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STRING-LIKE SOLUTIONS IN THE SKYRME MODEL†

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ABSTRACT: The equations of motion of the Skyrme model with a mass term are studied, and a variety of new solutions found, including a string-like solution having cylindrical symmetry, and a slab-like solution. In addition, it is demonstrated that the hedgehog (skyrmion) solves the equations. The new solutions are classically unstable, having vanishing topological winding density. The string-like solutions may be relevant to nuclear physics since one may speculate that these solutions are related to the QCD string. The string tension of the cylindrical solution is found to be approximately 0.85 GeV/fm. Although the slab-like solutions appear less experimentally accessible, they might be relevant to high-energy nuclear collisions. The slab-like solution has a counterpart in $2+1$ dimensional Skyrme models, where it appears as a string-like object. In this context, this solution may have applications in solid-state physics.

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1. INTRODUCTION

Interest in the large-N expansion of Q.C.D. [1,2] has lead to a revival [3,4,5] of the Skyrme model of the nucleon [6]. One can ask if there are large amplitude solutions in the model other than the Skyrmion, that might have some relevance to hadronic and nuclear physics. In the past, most efforts in this respect have been confined to solutions with higher baryon number [7]. In addition, a few investigations of the baryon-antibaron system have been made [8]. In this paper, we study the equations of motion, in particular we look for large-amplitude static solutions that have zero baryon density (although we will touch on the hedgehog solution briefly, and also find a time-dependent solution). By restricting the gradients of the field to be diagonalizable, the equations of motion simplify tremendously. It is then trivial to show *e.g.* that the hedgehog satisfies the equations of motion. If, further, we restrict the dimensional character of the field in internal space, we find that further simplifications occur. Utilizing these methods, we find a variety of new solutions to the equations of motion, including a string-like object and a slab-like object. Since these solutions have zero winding density, they are classically unstable. However, we will argue that the string solution is physically relevant to nuclear and hadronic physics, being closely related to the Q.C.D. string - in this connection the decay of the string solution is of special interest, and these arguments are presented in more detail elsewhere [9]. The string tension is found to be of the order 0.85 GeV/fm with the parameters of Jackson and Rho [4] . We investigate the limitations of the present form of the string, arising from its radial instability in the chiral limit. Further work aimed at remedying these defects is in progress [10]. It would be of considerable interest if the phenomenology of the Skyrme model could be widened to include the plethora of applications of Q.C.D. strings.

The slab-like solutions are not likely to be so relevant to nuclear physics, since the physical circumstances which might give rise to such solutions would be hard (although perhaps not impossible - see section 4) to create. However, a similar solution exists in the $2+1$ dimensional version of the Skyrme model, where this solution takes on a string-like character [11].

The paper is organized as follows, in section 2 we discuss the static equations of motion in full generality and outline conditions on the form of the field that enable one to simplify them.

We consider fields that cover subspaces of dimension different from that of the internal space, and derive the equations of motion for these cases.

In section 3 we discuss solutions to the equations derived in section 1. In particular we confirm that the hedgehog (skyrmion) solves the equations of motion. Then we discuss the string solution, and show that it is essentially a two-dimensional skyrmion embedded in three dimensions. Finally we discuss three solutions covering one-dimensional subspaces of the internal space. The first is the Sine-Gordon soliton embedded in three dimensions and consists of a wall of energy density, while the others have the character of "a particle spinning in a bowl".

In section 4 we discuss the physical relevance of the new solutions and point out some directions for future work.

2. THE EQUATIONS OF MOTION

We consider the static energy density arising from Skyrme's lagrangian [6] with a mass term

$$\mathcal{E} = \frac{f_\pi^2}{2} B^i_\mu B^i_\mu + 2\epsilon^2 C^i_{\mu\nu} C^i_{\mu\nu} + m_\pi^2 f_\pi^2 \frac{Tr}{2}(1-U) \ , \tag{2.1}$$

where

$$U = \cos(f) + i\tau^i \hat{\pi}^i \sin(f), \quad U^\dagger U = 1 \ , \tag{2.2}$$

$$B^i_\mu = -i\frac{Tr}{2}(\tau^i U^\dagger \partial_\mu U) \ , \tag{2.3}$$

$$C^i_{\mu\nu} = \epsilon^{ijk} B^j_\mu B^k_\nu , \tag{2.4}$$

and, f_π and m_π are the decay constant and mass of the pion, respectively. Unless stated, we use the convention that all greek indices are spatial and all latin indices are internal, and there are no minus signs associated with summation of the spatial indices.

The full equations of motion arising from (2.1) are

$$\partial_\mu[aB^i_\mu + bB^j_\nu(B^i_\mu B^j_\nu - B^j_\mu B^i_\nu)] - m^2_\pi f^2_\pi \sin(f)\hat{\pi}^i = 0 \ , \qquad (2.5)$$

where $a = f^2_\pi$ and $b = 8\epsilon^2$.

We first examine the form of the gradients of the field. The gradients (2.4) will in general be a three-by-three matrix, and we write

$$B^i_\mu = \hat{m}_\mu(u\hat{B}(1)^i + s\hat{B}(2)^i + o\hat{B}(3)^i) + \hat{n}_\mu(t\hat{B}(1)^i + v\hat{B}(2)^i + q\hat{B}(3)^i) + \hat{l}_\mu(p\hat{B}(1)^i + r\hat{B}(2)^i + w\hat{B}(3)^i) \ , \qquad (2.6)$$

where the vectors $\hat{m}_\mu$, $\hat{n}_\mu$, $\hat{l}_\mu$ and $\hat{B}(1)^i$, $\hat{B}(2)^i$ and $\hat{B}(3)^i$ form an orthonormal set in space and isospace, respectively, and $o, p, ..., v, w$ are scalar functions.

Since, for a given field, we are at liberty to use curvilinear coordinates especially suited to that configuration, we can get away with less parameters than those in (2.6). Assuming that U is sufficiently smooth, one can use a form with only six non-zero scalar functions

$$B^i_\mu = \hat{m}_\mu(u\hat{B}(1)^i + s\hat{B}(2)^i + o\hat{B}(3)^i) + \hat{n}_\mu(v\hat{B}(2)^i + q\hat{B}(3)^i) + \hat{l}_\mu(w\hat{B}(3)^i) \ . \qquad (2.7)$$

Our approach will be to exploit these facts, and search for even simpler solutions for which B^i_μ can be written in diagonal form. Solutions of more complex form may exist, requiring all six functions for their description, but we will not discuss these. We also make no claim to have exhausted the possibilities in the classes we study. To do this one could try a "little group" analysis [12]. For the moment, for the sake of generality, we live with the clumsy form (2.6). The full equations including mass terms are

$$\partial_\mu C_\mu + D_\mu \hat{B}(1)^i \partial_\mu \hat{B}(2)^i + E_\mu \hat{B}(1)^i \partial_\mu \hat{B}(3)^i - am^2_\pi \sin(f)\hat{B}(1)^i \hat{\pi}^i = 0 \ , \qquad (2.8a)$$

$$\partial_\mu D_\mu + E_\mu \hat{B}(2)^i \partial_\mu \hat{B}(3)^i + C_\mu \hat{B}(2)^i \partial_\mu \hat{B}(1)^i - am^2_\pi \sin(f)\hat{B}(2)^i \hat{\pi}^i = 0 \ , \qquad (2.8b)$$

$$\partial_\mu E_\mu + C_\mu \hat{B}(3)^i \partial_\mu \hat{B}(1)^i + D_\mu \hat{B}(3)^i \partial_\mu \hat{B}(2)^i - am^2_\pi \sin(f)\hat{B}(3)^i \hat{\pi}^i = 0 \qquad (2.8c) \ ,$$

where the functions C_μ, D_μ and E_μ are defined by:

$$[aB^i_\mu + bB^j_\nu(B^i_\mu B^j_\nu - B^j_\mu B^i_\nu)] = C_\mu \hat{B}(1)^i + D_\mu \hat{B}(2)^i + E_\mu \hat{B}(3)^i \ . \tag{2.8d}$$

In terms of the spatial vectors and functions in (2.6), these expressions are horrendous. However, in cases where, by suitable choice of coordinates, B^i_μ can be diagonalised, the expressions for C_μ etc. simplify considerably.

