Lawrence Berkeley National Laboratory
Recent Work

Title
ABERRATIONS AND FRINGING EFFECTS IN A l80| SIEGBAHN DOUBLE FOCUSING MAGNETIC SPECTROMETER

Permalink
https://escholarship.org/uc/item/0sh435n5

Authors
Judd, David L.
Bludman, Sidney A.

Publication Date
1956-09-01

UNIVERSITY OF CALIFORNIA

Radiation Laboratory

BERKELEY, CALIFORNIA

DISCLAIMER

UNIVERSITY OF CALIFORNIA

Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

ABERRATIONS AND FRINGING EFFECTS
IN A 180° SIEGBAHN DOUBLE FOCUSING MAGNETIC SPECTROMETER

David L. Judd and Sidney A. Bludman

September 1956

ABERRATIONS AND FRINGING EFFECTS

IN A 180° SIEGBAHN DOUBLE FOCUSING MAGNETIC SPECTROMETER

David L. Judd and Sidney A. Bludman

Radiation Laboratory, University of California
Berkeley, California

ABSTRACT

Second and third order effects, including fringing, are considered for a 180° symmetrical double focusing spectrometer. The optic axis and focal points for particles of a definite momentum are found. Values are calculated for β and γ, the coefficients of the quadratic and cubic terms in the expansion of the magnetic field, which eliminate second and third order aberrations due to radial motion. The linear dispersion and the orientation of the focal plane for paraxial trajectories of slightly different momenta are determined. Numerical values are presented for a high transmission alpha-particle spectrometer designed for this laboratory.

ABERRATIONS AND FRINGING EFFECTS

IN A 180° SIEGBAHN DOUBLE FOCUSING MAGNETIC SPECTROMETER

David L. Judd and Sidney A. Bludman

Radiation Laboratory, University of California
Berkeley, California

I. INTRODUCTION

Several years ago one of us[1] investigated some of the lowest

order magneto-optical properties of a class of magnetic spectrometers

generalized from that proposed by Siegbahn and Svartholm.[2] In these

instruments one or both of the conjugate foci may lie outside the region

of magnetic field, and it was recognized that the treatment of paraxial

rays, to determine the Gaussian optics of the system, could not be

extended to calculate aberrations without at the same time including a

discussion of the effects on the trajectories of magnetic fringing where

the particles enter and leave the field. This paper is devoted to an

examination of the second order aberrations and of fringing effects of

comparable magnitude, together with some calculations relating to third

order aberrations. This work was undertaken in connection with the design

and construction of a 180° symmetrical double focusing alpha-particle

spectrometer of the Siegbahn type by F. Asaro and I. Perlman at this

laboratory, and many of our calculations are specialized to this case.

This instrument is now complete but is still undergoing tests to determine

its optical quality. Its properties will be reported at a later date,

---

[1] D. L. Judd, Rev. Sci. Instr. 21, 213 (1950).

[2] K. Siegbahn and N. Svartholm, Nature 157, 872 (1946); Ark. f. Mat. Astr. o. Fys. 33A, No. 21 (1946).

but the present authors are honored to have this opportunity to contribute
an account of their theoretical calculations to this first issue of <u>Nuclear</u>
<u>Instruments</u>, together with their best wishes for its success and their
congratulations to Professor Siegbahn for initiating its publication.

II. SCOPE OF THE PRESENT CALCULATIONS

In building a 180° double-focusing magnetic spectrometer it is simplest to construct the magnet poles of exactly semicircular shape, with a gap separation depending only on the radial distance r from the axis of symmetry in cylindrical coordinates r, θ, z. This will generate a magnetic field, symmetric about z = 0, the midplane between the poles, and independent of the azimuth θ well inside the poles. Near θ = ± 90°, in the fringing region, the magnetic field decreases to zero continuously over a distance of the order of magnitude of the pole gap. Neglecting the finite extent of the fringing field and considering only paraxial rays, the optic axis of the system consists of the semicircle r = R, z = 0, inside the field $|\theta| \le \pi/2$ and of the rectilinear extensions of this semicircle outside the pole edges. The theory of paraxial rays[1] shows that, if the field inside the poles is shaped so that

