf{S}$ , where

$$\mathbf{N} = \frac{1}{2}\mathbf{b}[(b_{i,j}b_{j,i}) - (\nabla \cdot \mathbf{b})^2] + (\nabla \cdot \mathbf{b})\mathbf{b} \cdot \nabla\mathbf{b} - \mathbf{b} \cdot \nabla\mathbf{b} \cdot \nabla\mathbf{b}. \quad (24)$$

Thus, if  $\mathbf{N} \neq 0$ , it will be impossible to set up a field of triads which is rotationless along every path.

A complementary point of view is to imagine we are given an arbitrary field of triads  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$ , which no one has tried to make rotationless, and find what angle of rotation the triad suffers around a closed curve. By the argument surrounding (19)-(20), this is the line integral of  $\mathbf{R}$  around the closed curve, and Stokes' theorem can be applied:

$$\oint \mathbf{R} \cdot d\mathbf{x} = \int (\nabla \times \mathbf{R}) \cdot d\mathbf{S}. \quad (25)$$

From this it is clear that

$$\mathbf{N} = \nabla \times \mathbf{R}, \quad (26)$$

and indeed we see that  $\nabla \times \mathbf{R}$ , which is gyrogauge invariant by (21), can be expressed purely in terms of  $\mathbf{b}$ . One can also verify (26) directly, by taking the curl of (16) and using some arcane vector identities.

The vector  $\mathbf{N}$  is related to a 2-form  $\nu$  by

$$\nu = \frac{1}{2} \sum_{ijk} \epsilon_{ijk} N_k dx_i \wedge dx_j. \quad (27)$$

$\nu$  represents the angle of rotation a triad suffers on being transported around the boundary of a 2-dimensional region. If surfaces exist perpendicular to  $\mathbf{b}$  and the regions considered lie in these surfaces, then  $\nu$  is the curvature form (there is only one) of the surface. We note in (12) that  $\nu = d\rho$  forms the fourth and final contribution to the symplectic structure of the guiding center motion.

Finally, I would like to summarize some algebraic properties of the vectors  $\mathbf{R}$  and  $\mathbf{N}$ , which cannot be found anywhere else. By (26), we must have  $\nabla \cdot \mathbf{N} = 0$ . However, when we work this out explicitly from (24), we find

$$\nabla \cdot \mathbf{N} = \frac{2}{3} (\nabla \cdot \mathbf{b})(b_{i,j} b_{j,i}) - \frac{1}{2} (\nabla \cdot \mathbf{b})^2 - (b_{j,i} b_{k,j} b_{i,k}). \quad (28)$$

To see that this actually does vanish (it is not obvious), we call on the following formula. Let  $M$  be a  $3 \times 3$  matrix with components  $M_{ij}$ . Then

$$\det M = \frac{1}{3!} [(\text{Tr } M)^3 - 3(\text{Tr } M)(\text{Tr } M^2) + (\text{Tr } M^3)]. \quad (29)$$

Therefore by setting  $M_{ij} = b_{j,i}$ , we have

$$\nabla \cdot \mathbf{N} = -\frac{1}{3} \det(b_{i,j}). \quad (30)$$

And this in turn vanishes because the matrix  $b_{i,j}$  has an eigenvector with eigenvalue zero. This is none other than  $\mathbf{b}$  itself:

$$\nabla \mathbf{b} \cdot \mathbf{b} = \frac{1}{2} \nabla(\mathbf{b}^2) = 0, \quad (31)$$

since  $\mathbf{b}$  is a unit vector.

On the other hand,  $\mathbf{N}$  is closely related to the characteristic polynomial of  $b_{i,j}$ . Let the eigenvalues of  $b_{i,j}$  be  $\lambda_0, \lambda_1, \lambda_2$ , and suppose  $\lambda_0 = 0$  (one of them must be zero). Then it turns out that (24) can be written in the form

$$\mathbf{N} = -\lambda_1 \lambda_2 \mathbf{b} + (\lambda_1 + \lambda_2) \mathbf{b} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{b} \cdot \nabla \mathbf{b}. \quad (32)$$

Finally, we obtain another useful identity by multiplying  $\nabla \mathbf{b}$  on the left by this. The result must vanish, because any matrix satisfies its own secular equation. Thus,

$$\mathbf{N} \cdot \nabla \mathbf{b} = 0, \quad (33)$$

and we see that  $\mathbf{N}$  is a left eigenvector of  $\nabla \mathbf{b}$ , just as  $\mathbf{b}$  is a right eigenvector.

## 6. Gyrogauge Invariance: Dynamical Considerations

We have succeeded in dealing nicely with the interpretation of the gyrogauge dependent quantity  $\mathbf{R}$ , and in showing that it cannot be transformed or defined away. But if we were to carry the guiding center theory out to higher order, would we keep running into other gyrogauge dependent quantities which would have to be analyzed similarly, or can we settle the issue once and for all? In a similar vein, is it possible to find drift equations which are gyrogauge invariant to all orders?

