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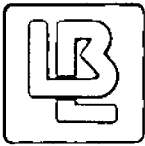
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Author

Wohl, C.G.

Publication Date

1981-05-01



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Physics, Computer Science & Mathematics Division

Submitted to the American Journal of Physics

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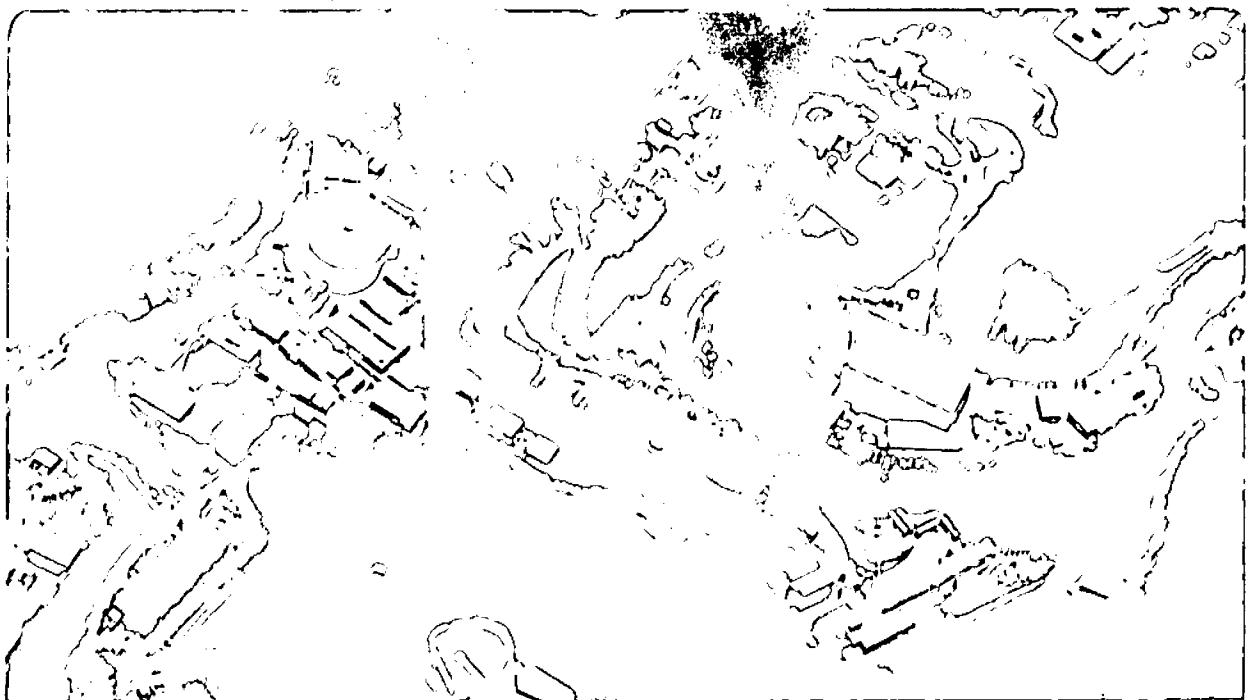
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ISOSPIN RELATIONS BY COUNTING

Charles G. Wohl
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

Abstract - A method is given for finding the relations between reaction cross sections or decay branching fractions that result from isospin conservation. The method was discovered long ago by Shmushkevich but is not widely known. It makes no call on the usual machinery of amplitude expansions and Clebsch-Gordan coefficients, but works instead by apportioning certain populations of particles according to a simple counting rule, the charge-uniformity rule. A number of examples, including several that require lengthy calculation to solve using Clebsch-Gordan coefficients, are here solved by inspection, usually without any equations at all.

This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract Number W-7405-ENG-48.

I. SHMUSHKEVICH'S METHOD (CHARGE UNIFORMITY)

Most of the strongly interacting particles (hadrons) occur in sets of two or three or four having nearly the same mass and differing in charge by unit steps of the electronic charge e . Examples are the nucleon doublet (proton p and neutron n) and the pion triplet (π^+ , π^0 , and π^-). There are also a number of single particles such as the ω meson. It is usual to consider the particles of a set to be the members of an isospin multiplet, the charge states being analogous to the m_s states (magnetic quantum number states) of a spin angular momentum multiplet. The analogy is only formal but it is nonetheless far-reaching, for it is found that the strong interactions, as opposed to the electromagnetic and weak interactions, conserve isospin. That is, in strong interactions the isospins of the particles making up a system combine according to the same vector coupling rules as do angular momenta; and the amplitudes for the various charge modes for a specific reaction or decay process (such as $NN \rightarrow NN\pi$ or $\omega \rightarrow \pi\pi\pi$) may be written, using Clebsch-Gordan coefficients, as linear superpositions of a common set of pure isospin amplitudes, one such amplitude for each independent way the particles can, according to the rules, couple together and have the same isospin in the initial state as in the final state. In all but trivial cases there are fewer isospin amplitudes than charge-mode amplitudes, and so the latter are not all independent. This leads to relations between the charge-mode cross sections or branching fractions, which are proportional to the squares of the magnitudes of the charge-mode

