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INTEGRATION OF TWISTED DIRAC BRACKETS

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Abstract

Given a Lie groupoid G over a manifold M, we show that multiplicative 2-forms on G relatively closed with respect to a closed 3-form ϕ on M correspond to maps from the Lie algebroid of G into T^*M satisfying an algebraic condition and a differential condition with respect to the ϕ-twisted Courant bracket. This correspondence describes, as a special case, the global objects associated to ϕ-twisted Dirac structures. As applications, we relate our results to equivariant cohomology and foliation theory, and we give a new description of quasi-Hamiltonian spaces and group-valued momentum maps.

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1. Introduction

The correspondence between Poisson structures and symplectic groupoids (see [7], [9], [14]), analogous to the correspondence between Lie algebras and Lie groups, plays an important role in Poisson geometry; it offers, in particular, a unifying framework for the study of Hamiltonian and Poisson actions (see, e.g., [31]). In this paper we extend this correspondence to the context of Dirac structures twisted by a closed 3-form.

Dirac structures were introduced by Courant in [10] and [11], motivated by the study of constrained mechanical systems. Examples of Dirac structures include Poisson structures, presymplectic forms, and regular foliations. Connections between Poisson geometry and topological sigma models (see [20], [26]) have led to the notion of Poisson structures twisted by a closed 3-form, which were described by Severa and Weinstein in [27] as special cases of twisted Dirac structures. Figure 1 describes these various generalizations of Poisson structures and the corresponding global objects.

In this paper we fill the gaps in Figure 1 by introducing the notion of a *twisted presymplectic groupoid* relative to a closed 3-form ϕ on a manifold M and establishing a bijective correspondence (up to natural isomorphisms) between source simply connected ϕ-twisted presymplectic groupoids over M and ϕ-twisted Dirac structures. Our results complete, and were inspired by, earlier work of Bursztyn and Radko [5] on gauge transformations of Poisson structures, and of Cattaneo and Xu [8], who used the method of sigma models (see [7]) to construct global objects attached to twisted Poisson structures. We remark that the degeneracies of presymplectic forms force our techniques to be different in nature from those used to establish the correspondence between Poisson structures and symplectic groupoids (as in [9], [29], [8]). In our search for the right notion of nondegeneracy which 2-forms on presymplectic groupoids must satisfy (Def. 2.1), the examples provided by [5] and [14] were essential. (In fact, our presymplectic groupoids are closely related to those introduced in [14].)

Twisted presymplectic groupoids turn out to be best approached through a general

$$\begin{tikzcd}
\boxed{\text{Poisson structures}} \arrow[r, leftrightarrow] \arrow[d] \arrow[ddd, bend right] & \boxed{\text{symplectic groupoids}} \arrow[d] \arrow[ddd, bend left] \\
\boxed{\text{Dirac structures}} \arrow[r, leftrightarrow] \arrow[d] & \boxed{?} \arrow[d] \\
\boxed{\text{twisted Dirac structures}} \arrow[r, leftrightarrow] & \boxed{\text{twisted } ?} \\
\boxed{\text{twisted Poisson structures}} \arrow[r, leftrightarrow] \arrow[u] & \boxed{\text{twisted symplectic groupoids}} \arrow[u]
\end{tikzcd}$$

Figure 1

study of multiplicative 2-forms on groupoids, which turn out to be extremely rigid. (On groups, they are all zero.) Given a closed 3-form ϕ on the base M of a groupoid G, we call a 2-form ω on G *relatively ϕ-closed* if $d\omega = s^*\phi - t^*\phi$, where s and t are the source and target maps of G. If G is source simply connected, we identify the infinitesimal counterparts of relatively ϕ-closed multiplicative 2-forms on G as those maps to T^*M from the Lie algebroid A of G which satisfy an algebraic condition and a differential condition related to the ϕ-twisted Courant bracket (see [27]). In fact, the (twisted) Courant bracket itself could be rediscovered from the properties of (relatively) closed 2-forms on groupoids. The reconstruction of multiplicative 2-forms out of the infinitesimal data is based on the constructions of [13, Sec. 4.2].

Motivated by the relationship between symplectic realizations of Poisson manifolds and Hamiltonian actions (see [9]), we study *presymplectic realizations* of twisted Dirac structures. Just as in usual Poisson geometry, these presymplectic realizations carry natural actions of presymplectic groupoids. In fact, it is this property that determines our definition of presymplectic realizations. An important example of twisted Dirac structure is described in [27, Exam. 4.2]: any nondegenerate invariant inner product on the Lie algebra $\mathfrak{h}$ of a Lie group H induces a natural Dirac structure on H, twisted by the invariant Cartan 3-form; we call such structures *Cartan-Dirac structures*. We show that presymplectic realizations of Cartan-Dirac structures are equivalent to the quasi-Hamiltonian $\mathfrak{h}$-spaces of Alekseev, Malkin, and Meinrenken [1] in such a way that the realization maps are the associated group-valued momentum maps. It also follows from our results that the transformation groupoid $H \ltimes H$ corresponding to the conjugation action carries a canonical twisted presymplectic structure, which we obtain explicitly by "integrating" the Cartan-Dirac structure. As a result, we recover the 2-form on the "double" $D(H)$ of [1] and the Alekseev-Malkin-Meinrenken (AMM) groupoid of [3]. (Closely related forms were introduced earlier in [18], [30].) A unifying approach to momentum map theories based on Morita equivalence of presymplectic groupoids has been developed by Xu in [34]; much of our motivation for considering quasi-Hamiltonian spaces comes from his work. Our results indicate that Dirac structures provide a natural framework for the common description of various notions of momentum maps (as, e.g., in [1], [22]; see also [31]).

We illustrate our results on multiplicative 2-forms and Dirac structures in many examples. In the case of action groupoids, we obtain an explicit formula for the natural map from the cohomology of the Cartan model of an H-manifold (see [2], [16]) to (Borel) equivariant cohomology (see [2], [4]) in degree three; for the monodromy groupoid of a foliation $\mathscr{F}$, we show that multiplicative 2-forms are closely related to the usual cohomology and spectral sequence of $\mathscr{F}$ (see [19]).

The entire discussion of relatively closed multiplicative 2-forms on groupoids may be embedded in the more general context of a van Est theorem for the "bar–de Rham" double complex of forms on the simplicial space of composable sequences in a groupoid G, whose total complex computes the cohomology of the classifying space BG. We reserve this more general discussion for a future paper.

Outline of the paper

In Section 2 we review basic definitions concerning Dirac structures and groupoids, we introduce various notions of "adapted" 2-forms on groupoids, and we state our two main results, which allow one to go back and forth between (twisted) presymplectic groupoids and their infinitesimal counterparts, (twisted) Dirac manifolds. Section 3 begins the work of proving the theorems with a study of the global objects, namely, multiplicative 2-forms on groupoids. In Section 4 we pass from the groupoids to the infinitesimal objects, and in Section 5 we go in the opposite direction, completing the proofs of the main theorems. Section 6 begins with simple examples. Then, after a discussion of 2-forms on groupoids that become presymplectic only after one passes to the quotient by a foliation, we show how twisted-multiplicative forms on action groupoids are related to equivariant cohomology. This leads us to Section 7, where we study the AMM groupoid and apply our results to quasi-Hamiltonian actions. Finally, Section 8 is devoted to multiplicative 2-forms on foliation groupoids, with applications to Dirac structures whose presymplectic leaves all have the same dimension.

2. Basic definitions and the main results

2.1. Twisted Dirac structures

We recall some basic concepts in Dirac geometry (see [10]). Let V be a finite-dimensional vector space, and equip $V \oplus V^*$ with the symmetric pairing

$$\big\langle (x,\xi),(y,\eta)\big\rangle_+ = \xi(y) + \eta(x). \tag{2.1}$$

A *linear Dirac structure* on V is a subspace $L \subset V \oplus V^*$ which is maximal isotropic with respect to $\langle\ ,\ \rangle_+$. The set of all Dirac structures on V is a smooth submanifold of a Grassmann manifold; we denote it by $\mathrm{Dir}(V)$.

Natural examples of vector spaces carrying linear Dirac structures are presymplectic and Poisson vector spaces (i.e, spaces equipped with a skew-symmetric bilinear form and a bivector, respectively). More precisely, a skew-symmetric bilinear form θ (resp., bivector π) corresponds to the Dirac structure L_θ (resp., L_π) given by the graph of the map $\widetilde{\theta} : V \longrightarrow V^*$, $\widetilde{\theta}(x)(y) = \theta(x,y)$ $\big($resp., $\widetilde{\pi} : V^* \longrightarrow V$, $\widetilde{\pi}(\alpha)(\beta) = \pi(\beta,\alpha)\big)$.

General Dirac structures can be described in terms of either bilinear forms or bivectors. Let $\mathrm{pr}_1 : V \oplus V^* \longrightarrow V$ and $\mathrm{pr}_2 : V \oplus V^* \longrightarrow V^*$ be the natural projections. A linear Dirac structure L has an associated range,

$$\mathscr{R}(L) = \mathrm{pr}_1(L) = \big\{v \in V : (v,\xi) \in L \text{ for some } \xi \in V^*\big\} \subset V,$$

and a skew-symmetric bilinear form θ_L on $\mathscr{R}(L)$,

$$\theta_L(v_1,v_2) = \xi_1(v_2), \quad \text{where } \xi_1 \in V^* \text{ is such that } (v_1,\xi_1) \in L. \tag{2.2}$$

For the description of L in terms of a bivector, let us define the *kernel* of L as the kernel of θ_L:

$$\mathrm{Ker}(L) := \mathrm{Ker}(\theta_L) = \mathrm{pr}_2(L)^\circ = \big\{v \in V : (v,0) \in L\big\} \subset V.$$

(Here $^\circ$ stands for the annihilator.) The bivector π_L, defined on $V/\mathrm{Ker}(L)$, is the one induced by a form analogous to (2.2) on $\mathrm{pr}_2(L) \subset V^*$. It is not difficult to see that L is completely characterized by the pair $(\mathscr{R}(L), \theta_L)$ or, analogously, by the pair $(\mathrm{Ker}(L), \pi_L)$. We observe that

(i) $R(L) = V$ if and only if $L = L_\theta$ for some skew-symmetric bilinear form θ on V;

(ii) $\mathrm{Ker}(L) = 0$ if and only if $L = L_\pi$ for some bivector π on V.

