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On the duality condition for quantum fields.

by

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Abstract.

A general quantum field theory is considered, in which the fields are assumed to be operator-valued tempered distributions. The system of fields may include any number of boson fields and fermion fields. A theorem which relates certain complex Lorentz transformations to the TCP-transformation is stated and proved. With reference to this theorem duality conditions are considered, and it is shown that such conditions hold under various physically reasonable assumptions about the fields. Extensions of the algebras of field operators are discussed with reference to the duality conditions. Local internal symmetries are discussed, and it is shown that these commute with the Poincaré group and with the TCP-transformation.

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I. Introduction.

In an earlier publication [1], hereafter referred to as BW I, the authors have discussed the duality condition for a Hermitian scalar field. It is the purpose of the present paper to extend the results in BW I to a general field theory, within the framework described in the monographs by Streater and Wightman [2], and by Jost [3]. We thus consider a theory in which there appears an arbitrary set of local and relatively local spinor- and tensor fields. Each field has a finite number of components, and is assumed to be an operator-valued tempered distribution. In contrast to the situation in BW I we now have to consider fermion fields, and their characteristic anticommutation relations, which necessitates an obvious modification in the definitions of the duality conditions.

As we shall see, however, much of the reasoning in BW I applies in almost unchanged form to the issues in the present study. When this is the case we shall rely heavily on BW I, and not repeat arguments already given in that paper. The notation and terminology in BW I will be followed whenever applicable. We also refer to BW I for additional references to related work.

In Sec. II we review some aspects of the geometry of Minkowski space, andwe also review some well-known facts about the quantum mechanical Poincaré group and its complex extension. In Sec. III we state our assumptions about the quantum fields, which are more or less standard. In these two sections we also explain the notation which we follow in the subsequent discussion.

The locality condition for the quantum fields is expressed in terms of the familiar (normal) commutation- and anticommutation relations. For our purposes it would be extremely cumbersome to have to consider commutation- and anticommutation relations simultaneously, and we therefore find it advantageous to restate the locality conditions in terms of the vanishing of certain commutators. The simple device through which this can be achieved is explained in Lemma 1 with reference to the field operators, and more generally, in Theorem 2 in Sec. V.

In Sec. IV we discuss the relationship between complex Lorentz transformations and the TCP-transformation. The considerations are analogous to the considerations in Secs. III and IV in BW I, except that we now deal with spinor- and tensor fields rather than with a single scalar field as in BW I. The main result in this section is presented in Theorem 1; this theorem is analogous to Theorem 1 in BW I. The form of this theorem is hardly surprising, in view of the analogous result in BW I, and some readers might feel that it would have been enough just to state the theorem. We felt, however, that an outline of the reasoning was in order, and that some of the cumbersome details should be presented explicitly in writing and not left entirely to the imagination of the reader.

Sec. V in BW I was devoted to a discussion of some algebraic questions relating to Theorem 1. This discussion applies as such to the present study, and we do not repeat it here.

In Sec. V of the present paper we discuss the duality condition for the wedge regions W_R and W_L , where W_R = $\{ x \mid x^3 > |x^4| \}$ and $W_L = \{ x \mid x^3 < -|x^4| \}$. This discussion is analogous to the discussion in Sec. VI in BW I. The issue is the following. We wish to find two von Neumann algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ such that $\mathcal{A}(W_R)$ can be regarded as locally associated with W_R and $\mathcal{A}(W_L)$ can be regarded as locally associated with W_L . Furthermore the association should be consistent with the well-known TCP-symmetry of the quantum fields. These notions are defined precisely in Definition 2 in Sec. V. If there are no fermion fields, then one aspect of locality is that $\mathcal{A}(W_R)$ is contained in the commutant $\mathcal{A}(W_L)'$ of $\mathcal{A}(W_L)$, and the <u>condition of duality</u> is that $\mathcal{A}(W_R) = \mathcal{A}(W_L)'$. In a theory in which fermion fields do occur these conditions have to be modified in an obvious way. The condition of duality is now that $\mathcal{A}(W_R) = (Z\mathcal{A}(W_L) Z^{-1})'$, where Z is the unitary operator defined by $Z = (I + iU_0)/(1+i)$ in terms of the unitary operator U_0 which represents a rotation by angle 2π about any axis. In this paper we employ the notation $\mathcal{A}(W_L)^q = (Z\mathcal{A}(W_L) Z^{-1})'$, and we call $\mathcal{A}(W_L)^q$ the <u>quasicommutant</u> of the algebra $\mathcal{A}(W_L)$. The modified conditions of locality and duality are thus stated in terms of the notion of a quasi-commutant. We note here that the second iterated quasicommutant is equal to the second iterated commutant, and that the quasi-commutant is equal to the commutant whenever $U_0 = I$, and hence $Z = I$. The reader who feels temporarily bewildered by the appearance of the superscript q in Secs. V and VI might find it helpful to ignore, at first, the distinction between a quasicommutant and a commutant, and hence to read the superscript q as the familiar von Neumann prime. This corresponds to the special case of no fermion fields. We feel that the modifications occasioned by the presence of fermion fields are really utterly trivial, although perhaps slightly distractive at first.

In a quantum field theory the local von Neumann algebras must be appropriately related to the field operators. Let $\mathcal{P}(W_R)$ denote the algebra of (in general unbounded) operators constructed from fields averaged with test functions with support in W_R, and let $\mathcal{P}(W_L)$ be analogously defined. A natural relationship between $\mathcal{A}(W_R)$ and $\mathcal{P}(W_R)$ is that the operators in the latter algebra shall have closed extensions affiliated to $\mathcal{A}(W_R)$, with the analogous relationship between $\mathcal{A}(W_L)$ and $\mathcal{P}(W_L)$. We have <u>not</u> been able to show that von Neumann algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ with the above properties do exist for a <u>general</u> field theory, i.e., without further assumptions about the fields which go beyond the usual minimal assumptions. Hence we consider some special conditions on the fields which guarantee the existence of algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ with physically satisfactory properties. Our conditions on the fields are not as such physically unreasonable, but it would clearly be desirable to settle the question of whether they are in fact necessary. The main results in Sec. V are presented in Theorem 3 and 4. We note here that these results, in the special case of a single Hermitian scalar field, are considerably stronger than our results in BW I.

In Sec. VI we discuss the construction of local von Neumann algebras associated with other regions than wedge regions in terms of algebras associated with W_R and W_L, and we show that the extended system of local algebras satisfy a condition of duality if the algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ do. For reasons of simplicity we restrict our considerations to very special regions: double cones and their causal complements. Our results concerning the properties of the extended system of algebras in general are stated in Theorems 5 and 6. Theorem 7 describes the situation under specific assumptions about the fields. The discussion in Sec. VI is analogous to the discussion in Sec. VII in BW I, but the results in the present paper are considerably stronger than our earlier results. The paper concludes with Theorem 8, concerning local internal symmetries, in which we note that such symmetries commute with all Poincaré transformations and with the TCP-transformation.

II. Geometrical preliminaries. About the quantum mechanical Poincaré group.

Minkowski space $\mathcal{M}$ is parametrized by the customary Cartesian coordinates $x = (x^1, x^2, x^3, x^4)$. The Lorentz "metric" is so defined that $x \cdot y = x^4y^4 - x^1y^1 - x^2y^2 - x^3y^3$. The elements $\Lambda = \Lambda(M,y)$ of the proper Poincaré group $\bar{L}_o$ are parametrized by a four-by-four Lorentz matrix M, and a real four-vector y, such that the image Λx of a point $x \in \mathcal{M}$ under any $\Lambda \in \bar{L}_o$ is given by $\Lambda x \doteq \Lambda(M,y)\, x = Mx + y$. The image of any subset R of $\mathcal{M}$ under Λ is denoted ΛR.

The group of all four-by-four Lorentz matrices M, i.e., the group of all proper homogeneous Lorentz transformations, is denoted L_o. A rotation in L_o by angle Θ about the unit vector $\underset{\sim}{e}$ is denoted $R(\underset{\sim}{e},\Theta)$. We denote by $V(\underset{\sim}{e}_3, t)$ the velocity transformation (in L_o) in the 3-direction given by

$$V(\underset{\sim}{e}_3, t) = \begin{bmatrix} 1 & , & 0 & , & 0 & , & 0 \\ 0 & , & 1 & , & 0 & , & 0 \\ 0 & , & 0 & , & \cosh(t) & , & \sinh(t) \\ 0 & , & 0 & , & \sinh(t) & , & \cosh(t) \end{bmatrix} \qquad (1)$$

We define a "right wedge" W_R, and a "left wedge" W_L, as the following open subsets of Minkowski space:

$$W_R = \{ x \mid x^3 > |x^4| \}, \quad W_L = \{ x \mid x^3 < -|x^4| \} \qquad (2)$$

These two regions are bounded by two characteristic planes whose intersection is the 2-plane $\{ x \mid x^3 = x^4 = 0 \}$. We note that the one-parameter Abelian group of velocity transformations $V(\underset{\sim}{e}_3, t)$, t real, maps W_R onto itself and W_L onto itself.

We next consider an involutory mapping $x \to Jx$ of Minkowski space onto itself, defined by

$$Jx = - R(\underset{\sim}{e}_3, \pi)x \quad , \text{ or } \quad J(x^1, x^2, x^3, x^4) = (x^1, x^2, -x^3, -x^4) \qquad (3)$$

where $R(\underset{\sim}{e}_3, \pi)$ denotes the rotation by angle π about the 3-axis. We see that J maps W_R onto W_L, and the mapping can be described as a reflection in the common "edge" $\{ x \mid x^3 = x^4 = 0 \}$ of the pair of wedges W_R and W_L.

We note that $V(\underset{\sim}{e}_3, t)$, as given in (1), is an entire analytic function of t. It is easily seen that

$$J = V(\underset{\sim}{e}_3, i\pi) \qquad (4)$$

For any subset R of Minkowski space $\mathcal{M}$ we define the causal complement R^c of R by

$$R^c = \{ x \mid (x-y)\cdot(x-y) < 0 \text{ , all } y \in R \} \qquad (5)$$

We note that with this definition $W_R{}^c = \bar{W}_L$ and $W_L{}^c = \bar{W}_R$, where the bar denotes the closure. Two open regions R_1 and R_2 such that $R_1{}^c = \bar{R}_2$ and $R_2{}^c = \bar{R}_1$ form a pair of causally complementary open regions. Among such pairs the pair W_R and W_L is distinguished by the simple geometric relationships described above. Any pair of wedge-regions bounded by two non-parallel characteristic planes are distinguished in the same sense, and any such pair is in fact Poincaré-equivalent to the pair (W_R, W_L), i.e., of the form $(\Lambda W_R, \Lambda W_L)$ for some $\Lambda \in \bar{L}_0$. We shall here define $\mathcal{W}$ as the set of all (open) wedge-regions bounded by two intersecting characteristic planes, i.e.,

$$\mathcal{W} = \{ \Lambda W_R \mid \Lambda \in \bar{L}_0 \} \qquad (6)$$

Although we shall at first be explicitly concerned with W_R it is clear that analogous considerations apply to any $W \in \mathcal{W}$.

The regions W_R and W_L have further distinguishing properties, which are of crucial importance for our discussion, namely the following. Let $t = t_r + it_i$, with t_r, t_i real. If $x \in W_R$, then the complex four-vector $z(t) = V(\underset{\sim}{e}_3, t)x$ is an element of the forward imaginary tube in C^4, i.e., $\mathrm{Im}(z(t)) \in V_+$, for all complex t in the open strip $0 < t_i < \pi$, and $z(t)$ is in the closure of the forward imaginary tube for all t in the closed strip $0 \leq t_i \leq \pi$. We here denote the forward lightcone with the origin as apex by V_+; the backward lightcone is denoted V_-. Similarly: if $x \in W_L$, then $z(t)$ is in the forward imaginary tube for all complex t in the strip $-\pi < t_i < 0$, and in the closed forward imaginary tube for all t in the closure of the above strip. These assertions are easily established through a simple computation. (See formula(45b) in BW I). We note that the above facts were also of crucial importance in Jost's proof of the TCP-theorem [4].

For the reader's convenience we shall here review some well-known facts about the universal covering groups of the Lorentz- and Poincaré groups, and about the complex extension of the covering group of the Lorentz group [5].

The universal covering group of L_o, i.e., the group of all unimodular two-by-two complex matrices, is denoted $\mathcal{G}$. A specific two-to-one homomorphism of $\mathcal{G}$ onto L_o is given by

$$g \rightarrow M(g) \quad , \quad M_{rs}(g) = \frac{1}{2}\,\mathrm{Tr}(g^{\dagger}\sigma_r g \sigma_s) \tag{7}$$

where $\sigma_1,\sigma_2,\sigma_3$ are the usual Pauli-matrices, and where $\sigma_4 = I$. The rotations and velocity transformations in $\mathcal{G}$ are denoted

$$u(\underset{\sim}{e},\theta) = \exp(-\frac{i}{2}\,\theta\,\underset{\sim}{e}\cdot\underset{\sim}{\sigma}) \ , \quad v(\underset{\sim}{e},t) = \exp(\frac{t}{2}\,\underset{\sim}{e}\cdot\underset{\sim}{\sigma}) \tag{8}$$

and under the homomorphism (7) we thus have

$$R(\underset{\sim}{e},\theta) = M(u(\underset{\sim}{e},\theta)) \ , \quad V(\underset{\sim}{e},t) = M(v(\underset{\sim}{e},t)) \tag{9}$$

The group $\mathcal{G}$ can be regarded as the complex extension of the group SU(2) of all <u>unitary</u> matrices (rotations) $u \in \mathcal{G}$, and every irreducible(unitary) representation $u \rightarrow D^s(u)$ of SU(2) can be analytically extended to a representation $g \rightarrow D^s(g)$ of $\mathcal{G}$, such that the matrix elements of $D^s(g)$ are homogeneous polynomials of degree 2s in the matrix elements of g. The most general finite-dimensional irreducible representation of $\mathcal{G}$ is of the form

$$g \longrightarrow D^{s',s''}(g) = D^{s'}(g^r) \otimes D^{s''}(g) \tag{10}$$

where $g^r = (g^{\dagger})^{-1}$. The mapping $g \rightarrow g^r$ is an outer automorphism of $\mathcal{G}$ which preserves every element in the subgroup SU(2).

In view of the complex structure of $\mathcal{G}$ it follows that the complex extension $\mathcal{G}_c$ of $\mathcal{G}$ is the direct product of $\mathcal{G}$ with itself, i.e., the group $\mathcal{G}_c = \mathcal{G} \times \mathcal{G}$ of all ordered pairs (g_1, g_2) of elements in $\mathcal{G}$ with the law of composition $(g_1', g_2')(g_1'', g_2'') = (g_1'g_1'', g_2'g_2'')$. The group $\mathcal{G}$ can be identified with a particular "real subgroup" of $\mathcal{G}_c$ through the one-one correspondence

$$g \leftrightarrow (g^r, g) \tag{11}$$

To the set of all finite-dimensional irreducible representations $g \rightarrow D^{s',s''}(g)$ of $\mathcal{G}$ corresponds a particular family of finite-dimensional irreducible representations of $\mathcal{G}_c$, which can be regarded as the set of all finite-dimensional irreducible <u>analytic</u> representations of $\mathcal{G}_c$, namely the representations

$$(g_1, g_2) \longrightarrow D_c^{s',s''}(g_1, g_2) = D^{s'}(g_1) \otimes D^{s''}(g_2) \tag{12}$$

With reference to the above definitions we define, for any complex number t, the complex velocity transformation $v_c(\underset{\sim}{e}_3, t)$ in the 3-direction as the element

$$v_c(\underset{\sim}{e}_3, t) = (\exp(-\tfrac{1}{2}t\sigma_3),\ \exp(\tfrac{1}{2}t\sigma_3)) \tag{13a}$$

of the group $\mathcal{G}_c$, and it follows from (12) that

$$D_c^{s',s''}(v_c(\underset{\sim}{e}_3, t)) = D^{s'}(\exp(-\tfrac{1}{2}t\sigma_3)) \otimes D^{s''}(\exp(\tfrac{1}{2}t\sigma_3)) \tag{13b}$$

The matrix-valued function of t in (13b) is an entire analytic function of the complex variable t, and hence the unique analytic extension of the matrix-valued function $D^{s',s''}(v(\underset{\sim}{e}_3,t))$ of the real variable t. We note in particular that

$$D_c^{s',s''}(v_c(\underset{\sim}{e}_3,i\pi)) = (-1)^{2s''}\ D^{s',s''}(u(\underset{\sim}{e}_3,\pi)) \qquad (14a)$$

$$D_c^{s',s''}(v_c(\underset{\sim}{e}_3,-i\pi)) = (-1)^{2s'}\ D^{s',s''}(u(\underset{\sim}{e}_3,\pi)) \qquad (14b)$$

The formula $V(\underset{\sim}{e}_3,i\pi) = -R(\underset{\sim}{e}_3,\pi)$ is a special case of (14a) (with $s' = s'' = 1/2$), and with M_c denoting the analytic extension of the representation $g \longrightarrow M(g)$ to the complex group $\mathcal{G}_c$ we have $M_c(v_c(\underset{\sim}{e}_3,t)) = V(\underset{\sim}{e}_3,t)$ for all complex t.

The universal covering group of $\bar{L}_o$ is denoted $\bar{\mathcal{G}}$. The elements $\lambda = \lambda(g,x)$ are the ordered pairs consisting of any $g \in \mathcal{G}$ and any $x \in \mathcal{M}$, with the law of composition $\lambda(g',x')\lambda(g'',x'') = \lambda(g'g'', x'+M(g')x'')$. We define an explicit homomorphism $\lambda \rightarrow \Lambda(\lambda)$ by $\Lambda(\lambda(g,x)) = \Lambda(M(g),x)$.

The Hilbert space $\mathcal{H}$ of physical states is assumed to be separable. It is assumed to carry a strongly continuous unitary representation $\lambda \longrightarrow U(\lambda)$ of the quantum mechanical Poincare group $\bar{\mathcal{G}}$. We write $U(g,x) = U(\lambda(g,x))$, and we also employ the special notation $T(x) = U(I,x)$ for the translations. The translations have the common spectral resolution

$$T(x) = U(I,x) = \int e^{ix\cdot p}\ \mu(d^4p) \qquad (15)$$

and it is assumed that the support of the spectral measure μ is contained in the closed forward lightcone $\bar{V}_+$ (in momentum space). This assumption about the support of μ will be referred to as the "spectral condition" in what follows.

We assume the existence of a vacuum state, represented by the unit vector Ω , uniquely characterized by its invariance under all translations. The vacuum state then satisfies $U(\lambda)\Omega = \Omega$ for all $\lambda \varepsilon \tilde{\mathcal{G}}$. It is well-known that the spectral condition allows the extension of the representation of the translation subgroup to a unique representation $z \to T(z)$ of the semigroup of complex translations for which $Im(z) \varepsilon \bar{V}_+$,such that T(z) is a bounded and strongly continuous function of z in the closed forward imaginary tube, and a strongly analytic function of z in the open forward imaginary tube.