$H_z(r) = H_0(1 - \frac{1}{2}\rho + ...)$ in the midplane $\left(\rho \equiv (r - R)/R , \right.$ $\left. H_0 \equiv (p_0 c)/(e R) \right)$, then a point source of monoenergetic charged particles of momentum p_0 and charge e, placed on the optic axis outside the field, will produce a paraxial point image. Source and image will be symmetrically located if each lies a distance $d_0 = 2^{\frac{1}{2}} \cot(2^{-3/2} \pi)$ beyond the pole edges. If the source emits particles of slightly different momentum $p \equiv p_0(1 + \Delta)$ the paraxial image will be displaced by a distance $4 R \Delta$ normal to the optic axis.

In order to determine and optimize the resolution attainable with such a spectrometer, it is necessary to calculate the size and shape of the image produced by a pencil of rays occupying a finite solid angle. This involves expanding the field and the equations of the trajectories to

include terms of second order in the displacements normal to the optic axis, and then selecting the second field shape parameter β in the expansion

$$H_z(r)\Big|_{z=0} = H_0(1 - \tfrac{1}{2}\rho + \beta\rho^2 + \gamma\rho^3 + \ldots)$$

so as to minimize the resulting image size. It is known that it is impossible to select β so as to reduce both dimensions of the image to zero to second order, but it is of greater importance to eliminate aberrations in the radial direction than those normal to it, since a line image of a point source, oriented normal to the direction of dispersion, is acceptable. Furthermore, it is convenient to achieve a solid angle of acceptance large enough to be useful by admitting a pencil of rays having a radial extent an order of magnitude larger than its axial height; thus aberrations due to displacements in z are intrinsically much smaller than radial aberrations. We have therefore calculated and eliminated aberrations in the plane z = 0 and estimated the effects of z displacements on the image size and shape.

Since it is desired to use as wide a radial aperture as possible, we will also calculate third order terms in the displacements so as to arrive at an optimum value of γ , the third order field shape parameter, which will eliminate aberrations of the corresponding order in the plane. In a careful treatment of this plane problem, it is necessary to take account of the effects of fringing. The fringing effects will be measured by a parameter a , defined as the distance along the optic axis over which the fringing field falls by an order of magnitude, measured in units of R. The effects of fringing are as follows. (1) The optic axes outside the magnet are bent through equal angles of order a , bringing image and object closer to each other. (2) The symmetrical paraxial conjugate foci (source and image points) are moved along the optic axis by a distance of order a.

(3) The optimum values of β, γ, ;.. (defined as those eliminating aberrations in the midplane for $\Delta = 0$) are altered from their values in the absence of fringing by amounts of order a. Each of these effects may be expanded in a power series in a, the coefficients involving integrals of the field shape in the fringing region. Effects (1) and (2) are of no importance in magnet design, but the focusing effect (3) leads to a description of the midplane field shape as follows:

$$H_z(r)\Big|_{z=0} = H_0 \left[1 - \tfrac{1}{2}\rho + (\beta_0 + \beta_1 a + \beta_2 a^2 + \ldots)\rho^2 \right.$$

$$\left. + (\gamma_0 + \gamma_1 a + \ldots)\rho^3 + (\delta_0 + \ldots)\rho^4 + \ldots \right]$$

Since ρ and a are both regarded as small quantities, it is consistent to evaluate β_0, β_1, γ_0 but not β_2, γ_1, or to evaluate β_0, β_1, β_2, γ_0, γ_1, δ_0 but not β_3, γ_2, δ_1, etc. In this paper we make the former choice, since β_2 depends in a very detailed way on the shape of the fringing field. (It should also be noted that, under the assumptions made above, a will be considerably smaller than the largest values of ρ we hope to use, so that in future work it would be more important to evaluate δ_0 and γ_1 than to determine β_2.)