If V and W are vector spaces, any linear map $\psi : V \longrightarrow W$ induces a pushforward map $\mathfrak{F}\psi : \mathrm{Dir}(V) \longrightarrow \mathrm{Dir}(W)$ by

$$\mathfrak{F}\psi(L_V) = \big\{(\psi(x),\eta)\,\big|\, x \in V,\ \eta \in W^*,\ (x,\psi^*(\eta)) \in L_V\big\}, \tag{2.3}$$

where $L_V \in \mathrm{Dir}(V)$. We note that the map $\mathfrak{F}\psi$ is *not* continuous at every L_V.

Given $L_V \in \mathrm{Dir}(V)$ and $L_W \in \mathrm{Dir}(W)$, we call a linear map $\psi : V \longrightarrow W$ *forward Dirac* if $\mathfrak{F}\psi(L_V) = L_W$. There is a corresponding concept of a backward Dirac map (see, e.g., [5]), but we do not deal with it in this paper. Hence, for simplicity, we refer to forward Dirac maps just as *Dirac maps*. As an example, we recall that a Dirac map between Poisson vector spaces is just a Poisson map.

An almost Dirac structure on a smooth manifold M is a subbundle $L \subset TM \oplus T^*M$ defining a linear Dirac structure on each fiber. Note that the dimensions of the range $\mathscr{R}(L)$ and the kernel $\mathrm{Ker}(L)$ (defined fiberwise) may vary from a point to another.

A *Dirac structure* is an almost Dirac structure whose sections are closed under the Courant bracket* $[\ ,\ ] : \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \longrightarrow \Gamma(TM \oplus T^*M)$,

$$[(X, \xi), (Y, \eta)] = ([X, Y], \mathscr{L}_X \eta - i_Y d\xi). \tag{2.4}$$

For example, a bivector field π on M corresponds to a Dirac structure if and only if it defines a Poisson structure.

As observed in [27], one can use a closed 3-form ϕ on M to modify the standard Courant bracket as follows:

$$[(X, \xi), (Y, \eta)]_\phi = \big([X, Y], \mathscr{L}_X \eta - i_Y d\xi + \phi(X, Y, \cdot)\big). \tag{2.5}$$

A Dirac structure twisted by ϕ, or simply a *ϕ-twisted Dirac structure*, is an almost Dirac structure whose sections are closed under $[\cdot, \cdot]_\phi$. A bivector π defines a ϕ-twisted Dirac structure if and only if

$$[\pi, \pi] = 2(\wedge^3)\tilde{\pi}(\phi),$$

where $[\ ,\ ]$ is the Schouten bracket. Similarly, a 2-form ω defines a ϕ-twisted Dirac structure if and only if $d\omega + \phi = 0$. These two kinds of Dirac structures are called *ϕ-twisted Poisson structures* and *ϕ-twisted presymplectic structures*.

Let (M, L_M, ϕ_M) and (N, L_N, ϕ_N) be twisted Dirac manifolds. A smooth map $\psi : M \longrightarrow N$ is called a (forward) *Dirac map* if $\mathfrak{F}(d\psi)_x((L_M)_x) = (L_N)_{\psi(x)}$ for all $x \in M$. (Here, and throughout this paper, we denote by $(df)_x : T_xM \longrightarrow T_{f(x)}N$ the differential of a smooth function $f : M \longrightarrow N$ at the point $x \in M$.)

We observe that the pushforward operation between Dirac structures defined in the linear case (2.3) is not well defined for manifolds in general. For instance, if $\psi : M \longrightarrow N$ is a surjective submersion and L_M is a Dirac structure on M, the pointwise pushforward structures $\mathfrak{F}d_x\psi((L_M)_x)$ may differ for points x along the same ψ-fiber;

*This is the non-skew-symmetric version of Courant's [10] original bracket, as introduced in [21] and used in [27].

even if they coincide, the resulting family of vector spaces might not be a subbundle because of discontinuities of the map $\mathfrak{F}$.

2.2. Groupoids and algebroids

Throughout the text, G denotes a Lie groupoid. We denote the unit map by $\varepsilon : M \longrightarrow G$, the inversion by $i : G \longrightarrow G$, and the source (resp., target) map by $s : G \longrightarrow M$ (resp., $t : G \longrightarrow M$). We denote the set of composable pairs of groupoid elements by G_2 $\big($adopting the convention that $(g, h) \in G \times G$ is composable if $s(g) = t(h)\big)$, and we write $m : G_2 \longrightarrow G$ for the multiplication operation. We often identify M with the submanifold of G of identity arrows. In particular, given $x \in M$ and $v \in T_xM$, $\epsilon(x) = 1_x \in G$ is identified with x, and the tangent vector $(d\epsilon)_x(v) \in T_xG$ with $v \in T_xM$.

We emphasize that the Lie groupoids we consider may be non-Hausdorff. Basic important examples come from foliation theory and from integration of bundles of Lie algebras. However, M and the fibers of s are always assumed to be Hausdorff manifolds, and s-fibers are assumed to be connected. We say that G is s-simply connected if the s-fibers are simply connected. We use the notation $G(-, x) = s^{-1}(x)$ (arrows starting at $x \in M$), and, for $y \overset{g}{\leftarrow} x$, we write the corresponding right multiplication map as $R_g : G(-, y) \longrightarrow G(-, x)$, $R_g(a) = ag$.

The infinitesimal version of a Lie groupoid is a Lie algebroid. To fix our notation, we recall that a Lie algebroid A over M is a vector bundle A over M together with a Lie bracket $[\cdot, \cdot]$ on the space of sections $\Gamma(A)$, and a bundle map $\rho : A \longrightarrow TM$ satisfying the Leibniz rule

$$[\alpha, f\beta] = f[\alpha, \beta] + \mathscr{L}_\alpha(f)\beta.$$

Here and elsewhere in this paper, we use the notation $\mathscr{L}_\alpha$ for the Lie derivative with respect to the vector field $\rho(\alpha)$.

Given a Lie groupoid G, the associated Lie algebroid $A = \mathrm{Lie}(G)$ has fibers $A_x = \mathrm{Ker}(ds)_x = T_x(G(-, x))$ for $x \in M$. Any $\alpha \in \Gamma(A)$ extends to a unique right-invariant vector field on G, which is denoted by the same letter α. This correspondence identifies $\Gamma(A)$ with the space $\mathscr{X}^s_{\mathrm{inv}}(G)$ of vector fields on G which are tangent to the s-fibers and right invariant. The usual Lie bracket on vector fields induces the bracket on $\Gamma(A)$, and the anchor is given by $\rho = dt : A \longrightarrow TM$.

Given a Lie algebroid A, an *integration* of A is a Lie groupoid G *together with* an isomorphism $A \cong \mathrm{Lie}(G)$. By abuse of notation, we often call G alone an integration of A. If such a G exists, we say that A is *integrable*. In contrast with the case of Lie algebras, not all Lie algebroids are integrable (see [13] and references therein). However, as with Lie algebras, integrability implies the existence of a canonical s-simply connected integration $G(A)$ of A, and any other s-simply connected integration of

A is isomorphic to $G(A)$. Roughly speaking, $G(A)$ consists of A-homotopy classes of A-paths, where an A-path consists of a path $\gamma : I \longrightarrow M$ together with an "A-derivative of γ," that is, a path $a : I \longrightarrow A$ above γ with the property that $\rho(a)$ is the usual derivative $d\gamma/dt$. In general, $G(A)$ is a topological groupoid carrying an additional "smooth structure" as the leaf space of a foliation, called in [13] the *Weinstein groupoid of* A, and its being a manifold is equivalent to the integrability of A. Further details and the precise obstructions to integrability can be found in [13], though some facts about $G(A)$ are recalled in Section 5.

A ϕ-twisted Dirac structure L has an induced Lie algebroid structure (see [10], [21], [27]): the bracket on $\Gamma(L)$ is the restriction of the Courant bracket $[\cdot,\cdot]_\phi$, and the anchor is the restriction of the projection $\mathrm{pr}_1 : TM \oplus T^*M \longrightarrow TM$. We denote by $G(L)$ the groupoid associated to this Lie algebroid structure. The main theme in this paper is the description of the extra structure on $G(L)$ induced by the Dirac structure. For instance, if $L = L_\pi$ is the Dirac structure coming from a Poisson tensor $\pi \in \Gamma(\Lambda^2 TM)$, then $G(L_\pi)$ carries a canonical symplectic structure making it into a *symplectic groupoid* (also interpreted as the phase space of an associated Poisson sigma-model; see [7], [14]). If π is a twisted Poisson structure, $G(L_\pi)$ becomes a twisted symplectic groupoid (see [8]).

2.3. Dirac $\leftrightarrow$ presymplectic

Symplectic structures appear in several ways in connection with Poisson manifolds, for example, as symplectic leaves, symplectic realizations, and symplectic groupoids. With the relationship (Poisson manifolds) $\leftrightarrow$ (symplectic structures) in mind, we briefly discuss in the remainder of this section the analogous correspondence for (twisted) Dirac manifolds.