amplitudes.¹

About 25 years ago, the Russian physicist Shmushkevich discovered a way to get the relations between cross sections or branching fractions without having to use amplitude expansions or Clebsch-Gordan coefficients.² Although it is occasionally mentioned in the literature, Shmushkevich's method is known to few of even those physicists who have often to make isospin calculations. The starting point is a definition: a population of a particle multiplet is uniform if and only if it is made up of equal numbers of the members of the multiplet. Thus a uniform population of 100 nucleons is made up of 50 protons and 50 neutrons, and a uniform population of 300 pions is made up of 100 π^+ 's, 100 π^0 's, and 100 π^- 's. Any population of singlets is automatically uniform. Shmushkevich's method then consists in applying the following rule:

All particle populations involved in isospin-conserving reactions or decays of initially uniform populations are uniform at all times.

This is here dubbed the charge-uniformity rule. It applies separately to each set of charge modes (the modes for a specific reaction or decay process).

Examples take up nearly the whole of this paper, Sect. II on decays and Sect. III on reactions, and these will make the meaning of the above rule evident. The relation between charge uniformity and the usual isospin formalism is discussed in the Appendix, but until then any reference to this formalism is studiously avoided. Here it need only be noted that while charge uniformity is a consequence of

isospin conservation, it is not the whole thing. It has, for example, nothing to say about the phase relations between amplitudes, and in fact it deals not at all with amplitudes as such but only with relations between their squared magnitudes. And it sometimes needs help from the well-known charge-symmetry rule, which is introduced in Sect. II·B. For each set of reaction or decay modes, the relations between the cross sections or branching fractions come directly from arranging, in a manner specified by the rules, a kind of balanced or harmonious relation of the parts to the whole. Few equations are needed, for from the point of view advanced here a moment's thought usually suffices to make the relations obvious.

II. ISOSPIN-CONSERVING DECAYS

A. First examples

The ω meson, a neutral isospin singlet, decays mainly into three pions. As far as conservation of charge is concerned, the possible 3-pion charge modes are

$$\begin{aligned}\omega^0 &\rightarrow \pi^+ \pi^0 \pi^- \\ &\rightarrow \pi^0 \pi^0 \pi^0 .\end{aligned}$$

Suppose 100 ω mesons decay this way, producing 300 pions. The ω population is uniform because the ω is a singlet. If all 100 ω mesons decay into $\pi^+ \pi^0 \pi^-$, then afterwards there are 100 of each pion charge state, and the pion population is uniform too. If, however, even one ω decays into $\pi^0 \pi^0 \pi^0$, then after all the decays there are too many π^0 's and not enough π^+ 's or π^- 's (count them). Therefore $\omega^0 \rightarrow \pi^0 \pi^0 \pi^0$ decay is forbidden by isospin conservation.

The f meson, another neutral singlet, decays mainly into two pions:

$$\begin{aligned}f^0 &\rightarrow \pi^+ \pi^- \\ &\rightarrow \pi^0 \pi^0 .\end{aligned}$$

Suppose 150 f mesons decay this way, producing 300 pions. The f population is uniform, and the only way to get 100 of each pion charge state, thereby making the pion population uniform too, is for 100 of the f mesons to decay into $\pi^+ \pi^-$ and the other 50 to decay into $\pi^0 \pi^0$. Therefore, if isospin is conserved in the decay, the $\pi^+ \pi^-$ and $\pi^0 \pi^0$ branching fractions, normalized so that their sum is unity, are 2/3

and $1/3$.³

The mass of the f meson is large enough for $f \rightarrow \omega\pi$ decay to occur, but the only way to conserve charge is

$$f^0 \rightarrow \omega^0 \pi^0 .$$

And this mode is then forbidden by isospin conservation since there are no charged pions available to match the π^0 's the decays would produce.

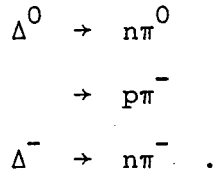
The ρ meson like the f meson decays into two pions, but the ρ is an isospin triplet. The possible charge modes are

$$\begin{aligned} \rho^+ &\rightarrow \pi^+ \pi^0 \\ \rho^0 &\rightarrow \pi^+ \pi^- \\ &\rightarrow \pi^0 \pi^0 \\ \rho^- &\rightarrow \pi^- \pi^0 . \end{aligned}$$

Suppose a ρ population made up of 50 of each of the three ρ charge states decays this way, producing altogether 300 pions. The ρ population is uniform by construction. The 50 ρ^+ and 50 ρ^- decays together produce the required 100 π^0 's, and all 50 ρ^0 's have to decay into $\pi^+ \pi^-$ to get enough charged pions. Therefore $\rho^0 \rightarrow \pi^0 \pi^0$ decay is forbidden by isospin conservation.