To begin, the Poincaré-Cartan form of the particle, shown in (7), is certainly gyro-gauge invariant, as it must be. Similarly, the Poincaré-Cartan form of the guiding center, shown in (12) is also gyro-gauge invariant, although here some discussion is called for. The components of  $\bar{\theta}$  are not individually gyro-gauge invariant, because the vector  $\mathbf{R}$  appears. But the coordinate differential  $d\zeta$  is not gyro-gauge invariant either, for by (15) we have

$$d\zeta' = d\zeta + \mathbf{R} \cdot d\mathbf{X}. \quad (34)$$

However, when we examine the behavior of  $\bar{\theta}$  under a gyro-gauge transformation, we find that the transformation of  $\mathbf{R}$  and that of  $d\zeta$  exactly cancel one another, showing overall gyro-gauge invariance for  $\bar{\theta}$ .

The result is that the drift equations coming from  $\bar{\theta}$  are gyro-gauge invariant. For example, the equations of motion for  $\mathbf{X}$  and  $U$  involve the vector  $\mathbf{R}$  only through its curl, which is given purely in terms of  $\mathbf{b}$  by (24). Similarly, the equation for  $\zeta$ , shown in (18), is form invariant under a gyro-gauge transformation, even though the vector  $\mathbf{R}$  appears in it; the transformation properties of the two sides of the equation cancel one another.

But how did this gyro-gauge invariance come about in the averaged Poincaré-Cartan form of (12), and would it persist to higher order? This gyro-gauge invariance was not automatic; rather, it came about by using gyro-gauge invariant generators  $G$  in the Lie transforms. As long as such generators are used in (10) for the perturbation transformations, the result will be gyro-gauge invariant

guiding center dynamics to arbitrary order.

A generating vector field  $G$  appearing in (10) can be written in the form

$$G = G_{\mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{X}} + G_U \frac{\partial}{\partial U} + G_{\mu} \frac{\partial}{\partial \mu} + G_{\zeta} \frac{\partial}{\partial \zeta}. \quad (35)$$

All of the partial derivative operators appearing here are gyrogauged invariant with the exception of  $\partial/\partial \mathbf{X}$ , which transforms according to

$$\left( \frac{\partial}{\partial \mathbf{X}} \right)' = \frac{\partial}{\partial \mathbf{X}} - \nabla \psi \frac{\partial}{\partial \zeta}. \quad (36)$$

Therefore the overall vector field  $G$  will be gyrogauged invariant if the component  $G_{\zeta}$  compensates for (36) by transforming according to

$$G'_{\zeta} = G_{\zeta} + \nabla \psi \cdot G_{\mathbf{x}}. \quad (37)$$

And this will be the case if all the components of  $G$  consist of gyrogauged invariant quantities, except for  $G_{\zeta}$  which must contain a term equal to  $\mathbf{R} \cdot G_{\mathbf{x}}$ . By this definition of a gyrogauged invariant generator, one can construct a gyrogauged invariant guiding center theory to arbitrary order.

It is interesting that the gyrogauged transformation (17) is mathematically identical to an ordinary magnetic gauge transformation. Thus, the vector  $\mathbf{R}$  is analogous to the vector potential  $\mathbf{A}$ , and the divergence free and completely physical vector  $\mathbf{N} = \nabla \times \mathbf{R}$  is analogous to the magnetic field  $\mathbf{B}$ . Furthermore, these two types of gauge fields are coupled to each other in the guiding center Poincaré-Cartan form of (12), and the coupling constant is the magnetic moment.

This analogy is even more striking when time-dependent fields are considered. Then the vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$  depend on time as well as space, and one finds a gyrogauged dependent scalar,

$$\sigma = \frac{\partial \mathbf{e}_2}{\partial t} \cdot \mathbf{e}_1. \quad (38)$$

Under a time-dependent gyrogauged transformation, one has

$$\sigma' = \sigma - \frac{\partial \psi}{\partial t}, \quad (39)$$

which taken with (17) shows that  $\sigma$  is like an electric potential. Indeed, just as  $\mathbf{N} = \nabla \times \mathbf{R}$  is gyrogauged invariant, so now is the vector  $\mathbf{F}$ , given by

$$\mathbf{F} = -\nabla \sigma - \frac{\partial \mathbf{R}}{\partial t}. \quad (40)$$