Recall that a *presymplectic manifold* is a manifold M equipped with a closed 2-form ω. When $\mathrm{Ker}(\omega)$ has constant rank, it defines a foliation on M; if this foliation is *simple* (i.e., the space of leaves $M/\mathrm{Ker}(\omega)$ is smooth and the quotient map is a submersion), then $M/\mathrm{Ker}(\omega)$ becomes a symplectic manifold with symplectic form induced by ω. Hence modulo global regularity issues, presymplectic manifolds can be reduced to symplectic manifolds. In the case of a ϕ-twisted presymplectic manifold, the reduction mentioned above works only when $\mathrm{Ker}(\omega) \subset \mathrm{Ker}(\phi)$.

Just as Poisson structures can be viewed as singular foliations whose leaves are symplectic manifolds, ϕ-twisted Dirac structures L are singular foliations whose leaves are ϕ-twisted presymplectic manifolds. Given L, the foliation is defined at each $x \in M$ by its range $\mathscr{R}(L_x)$. Note that this (singular) distribution coincides with the image of the anchor map of the Lie algebroid structure of L and hence is necessarily integrable. In particular, the leaves of the foliation are the orbits of this Lie algebroid. For each such leaf S, the $\phi|_S$-presymplectic form $\theta_S \in \Omega^2(S)$ is, at each point, just the 2-form θ_{L_x} associated to the linear Dirac space L_x. As above, under certain regularity conditions (e.g., if $\mathrm{Ker}(\omega) \subset \mathrm{Ker}(\phi)$ and $\mathrm{Ker}(L)$ is simple) one can quotient out M by $\mathrm{Ker}(L)$ and reduce (M, L) to a twisted Poisson manifold $M/\mathrm{Ker}(L)$ whose symplectic leaves are precisely the reductions of the presymplectic leaves of L. This suggests that Dirac structures could very well be called "pre-Poisson structures."

The notion of a *presymplectic realization* of a ϕ-twisted Dirac structure L on M is more subtle: it is a Dirac map

$$\mu : (P, \eta) \longrightarrow (M, L),$$

where $\eta \in \Omega^2(P)$ is a $\mu^*\phi$-twisted presymplectic form ($d\eta + \mu^*\phi = 0$) with the extra property that $\mathrm{Ker}(d\mu) \cap \mathrm{Ker}(\eta) = \{0\}$. This "nondegeneracy condition" for η is explained in more detail in Section 6. As we will see, presymplectic realizations are the *infinitesimal* counterparts of the actions studied in [34], from where much of our motivation comes.

The only things still to be explained in the correspondence between Dirac structures and presymplectic structures are presymplectic groupoids, and this leads us to the main results of the paper.

2.4. The main results

A 2-form ω on a Lie groupoid G is called *multiplicative* if the graph of $m : G_2 \longrightarrow G$ is an isotropic submanifold of $(G, \omega) \times (G, \omega) \times (G, -\omega)$ or, equivalently, if

$$m^*\omega = \mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\omega, \tag{2.6}$$

where $\mathrm{pr}_i : G_2 \longrightarrow G$, $i = 1, 2$, are the natural projections.

Let G be a Lie groupoid over M, let ϕ be a closed 3-form on M, and let ω be a multiplicative 2-form on G. We call ω *relatively ϕ-closed* if $d\omega = s^*\phi - t^*\phi$.

Definition 2.1

We call (G, ω, ϕ) a presymplectic groupoid twisted by ϕ, or a *ϕ-twisted presymplectic groupoid*, if ω is relatively ϕ-closed, if $\dim(G) = 2\dim(M)$, and if

$$\mathrm{Ker}(\omega_x) \cap \mathrm{Ker}(ds)_x \cap \mathrm{Ker}(dt)_x = \{0\} \tag{2.7}$$

for all $x \in M$.

Condition (2.7) provides a restriction on how degenerate ω can be; we discuss it further in Section 4. If $\phi = 0$ and ω is nondegenerate, the groupoid in Definition 2.1 is just a symplectic groupoid in the usual sense (see [28]).

We find that, modulo integrability issues, (twisted) Dirac structures on M are basically the same thing as (twisted) presymplectic groupoids over M. This extends the relationship between integrable Poisson manifolds and symplectic groupoids (see, e.g., [9]). More precisely, we have two main results on this correspondence. The first starts with the groupoid.

THEOREM 2.2
Let (G, ω, ϕ) *be a* ϕ*-twisted presymplectic groupoid. Then*
(i) *there is a (canonical and unique)* ϕ*-twisted Dirac structure* L *on* M*, such that* $t : G \longrightarrow M$ *is a Dirac map, while* s *is anti-Dirac;*
(ii) *there is an induced isomorphism between the Lie algebroid* $\mathrm{Lie}(G)$ *of* G *and the Lie algebroid of* L*.*

We also see that known properties of symplectic groupoids naturally extend to our setting. For instance, $(\mathrm{Ker}(dt)_g)^\perp = \mathrm{Ker}(ds)_g + \mathrm{Ker}(\omega_g)$ for all $g \in G$, and $(TM)^\perp = TM + \mathrm{Ker}(\omega)$ (where $\perp$ denotes the orthogonal with respect to ω).

Remark 2.3
The alternative convention of identifying the Lie algebroid of G with left-invariant vector fields would lead to s being Dirac and t being anti-Dirac. This is the convention adopted in [9] and [28].

In the situation of Theorem 2.2, we say that (G, ω) is an *integration* of the ϕ-twisted Dirac structure L. Note that such an integration immediately gives rise to an integration of the Lie algebroid associated with L (in the sense of Sec. 2.2), namely, the Lie groupoid G together with the isomorphism ensured by Theorem 2.2(ii) (which does depend on ω!).

Our second result starts with a Dirac structure.

THEOREM 2.4
Let L *be a* ϕ*-twisted Dirac structure on* M *whose associated Lie algebroid is integrable, and let* $G(L)$ *be its* s*-simply connected integration. Then there exists a unique* 2*-form* ω_L *such that* $(G(L), \omega_L)$ *is an integration of the* ϕ*-twisted Dirac structure* L*.*

Hence we obtain a one-to-one correspondence between integrable ϕ-twisted Dirac structures on M and ϕ-twisted presymplectic groupoids over M by

$$L \leftrightarrow \big(G(L), \omega_L\big).$$

In order to prove Theorems 2.2 and 2.4, we have to understand the intricacies of multiplicative 2-forms, starting with their infinitesimal counterpart. We prove the following result, which we expect to be useful in other settings as well.

THEOREM 2.5
Let G be an s-simply connected Lie groupoid over M with Lie algebroid A, and let $\phi \in \Omega^3(M)$ be a closed 3-form. Then there is a one-to-one correspondence between
(i) *multiplicative 2-forms $\omega \in \Omega^2(G)$ with $d\omega = s^*\phi - t^*\phi$ and*
(ii) *bundle maps $\rho^* : A \longrightarrow T^*M$ with the properties*

$$\langle \rho^*(\alpha), \rho(\beta) \rangle = -\langle \rho^*(\beta), \rho(\alpha) \rangle,$$
$$\rho^*([\alpha, \beta]) = \mathscr{L}_\alpha\big(\rho^*(\beta)\big) - \mathscr{L}_\beta\big(\rho^*(\alpha)\big) + d\langle \rho^*(\alpha), \rho(\beta) \rangle + i_{\rho(\alpha)\wedge\rho(\beta)}(\phi).$$

In fact, for a given ω, the corresponding ρ^* is $\rho^*_\omega(\alpha)(X) = \omega(\alpha, X)$.

3. Multiplicative 2-forms on groupoids

In this section we discuss general properties of multiplicative 2-forms on Lie groupoids.

LEMMA 3.1
If $\omega \in \Omega^2(G)$ is multiplicative, then
(i) *$\varepsilon^*\omega = 0$, and $i^*\omega = -\omega$;*
(ii) $\mathrm{Ker}(ds)_g + \mathrm{Ker}(\omega_g) \subset (\mathrm{Ker}(dt)_g)^\perp$ *for all $g \in G$;*
(iii) *for all arrows $g : y \longleftarrow x$, the map $(di)_g$ induces an isomorphism*

$$\mathrm{Ker}(\omega_g) \longrightarrow \mathrm{Ker}(\omega_{g^{-1}}),$$

and $(dR_g)_y$ induces isomorphisms

$$\mathrm{Ker}(\omega_y) \cap \mathrm{Ker}(ds)_y \longrightarrow \mathrm{Ker}(\omega_g) \cap \mathrm{Ker}(ds)_g,$$
$$\mathrm{Ker}(\omega_y) \cap \mathrm{Ker}(ds)_y \cap \mathrm{Ker}(dt)_y \longrightarrow \mathrm{Ker}(\omega_g) \cap \mathrm{Ker}(ds)_g \cap \mathrm{Ker}(dt)_g,$$
$$\mathrm{Ker}(ds)_y \cap \big(\mathrm{Ker}(ds)_y\big)^\perp \longrightarrow \mathrm{Ker}(ds)_g \cap \big(\mathrm{Ker}(ds)_g\big)^\perp;$$

(iv) *on each orbit S of G, there is an induced 2-form θ_S, uniquely determined by the formula $\omega|_{G_S} = t^*\theta_S - s^*\theta_S$, where $G_S = s^{-1}(S) = t^{-1}(S)$ is the restriction of G to S; moreover, if ω is relatively ϕ-closed, then $d\theta = -\phi|_S$; hence each orbit of G becomes a $(\phi|_S)$-twisted presymplectic manifold.*

Note that, in Lemma 3.1(iii), $\mathrm{Ker}(\omega_y)$ and $\mathrm{Ker}(\omega_g)$ are not isomorphic in general.

Remark 3.2
The statements in Lemma 3.1(ii) and (iii) clearly hold with the roles of s and t interchanged (and right multiplication replaced by left multiplication).