Assuming that B^i_μ can be expressed in diagonal form we write

$$B^i_\mu = \hat{m}_\mu u \hat{B}(1)^i + \hat{n}_\mu v \hat{B}(2)^i + \hat{l}_\mu w \hat{B}(3)^i \ , \tag{2.9}$$

we then find

$$C_\mu = \hat{m}_\mu u(a + b[v^2 + w^2]) \ , \tag{2.10a}$$

$$D_\mu = \hat{n}_\mu v(a + b[w^2 + u^2]) \ , \tag{2.10b}$$

$$E_\mu = \hat{l}_\mu w(a + b[u^2 + v^2]) \ . \tag{2.10c}$$

Note that each of these vectors is now proportional to one of the functions u, v, w.

If the field covers a two-dimensional sub-manifold of the internal space, we can always find a local direction in which the internal variables remain stationary. We define $\hat{l}_\mu$ to be in this direction. We then define $\hat{m}_\mu$ and $\hat{n}_\mu$, such that, together with $\hat{l}_\mu$, they form an orthonormal set. In this case, the gradients B^i_μ can be written

$$B^i_\mu = \hat{m}_\mu(u\hat{B}(1)^i + s\hat{B}(2)^i) + \hat{n}_\mu(t\hat{B}(1)^i + v\hat{B}(2)^i) \ , \tag{2.11}$$

where $\hat{B}(1)^i$ and $\hat{B}(2)^i$ are two orthogonal unit vectors in internal space. A third can be defined as their cross product. This choice is equivalent to putting o, p, q, r and $w = 0$ in (2.6). (Note again that by suitable choice of coordinate system, t may be set zero). The equations of motion are then the same as (2.8a-c), save that C_μ, D_μ and E_μ are now given by

$$C_\mu = a(u\hat{m}_\mu + t\hat{n}_\mu) - b(uv - st)(s\hat{n}_\mu - v\hat{m}_\mu) \ , \tag{2.12a}$$

$$D_\mu = a(s\hat{m}_\mu + v\hat{n}_\mu) - b(uv - st)(t\hat{m}_\mu - u\hat{n}_\mu) \ , \tag{2.12b}$$

$$E_\mu = 0 \ . \tag{2.12c}$$

If in addition the gradients B^i_μ can again be diagonalised such that

$$s = t = 0 \ , \tag{2.13}$$

then the expressions for C_μ and D_μ become

$$C_\mu = u(a + bv^2)\hat{m}_\mu \ , \tag{2.14a}$$

$$D_\mu = v(a + bu^2)\hat{n}_\mu \ , \tag{2.14b}$$

which is equivalent to putting $w = 0$ in (2.10).

If the field covers a one-dimensional sub-manifold of the internal space, then there will be only one direction in space in which the field changes, and we define this direction to be $\hat{m}_\mu$. In this case, we can write

$$B^i_\mu = \hat{m}_\mu u \hat{B}(1)^i \ . \tag{2.15}$$

We then find that $D_\mu = 0$ and the equations of motion (2.8a-c) can be rewritten as

$$\partial_\mu C_\mu - am_\pi^2 \sin(f)\hat{B}(1)^j \hat{\pi}^j = 0 \ , \tag{2.16a}$$

$$C_\mu \partial_\mu \hat{B}(1)^i - am_\pi^2 \sin(f)\hat{\pi}^j(\delta^{ji} - \hat{B}(1)^j \hat{B}(1)^i) = 0 \ , \tag{2.16b}$$

where now:

$$C_\mu = \hat{m}_\mu a u \ . \tag{2.17}$$

Notice that the contributions from the fourth-order term have disapeared from the equations of motion. This is a simple consequence of the fact that the field must cover a sub-manifold of the internal space with dimension ≥ 2 before the fourth-order term makes any contribution to the energy density.

3. SOLUTIONS

We have found new solutions to the Skyrme model by solving equations (2.8a-c,2.12a,2.14a-b) and equations (2.16a-b,2.17). As a warm up we first discuss the venerable hedgehog ansatz and show that it is a true solution. The properties of this solution are useful as a guide to finding the others.

A. The Hedgehog

The hedgehog ansatz is:

$$U_H = \cos(f(r)) + i\tau^i \hat{r}^i \sin(f(r)) \ . \tag{3.1}$$

The gradients of U_H can be written

$$B^i_\mu = f' \hat{r}_\mu \hat{r}^i + \frac{\sin f}{r}(\hat{\phi}_\mu[\cos f \hat{\phi}^i - \sin f \hat{\theta}^i] + \hat{\theta}_\mu[\sin f \hat{\phi}^i + \cos f \hat{\theta}^i]) \ . \tag{3.2}$$

This is clearly of the diagonal form (2.9), so identifying

$$u = f', \quad v = \frac{\sin f}{r}, \quad w = \frac{\sin f}{r} \ ,$$

$$\hat{m}_\mu = \hat{r}_\mu, \quad \hat{n}_\mu = \hat{\phi}_\mu \quad \hat{l}_\mu = \hat{\theta}_\mu \ ,$$

$$\hat{B}(1)^i = \hat{r}^i, \quad \hat{B}(2)^i = [\cos f \hat{\phi}^i - \sin f \hat{\theta}^i], \quad \hat{B}(3)^i = [\sin f \hat{\phi}^i + \cos f \hat{\theta}^i] \ , \tag{3.3}$$

we find in (2.10a-c):

$$C_\mu = \hat{r}_\mu f'(a + 2b\frac{\sin f}{r}) ,$$

$$D_\mu = \hat{\phi}_\mu \frac{\sin f}{r}(a + b(f'^2 + \frac{\sin f}{r})) ,$$

$$E_\mu = \hat{\theta}_\mu \frac{\sin f}{r}(a + b(f'^2 + \frac{\sin f}{r})) . \quad (3.4)$$

Using (3.4),(3.3) and (3.1), we find immediately

$$C_\mu \hat{B}(2)^i \partial_\mu \hat{B}(1)^i = C_\mu \hat{B}(3)^i \partial_\mu \hat{B}(1)^i = 0 , \quad (3.5a)$$

$$D_\mu \hat{b}(3)^i \partial_\mu \hat{B}(2)^i = E_\mu \hat{B}(2)^i \partial_\mu \hat{B}(3)^i = 0 , \quad (3.5b)$$

$$\partial_\mu D_\mu = \partial_\mu E_\mu = 0 , \quad (3.5c)$$

$$\hat{B}(2)^i \hat{\pi}^i = \hat{B}(3)^i \hat{\pi}^i = 0 . \quad (3.5d)$$

Hence, (2.8b) and (2.8c) are satisfied trivially, since all terms in the equations are individually zero. However, (2.8a) yields a non-trivial equation for f, since from (3.4), (3.3) and (3.1)

$$\partial_\mu C_\mu = f''(a + 2b\frac{\sin^2 f}{r^2}) + 4bf'^2 \frac{\cos f \sin f}{r^2} + \frac{2a}{r} f' , \quad (3.6a)$$

$$D_\mu \hat{B}(1)^i \partial_\mu \hat{B}(2)^i = E_\mu \hat{B}(1)^i \partial_\mu \hat{B}(3)^i = -\frac{\cos f \sin f}{r^2}(a + b(f'^2 + \frac{\sin^2 f}{r^2})) , \quad (3.6b)$$

$$\hat{B}(1)^i \hat{\pi}^i = 1 . \quad (3.6c)$$

This gives

$$f''(a + 2b\frac{\sin^2 f}{r^2}) + \frac{2a}{r} f' - 2\frac{\sin f \cos f}{r^2}(a + b(-f'^2 + \frac{\sin^2 f}{r^2})) - am_\pi^2 \sin f = 0 , \quad (3.7)$$

which is just the same equation for the chiral angle given the ansatz (3.1) that was used by Adkins et al. [3] and Jackson and Rho [4]. The usual derivation of (3.7), however, consists of treating the form (3.1) as a variational ansatz, and solving the resulting functional variational equation for f. In contrast, we have shown that the hedgehog constitutes a real solution to the equations of motion.