The one-parameter group of velocity transformations in the 3-direction, as well as its analytic extension to the complex domain, will be of particular interest, and we shall therefore employ the shorter notation $V(t) = U(v(\underset{\sim}{e}_3, t), 0)$ for the representatives of these velocity transformations. More generally we shall write

$$V(\tau) = \exp(-i\tau K_3) = \int e^{-i\tau s} \mu_K(ds) \qquad (16)$$

for any complex τ. Here μ_K is the spectral measure in the simultaneous spectral resolution of the group of all V(t), t real, and K_3 is the unique self-adjoint operator, with domain D_K, such that $V(t) = \exp(-itK_3)$. For a discussion of the domains of the normal operators $V(\tau)$ we refer to Sec. IV in BW I. We denote (as in BW I) by D_+ the domain on which $V(i\pi)$ is self-adjoint, and by D_- the domain on which $V(-i\pi)$ is self-adjoint.

III. Assumptions about the quantum fields.

We denote by $\mathcal{D}(R^n)$ the set of all complex-valued infinitely differentiable functions of compact support on n-dimensional Euclidean space R^n, and we denote by $\mathcal{S}(R^n)$ the space of test functions on R^n in terms of which tempered distributions are defined. The space $\mathcal{S}(R^n)$ is regarded as endowed with the particular topology appropriate to the definition of tempered distributions [6].

For an unbounded linear or antilinear operator X defined on a domain D we shall employ the unorthodox notation (X,D), as in BW I. The adjoint of (X,D) is denoted $(X,D)^* = (X^*, D(X^*))$, where $D(X^*)$ is the domain of the adjoint. This notation will not be employed for manifestly bounded operators, for which the domain is taken to be the entire Hilbert space $\mathcal{H}$.

We shall next state the basic assumptions about the quantum fields. It is not our aim here to state a set of minimal independent assumptions for a field theory, but rather to describe the situation which prevails in a standard field theory.

a) We assume the existence of a set of <u>boson fields</u> $\beta^{(b)}(x)$, where b is an element in an index set I_B, and a set of <u>fermion fields</u> $\phi^{(f)}(x)$, where f is an element in an index set I_F. The index sets are regarded as disjoint, and it is assumed that at least one of these sets is nonempty; otherwise they are arbitrary. We thus admit as possible special cases the cases when either I_B, or else I_F is empty. Each field $\beta^{(b)}(x)$ or $\phi^{(f)}(x)$ has a <u>finite</u> number of components, denoted $\beta^{(b)}_\mu(x)$, respectively $\phi_{\mu}^{(f)}(x)$, where μ is a suitable index distinguishing between the components.

b) We also consider the set of all components of all the fermion fields and all the boson fields. An element in this set is denoted $\varphi_{\mu}(x)$, where μ is an element in an index set I_T such that when μ runs through I_T each component of each field is obtained once and once only. Each component $\varphi_{\mu}(x)$ is an operator-valued tempered distribution in the following sense. To each $f(x) \varepsilon \mathcal{S}(R^4)$, and each $\mu \varepsilon I_T$, corresponds a closable linear operator $(\varphi_{\mu}[f], D_1)$ on a dense domain D_1 (independent of f and μ) such that $\varphi_{\mu}[f]D_1 \subset D_1$. The mapping $f \rightarrow (\varphi_{\mu}[f], D_1)$ is linear, and for any $\xi \varepsilon D_1$ the vector $\varphi_{\mu}[f]\xi$ is a strongly continuous function of f on $\mathcal{S}(R^4)$.

Furthermore, if $\sigma = (\mu 1, \mu 2, \ldots, \mu n)$ is any ordered n-tuplet of indices from I_T, then there corresponds to every $f(x_1, x_2, \ldots, x_n) \varepsilon \mathcal{S}(R^{4n})$ a closable linear operator $(\varphi\{f;\sigma\}, D_1)$ on D_1 such that $\varphi\{f;\sigma\}D_1 \subset D_1$. The mapping $f \rightarrow (\varphi\{f;\sigma\}, D_1)$ is linear, and for any $\xi \varepsilon D_1$ the vector $\varphi\{f;\sigma\}\xi$ is a strongly continuous function of f on $\mathcal{S}(R^{4n})$. If f is of the particular form $f(x_1, x_2, \ldots, x_n) = f_1(x_1)f_2(x_2)\ldots f_n(x_n)$, with $f_k \varepsilon \mathcal{S}(R^4)$ for $k = 1, \ldots, n$, then, on D_1,

$$\varphi\{f;\sigma\} = \varphi_{\mu 1}[f_1]\, \varphi_{\mu 2}[f_2] \ldots \varphi_{\mu n}[f_n] \tag{17}$$

This is consistent with the common notation for $\varphi\{f;\sigma\}$ in terms of the symbolic integral at right in

$$\varphi\{f;\sigma\} =$$

$$\int_{(\infty)} d^4(x_1)d^4(x_2)\ldots d^4(x_n)f(x_1,x_2,\ldots,x_n)\,\varphi_{\mu 1}(x_1)\,\varphi_{\mu 2}(x_2)\ldots\varphi_{\mu n}(x_n) \tag{18}$$

c) Let $\mathcal{P}(\mathcal{M})$ be the algebra, defined on D_1, which is the linear span of the identity I and all operators $(\varphi\{f;\sigma\},D_1)$. The dense domain D_1 is assumed to be <u>precisely</u> equal to $\mathcal{P}(\mathcal{M})\Omega$.

d) For any field component $\varphi_\mu(x)$ there exists a field component $\varphi_{\mu'}(x)$ such that for any $f \in \mathcal{S}(R^4)$

$$(\varphi_{\mu'}[f^*]\,,D_1)^* \supset (\varphi_\mu[f]\,,D_1) \tag{19}$$

The field component $\varphi_{\mu'}(x)$ is then also denoted $\varphi_\mu^\dagger(x)$.

e) The domain D_1 is invariant under $\bar{\mathcal{g}}$, i.e., $U(\lambda)D_1 = D_1$,for any $\lambda \in \bar{\mathcal{g}}$. The action of $U(\lambda)$ by conjugation on the elements of $\mathcal{P}(\mathcal{M})$ is specified by the conditions

α)
$$T(x')\varphi_\mu(x)\,T(x')^{-1} = \varphi_\mu(x+x') \tag{20a}$$

for any field component $\varphi_\mu(x)$.

β) For each $b \in I_B$,

$$U(g,0)\,\beta_\mu^{(b)}(x)\,U(g,0)^{-1} = \sum_{\mu'} \Gamma_{\mu\mu'}^{(b)}(g^{-1})\,\beta_{\mu'}^{(b)}(M(g)x) \tag{20b}$$

where $g \rightarrow \Gamma^{(b)}(g)$ is similar to one of the representations $g \rightarrow D^{s',s''}(g)$ for which $2(s'+s'')$ is an <u>even</u> integer.

γ) For each $f \in I_F$,

$$U(g,0)\,\phi_\mu^{(f)}(x)\,U(g,0)^{-1} = \sum_{\mu'} \Gamma_{\mu\mu'}^{(f)}(g^{-1})\,\phi_{\mu'}^{(f)}(M(g)x) \tag{20c}$$

where $g \to \Gamma^{(f)}(g)$ is similar to one of the representations $g \to D^{s',s''}(g)$ for which $2(s'+s'')$ is an odd integer.

The sums at right in (20b) and (20c) extend over all the components of the field $\beta^{(b)}(x)$, respectively the field $\phi^{(f)}(x)$.

f) All the fields satisfy the normal conditions of locality, i.e., they satisfy the conditions (in the sense of distributions),

$$[\beta_{\mu}^{(b)}(x), \beta_{\mu'}^{(b')}(x')] = 0, \; [\beta_{\mu}^{(b)}(x), \phi_{\mu'}^{(f')}(x')] = 0$$

$$\{\phi_{\mu}^{(f)}(x), \phi_{\mu'}^{(f')}(x')\} = 0 \qquad (21)$$

on D_1 for all spacelike $x-x'$. Here the curly bracket denotes the anticommutator, i.e., $\{X,X'\} = XX' + X'X$.

The above formulation of the basic assumptions about the fields is more or less standard. The essence of the notion of a set of quantum fields is a certain kind of representation of a tensor algebra of multicomponent test functions by an operator algebra $\mathcal{P}(\mathcal{M})$. The precise formulation of a general field theory is unfortunately beset by cosiderable notational difficulties. We have tried to select a notation which is convenient for our particular purposes. Let us now elaborate further on the basic assumptions, and on some well-known immediate consequences.

g) Whether the number of fields is finite, countably infinite, or uncountably infinite is immaterial for the conclusions which we shall draw. That each field $\beta^{(b)}(x)$ or $\phi^{(f)}(x)$ has only a finite number of components, where the notion of "component" of course refers specifically to the transformation laws (20b) and (20c), is, however, essential. Our purpose with introducing the specific "irreducible fields" $\beta^{(b)}(x)$ and $\phi^{(f)}(x)$ was to be able to state the transformation laws (20b) and (20c), as well as the locality conditions (21), with maximum clarity. For the subsequent discussion it will, however, be more convenient to employ a unified notation, in terms of the symbols $\varphi_\mu(x)$, for all the field components, and we shall therefore restate the conditions (20b) and (20c) in the form

$$U(g,0)\varphi_\mu(x)\, U(g,0)^{-1} = \sum_{\mu'} \Gamma_{\mu\mu'}(g^{-1})\, \varphi_{\mu'}(M(g)x) \qquad (22)$$

The "matrix" $\Gamma(g)$ can be regarded as the direct sum of the finite-dimensional matrices $\Gamma^{(b)}(g)$ and $\Gamma^{(f)}(g)$ in an obvious sense. The sum in (22) is always a finite sum, and for each fixed μ (or each fixed μ') there is only a finite number of values of μ' (respectively of μ) for which $\Gamma_{\mu\mu'}$ is different from zero. We shall also consider the analytic extension of the representation $g \rightarrow \Gamma(g)$ of $\mathcal{G}$ to a representation $(g_1,g_2) \longrightarrow \Gamma(g_1,g_2)$ of $\mathcal{G}_c$, defined as the direct sum of the corresponding analytic extensions of the representations $\Gamma^{(b)}(g)$ and $\Gamma^{(f)}(g)$ as described in Sec.II. To the complex velocity transformation $v_c(\underset{\sim}{e}_3,t)$ thus corresponds the representative $\Gamma(v_c(\underset{\sim}{e}_3,t))$, each matrix element of which is an entire analytic function of the complex variable t. With reference to this extension we thus define the diagonal "matrix" Γ'' (with eigenvalues +1 and -1) by

$$\Gamma'' = \Gamma(v_c(\underset{\sim}{e}_3,-i\pi))\; \Gamma(u(\underset{\sim}{e}_3,\pi)) \qquad (23)$$

That Γ'' has the stated properties follows at once from (14a).

h) The domain D_1 on which the "averaged fields," and the operators in $\mathcal{P}(\mathcal{M})$ are defined should be carefully noted. It follows readily from our assumptions that for any $(X,D_1) \in \mathcal{P}(\mathcal{M})$ the domain of the adjoint $(X,D_1)^*$ contains D_1. The restriction of the adjoint to D_1 shall be denoted $(X^\dagger,D_1)$, and called the Hermitian conjugate of X ; the notion of the Hermitian conjugate of a field operator thus depends on the specific choice of D_1. It also follows from our assumptions that $(X^\dagger,D_1) \in \mathcal{P}(\mathcal{M})$ for all $(X,D_1) \in \mathcal{P}(\mathcal{M})$. In particular the hermitian conjugate $\varphi_\mu[f]^\dagger$ of the averaged field $\varphi_\mu[f]$ is the averaged field $\varphi_\mu^\dagger[f^*]$. The mapping $(X,D_1) \rightarrow (X^\dagger,D_1)$ is an antilinear involution of $\mathcal{P}(\mathcal{M})$ (such that $(X_1X_2)^\dagger = X_2{}^\dagger X_1{}^\dagger$).

We note that every operator $(X,D_1) \in \mathcal{P}(\mathcal{M})$ satisfies

$$(X^\dagger,D_1)^{**} \subset (X,D_1)^* \tag{24}$$

It is a hitherto unsolved problem whether the assumptions which we have made imply that the inclusion in (24) can be replaced by equality for some non-trivial set of operators in $\mathcal{P}(\mathcal{M})$.

i) Let R be any subset of Minkowski space $\mathcal{M}$.
We define $\mathcal{P}_0(R)$ as the polynomial algebra generated by the identity operator I and all operators $(\varphi_\mu[f],D_1)$, with $\mu \in I_T$, $f(x) \in \mathcal{S}(R^4)$ and $\text{supp}(f) \subset R$. We define the algebra $\mathcal{P}(R)$ as the linear span of I and all operators $(\varphi\{f;\sigma\},D_1)$, where $\sigma = (\mu 1,\mu 2,\ldots,\mu n)$ is any n-tuplet of indices in I_T, and where $f(x_1,x_2,\ldots,x_n) \in \mathcal{S}(R^{4n})$ with $\text{supp}(f) \subset (\times R)^n$.

It is easily seen that $(X,D_1) \rightarrow (X^\dagger,D_1)$ is an involution of both $\mathcal{P}_0(R)$ and $\mathcal{P}(R)$. From the conditions (20a)-(20c) it follows that

$$U(\lambda)\,\mathcal{P}_0(R)\,U(\lambda)^{-1} = \mathcal{P}_0(\Lambda(\lambda)R)\,,$$

$$U(\lambda)\,\mathcal{P}(R)\,U(\lambda)^{-1} = \mathcal{P}(\Lambda(\lambda)R) \tag{25}$$

for any $\lambda \in \overline{\mathcal{G}}$ and any R.

We trivially have $\mathcal{P}_0(R) \subset \mathcal{P}(R) \subset \mathcal{P}(\mathcal{M})$. According to a well-known theorem of Reeh and Schlieder [7] the linear manifold $\mathcal{P}_0(R)\Omega$ is dense in $\mathcal{H}$ for any open nonempty R.

j) Let the unitary operators U_0 and Z be defined by

$$U_0 = U(-I,0)\quad,\quad Z = (I + iU_0)/(1 + i) \tag{26}$$

These operators trivially satisfy

$$U_0^{\,2} = I\;,\; Z^2 = U_0\;,\;\; U(\lambda)U_0U(\lambda)^{-1} = U_0,\; U(\lambda)ZU(\lambda)^{-1} = Z \tag{27a}$$

and

$$U_0\Omega = Z\Omega = \Omega\quad,\;\; U_0D_1 = D_1\;,\;\; ZD_1 = D_1 \tag{27b}$$

Furthermore it follows from the assumptions in e) above that

$$U_0\,\beta^{(b)}_\mu(x) = \beta^{(b)}_\mu(x)\,U_0\;,\quad Z\,\beta^{(b)}_\mu(x)\,Z^{-1} = \beta^{(b)}_\mu(x) \tag{28a}$$

$$U_0\,\phi^{(f)}_\mu(x) = -\,\phi^{(f)}_\mu(x)\,U_0,\quad Z\,\phi^{(f)}_\mu(x)\,Z^{-1} = iU_0\,\phi^{(f)}_\mu(x) \tag{28b}$$

for all boson fields $\beta^{(b)}(x)$ and all fermion fields $\phi^{(f)}(x)$.

The fact that the involution U_0 commutes with all boson fields, but anticommutes with all fermion fields permits a unique resolution of any field operator into a sum of a "boson operator" and a "fermion operator," and it also permits a re-statement of the locality conditions (21) in terms of the vanishing of certain commutators. We shall state the important facts in the matter in the form of a lemma for later reference.

<u>Lemma 1</u>.a) Let U_0 and Z be defined as in (26). For any subset R of $\mathcal{M}$, let

$$\mathcal{P}_B(R) = \{ (X,D_1) \mid U_0XU_0 = X \ , \ (X,D_1) \in \mathcal{P}(R) \} \tag{29a}$$

$$\mathcal{P}_F(R) = \{ (X,D_1) \mid U_0XU_0 = -X \ , \ (X,D_1) \in \mathcal{P}(R) \} \tag{29b}$$

Then every $(X,D_1) \in \mathcal{P}(R)$ has a unique resolution of the form

$$X = X_b + X_f \ , \quad X_b \in \mathcal{P}_B(R) \ , \ X_f \in \mathcal{P}_F(R) \tag{30a}$$

where, in fact,

$$X_b = \tfrac{1}{2}(X + U_0XU_0) \ , \quad X_f = \tfrac{1}{2}(X - U_0XU_0) \tag{30b}$$

The sets $\mathcal{P}_B(R)$ and $\mathcal{P}_F(R)$ are mapped onto themselves under the involution $(X,D_1) \to (X^\dagger,D_1)$. Furthermore,

$$ZX_bZ^{-1} = X_b \ , \quad ZX_fZ^{-1} = iU_0X_f \tag{31}$$

for all $X_b \in \mathcal{P}_B(R)$ and all $X_f \in \mathcal{P}_F(R)$.

b) For any $(X,D_1) \in \mathcal{P}(R)$, let (X^z,D_1) be defined by

$$(X^z,D_1) = Z(X,D_1)Z^{-1} = (ZXZ^{-1},D_1) \tag{32}$$

If R_1 and R_2 are two open subsets of $\mathcal{M}$ such that $R_1 \subset R_2{}^c$, then it follows from the locality conditions in f) above that

$$[X_b,Y_b] = 0 \ , \ [X_b,Y_f] = 0, \ [X_f,Y_b] = 0, \ \{X_f,Y_f\} = 0 \tag{33a}$$

on D_1 for all $X_b \in \mathcal{P}_B(R_1)$, $X_f \in \mathcal{P}_F(R_1)$, $Y_b \in \mathcal{P}_B(R_2)$ and $Y_f \in \mathcal{P}_F(R_2)$. The conditions (33a) are equivalent to the condition

$$[X, Y^z] = 0 \tag{33b}$$

on D_1 for all $X \in \mathcal{P}(R_1)$, $Y \in \mathcal{P}(R_2)$.

We omit the completely trivial proof. We note that the Lemma is vacuous if $U_0 = I$, which is the case if and only if there is no fermion field.

IV. Complex Lorentz transformations and the TCP transformation.

In this section we shall present the generalizations appropriate for the present situation of the considerations in Secs. III and IV in BW I. The main result is presented in Theorem 1, which corresponds to Theorem 1 in BW I. As in BW I we arrive at the main conclusion through a sequence of lemmas, arranged in such a way that the similarities with the discussion in BW I are pretty obvious.

For any $f(x_1, x_2, \ldots, x_n) \in \mathcal{S}(R^{4n})$ we define a Fourier transform $\tilde{f}$ by

$$\tilde{f}(p_1, \ldots, p_n) =$$

$$\int_{(\infty)} d^4(x_1) \ldots d^4(x_n) f(x_1, \ldots, x_n) \exp\left(i \sum_{r=1}^{n} x_r \cdot p_r \right) \qquad (34)$$

For any positive integer n we denote by T_n the open tube-region

$$T_n = \left\{ (z_1, z_2, \ldots, z_n) \;\middle|\; \operatorname{Im}(z_k) \in V_+ \,,\ k = 1, \ldots, n \right\} \qquad (35)$$

in complex 4n-dimensional space, regarded as a direct sum of n replicas of complex Minkowski space, and parametrized by an n-tuplet $(z_1, z_2, \ldots, z_n)$ of complex four-vectors. The closure of T_n is denoted $\bar{T}_n$.