Proof
We need the following simple identities:

$$v_g = (dm)_{y,g}\big((dt)_g(v_g), v_g\big) = (dm)_{g,x}\big(v_g, (ds)_g(v_g)\big) \quad \text{for all } v_g \in T_gG, \tag{3.1}$$

$$(dR_g)_y(\alpha_y) = (dm)_{y,g}(\alpha_y, 0) \quad \text{for all } \alpha_y \in \mathrm{Ker}(ds)_y, \tag{3.2}$$

$$(dt)_g(v_g) = (dm)_{g,g^{-1}}\big(v_g, (di)_g(v_g)\big) \quad \text{for all } v_g \in T_gG. \tag{3.3}$$

The identity in (3.1) is obtained by differentiating $id_G = m \circ (t, id_G) = m \circ (id_G, s)$. We verify the other two formulas similarly: for (3.2), we write $R_g : G(-, y) \longrightarrow G$ as $a \mapsto (a, g) \overset{m}{\mapsto} ag$, while for (3.3), we write the target map t as the composition of $G \longrightarrow G \times_M G$, $g \mapsto (g, g^{-1})$ with m.

We now prove the lemma. Consider the map $(id \times i) : G \longrightarrow G \times_M G$, $g \mapsto (g, g^{-1})$. Applying $(id \times i)^*$ to (2.6) and using (3.3), we deduce that $t^*\varepsilon^*\omega = \omega + i^*\omega$. Now applying ε^* to this equation, we get $\varepsilon^*\omega = \varepsilon^*\omega + \varepsilon^*\omega = 0$, and therefore $\omega + i^*\omega = 0$. This proves (i).

Using (3.1), (3.2), and the multiplicativity of ω, we get

$$\omega_g\big((dR_g)_y(\alpha_y), v_g\big) = \omega_y\big(\alpha_y, (dt)_g(v_g)\big) \tag{3.4}$$

for all $\alpha_y \in \mathrm{Ker}(ds)_y$ and $v_g \in T_gG$. When $v_g \in \mathrm{Ker}(dt)_g$, since $(dR_g)_y$ maps $\mathrm{Ker}(ds)_y$ isomorphically into $\mathrm{Ker}(ds)_g$, it follows that $\mathrm{Ker}(ds)_g \subset (\mathrm{Ker}(dt)_g)^{\perp}$. Since $\mathrm{Ker}(\omega)$ is inside all orthogonals, (ii) follows.

The first isomorphism in (iii) follows from $i^*\omega = -\omega$. In order to check the following two, note that $(dR_g)_y$ maps $\mathrm{Ker}(ds)_y$ isomorphically onto $\mathrm{Ker}(ds)_g$, and $\mathrm{Ker}(ds)_y \cap \mathrm{Ker}(dt)_y$ isomorphically onto $\mathrm{Ker}(ds)_g \cap \mathrm{Ker}(dt)_g$. So it suffices to prove the first isomorphism induced by $(dR_g)_y$ (as the second follows from it). Let $\alpha_y \in \mathrm{Ker}(ds)_y$. Using (3.2) and the multiplicativity of ω, we have

$$\omega_g\big((dR_g)_y(\alpha_y), (dm)_{y,g}(v_y, w_g)\big) = \omega_y(\alpha_y, v_y)$$

for all (v_y, w_g) tangent to the graph of the multiplication, which shows that $\alpha_y \in \mathrm{Ker}(ds)_y \cap \mathrm{Ker}(\omega_y)$ if and only if $(dR_g)_y(\alpha_y) \in \mathrm{Ker}(\omega_g)$. The last isomorphism in (iii) is implied by

$$\omega_g\big((dR_g)_y(\alpha_y), (dR_g)_y(u_y)\big) = \omega_y(\alpha_y, u_y)$$

for all $\alpha_y, u_y \in \mathrm{Ker}(ds)_y$, which follows from (3.2) and the multiplicativity of ω.

Part (iv) is a statement about transitive groupoids (namely, $G|_S$), so we may assume that $S = M$ and G is transitive. If G is the pair groupoid $M \times M$ over M, the proof is straightforward. Indeed, a multiplicative 2-form ω satisfies

$$\omega_{(x,y)}\big((u_x, u_y), (v_x, v_y)\big) = \omega_{(x,z)}\big((u_x, w_z), (v_x, w'_z)\big) + \omega_{(z,y)}\big((w_z, u_y), (w'_z, v_y)\big) \tag{3.5}$$

for $u_x, v_x \in T_xM$, $u_y, v_y \in T_yM$, and $w_z, w'_z \in T_zM$. In particular, we can write

$$\omega_{(x,y)}\big((u_x,u_y),(v_x,v_y)\big) = \omega_{(x,z)}\big((u_x,0_z),(v_x,0_z)\big) + \omega_{(z,y)}\big((0_z,u_y),(0_z,v_y)\big). \tag{3.6}$$

A direct application of (3.5) shows that the first factor in the right-hand side of (3.6) is of the form $t^*\theta$, and the second is of the form $s^*\theta'$ for $\theta, \theta' \in \Omega^2(M)$. The fact that $\theta = -\theta'$ follows from the first assertion of part (i).

In general, a transitive groupoid can be written as $G = (P \times P)/K$, the quotient of the pair groupoid $G(P) = P \times P$ by the action of a Lie group K, where $P \longrightarrow M$ is a principal K-bundle. (Fix $x \in M$, and take P as the set of arrows starting at x.) Let $p_1 : G(P) \longrightarrow G$ and $p_2 : P \longrightarrow M$ be the natural projections. (By abuse of notation, we denote source and target maps on either $G(P)$ or G by s and t since the context should avoid any confusion.) If $\omega \in \Omega^2(G)$ is multiplicative, then so is $p_1^*\omega$, and we can write $p_1^*\omega = t^*\theta_0 - s^*\theta_0$ for some $\theta_0 \in \Omega^2(P)$. Since the correspondence $\theta_0 \mapsto t^*\theta_0 - s^*\theta_0$ is injective, the fact that $p_1^*\omega$ is basic implies that θ_0 is basic. Hence $\theta_0 = p_2^*\theta$ for some $\theta \in \Omega^2(M)$. Since p_1 is a submersion, it follows that $\omega = t^*\theta - s^*\theta$. Similarly, $s^*\phi - t^*\phi = d\omega = t^*d\theta - s^*d\theta$ implies that $d\theta = -\phi$. □

We now look at what happens at points $x \in M$. The following is a first sign of the rigidity of multiplicative 2-forms.

LEMMA 3.3

If $\omega \in \Omega^2(G)$ is multiplicative, then at points $x \in M$,

$$\begin{aligned}
\mathrm{Ker}(ds)_x + \mathrm{Ker}(\omega_x) &= \big(\mathrm{Ker}(dt)_x\big)^{\perp},\\
T_xM + \mathrm{Ker}(\omega_x) &= (T_xM)^{\perp},\\
\mathrm{Ker}(\omega_x) &= \mathrm{Ker}(\omega_x) \cap \mathrm{Ker}(ds)_x \oplus \mathrm{Ker}(\omega_x) \cap T_xM.
\end{aligned}$$

(The first and third identities also hold for s and t interchanged.)

Furthermore, the following identities hold:

$$\begin{cases}
\dim\big(\mathrm{Ker}(\omega_x) \cap T_xM\big) = \frac{1}{2}\big(\dim(\mathrm{Ker}(\omega_x)) + 2\dim(M) - \dim(G)\big),\\
\dim\big(\mathrm{Ker}(\omega_x) \cap \mathrm{Ker}(ds)_x\big) = \frac{1}{2}\big(\dim(\mathrm{Ker}(\omega_x)) - 2\dim(M) + \dim(G)\big).
\end{cases}$$

Proof

Let us first prove the third equality. Let $u_x \in \mathrm{Ker}(\omega_x)$, and write

$$u_x = \big(u_x - (ds)_x(u_x)\big) + (ds)_x(u_x).$$

It then suffices to show that $(ds)_x(u_x) \in \mathrm{Ker}(\omega_x)$. Using (3.1) (the one involving ds) and the multiplicativity of ω, we get

$$\omega_x\big(u_x, (dm)_{x,x}(v_x, w_x)\big) = \omega_x(u_x, v_x) + \omega_x\big((ds)_x(u_x), w_x\big)$$

for all (v_x, w_x) tangent to the graph of m at (x, x). This shows that, indeed, $\omega_x((ds)_x(u_x), w_x) = 0$ for all $w_x \in T_xG$. Now, for the first two equalities, note that the direct inclusions are consequences of Lemma 3.1. We now compute the dimensions of the spaces involved. First, recall that, for any subspace W of a linear presymplectic space (V, ω), we have

$$\dim(W^{\perp}) = \dim(V) - \dim(W) + \dim\big(W \cap \operatorname{Ker}(\omega)\big). \tag{3.7}$$

Then, comparing dimensions in $\operatorname{Ker}(ds) + \operatorname{Ker}(\omega) \subset (\operatorname{Ker}(dt))^{\perp}$, we get

$$\begin{aligned}\dim\big(\operatorname{Ker}(\omega)\big) \le{}& \dim\big(\operatorname{Ker}(\omega) \cap \operatorname{Ker}(ds)\big) + \dim\big(\operatorname{Ker}(\omega) \cap \operatorname{Ker}(dt)\big)\\ &+ 2\dim(M) - \dim(G)\end{aligned}$$

at all $g \in G$. At a point $x \in M$,

$$\dim\big(\operatorname{Ker}(\omega) \cap \operatorname{Ker}(ds)\big) = \dim\big(\operatorname{Ker}(\omega) \cap \operatorname{Ker}(dt)\big)$$

(since $(di)_x$ is an isomorphism between these spaces), so

$$\dim\big(\operatorname{Ker}(\omega_x)\big) \le 2\dim\big(\operatorname{Ker}(\omega_x) \cap \operatorname{Ker}(ds)_x\big) + 2\dim(M) - \dim(G). \tag{3.8}$$

Comparing dimensions in $T_xM + \operatorname{Ker}(\omega_x) \subset (T_xM)^{\perp}$, we get

$$\dim\big(\operatorname{Ker}(\omega_x)\big) \le \dim\big(\operatorname{Ker}(\omega_x) \cap T_xM\big) + \dim(G) - 2\dim(M).$$

Since we already proved the third equality in the statement, we do know that

$$\dim\big(\operatorname{Ker}(\omega_x) \cap T_xM\big) = \dim\big(\operatorname{Ker}(\omega_x)\big) - \dim\big(\operatorname{Ker}(\omega_x) \cap \operatorname{Ker}(ds)_x\big). \tag{3.9}$$

Plugging this into the last inequality, we get precisely the opposite of (3.8). This shows that inequality (3.8) and the direct inclusions for the first two relations in the statement must become equalities. This immediately implies the dimension identities in the statement of the lemma and completes the proof. □

Note that the first and third identities in the lemma immediately imply that

$$\big(\operatorname{Ker}(dt)_x\big)^{\perp} = \operatorname{Ker}(ds)_x + \operatorname{Ker}(\omega_x) \cap T_xM, \quad x \in M. \tag{3.10}$$

COROLLARY 3.4
Two multiplicative forms $\omega, \omega' \in \Omega^2(G)$ which have the same differential $d\omega = d\omega'$, and which coincide at all $x \in M$, must coincide globally.