III. TRAJECTORY CALCULATIONS INSIDE THE MAGNET

We use the cylindrical coordinates introduced above in the region where the field has axial symmetry, and choose units of length, field strength, and momentum such that $R = H_0 = p_0 = 1$. The exact orbit equations then are

$$(1+\rho)\rho'' - 2\rho'^2 - (1+\rho)^2 =$$

$$\left[(1+\rho)^2 + \rho'^2 + z'^2\right]^{\frac{1}{2}} \left\{ -\left[(1+\rho)^2 + \rho'^2\right] H_z + \rho' z' H_r \right\} (1+\Delta)^{-1},$$

$$(1+\rho)z'' - 2\rho'z' =$$

$$\left[(1+\rho)^2 + \rho'^2 + z'^2\right]^{\frac{1}{2}} \left\{ \left[(1+\rho)^2 + z'^2\right] H_r - \rho' z' H_z \right\} (1+\Delta)^{-1},$$

where primes denote differentiations with respect to θ. The sign of the magnetic field has been chosen so that a positively charged particle will move in the direction of increasing θ. The field components H_z and H_r are related by $\nabla \cdot \vec{H} = \nabla \times \vec{H} = 0$, and can be expanded about the reference circle as follows:

$$H_z = \left[1 - \tfrac{1}{2}\rho + \beta\rho^2 + \gamma\rho^3 + \ldots\right]$$
$$+ z^2\left[(\tfrac{1}{4} - \beta) + (-\tfrac{1}{4} - \beta - 3\gamma)\rho + \ldots\right] + \ldots,$$

$$H_r = z\left[-\tfrac{1}{2} + 2\beta\rho + \ldots\right] + \ldots.$$

This scheme may be readily continued by use of the definitions

$$H_z = \sum_{k \geq 0} \sum_{\ell \text{ even}} A_{k\ell}\, \rho^k z^\ell \,, \qquad H_r = \sum_{k \geq 0} \sum_{\ell \text{ odd}} C_{k\ell}\, \rho^k z^\ell \,,$$

$$A_{00} = 1, \quad A_{10} = -\tfrac{1}{2}, \quad A_{20} = \beta \,, \quad A_{30} = \gamma \,, \quad C_{00} = 0, \quad A_{k\ell} = 0 \text{ for } k < 0 \,,$$

and the relations

$$(\ell + 1)C_{k,\ell+1} = (k+1)A_{k+1,\ell} \qquad\qquad (\text{even } \ell)$$

$$(k+1)(C_{k\ell} + C_{k+1,\ell}) = -(\ell+1)(A_{k-1,\ell+1} + A_{k,\ell+1}) \qquad (\text{odd } \ell).$$

We will retain in the equations of motion terms linear in Δ, of second order in z and z', and of third order in ρ and ρ', obtaining the expanded orbit equations

$$\rho'' + \tfrac{1}{2}\rho = \Delta(1 + 3\rho/2) - (1 - \Delta)\beta\rho^2 + \tfrac{1}{2}(1 + 3\Delta)\rho'^2$$

$$+ (1 - \Delta)(\beta - \tfrac{1}{4})z^2 - \tfrac{3}{2}(1 - \Delta)z'^2 + (1 - \Delta)(\tfrac{1}{2} - 2\beta - \gamma)\rho^3$$

$$- \left[2 - \tfrac{3}{2}(1 - \Delta) \right] \rho\rho'^2 + \cdots$$

$$z'' + \tfrac{1}{2}z = \tfrac{1}{2}\Delta z + (1 - \Delta)(2\beta - 1)\rho z + (1 + \Delta)\rho' z' + \cdots \,.$$

To optimize β and γ, for $\Delta = 0$, we will deal only with trajectories symmetric about $\theta = 0$. A point source and its image will thus be symmetrically arranged with respect to the magnet in every order calculated. The solution for this case, obtained by iteration, is

$$\rho(\theta) = \rho_m \cos\psi + (\rho_m^2/12)\left[\, 3(1 - 4\beta)(1 - \cos\psi)\right.$$

$$\left. + (1 + 4\beta)(\cos 2\psi - \cos\psi)\right]$$

$$+ \rho_m^3\left\{ \frac{1}{32}\left[\left(\frac{9}{4} - 4\beta - 2\gamma\right) - \frac{1}{3}\left(1 + 2\beta\right)\left(1 + 4\beta\right)\right](\cos\psi - \cos 3\psi)\right.$$

$$+ \frac{1}{6}(1 - 2\beta)(1 - 4\beta)(\cos\psi - 1) + \frac{1}{18}\left(1 - 2\beta\right)\left(1 + 4\beta\right)(\cos\psi - \cos 2\psi)$$