<u>Lemma 2</u>: Let $z \in \bar{T}_1$, i.e., z is any complex four-vector in the closed forward imaginary tube. Then:

a)
$$T(z)\, D_1 \subset D_1 \qquad (36a)$$

b) If $f \in \mathcal{S}(R^{4n})$ there exists an $f_z \in \mathcal{S}(R^{4n})$ such that

$$\tilde{f}_z(p_1,\ldots,p_n) = \tilde{f}(p_1,\ldots,p_n) \exp(iz\cdot\sum_{r=1}^{n} p_r) \tag{36b}$$

for $(p_1,\ldots,p_n) \varepsilon V_n$, where V_n is the subset of R^{4n} defined by

$$V_n = \{ (p_1,\ldots,p_n) \mid \sum_{r=k}^{n} p_r \varepsilon \bar{V}_+ , k = 1,\ldots,n \} \tag{36c}$$

and for every such f_z we have

$$T(z)\varphi\{f;\sigma\}\Omega = \varphi\{f_z;\sigma\}\Omega \tag{36d}$$

where σ is any ordered n-tuplet $(\mu 1,\mu 2,\ldots,\mu n)$ of indices from I_T.

Lemma 3. a) For each $n \geq 1$, let E_n be the set of all functions $f(x_1,\ldots,x_n;z_1,\ldots,z_n)$ defined for $(x_1,\ldots,x_n) \varepsilon R^{4n}$ and $(z_1,\ldots,z_n) \varepsilon T_n$, and such that $f \varepsilon \mathcal{S}(R^{4n})$ and such that the Fourier transform $\tilde{f}$ of f relative to the variables $(x_1,\ldots,x_n)$ satisfies the condition

$$\tilde{f}(p_1,\ldots,p_n;z_1,\ldots,z_n) = \exp\left(i\sum_{k=1}^{n} \sum_{r=k}^{n} z_k\cdot p_r \right) \tag{37a}$$

for all $(p_1,\ldots,p_n) \varepsilon V_n$, with V_n defined as in (36c). The set E_n is non-empty, and to every n-tuplet $\sigma = (\mu 1,\mu 2,\ldots,\mu n)$ of indices from I_T corresponds a unique vector-valued function $\phi(z_1,z_2,\ldots,z_n;\sigma)$ on T_n, defined by

$$\phi(z_1,z_2,\ldots,z_n;\sigma) = \varphi\{f;\sigma\}\Omega \tag{37b}$$

where f is any element of E_n.

b) The vector-valued function $\phi(z_1, z_2, \ldots, z_n; \sigma)$ is a strongly analytic function of $(z_1, z_2, \ldots, z_n)$ on T_n, and for each point in this domain it is an analytic vector for the Lie algebra of the group $U(\overline{\mathcal{G}})$.

c) For any element $\lambda = \lambda(g,x)$ of the quantum mechanical Poincare group $\overline{\mathcal{G}}$,

$$U(\lambda)\, \phi(z_1, z_2, \ldots, z_n; \sigma) = \sum_{\sigma'} \hat{\Gamma}_{\sigma,\sigma'}(g^{-1})\, \phi(Mz_1+x, Mz_2, Mz_3, \ldots, Mz_n; \sigma') \quad (37c)$$

where $M = M(g)$, and where the sum is over the <u>finite</u> number of n-tuplets $\sigma' = (\mu 1', \mu 2', \ldots, \mu n')$ of indices from I_T for which

$$\hat{\Gamma}_{\sigma,\sigma'}(g) = \Gamma_{\mu 1,\mu 1'}(g)\, \Gamma_{\mu 2,\mu 2'}(g) \ldots \Gamma_{\mu n,\mu n'}(g) \quad (37d)$$

is not identically zero (as a function of g).

It may here be noted that

$$\varphi_{\mu 1}(z_1)\, \varphi_{\mu 2}(z_1+z_2) \ldots \varphi_{\mu n}(z_1+z_2+\ldots+z_n)\, \Omega \quad (37e)$$

is a defensible notation (within the framework of distribution theory) for the vector $\phi(z_1, z_2, \ldots, z_n; \sigma)$.

<u>Lemma 4.</u> a) Let $\{ f_k \mid f_k \in \mathcal{S}(R^4),\ k = 1, \ldots, n \}$ be any n-tuplet of test functions, and let $\sigma = (\mu 1, \mu 2, \ldots, \mu n)$ be any ordered n-tuplet of indices from I_T. For $k = 1, \ldots, n$, let $X_k = \varphi_{\mu k}[f_k]$. Then the vector

$$T(z_1)\, X_1\, T(z_2)\, X_2 \ldots T(z_n)\, X_n\, \Omega \quad (38a)$$

is well defined (through successive left multiplications) for all $(z_1,z_2,\ldots,z_n) \in \bar{T}_n$, and it is a strongly continuous function of the variables $(z_1,z_2,\ldots,z_n)$ on $\bar{T}_n$, and a strongly analytic function of these variables on T_n.

b) There exist functions $f(x_1,\ldots,x_n;z_1,\ldots,z_n)$ defined for $(x_1,\ldots,x_n) \in R^{4n}$ and $(z_1,\ldots,z_n) \in \bar{T}_n$, and such that $f \in \mathcal{S}(R^{4n})$ and such that the Fourier transform $\tilde{f}$ of f relative to the variables $(x_1,\ldots,x_n)$ satisfies the condition

$$\tilde{f}(p_1,\ldots,p_n;z_1,\ldots,z_n) = \exp\left(i\sum_{k=1}^{n}\sum_{r=k}^{n} z_k\cdot p_r\right)\prod_{k=1}^{n}\tilde{f}_k(p_k) \qquad (38b)$$

for all $(p_1,\ldots,p_n) \in V_n$, with V_n defined as in (36c), and for all $(z_1,z_2,\ldots,z_n) \in \bar{T}_n$. For any such function f,

$$\varphi\{f;\sigma\}\Omega = T(z_1)\,X_1\,T(z_2)\,X_2\ldots T(z_n)\,X_n\,\Omega \qquad (38c)$$

c) If $f_k \in \mathcal{D}(R^4)$ for $k = 1,2,\ldots,n$, and $(z_1,z_2,\ldots,z_n) \in T_n$, then,

$$\int_{(\infty)} d^4(x_1)\ldots d^4(x_n)\; f_1(x_1)f_2(x_2)\ldots f_n(x_n) \times$$

$$\phi(z_1+x_1, z_2+x_2-x_1, z_3+x_3-x_2, \ldots, z_n+x_n-x_{n-1};\sigma) =$$

$$T(z_1)\,X_1\,T(z_2)\,X_2\ldots T(z_n)\,X_n\,\Omega \qquad (38d)$$

d) Let $\{R_n \mid n = 1,\ldots,\infty\}$ be any set of open, non-empty subsets of Minkowski space. For such a set, and for any $n \geq 1$, let S_n denote the linear span of all vectors of the form $X_1X_2\ldots X_n\Omega$, with X_k defined as in a) above, and with $f_k \in \mathcal{S}(R^4)$, $\text{supp}(f_k) \subset R_k$, for $k = 1,\ldots,n$.

Then the linear span of the vacuum vector Ω and the union of all the linear manifolds S_n is dense in the Hilbert space $\mathcal{H}$.

About the proofs: The Lemmas 2-4 in the present paper correspond to the Lemmas 2-6 in Sec. III of BW I, and the reasoning there presented applies with very trivial modifications. The conclusions in Lemmas 2 and 4; the conclusion in part a) of Lemma 3, and the conclusion (in part b) of Lemma 3) that $\phi(z_1,z_2,\ldots,z_n;\sigma)$ is analytic as asserted, follow from the spectral condition, the action of the translation group by conjugation on the fields, and the assumption that the fields are tempered distributions on the domain D_1. That we now deal with an arbitrary number of field components instead of with a single field as in BW I is immaterial in the proofs. The formula (37c) is the trivial generalization of the formula (34) in BW I. Since the matrix $\hat{\Gamma}(g^{-1})$ in (37c) is in effect similar to a finite direct sum of matrices $D^{s',s''}(g^{-1})$, and hence an entire analytic function of g , it follows that $\phi(z_1,z_2,\ldots,z_n;\sigma)$ is an analytic vector for the Lie algebra of the group $U(\mathcal{G},0)$, and hence also for the Lie algebra of the group $U(\bar{\mathcal{G}})$.

We next consider the action of the complex velocity transformations $V(t) = \exp(-itK_3)$, where t is complex, on the vectors $\phi(z_1,z_2,\ldots,z_n;\sigma)$. We denote by $D_V(\pi/2)$ the domain on which $V(i\pi/2)$ is self-adjoint, and by $D_V(-\pi/2)$ the domain on which $V(-i\pi/2)$ is self-adjoint. The domain $D_V(\pi/2)$ is then a core for all operators $V(t)$ with $0 \leq \text{Im}(t) \leq \pi/2$, and the domain

$D_V(-\pi/2)$ is a core for all operators $V(t)$ with $0 \geq \mathrm{Im}(t) \geq -\pi/2$. The next Lemma corresponds to Lemmas 8 and 9 in BW I, and it is proved, on the basis of Lemma 3, by a very trivial modification of the reasoning in BW I.

<u>Lemma 5.</u> Let $(z_1,\ldots,z_n)$ be an n-tuplet of complex four-vectors $z_k = x_k+iy_k$, where x_k, y_k, real; $y_k^1 = y_k^2 = 0$; $y_k^4 > |y_k^3|$, for $k = 1,\ldots,n$. Let $\sigma = (\mu1,\mu2,\ldots,\mu n)$ be any ordered n-tuplet of indices from I_T. For any k, and any complex t we define $z_k(t)$ by

$$z_k(t) = V(\underset{\sim}{e}_3, t)\, z_k \tag{39a}$$

a) If $x_k \in W_R$ (i.e., $x_k^3 > |x_k^4|$), for $k = 1,\ldots,n$, then $(z_1(i\tau),\ldots,z_n(i\tau)) \in T_n$ for all $\tau \in [0,\pi/2]$. The vector $\phi(z_1,\ldots,z_n;\sigma)$ is in the domain $D_V(\pi/2)$, and

$$V(i\tau)\phi(z_1,\ldots,z_n;\sigma) = \sum_{\sigma'} \hat{\Gamma}_{\sigma,\sigma'}(v_c(\underset{\sim}{e}_3,-i\tau))\, \phi(z_1(i\tau),\ldots,z_n(i\tau);\sigma') \tag{39b}$$

for all $\tau \in [0,\pi/2]$, where $\hat{\Gamma}$ is defined as in (37d).

b) If $x_k \in W_L$ (i.e., $x_k^3 < -|x_k^4|$), for $k = 1,\ldots,n$, then $(z_1(i\tau),\ldots,z_n(i\tau)) \in T_n$ for all $\tau \in [-\pi/2, 0]$. The vector $\phi(z_1,\ldots,z_n;\sigma)$ is in the domain $D_V(-\pi/2)$, and the relation (39b) holds for all $\tau \in [-\pi/2, 0]$.

c) Let $(x_1,\ldots,x_n)$ be such that $x_k \in W_R$ for $k = 1,\ldots,n$. Let v be the real forward timelike four-vector with components $v = (0,0,0,1)$, and let t be a real variable.

Then

$$\underset{t \to 0+}{\text{s-lim}} \sum_{\sigma'} \hat{\Gamma}_{\sigma,\sigma'}(c_+)\, V(i\pi/2)\, \phi(x_1+itv, x_2+itv, \ldots, x_n+itv;\sigma') =$$

$$\underset{t \to 0+}{\text{s-lim}} \sum_{\sigma'} \hat{\Gamma}_{\sigma,\sigma'}(c_-)\, V(-i\pi/2)\, \phi(Jx_1+itv,\ Jx_2+itv, \ldots,\ Jx_n+itv;\sigma') =$$

$$= \phi(z_1, \ldots, z_n;\sigma) \qquad (39c)$$

where $z_k = (x_k^1, x_k^2, ix_k^4, ix_k^3)$, for $k = 1,\ldots,n$, and where c_+ and c_- are the elements $c_+ = v_c(\underset{\sim}{e}_3, i\pi/2)$, $c_- = v_c(\underset{\sim}{e}_3, -i\pi/2)$, of the group $\mathcal{G}_c$. Here J is defined as in (3).

The next Lemma corresponds to Lemma 10 in BW I.

<u>Lemma 6.</u> Let R_1 be a bounded, open, non-empty subset of W_R , and let $x_0 \in W_R$ be such that $(x-x_0) \in W_L$ for all $x \in \bar{R}_1$. For any integer $n > 1$ we define the set R_n by

$$R_n = \{ x + (n-1)x_0 \mid x \in R_1 \} \qquad (40a)$$

a) Then $R_n \subset W_R$ for all n, and if $n > k$, then $(x'-x'') \in W_R$ for all $x' \in R_n$, $x'' \in R_k$. In particular R_n is space-like separated from R_k (i.e., $R_n \subset R_k^c$) if $n \neq k$.

b) Let $\{ f_k \mid k = 1,\ldots,n \}$ be an n-tuplet of test functions such that $f_k \in \mathcal{S}(R^4)$ and $\text{supp}(f_k) \subset R_k$, for $k = 1,\ldots,n$. Let f_k^i denote the test function defined by $f_k^i(x) = f_k(-x)$. Let $\sigma = (\mu1, \mu2, \ldots, \mu n)$ be any ordered n-tuplet of indices from I_T. Let $c(s) \in \mathcal{D}(R^1)$. Then

$$V(i\pi)c(K_3)\,\varphi_{\mu 1}[f_1]\;\varphi_{\mu 2}[f_2]\;\cdots\;\varphi_{\mu n}[f_n]\,\Omega =$$

$$\hat{\Gamma}''_{\sigma,\sigma}\;U(u(\underset{\sim}{e}_3,\pi),0)c(K_3)\,\varphi_{\mu 1}[f_1{}^i]\;\varphi_{\mu 2}[f_2{}^i]\;\cdots\;\varphi_{\mu n}[f_n{}^i]\,\Omega \qquad (40b)$$

where $\hat{\Gamma}''$ is the diagonal matrix given by

$$\hat{\Gamma}'' = \hat{\Gamma}(v_c(\underset{\sim}{e}_3,-i\pi))\;\hat{\Gamma}(u(\underset{\sim}{e}_3,\pi)) \qquad (40c)$$

This Lemma can be proved, on the basis of Lemmas 4 and 5, by a trivial modification of the reasoning by which we proved Lemma 10 in BW I; the modification, of course, has to do with the appearance of the matrices $\hat{\Gamma}$ in the formulas. To bring out the similarities with the discussion in BW I we define the test function $f_k{}^j$ by $f_k{}^j(x) = f_k(\jmath x)$, and we then have

$$U(u(\underset{\sim}{e}_3,\pi),0)\;\varphi_{\mu k}[f_k{}^i]\;U(u(\underset{\sim}{e}_3,\pi),0)^{-1} =$$

$$\sum_{\mu'}\Gamma_{\mu k,\mu'}(u(\underset{\sim}{e}_3,-\pi))\;\varphi_{\mu'}[f_k{}^j] \qquad (40d)$$

With reference to this formula it is easily seen that the formula (52) in BW I is a special case of (40b).

That the matrix $\hat{\Gamma}''$ in (40c) is diagonal (with diagonal elements +1 or -1) follows at once from the fact that the matrix Γ'' in (23) is diagonal (with diagonal elements +1 or -1).

Our conclusions up to this point in this section are completely independent of the locality conditions f) in Section III. We shall now draw some further conclusions, in which we take the locality conditions into account. Before we state the relevant lemma we recall that the domain of the closed and normal operarator $V(t)$, t complex, depends only on $\mathrm{Im}(t)$. We write the operator as $(V(t), D_V(\mathrm{Im}(t)))$ when we wish to exhibit the domain explicitly.

<u>Lemma 7.</u> Let $\{ R_n \mid n = 1,\dots,\infty \}$ be a <u>fixed</u> set of bounded, open, non-empty subsets of W_R, constructed as in Lemma 6. Let $\mathcal{Q}$ be the linear span of the identity operator I and all operators (Q, D_1) of the form

$$Q = \varphi_{\mu 1}[f_1]\, \varphi_{\mu 2}[f_2] \cdots \varphi_{\mu n}[f_n] \tag{41a}$$

where $\{ f_k \mid k = 1,\dots,n \}$ is any n-tuplet of test functions such that $f_k \in \mathcal{S}(R^4)$ and $\mathrm{supp}(f_k) \subset R_k$, for $k = 1,\dots,n$, and where $\sigma = (\mu 1, \mu 2, \dots, \mu n)$ is any ordered n-tuplet of indices from I_T. Then:

a) The linear manifold $D_q = \mathcal{Q}\,\Omega$ is dense in the Hilbert space $\mathcal{H}$, and $D_{qc} = \mathrm{span}\,\{ c(K_3) D_q \mid c(s) \in \mathcal{D}(R^1) \}$ is a core for every operator $(V(t), D_V(\mathrm{Im}(t)))$.

b) $(Q^*, D_1) \in \mathcal{Q}$ if $(Q, D_1) \in \mathcal{Q}$.

c) There exists a <u>unique</u> <u>antiunitary</u> operator J such that if $(Q, D_1) \in \mathcal{Q}$ and $c(s) \in \mathcal{D}(R^1)$, then

$$V(i\pi)\, c(K_3)\, Q\,\Omega = c(K_3)\, J\, Q^*\,\Omega \tag{41b}$$

The operator J is an involution, i.e.,

$$J^2 = I \tag{41c}$$

and it satisfies the conditions

$$J\,\Omega = \Omega \quad , \quad J\,D_1 = D_1 \quad , \quad ZJXJZ^{-1} \in \mathcal{P}(\mathcal{M}) \tag{41d}$$

for all $(X,D_1) \in \mathcal{P}(\mathcal{M})$, and

$$JZJ = Z^{-1}\ ,\ JU_oJ = U_o\ , \quad JV(t)J = V(t) \text{ for all real } t \tag{41e}$$

$$JD_+ = D_-\ , \quad J(V(i\pi),D_+)J = (V(-i\pi),D_-) \tag{41f}$$

$$JD_- = D_+\ , \quad J(V(-i\pi),D_-)J = (V(i\pi),D_+) \tag{41g}$$

d) The antiunitary operator Θ_0 defined by

$$J = Z\ U(u(\underset{\sim}{e}_3,\pi),0)\ \Theta_0 \tag{41h}$$

is a TCP-transformation which satisfies the conditions:

$$\Theta_0{}^2 = U_0 \quad , \quad \Theta_0\Omega = \Omega \quad , \quad \Theta_0\ U(g,x)\ \Theta_0{}^{-1} = U(g,-x) \tag{42a}$$

$$\Theta_0 D_1 = D_1 \quad , \quad \Theta_0\ \mathcal{P}(\mathcal{M})\ \Theta_0{}^{-1} = \mathcal{P}(\mathcal{M}) \tag{42b}$$

and

$$\Theta_0\,\varphi_\mu(x)\ \Theta_0{}^{-1} = \rho_\mu\ \Gamma''_{\mu,\mu}\ \varphi^\dagger_\mu(-x) \tag{42c}$$

where $\rho_\mu = +1$ if $\varphi_\mu(x)$ is a component of a boson field, and $\rho_\mu = -i$ if $\varphi_\mu(x)$ is a component of a fermion field.