Proof
We may clearly assume that $\omega' = 0$. Equation (3.4) implies that $i_v(\omega) = 0$ for all v tangent to the s-fibers. Since ω is closed, it follows that $\mathscr{L}_v(\omega) = 0$ for all such v's, and therefore $\omega = s^*\eta$ for some 2-form η on M. Since $\omega|_{TM} = 0$, we must have $\eta = 0$; hence $\omega = 0$. □

We now discuss the infinitesimal counterpart of multiplicative forms. A multiplicative 2-form $\omega \in \Omega^2(G)$ induces a bundle map $\rho_\omega^* : A \longrightarrow T^*M$ characterized by the equation

$$\rho_\omega^*(\alpha) \circ (dt) = i_\alpha(\omega). \tag{3.11}$$

This equation is, in fact, equivalent to $\rho_\omega^*(\alpha) = i_\alpha(\omega)|_M$, as a consequence of Lemma 3.1(ii).

PROPOSITION 3.5
The bundle map ρ_ω associated to a multiplicative 2-form $\omega \in \Omega^2(G)$ has the following properties:

(i) *for $\alpha, \beta \in \Gamma(A)$, we have $\langle \rho_\omega^*(\alpha), \rho(\beta)\rangle = -\langle \rho_\omega^*(\beta), \rho(\alpha)\rangle$;*

(ii) *if ϕ is a closed 3-form on M and $d\omega = s^*\phi - t^*\phi$, then*

$$\rho_\omega^*([\alpha,\beta]) = \mathscr{L}_\alpha\big(\rho_\omega^*(\beta)\big) - \mathscr{L}_\beta\big(\rho_\omega^*(\alpha)\big) + d\big\langle \rho_\omega^*(\alpha), \rho(\beta)\big\rangle + i_{\rho(\alpha)\wedge\rho(\beta)}(\phi)$$

for all $\alpha, \beta \in \Gamma(A)$;

(iii) *two multiplicative forms ω_1 and ω_2 coincide if and only if $\rho_{\omega_1}^* = \rho_{\omega_2}^*$ and $d\omega_1 = d\omega_2$.*

Proof
By the comment preceding the proposition, (i) is clear, and (iii) is just Corollary 3.4. For part (ii), we fix v a vector field on M, and we prove that the right- and left-hand sides of (ii) applied to v are the same. Let $\alpha, \beta \in \Gamma(A)$, and denote the vector fields on G tangent to the s-fibers induced by them by the same letters. We consider

$$\begin{aligned} d\omega(\alpha,\beta,\tilde{v}) = {} & -\omega([\alpha,\beta],\tilde{v}) + \omega([\alpha,\tilde{v}],\beta) - \omega([\beta,\tilde{v}],\alpha) \\ & + \mathscr{L}_\alpha\big(\omega(\beta,\tilde{v})\big) - \mathscr{L}_\beta\big(\omega(\alpha,\tilde{v})\big) + \mathscr{L}_{\tilde{v}}\big(\omega(\alpha,\beta)\big) \end{aligned} \tag{3.12}$$

at points $x \in M$, where $\tilde{v}$ is a vector field on G to be chosen. We pick $\tilde{v}$ so that it agrees with v on M and so that it is t-projectable; that is,

$$(dt)_g(\tilde{v}_g) = \tilde{v}_{t(g)} = v_{t(g)}.$$

Note that, at points $x \in M$,

$$d\omega(\alpha,\beta,\tilde{v}) = (s^*\phi - t^*\phi)(\alpha,\beta,\tilde{v}) = -\phi\big(\rho(\alpha),\rho(\beta),v\big).$$

We claim that this observation and (3.12) imply part (ii).

In order to check that, we need some remarks on t-projectable vector fields and t-projectable functions on G $\big($i.e., functions f with $f(g) = f(t(g))\big)$:

(1) any vector field v on M admits (locally if G is non-Hausdorff) a t-projectable extension $\tilde{v}$ to G;

(2) if X is a vector field on G, and f is a t-projectable function, then

$$\mathscr{L}_X(f)(x) = \mathscr{L}_{dt(X_x)}(f|_M)(x)$$

for all $x \in M$;

(3) if $\alpha, \beta \in \Gamma(A)$, then $\omega(\alpha, \beta)$, as a function on G, is t-projectable;

(4) if $\alpha \in \Gamma(A)$ and $\tilde{v}$ is t-projectable, then $\omega(\alpha, \tilde{v})$ is t-projectable;

(5) if $\alpha \in \Gamma(A)$ and $\tilde{v}$ is t-projectable, then $(dt)_x[\alpha, \tilde{v}]_x = [\rho(\alpha), v]_x$ for all $x \in M$; note that, if $\tilde{v}$ extends v, then this implies that

$$\omega([\alpha, \tilde{v}], \beta) = \omega([\rho(\alpha), v], \beta)$$

at all $x \in M$ and for all $\beta \in \Gamma(A)$ (as a result of s- and t-fibers being orthogonal with respect to ω).

Taking these remarks into account in (3.12), we immediately obtain

$$\begin{aligned}\rho_\omega^*([\alpha, \beta])(v) = i_{[\rho(\beta), v]}\rho_\omega^*(\alpha) - i_{[\rho(\alpha), v]}\rho_\omega^*(\beta) + \mathscr{L}_{\rho(\alpha)} i_v \rho_\omega^*(\beta) \\ - \mathscr{L}_{\rho(\beta)} i_v \rho_\omega^*(\alpha) + i_v d\big(\langle \rho_\omega^*(\beta), \rho(\alpha)\rangle\big) + \phi\big(\rho(\alpha), \rho(\beta), v\big).\end{aligned}$$

$\big($Note that, in the statement of (ii), we follow the convention that $\mathscr{L}_\alpha$ means $\mathscr{L}_{\rho(\alpha)}$, and the same holds for $\mathscr{L}_\beta$.$\big)$ Now, using the identity $i_{[X,Y]} = \mathscr{L}_X i_Y - i_Y \mathscr{L}_X$ on the first two terms of the right-hand side, we obtain (ii).

Let us briefly discuss the remarks above.

In order to prove (1), let σ be a splitting of the bundle map $dt : TG \longrightarrow t^*TM$. $\big($In general, if G is non-Hausdorff, we can find such a splitting only locally. Since the formula to be proven is local, this is enough.$\big)$ Then $u_x = v_x - \sigma_x(v_x)$ is in $\mathrm{Ker}(dt)_x$. Now we extend it to a vector field on G tangent to the t-fibers by left translations, and we set $\tilde{v}_g = \sigma_g(v_{t(g)}) + u_g$.

Remark (2) is clear, while (3) and (4) follow from equation (3.4).

To check (5), we first look at $(dt)_x[\alpha, \tilde{v}]_x$. We see that

$$(dt)_x \frac{d}{d\epsilon}\Big|_{\epsilon=0} (d\Phi_\alpha^\epsilon)_{\Phi_\alpha^{-\epsilon}(x)}(\tilde{v}_{\Phi_\alpha^{-\epsilon}(x)}) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} d(t \circ \Phi_\alpha^\epsilon)_{\Phi_\alpha^{-\epsilon}(x)}(\tilde{v}_{\Phi_\alpha^{-\epsilon}(x)}), \tag{3.13}$$

where Φ_α^ϵ is the flow of α viewed as a vector field on G (extended by right translation). We have $t \circ \Phi_\alpha^\epsilon = \Phi_{\rho(\alpha)}^\epsilon \circ t$, and, since $\tilde{v}$ is t-projectable, $(dt)(\tilde{v}_{\Phi_\alpha^{-\epsilon}(x)}) = v_{\Phi_{\rho(\alpha)}^{-\epsilon}(x)}$. Hence the last term in (3.13) equals

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} d(t \circ \Phi_\alpha^\epsilon)_{\Phi_\alpha^{-\epsilon}(x)}(v_{\Phi_{\rho(\alpha)}^{-\epsilon}(x)}),$$

and this gives us $[\rho(\alpha), v]_x$. $\square$

Remark 3.6
Note that the bundle map ρ_ω^* (see (3.11)) carries all the information about ω at units: at $x \in M$, one has a canonical decomposition

$$T_xG \cong T_xM \oplus A_x,\ v_x \mapsto \big(ds(v_x), v_x - (ds)_x(v_x)\big),$$

and with respect to this decomposition, it is easy to see that

$$\omega\big((X,\alpha),(Y,\beta)\big) = \rho_\omega^*(\alpha)(Y) - \rho_\omega^*(\beta)(X) + \langle \rho_\omega^*(\alpha), \rho(\beta)\rangle. \tag{3.14}$$

4. Multiplicative 2-forms and induced Dirac structures

In this section we discuss the relationship between multiplicative 2-forms and Dirac structures, proving in particular Theorem 2.2.