Evidently,  $\mathbf{F}$  is like an electric field, just as  $\mathbf{N}$  is like a magnetic field. Since  $\mathbf{F}$  is gyrogauged invariant, it can also (like  $\mathbf{N}$ ) be expressed purely in terms of  $\mathbf{b}$ ; the result is

$$\mathbf{F} = \nabla \mathbf{b} \cdot \left( \mathbf{b} \times \frac{\partial \mathbf{b}}{\partial t} \right). \quad (41)$$

Finally, there is another gyrogauged equivalent of a Maxwell equation, the complement of  $\nabla \cdot \mathbf{N} = 0$ :

$$\nabla \times \mathbf{F} = -\frac{\partial \mathbf{N}}{\partial t}. \quad (42)$$

Thus, by using the quantities  $\sigma$  and  $\mathbf{F}$ , it is straightforward to extend gyrogauged invariance to time-dependent systems. One finds, for example, the term  $\mu \sigma$  in the Hamiltonian.

It turns out that it is impossible to find a set of variables to describe the guiding center motion which are both canonical and gyrogauged invariant, if one of these variables is the magnetic moment and another is some gyrophase canonically conjugate to it. This was the principal esthetic difficulty with my earlier work on guiding center motion, which used Darboux's theorem to construct averaged variables. In the present formalism, the Poisson brackets  $\{\mathbf{X}, \zeta\}$  and  $\{U, \zeta\}$ , which can be derived from the 1-form of (12) in a manner described in Ref. 3, are nonzero. If one redefines these variables so that these brackets vanish, then the variables  $(\mathbf{X}, U)$  cannot be made gyrogauged invariant. This conclusion may have interesting consequences for the applicability of canonical coordinates and canonical transformation theory to other dynamical systems with a symmetry and for the reduction process in general.

## 7. Conclusions

The principal practical goal of guiding center theory has been to address specific problems in plasma physics, and therefore I have not considered possible mathematical generalizations. Let me now suggest a few of these, and raise some questions. Some of these are vague and not clearly thought out, but perhaps they will be suggestive.

In guiding center motion, the reduction of the phase space by means of the symmetry associated with the ignorable coordinate  $\zeta$  has produced a reduced

phase space, with coordinates  $(\mathbf{X}, U)$ , which is the guiding center phase space. Each point of this space can be considered to have a fiber attached to it, for which the ignorable coordinate  $\zeta$  serves as a coordinate. That is, we have a circle bundle on the reduced phase space, each circle being a copy of the symmetry group  $U(1)$ . For an arbitrary dynamical system with a symmetry, would we in the same manner obtain a group bundle on the reduced phase space?

If so, then the identity element in each copy of the group could be redefined from point to point on the reduced phase space with no physical effect, just as we used the field  $\psi$  to redefine the origin of the gyrophase.  $\psi$  would now be a field of group elements. Similarly, the 1-form  $\rho = \mathbf{R} \cdot d\mathbf{X}$  would generalize to a Lie algebra-valued 1-form.

Would there then be a connection? The metrical structure of Euclidean  $\mathbb{R}^3$  seemed to play an essential role in the connection we have discovered here; can this be generalized? What role does the curvature form of the group bundle generally play in the symplectic structure of the dynamical system? It seems to have played a role in guiding center theory, through the form  $\nu = d\rho$ , but then here  $\nu$  is closed. I will leave these and further issues to my mathematical colleagues.



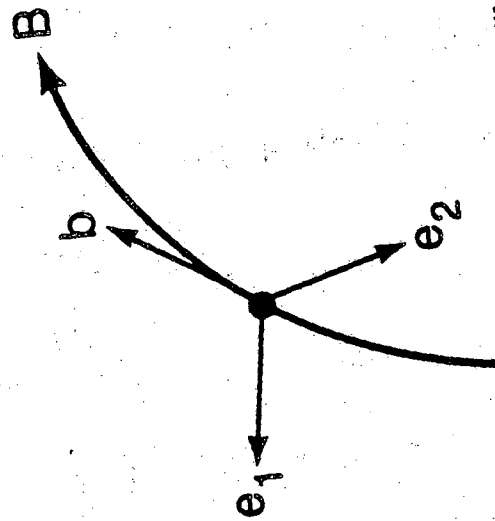
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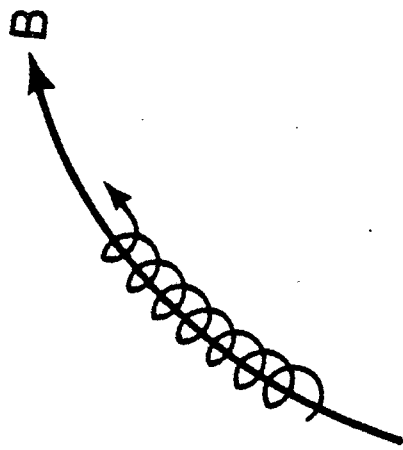
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Fig. 2



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Fig. 1

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