<u>Proof</u>: 1) This Lemma corresponds to Lemma 11 in BW I. The reasoning in its proof is similar to our reasoning in BW I, but there are some important differences of detail which have to be discussed. We first note that the assertions a) and b) are trivial. The remaining assertions might be proved in the stated order, which in particular yields a proof of the TCP theorem. In order to shorten the discussion we shall, however, base our proof of the assertion c) on the well-known fact that under our general assumptions about the fields a TCP-transformation Θ_0

which satisfies the conditions (42a)-(42c) does exist.[8] The relations (42a)-(42c) will thus be assumed, and we define the antiunitary operator J by (41h), where Z is given by (26). It is then trivial to show that J satisfies the relations (41c)-(41g).

2) The formula (41b) holds trivially if Q is a multiple of I. Suppose now that Q is of the form (41a). We write $X_k = \varphi_{\mu k}[f_k]$ and $Y_k = \varphi_{\mu k}[f_k{}^i]$ for $k = 1,\ldots,n$, and we then have

$$J\, Q^* \Omega \;=\; J\, X_n{}^\dagger \ldots X_2{}^\dagger X_1{}^\dagger \Omega \;=$$

$$\hat{\rho}_\sigma^* \,\hat{\Gamma}''_{\sigma,\sigma}\, Z\, U(u(\underset{\sim}{e}_3,\pi),0)\, Y_n \ldots Y_2\, Y_1\, \Omega \qquad (43a)$$

where $\hat{\rho}_\sigma = \rho_{\mu 1}\,\rho_{\mu 2} \cdots \rho_{\mu n}$, in view of (41h) and (42c). For any two operators Y_r and Y_s in the set $\{Y_1, Y_2, \ldots, Y_n\}$ the supports of the corresponding test functions $f_r{}^i$ and $f_s{}^i$ are space-like separated, and hence Y_r anticommutes with Y_s if both operators are averaged fermion fields, whereas Y_r commutes with Y_s in all other cases. It is easily shown that under these circumstances

$$\hat{\rho}_\sigma^*\, Z\, Y_n \ldots Y_2\, Y_1\, \Omega \;=\; Y_1\, Y_2 \ldots Y_n\, \Omega \qquad (43b)$$

and hence

$$J\, Q^* \Omega \;=\; \hat{\Gamma}''_{\sigma,\sigma}\, U(u(\underset{\sim}{e}_3,\pi),0)\, Y_1\, Y_2 \ldots Y_n\, \Omega \qquad (43c)$$

From this it follows, in view of (40b) in Lemma 6, that the operator Q satisfies (41b). From this it trivially follows that (41b) holds for all $Q \in \mathcal{Q}$.

We are now prepared to state the main theorem of this section. It will be convenient for the subsequent discussion to introduce the following notation. For any subset R of $\mathcal{M}$ we define the algebra $\mathcal{P}(R)^z$ by

$$\mathcal{P}(R)^z = \{ (ZXZ^{-1},D_1) \mid (X,D_1) \in \mathcal{P}(R) \} \tag{44}$$

where Z is given by (26).

<u>Theorem 1</u>: a) The algebras $\mathcal{P}(W_R)$ and $\mathcal{P}(W_L)^z$ are *-algebras with the antilinear involution $(X,D_1) \longrightarrow (X^*,D_1)$. They commute on D_1, i.e.,

$$[X,Y]\psi = 0 \tag{45a}$$

for all $\psi \in D_1$ and for all $X \in \mathcal{P}(W_R)$, $Y \in \mathcal{P}(W_L)^z$.

b) The vacuum vector Ω is cyclic and separating for both $\mathcal{P}(W_R)$ and $\mathcal{P}(W_L)^z$.

c) With $V(t) = U(v(\underset{\sim}{e}_3,t),0)$ (a velocity transformation in the 3-direction),

$$V(t)\mathcal{P}(W_R)V(t)^{-1} = \mathcal{P}(W_R) \quad , \quad V(t)\mathcal{P}(W_L)^z V(t)^{-1} = \mathcal{P}(W_L)^z \tag{45b}$$

for all real t, and with J defined as in Lemma 7,

$$J\mathcal{P}(W_R)J = \mathcal{P}(W_L)^z \tag{45c}$$

d) With the domains D_+ and D_- such that the operators $(V(i\pi),D_+)$ and $(V(-i\pi),D_-)$ are self-adjoint,

$$\mathcal{P}(W_R)\Omega \subset D_+ \; , \qquad V(i\pi)X\Omega = JX^*\Omega \tag{45d}$$

for any $X \varepsilon \mathcal{P}(W_R)$, and

$$\mathcal{P}(W_L)^z \Omega \subset D_- \quad , \quad V(-i\pi)\, Y\Omega = J\, Y^* \Omega \tag{45e}$$

for any $Y \varepsilon \mathcal{P}(W_L)^z$.

e) The condition

$$C_R\, X\Omega = X^* \Omega \quad , \text{ all } X \varepsilon \mathcal{P}(W_R) \tag{46a}$$

defines an <u>antilinear</u> operator $(C_R, \mathcal{P}(W_R)\Omega)$, and the condition

$$C_L{}^z\, Y\Omega = Y^* \Omega \quad , \text{ all } Y \varepsilon \mathcal{P}(W_L)^z \tag{46b}$$

defines an <u>antilinear</u> operator $(C_L{}^z, \mathcal{P}(W_L)^z \Omega)$.

These two operators satisfy the relations

$$(C_R, \mathcal{P}(W_R)\Omega)^{**} = (C_L{}^z, \mathcal{P}(W_L)^z \Omega)^* = (JV(i\pi), D_+) \tag{46c}$$

$$(C_L{}^z, \mathcal{P}(W_L)^z \Omega)^{**} = (C_R, \mathcal{P}(W_R)\Omega)^* = (J\,V(-i\pi), D_-) \tag{46d}$$

This theorem corresponds to Theorem 1 in BW I. The proof is <u>identical</u> with our proof in BW I, provided that we consistently substitute the operator $C_L{}^z$ for the operator C_L , and the algebra $\mathcal{P}(W_L)^z$ for the algebra $\mathcal{P}(W_L)$. In the particular case that there is <u>no</u> fermion field among the quantum fields we have $U_0 = I$ and $Z = I$, and hence $\mathcal{P}(W_L)^z = \mathcal{P}(W_L)$,in which case the present theorem is identical with Theorem 1 in BW I.

The algebra $\mathcal{P}(W_R)$, respectively the algebra $\mathcal{P}(W_L)$, can be regarded as consisting of field operators locally associated with the wedge-region W_R, respectively the region W_L. We note that the role of these algebras is not quite as symmetric in the present theorem as in BW I, in the sense that the assertions are about the pair $(\mathcal{P}(W_R), \mathcal{P}(W_L)^z)$ rather than about the pair $(\mathcal{P}(W_R), \mathcal{P}(W_L))$. It is, however, easily seen that there is a completely equivalent formulation in terms of the pair $(\mathcal{P}(W_L), \mathcal{P}(W_R)^z)$, and we note, for instance, that

$$\mathcal{P}(W_L)\,\Omega \subset D_- \;, \qquad V(-i\pi)\, Y\Omega = J_L\, Y^*\Omega \tag{47a}$$

for any $Y \in \mathcal{P}(W_L)$, and

$$\mathcal{P}(W_R)^z\,\Omega \subset D_+ \;, \qquad V(i\pi)\, X\Omega = J_L\, X^*\Omega \tag{47b}$$

for any $X \in \mathcal{P}(W_R)^z$, where

$$J_L = ZJZ^{-1} = U_0 J = JU_0 \tag{47c}$$

Furthermore,

$$J_L\, \mathcal{P}(W_L)\, J_L = \mathcal{P}(W_R)^z \tag{47d}$$

We conclude this section with the remark that <u>all</u> the considerations in Section V in BW I also apply to the present situation, provided that $\mathcal{P}(W_L)$ is replaced by $\mathcal{P}(W_L)^z$ and that $\mathcal{P}_0(W_L)$ is replaced by $\mathcal{P}_0(W_L)^z = Z\,\mathcal{P}_0(W_L)\,Z^{-1}$ everywhere in the discussion. In order to have a more suggestive notation it is then convenient to change the notation in BW I according to the scheme: $\mathcal{U}(W_L) \to \mathcal{U}(W_L)^z$, $\mathcal{A}_L \to \mathcal{A}_L^{\,z}$, etc.

V. The duality condition for the wedge regions W_R and W_L.

The discussion in this section corresponds to the discussion in Sec. VI in BW I. We are thus concerned with the question of how the field operators in $\mathcal{P}(W_R)$ might generate a von Neumann algebra of bounded operators which can be regarded as being locally associated with the region W_R. We must, of course, here define the term "locally asscoeiated with" precisely, and in a manner appropriate for a field theory in which fermion fields might occur. To set the stage for the discussion we begin with some algebraic considerations.

Definition 1: If $\mathcal{A}$ is a von Neumann algebra such that $U_0 \mathcal{A} U_0^{-1} = \mathcal{A}$, and if $\mathcal{A}^z = Z \mathcal{A} Z^{-1}$ with Z defined as in (26), then the quasicommutant $\mathcal{A}^q$ of $\mathcal{A}$ is defined as the von Neumann algebra $\mathcal{A}^q = (\mathcal{A}^z)'$.

In a theory in which fermion operators, i.e., operators X which satisfy $U_0 X U_0^{-1} = -X$, occur the notion of quasicommutant [9] is the proper notion in terms of which one may formulate the conditions of locality and of duality. As an algebraic notion the notion of a quasicommutant is less general than the notion of a commutant in the sense that the former notion refers to a specific unitary involution U_0.

We formulate the pertinent facts about the notion of a quasicommutant as follows.

Theorem 2: Let $\mathcal{A}$ be a von Neumann algebra such that $U_0 \mathcal{A} U_0^{-1} = \mathcal{A}$, and let $\mathcal{A}^q = (Z \mathcal{A} Z^{-1})'$ be its quasicommutant. Let

$$\mathcal{A}_B = \{ X \mid U_0 X U_0^{-1} = X , X \in \mathcal{A} \},$$

$$\mathcal{A}_F = \{ X \mid U_0 X U_0^{-1} = -X, X \in \mathcal{A} \} \tag{48a}$$

and

$$(\mathcal{A}^q)_B = \{ Y \mid U_0 Y U_0^{-1} = Y \ , \ Y \in \mathcal{A}^q \} \ ,$$

$$(\mathcal{A}^q)_F = \{ Y \mid U_0 Y U_0^{-1} = -Y \ , \ Y \in \mathcal{A}^q \} \qquad (48b)$$

Then:

a) $$U_0 \mathcal{A}^q U_0^{-1} = \mathcal{A}^q \ , \ \mathcal{A}^q = Z(\mathcal{A}')Z^{-1} \ , \ (\mathcal{A}^q)^q = \mathcal{A} \qquad (49)$$

b) Every operator $X \in \mathcal{A}$ has the unique representation

$$X = X_b + X_f \ , \quad \text{with} \quad X_b \in \mathcal{A}_B \ , \ X_f \in \mathcal{A}_F \qquad (50a)$$

where, in fact,

$$X_b = \tfrac{1}{2}(X + U_0 X U_0^{-1}) \ , \quad X_f = \tfrac{1}{2}(X - U_0 X U_0^{-1}) \qquad (50b)$$

Every operator $Y \in \mathcal{A}^q$ has the unique representation

$$Y = Y_b + Y_f \ , \quad \text{with} \quad Y_b \in (\mathcal{A}^q)_B \ , \ Y_f \in (\mathcal{A}^q)_F \qquad (50c)$$

where, in fact,

$$Y_b = \tfrac{1}{2}(Y + U_0 Y U_0^{-1}) \ , \quad Y_f = \tfrac{1}{2}(Y - U_0 Y U_0^{-1}) \qquad (50d)$$

c) The elements $X_b \in \mathcal{A}_B$; $X_f \in \mathcal{A}_F$; $Y_b \in (\mathcal{A}^q)_B$ and $Y_f \in (\mathcal{A}^q)_F$ satisfy the conditions

$$[X_b, Y_b] = 0 \qquad (51a)$$

$$[X_b, Y_f] = 0 \qquad (51b)$$

$$[X_f, Y_b] = 0 \qquad (51c)$$

$$\{X_f, Y_f\} = X_f Y_f + Y_f X_f = 0 \qquad (51d)$$

The set $(\mathcal{A}^q)_B$ is a von Neumann algebra, precisely equal to the set of all bounded operators Y_b which satisfy the condition $U_0 Y_b U_0^{-1} = Y_b$, and the conditions (51a) and (51c) for all $X_b \varepsilon \mathcal{A}_B$, $X_f \varepsilon \mathcal{A}_F$. The set $\mathcal{A}_B$ is a von Neumann algebra, precisely equal to the set of all bounded operators X_b which satisfy the condition $U_0 X_b U_0^{-1} = X_b$, and the conditions (51a) and (51b) for all $Y_b \varepsilon (\mathcal{A}^q)_B$, $Y_f \varepsilon (\mathcal{A}^q)_F$. The set $(\mathcal{A}^q)_F$ is precisely equal to the set of all bounded operators Y_f which satisfy the condition $U_0 Y_f U_0^{-1} = - Y_f$, and the conditions (51b) and (51d) for all $X_b \varepsilon \mathcal{A}_B$, $X_f \varepsilon \mathcal{A}_F$. The set $\mathcal{A}_F$ is precisely equal to the set of all bounded operators X_f which satisfy the condition $U_0 X_f U_0^{-1} = - X_f$, and the conditions (51c) and (51d) for all $Y_b \varepsilon (\mathcal{A}^q)_B$, $Y_f \varepsilon (\mathcal{A}^q)_F$.

d) The vector Ω is cyclic (respectively separating) for $\mathcal{A}$ if and only if it is separating (respectively cyclic) for $\mathcal{A}^q$.

We omit the very trivial proofs of these assertions. We stated the above facts in the form of a formal theorem in view of their importance for our discussion. The situation might be illustrated as follows. Suppose that two von Neumann algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are "locally associated with" two regions R_1, respectively R_2, which are causally independent. The "local" nature of the association can then be expressed through the relation $\mathcal{A}_1 \subset \mathcal{A}_2^q$, which, in view of the theorem, is equivalent to the customary conditions in terms of commutators and anticommutators, i.e., the fermion operators in $\mathcal{A}_1$ anticommute with the fermion operators in $\mathcal{A}_2$ and commute with the boson operators in $\mathcal{A}_2$, whereas the boson operators in $\mathcal{A}_1$ commute with all operators in $\mathcal{A}_2$. Now $\mathcal{A}_1 \subset \mathcal{A}_2^q$ is equivalent to the condition that $[X,Y] = 0$ for all $X \varepsilon \mathcal{A}_1$ and all $Y \varepsilon \mathcal{A}_2^z = Z \mathcal{A}_2 Z^{-1}$, which means that the locality conditions are expressible in terms of the vanishing of certain commutators, irrespective of whether fermion operators occur or do not occur in the theory. This has the important practical consequence, from our point of view, that we do not have to create a new algebraic theory in order to deal with the case of fermion operators; as in BW I it suffices to consider the relationships between von Neumann algebras and their commutants. [10] Let us also note here that according to the fermion-superselection principle only a boson operator can be a physical observable. This means, with reference to our illustration above, that the observables in $\mathcal{A}_2$ and $\mathcal{A}_2{}^z$ are precisely the same, and thus that the observables associated with the region R_1 commute with the observables associated with R_2.

Definition 2: a) A set $\mathcal{K}(W_R)$ of bounded operators such that $X^* \varepsilon \mathcal{K}(W_R)$ for all $X \varepsilon \mathcal{K}(W_R)$ shall be said to be covariantly associated with W_R if and only if

$$U(\lambda)\, \mathcal{K}(W_R)\, U(\lambda)^{-1} \subset \mathcal{K}(W_R) \tag{52a}$$

for all elements λ in the semigroup $\sigma(W_R)$ consisting of all $\lambda \varepsilon \bar{\mathcal{g}}$ such that $\Lambda(\lambda)W_R \subset W_R$. In particular,

$$V(t)\, \mathcal{K}(W_R)\, V(t)^{-1} = \mathcal{K}(W_R) \text{ , all real } t, \tag{52b}$$

and, more generally,

$$U(\lambda)\, \mathcal{K}(W_R)\, U(\lambda)^{-1} = \mathcal{K}(W_R) \text{ , all } \lambda \varepsilon \bar{\mathcal{g}}(W_R) \tag{52c}$$

where $\bar{\mathcal{g}}(W_R)$ is the group of all elements $\lambda \varepsilon \bar{\mathcal{g}}$ such that $\Lambda(\lambda)W_R = W_R$,i.e., all Poincaré transformations which map W_R onto W_R.

b) A set $\mathcal{K}(W_L)$ of bounded operators such that $Y^* \varepsilon \mathcal{K}(W_L)$ for all $Y \varepsilon \mathcal{K}(W_L)$ shall be said to be <u>covariantly associated with</u> W_L if and only if

$$\mathcal{K}(W_L) = U(u(\underset{\sim}{e}_1,\pi),0)\,\mathcal{K}(W_R)\,U(u(\underset{\sim}{e}_1,\pi),0)^{-1} \qquad (53)$$

where $\mathcal{K}(W_R)$ is a set covariantly associated with W_R.

c) Let $\mathcal{K}(W_R)$ be a set of bounded operators, covariantly associated with W_R as above. The association shall be said to be <u>TCP-symmetric</u> if and only if

$$\theta_0\,\mathcal{K}(W_R)\,\theta_0^{-1} = \mathcal{K}(W_L) \qquad (54a)$$

or, equivalently,

$$J\,\mathcal{K}(W_R)\,J^{-1} = \mathcal{K}(W_L)^z \qquad (54b)$$

where $\mathcal{K}(W_L)$ is given by (53).

d) A set $\mathcal{K}(W_R)$ of bounded operators which contains X^* if it contains X shall be said to be <u>locally associated with</u> W_R if and only if $\mathcal{K}(W_R)$ is covariantly associated with W_R <u>and</u>

$$\mathcal{K}(W_R) \subset \mathcal{K}(W_L)^q \qquad (55)$$

where $\mathcal{K}(W_L)$ is given by (53), and where the von Neumann algebra $\mathcal{K}(W_L)^q$ is defined as $(\mathcal{K}(W_L)^z)'$.

e) A von Neumann algebra $\mathcal{A}(W_R)$, locally associated with W_R, shall be said to satisfy the <u>condition of duality</u> if and only if

$$\mathcal{A}(W_R) = \mathcal{A}(W_L)^q \qquad (56)$$

where $\mathcal{A}(W_L)$ is defined in terms of $\mathcal{A}(W_R)$ in analogy with (53).

We present these formal definitions for later reference as we will repeatedly encounter sets which satisfy one, or several, of these defining relations. The geometrical significance of these definitions is obvious and need not be discussed here. Concerning the physical interpretation we note that the conditions in d) are minimum conditions which a set of "local observables for W_R" would have to satisfy. In a quantum field theory these conditions are not, however, by themselves enough; the bounded local operators should also satisfy some condition of locality relative to the local field operators.