To solve the full set of equations it was vital that all terms in (2.8b-c) were zero. Unless this occurs, it is difficult to see how one could satisfy all three equations (2.8a-c), with an ansatz that contains only one function of one variable, $f(r)$. If f is not to be overconstrained, (2.8a-c) must be satisfied without putting extra conditions on f. Below, we will use this property to find other solutions to the equations.

B. The String

We now turn to the case where the field covers a two-dimensional sub-manifold of the internal space.

The simplest guess that depends on one function of one variable and for which two of the equations of motion are trivial is the cylindrical form

$$U_S = \cos(f(\rho)) + i\tau^i \hat{\rho}^i \sin(f(\rho)) \; , \tag{3.8a}$$

together with the boundary conditions

$$f(0) = n\pi; \quad f(\rho) \to 0, \quad \rho \to \infty \; . \tag{3.8b}$$

We will only consider the case where $n = 1$. The form of this ansatz is shown in Fig. 1.

We can expect configurations such as (3.8) to be important in circumstances where baryon density is distributed so that parts of a baryon are separated in space. For example if a baryon is cut in half and the halves placed at positions $\pm l$ along the z axis, then it is natural to impose the boundary condition that the field take the form given by (3.8) on the planes $z = \pm l$. The form (3.8) interpolates between two half baryons in a continuous way. Such configurations may occur when a baryon is struck in a collision and momentum is transferred to a part of it, or when a baryon spins at high angular momentum. Actually, we need not insist that the material at the ends of the string constitute a baryon. It is just as easy to place half an anti-baryon (appropriately rotated) at one end, and thus the complete configuration of string plus ends can be either mesonic or baryonic.

Thus configurations similar in form to (3.8) can be expected to play a role in those situations where one uses the Q.C.D. string models - such as the Lund model or the string picture of the

Regge trajectories. Thus the boundary conditions that produce string-like configurations such as (3.8) are very common in nuclear and hadronic physics, and we expect this string to have many applications.

We show elsewhere [9] that the configuration (3.8) is unstable against decay via rotation of the pion field vectors into the $\hat{z}^i$ direction. We show there that this process is accompanied by a baryon-number current flowing along the string, resulting in the creation of baryon or anti-baryon density at the ends of the string. Thus when the string decays, it produces particle and anti-particle density, as is usually envisioned in the decay of the Q.C.D. string. If the decay direction for the pion field vectors is chosen differently along the length of the string we can get a variety of different decay products (*e.g.* the string can decay into mesons, or into baryon anti-baryon pairs, etc...).

There are several reasons to believe that (3.8) will solve the equations. Consider the $O(3)$ non-linear sigma model with a Skyrme term and a mass term in $2+1$ dimensions. It is well known that the soliton solution has the form

$$\Phi^i(\rho,\phi) = \hat{z}^i \cos(f(\rho)) + \hat{\rho}^i \sin(f(\rho)) \ , \tag{3.9}$$

with the same boundary conditions as (3.8b). The ansatz (3.8a) has the same form as (3.9) for constant z. (3.8) can be thought of as an embedding of (3.9) in $3+1$ dimensions along a straight line in the z direction. Another reason to expect that this will work is that at constant z, (3.8) has the same form as an equatorial cross section through the hedgehog (3.1). Therefore the condition that gradient combinations in (2.8b-c) vanish individually will be satisfied for the combinations taken within this cross section. Whether the string-like form of the embedding is correct remains to be shown, but one expects that it is, since the ansatz is translationally invariant in the z direction.

Explicitly, the gradients of (3.8) are

$$B^i_\mu = f'\hat{\rho}_\mu\hat{\rho}^i + \frac{\sin(f)}{\rho}[\cos(f)\hat{\phi}^i + \sin(f)\hat{z}^i]\hat{\phi}_\mu \ . \tag{3.10}$$

Substituting in (2.1) gives an energy density of

$$\mathcal{E} = \frac{a}{2}[f'^2 + \frac{\sin^2(f)}{\rho^2}] + \frac{b}{2}[f'^2 \frac{\sin^2(f)}{\rho^2}] \ .$$

From (3.10), we see that as expected, we can write B^i_μ in the diagonal form (2.11) by choosing

$$\hat{B}(1)^i = \hat{\rho}^i \ , \ \hat{m}_\mu = \hat{\rho}_\mu \ , \ u = f' = \hat{\rho}_\mu \partial_\mu f \ , \tag{3.11a}$$

$$\hat{B}(2)^i = [\cos(f)\hat{\phi}^i + \sin(f)\hat{z}^i] \ , \ \hat{l}_\mu = \hat{\phi}_\mu \ , \ v = \frac{\sin(f)}{\rho} \ , \tag{3.11b}$$

$$\hat{B}(3)^i = [-\sin(f)\hat{\phi}^i + \cos(f)\hat{z}^i] \ , \tag{3.11c}$$

clearly (2.13) is satisfied and we get, in (2.10) and (2.14)

$$C_\mu = f'(a + b\frac{\sin^2(f)}{\rho^2})\hat{\rho}_\mu \ , \tag{3.12a}$$

$$D_\mu = \frac{\sin(f)}{\rho}(a + bf'^2)\hat{\phi}_\mu \ , \tag{3.12b}$$

$$E_\mu = 0 \ . \tag{3.12c}$$

In consequence, we can easily verify that all eight terms in (2.8b) and (2.8c) are zero individually.

However, (2.8a) yields a non trivial equation for f

$$f''(a + b\frac{\sin^2(f)}{\rho^2}) + \frac{f'}{\rho}(a - b\frac{\sin^2(f)}{\rho^2}) - \frac{\sin(f)\cos(f)}{\rho^2}(a - bf'^2) - m_\pi^2 a \sin(f) = 0 \ . \tag{3.13}$$

This equation is precisely the same as the one obtained for the soliton in the two-dimensional Skyrme model with a mass term. This should not come as a surprise, considering our previous discussion of (3.8) and (3.9). This being the case, several properties of the two-dimensional model follow for the solution of (3.13). The most important of these is that when either the fourth-order or the mass term in (2.1) is turned off, the energy of the solution unstable w.r.t. contraction or expansion, respectively. If *both* terms are turned off the energy is insensitive to the scale of the solution [13]. To illustrate this, consider varying the scale, Λ, of the chiral functions for the hedgehog and the string, for fixed values of the parameters f_π, m_π and ϵ^2. For the hedgehog we find that the energy of the resulting configuration is

$$E_h(\Lambda) = \Lambda E_{h,2} + 1/\Lambda E_{h,4} + \Lambda^3 E_{h,m_\pi} \ , \tag{3.14}$$

whereas the energy per unit length of the string is:

$$\sigma_s(\Lambda) = \sigma_{s,2} + 1/\Lambda^2 \sigma_{s,4} + \Lambda^2 \sigma_{s,m_\pi} \ , \tag{3.15}$$

where we have kept the contributions of the three terms in (2.1) distinct, and *e.g.* $\sigma_{s,2}$ is the contribution from the second-order term for the string. We now minimize (3.14) and (3.15) w.r.t. Λ and find

$$\Lambda_h = \sqrt{E_{h,4}/E_{h,2}} \ , \tag{3.16}$$

$$\Lambda_s = \sqrt[4]{\sigma_{s,4}/\sigma_{s,m_\pi}} \ . \tag{3.17}$$

In getting (3.16) we have neglected the pion mass term in (3.14), since it is small. We cannot do this in (3.17), since then Λ_s diverges. Substituting back in (3.14) and (3.15), we find

$$E_h \approx 2\sqrt{E_{h,2} E_{h,4}} \ . \tag{3.18}$$

$$\sigma_s = \sigma_{s,2} + 2\sqrt{\sigma_{s,4}\sigma_{s,m_\pi}} \ . \tag{3.19}$$

In (3.18) we have ignored the pion mass term again, and in (3.19) we retain the second-order piece, which is not negligible. Note that (3.17) and (3.15) imply the relation $\sigma_{s,4} = \sigma_{s,m_\pi}$.