Let G be a Lie groupoid over M, and let $\phi \in \Omega^3(M)$ be a closed 3-form. Given $\omega \in \Omega^2(G)$ multiplicative, we first look at when the Dirac structure L_ω associated to ω can be linearly pushed forward by the target map t. For $g \in G$, let $t_*(L_{\omega,g})$ denote the pushforward of $L_{\omega,g}$ by $(dt)_g$:

$$t_*(L_{\omega,g}) = \big\{((dt)_g(v_g), \xi_x) : i_{v_g}(\omega) = \xi_x \circ (dt)_g\big\} \subset T_xM \oplus T_x^*M, \tag{4.1}$$

where $x = t(g)$. In particular, restricting (4.1) to points in M, we get a (possibly nonsmooth but of constant rank) bundle of linear Dirac structures L_M on M:

$$L_{M,x} = \big\{((dt)_x(v_x), \xi_x) : i_{v_x}(\omega) = \xi_x \circ (dt)_x\big\} \subset T_xM \oplus T_x^*(M).$$

The problem is to understand when this bundle agrees with (4.1) for all $g \in G$.

Definition 4.1
We say that a multiplicative 2-form ω on G is of *Dirac type* if $t_*(L_{\omega,g}) = L_{M,t(g)}$ for all $g \in G$.

The next result provides alternative characterizations of 2-forms of Dirac type.

LEMMA 4.2
Given a multiplicative 2-form ω on G, one has

$$L_M = \big\{(\rho(\alpha) + u, \rho_\omega^*(\alpha)) : \alpha \in A, u \in \mathrm{Ker}(\omega) \cap TM\big\}, \tag{4.2}$$

and the following are equivalent:
(i) *ω is of Dirac type,*
(ii) $\mathrm{Ker}(ds)_g^\perp = \mathrm{Ker}(dt)_g + \mathrm{Ker}(\omega_g)$ *for all $g \in G$,*
(iii) $(dt)_g : \mathrm{Ker}(\omega_g) \longrightarrow \mathrm{Ker}(\omega_{t(g)}) \cap TM$ *is surjective for all $g \in G$,*

(iv) $\dim(\mathrm{Ker}(\omega_g)) = (1/2)\big(\dim(\mathrm{Ker}(\omega_x)) + \dim(\mathrm{Ker}(\omega_y))\big)$ *for all* $g \in G$.

Proof
Let $x \in M$. Using the first and third equalities in Lemma 3.3, we see that $\mathrm{Ker}(dt)_x^{\perp} = \mathrm{Ker}(ds)_x + \mathrm{Ker}(\omega_x) \cap TM$. Since the equation $i_v(\omega) = \xi \circ (dt)_x$ in the definition of $L_{M,x}$ implies that $v \in \mathrm{Ker}(dt)_x^{\perp}$, the elements of $L_{M,x}$ are pairs $((dt)_x(\alpha) + u, \xi)$, where $\alpha \in \mathrm{Ker}(ds)_x = A_x$, $u \in \mathrm{Ker}(\omega) \cap TM$, and $i_\alpha(\omega) = \xi_x \circ (dt)_x$. Since this last equation is exactly the one defining $\rho_\omega^*(\alpha)$, (4.2) is proven.

Let $g \in G$ with $s(g) = x$, $t(g) = y$. First, let us compute the codimension of $\mathrm{Ker}(dt)_g + \mathrm{Ker}(\omega_g)$ in $\mathrm{Ker}(ds)_g^{\perp}$. Using (3.7), and the fact that $\mathrm{Ker}(ds)_g \cap \mathrm{Ker}(\omega_g) \cong \mathrm{Ker}(ds)_y \cap \mathrm{Ker}(\omega_y)$, and the analogous equation with s replaced by t (cf. Lem. 3.1), we find that the codimension equals

$$2\dim(M) - \dim(G) + \dim\big(\mathrm{Ker}(\omega_x) \cap \mathrm{Ker}(dt)_x\big) + \dim\big(\mathrm{Ker}(\omega_y) \cap \mathrm{Ker}(ds)_y\big) - \dim\big(\mathrm{Ker}(\omega_g)\big).$$

Using the last two formulas of Lemma 3.3, we get

$$\dim\big(\mathrm{Ker}(ds)_g^{\perp}\big) - \dim\big(\mathrm{Ker}(dt)_g + \mathrm{Ker}(\omega_g)\big) = -\dim\big(\mathrm{Ker}(\omega_g)\big) + \frac{1}{2}\big(\dim(\mathrm{Ker}(\omega_y)) + \dim(\mathrm{Ker}(\omega_x))\big). \quad (4.3)$$

Next, we compute the corank of the map in (iii) $\big($note that $(dt)_g(\mathrm{Ker}(\omega_g)) \subseteq \mathrm{Ker}(\omega_y) \cap TM$ by (3.4)$\big)$:

$$\dim\big(\mathrm{Ker}(\omega_y \cap T_yM)\big) - \dim\big(\mathrm{Ker}(\omega_g)\big) + \dim\big(\mathrm{Ker}(\omega_g) \cap \mathrm{Ker}(dt)_g\big),$$

where, as above, we may replace g by x in the last term; replacing the first and last terms by the last two formulas of Lemma 3.3, we obtain

$$\dim\big(\mathrm{Ker}(\omega_y \cap T_xM)\big) - \dim\big((dt)_g(\mathrm{Ker}(\omega_g))\big) = -\dim\big(\mathrm{Ker}(\omega_g)\big) + \frac{1}{2}\big(\dim(\mathrm{Ker}(\omega_y)) + \dim(\mathrm{Ker}(\omega_x))\big). \quad (4.4)$$

Equations (4.3) and (4.4) immediately imply the equivalence of (ii), (iii), and (iv).

To see that (i) implies (iii), assume that $v \in \mathrm{Ker}(\omega_y) \cap TM$. Since $(v, 0) \in t_*(L_{\omega,y})$, we must have $(v, 0) \in t_*(L_{\omega,g})$, that is, $v = (dt)_g(w)$ with $w \in \mathrm{Ker}(\omega_g)$, and this proves (iii). For the converse, since $L_{M,y}$ and $t_*(L_{\omega,g})$ have the same dimension, it suffices to prove that $L_{M,y} \subset t_*(L_{\omega,g})$. Let $(\rho(\alpha) + u, \rho_\omega^*(\alpha))$ be an element in $L_{M,y}$. Write $\rho(\alpha) = (dt)_g(\tilde{\alpha})$, and $u = (dt)_g(\tilde{u})$, where $\tilde{\alpha} = (dR_g)_y(\alpha)$, and $\tilde{u} \in \mathrm{Ker}(\omega_g)$. By formula (3.4), we have $i_{\tilde{\alpha}}(\omega) = i_\alpha(\omega) \circ (dt)_g = \rho^*(\alpha) \circ (dt)_g$, and we now see that $(\rho(\alpha) + u, \rho^*(\alpha))$ is in $t_*(L_{\omega,g})$. □

Note that, in contrast to the Poisson bivectors on Poisson groupoids (see [29]), which always satisfy a condition like that in Definition 4.1, a form ω can be multiplicative and closed without being of Dirac type (see Example 6.4).

It can also happen that ω is multiplicative and of Dirac type, but the "leaves of L_M" do not coincide with the orbits of G (take, e.g., ω to be zero on a nontransitive groupoid). When they do coincide, the situation becomes much more rigid, as we now discuss. Recall that the orbits S of G are (twisted) presymplectic manifolds (see Lem. 3.1(iv)); we denote the corresponding family of 2-forms by $\theta = \{\theta_S\}$, and we consider the bundle L_θ of linear Dirac structures on M induced by θ,

$$L_{\theta,x} := \big\{(v_x, \xi_x) : v_x \in T_xS,\ \xi_x|_S = i_{v_x}(\theta)\big\} \subset T_xM \oplus T_x^*M.$$

LEMMA 4.3

Given a multiplicative 2-form $\omega \in \Omega^2(G)$, and $x \in M$, the following are equivalent:

(i) *the range $\mathscr{R}(L_{M,x})$ equals the tangent space to the orbit of G through x;*

(ii) *the conormal bundle to the orbit through x, $(\mathrm{Im}(\rho_x))^\circ$, sits inside $\mathrm{Im}(\rho_\omega^*)$;*

(iii) $\mathrm{Ker}(\omega_x) \cap T_xM \subset \mathrm{Im}(\rho)$*;*

(iv) $\mathrm{Ker}(\omega_x) \cap T_xM = \mathrm{Ker}(\theta_x)$*;*

(v) $L_{M,x} = L_{\theta,x}$.

Moreover, if these hold at all $x \in M$, then ω is of Dirac type.