Lemma 8: Let $\mathcal{F}$ be a set of closable operators, such that $U_0 \mathcal{F} U_0^{-1} = \mathcal{F}$. We define the set $\mathcal{F}^q$ as the set of all bounded operators X such that

$$X^z (Y,D(Y))^* \subset (Y,D(Y))^* X^z ,$$

$$X^z (Y,D(Y))^{**} \subset (Y,D(Y))^{**} X^z \qquad (57)$$

for all $(Y,D(Y)) \varepsilon \mathcal{F}$. Or, equivalently, the set $\mathcal{F}^q$ is precisely equal to the set of all bounded operators X such that for all $(Y,D(Y)) \varepsilon \mathcal{F}$,

$$X^z (Y,D(Y)) \subset (Y,D(Y))^{**} X^z ,$$

$$(X^z)^* (Y,D(Y)) \subset (Y,D(Y))^{**} (X^z)^* \qquad (58)$$

a) The set $\mathcal{F}^q$ is a von Neumann algebra, and it satisfies the relation $U_0(\mathcal{F}^q)\,U_0^{-1} = \mathcal{F}^q$.

b) Let the set $\mathcal{F}^{qq}$ of bounded operators be defined by

$$\mathcal{F}^{qq} = (\mathcal{F}^q)^q \tag{59}$$

Then $\mathcal{F}^{qq}$ is a von Neumann algebra precisely equal to the von Neumann algebra generated by the operators V and the spectral projections of the operators K for all pairs of operators $\{V, K\}$,where V is the unique partial isometry, and K is the unique non-negative definite self-adjoint operator, defined through the polar decomposition

$$(Y,D(Y))^{**} = V\,(K, D(Y^{**})) \tag{60}$$

of the closure of any $(Y,D(Y)) \varepsilon\, \mathcal{F}$.

This lemma is a paraphrase of well-known facts about the commutant in the sense of von Neumann [11] of a set of closed operators. An equivalent definition for $\mathcal{F}^q$ is thus

$$\mathcal{F}^q = (Z\,\mathcal{F}^{**}\,Z^{-1})' \tag{61a}$$

with the prime-notation of von Neumann, and the set $\mathcal{F}^{qq}$ is then given by

$$\mathcal{F}^{qq} = (\mathcal{F}^{**})'' \tag{61b}$$

where $\mathcal{F}^{**}$ denotes the set of all closures of the operators in $\mathcal{F}$. That the assertion in b) above about the algebra $\mathcal{F}^{qq}$ (regarded as given by (61b)) holds is well-known [12] (and easily proved). That $\mathcal{F}^q$ (and hence $\mathcal{F}^{qq}$) is invariant under conjugation by U_0 follows trivially from the corresponding property of $\mathcal{F}$.

We shall call $\mathfrak{F}^q$ the quasicommutant of the set of adjoints and closures of the possibly unbounded operators in $\mathfrak{F}$; this is consistent with our earlier terminology in the case that $\mathfrak{F}$ is actually a von Neumann algebra. We shall say that the von Neumann algebra $\mathfrak{F}^{qq}$ is <u>generated</u> by the set $\mathfrak{F}$.

We shall next consider some special sets of bounded operators defined in terms of field operators in $\mathcal{P}(R)$, where R is any subset of $\mathcal{M}$. In this section we are primarily interested in the wedge-regions W_R and W_L , but for later reference it will be convenient to consider other regions R as well. We note here that it would be reasonable to restrict the regions R such that they satisfy the condition $R^{cc} = R$, but we shall not do so since we do not here wish to investigate the geometrical implications of this restriction.

<u>Definition 3:</u> Let R be any subset of Minkowski space, and let R^c be its causal complement (as defined in (5)).

a) The set $\mathfrak{L}(R)$ is defined as the set of all finite linear combinations of operators of the form $(\varphi_\mu[f], D_1)$, where where $\mu \in I_T$, and where $f \in \mathcal{S}(R^4)$, with $\mathrm{supp}(f) \subset R$.

b) The set $\mathcal{G}(R)$ is defined as the von Neumann algebra <u>generated by</u> $\mathfrak{L}(R)$, i.e.,

$$\mathcal{G}(R) = \mathfrak{L}(R)^{qq} \qquad (62)$$

where the superscript "qq" denotes the mapping $\mathfrak{F} \rightarrow \mathfrak{F}^{qq}$

defined in Lemma 8.

c) The von Neumann algebra $\mathcal{C}(R)$ is defined as the <u>quasicommutant of</u> $\mathcal{L}(R^c)$, i.e.,

$$\mathcal{C}(R) = \mathcal{L}(R^c)^q = \mathcal{G}(R^c)^q \tag{63}$$

where the superscript "q" denotes the mapping $\mathcal{F} \rightarrow \mathcal{F}^q$ defined in Lemma 8.

d) The <u>weak quasicommutant</u> $\mathcal{C}_w(R)$ of $\mathcal{P}(R^c)$ is defined as the set of all bounded operators X such that

$$\langle Y^*\phi \mid X\psi \rangle = \langle X^*\phi \mid Y\psi \rangle \tag{64}$$

for all $\phi, \psi \in D_1$, and all $(Y, D_1) \in \mathcal{P}(R^c)^z = Z\,\mathcal{P}(R^c)\,Z^{-1}$.

We introduce the new term "weak quasicommutant" with some reluctance, but it does seem fairly appropriate to describe the nature of the sets $\mathcal{C}_w(R)$. The adjective "weak" is here intended to convey an impression of the "weak" nature of the "commutation relations" (64), as contrasted with the more restrictive conditions (57). It should be noted, however, that the operators in $\mathcal{C}_w(R)$ commute in the weak sense of (64) with <u>all</u> the operators in $\mathcal{P}(R^c)^z$, whereas the operators in $\mathcal{C}(R)$ commute in the strong sense of (57) only with the operators in the subset $\mathcal{L}(R^c)^z$ of $\mathcal{P}(R^c)^z$.

We shall next consider some fairly elementary properties of the sets defined above.

<u>Lemma 9</u>: Let R be any subset of Minkowski space, and let the sets $\mathcal{L}(R)$, $\mathcal{C}(R)$, $\mathcal{C}_w(R)$ and $\mathcal{G}(R)$ be defined as in Definition 3. Then:

a) Each one of these four sets satisfies the condition (65a) of <u>covariance</u>, the condition (65b) of <u>TCP-symmetry</u>, and the condition (65c) of <u>isotony</u>, i.e., if $\mathcal{Q}(R)$ is any one of the sets $\mathcal{L}(R)$, $\mathcal{C}(R)$, $\mathcal{C}_w(R)$ or $\mathcal{G}(R)$, then

$$U(\lambda)\, \mathcal{Q}(R)\, U(\lambda)^{-1} = \mathcal{Q}(\Lambda(\lambda)R) \quad , \text{ all } \lambda \in \tilde{\mathcal{G}} \tag{65a}$$

$$\theta_0\, \mathcal{Q}(R)\, \theta_0^{-1} = \mathcal{Q}(-R) \tag{65b}$$

where $-R$ denotes the set $-R = \{ -x \mid x \in R \}$.

$$\mathcal{Q}(R) \supset \mathcal{Q}(R_1) \quad , \text{ whenever } R \supset R_1 \tag{65c}$$

b) The set $\mathcal{C}_w(R)$ is a weakly closed linear manifold, closed under the *-operation, i.e., it contains X^* if it contains X.

A bounded operator X is in $\mathcal{C}_w(R)$ if and only if

$$X\,(Y^*,D_1) \subset (Y,D_1)^*\, X \tag{66}$$

for all $(Y,D_1) \in \mathcal{P}(R^c)^z$.

c) A bounded operator X is in $\mathcal{C}_w(R)$ if and only if the condition (64) holds for all ϕ, $\psi \in D_1$, and all $(Y,D_1) \in \mathcal{L}(R^c)^z$, or, equivalently, if and only if the condition (66) holds for all $(Y,D_1) \in \mathcal{L}(R^c)^z$.

d)

$$X_1\, X\, X_2 \in \mathcal{C}_w(R) \tag{67a}$$

for all $X \in \mathcal{C}_w(R)$ and all X_1, $X_2 \in \mathcal{C}(R)$. In particular,

$$\mathcal{C}(R) \subset \mathcal{C}_w(R) \tag{67b}$$

e) If R^c has a nonempty interior, then Ω is separating for $\mathcal{C}_w(R)$, i.e., if $X \varepsilon \mathcal{C}_w(R)$ and $X\Omega = 0$, then $X = 0$.

If R has a nonempty interior, then $\mathcal{G}(R)\Omega$ is dense in the Hilbert space $\mathcal{H}$.

f) If (for a particular subset R) the "linear field operators" in the set $\mathcal{L}(R^c)$ satisfy the condition that D_1 is a core for the adjoints of the operators in the set, i.e., $(Y^\dagger, D_1)^* = (Y, D_1)^{**}$ for all $(Y, D_1) \varepsilon \mathcal{L}(R^c)$, then $\mathcal{C}(R) = \mathcal{C}_w(R)$.

<u>Proof</u>: 1) The assertions a) and b) are trivial. We note here that the condition (66) (which is a trivial restatement of the condition (64)) is equivalent to the condition that

$$X\,(Y^*, D_1)^{**} \subset (Y, D_1)^*\, X \qquad (68)$$

for all $(Y, D_1) \varepsilon \mathcal{P}(R^c)^z$.

2) To prove the assertion c) we assume that X is a bounded operator which satisfies the condition (64) for all $\phi, \psi \varepsilon D_1$, and all $(Y, D_1) \varepsilon \mathcal{L}(R^c)^z$. It follows at once that the condition (64) then also holds for all $(Y, D_1) \varepsilon \mathcal{P}_0(R^c)^z$. For such an X, let $\phi, \psi \varepsilon D_1$, and let $(Y, D_1) \varepsilon \mathcal{P}(R^c)^z$. Since we have $ZD_1 = D_1$, and since the quantum fields are operator-valued tempered distributions, it follows from the fact that $(\otimes \mathcal{D}(R^4))^n$ is dense in $\mathcal{S}(R^{4n})$ that there exists a sequence $\{(Y_k, D_1) \mid (Y_k, D_1) \varepsilon \mathcal{P}_0(R^c)^z, k = 1, \ldots, \infty\}$ of operators such that

$$\underset{k\to\infty}{\text{s-lim}}\, Y_k \psi = Y\psi \quad , \quad \underset{k\to\infty}{\text{s-lim}}\, Y_k^* \phi = Y^* \phi \qquad (69)$$

It readily follows that the relation (64) holds for the above operator (Y,D_1), and hence $X \varepsilon \mathcal{C}_w(R)$ as asserted.

3) We consider the assertion d). Let $X \varepsilon \mathcal{C}(R)$, $X_w \varepsilon \mathcal{C}_w(R)$, and $(Y,D_1) \varepsilon \mathcal{L}(R^c)^z$. We then have, in view of (57) and (68),

$$X X_w (Y^*,D_1)^{**} \subset X (Y,D_1)^* X_w \subset (Y,D_1)^* X X_w \qquad (70)$$

which means that $XX_w \varepsilon \mathcal{C}_w(R)$. From this (67a) follows readily, and since $I \varepsilon \mathcal{C}_w(R)$ the relation (67b) follows.

4) If $X \varepsilon \mathcal{C}_w(R)$, then $X\Omega = 0$ implies that

$$\langle Y_1 \Omega | X Y_2 \Omega \rangle = \langle Y_2^* Y_1 \Omega | X \Omega \rangle = 0 \qquad (71)$$

for all $Y_1, Y_2 \varepsilon \mathcal{P}(R^c)^z$. By the Reeh-Schlieder theorem the set $\mathcal{P}(R^c)^z \Omega$ is dense if R^c has a nonempty interior, which implies that in this case $X = 0$ if (71) holds. This proves the first assertion in e), and in view of (67b) it follows that Ω is a separating vector for the von Neumann algebra $\mathcal{C}(R)$, and hence a cyclic vector for its quasicommutant $\mathcal{G}(R^c)$ whenever the interior of R^c is nonempty. It readily follows, since $\mathcal{G}(R)$ satisfies the condition of isotony (65c), that $\mathcal{G}(R)\Omega$ is dense whenever R has a nonempty interior.

5) We consider the assertion f). If $(Y^*,D_1)^* = (Y,D_1)^{**}$ for all $(Y,D_1) \varepsilon \mathcal{L}(R^c)$, and if $X \varepsilon \mathcal{C}_w(R)$, then the relation (68) implies that $X \varepsilon \mathcal{C}(R)$. In view of (67b) this implies that $\mathcal{C}_w(R) = \mathcal{C}(R)$, as asserted. This completes the proof.

We note that it does not follow from the definition of $\mathcal{C}_w(R)$ as a weak quasicommutant of an algebra $\mathcal{P}(R^c)$ of unbounded operators (or equivalently as the "weak commutant" of the operator algebra $\mathcal{P}(R^c)^z$) that $\mathcal{C}_w(R)$ is a von Neumann algebra; the set need not be closed under multiplication. What the actual situation is in quantum field theory we do not know. In the case of free fields the premises in part f) of the lemma are trivially satisfied, and $\mathcal{C}_w(R)$ is then identical with the von Neumann algebra $\mathcal{C}(R)$. In this connection we refer to the work of Powers on algebras of unbounded operators, their "weak commutants," and related subjects.[13)]

Lemma 10: Let R be any subset of Minkowski space, and let the notation be as in Definition 3 and Lemma 9. Let $\mathcal{A}_0(R)$ be defined as the set of all bounded operators X such that XX_w and X_wX are both in $\mathcal{C}_w(R)$ for all $X_w \in \mathcal{C}_w(R)$. Then:

a) The set $\mathcal{A}_0(R)$ is a von Neumann algebra, and

$$\mathcal{C}(R) \subset \mathcal{A}_0(R) \subset \mathcal{C}_w(R) \tag{72}$$

b) The mapping $R \to \mathcal{A}_0(R)$ satisfies the condition of covariance (65a) and the condition of TCP-symmetry (65b) in Lemma 9. In particular $U_0\ \mathcal{A}_0(R)\ U_0^{-1} = \mathcal{A}_0(R)$.

c) All operators $(Y,D_1) \in \mathcal{P}(R^c)$ have closable extensions defined by

$$(Y,D_1) \rightarrow (a(Y),D_a) = (Y^{\dagger *}, D_a) \tag{73a}$$

where D_a is the domain defined by

$$D_a = \text{span}\ \{\ X\phi\ \mid X \in \mathcal{A}_0(R),\ \phi \in D_1\ \} \tag{73b}$$

These extensions satisfy the conditions

$$(Y^*,D_1)^* \supset (a(Y)^*,D_a)^* \supset (a(Y),D_a) \supset (Y,D_1) \tag{73c}$$

d) Let $\mathcal{P}_a(R^c)$ be the set of all operators $(a(Y),D_a)$ with $(Y,D_1) \in \mathcal{P}(R^c)$. Then, with the notation in Lemma 8,

$$\mathcal{A}_0(R) = \mathcal{P}_a(R^c)^q \quad , \mathcal{P}_a(R^c)^{qq} = \mathcal{P}_a(R^c)'' = \mathcal{A}_0(R)^q \subset \mathcal{G}(R^c) \tag{74a}$$

and the closures and adjoints of the operators $(a(Y),D_a)$ in $\mathcal{P}_a(R^c)$ are thus affiliated to the von Neumann algebra $\mathcal{A}_0(R)^q$.

The weak quasicommutant of $\mathcal{P}_a(R^c)$ relative to the domain D_a, i.e., the set of all bounded operators X such that

$$\langle X^* \phi | a(Y)^z \psi \rangle = \langle (a(Y)^z)^* \phi \,|\, X \psi \rangle \tag{74b}$$

for all $\phi, \psi \in D_a$, all $(a(Y),D_a) \in \mathcal{P}_a(R^c)$, is precisely equal to the set $\mathcal{C}_w(R)$.

e) The mapping $(Y,D_1) \to (a(Y),D_a)$ of the algebra $\mathcal{P}(R^c)$ onto $\mathcal{P}_a(R^c)$ is a representation, and it is a *-representation of the *-algebra $\mathcal{P}(R^c)$ in the sense that

$$(a(Y^\dagger),D_a) = (a(Y)^*,D_a) \tag{75a}$$

The representation is continuous in the sense that

$$\underset{k\to\infty}{\text{s-lim}} \; a(Y_k)\psi = 0 \tag{75b}$$

for all $\psi \in D_a$ whenever

$$\underset{k\to\infty}{\text{s-lim}} \; Y_k \phi = 0 \tag{75c}$$

for all $\phi \in D_1$.

Proof: 1) $\mathcal{A}_0(R)$ is trivially a *-algebra since $\mathcal{C}_w(R)$ is closed under the *-operation. From the fact that $\mathcal{C}_w(R)$ is weakly closed it follows that $\mathcal{A}_0(R)$ is also weakly closed, and hence a von Neumann algebra. The relation (72) is trivial in view of (67b). The assertions b) are obvious.

2) It follows from (66) that if $X \varepsilon \mathcal{C}_w(R)$ and $\phi \varepsilon D_1$, then $X\phi \varepsilon D(Y^*)$, for any $(Y,D_1) \varepsilon \mathcal{P}(R^c)^z$. In view of (72) this implies that D_a, as defined in (73b), is contained in the domain of the adjoint of any operator (Y,D_1) in $\mathcal{P}(R^c)^z$ or in $\mathcal{P}(R^c)$, since $ZD_a = D_a$. It follows that the extensions $(a(Y),D_a)$ are well-defined by (73a). Furthermore (73a) also defines an extension of every operator $(Y^z,D_1) \varepsilon \mathcal{P}(R^c)^z$, and we have

$$(a(ZYZ^{-1}),D_a) = Z\ (a(Y),D_a)\ Z^{-1} \tag{76a}$$

for all $(Y,D_1) \varepsilon \mathcal{P}(R^c)$.

3) Let $X_1, X_2 \varepsilon \mathcal{A}_0(R)$; $\phi \varepsilon D_1$ and $(Y,D_1) \varepsilon \mathcal{P}(R^c)^z$. Then $X_1X_2 \varepsilon \mathcal{A}_0(R)$, and since $\mathcal{A}_0(R) \subset \mathcal{C}_w(R)$ we have

$$\begin{aligned} a(Y)X_1X_2\phi &= Y^{\dagger *} X_1X_2\phi = X_1X_2 Y\phi = \\ &= X_1Y^{\dagger *}X_2\phi = X_1a(Y) X_2\phi \end{aligned} \tag{76b}$$

which implies that X_1 commutes with $(a(Y),D_a)^{**}$ in the strong sense of (57), and we have thus proved that $\mathcal{A}_0(R) \subset \mathcal{P}_a(R^c)^q$. It furthermore readily follows that the relations (73c) hold for all $(Y,D_1) \varepsilon \mathcal{P}(R^c)^z$, and hence for all $(Y,D_1) \varepsilon \mathcal{P}(R^c)$. The relation (75a) is then trivial.

4) We next consider the weak quasicommutant $\mathcal{C}_{wa}(R)$ of $\mathcal{P}_a(R^c)$ relative to the domain D_a. It is easily seen from the condition (74b) that a bounded operator X is in $\mathcal{C}_{wa}(R)$ if and only if $X_1XX_2 \varepsilon \mathcal{C}_w(R)$ for all $X_1,X_2 \varepsilon \mathcal{A}_0(R)$. This implies that $\mathcal{C}_{wa}(R) = \mathcal{C}_w(R)$, as asserted. We obviously have

XX_w, $X_wX \in \mathcal{C}_{wa}(R)$ for all $X_w \in \mathcal{C}_{wa}(R)$, $X \in \mathcal{P}_a(R^c)^q$, and in view of the results in step 3) above the first relation (74a) follows. The remaining relations (74a) then follow trivially, in view of (72).