In the present context, the stabilization of the solution against expansion by the mass term must be viewed as an unsatisfactory feature, since the width of the Q.C.D. string should not be controlled by the scale of chiral symmetry breaking! (Although it is amusing to note that the pion mass term acts like an effective bag constant in three dimensions, since its contribution to the energy scales as the volume over which f is appreciably non-zero. The resulting effective bag constant is $B^{\frac{1}{4}} \sim (f_\pi m_\pi)^{\frac{1}{2}} = 113$ MeV.) Therefore we expect there to be important corrections to the size of the string when other meson fields are included in the effective lagrangian.

The form of the embedding of the solution in three dimensions is not arbitrary, since forms other than (3.8) generally do not satisfy the full coupled set of equations (2.8a-c). As before, the vanishing of each of the terms in (2.8b-c) separately is vital in finding a solution. If the solution (3.9) is embedded along a curved line, then the curvature will induce extra conditions on f from(2.8b-c). In general, these conditions cannot be met if f is a function of only one variable. (Note that one might get around this problem if f were a function of more than one variable, *e.g.* one could have solutions of the toroidal sort envisioned by de Vega [14]. Such solutions may be related to the dinotor configurations of Jackson, Castillejo and Goldhaber in the bag model [15].)

One can solve (3.13) asymptotically with a cylindrical Bessel function, $f \sim K_1(m_\pi \rho)$. We then numerically solved the full equation of motion using the parameters of Jackson and Rho [4] for a and b (fitting f_π and g_a) yielding $a = f_\pi^2 = 93^2$ MeV^2 and $b = 8 \times 0.00552$. The solution is shown in Fig. 2.

We found that the solution follows the Bessel function to within 1% up to about 2 fm when the chiral angle, f, is about 0.39. The energy per unit length for this object, σ, is 858 MeV/fm. The separate contributions to the string tension are $\sigma_{s,2} = 590$ MeV/fm and $\sigma_{s,4} = \sigma_{s,\pi} = 134$ MeV/fm. The string is rather fat in comparison with the hedgehog, *e.g.* the rms string tension radius,

$$< r^2 >_\sigma^{1/2} = (2\pi \int_0^\infty d\rho \rho^3 \mathcal{E}(\rho)/\sigma_s)^{1/2} ,$$

is 1.05 fm, and the chiral angle is still ≈ 0.15 at $\rho = 3$ fm, presumably because the pion mass term controls the radial scale. The energy density is about 405 MeV/fm^3 in the centre of the string and dies away by $\rho \sim 2$ fm to ~ 5 MeV/fm^3. The energy density as a function of ρ is shown in Fig. 3. The baryon density is zero everywhere, since the ansatz (3.8) covers a two-dimensional sub-manifold of the internal space.

Before one gets too exited by the value of the string tension, 0.86 GeV/fm, it is well to consider that with the parameters used here, the unquantized skyrmion mass is about 1430 MeV. The dimensionless ratio of the square root of the string tension to the hedgehog mass is therefore $\sqrt{\sigma_s}/M_H = 0.29$. The empirical ratio is $\sqrt{\sigma}/M_N \sim 0.45$, for $\sigma \sim 0.92$ GeV/fm and $M_N \sim 0.94$ GeV. This ratio is a little better if one uses the Adkins-Nappi-Witten [3] parameters: $f_\pi = 64.5$ MeV, $\epsilon^2 = 0.00421$. The string tension is then $\sigma_s = 0.445$ GeV/fm and the unquantized hedgehog mass is 0.86 GeV, giving $\sqrt{\sigma_s}/M_H = 0.34$. $< r^2 >_{\sigma_s}^{1/2}$ increases slightly to 1.15 fm. Most of the change in the string tension comes about from the reduction in the contribution from the second-order term given this lower value of f_π. We have studied the string tension as a function of f_π (for fixed ϵ^2 and m_π), and found that the second-order contribution $\sigma_{s,2}$ scales as f_π^2 to a few percent. This is expected from the fact that $\sigma_{s,2}$ is insensitive to the precise form of $f(\rho)$ [13].

Alternatively, regarding f_π and m_π as fixed, the ratio of the mass of the skyrmion to the square root of the string tension can be changed by adjusting ϵ^2 in (2.1); this will also change the relative sizes of the two objects. Taking $f_\pi = 93$ MeV and $m_\pi = 137$ MeV, we have studied the properties of the string as functions of ϵ^2. In Figs. (4) and (5) we show the variation of σ_s and $< r^2 >_{\sigma_s}^{1/2}$ with ϵ^2. Note that $\sigma_{s,2}$ increases by only $\sim 2\%$ for a 75% increase in ϵ^2, whereas $\sigma_{s,4}$ and $\sigma_{s,\pi}$ change by 30% over the same range in ϵ^2. From equations (3.15), (3.17) and (3.19), the variation in $\sigma_{s,4}$ can be guessed to be $\sigma_{s,4}(\epsilon^2) \sim \sqrt{\epsilon^2}$. This behaviour is not exact, since the solution will change in form as ϵ^2 is varied.

C. The Slab and Other Configurations

Now we turn to the simplest solutions, those that cover a one-dimensional sub-space of the internal space. Equations (2.16) and (2.17) are comparatively easy to solve (the first appearance of these equations is essentially in Skyrme's original papers [6], although he concentrated on small-amplitude solutions). The fourth-order term makes no contribution to equations (2.16) and (2.17). In consequence, all the solutions of the ordinary non-linear sigma model that cover a one-dimensional sub-space of the internal space are also solutions of the Skyrme model. We present here two new solutions, and finally a third solution which is time dependent and whose form is strongly suggested by the second solution.

The first of these solutions is essentially the Sine-Gordon soliton extended into three dimensions like a plane wall or slab. It is difficult to imagine the physical circumstances where such a solution might occur in nuclear or hadronic physics. The boundary conditions required to produce it are very unusual. Nevertheless, in solid-state physics, Skyrme models can be used as effective lagrangians for order parameters [16]. By appropriate manipulation of boundary conditions, it might be possible to create such slab-like configurations, and the others described below. For example, a magnetic field could be used to set up large amplitude spin waves.