$$\left. + \frac{1}{8}\left[3(1 - 4\beta - 2\gamma) - \frac{5}{4} + \frac{1}{3}\left(1 - 10\beta + 40\beta^2\right)\right]\psi\sin\psi \right\} + \mathcal{O}(\rho_m^4)\ .$$

where $\psi \equiv 2^{-\frac{1}{2}}\theta$ and ρ_m is the displacement at $\theta = 0$. This function
will later be joined to a solution obtained separately in the region of
fringing field in order to optimize β and γ.

To discuss dispersive effects arising from small but finite Δ,
it will be necessary to consider trajectories that are not symmetric about
$\theta = 0$. Here, however, we will be content with second order terms in ρ.
Let ρ_i and ρ_i' be the displacement and its derivative at an arbitrary
initial value θ_i. Then for $z \equiv 0$ we obtain by iteration

$$\rho(\theta - \theta_i) = \rho_i \cos\psi + 2^{\frac{1}{2}}\rho_i'\sin\psi + 2\Delta(1 - \cos\psi)$$

$$+ \Delta\left\{ -(1 - 4\beta)\rho_i + \left(\frac{3}{2} - 4\beta\right)\left[\rho_i\,\psi\sin\psi - 2^{\frac{1}{2}}\rho_i'(\psi\cos\psi - \sin\psi)\right]\right.$$

$$\left. -\left(\frac{1 + 4\beta}{3}\right)\left[\rho_i(\cos 2\psi - \cos\psi) + 2^{\frac{1}{2}}\rho_i'(\sin 2\psi - 2\sin\psi)\right]\right\}$$

$$+ \left(\frac{1 - 4\beta}{4}\right)(\rho_i^2 + 2\rho_i'^2)(1 - \cos\psi) + \left(\frac{1 + 4\beta}{12}\right)(\rho_i^2 - 2\rho_i'^2)(\cos 2\psi - \cos\psi)$$

$$+ \left(\frac{1+4\beta}{6}\right) 2^{\frac{1}{2}} \rho_i \rho_i'(\sin 2\psi - 2\sin\psi)$$

$$+ \left(\frac{1+4\beta}{3}\right) 2^{\frac{1}{2}} \rho_i \rho_i'(\sin 2\psi - 2\sin\psi) + \mathcal{O}(\Delta\rho_i^2, \rho_i^3),$$

where

$$\psi \equiv 2^{-\frac{1}{2}}(\theta - \theta_i).$$

To study the influence of displacements in z on the focal properties with respect to ρ, we may add to the foregoing equation those terms of second order in z_i and z_i', and independent of Δ, which arise from iterating the coupled equations for ρ and z through second order. These added terms appear on the right side as

$$z_i^2 \left[(\beta - \tfrac{1}{2})(1 - \cos\psi) - \frac{\beta}{3}(\cos 2\psi - \cos\psi)\right]$$

$$- 2^{\frac{1}{2}} z_i z_i' \frac{2\beta}{3}(\sin 2\psi - 2\sin\psi)$$

$$+ z_i'^2 \left[(2\beta - 1)(1 - \cos\psi) + \frac{2\beta}{3}(\cos 2\psi - \cos\psi)\right].$$

The results obtained above agree with the second order formulas of Shull and Dennison[3] for the special case $\psi = \pi$, $\Delta = 0$ and with the results of Svartholm[4] for the special case $\rho_i = z_i = \Delta = 0$ in the absence of his electric field but for arbitrary ψ.

---

[3] F. B. Shull and D. M. Dennison, Phys. Rev. 71, 681 (1947), Eqs. (25) and (26), as corrected in Phys. Rev. 72, 256 (1947).

[4] N. Svartholm, Arkiv f. Fys. 2, 195 (1950).