5) The remaining assertions in part e) of the lemma are trivial, and we omit the detailed proofs.

We must here state that we know much less about the relationships between the sets $\mathcal{C}(R)$, $\mathcal{C}_w(R)$ and $\mathcal{A}_0(R)$ than we would like to know. We note here that $\mathcal{C}(R)$ was defined as the quasicommutant of the <u>subset</u> $\mathcal{L}(R^c)$ of $\mathcal{P}(R^c)$, which means that the closures and adjoints of the operators in $\mathcal{L}(R^c)$ are affiliated to the von Neumann algebra $\mathcal{G}(R^c) = \mathcal{C}(R)^q$, but we see no obvious reason why this would imply that the closures and adjoints of the operators in $\mathcal{P}(R^c)$ are also affiliated to this same von Neumann algebra. The lemma now shows that there exists a "natural" extension $(a(Y), D_a)$ of all the operators in $\mathcal{P}(R^c)$ such that the closures and adjoints of the extended operators are affiliated to $\mathcal{G}(R^c)$, or to the possibly smaller von Neumann algebra $\mathcal{A}_0(R)^q$. It is here important to note that this extension depends on the set R^c, although this is not shown explicitly in our notation. A field operator which can be associated with different regions might thus have different extensions constructed as in the lemma.

In view of our present lack of understanding of the general structure of a quantum field theory the possible physical interpretation of the weak quasicommutant $\mathcal{C}_w(R)$ of $\mathcal{P}(R^c)$ is far from clear. With reference to the discussion by Licht of strict localization [14] we note here the following. Let V be a partial isometry in $\mathcal{C}_w(R)^z$ such that $V^*V = I$, and let $\psi = V\Omega$. Then ψ is in the domain of $(Y,D_1)^*$ for any $(Y,D_1) \in \mathcal{P}(R^c)$ and we have, for any such (Y,D_1),

$$\langle \psi | Y^{\dagger *} \psi \rangle = \langle \Omega | Y \Omega \rangle \tag{77a}$$

and, more generally,

$$\langle Y_1{}^{\dagger *} \psi \mid Y_2{}^{\dagger *} \psi \rangle = \langle Y_1 \Omega \mid Y_2 \Omega \rangle \tag{77b}$$

for any two (Y_1,D_1), $(Y_2,D_1) \in \mathcal{P}(R^c)$. We here assume that both R and R^c have nonempty interiors. It is then not hard to show that if a vector ψ satisfies the conditions (77b), then ψ is of the above form.

The expression at left in (77a) might be loosely regarded as the "expectation value of the field operator Y in the state ψ", and the "local character" of the state then manifests itself in the fact that the expectation value in the state equals the vacuum expectation value, for all operators $(Y,D_1) \in \mathcal{P}(R^c)$. Note, however, that the operator $Y^{\dagger *}$ at left in (77a) cannot in general be replaced by Y^{**} or by Y, as ψ might not be in the domains of these operators. We furthermore note that the condition (77a) also holds for all the bounded operators in the von Neumann algebra $\mathcal{C}_w(R)^q$, but not necessarily for the operators in $\mathcal{G}(R^c)$. In our opinion (77a) is a necessary condition for a local state (localized in the complement of R^c) but by no means a sufficient condition.

We shall next consider the properties of the sets $\mathcal{C}(R)$, $\mathcal{A}_0(R)$, $\mathcal{C}_w(R)$ and $\mathcal{G}(R)$ for the special case that $R \in \mathcal{W}$. The

We shall call $\mathfrak{F}^q$ the quasicommutant of the set of adjoints and closures of the possibly unbounded operators in $\mathfrak{F}$; this is consistent with our earlier terminology in the case that $\mathfrak{F}$ is actually a von Neumann algebra. We shall say that the von Neumann algebra $\mathfrak{F}^{qq}$ is <u>generated</u> by the set $\mathfrak{F}$.

We shall next consider some special sets of bounded operators defined in terms of field operators in $\mathcal{P}(R)$, where R is any subset of $\mathcal{M}$. In this section we are primarily interested in the wedge-regions W_R and W_L, but for later reference it will be convenient to consider other regions R as well. We note here that it would be reasonable to restrict the regions R such that they satisfy the condition $R^{cc} = R$, but we shall not do so since we do not here wish to investigate the geometrical implications of this restriction.

<u>Definition 3:</u> Let R be any subset of Minkowski space, and let R^c be its causal complement (as defined in (5)).

a) The set $\mathcal{L}(R)$ is defined as the set of all finite linear combinations of operators of the form $(\varphi_\mu[f], D_1)$, where where $\mu \in I_T$, and where $f \in \mathcal{S}(R^4)$, with $\mathrm{supp}(f) \subset R$.

b) The set $\mathcal{G}(R)$ is defined as the von Neumann algebra <u>generated by</u> $\mathcal{L}(R)$, i.e.,

$$\mathcal{G}(R) = \mathcal{L}(R)^{qq} \qquad (62)$$

where the superscript "qq" denotes the mapping $\mathfrak{F} \rightarrow \mathfrak{F}^{qq}$

defined in Lemma 8.

c) The von Neumann algebra $\mathcal{C}(R)$ is defined as the <u>quasicommutant of</u> $\mathcal{L}(R^c)$, i.e.,

$$\mathcal{C}(R) = \mathcal{L}(R^c)^q = \mathcal{G}(R^c)^q \tag{63}$$

where the superscript "q" denotes the mapping $\mathcal{F} \rightarrow \mathcal{F}^q$ defined in Lemma 8.

d) The <u>weak quasicommutant</u> $\mathcal{C}_w(R)$ of $\mathcal{P}(R^c)$ is defined as the set of all bounded operators X such that

$$\langle Y^*\phi \mid X\psi \rangle = \langle X^*\phi \mid Y\psi \rangle \tag{64}$$

for all $\phi, \psi \in D_1$, and all $(Y, D_1) \in \mathcal{P}(R^c)^z = Z\,\mathcal{P}(R^c)\,Z^{-1}$.

We introduce the new term "weak quasicommutant" with some reluctance, but it does seem fairly appropriate to describe the nature of the sets $\mathcal{C}_w(R)$. The adjective "weak" is here intended to convey an impression of the "weak" nature of the "commutation relations" (64), as contrasted with the more restrictive conditions (57). It should be noted, however, that the operators in $\mathcal{C}_w(R)$ commute in the weak sense of (64) with <u>all</u> the operators in $\mathcal{P}(R^c)^z$, whereas the operators in $\mathcal{C}(R)$ commute in the strong sense of (57) only with the operators in the subset $\mathcal{L}(R^c)^z$ of $\mathcal{P}(R^c)^z$.

We shall next consider some fairly elementary properties of the sets defined above.

<u>Lemma 9</u>: Let R be any subset of Minkowski space, and let the sets $\mathcal{L}(R)$, $\mathcal{C}(R)$, $\mathcal{C}_w(R)$ and $\mathcal{G}(R)$ be defined as in Definition 3. Then:

lemma which follows corresponds in part to our Theorem 3 in BW I, with some added refinements which we overlooked before.

<u>Lemma 11</u>: Let $\mathcal{C}(R)$, $\mathcal{C}_w(R)$, $\mathcal{A}_0(R)$ and $\mathcal{G}(R)$ be defined as in Definition 3 and Lemma 10. Then:

a)

$$\mathcal{C}(W_R) = \mathcal{C}(\bar{W}_R) \quad , \quad \mathcal{C}_w(W_R) = \mathcal{C}_w(\bar{W}_R) \; ,$$

$$\mathcal{G}(W_R) = \mathcal{G}(\bar{W}_R) \; , \quad \mathcal{A}_0(W_R) = \mathcal{A}_0(\bar{W}_R) \tag{78a}$$

with analogous identities for the corresponding objects associated with W_L.

$$\mathcal{C}(W_R) \subset \mathcal{A}_0(W_R) \subset \mathcal{C}_w(W_R) \subset \mathcal{G}(W_R) = \mathcal{C}(W_L)^q \tag{78b}$$

b) The von Neumann algebra $\mathcal{C}(W_R)$ is <u>locally associated with</u> W_R, and the association is <u>TCP-symmetric</u>, in the sense of Definition 2.

c) The set $\mathcal{C}_w(W_R)$ and the von Neumann algebra $\mathcal{G}(W_R)$ are <u>covariantly associated with</u> W_R, and the association is <u>TCP-symmetric</u>, in the sense of Definition 2.

d) For every $X \in \mathcal{C}_w(W_R)$ (and hence for every X in $\mathcal{C}(W_R)$ or $\mathcal{A}_0(W_R)$) we have

$$X\Omega \in D_+ \quad , \quad V(i\pi)\, X\,\Omega = J\, X^*\Omega \tag{79}$$

e) The von Neumann algebra $\mathcal{A}_0(W_R)$ satisfies the conditions:

$$\mathcal{A}_0(W_L) = \Theta_0 \mathcal{A}_0(W_R)\, \Theta_0^{-1} = U(u(\underset{\sim}{e}_1,\pi),0)\mathcal{A}_0(W_R)\; U(u(\underset{\sim}{e}_1,\pi),0)^{-1} \tag{80a}$$

and

$$U(\lambda)\mathcal{A}_0(W_R)\, U(\lambda)^{-1} = \mathcal{A}_0(W_R) \tag{80b}$$

for all $\lambda \varepsilon \bar{g}$ such that $\Lambda(\lambda)\, W_R = W_R$, i.e., for all Poincaré transformations which map W_R onto W_R.

f)

$$[X, JX_wJ]\,\Omega = 0 \tag{81}$$

for all $X \varepsilon \mathcal{A}_0(W_R)$, $X_w \varepsilon \mathcal{C}_w(W_R)$.

Proof: 1) We consider the identities (78a). Let $x \varepsilon W_R$. Then we have $\mathcal{C}(\bar{W}_R) \supset \mathcal{C}(W_R) \supset T(x)\mathcal{C}(\bar{W}_R)\, T(x)^{-1}$, in view of the fact that $\mathcal{C}(R)$ satisfies the condition of isotony. Since $\mathcal{C}(R)$ is weakly closed, and since $T(x)$ is a strongly continuous function of x, it follows at once that the first identity in (78a) holds. The next two identities are proved by exactly the same reasoning. The fourth identity follows from the second, and from the definition of $\mathcal{A}_0(R)$ in terms of $\mathcal{C}_w(R)$.

2) The inclusion relations between the first three sets at left in (78b) correspond to (72) in Lemma 10. The assertions e) also follow from Lemma 10. (Note that we do not assert that (80b) holds for all Poincaré transformations λ which map W_R into W_R). The assertion c) is trivial.

3) The relation $\mathcal{C}_w(W_R) \subset \mathcal{C}(W_L)^q$ is not trivial; it is equivalent to the condition that all operators in $\mathcal{C}_w(W_R)$ commute with all operators in $\mathcal{C}(W_L)^z$. To prove this relation we first consider the assertion d) of the lemma. The relations (79) follows readily from the definition of $\mathcal{C}_w(W_R)$, and Lemma 13 in BW I.(In this argument we depend, of course, ulti-

mately on Theorem 1 of the present paper in place of Theorem 1 in BW I.)

4) Let $X \varepsilon \mathcal{A}_0(W_R)$ and let $X_w \varepsilon \mathcal{C}_w(W_R)$. Since, by c) above, $\mathcal{C}_w(W_R)$ is invariant under conjugation by $V(t)$, it follows that $X\, V(t)\, X_w^*\, V(t)^{-1} \;\varepsilon\; \mathcal{C}_w(W_R)$ for all real t. In view of d) above it then follows from Lemma 14 in BW I that the relation (81) holds.

5) Let $X \varepsilon \mathcal{C}(W_R)$, and let $X_w \varepsilon \mathcal{C}_w(W_R)$. We write $Y = ZJX_wJZ^{-1}$, and we then have $Y \varepsilon \mathcal{C}_w(W_L)$. Let $x \varepsilon W_R$, and let $X(x) = T(x)\, X\, T(x)^{-1}$. Then
$X(x) \varepsilon \mathcal{C}(W_R)$, and (81) holds with X replaced by $X(x)$. We consider the special cases when each one of the operators X and Y is either a boson operator (i.e., a bounded operator which commutes with U_0), or else a fermion operator (i.e., a bounded operator which anti-commutes with U_0). The relation (81) then implies that

$$(\,X(x)\,Y \;+ s\,Y\,X(x))\,\Omega \;= 0 \qquad (82)$$

where $s = +1$ if both X and Y are fermion operators, and $s = -1$ if at least one of the operators X and Y is a boson operator.

We note that the operator $Q(x) = X(x)\,Y + s\,Y\,X(x)$ is included in the set $\mathcal{C}_w(R)$, where $R = W_L \cup \Lambda(I,x)W_R$; this follows from Lemma 9 since $X(x) \,\varepsilon\, \mathcal{C}(\Lambda(I,x)W_R) \subset \mathcal{C}(R)$ and $Y \varepsilon \mathcal{C}_w(W_L) \subset \mathcal{C}_w(R)$. Since the interior of R^c is nonempty it follows from Lemma 9 that $Q(x) = 0$. Since $Q(x)$ is a strongly continuous function of x we conclude that $(XY + sYX) = Q(0) = 0$. This in turn implies that

$[X, JX_wJ] = 0$. From the fact that this relation holds in the special cases considered it readily follows that it holds for all $X \varepsilon \mathcal{C}(W_R)$, $X_w \varepsilon \mathcal{C}_w(W_R)$. This means that

$\mathcal{C}_w(W_R) \subset \mathcal{C}(W_L)^q = \mathcal{G}(W_R)$, as asserted in (78b). This completes the proof of the lemma.

The relations (78a) should be carefully noted. The algebraic objects appearing in these relations are thus the same for the closed wedge $\bar{W}_R$ as for the open wedge W_R, which fact leads to a considerable simplification of the subsequent discussion. We employ a notation in the following according to which the objects are labeled by the open wedges W_R and W_L.

The facts stated in part b) of the lemma correspond, in a sense, to a well-known result of Borchers concerning the local nature of quantum fields which are local relative to an irreducible set of local fields. [15)]

Theorem 3: Let the notation be as in Definition 3, and Lemmas 10 and 11.

A) If the quantum fields are such that $\mathcal{A}_0(W_R)\Omega$ is dense in the Hilbert space $\mathcal{H}$, then $\mathcal{A}_0(W_R)$ is locally associated with W_R, and the association is TCP-symmetric, in the sense of Definition 2. Furthermore $\mathcal{A}_0(W_R)$ satisfies the condition of duality, and

$$\mathcal{C}(W_R) \subset \mathcal{A}_0(W_R) = \mathcal{C}_w(W_R) = \mathcal{A}_0(W_L)^q \subset \mathcal{G}(W_R) \qquad (83)$$

B) If the quantum fields are such that there exists a von Neumann algebra $\mathcal{A}(W_R) \subset \mathcal{C}_w(W_R)$ such that $\mathcal{A}(W_R)\Omega$ is dense, and such that $\mathcal{A}(W_R)$ is either locally associated with W_R, or else covariantly and TCP-symmetrically associated with W_R, in the sense of Definition 2, then:

a) The algebra $\mathcal{A}(W_R)$ is locally, and TCP-symmetrically, associated with W_R. Furthermore $\mathcal{A}(W_R)$ satisfies the condition of duality, and

$$\mathcal{A}_0(W_R) \subset \mathcal{A}(W_R) = \mathcal{A}(W_L)^q \subset \mathcal{C}_w(W_R) \tag{84a}$$

where

$$\mathcal{A}(W_L) = U(u(\underset{\sim}{e}_1,\pi),0)\ \mathcal{A}(W_R)\ U(u(\underset{\sim}{e}_1,\pi),0)^{-1} \tag{84b}$$

as in Definition 2. The relation

$\mathcal{A}_0(W_R) = \mathcal{A}(W_R)$ holds if and only if $\mathcal{A}_0(W_R)\Omega$ is dense.

b) The algebra $\mathcal{A}(W_R)$ is a factor, with Ω as a cyclic and separating vector. For any $X \varepsilon \mathcal{A}(W_R)$,

$$X\Omega \ \varepsilon\ D_+ \quad , \quad V(i\pi)\, X\Omega = J\, X^*\Omega \tag{85a}$$

and

$$J\, \mathcal{A}(W_R)\, J = \mathcal{A}(W_R)' \tag{85b}$$

c) There exists an extension of the operators in $\mathcal{P}(W_R)$ defined by

$$(X,D_1) \longrightarrow (e_R(X),D_{1R}) = (X^{\dagger *},\ D_{1R}) \tag{86a}$$

where

$$D_{1R} = \text{span}\ \{\ Y\phi\ \big|\ Y\ \varepsilon\ \mathcal{A}(W_L),\quad \phi\ \varepsilon\ D_1\ \} \tag{86b}$$

such that the extension satisfies the conditions

$$(X^\dagger,D_1)^* \supset (e_R(X)^*,D_{1R})^* \supset (e_R(X),D_{1R}) \supset (X,D_1) \tag{86c}$$

The mapping $(X,D_1) \rightarrow (e_R(X),D_{1R})$ of $\mathcal{P}(W_R)$ onto the set $\mathcal{P}_e(W_R)$ of the extended operators is a continuous *-representation in the sense described in Lemma 10.

The closures and adjoints of all operators $(e_R(X),D_{1R}) \in \mathcal{P}_e(W_R)$ are affiliated to the von Neumann algebra $\mathcal{A}(W_R)$.

d) The weak quasicommutant $\mathcal{C}_{we}(W_L)$ of $\mathcal{P}_e(W_R)$ relative to the domain D_{1R}, i.e., the set of all bounded operators Y such that for all $(X,D_1) \in \mathcal{P}(W_R)$,

$$Y^z \, (e_R(X)^*,D_{1R}) \subset (e_R(X),D_{1R})^* \, Y^z \tag{87}$$

is precisely equal to the quasicommutant $\mathcal{A}(W_L)$ of $\mathcal{P}_e(W_R)$.

Proof: 1) Let $\mathcal{A}(W_R)$ be a von Neumann algebra such that $\mathcal{A}(W_R) \subset \mathcal{C}_w(W_R)$ and $V(t)\mathcal{A}(W_R)\,V(t)^{-1} = \mathcal{A}(W_R)$ for all real t. The algebra $\mathcal{A}_0(W_R)$, in particular, satisfies these conditions, in view of Lemma 11. If now $\mathcal{A}(W_R)\Omega$ is dense, then it follows from Theorem 2 in BW I that (85a) and (85b) hold. It furthermore follows from Lemma 15 in BW I that $\mathcal{A}(W_R)$ is a factor. We have thus proved the assertions Bb).