The fact that this slab-like solution exists is strongly suggested by the fact that the energy density reduces to the form of the Sine-Gordon energy density when the field moves over a one-dimensional sub-space of the internal space. The extra equation (2.16b) is easily solved by a configuration which consists of a great circle in internal space, provided we make the great circle pass through the point $U = 1$, then

$$\partial_\mu \hat{B}(1)^i = 0 \quad \text{and} \quad \hat{\pi}^j(\delta^{ji} - \hat{B}(1)^j \hat{B}(1)^i) = 0 \; . \tag{3.20}$$

The equations of motion then reduce to (2.16a), which for constant $\hat{m}_\mu$, reduce to

$$\hat{m}_\mu \partial_\mu u - m_\pi^2 \sin(f) = 0 \; . \tag{3.21}$$

An easy way to satisfy all the above conditions is to take

$$U_W = \cos(f(\zeta)) + i\tau^i \hat{\Pi}^i \sin(f(\zeta)) \; , \tag{3.22}$$

where:

$$\zeta = \hat{m}_\mu x_\mu \; ,$$

for constant $\hat{m}_\mu$, and $\hat{\Pi}^i$ is a constant unit vector in internal space. For concreteness, take for the moment, $\hat{m}_\mu = \hat{z}_\mu$ and $\hat{\Pi}^i = \hat{z}^i$, we then have

$$u = f' = \hat{z}_\mu \partial_\mu f(z) \ , \tag{3.23}$$

and we see that (2.16a) reduces to the Sine-Gordon equation, *exactly*. Therefore we can solve it with, *e.g.*:

$$f(z) = 2\sin^{-1}(\mathrm{sech}(m_\pi z)) \ . \tag{3.24}$$

This solution consists of a plane wall of energy density at $z = 0$, with thickness controlled by $1/m_\pi = 1.44$ fm. The energy per unit area of this solution is then given by the standard result for the Sine-Gordon soliton

$$E/A = 8 m_\pi f_\pi^2 = 244 \ \mathrm{MeV/fm}^2 \ . \tag{3.25}$$

The numerical value is obtained using $f_\pi = 93$ MeV, and $m_\pi = 137$ MeV. Notice that, in the chiral limit, the solution dilates indefinitely and the energy density goes to zero.

The internal space in much larger than that of the Sine-Gordon model, and we can look for solutions are not restricted to motion on a great circle through $U = 1$. To find such solutions, it is useful to think of the problem with the "inverted potential" method [17]. If one thinks of the static problem as the analogue of a time dependent problem in an inverted potential, one can view the problem at hand as that of a massive particle moving in a spherical bowl under gravity. The sine-gordon solution then corresponds to vertical motion over the top of the spherical bowl. One's intuition about the problem of the spherical bowl tells one that there is a whole host of solutions corresponding to a particle spinning horizontally. To illustrate the simplest of these we look for solutions of the form

$$U = \cos f + i\tau^i[\hat{x}^i \cos(\phi(z)) + \hat{y}^i \sin(\phi(z))] \sin f \ , \tag{3.26}$$

where now f is a constant. Since f is not necessarily $\pi/2$ the motion is no longer on a great circle in internal space. However, common sence tells us there should be closed orbits where the "angular momentum" of the particle spinning balances the "gravitational pull" of the mass term. The gradients are then

$$B^i_\mu = \hat{z}_\mu a \sin(f)\phi'[\cos(f)(\hat{x}^i\cos(\phi) - \hat{y}^i\sin(\phi)) - \sin(f)\hat{z}^i] \ , \tag{3.27}$$

where $\phi' = \hat{z}_\mu \partial_\mu \phi(z)$, is the rate at which the particle spins round the bowl. We identify

$$C_\mu = a\sin(f)\phi'\hat{z}_\mu, \quad \hat{B}(1)^i = [\cos(f)(\hat{x}^i\cos(\phi) - \hat{y}^i\sin(\phi)) - \sin(f)\hat{z}^i] \ . \tag{3.28}$$

The mass term contributes only to (2.16b), since

$$\hat{\pi}^j\hat{B}(1)^j = 0, \quad \text{and} \quad \hat{\pi}^j(\delta^{ji} - \hat{B}(1)^j\hat{B}(1)^i) = \hat{\pi}^i \ , \tag{3.29}$$

hence we get for the equations of motion, (2.16a-b)

$$\partial_\mu\phi'\hat{z}_\mu = \phi'' = 0 \ , \tag{3.30}$$

and

$$\phi'\hat{z}_\mu\partial_\mu\hat{B}(1)^i - m_\pi^2\hat{\pi}^i = -(\phi'^2\cos(f) + m_\pi^2)\hat{\pi}^i = 0 \ . \tag{3.31}$$

We can solve (3.30) with a constant circulation rate

$$\phi = \alpha + kz, \quad \phi' = k \ , \tag{3.32}$$

equation (3.31) then constrains the momentum k in terms of the height at which the particle spins

$$k = \sqrt{-\frac{m_\pi^2}{\cos(f)}} \ . \tag{3.33}$$

Note that $k > m_\pi$ and is real only if $f > \pi/2$ - this corresponds to motion around the "bottom of the bowl" in the inverted potential. In the limit $f \to \pi/2, \quad k \to \infty$. Corresponding to the fact that in the absence of friction, motion corresponding to a partical spinning on a vertical wall (a kind of "wall of death" solution) is forbidden unless $m_\pi = 0$. In the chiral limit, $m_\pi \to 0, \quad k \to 0$, and the solution becomes a constant unless $\cos f = 0$. The energy density of this solution is constant, and given by

$$\mathcal{E} = f_\pi^2(\frac{k^2}{2}\sin^2(f) + m_\pi^2[1-\cos(f)]) = f_\pi^2(\frac{k^2}{2} + m_\pi^2 + \frac{m_\pi^4}{2k^2}) \ . \tag{3.34}$$

In addition to this solution, the analogous problem tells us intuitively that there are an infinite set of solutions corresponding to trajectories in which the height of the particle oscillates as it spins round the bowl. For these trajectories, k is not constant. We have not investigated this "wobbling" class of solutions further.

If we look for time-dependent solutions in the Skyrme model, then for every static solution to (2.16,2.17), we can find an equivalent time-dependent solution. The solution equivalent to (3.26) and (3.32,3.33) is a rigid rotation of the field vector

$$U = \cos(f) + i\tau^i[\hat{x}^i\cos(\phi(t)) + \hat{y}^i\sin(\phi(t))]\sin(f) \ , \tag{3.35}$$

where

$$\phi(t) = \gamma + \omega t, \quad \omega = \sqrt{\frac{m_\pi^2}{\cos(f)}} \ . \tag{3.36}$$

The condition for real ω is $f < \pi/2$, again there is no solution for $f = \pi/2$ unless $m_\pi = 0$. In the chiral limit $m_\pi \to 0$ the solution becomes static.

Both (3.36) and (3.33) indicate that, in the chiral limit there is another class of solutions that do not exist for $m_\pi \neq 0$. These are trivial to find, they are constant gradient trajectories on any great circle in internal space. For example, one such (static) trajectory is

$$U = \cos(f(\hat{m}_\mu x_\mu)) + i\tau^i\hat{\Pi}^i\sin(f(\hat{m}_\mu x_\mu)) \ , \tag{3.37}$$

where, $f = \epsilon + k\hat{m}_\mu x_\mu$ for constant $\hat{m}_\mu$, and $\hat{\Pi}^i$ is a constant unit vector in internal space.

It should be clear by now that using the analogy with a particle in a bowl, one can generate a huge number of large amplitude solutions to equations (1.16a-b).

4. DISCUSSION AND CONCLUSIONS

We have found a number of new solutions of the static equations of motion. These solutions are of the form that the gradients can be diagonalised in a suitably chosen orthonormal set of curvilinear coordinates. These properties have enabled us to solve the equations of motion with simple ansatze. The size of the pion field in these solutions can be as large as $\sim f_\pi$. Nevertheless, due to the dimensional character of the solutions, they have baryon density zero everywhere. In consequence, all of the solutions presented here are classically unstable and can be expected to decay. Therefore, they require special physical circumstances (in which appropriate boundary conditions occur) to create them.

In the case of the string, the circumstances are similar to those where one normally uses Q.C.D. strings. For example, when the baryon density of a nucleon is separated into two pieces, unless baryon density is produced between these pieces one is forced (by continuity of the pion field) to have a tube of energy density between them. The character of this tubular field is similar to that of the string solution. The string solution is therefore expected to have a wide variety of potential applications in nuclear and hadron physics.