IV. THE FRINGING REGION

To discuss fringing we employ a separate Cartesian coordinate system in and beyond each fringing region. In what follows we refer explicitly to the image side; quantities referring to the source side are related to these by obvious changes of sign; for $\Delta = 0$ the trajectories from a point source are completely symmetrical about the plane $\theta = 0$. The Cartesian and cylindrical z-coordinates coincide. The origin of Cartesian coordinates is at $r = R$, $\theta = \dfrac{\pi}{2} - b$. The y axis is directed radially outwards in the plane $z = 0$, and the x axis is directed so that particles move with increasing x. Displacements and slopes of trajectories will be matched between polar and Cartesian systems along the y axis, which separates their domains of applicability. The same unit of length will be adopted in these systems so that, at $\theta = \dfrac{\pi}{2} - b$ or $x = 0$, $\rho = y$ and $\rho'/(1 + \rho) = \dfrac{dr}{rd\theta} = y' \equiv dy/dx$. Both b, the angle between the y axis and the magnet edge, and a, the field fall-off distance, are of the order of the physical separation D of the magnet poles at $r = R$.

The rigorous trajectory equation for particles in the plane $z = 0$ having $\Delta = 0$ may be written in Cartesian coordinates as

$$\frac{d}{dx}\left[y'/(1 + y'^2)^{\frac{1}{2}} \right] = - h(x, y) \quad ,$$

where $h(x, y) = H_z(x, y)/H_0$. Now we assume that the fringing may be characterized by a single function $h(x)$ multiplied by the radial shape factor $(1 - \frac{1}{2} y + \beta y^2)$; this assumption contains two approximations which are believed to lead to negligible error but which could be corrected, if fringing field measurements showed it to be necessary: the factorization approximation seems to describe the empirical variation with gap height to

within a few percent, while variations in the orientation of the axes within the small angle b seem to lead to effects of higher order than those considered here.

Under this assumption we integrate once to obtain

$$y'/(1+y'^2)^{\frac{1}{2}} = \rho'/\left[(1+\rho)^2 + \rho'^2\right]^{\frac{1}{2}} - (1 - \tfrac{1}{2}\rho + \beta\rho^2)f(x) + \tfrac{1}{2}\rho' g(x) ,$$

where

$$f(x) \equiv \int_0^x h(x')dx' \qquad \text{and} \qquad g(x) \equiv \int_0^x x'h(x')dx' .$$

In the term linear in y on the right side, we have approximated $y(x)$ by $\rho + \rho' x$, and y^2 by ρ^2, before integrating. In these and the following equations ρ and ρ' are evaluated at the matching lines $x = 0$. We next solve this equation for y' by iteration and expand in powers of ρ and ρ', obtaining

$$y'(x) = \rho'(1 - \rho + \rho^2) - (1 - \tfrac{1}{2}\rho + \beta\rho^2 + \tfrac{3}{2}\rho'^2)f + (\tfrac{3}{2}\rho' - 3\rho\rho')f^2$$

$$+ \tfrac{1}{2}\rho'g - \tfrac{1}{2}f^3 + \cdots .$$

Integrating once more, we have

$$y(x) = \rho + \rho'(1 - \rho + \rho^2)x - (1 - \tfrac{1}{2}\rho + \beta\rho^2 + \tfrac{3}{2}\rho'^2)\int_0^x f\, dx$$

$$+ (\tfrac{3}{2}\rho' - 3\rho\rho')\int_0^x f^2\, dx + \tfrac{1}{2}\rho'\int_0^x g\, dx - \tfrac{1}{2}\int_0^x f^3\, dx + \cdots .$$

The optic axis $y_{OA}(x)$ is obtained by setting $\rho = \rho' = 0$.
The paraxial focal point x_0 is obtained by setting $y(x_0) = y_{OA}(x_0)$ and retaining only the terms linear in ρ_m; the field shape parameters β,

γ , ... may be selected by requiring $y(x_0) = y_{OA}(x_0)$ to higher orders in ρ .

Since $h(x)$ is negligible at $x \sim x_0$ the integrals involved may be expressed as

$$\int_0^x f(x')dx' = C_1\, a\, x + C_2\, a^2 \quad ;$$

$$\int_0^x f^2\, dx = C_1^{\,2}\, a^2\, x + \mathcal{O}(a^3) \quad ; \qquad\qquad x \gg a$$

$$\int_0^x g(x')dx' = C_3\, a^2\, x + \mathcal{O}(a^3) \quad ,$$

with/coefficients C_1, C_2, and C_3 of order unity. Integration by parts shows that $C_2 = -C_3$; since in this paper we retain only terms of first order in a, C_2 and C_3 will not appear in the present results for β and γ.