The Δ baryon resonances, which are isospin quartets, decay partly into a nucleon and a pion:

$$\begin{aligned} \Delta^{++} &\rightarrow p\pi^+ \\ \Delta^+ &\rightarrow n\pi^+ \\ &\rightarrow p\pi^0 \end{aligned}$$



Suppose a Δ population made up of 75 of each of the four Δ charge states decays this way, producing altogether 300 nucleons and 300 pions. The 75 Δ^{++} decays produce 75 π^+ 's, and the only way to get the required total of 100 π^+ 's is for 25 of the Δ^+ 's to decay into $n\pi^+$. Similarly, to get 100 π^- 's, 25 of the Δ^0 's have to decay into $p\pi^-$. Thus, going down the list, the numbers of decays are 75, 25, 50, 50, 25, and 75, and this distribution makes the nucleon population uniform too. The branching fractions, normalized so that their sum for each of the Δ charge states is unity, are 1, 1/3, 2/3, 2/3, 1/3, and 1.⁴

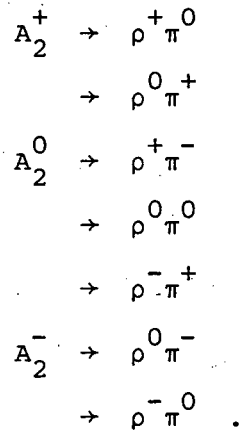
B. Charge symmetry

Let the members of an isospin multiplet be listed in order of decreasing charge; for example, pn , or $\pi^+\pi^0\pi^-$, or $\Delta^{++}\Delta^+\Delta^0\Delta^-$, or ω^0 . The charge-symmetry transformation replaces a particle so many steps from one end of such a list with the particle that is the same number of steps from the other end: $p \rightarrow n$, $n \rightarrow p$, $\pi^+ \rightarrow \pi^-$, $\pi^0 \rightarrow \pi^0$, $\Delta^0 \rightarrow \Delta^+$, $\omega^0 \rightarrow \omega^0$, etc. The transform of a reaction or decay mode is obtained by transforming each of the particles of the mode in this way. Thus the $\Delta^{++} \rightarrow p\pi^+$ mode turns into the $\Delta^- \rightarrow n\pi^-$ mode, and conversely, while the $f^0 \rightarrow \pi^+\pi^-$ mode is its own transform. Another consequence besides charge uniformity of isospin conservation

is the charge-symmetry rule:

Cross sections or branching fractions are equal for modes that transform into one another under charge symmetry.

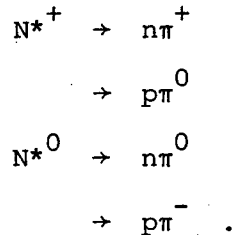
Sometimes the requirements of charge symmetry get satisfied just by making all the populations uniform. This was the case in the $\Delta \rightarrow N\pi$ example, where the six modes divide into three pairs of charge-symmetric modes, and the branching fractions of the paired modes, determined above without any reference to charge symmetry, are indeed equal. In general, however, charge symmetry and charge uniformity give independent constraints. For example, the A_2 meson, another isospin triplet, decays mainly into a ρ meson and a pion:



One way to make all the populations uniform is to have 100 of each of the first, fifth, and sixth decays in the list, and no others; but then there are no decays in the modes charge symmetric to these given three. There are many other ways to make all the populations uniform, but there is only one way to at the same time satisfy the charge-symmetry rule. It is easy to show that $A_2^0 \rightarrow \rho^0 \pi^0$ decay is

forbidden (otherwise there are too many neutral particles) and that all the other branching fractions are 1/2.

The next example illustrates an approach that is useful in a large class of problems. The N^* baryon resonances, which like the nucleon are isospin doublets, decay partly into a nucleon and a pion:



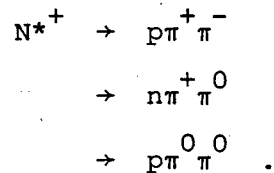
Each N^{*+} mode is paired off by charge symmetry with an N^{*0} mode, and the equality of branching fractions for paired modes ensures that an initially uniform N^* population will produce as many protons as neutrons and as many π^+ 's as π^- 's. The only remaining requirement is that there be as many π^+ 's or π^- 's as π^0 's, or, equivalently, twice as many charged pions as π^0 's. But since a charged pion in one half of the full set of modes is mirrored by a charged pion in the other half, and a π^0 is mirrored by a π^0 , this 2-to-1 ratio holds not only for the full set, but also for either half. It follows at once that the N^{*+} and N^{*0} each branch 2 to 1 in favor of the available mode with a charged pion, so that, going down the list, the branching fractions are 2/3, 1/3, 1/3, and 2/3.