For the solutions that cover a one-dimensional sub-space of the internal space, the situation is more awkward. The typical form of the boundary values needed to produce these solutions are non trivial conditions imposed on the field on large surfaces or volumes. For example, to create the wall solution one must rotate the field vectors in concert over a large surface. The rotation must swing the field vectors over the top of the internal space. It is difficult to think of circumstances accessible to experiment in nuclear physics that fit the bill. Two possibilities come to mind. The first is in stopped collisions of ultra-relativistic heavy ions, where the stopped nucleons might be expected to throw off a large-amplitude pion field with a slab-like form. The second involves star-quakes on the surface of a nutron star. The position is more hopeful in solid-state physics, where the boundary conditions of the systems under investigation are more easily manipulated.

However, the solutions in section 3.c have counterparts in other lower-dimensional models (this statement is obviously true in the case of the Sine-Gordon model!), and our results suggest a more general principle: topological solitons in lower-dimensional sigma models become unstable solutions in higher-dimensional models when appropriately embedded. In Table 1 we show a hierarchy of solutions to sigma models in various dimensions. Note that the soliton in the Sine-Gordon model can be embedded in an $O(3)$ sigma model in $2+1$ dimensions, where it plays the role of a string-like solution, [11]. We therefore expect it to play an analogous role to the Skyrme string in the Skyrme model.

Having considered the problem of producing our unstable solutions, we now turn to the question of their decay. The instability that causes the string solution to decay has interesting properties in its own right. From continuity arguments, one can see that to produce the vacuum between the ends of the string one must produce (minimally) half a skyrmion and half an anti-skyrmion. The decay of the string is therefore accompanied by the production of baryon density, a feature reminiscent of the decay of a Q.C.D. string. These properties are investigated more fully elsewhere [9].

The Skyrme string and the Q.C.D. string occur in similar physical circumstances and have intriguing properties in common. However, there are several qualitative differences and problems that prevent one from making a firm identification between the two without modification of the Skyrme string. The first of these is the absence of a chiral limit for the Skyrme string: we have shown that the width of the Skyrme string grows indefinitely as one takes the chiral limit. (Note though, that while the radius of the string dilates indefinitely as $m_\pi \to 0$, the string tension does not, due to the Bogomol'ny bound [13].)

Second, one does not expect that the Skyrme lagrangian would describe the gluonic sector of Q.C.D. at all, and indeed there are obvious circumstances when one can have a Q.C.D. string without a Skyrme string. For example, when colour is exchanged between two nucleons via gluon exchange, there is no Skyrme string. Clearly the Skyrme string is generated by separation of baryon density, whereas the Q.C.D. string is generated by separation of colour charges. Work to resolve these problems is in progress [10].

Nevertheless, we feel that the decay properties of the Skyrme string are sufficiently intriguing as to merit further investigation. If one can make the identification of the Skyrme string with the Q.C.D. string more solid, a number of interesting problems can be addressed. The asymptotic interaction between parallel Skyrme strings can then be worked out using a simple modification of the methods for skyrmions [18]. At a sufficiently high density of strings, one expects a phase transition to occur to a state where the strings merge - in analogy with results for skyrmions and their two-dimensional counterparts [19]. The decay properties of the string are sufficiently rich that problems such as the relative decay rate to $B\overline{B}$ could be addressed.

Finally, we note that Boguta has found a string-like object in an effective field theory containing σ and ω fields [20]. The corresponding string tension was found to be 190 MeV/fm. It does not seem that this string is related to ours since there is no pion field in Bogutas string, and we do not expect the ω field to couple to our static pion field, since the baryon density is zero everywhere in our solution.

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Table 1

	Real Space Dimension	Internal Space	Lagrangian Structure	Solutions (1 – D)	Solutions (2 – D)	Solutions (3 – D)
Sine–Gordon	$1+1$	S^1	$\mathcal{L}_2 + \mathcal{L}_{m_\pi}$	point (S. – G. Soliton) ↓	–	–
	$2+1$	S^2	$\mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_{m_\pi}$	string* ↓	point ↓	–
Skyrme Model	$3+1$	S^3	$\mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_{m_\pi}$	slab*	string*	point (Skyrmion)

Table Caption

Table.1. Hierarchy of static solutions to sigma models, based on embedding topological solitons in higher dimensional models . The brackets (1 - D) etc. indicate the dimension of the internal space manifold covered by the solution. Asterisks indicate unstable solutions. Note that all the embedded solutions are unstable.