2) We consider the relation (81) in Lemma 11, with $X_w = X_1X_2$, where X_1 and X_2 are elements of a von Neumann algebra $\mathcal{A}(W_R)$ which satisfies the premises in step 1) above, and where $X \in \mathcal{A}_0(W_R)$. By repeated application of (81) it readily follows that $[X, JX_1J]JX_2\Omega = 0$, and if $\mathcal{A}(W_R)\Omega$ is dense it follows that $[X, JX_1J] = 0$ for all $X \in \mathcal{A}_0(W_R)$, $X_1 \in \mathcal{A}(W_R)$. In view of (85b) this implies that $\mathcal{A}_0(W_R) \subset \mathcal{A}(W_R)$, as asserted in (84a).

3) We consider again the relation (81), with $X = X_3X_4$, where

X_3, $X_4 \in \mathcal{A}_0(W_R)$, and $X_w \in \mathcal{C}_w(W_R)$. By repeated application of (81) we easily show that

$$[\, X_3, JX_wJ \,]\, X_4\,\Omega \quad = 0 \tag{88}$$

In the particular case that $\mathcal{A}_0(W_R)\Omega$ is dense the relation (88) implies that $\mathcal{C}_w(W_R) \subset (J\mathcal{A}_0(W_R)\,J)' = \mathcal{A}_0(W_R)$, where the equality between the last two members follows from step 1) above. In view of (78b) in Lemma 11 it then follows that the relations (83) hold. We have thus shown that the premises in A) imply the relations (83). Since $\mathcal{C}_w(W_R)$ is covariantly associated with W_R we then conclude that $\mathcal{A}_0(W_R)$ is locally associated with W_R. We have thus proved the assertions A).

4) We consider a von Neumann algebra $\mathcal{A}(W_R)$ which satisfies the premises in part B). If $\mathcal{A}(W_R)$ is locally associated with W_R, then $\mathcal{A}(W_L) \subset \mathcal{A}(W_R)^q = (\mathcal{A}(W_R)')^z = (J\mathcal{A}(W_R)\,J)^z$ in view of (85b), and this means that the association of $\mathcal{A}(W_R)$ with W_R is TCP-symmetric. Conversely, if $\mathcal{A}(W_R)$ is TCP-symmetrically associated with W_R, then (85b) implies at once that $\mathcal{A}(W_R) = \mathcal{A}(W_L)^q$, and in particular the association is local. It readily follows from the results in steps 2) and 3) above that $\mathcal{A}_0(W_R) = \mathcal{A}(W_R)$ if and only if $\mathcal{A}_0(W_R)\Omega$ is dense. We have thus proved the assertions Ba).

5) The assertions Bc) are proved in the same manner as the corresponding assertions about the extension $(Y,D_1) \longrightarrow (a(Y),D_a)$ in Lemma 10, and we need not repeat the arguments.

6) We finally consider the assertion d). It readily follows

from (87) that a bounded operator Y_w is in $\mathcal{C}_{we}(W_L)$ if and only if $Y_1 Y_w Y_2 \; \varepsilon \; \mathcal{C}_w(W_L)$ for all $Y_1, Y_2 \; \varepsilon \; \mathcal{A}(W_L)$. We can restate this as follows. The operator X_w is in $(J\,\mathcal{C}_{we}(W_L)\,J)^z$ if and only if $X_1 X_w X_2 \; \varepsilon \; \mathcal{C}_w(W_R)$ for all $X_1, X_2 \; \varepsilon \; \mathcal{A}(W_R)$.

An operator X_w which satisfies the above condition is thus included in $\mathcal{C}_w(W_R)$. By the same reasoning as in the proof of (81) in Lemma 11 we show that $[\,X, JX_wJ\,]\,\Omega = 0$ for all $X \; \varepsilon \; \mathcal{A}(W_R)$, $X_w \; \varepsilon \; (J\,\mathcal{C}_{we}(W_L)\,J)^z$. By the same reasoning as in step 3) in the present proof we conclude that $[\,X, JX_wJ\,] = 0$, which means that $\mathcal{C}_{we}(W_L)^z \subset \mathcal{A}(W_R)' = \mathcal{A}(W_L)^z$. Since the set $\mathcal{A}(W_L)$ is trivially included in $\mathcal{C}_{we}(W_L)$ it follows that the two sets are equal, as asserted.

This completes the proof of the theorem. We postpone the discussion of this result until after the next theorem.

Theorem 4: Let the notation be as in Theorem 3 (i.e., as in Definition 3 and Lemma 10).

a) The following six conditions are equivalent:

1) $$\mathcal{G}(W_R) \subset \mathcal{G}(W_L)^q \tag{89a}$$

2) $$\mathcal{C}(W_R) = \mathcal{C}(W_L)^q \tag{89b}$$

3) $$\mathcal{G}(W_R) \subset \mathcal{C}_w(W_R) \tag{89c}$$

4) Ω is a cyclic vector for $\mathcal{C}(W_R)$.

5) Ω is a separating vector for $\mathcal{G}(W_R)$.

6) $\mathcal{G}(W_R)\Omega \subset D_+$, and

$$V(i\pi)\, X\Omega \quad = \quad J\, X^{*}\Omega \tag{89d}$$

for all $X \varepsilon \mathcal{G}(W_R)$.

b) If these conditions are satisfied, then

$$\mathcal{A}_0(W_R) = \mathcal{C}(W_R) = \mathcal{C}_w(W_R) = \mathcal{G}(W_R) \tag{90}$$

The von Neumann algebra $\mathcal{A}_0(W_R)$ satisfies the premises of part A) of Theorem 3, and all the conclusions of that theorem apply. In particular $\mathcal{A}_0(W_R)$ is a factor with Ω as a cyclic and separating vector. It is locally and TCP-symmetrically associated with W_R, and it satisfies the condition of duality.

Proof: 1) We first note that since $\mathcal{G}(W_R)\Omega$ is dense by part e) of Lemma 9, the relations (90) imply that $\mathcal{A}_0(W_R)$ satisfies the premises of part A) of Theorem 3, and it then follows trivially from that theorem that the six conditions in part a) of the present theorem are satisfied.

2) Since $\mathcal{G}(W_L)^q = \mathcal{C}(W_R)$ the condition (89a), in view of (78b) in Lemma 11, at once implies the conditions (90). Similarly (89b) implies (90). The condition (89c) implies, in view of (78b), that $\mathcal{C}_w(W_R) = \mathcal{G}(W_R)$, and hence $\mathcal{C}_w(W_R)$ is a von Neumann algebra, which, by the definition of $\mathcal{A}_0(W_R)$ must be equal to $\mathcal{A}_0(W_R)$. Since this von Neumann algebra now has Ω as a cyclic vector it readily follows from Theorem 3 that all the conditions (90) hold.

3) The conditions 4) and 5) in part a) of the theorem are obviously equivalent. In condition 4) holds, then $\mathcal{A}(W_R) = \mathcal{C}(W_R)$ satisfies the premises of part B) of Theorem 3, and it follows tri-

vially that the conditions (90) are satisfied.

4) If condition 6) is satisfied it follows from Theorem 2 in BW I that $J \mathcal{G}(W_R) J = \mathcal{G}(W_R)'$, which implies (89b), and hence (90). This completes the proof.

As the symbolism in Theorems 3 and 4, and in the preceeding Lemmas, might appear bewildering we shall now discuss the situation in plain English. Part b) of Theorem 4 describes what we regard as highly desirable properties of a quantum field theory, and these

properties are thus implied by either one of the six equivalent conditions in part a). We consider the first of these, namely the relation (89a). The von Neumann algebra $\mathcal{G}(W_R)$ is "generated" by the quantum fields $(\varphi_\mu[f], D_1)$ with the support of f in the right wedge W_R , and $\mathcal{G}(W_L)$ is defined analogously. The condition (89a) is simply the condition that these algebras are local, i.e., one is contained in the quasicommutant of the other. These algebras are always sufficiently "large" in the sense that each one of them has the vacuum vector as a cyclic vector, and according to (78) in Lemma 11 it is always the case that the quasicommutant of either one is contained in the other. We do not know, however, whether (89a) holds generally; in a particular field theory it could be the case that these algebras are "too large" in the sense that they fail to be locally associated with the wedges. The theorem now shows that the condition that the algebra $\mathcal{G}(W_R)$ not be too large in the above sense is precisely the condition that Ω is a separating vector for $\mathcal{G}(W_R)$, i.e., the condition that $\mathcal{G}(W_R)$

does not contain any nonzero operators which annihilate the vacuum vector.

The algebra $\mathcal{C}(W_R)$ is defined as a "strong" quasicommutant of the field operators $(\varphi_\mu[f], D_1)$, with $supp(f) \subset W_L$, i.e., $\mathcal{C}(W_R)$ is precisely equal to the set of all bounded operators which commute with the closures of the operators $(\varphi_\mu[f], D_1)^z$, $supp(f) \subset W_L$, in the strong sense of von Neumann. The algebra $\mathcal{C}(W_R)$ is then trivially equal to the quasicommutant of $\mathcal{G}(W_L)$. According to Lemma 11 the algebra $\mathcal{C}(W_R)$ is always locally associated with W_R, and the association is furthermore TCP-symmetric. These circumstances correspond to a well-known result of Borchers which we referred to earlier.[15)] The algebra $\mathcal{C}(W_R)$ is a reasonable choice for "the algebra of all bounded operators locally associated with W_R" unless it so happens that this algebra is "too small" in the sense that it fails to satisfy the duality condition. By the theorem the algebra is too small in the above sense if and only if it does not have the vacuum vector as a cyclic vector, i.e., if and only if $\mathcal{C}(W_R)\Omega$ is a proper subspace of the Hilbert space $\mathcal{H}$.

We have already discussed (following Lemma 10) the possible physical interpretation of the set $\mathcal{C}_w(W_R)$, defined (in Definition 3) as the "weak quasicommutant" of all the operators in $\mathcal{P}(W_L)$. Now it is interesting to note that, by Lemma 11, the wedge-region W_R has the special property that $\mathcal{C}_w(W_R)$ is included in $\mathcal{G}(W_R)$. This result, which we derived on the basis of Theorem 1, is not a triviality in our opinion. We also know that an analogous inclusion relation does not hold for arbitrary open regions R. It is furthermore interesting to note that, by Theorem 4, the seemingly weak condition $\mathcal{G}(W_R) \subset \mathcal{C}_w(W_R)$,i.e., the condition that the operators in $\mathcal{G}(W_R)$ commute at least in the weak sense of (64) with the operators $(\varphi_\mu[f], D_1)^z$ for which $\text{supp}(f) \subset W_L$, in fact implies that $\mathcal{C}(W_R) = \mathcal{C}_w(W_R) = \mathcal{G}(W_R)$, i.e., that $\mathcal{C}_w(W_R)$ is a von Neumann algebra, identical with $\mathcal{G}(W_R)$, and that $\mathcal{G}(W_R)$ is locally associated with W_R and satisfies the condition of duality. This result is also ultimately based on Theorem 1, and it does not seem to follow from some more trivial considerations.

We do not know at this time whether $\mathcal{C}_w(W_R)$ is always a von Neumann algebra, i.e., closed under multiplication, without further conditions on the quantum fields. The set $\mathcal{C}_w(W_R)$ is trivially equal to the von Neumann algebra $\mathcal{C}(W_R)$ if $(X^\dagger, D_1)^* = (X, D_1)^{**}$ for all $(X, D_1) \in \mathcal{L}(W_L)$. One might thus say that the relation $\mathcal{C}_w(W_R) \neq \mathcal{C}(W_R)$ (if there are quantum field theories for which this is the case) in some sense reflects the inadequacy of the domain D_1 for the definition of the field operators. Let us here note that with our present understanding of the situation the equality $\mathcal{C}_w(W_R) = \mathcal{C}(W_R)$ does not by itself seem to imply the duality condition. In particular we have not shown that it might not happen that $\mathcal{C}_w(W_R)$ consists of multiples of the identity only.

The sixth condition in part a) of Theorem 4 is of a "technical" nature, without any immediate physical interpretation. We stated this condition because its form suggests a possible direct connection with Theorem 1. We note, for instance, that in the very special case that the vacuum vector is an analytic vector for the field operators $(\varphi_\mu[f] , D_1)$ (as is the case for a free field) then the sixth condition follows trivially from the facts in Theorem 1. We are not, however, here conjecturing that the sixth condition follows in general from Theorem 1 alone.

Even if the premises of Theorem 4 are not satisfied it is conceivable, according to Theorem 3, that the quantum fields nevertheless have extensions which are affiliated to von Neumann algebras which satisfy a duality condition, at least for the wedge-regions in $\mathcal{W}$. It is easily seen that if $(X, D_1) \longrightarrow (e_R(X), D_{1R})$ is an extension of a set of field operators which satisfies the condition (86c), then the weak quasicommutant (relative to D_{1R}) of the set of extended operators is necessarily contained in the weak quasicommutant of the original set. The premises in part B) of Theorem 3 thus seem to us to express minimal conditions which a "local" algebra "generated" by the fields must satisfy.

In Sec. VI of BW I we considered four particular conditions on the quantum field, called Conditions I-IV, which were shown to imply the duality condition for the wedge-regions. We shall not state the generalizations of these conditions here, but we assert that our earlier Conditions I, II and IV trivially imply the premises of Theorem 4, and that our Condition III implies the premises of part B) of Theorem 3.

VI. The duality condition for von Neumann algebras associated with double cones and their causal complements.

In this section we shall generalize the discussion in Sec. VII of BW I. We shall thus consider the construction of von Neumann algebras locally associated with a particular family of regions, namely double cones and their causal complements, in terms of a von Neumann algebra $\mathcal{A}(W_R)$ locally associated with W_R. The scheme is the same as in BW I.

Definition 4: Let the von Neumann algebra $\mathcal{A}(W_R)$ be locally associated with W_R, in the sense of Definition 2.

a) For any $W \varepsilon \mathcal{W}$, i.e., for any wedge-region W bounded by two non-parallel characteristic planes, we define a von Neumann algebra $\mathcal{A}(W)$ by

$$\mathcal{A}(\Lambda(\lambda)W_R) = U(\lambda)\mathcal{A}(W_R)\,U(\lambda)^{-1} \quad , \text{ any } \lambda \varepsilon \bar{\mathcal{G}} \tag{91}$$

b) For any two points x_1 and x_2 in Minkowski space such that $x_2 \varepsilon V_+(x_1)$, (where $V_+(x_1)$ is the forward light cone with x_1 as apex), we define the double cone $C = C(x_1,x_2)$ by

$$C(x_1,x_2) = V_+(x_1) \cap V_-(x_2) \tag{92}$$

where $V_-(x_2)$ is the backward light cone with x_2 as apex. The double cones so defined are thus open and nonempty. We denote by $\mathcal{D}_c$ the set of all double cones.

For any double cone C we define a von Neumann algebra $\mathcal{B}(\bar{C})$ by

$$\mathcal{B}(\bar{C}) = \cap\{\mathcal{A}(W) \mid W \in \mathcal{W}, \; W \supset \bar{C}\} \tag{93}$$

c) For any $C \in \mathcal{D}_c$ we define the von Neumann algebra $\mathcal{A}(\bar{C}^c)$ by

$$\mathcal{A}(\bar{C}^c) = \{\mathcal{A}(W) \mid W \in \mathcal{W}, \; W \subset \bar{C}^c\}'' \tag{94}$$

d) A set of von Neumann algebras, defined as above, shall be called a <u>local AB-system</u>.

It is easily seen that the definition in part a) above is consistent, i.e., that the algebras defined by the right-hand side of (91) for two different λ', λ'', are equal whenever $\Lambda(\lambda')W_R = \Lambda(\lambda'')W_R$. We remark here that, as in BW I, we prefer to regard $\mathcal{B}(\bar{C})$ as associated with the closed set $\bar{C}$, and hence the above notation.

We shall next state a theorem corresponding to Theorem 5 and part of Theorem 6 in BW I.

<u>Theorem 5:</u> Given a local AB-system, defined as in Definition 4 in terms of a von Neumann algebra $\mathcal{A}(W_R)$ locally associated with W_R. Then:

a) The algebras in the AB-system satisfy the conditions of <u>covariance</u> and <u>isotony</u>, i.e., if $\mathcal{Q}(R)$ denotes $\mathcal{A}(R)$ or $\mathcal{B}(R)$, with the appropriate restriction on R, then the conditions (65a) and (65c) hold. Furthermore,

$$\mathcal{B}(\bar{C}_1) \subset \mathcal{A}(W) \subset \mathcal{A}(\bar{C}_2{}^c) \tag{95}$$

for all $W \in \mathcal{W}$, $C_1, C_2 \in \mathcal{D}_c$, such that $C_1 \subset W \subset \bar{C}_2^{\,c}$.

b) The algebras $\mathcal{B}(\bar{C})$ are <u>local</u>, in the sense that

$$\mathcal{B}(\bar{C}_1) \subset \mathcal{B}(\bar{C}_2)^q \tag{96a}$$

for any $C_1, C_2 \in \mathcal{D}_c$, such that $C_1 \subset \bar{C}_2^{\,c}$. Furthermore,

$$\mathcal{B}(\bar{C})^q \supset \mathcal{A}(\bar{C}^c) \tag{96b}$$

for any $C \in \mathcal{D}_c$.

c) The mapping $W \rightarrow \mathcal{A}(W)$ is <u>continuous from the outside</u> in the sense that

$$\mathcal{A}(W) = \cap \{ \mathcal{A}(W_0) \mid W_0 \in \mathcal{W} \, , \, W_0 \supset \bar{W} \} \tag{97a}$$

and it is <u>continuous from the inside</u> in the sense that

$$\mathcal{A}(W) = \{ \mathcal{A}(W_i) \mid W_i \in \mathcal{W} \, , \, \bar{W}_i \subset W \}'' \tag{97b}$$

The mapping $\bar{C} \rightarrow \mathcal{B}(\bar{C})$ is continuous from the outside in the sense that

$$\mathcal{B}(\bar{C}) = \cap \{ \mathcal{B}(\bar{C}_0) \mid C_0 \in \mathcal{D}_c \, , \, \bar{C} \subset C_0 \} \tag{97c}$$

The mapping $\bar{C}^c \rightarrow \mathcal{A}(\bar{C}^c)$ is continuous from the inside in the sense that

$$\mathcal{A}(\bar{C}^c) = \{ \mathcal{A}(\bar{C}_1{}^c) \mid C_1 \varepsilon \mathcal{D}_c , \; C_1 \supset \bar{C} \}'' \qquad (97d)$$

d) If the algebra $\mathcal{A}(W_R)$ satisfies, in addition, the condition of TCP-symmetry, as stated in Definition 2, then the AB-system is TCP-symmetric in the sense that

$$\Theta_0 \mathcal{A}(W) \Theta_0{}^{-1} = \mathcal{A}(-W) \quad , \quad \Theta_0 \mathcal{B}(\bar{C}) \Theta_0{}^{-1} = \mathcal{B}(-\bar{C})$$

$$\Theta_0 \mathcal{A}(\bar{C}^c) \Theta_0{}^{-1} = \mathcal{A}(-\bar{C}^c) \qquad (98)$$

for all $W \varepsilon \mathcal{W}$, $C \varepsilon \mathcal{D}_c$, and where $-R = \{ x \mid -x \varepsilon R \}$ for any subset R of Minkowski space.

e) If the algebra $\mathcal{A}(W_R)$ satisfies, in addition, the condition of duality, as stated in Definition 2, then the algebras $\mathcal{B}(\bar{C})$ satisfy a condition of duality in the sense that

$$\mathcal{B}(\bar{C})^q = \mathcal{A}(\bar{C}^c) \qquad (99)$$

for any $C \varepsilon \mathcal{D}_c$.