The utility of the approach used above — the $N^* \rightarrow N\pi$ branching fractions could have been found using just charge uniformity — is that it applies to any set of modes that are all paired off by charge symmetry: the equality of rates for charge-symmetric modes

makes equal the numbers of charge-symmetric particles, and what remains to be done can be put in a "folded" form that applies to either half of the full set of modes. (Thus there is no reason to even write down more than half the modes if those of the other half are not of immediate interest.) For pions (or ρ mesons, or A_2 mesons), this folded form is, as was shown above, twice as many charged as neutral. For Δ resonances, it is, as is easy to show, as many Δ^{++} 's plus Δ^- 's as Δ^+ 's plus Δ^0 's. These results will be often used below.

C. A more complicated example

The N^* resonances also sometimes decay into a nucleon and two pions, the modes of an N^{*+} being



Suppose some number n_{tot} of $N^{*+} \rightarrow N\pi\pi$ decays to occur, and let n_{cc} , n_{co} , and n_{oo} , where the subscripts tell whether the pions produced are charged or neutral, be how n_{tot} is divided up among the three modes. Then the total number of charged pions produced by the decays is $2n_{cc} + n_{co}$, the number of neutral pions produced is $n_{co} + 2n_{oo}$, and equating the first of these to twice the second gives

$$2n_{cc} = n_{co} + 4n_{oo} \quad (1)$$

Dividing through by $n_{tot} = (n_{cc} + n_{co} + n_{oo})$ gives

$$2f_{cc} = f_{co} + 4f_{oo} \quad (2)$$

where $f_{cc} = n_{cc}/n_{tot}$, etc., are the $N^{*+} \rightarrow N\pi\pi$ branching fractions normalized so that $(f_{cc} + f_{co} + f_{oo}) = 1$. By charge symmetry, Eq. (2) also applies to the $N^{*0} \rightarrow N\pi\pi$ branching fractions. For a particular one of the many N^* resonances, the values of f_{cc} , f_{co} , and f_{oo} are the same for the N^{*+} and N^{*0} .

This is the first example in which isospin conservation does not fully determine the branching fractions; any three nonnegative numbers that satisfy Eq. (2) and add up to unity are allowed. The reason for this lack of definiteness is that this is the first set of modes in which the decay products can couple together in more than one way to form the decaying particle. For example, the two pions can form either a singlet (f-like) or a triplet (ρ -like) object, either of which with the nucleon can form a doublet, the N^* .⁵ Or the nucleon and one pion can form either a doublet (N^* -like) or a quartet (Δ -like) object, either of which with the second pion can form the N^* . The branching fractions in these special "pure" cases are easily found by considering the $N^* \rightarrow N\pi\pi$ decay to be a 2-step process; for example, $N^* \rightarrow Nf \rightarrow N\pi\pi$. The results are⁶

$$\begin{aligned}
 f_{cc} : f_{co} : f_{oo} &= 2/3 : 0 : 1/3 && \text{for } Nf \\
 &= 1/3 : 2/3 : 0 && \text{" } N\rho \\
 &= 4/9 : 4/9 : 1/9 && \text{" } N^*\pi \\
 &= 5/9 : 2/9 : 2/9 && \text{" } \Delta\pi
 \end{aligned}$$

Each set of branching fractions here of course satisfies Eq. (2).

If, in observing the $N\pi\pi$ decays of an N^{*+} , any two of n_{cc} , n_{co} ,

and n_{oo} are determined, then the third can be found from Eq. (1), and the sum of the three equals n_{tot} , the total number of $N\pi\pi$ decays there were. Or one can use Eq. (1) to write n_{tot} directly in terms of any two of its components:

$$n_{tot} = 3(n_{cc} - n_{oo}) \quad (3a)$$

$$= 3(n_{co} + 2n_{oo})/2 \quad (3b)$$

$$= 3(n_{co} + 2n_{cc})/4 \quad (3c)$$

Experimentally, it is much easier to see $N^{*+} \rightarrow p\pi^+\pi^-$ decays, where all the decay products are charged, than $N^{*+} \rightarrow n\pi^+\pi^0$ or $N^{*+} \rightarrow p\pi^0\pi^0$ decays. From Eqs. (3a) and (3c) and the fact that the components of n_{tot} are nonnegative, it follows that

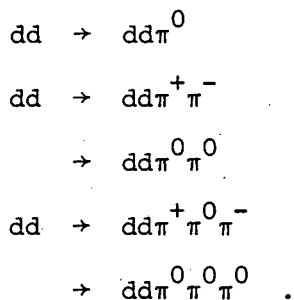
$$\frac{3}{2}n_{cc} \leq n_{tot} \leq 3n_{cc} \quad (4)$$

Thus upper and lower bounds can be placed on the total number of $N^{*+} \rightarrow N\pi\pi$ decays that occurred even when only the $N^{*+} \rightarrow p\pi^+\pi^-$ decays are observed. Bounds such as these are sometimes quite useful.