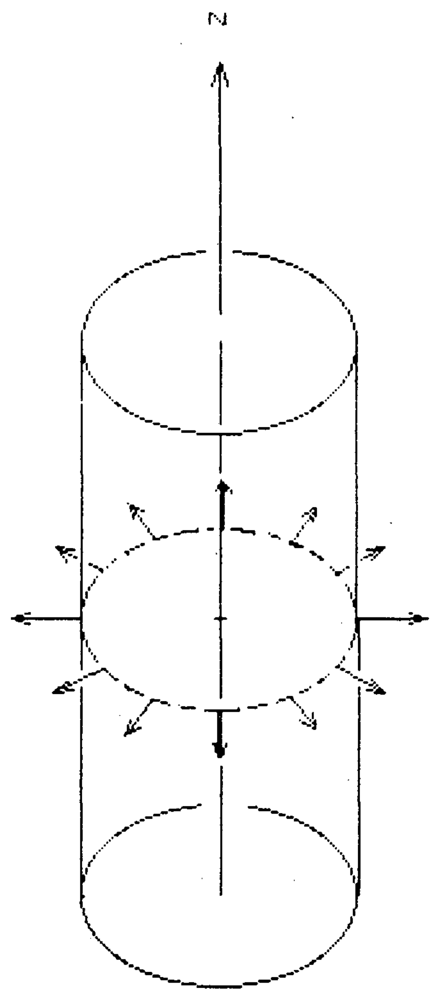


Fig. 1

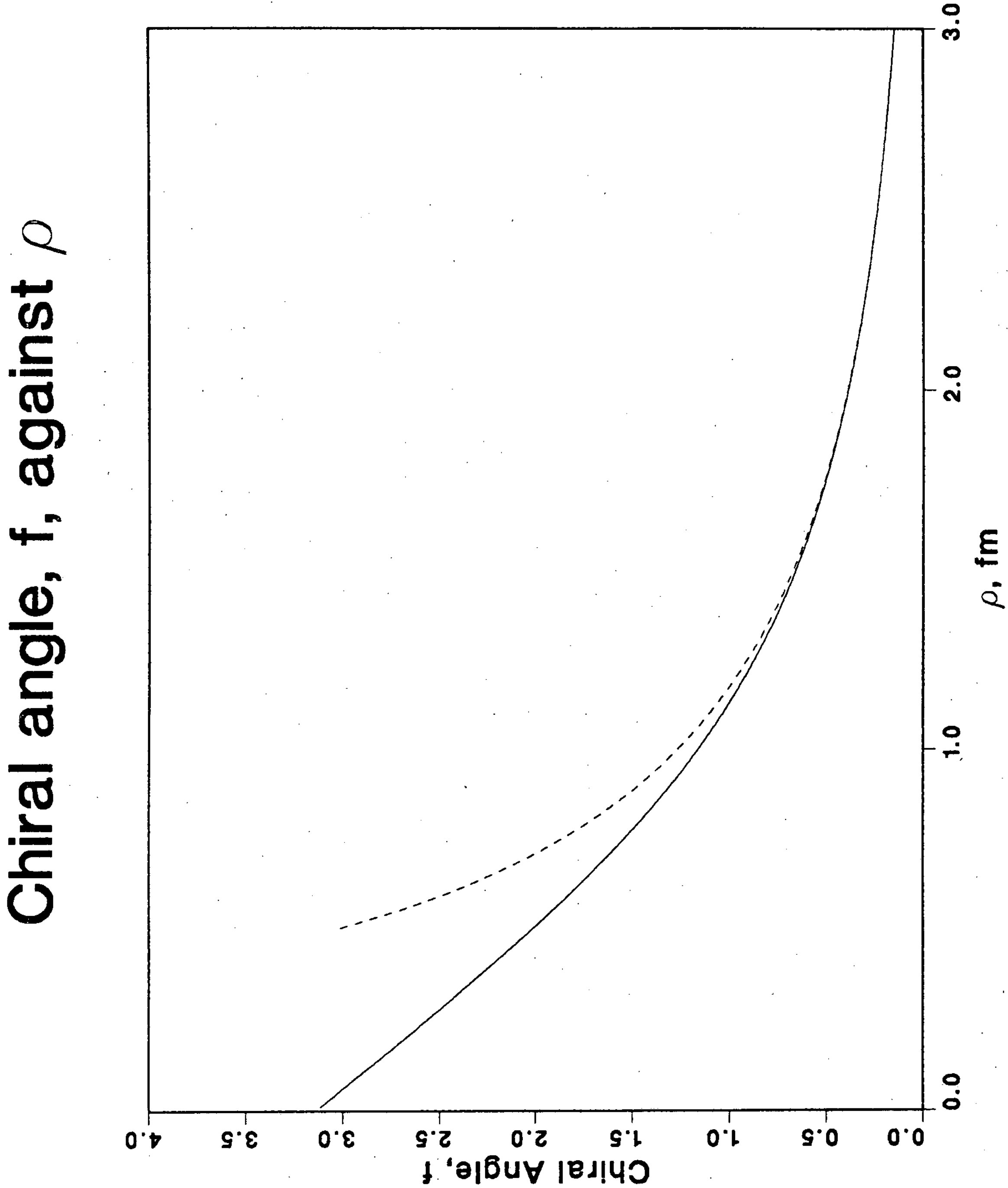

Chiral angle, f, against ρ
Chiral Angle, f
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
ρ, fm
0.0
1.0
2.0
3.0