Proof

Let us first show that

$$\mathrm{Ker}(\theta_x) = \mathrm{Ker}(\omega_x) \cap \mathrm{Im}(\rho) \qquad \text{and} \qquad \mathrm{Ker}(\omega_x) \cap T_xM = \mathrm{Im}(\rho_\omega^*)^\circ. \tag{4.5}$$

Given $v \in T_xS = \mathrm{Im}(\rho)$, we write $v = \rho(\alpha)$ with $\alpha \in \mathrm{Ker}(ds)_x$. From the defining property for $\theta = \theta_S$, we have $\theta(v, w) = \theta(\rho(\alpha), \rho(\beta)) = \omega(\alpha, \beta)$ for all $w = \rho(\beta) \in T_xS$. Hence a vector v is in $\mathrm{Ker}(\theta_x)$ if and only if $v = \rho(\alpha)$ has the property that $\omega(\alpha, \beta) = 0$ for all $\beta \in \mathrm{Ker}(ds)_x$, that is, $\alpha \in \mathrm{Ker}(ds)_x \cap (\mathrm{Ker}(ds)_x)^\perp = \mathrm{Ker}(ds)_x \cap (\mathrm{Ker}(dt)_x + \mathrm{Ker}(\omega_x) \cap T_xM)$, where for the last equality we used (3.10). Hence

$$\begin{aligned}\mathrm{Ker}(\theta_x) &= (dt)_x\big(\mathrm{Ker}(ds)_x \cap (\mathrm{Ker}(ds)_x)^\perp\big)\\ &= (dt)_x\big(\mathrm{Ker}(ds)_x \cap (\mathrm{Ker}(dt)_x + \mathrm{Ker}(\omega_x) \cap T_xM)\big),\end{aligned}$$

which is easily seen to be $\mathrm{Ker}(\omega_x) \cap (dt_x)(\mathrm{Ker}(ds)_x) = \mathrm{Ker}(\omega_x) \cap \mathrm{Im}(\rho)$. The other identity in (4.5) easily follows from the explicit formula for ω_x in terms of ρ and ρ_ω^* mentioned in Remark 3.6. Now, note that (4.5) proves the equivalence of (iii) and (iv); and (4.5) also shows that (ii) and (iii) are just dual to each other. Hence we are left with proving that (i) is equivalent to (iii). But this is immediate since $\mathscr{R}(L_{M,x}) = \mathrm{Im}(\rho_x) + \mathrm{Ker}(\omega_x) \cap TM$ (e.g., cf. (4.2)).

We now prove the last assertion of the lemma. We do that by showing that $\mathrm{Ker}(ds)_g^\perp = \mathrm{Ker}(dt)_g + \mathrm{Ker}(\omega_g)$ for all $g \in G$ (and using Lem. 4.2(ii)). So let $g : y \leftarrow x$ be in G.

CLAIM 1
Given $v \in T_g(G)$, one has $(dt)_g(v) \in \mathrm{Ker}(\omega_y)$ if and only if $v \in \mathrm{Ker}(ds)_g^\perp$.

Proof
This follows immediately from (3.4) and Lemma 3.1(i). □

CLAIM 2
If (i)–(v) hold at all $x \in M$, then

$$(dt)_g\big(\mathrm{Ker}(dt)_g^\perp\big) = \mathrm{Im}(\rho_y).$$

Proof
Clearly, the left-hand side contains the right-hand side, and we compare the dimensions of the two spaces. Recalling that $\dim(M) = \dim(G) - \dim(\mathrm{Ker}(dt)_g)$ and using (3.7), we find that the dimension of the left-hand side is

$$\begin{aligned}\dim\big(\mathrm{Ker}(dt)_g^\perp\big) - \dim\big(\mathrm{Ker}(dt)_g^\perp \cap \mathrm{Ker}(dt)_g\big) = {} & \dim\big(\mathrm{Ker}(dt)_g \cap \mathrm{Ker}(\omega_g)\big) \\ & - \dim\big(\mathrm{Ker}(dt)_g^\perp \cap \mathrm{Ker}(dt)_g\big) \\ & + \dim(M).\end{aligned}$$

Note that, by Lemma 3.1(iii) (see Rem. (3.2)), we can replace g in the right-hand side of the last formula by $x = s(g)$. Hence the dimension we are interested in equals the one of $(dt)_x(\mathrm{Ker}(dt)_x^\perp)$. Using again the fact that $\mathrm{Ker}(dt)_x^\perp = \mathrm{Ker}(ds)_x + \mathrm{Ker}(\omega_x) \cap T_xM$, together with (iii), we obtain $(dt)_x(\mathrm{Ker}(dt)_x^\perp) = \mathrm{Im}(\rho_x)$. But $\dim(\mathrm{Im}(\rho_x)) = \dim(\mathrm{Im}(\rho_y))$ because x and y are on the same orbit, and this concludes the proof of the claim. □

CLAIM 3
If (i)–(v) hold at all $x \in M$, then the following is a short exact sequence:

$$0 \longrightarrow \mathrm{Ker}(dt)_g \cap \mathrm{Ker}(dt)_g^\perp \longrightarrow \mathrm{Ker}(ds)_g^\perp \cap \mathrm{Ker}(dt)_g^\perp \xrightarrow{(dt)_g} \mathrm{Ker}(\omega_y) \cap TM \longrightarrow 0. \tag{4.6}$$

Proof
The claim easily follows if one checks the surjectivity of the last map. To see that, note that if $u \in \mathrm{Ker}(\omega_y) \cap TM$, then $u \in \mathrm{Im}(\rho)$ by (iii); so Claim 2 implies that

$u = dt_g(v)$ for some $v \in \operatorname{Ker}(dt)_g^{\perp}$. But by Claim 1, $v \in \operatorname{Ker}(ds)_g^{\perp}$. Hence the last map is surjective. □

As a result of Claim 3, we get

$$\begin{aligned}\dim\big(\operatorname{Ker}(ds)_g^{\perp} \cap \operatorname{Ker}(dt)_g^{\perp}\big) &= \dim\big(\operatorname{Ker}(dt)_g \cap \operatorname{Ker}(dt)_g^{\perp}\big)\\ &\quad + \dim\big(\operatorname{Ker}(\omega_y) \cap TM\big).\end{aligned} \tag{4.7}$$

Let us recall again that

$$\dim\big(\operatorname{Ker}(dt)_g^{\perp} \cap \operatorname{Ker}(dt)_g\big) = \dim\big(\operatorname{Ker}(dt)_x^{\perp} \cap \operatorname{Ker}(dt)_x\big).$$

Using the short exact sequence

$$0 \longrightarrow \operatorname{Ker}(dt)_x^{\perp} \cap \operatorname{Ker}(dt)_x \longrightarrow \operatorname{Ker}(dt)_x^{\perp} \xrightarrow{(dt)_x} \operatorname{Im}(\rho_x) \longrightarrow 0$$

and (3.7) to compute $\dim(\operatorname{Ker}(dt)_x^{\perp})$, we find

$$\begin{aligned}\dim\big(\operatorname{Ker}(dt)_g^{\perp} \cap \operatorname{Ker}(dt)_g\big) &= \dim(M)\\ &\quad + \dim\big(\operatorname{Ker}(dt)_x \cap \operatorname{Ker}(\omega_x)\big) - \dim\big(\operatorname{Im}(\rho_x)\big).\end{aligned} \tag{4.8}$$

Let us now compute the dimension of $\operatorname{Ker}(ds)_g^{\perp} \cap \operatorname{Ker}(dt)_g^{\perp} = (\operatorname{Ker}(ds)_g + \operatorname{Ker}(dt)_g)^{\perp}$. Using (3.7) and replacing $\dim(\operatorname{Ker}(ds)_g + \operatorname{Ker}(dt)_g)$ by $\dim(\operatorname{Im}(\rho_x)) + \dim(G) - \dim(M)$, which is possible due to the exact sequence

$$0 \longrightarrow \operatorname{Ker}(ds)_g \longrightarrow \operatorname{Ker}(ds)_g + \operatorname{Ker}(dt)_g \xrightarrow{(ds)_g} \operatorname{Im}(\rho_x) \longrightarrow 0,$$

we find

$$\begin{aligned}\dim\big(\operatorname{Ker}(ds)_g^{\perp} \cap \operatorname{Ker}(dt)_g^{\perp}\big) &= \dim(M) - \dim\big(\operatorname{Im}(\rho_x)\big)\\ &\quad + \dim\big((\operatorname{Ker}(ds)_g + \operatorname{Ker}(dt)_g) \cap \operatorname{Ker}(\omega_g)\big).\end{aligned} \tag{4.9}$$

Plugging (4.8) and (4.9) into (4.7), we find

$$\begin{aligned}\dim\big((\operatorname{Ker}(ds)_g + \operatorname{Ker}(dt)_g) \cap \operatorname{Ker}(\omega_g)\big) &= \dim\big(\operatorname{Ker}(dt)_x \cap \operatorname{Ker}(\omega_x)\big)\\ &\quad + \dim\big(\operatorname{Ker}(\omega_y) \cap TM\big)\\ &= \frac{1}{2}\big(\dim(\operatorname{Ker}(\omega_x)) + \dim(\operatorname{Ker}(\omega_y))\big),\end{aligned}$$

where the last identity follows from the last two formulas in Lemma 3.3. In particular,

$$\frac{1}{2}\big(\dim(\operatorname{Ker}(\omega_x)) + \dim(\operatorname{Ker}(\omega_y))\big) - \dim\big(\operatorname{Ker}(\omega_g)\big) \leq 0.$$

On the other hand, we have seen in the proof of Lemma 4.2 that this number is the codimension of $\operatorname{Ker}(dt)_g + \operatorname{Ker}(\omega_g)$ in $\operatorname{Ker}(ds)_g^{\perp}$ and hence positive. So it must be zero, and this completes the proof of Lemma 4.3. □

Note that, if L_θ is smooth, then it is automatically a ϕ-twisted Dirac structure since the 2-forms θ_S satisfy $d\theta_S = -\phi|_S$. In particular, we obtain the following.

COROLLARY 4.4
Let ω be a relatively ϕ-closed, multiplicative 2-form. Then the following are equivalent:
(i) *the bundle L_M is smooth and* $\mathrm{Ker}(\omega) \cap TM \subset \mathrm{Im}(\rho)$ *(or any of the equivalent conditions in Lem. 4.3);*
(ii) *there is a ϕ-twisted Dirac structure L on M so that $t : G \longrightarrow M$ is a Dirac map, and the presymplectic leaves of L coincide with the orbits of G.*
If these conditions hold, then L coincides with

$$L_M = \big\{(\rho(\alpha) + u, \rho^*_\omega(\alpha)) : \alpha \in A, u \in (\mathrm{Im}(\rho^*_\omega))^\circ\big\},$$

and L_M is a ϕ-twisted Dirac structure with presymplectic leaves (S, θ_S) and kernel $\mathrm{Ker}(\omega) \cap TM = (\mathrm{Im}(\rho^*_\omega))^\circ$.