The assertions a)-d) in the theorem correspond to Theorem 5 in BW I, and the assertion e) to the assertion a) in Theorem 6 in BW I. The above assertions are proved by a very trivial modification of the reasoning whereby we proved the corresponding assertions in BW I, and we do not feel that it is necessary to repeat the arguments here. The modifications, of course, have to do with the circumstance that the locality conditions in the present theorem refer to the notion of a quasicommutant, rather than to the notion of a commutant as in BW I.

The above theorem is of interest because it shows how a "wedge-algebra" $\mathcal{A}(W_R)$ with physically desirable properties gives rise to a <u>system</u> of algebras(associated with other regions) with physically desirable properties, such as covariance, isotony, TCP-symmetry and duality. In our study of a general quantum field theory the crux of the matter is thus to establish the existence of an algebra $\mathcal{A}(W_R)$ which is locally associated with W_R, and which satisfies the conditions of TCP-symmetry and duality.

Now it should be noted that nothing said so far guarantees that $\mathcal{B}(\bar{C})$, for some particular $C \in \mathcal{D}_c$, contains other elements than multiples of the identity. In a physically satisfactory "local" theory it must clearly be the case that at least <u>some</u> of the algebras $\mathcal{B}(\bar{C})$ are nontrivial. In a quantum field theory one might in fact demand that <u>all</u> the algebras $\mathcal{B}(\bar{C})$ are nontrivial, and furthermore one might demand that the algebras $\mathcal{B}(\bar{C})$ associated with all $\bar{C} \subset \bar{C}_o^c$, for some C_o, should generate the algebra $\mathcal{A}(\bar{C}_o^{\,c})$. We shall show that this is in fact the case if the quantum fields satisfy the conditions in part a) of Theorem 4. We do not have a corresponding result for fields which merely satisfy the premises of Theorem 3. The situation in the latter case is complicated by the fact that the extensions of the field operators described in Theorem 3 depend on the region with which the operators are associated, and to clarify the situation it would be necessary to investigate the relationship between the domains of the extensions for different regions. This we have not done, and we shall therefore restrict our considerations to the case when the premises of Theorem 4 are satisfied. We note, however, that we do obtain a satisfactory local theory if the fields satisfy the premises of Theorem 3, and some additional condition which guarantees that $\mathcal{B}(\bar{C})\Omega$ is dense. We refer here to the assertions b) and d) in Theorem 6 in BW I, which can readily be generalized to the present situation. It is of interest to state the generalization of the first one of these assertions as follows.

<u>Theorem 6</u>: Let the von Neumann algebra $\mathcal{A}(W_R)$ satisfy the premises of Theorem 5, and let a local AB-system be defined in terms of $\mathcal{A}(W_R)$ as in Definition 4. Let $\mathcal{A}(W_R)$ satisfy the condition of duality, as well as the additional condition that

$$X\Omega \;\varepsilon\; D_+ \;, \qquad V(i\pi)\, X\Omega = J X^* \Omega \tag{100}$$

for all $X \;\varepsilon\; \mathcal{A}(W_R)$.

If there exists a double cone C_0 such that $\mathcal{B}(\bar{C}_0)\Omega$ is dense in the Hilbert space $\mathcal{H}$, then:

$$\mathcal{A}(\bar{C}_1{}^c) = \{\mathcal{B}(\bar{C}) \mid C \;\varepsilon\; \mathcal{D}_c \;,\; \bar{C} \subset \bar{C}_1{}^c\}'' \tag{101a}$$

for every $C_1 \;\varepsilon\; \mathcal{D}_c$, and

$$\mathcal{A}(W) = \{ \mathfrak{B}(\Lambda \bar{C}_0) \mid \Lambda \in \bar{L}_0 \ , \Lambda \bar{C}_0 \subset W \}'' \qquad (101b)$$

$$\mathcal{A}(\bar{C}_1{}^c) = \{ \mathfrak{B}(\Lambda \bar{C}_0) \mid \Lambda \in \bar{L}_0 \ , \Lambda \bar{C}_0 \subset \bar{C}_1{}^c \}'' \qquad (101c)$$

for every $C_1 \in \mathcal{D}_c$, $W \in \mathcal{W}$. If furthermore $\bar{C}_0 \subset W_R$, then

$$\mathcal{A}(W_R) = \{ V(t) \mathfrak{B}(\bar{C}_0) V(t)^{-1} \mid t \in R^1 \}'' \qquad (101d)$$

These assertions are proved by the same reasoning as in our proof of the corresponding assertions in Theorem 6 in BW I, and we shall not repeat the arguments. We note here that the premises of the theorem at once imply that Ω is a cyclic and separating vector for $\mathcal{A}(W_R)$, as well as for $\mathfrak{B}(\bar{C}_0)$. We furthermore note that the condition (100) is not required for the conclusion in part e) of Theorem 5. It is, however, essential for the present theorem, and in particular for the conclusion (101d). We refer here to our discussion in Sec. V of BW I of the connection between our considerations and the Tomita-Takesaki theory of modular Hilbert algebras. [16] The relation (101d) can thus be understood with reference to the fact that because of (100) the group $\{ V(t) \mid t \in R^1 \}$ is precisely the modular automorphism group for $\mathcal{A}(W_R)$.

In preparation for Theorem 7 we prove a lemma about the nature of the weak quasicommutant $\mathcal{C}_w(R)$ in the special case that R is the closure of a double cone in $\mathcal{D}_c$.

<u>Lemma 12</u>: Let $C \in \mathcal{D}_o$. Then

$$\mathcal{C}_w(\bar{C}) = \cap \{ \mathcal{C}_w(W) \mid W \in \mathcal{W} \ , \ W \supset \bar{C} \} \tag{102}$$

<u>Proof</u>. 1) Let $\mathcal{C}_1$ denote the set defined by the right side of (102). It is at once obvious that $\mathcal{C}_w(\bar{C}) \subset \mathcal{C}_1$, and we thus have to prove that if $X \in \mathcal{C}_1$, then $X \in \mathcal{C}_w(\bar{C})$.

2) Let $\mu \in I_T$, and let $f(x) \in \mathcal{D}(R^4)$ such that $\text{supp}(f) = R_o \subset \bar{C}^c$. The support R_o of the test function f is thus a <u>compact</u> subset of the open set $\bar{C}^c$. For any x we denote by $b(x;\rho)$ the open ball of radius $\rho > 0$ centered at x (where Minkowski space is regarded as a Euclidean space with Cartesian coordinates $x = (x^1, x^2, x^3, x^4)$). Now, for each $x \in R_o$ we can select a $\rho(x) > 0$ such that $b(x;2\rho(x)) \subset W$ for some $W \in \mathcal{W}$ such that $W \subset \bar{C}^c$. The set $\{ b(x;\rho(x)) \mid x \in R_o \}$ of open balls covers R_o , and since R_o is compact this open covering contains a <u>finite</u> subcovering. There thus exists a finite set $\{ x_k \mid x_k \in R_o \ , \ k = 1,\ldots,n \}$ of points, and a set $\{ W_k \mid W_k \in \mathcal{W} \ , \ k = 1,\ldots,n \}$ of wedges, such that

$$R_o \subset \cup \{ b(x_k;\rho(x_k)) \mid k = 1,\ldots,n \} \tag{103a}$$

$$b(x_k;2\rho(x_k)) \subset W_k \subset \bar{C}^c \quad , \ k = 1,\ldots,n \tag{103b}$$

In view of (103a) there then exists a set $\{ g_k(x) \mid g_k \in \mathcal{D}(R^4) \ , \ k = 1,\ldots,n \}$ of functions such that $\text{supp}(g_k) \subset b(x_k;2\rho(x_k))$ for $k = 1,\ldots,n$, and

$$\sum_{k=1}^{n} g_k(x) = 1 \quad , \text{all} \quad x \in R_o \tag{103c}$$

Let $(Y,D_1) = (\varphi_\mu[f], D_1)$ and $(Y_k,D_1) = (\varphi_\mu[fg_k], D_1)$ for $k = 1,\ldots,n$. We then have

$$(Y,D_1) = \sum_{k=1}^{n} (Y_k,D_1) \tag{103d}$$

where $(Y_k,D_1) \in \mathcal{L}(W_k)$. If now $X \in \mathcal{C}_1$, then $X \in \mathcal{C}_w(\bar{W}_k{}^c)$ and hence X commutes in the weak sense (64) with $(Y_k,D_1)^z$ for $k = 1,\ldots,n$. It follows, in view of (103d) that

$$\langle (Y^\dagger)^z \phi \,|\, X\psi \rangle = \langle X^* \phi \,|\, Y^z \psi \rangle \tag{103e}$$

for all ϕ, $\psi \in D_1$.

3) For any $X \in \mathcal{C}_1$ the relation (103e) thus holds for all $(Y,D_1) = (\varphi_\mu[f], D_1) \in \mathcal{L}(\bar{C}^c)$ such that supp(f) is compact. The set $\mathcal{D}(R^4)$ is dense in $\mathcal{S}(R^4)$ in the topology of the space of tempered test functions, and since the quantum fields are operator-valued tempered distributions it readily follows that (103e) holds for <u>all</u> $(Y,D_1) = (\varphi_\mu[f], D_1) \in \mathcal{L}(\bar{C}^c)$ such that $f \in \mathcal{S}(R^4)$, $\text{supp}(f) \subset \bar{C}^c$, i.e., for all elements of $\mathcal{L}(\bar{C}^c)$. It then follows, in view of Lemma 9, part c), that $X \in \mathcal{C}_w(\bar{C})$. This, in effect, completes the proof of the Lemma.

We are now prepared to present the main result of this section.

<u>Theorem 7</u>: Let the quantum fields be such that the conditions in part a) of Theorem 4 are satisfied, i.e., the von Neumann algebra $\mathcal{A}(W_R) = \mathcal{A}_0(W_R)$ satisfies the relations

$$\mathcal{A}(W_R) = \mathcal{C}(W_R) = \mathcal{C}_w(W_R) = \mathcal{G}(W_R) \tag{104}$$

and hence the algebra is locally and TCP-symmetrically associated with W_R. Furthermore $\mathcal{A}(W_R)$ satisfies the condition of duality, and the conditions (100). Let a local AB-system be constructed from $\mathcal{A}(W_R)$, as in Definition 4. Then:

a) The algebra $\mathcal{A}(W_R)$ satisfies all the general and special premises of Theorems 5 and 6, and all the conclusions of these theorems apply. In particular $\mathcal{B}(\bar{C}_0)\Omega$ is dense for <u>any</u> $C_0 \varepsilon \mathcal{D}_c$. Furthermore, for any $C_0 \varepsilon \mathcal{D}_c$ such that $\bar{C}_0 \subset W_R$,

$$\mathcal{A}(W_R) = \{ V(t)\,\mathcal{G}(C_0)\,V(t)^{-1} \mid t \,\varepsilon\, R^1 \}'' \tag{105a}$$

$$\mathcal{A}(\bar{C}_1{}^c) = \{ \mathcal{G}(\Lambda C_0) \mid \Lambda \,\varepsilon\, \bar{L}_0 \;,\; \Lambda \bar{C}_0 \subset \bar{C}_1{}^c \}'' \tag{105b}$$

b) For any $C \,\varepsilon\, \mathcal{D}_c$,

$$\mathcal{C}(\bar{C}) \subset \mathcal{C}_w(\bar{C}) = \mathcal{B}(\bar{C}) \;, \qquad \mathcal{G}(\bar{C}) \subset \mathcal{B}(\bar{C}) \tag{106a}$$

$$\mathcal{C}_w(\bar{C}^c) \supset \mathcal{C}(\bar{C}^c) \supset \mathcal{A}(\bar{C}^c) \;, \qquad \mathcal{G}(\bar{C}^c) \supset \mathcal{A}(\bar{C}^c) \tag{106b}$$

c) With the notation of Lemma 10, $\mathcal{A}_0(\bar{C}) = \mathcal{C}_w(\bar{C}) = \mathcal{B}(\bar{C})$ for all $C \,\varepsilon\, \mathcal{D}_c$. For any such C the operators in $\mathcal{P}(\bar{C}^c)$ have extensions constructed as in part c) of Lemma 10, and these extensions have the properties described in the lemma. In particular the closures and adjoints of the extended operators are affiliated to the von Neumann algebra $\mathcal{A}(\bar{C}^c)$.

d) With the notation of Lemma 10, $\mathcal{C}_w(\bar{C}^c) \supset \mathcal{A}_0(\bar{C}^c) \supset \mathcal{C}(\bar{C}^c)$ for all $C \,\varepsilon\, \mathcal{D}_c$. For any such C the operators in $\mathcal{P}(\bar{C})$ have extensions constructed as in part c) of Lemma 10, and these extensions have the properties described in the lemma. In particular the closures and adjoints of the extended operators are affiliated to the von Neumann algebra $\mathcal{A}_0(\bar{C}^c)^q \subset \mathcal{G}(\bar{C}) \subset \mathcal{B}(\bar{C})$.

<u>Proof</u>:1)The algebra $\mathcal{A}(W_R)$ trivially satisfies the general premises of Theorem 5. From the construction of the AB-system, and from (104), it follows, in view of Lemma 12, that $\mathcal{C}_W(\bar{C}) = \mathcal{B}(\bar{C})$.

Since the mapping $R \rightarrow \mathcal{G}(R)$ satisfies the condition of isotony, the inclusion relation at right in (106a) follows from (104). The remaining relations (106a) and (106b) are then trivial.

2) Since, by Lemma 9, $\mathcal{G}(C)\Omega$ is dense for any $C \in \mathcal{D}_c$ it follows that $\mathcal{B}(\bar{C})\Omega$ is dense, as asserted in part a) of the theorem. Let now $C_0 \in \mathcal{D}_c$ and $\bar{C}_0 \subset W_R$. Let $\mathcal{A}_R$ denote the von Neumann algebra defined by the right member in (105a). The vector Ω is then a cyclic vector for $\mathcal{A}_R$, and in view of the construction we have $V(t)\mathcal{A}_R V(t)^{-1} = \mathcal{A}_R$ for all real t. Furthermore it is trivially the case that $\mathcal{A}(W_R) \supset \mathcal{A}_R$. It then follows from Theorem 2 in BW I that $\mathcal{A}(W_R) = \mathcal{A}_R$, as asserted in (105a). The relation (105b) follows trivially from the relation (105a).

3) The assertions c) and d) of the theorem are trivial in view of Lemma 10.

As we see from this theorem, a very satisfactory "local" theory results if the quantum fields satisfy the premises of Theorem 4, i.e., any one of the six conditions in part a) of that theorem. There thus exists a local AB-system which satisfies the condition of TCP-symmetry, and the condition of duality $\mathfrak{B}(\bar{C})^q = \mathcal{A}(\bar{C}^c)$. Furthermore, for any $C \varepsilon \mathcal{D}_c$, the von Neumann algebra $\mathfrak{B}(\bar{C})$ has Ω as a cyclic and separating vector. The relations (101a)-(101d) hold, which means that the set of local operators associated with the bounded regions C is sufficiently large in the sense that these operators generate all the algebras of the AB-system, as described by the relations (101a)-(101d). Now it is interesting to note that the algebra $\mathfrak{B}(\bar{C})$ is in fact equal to the weak quasicommutant $\mathcal{C}_w(\bar{C})$ of the set of all field operators of the form $(\varphi_\mu[f], D_1)$, where $f \varepsilon \mathcal{S}(R^4)$, $\mathrm{supp}(f) \subset \bar{C}^c$. We thus have a conceptually simple prescription for "finding" the algebras $\mathfrak{B}(\bar{C})$ provided that it has first been established that the quantum fields do satisfy the premises of Theorem 4.

We note here that this is the case under what we called Condition I in BW I, because this condition says that $\mathcal{C}(W_R)\Omega$ is dense. It follows that all the conclusions in Theorem 7 hold under our earlier Condition I. We overlooked this fact in our previous paper.

We infer from the work of Landau [17)] that $\mathcal{G}(\bar{C})$ is in general smaller than $\mathfrak{B}(\bar{C})$. The study of Landau is concerned with generalized free fields, in which case we have the further simplification that $\mathcal{C}_w(R) = \mathcal{C}(R)$ for any subset R of $\mathcal{M}$. We then have $\mathcal{A}(\bar{C}^c) = \mathcal{G}(\bar{C}^c)$ and $\mathfrak{B}(\bar{C}) = \mathcal{C}(\bar{C})$, but it can well happen that $\mathcal{G}(\bar{C}) \neq \mathfrak{B}(\bar{C})$.

We conclude by stating a theorem about local internal symmetries.

<u>Theorem 8:</u> Let $\mathcal{A}(W_R)$ be a von Neumann algebra locally and TCP-symmetrically associated with W_R. It is assumed that $\mathcal{A}(W_R)$ satisfies the condition of duality, and that furthermore

$$X\,\Omega \subset D_+ \quad , \quad V(i\pi)\,X\,\Omega = J\,X^*\,\Omega \tag{107}$$

for all $X \in \mathcal{A}(W_R)$. Let a local AB-system be constructed in terms of $\mathcal{A}(W_R)$ as in Definition 4.

Let G be a unitary operator such that

$$G\Omega = \Omega \quad , \quad G\,\mathcal{A}(W)\,G^{-1} = \mathcal{A}(W)\ , \text{ all } W \in \mathcal{W} \tag{108a}$$

Then:

a) The operator G commutes with the TCP-transformation, and with all Poincaré transformations, i.e.,

$$\Theta_0 G\,\Theta_0 = G \quad , \quad U(\lambda)\,G\,U(\lambda)^{-1} = G \quad , \text{ all } \lambda \in \bar{\mathcal{G}} \tag{108b}$$

b) For all double-cones C,

$$G\,\mathcal{B}(\bar{C})G^{-1} = \mathcal{B}(\bar{C}) \quad , \quad G\mathcal{A}(\bar{C}^c)G^{-1} = \mathcal{A}(\bar{C}^c) \tag{108c}$$

c) The set of all unitary operators G which satisfy the conditions (108a) forms a group, <u>the group of all local internal symmetries</u>.

This theorem is proved by the same reasoning as in our proof of the corresponding Theorem 7 in BW I, and it is not necessary to repeat the arguments here. We note here that the conclusions of the theorem do not follow (as far as we know) merely from the assumptions that $\mathcal{A}(W_R)$ satisfies the condition of duality and is locally and TCP-symmetrically associated with W_R. Our proof in BW I depends on the specific conditions (107), which presumably characterize local von Neumann algebras in a quantum field theory. Without the conditions (107) it can be shown [18] that G commutes with all translations, but it appears that further assumptions are necessary for the conclusion that G also commutes with homogeneous Lorentz transformations. [19]

We finally note that the "group of all local internal symmetries," as defined above, will in general include superselection symmetries with no observable physical effects.

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References and footnotes.

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name: in certain kinds of field theories there could conceivably exist other antiunitary transformations which deserve the name better.

9) Our term "quasicommutant" in Definition 1 has no official standing in the literature. We felt a need for a term to describe the relationship in question, and we selected the term quasicommutant (not without some misgivings) as one which to some extent suggests the relationship, and as one unencumbered by well-established mathematical connotations.

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