Theoretical studies[5] for plane parallel poles and experimental measurements[6] on various magnets have shown that the fringing field may be roughly described by a decreasing exponential

$$h(s) \simeq \begin{cases} 1 & s < -b \\ \exp\left[-(s-b)/a\right] & s > -b \end{cases} .$$

The approximate expression does not apply in a very small region near $s = -b$, where the sharp corner of the approximate formula must actually round off smoothly. For large distances, the already weak field actually

[5] N. D. Coggeshall, J. App. Phys. 18, 855 (1947).

[6] E.g., C. Mileikowsky, Arkiv f. Fys. 7, 33 (1953), Fig. 2, and unpublished measurements by C. Dols at this laboratory.

decays somewhat slower than exponentially. Here s is the distance
measured outward from the pole edge; in our coordinate system it would be
replaced by $x + b$. While for orientation one could adopt this exponential x
dependence for the slanted poles, we have preferred to evaluate C_1a by
numerical integration of a typical fringing field, taking $b = \frac{1}{2}D$, and
obtaining $C_1a \simeq (4/3)D$. Careful measurements of the actual fringing
field of a completed instrument could be used to refine this value of C_1a,
to examine the validity of the fringing assumptions made above, and to
determine C_3a^2. This information could then be used as a guide in making
minor pole tip changes in the fringing region if necessary.

V. EVALUATION OF β AND γ

Using the forms defined above for the fringing integrals, we may write

$$y(x_0) - y_{OA}(x_0) = \rho + (\tfrac{1}{2}\rho + \beta\rho^2 + \tfrac{3}{2}\rho'^2)c_3\, a^2$$

$$+ \left[\rho' - \rho\rho' + \rho^2\rho' - (\tfrac{1}{2}\rho + \beta\rho^2 + \tfrac{3}{2}\rho'^2)c_1\, a + (\tfrac{3}{2}\rho' - 3\rho\rho')c_1^2\, a^2\right.$$

$$\left. + \tfrac{1}{2}\rho'\, c_3 a^2\right] x_0 + \mathcal{O}(a^3,\ \rho^3 a,\ \rho^4)\ .$$

Here $\rho = \rho_0 + \rho_1 + \rho_2$, $\quad \rho' = \rho_0' + \rho_1' + \rho_2'$, (where ρ_0 is of order ρ_m, ρ_1 of order ρ_m^2, etc.) are obtained from the first trajectory solution given in Section III and its derivative, and are evaluated at $\theta = \tfrac{1}{2}\pi - b$. Expanding in powers of ρ_m, we obtain first

$$0 = \rho_0(1 - \tfrac{1}{2}c_3\, a^2) + \left\{ \rho_0'\left[1 + \tfrac{1}{2}(c_3 + 3\, c_1^2)a^2\right] + \tfrac{1}{2}\rho_0\, c_1 a\right\} x_0 = 0\ ;$$

hence

$$x_0 = d\left[1 + \tfrac{1}{2}c_1\, d\, a - \tfrac{1}{4}(6 - d^2)c_1^2\, a^2 + \dots\right]\ ,$$

where

$$d \equiv -\rho_0/\rho_0' = 2^{\tfrac{1}{2}}\cot 2^{-3/2}(\pi - 2b)$$

is the distance from focal point to the joining line/ if the fringing integrals vanish. We next evaluate y_{OA} at the focal point, obtaining

$$y_{OA}(x_0) = -c_1\, a\, x_0 + c_3\, a^2 + \mathcal{O}(a^3)\ ,$$

which displays the additional bending of the optic axes due to fringing.