III. ISOSPIN-CONSERVING REACTIONS

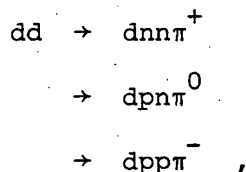
A. Analogies and extensions

The deuteron, the bound state of a proton and a neutron, is an isospin singlet. Consider collisions of deuterons with deuterons in which one or two or three pions are produced and the deuterons remain intact. As far as conservation of charge is concerned, the possible charge modes for the three reaction sets are



Since deuteron populations are automatically uniform, it remains only to choose, separately for each reaction set, some number of reactions and balance the pion populations. However, the pion charge combinations here are exactly those that occurred in the $f \rightarrow \omega\pi$, $f \rightarrow \pi\pi$, and $\omega \rightarrow \pi\pi\pi$ examples (why?), and arguments along the same lines as in those examples must lead to analogous conclusions: the first reaction is forbidden, the second and third occur in the ratio 2 to 1 (the cross sections are in this ratio), and the fifth reaction is forbidden.

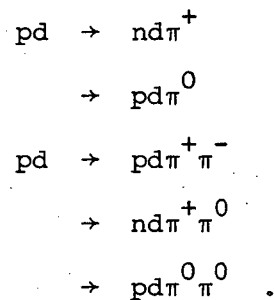
Production of a single pion becomes possible if one (at least) of the deuterons breaks up. The charge modes then are



and all the populations are uniform if and only if the three reactions occur at the same rate (have equal cross sections).

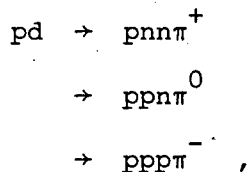
Clearly, no matter what the reaction set, be it $dd \rightarrow dd\pi\pi$ or $dd \rightarrow dNN\rho\pi\pi$ or $dd \rightarrow NNN^*\Delta\rho\omega\pi\pi\pi$, in deuteron-deuteron collisions each of the final-state populations will be uniform. In particular, the pion population will be uniform, and this is true not only for the pions that are produced directly, but also, provided that the decays conserve isospin, for those that come from the decays of resonances — because the populations of resonances are themselves born uniform. And since for each reaction set all the populations are uniform, they are uniform for any sum over reaction sets, and thus also overall.

Next, consider collisions of protons with deuterons in which one or two pions are produced and the deuterons remain intact:



For each of these reactions there is a charge symmetric neutron-deuteron reaction, so it remains only to find the relations between cross sections that yield twice as many charged pions as π^0 's; and the pion charge combinations here are just those that occurred in the $N^{*+} \rightarrow N\pi$ and $N^{*+} \rightarrow N\pi\pi$ examples. By analogy, therefore, the cross sections σ_c and σ_o for the first two reactions are related by $\sigma_c = 2\sigma_o$, and the cross sections σ_{cc} , σ_{co} , and σ_{oo} for the last three

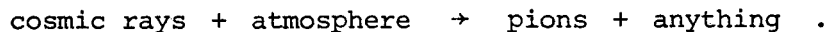
are related by $2\sigma_{cc} = \sigma_{co} + 4\sigma_{oo}$, where the subscripts tell whether the pions produced by the reactions are charged or neutral. When the deuteron breaks up, the one-pion modes are



and the relation between the cross sections may still be written as $\sigma_c = 2\sigma_o$, if now σ_c is the sum of the cross sections for the first and third of the reactions.

In, say, the reaction half set $pd \rightarrow NN\Delta\rho\omega\pi\pi\pi\pi$, there will be produced twice as many charged as neutral pions, twice as many charged as neutral ρ mesons, and as many Δ^{++} 's plus Δ^- 's as Δ^+ 's plus Δ^0 's. It is easy to show that decays of ρ and Δ populations constrained in this way also produce twice as many charged as neutral pions, so that the 2-to-1 pion production ratio holds not only for the pions that are produced directly, but also for those that come from isospin-conserving decays. This 2-to-1 ratio holds for any pd (or nd) reaction half set, so it holds for any sum over such half sets, and thus overall.

High energy cosmic rays striking the Earth's atmosphere interact with the atmospheric elements and produce, among other things, pions:



It so happens that both the "beam" and the "target" have, to good approximations, very simple isospin properties: about 90% of cosmic

rays are protons and nearly all the rest are alpha particles or other isospin-singlet nuclei; and more than 99% of the atoms in the dry atmosphere are nitrogen or oxygen, the nuclei of which are isospin singlets. It then follows from a superposition of results obtained above that, to the extent all the cosmic rays are protons or singlets and the atmosphere contains only singlets, charged and neutral pions are produced in the ratio 2 to 1.