Fig. 2

Energy density, $\mathscr{E}$, MeV/fm^3

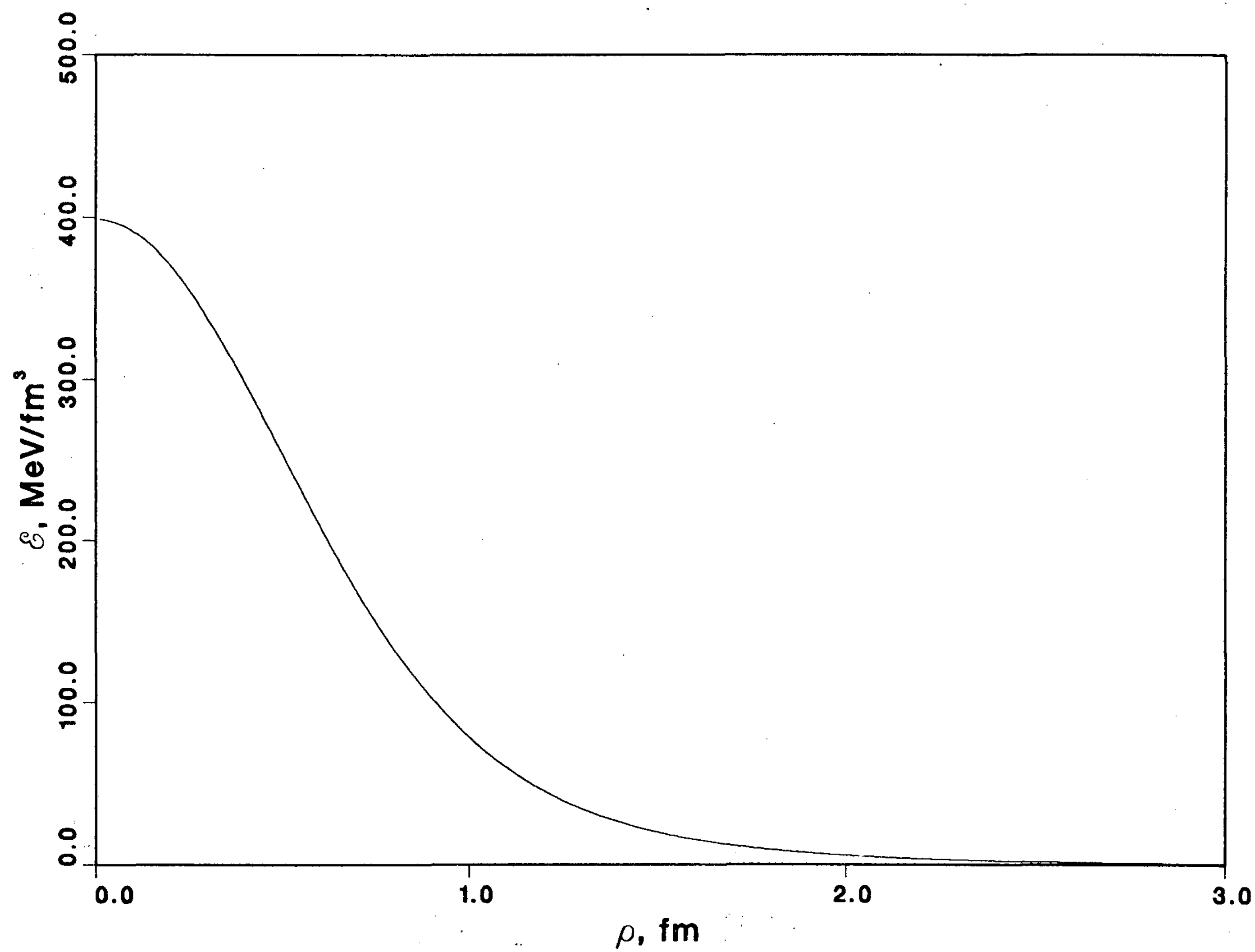


Fig. 3

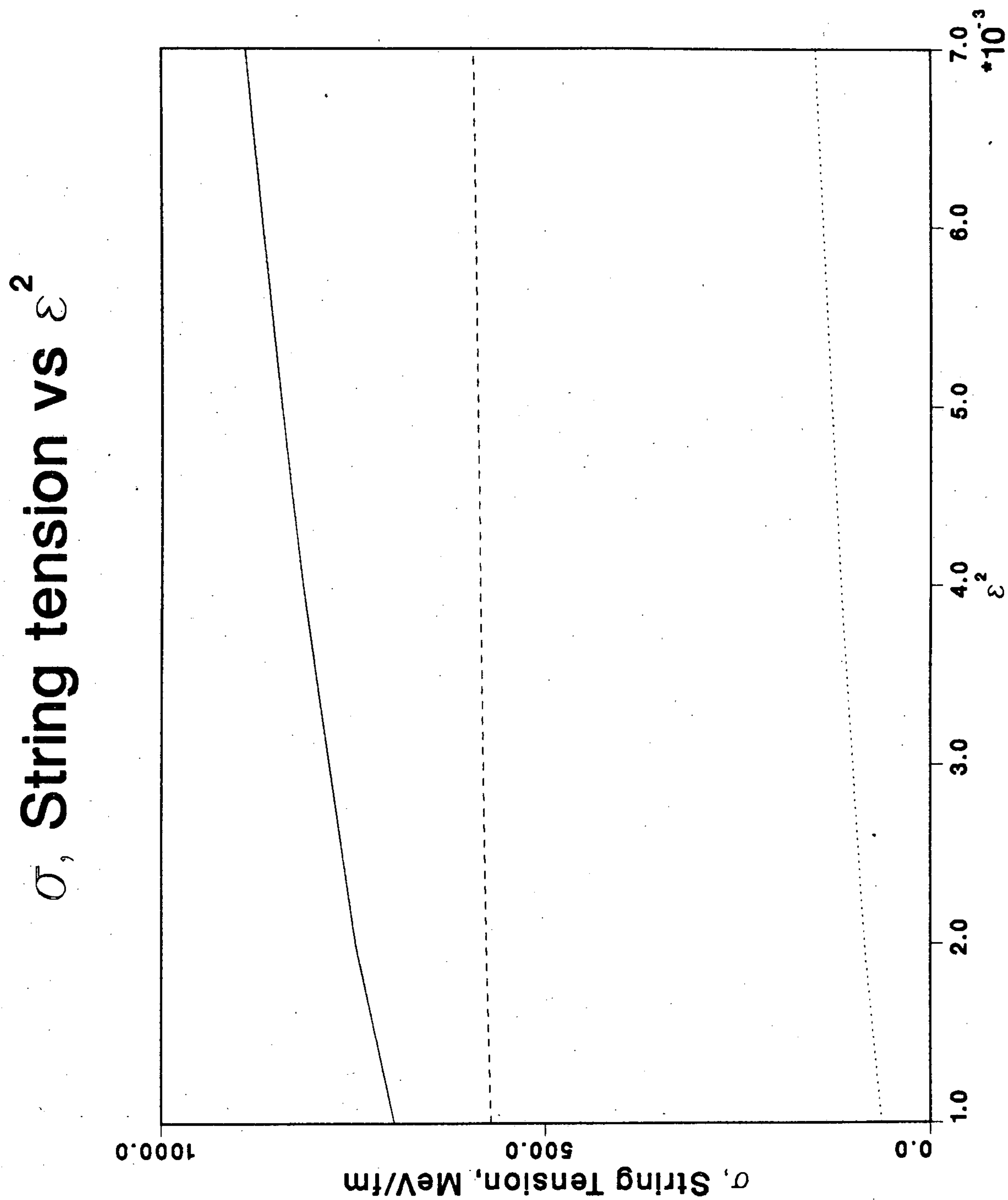
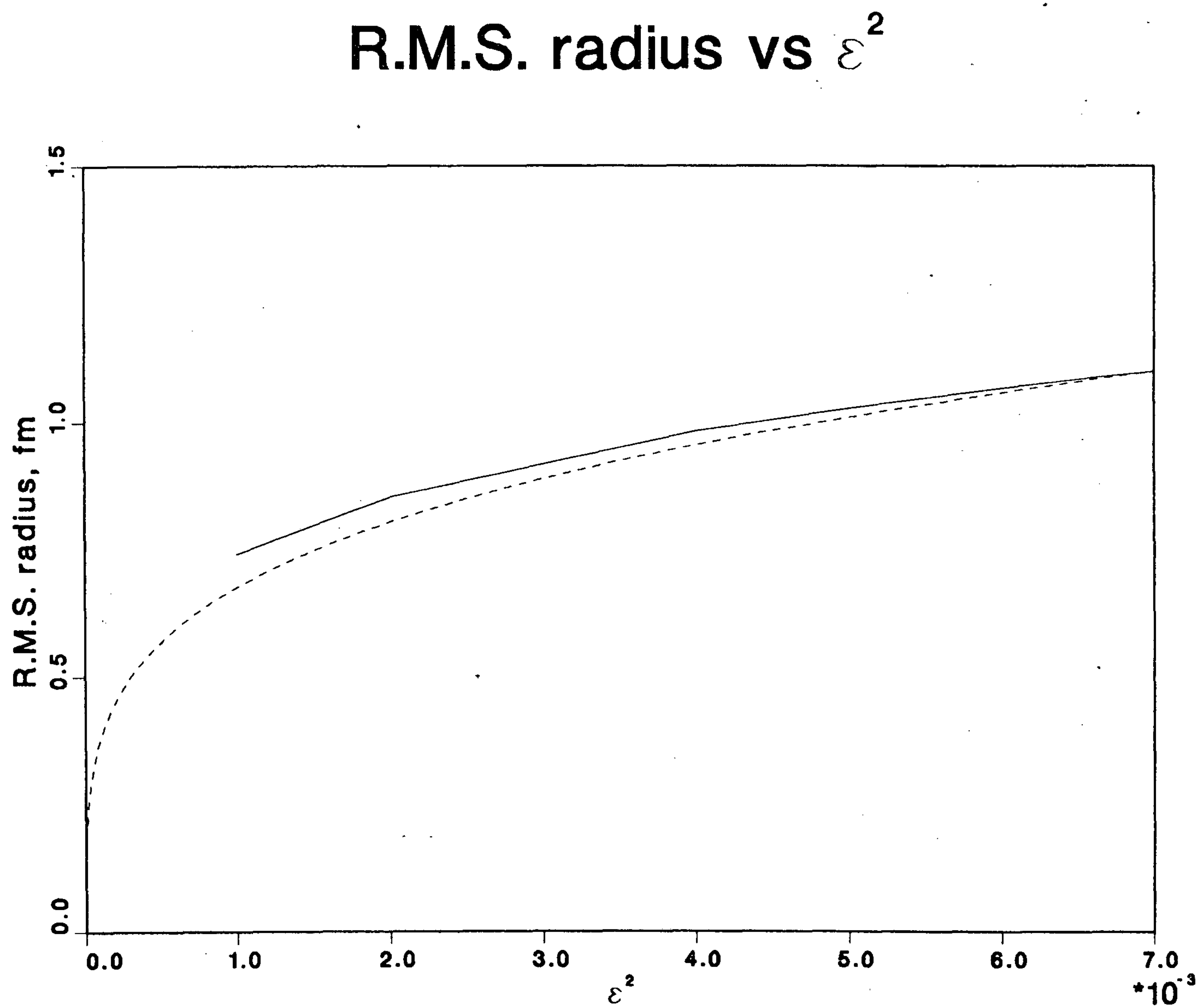

σ, String tension vs ε²
σ, String Tension, MeV/fm
1000.0
500.0
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
*10⁻³
ε²

Fig. 4



R.M.S. radius vs ε^2
R.M.S. radius, fm
0.0
0.5
1.0
1.5
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
ε^2
$*10^{-3}$

Fig. 5

Figure Captions

Fig. 1. The geometrical form of the string solution. The outline represents a section of the cylindrical surface for which $U = i\tau^i \hat{\rho}^i$, the arrows represent the direction of the pion field vector, and the line through the center of the cylinder represents the line where $U = -1$.

Fig. 2. The chiral angle, f, for the solution to equation (3.13), using $m_\pi = 137$ MeV, and the parameters of Jackson and Rho [4], $\epsilon^2 = 0.00552$, $f_\pi = 93$ MeV. The solid curve is the numerical solution, the dashed curve is the asymptotic form $f(\rho) = 1.1972\ K_1(m_\pi \rho)$.

Fig. 3. The energy density from equation (2.1), for the solution of equation (3.13) using the same parameters as fig.(2).

Fig. 4. The string tension as a function of ϵ^2, for values f_π and m_π fixed at their empirical values, 93 and 137 MeV, respectively. The solid curve represents the total, σ_s, the dashed and dotted curves represent the second and fourth-order contributions ($\sigma_{s,2}$ and $\sigma_{s,4}$), respectively. The contribution from the pion mass term is equal to that of the fourth-order term, as required by equations (3.15) and (3.17).

Fig. 5. The r.m.s. string tension radius, $< r^2 >_\sigma^{1/2}$, as a function of ϵ^2, for values f_π and m_π fixed as in fig.(4). Also shown is the form $< r^2 >_\sigma^{1/2} = \kappa\ {}^4\sqrt{\epsilon^2}$ that would result if changing ϵ^2 resulted *only* in a rescaling of the solution. The constant κ is determined to be 3.8 from the point $\epsilon^2 = 0.007$, making no attempt to produce a best fit.

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