Terms of the next order in ρ_m, when collected, yield

$$\frac{\rho'}{\rho_0} - \frac{\rho_1'}{\rho_0'} + \rho_0 \left[1 - (\beta + \tfrac{3}{4}\tan^2\psi)c_1\,d\,a\right] + \mathcal{O}(a^2) = 0 \ .$$

The terms in this equation which are independent of a determine β_0; those linear in a fix β_1, etc. We have for β_0 the equation $\rho_1 + d\,\rho_1' = -\rho_0^2$, where from Section III

$$\rho_0 = \rho_m \cos\psi, \qquad \rho_0' = 2^{-\tfrac{1}{2}}\rho_m \sin\psi,$$

$$\rho_1 = (\rho_m^2/12)\left[3(1-4\beta)(1-\cos\psi) + (1+4\beta)(\cos 2\psi - \cos\psi)\right] \ ,$$

$$\rho_1' = 2^{-\tfrac{1}{2}}(\rho_m^2/12)\left[3(1-4\beta)\sin\psi - 2(1+4\beta)(\sin 2\psi - \sin\psi)\right] \ .$$

Substituting and solving for β_0 we obtain

$$\beta_0 = \tfrac{1}{4}\,\frac{7+5\cos 2\psi}{5+\cos 2\psi} \ ,$$

where $2^{\tfrac{1}{2}}\psi = \tfrac{1}{2}\pi - b$, and would be $\tfrac{1}{2}\pi$ in the absence of fringing. If we repeat, retaining terms linear in a, we obtain $\beta = \beta_0 + \beta_1 a + \mathcal{O}(a^2)$, with

$$\beta_1 = -\frac{3}{2}\,\frac{1+\cos 2\psi}{5+\cos 2\psi}\left(\beta_0 + \tfrac{3}{4}\tan^2\psi\right)c_1\,d \ .$$

In a similar manner one could evaluate β_2 if desired.

In collecting terms of third order in ρ_m to evaluate γ we will, as explained earlier, retain only those independent of a, leading to an evaluation of γ_0 but neglecting γ_1. This equation is

$$\rho_2 + d\,\rho_2' = -2\rho_0\rho_1 \ .$$

After substituting, rearranging terms, and separating those involving γ_0, the equation is

$$\frac{\gamma_0}{8} (9 \cos \psi + \cos 3 \psi + 6 \psi / \sin \psi) + \frac{1}{18} (7 - 34\beta + 40\beta^2)$$

$$+ (- \frac{231}{192} + \frac{113}{24} \beta - \frac{7}{6} \beta^2) \cos \psi + (\frac{5}{18} - \frac{7}{9} \beta + \frac{4}{9} \beta^2) \cos 2 \psi$$

$$+ (- \frac{39}{192} + \frac{1}{24} \beta + \frac{1}{6} \beta^2) \cos 3 \psi + (- \frac{25}{96} + \frac{23}{12} \beta - \frac{5}{3}\beta^2)(\psi / \sin \psi) = 0 .$$

Numerical values for β_0, β_1, and γ_0 will be presented in Section VII.

VI. DISPERSION

To determine the linear dispersion properties of the field we neglect fringing effects and examine those terms of the solution found in Section III for $\Delta \neq 0$ which are of order ρ, Δ, and $\rho\Delta$. We now regard $2^{\frac{1}{2}} \psi_i = -\pi/2$ as the point of entry into the magnet and $2^{\frac{1}{2}}\psi_f = \pi/2$ as the point of exit. If a point source is located on the optic axis a distance d_0 from the magnet, the particles will travel in straight lines to the pole edge where their displacement and slope will be related by $\rho_i = d_0 \rho_i{}'$. By inserting this relation into the terms to be studied and their derivatives, we obtain the displacement and slope of a trajectory at any angle inside the magnet. We evaluate these expressions at the point of exit and, after some reduction, obtain

$$\rho_e = \rho_i(1 + \alpha_1 \Delta / d_0) + \alpha_0 \Delta \quad ,$$

$$\rho_e{}' = -\rho_i{}'(1 - \bar{\alpha}_1 \Delta d_0/2) + \alpha_0 \Delta d_0/2 \quad ,$$

where

$$\alpha_0 = 2(1 - \cos 2\psi_f) \quad ,$$

$$\alpha_1 = 2^{\frac{1}{2}} \left[\frac{3}{2} (2\psi_f + \sin 2\psi_f) + 4\beta (\sin 2\psi_f - 2\psi_f) \right] \quad ,$$