B. More complicated initial states

In the reactions considered so far, at least one of the colliding particles was an isospin singlet. Since singlet populations are automatically uniform, the initial state in such reactions is no more complicated than it is in a decay process. It remains to consider reactions in which neither of the colliding particles is a singlet. Suppose, for example, a beam each portion of which contains equal numbers of π^+ 's, π^0 's, and π^- 's is incident upon a target each portion of which contains equal numbers of protons and neutrons.⁷ The beam population will remain uniform only if its interactions as it traverses the target remove π^+ 's, π^0 's, and π^- 's at the same rate; and since these depletion rates are proportional to cross sections, the cross sections (either overall, or for any specific reaction set) must be related by

$$\sigma_{\pi^+p} + \sigma_{\pi^+n} = \sigma_{\pi^0p} + \sigma_{\pi^0n} = \sigma_{\pi^-p} + \sigma_{\pi^-n} , \quad (5)$$

where, for example, σ_{π^+p} is the sum of all the cross sections for the π^+p modes. Now any particular π^+p mode is charge symmetric to

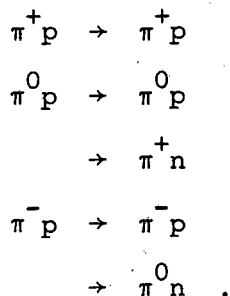
a π^-n mode, and thus $\sigma_{\pi^+p} = \sigma_{\pi^-n}$ by the charge-symmetry rule. Similarly, $\sigma_{\pi^0p} = \sigma_{\pi^0n}$ and $\sigma_{\pi^-p} = \sigma_{\pi^+n}$, so that, from Eq. (5),

$$\sigma_{\pi^+p} + \sigma_{\pi^-p} = 2\sigma_{\pi^0p} \quad (6)$$

Thus the 2-to-1 charged-to-neutral rule for pions in a reaction half set (here the πp reactions) applies to the initial state as well as to the final state. Note that charge uniformity does not imply that σ_{π^+p} , σ_{π^0p} , and σ_{π^-p} are all equal.

As an example, consider pion-proton collisions in which no additional particles are created (elastic and charge-exchange scattering).

The charge modes are



According to Eq. (6),

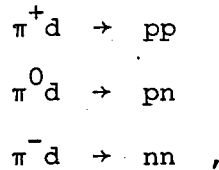
$$\sigma_{++} + \sigma_{--} + \sigma_{-0} = 2(\sigma_{00} + \sigma_{0+}) ,$$

where the first and second subscripts give the charges of the initial- and final-state pions. And only if

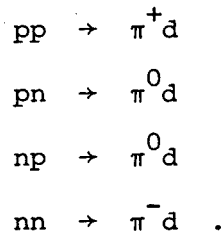
$$\sigma_{++} + \sigma_{--} + \sigma_{0+} = 2(\sigma_{00} + \sigma_{-0})$$

will twice as many charged pions as π^0 's be produced. These equations may be solved to give the π^0p cross sections in terms of the others.

A last point, to be made by comparing the reactions $\pi d \rightarrow NN$ and $NN \rightarrow \pi d$, concerns reactions in which the beam and target particles are from the same multiplet. The $\pi d \rightarrow NN$ charge modes are



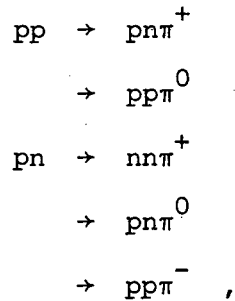
and a uniform beam of pions incident upon a target of deuterons will remain uniform only if the cross sections for the three modes are equal. Now consider the reactions $NN \rightarrow \pi d$, with a uniform beam of nucleons incident upon a uniform target of nucleons. If, by the order of the initial-state particles, account is taken of whether the $\pi^0 d$ final state is produced by protons incident upon neutrons or neutrons upon protons, there are four charge modes:



For the pion population to be uniform, the sum of the $pn \rightarrow \pi^0 d$ and $np \rightarrow \pi^0 d$ cross sections must equal either of the other cross sections. But an actual experiment measures either the $pn \rightarrow \pi^0 d$ or the $np \rightarrow \pi^0 d$ cross section, not the undifferentiated sum. Since these two cross sections are equal by charge symmetry, the cross-section ratios of interest here are, going down the list, 2:1:1:2.⁸

In any nucleon-nucleon reaction set, the pN and nN (or the Np

and Nn) reactions form charge-symmetric half sets. Thus for the pN reactions in which a single pion is produced,



the cross sections are related by $\sigma_c = 2\sigma_o$, where σ_c and σ_o are the sums of the cross sections for the reactions in which a charged or neutral pion is produced. And for the pN reactions in which two pions are produced, the relation is, with the usual notation, the by now familiar $2\sigma_{cc} = \sigma_{co} + 4\sigma_{oo}$.

Shane Burns and Thomas Trippe (both of Lawrence Berkeley Laboratory) forced me to be clearer. This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract No. W-7405-ENG-48.

APPENDIX

Some familiarity with the elementary formalism of isospin (or angular-momentum) conservation and Clebsch-Gordan coefficients is assumed in what follows. Charge uniformity implies a certain relation between Clebsch-Gordan coefficients, and this relation is shown to follow from a symmetry property of the coefficients. The proof here directly covers only the basic 2-body coupling $c \rightarrow a+b$, where, as in $\Delta \rightarrow N\pi$, $N^* \rightarrow N\pi$, and $A_2 \rightarrow \rho\pi$ decays, a and b are particles belonging to different multiplets.⁹

Two quantum numbers, the isospin I and the third component m , specify the isospin properties of a particle. The value of I , which is the same for all the members of a multiplet, is given by the fact that the number of members is $(2I+1)$; and in order of decreasing charge the members take on the m values $I, (I-1), (I-2), \dots, -I$. Thus the quantum numbers (I,m) of the proton, π^- , and Δ^0 are $(\frac{1}{2}, +\frac{1}{2})$, $(1, -1)$, and $(\frac{3}{2}, -\frac{1}{2})$, respectively. If isospin is conserved in the decay $c \rightarrow a+b$, where the particle quantum numbers are (I_c, m_c) , etc., then I_c can only be one of the values $(I_a + I_b), (I_a + I_b - 1), \dots, |I_a - I_b|$, and m_c must equal $(m_a + m_b)$. The branching fraction for this decay is equal to the square of the Clebsch-Gordan coefficient $C(I_a I_b I_c; m_a m_b m_c)$. Thus, since the $\Delta^0 \rightarrow p\pi^-$ branching fraction is $1/3$, the magnitude of $C(\frac{1}{2}, 1, \frac{3}{2}; +\frac{1}{2}, -1, -\frac{1}{2})$ is $1/\sqrt{3}$. The standard normalization for Clebsch-Gordan coefficients,

$$\sum_{m_a} C^2(I_a I_b I_c; m_a, m_c - m_a, m_c) = 1, \quad (A1)$$

where the summation is over all the m_a (or $m_b = m_c - m_a$) states accessible from a given m_c state, corresponds to having the branching fractions for each charge state of multiplet c add up to unity.

Since the sum of branching fractions for each m_c state is one, the sum of branching fractions for all $(2I_c + 1)$ m_c states is $(2I_c + 1)$. This sum of all the branching fractions may be split up into $(2I_a + 1)$ pieces, one for each of the charge states of multiplet a : the six $\Delta \rightarrow N\pi$ branching fractions divide into two sets, one for those associated with a proton, the other for those associated with a neutron. If the multiplet- c population is initially uniform, then the total number of a particular m_a state produced by the decays is proportional to the sum of the branching fractions associated with this m_a state. Thus the multiplet- a population will only be uniform if the $(2I_a + 1)$ partial sums of branching fractions are all equal, independent of the value of m_a . For $(2I_a + 1)$ equal pieces to add up to $(2I_c + 1)$, they must each be equal to $(2I_c + 1)/(2I_a + 1)$. Thus, in terms of Clebsch-Gordan coefficients, charge uniformity implies that

$$\sum_{m_c} C^2(I_a I_b I_c; m_a, m_c - m_a, m_c) = \left(\frac{2I_c + 1}{2I_a + 1} \right) \quad (A2)$$

for each value of m_a .

There are a number of so-called symmetry relations between Clebsch-Gordan coefficients, and one of them tells how the coefficients for the coupling together of isospins I_a and I_b to get isospin I_c are related to those for the coupling of isospins I_c and

I_b to get isospin I_a :¹⁰

$$C(I_a I_b I_c; m_a m_b m_c) = (-1)^{I_b + m_b} \left(\frac{2I_c + 1}{2I_a + 1} \right)^{\frac{1}{2}} C(I_c I_b I_a; -m_c, m_b, -m_a) \quad (A3)$$

Squaring both sides of this and summing over m_c while keeping m_a fixed gives

$$\sum_{m_c} C^2(I_a I_b I_c; m_a, m_c, -m_a, m_c) = \left(\frac{2I_c + 1}{2I_a + 1} \right) \sum_{m_c} C^2(I_c I_b I_a; -m_c, m_c, -m_a, -m_a) \quad (A4)$$

The summation factor on the right-hand side is by Eq. (A1) equal to unity, and what then remains of Eq. (A4) is Eq. (A2). Thus the relation implied by charge uniformity is indeed satisfied by the Clebsch-Gordan coefficients.