$$\bar{\alpha}_1 = \alpha_1 + (4/3)(1 + 4\beta)(1 - \cos 2\psi_f)/d_0 \quad .$$

The trajectories after leaving the magnet are straight lines whose equations are $y = \rho_e{}' x + \rho_e$, or $y - y_0 = \rho_e{}' (x - x_0)$, where y_0 and x_0 are the coordinates of the paraxial image, and depend linearly on Δ. An elementary calculation yields

$$y_0 = \alpha_0(1 + \frac{d_0{}^2}{2})\Delta = 4\Delta \quad ,$$

or the linear dispersion $y_0/\Delta = 4$, as expected for any symmetrical instrument. For finite Δ the image is also displaced in x, by the amount

$$x_0 - d_0 = (q_1' + \tfrac{1}{2} d_0^2 \overline{q}_1') \Delta .$$

The focal plane for paraxial rays is obtained by eliminating Δ between these two equations, and is not normal to the optic axis, but is turned in the sense of increasing θ through an angle $\emptyset$ from this position.

$$\tan \emptyset = \tfrac{1}{4}(q_1' + \tfrac{1}{2} d_0^2 \overline{q}_1') .$$

Aberrations due to terms of order ρ^2 will not vanish for $\Delta \neq 0$, but will produce a cross-fire of trajectories leading to images of finite radial extent. These will not lie in the focal plane obtained above, but in a curved focal surface tangent to this plane at the image point for $\Delta = 0$. Estimates, not to be presented here, indicate that for large radial apertures the range of momenta in good focus is quite small.

VII. NUMERICAL RESULTS

The spectrometer of Asaro and Perlman at this laboratory has been
designed with $R = 35$ cm; the pole gap DR at this radius is 2.54 cm, so
that $D = 0.0726$. Having chosen $b = \frac{1}{2}D$, we use $2^{\frac{1}{2}} \psi = \frac{1}{2}\pi - 0.0363$,
to calculate $\beta_0 = 0.236$, and $\beta_1 a = -0.032$, where we have inserted
$c_1 a = (4/3)D$. We thus obtain $\beta \approx 0.204 + \mathcal{O}(a^2)$. The actual value
chosen for this instrument was $\beta = 0.209$, based on estimates of the
neglected terms, although this difference amounts to a change in field
strength of only about 0.05% for $|\rho|$ as large as 0.3. Using this
value of β, but inserting $\psi = 2^{-3/2}\pi$, we calculate $\gamma_0 = -0.15$.
An uncertainty in γ of 0.04 corresponds to a change in field strength
of 0.1% for $|\rho|$ as large as 0.3. Just as $\beta_1 a$ is an eighth of β_0,
one might expect $\gamma_1 a$ to be an order of magnitude smaller than γ_0,
so that the neglected terms correspond to higher accuracy than can be
attained in field shaping. Terms higher than the third order have not been
considered; their effects are difficult to predict and should perhaps
already be considered for this large an aperture.

The angle of rotation ϕ of the paraxial focal plane, calculated
using the full sector angle, is found to be $\tan^{-1} 1.78 \approx 60^\circ$.

The neglect of detailed calculations of motion in z requires
justification here. The terms quadratic in z_i and z_i' given in
Section III have been evaluated for $z_i = d_0 z_i'$ and extrapolated into
the image space, where they produce a radial spread of the image. This
spread is of order $\left[2^{-3/2} D \sin (2^{-3/2}\pi) \right]^2$, is extremely small for
the small D used here, and contributes a negligible amount to the image

size. The resolution of the instrument will therefore be determined
primarily by the source size and by residual magnetic field errors. The
image height can be estimated from the terms of order ρz in the z
equation of motion, and is of order 0.1 cm for the parameters used above.

The solid angle accepted by the magnet is approximately
$\frac{1}{2} \rho_m D \sin^2 2^{-3/2} \eta'$, which is 0.017 steradians for $\rho_m = 0.3$ and the
D used above.

Preliminary measurements on this spectrometer have indicated a
resolution of 0.1% in energy with a solid angle of about 0.009 steradians.

The authors are grateful to Dr. Asaro for many helpful discussions
and for his unpublished performance data, and to Mr. James Baker for
extensive assistance with the calculations. This work was performed
under the auspices of the United States Atomic Energy Commission.