NOTES AND REFERENCES

1. See almost any text on elementary particle physics for examples of such calculations. The only two (both quite old) that also discuss Shmushkevich's method are P. Roman, Theory of Elementary Particles (North-Holland, Amsterdam, 1960), and R.E. Marshak and E.C.G. Sudarshan, Introduction to Elementary Particle Physics (Interscience, New York, 1961).
2. The original papers are I. Shmushkevich, Dokl. Akad. Nauk SSSR 103, 235 (1955) [translation: AEC-tr-2270 (unpublished)], and N. Dushin and I. Shmushkevich, Dokl. Akad. Nauk SSSR 106, 801 (1956) [translation: Soviet Phys. - Dokl. 1, 94 (1956)]. The most complete treatment is G. Pinski, A.J. Macfarlane, and E.C.G. Sudarshan, Phys. Rev. 140, B1045 (1965). See also the two books mentioned in Ref. 1.
3. Two comments should perhaps be made here. (1) Although strict charge uniformity is assumed in getting branching fractions, it is in fact only statistically true. Each f meson that decays into two pions "decides" with probabilities $2/3$ and $1/3$ between the $\pi^+\pi^-$ and $\pi^0\pi^0$ modes. Thus, in accord with ordinary statistics, a finite sample of $f \rightarrow \pi\pi$ decays is unlikely to split exactly in the 2-to-1 ratio: a sample of 150 might split 102 to 48, or 96 to 54, etc.
(2) Small corrections need to be made to branching fractions to take into account the small differences in masses between the particles of a multiplet. The π^0 is slightly less massive than the π^+ and π^- , so slightly more energy is released in $f^0 \rightarrow \pi^0\pi^0$ decay than in $f^0 \rightarrow \pi^+\pi^-$ decay, and the corresponding branching fractions are

expected to be shifted slightly up from 1/3 and down from 2/3.

4. As the examples have shown, branching fractions are found by making all the particle populations uniform first before any decays and then again after them all. A related result comes from keeping the populations uniform while the decays are going on. In the $\rho \rightarrow \pi\pi$ example, the initially uniform ρ population will remain uniform at all times only if the ρ^+ , ρ^0 , and ρ^- all decay at the same rate: by the time any given fraction of the ρ^+ 's have decayed, so also have decayed just the same fraction of the ρ^0 's and ρ^- 's. Thus the lifetime distributions of the ρ^+ , ρ^0 , and ρ^- are all the same. In the $f \rightarrow \pi\pi$ example, the pion population will be uniform at all times only if the 2-to-1 ratio of $\pi^+\pi^-$ and $\pi^0\pi^0$ decays is maintained in every time interval. Thus the ratio of the branching fractions is the ratio of the decay rates, and the lifetime distributions of the two modes have the same time dependence. It is easy to show that the same is true for the six $\Delta \rightarrow N\pi$ modes.
5. These statements follow from the rules for coupling isospin vectors together. It is, however, more in the spirit of the present work to remark (without proof) that a process such as $N^* \rightarrow N\pi$ is allowed by isospin conservation if and only if all the charge states of all the particle multiplets appear when the charge modes are written down; if they do not all appear, as in the $f \rightarrow \omega\pi$ example where the π^+ and π^- were absent, then there is no way to make all the populations uniform. By this rule, both $N^* \rightarrow N\pi$ and $N^* \rightarrow N\rho$ decays are allowed. Similar comments apply to the next sentence in the text.

6. The $\Delta\pi$ case is a little more complicated than the others. The charge modes for the first step are

$$\begin{aligned} N^{*+} &\rightarrow \Delta^{++}\pi^{-} \\ &\rightarrow \Delta^{+}\pi^{0} \\ &\rightarrow \Delta^{0}\pi^{+} \end{aligned}$$

Let the corresponding branching fractions be f_{++} , f_{+} , and f_0 , where the subscripts are the Δ charges. The normalization constraint is $f_{++} + f_{+} + f_0 = 1$. The requirement that there be twice as many charged pions as π^0 's is $f_{++} + f_0 = 2f_{+}$. The requirement that there be as many Δ^{++} 's plus Δ^{-} 's as Δ^{+} 's plus Δ^0 's is $f_{++} = f_{+} + f_0$. The solution of these equations is $f_{++} = 1/2$, $f_{+} = 1/3$, and $f_0 = 1/6$. Then using the $\Delta \rightarrow N\pi$ branching fractions derived in Sect. II·A and summing up the contributions to f_{cc} , f_{co} , and f_{oo} gives the results in the text.

7. Beams of π^0 's are in fact not obtainable because the π^0 lifetime is very short (isospin is not conserved in pion decays).
8. Thus the order of the initial-state particles tells which is beam and which is target, whereas the list of final-state particles is just that, a list of the particles produced, with the order being of no significance (so that different orderings are not counted as distinct). The order of the final-state particles does take on significance when differential distributions are considered — it becomes necessary to distinguish, for example, between $\pi^0 d \rightarrow pn$ and $\pi^0 d \rightarrow np$, or between $f^0 \rightarrow \pi^{+}\pi^{-}$ and $f^0 \rightarrow \pi^{-}\pi^{+}$ — but this complication will not be gone into here.

9. For a more general and sophisticated treatment, see the article by G. Pinski et al. cited in Ref. 2.
10. This is Eq. (3.17a) of M.E. Rose, Elementary Theory of Angular Momentum (Wiley, New York, 1957). It may be obtained from Eq. (3.5.15) of A.R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton, Princeton, 1960), by using his Eqs. (3.5.14) and (3.5.17).

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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