Another interesting consequence is the following.

PROPOSITION/DEFINITION 4.5
Given a relatively ϕ-closed, multiplicative 2-form ω on G, the following are equivalent:
(i) *there exists a ϕ-twisted Poisson structure on M such that $t : G \longrightarrow M$ is a Dirac map;*
(ii) $\dim(\mathrm{Ker}(\omega_x)) = \dim(G) - 2\dim(M)$ *for all $x \in M$;*
(iii) $\mathrm{Ker}(dt_g)^\perp = \mathrm{Ker}(ds_g)$*;*
(iv) $\mathrm{Ker}(\omega_x) \subset \mathrm{Ker}(ds)_x \cap \mathrm{Ker}(dt)_x$ *for all $x \in M$ (or, equivalently,* $\mathrm{Ker}(\rho^*_\omega) \subset \mathrm{Ker}(\rho)$*).*
In this case we say that (G, ω) is a twisted over-symplectic *groupoid.*

Proof
Condition (i) is equivalent to L_M being smooth and having zero kernel, that is, $\mathrm{Ker}(\omega) \cap TM = \{0\}$. Hence, by (3.10), $(\mathrm{Ker}(dt)_x)^\perp = \mathrm{Ker}(ds)_x$ for all $x \in M$. A simple dimension counting, using (3.7) and $\mathrm{Ker}(\omega_x) = \mathrm{Ker}(\omega_x) \cap \mathrm{Ker}(dt)_x$, directly shows (ii). Using (3.7) once again to compute $(\dim(dt)_g)^\perp$, and recalling that $\dim(\mathrm{Ker}(\omega_g) \cap \mathrm{Ker}(dt)_g) = \dim(\mathrm{Ker}(\omega_x) \cap \mathrm{Ker}(dt)_x)$ for $x = s(g)$, another dimension counting shows that (ii) implies (iii). Note that (iii) implies that $\mathrm{Ker}(\omega_x) \subset \mathrm{Ker}(ds)_x$. Since (iii) also holds for t and s interchanged, we get $\mathrm{Ker}(\omega_x) \subset \mathrm{Ker}(ds)_x \cap \mathrm{Ker}(dt)_x$, and (iv) follows. Finally, if (iv) holds, then (4.2) implies that $(\rho, \rho^*_\omega) : A \longrightarrow L_M$ is surjective; thus L_M is smooth. Since $\mathrm{Ker}(\omega_x) \cap T_xM =$

$\mathrm{Ker}(ds)_x \cap \mathrm{Ker}(dt)_x \cap T_x M = \{0\}$, ω is of Dirac type, and the Dirac structure induced on M is Poisson. □

Definition 4.6
Let G be a Lie groupoid over M, and let ϕ be a closed 3-form on M. A multiplicative 2-form ω on G is called *robust* if the isotropy bundle $\mathfrak{g}(\omega)$, defined by

$$\mathfrak{g}_x(\omega) = \mathrm{Ker}(\omega_x) \cap \mathrm{Ker}(ds)_x \cap \mathrm{Ker}(dt)_x, \quad x \in M,$$

has constant rank equal to $\dim(G) - 2\dim(M)$. If ω is also relatively ϕ-closed, we say that (G, ω) is a *ϕ-twisted over-presymplectic groupoid.*

We explain this "over" terminology in Remark 4.9.

A *ϕ-twisted presymplectic groupoid* (Def. 2.1) is a ϕ-twisted over-presymplectic groupoid (G, ω) with $\dim(G) = 2\dim(M)$. If ω is nondegenerate, then (G, ω) is called a *ϕ-twisted symplectic groupoid* (see [8]).

Note that, if (G, ω) is a ϕ-twisted over-presymplectic groupoid, then $\mathfrak{g}(\omega)$ is a *smooth* bundle of Lie algebras.

LEMMA 4.7
Let ω be a multiplicative 2-form on G. The following statements are equivalent:
(i) *ω is robust;*
(ii) *$(\rho, \rho^*_\omega) : A \longrightarrow L_M$ is surjective;*
(iii) *$\mathrm{Ker}(dt)_g^{\perp} = \mathrm{Ker}(ds)_g + \mathrm{Ker}(\omega_g) \cap \mathrm{Ker}(dt)_g$;*
(iv) *$(dt)_g : \mathrm{Ker}(\omega_g) \cap \mathrm{Ker}(ds)_g \longrightarrow \mathrm{Ker}(\omega_{t(g)}) \cap TM$ is onto;*
(v) *$\mathrm{Ker}(\omega_g) = \mathrm{Ker}(\omega_g) \cap \mathrm{Ker}(ds)_g + \mathrm{Ker}(\omega_g) \cap \mathrm{Ker}(dt)_g$.*
Note that (iii), (iv), and (v) are required to hold for all $g \in G$ or, equivalently, for all $g = x \in M$. In this case, ω is of Dirac type, and L_M is a ϕ-twisted Dirac structure on M.

Proof
The equivalence of (i) and (ii) is immediate by a dimension counting. Using (3.7) and noticing that (iii) holds if and only if the dimensions of the right- and left-hand sides coincide (due to the inclusion in Lem. 3.1(ii)), we conclude that (iii) is equivalent to

$$\dim\big(\mathrm{Ker}(ds)_g\big) - \dim(M) = \dim\big(\mathfrak{g}_g(\omega)\big). \tag{4.10}$$

Also, (ρ, ρ^*_ω) is surjective if and only if the dimension of its image $\big($which equals $\dim(\mathrm{Ker}(ds)) - \dim(\mathrm{Ker}(\omega) \cap \mathrm{Ker}(ds) \cap \mathrm{Ker}(dt))\big)$ coincides with the dimension of L_M (which equals $\dim(M)$), and this is precisely (4.10) at points in M. On the other hand, Lemma 3.1(iii) shows that (4.10) is equivalent to the same relation at the point

$t(g) \in M$. So we conclude that (i), (ii), and (iii) are equivalent to each other. A direct dimension counting, allied with Lemma 3.1 and then the last two formulas of Lemma 3.3, shows that (iv) is also equivalent to (i).

Finally, a similar argument (i.e., dimension counting together with Lem. 3.1 and the last formula of Lem. 3.3) shows that the equality in (v) is equivalent to

$$\dim(\operatorname{Ker}\omega_g) = \frac{1}{2}\big(\dim(\operatorname{Ker}(\omega_x)) + \dim(\operatorname{Ker}(\omega_y))\big) + \dim(G) - 2\dim(M) - \dim\big(\mathfrak{g}_x(\omega)\big) \tag{4.11}$$

at all $g \in G$ (where $x = s(g)$, $y = t(g)$). Evaluating this expression at $g = x \in M$ immediately implies that ω is robust (i.e., (i)). Conversely, if (iii) holds, then the equivalence of (ii) and (iv) of Lemma 4.2 implies (4.11). □

COROLLARY 4.8
Let G be a Lie groupoid over M, and let $\omega \in \Omega^2(G)$ be multiplicative. The following statements are equivalent:
(i) $\dim(G) = 2\dim(M)$*, and ω is robust;*
(ii) $(\rho, \rho^*_\omega) : A \longrightarrow L_M$ *is an isomorphism;*
(iii) $\operatorname{Ker}(dt)^{\perp}_g = \operatorname{Ker}(ds)_g \oplus \operatorname{Ker}(\omega_g) \cap \operatorname{Ker}(dt)_g$*;*
(iv) $\operatorname{Ker}(\omega_g) = \operatorname{Ker}(\omega_g) \cap \operatorname{Ker}(ds)_g \oplus \operatorname{Ker}(\omega_g) \cap \operatorname{Ker}(dt)_g$*;*
(v) $(dt)_g : \operatorname{Ker}(\omega_g) \cap \operatorname{Ker}(ds)_g \longrightarrow \operatorname{Ker}(\omega_{t(g)}) \cap TM$ *is an isomorphism,*
where (iii), (iv), and (v) are required to hold for all $g \in G$ or, equivalently, for all $g = x \in M$.

Moreover, if ω is relatively ϕ-closed (i.e., (G, ω) is a ϕ-twisted presymplectic groupoid), then L_M is a ϕ-twisted Dirac structure on M, characterized by the properties that t is a Dirac map and that (ρ, ρ^*_ω) in (ii) above is an isomorphism of algebroids.

Proof
The equivalence of (i) and (ii) follows immediately from (4.10). The rest is a direct consequence of Lemma 4.7. □

Clearly, Corollary 4.8 proves Theorem 2.2.

Remark 4.9
(i) For any ϕ-twisted over-presymplectic groupoid (G, ω), the isotropy bundle $\mathfrak{g}(\omega)$ is a smooth bundle of Lie algebras, and a subbundle of the (possibly nonsmooth) isotropy Lie algebra bundle of G, $\mathfrak{g}(G) = \operatorname{Ker}(\rho)$. Integrating $\mathfrak{g}(\omega)$ to a bundle of simply connected Lie groups $G(\omega)$, assuming that the quotient $G/G(\omega)$ is smooth,

we see that ω reduces to a multiplicative 2-form $\overline{\omega}$ on $G/G(\omega)$, and $(G/G(\omega), \overline{\omega})$ becomes a ϕ-twisted presymplectic groupoid, over the same manifold M, which induces the same ϕ-twisted Dirac structure on M. (Of course, if (G, ω) is over-symplectic, then $(G/G(\omega), \overline{\omega})$ is a symplectic groupoid.) Although this does not always work (i.e., $G/G(\omega)$ may be nonsmooth), this explains our "over" terminology, and it shows that $\mathfrak{g}(\omega)$ can be viewed as an obstruction to our final goal of obtaining a one-to-one correspondence between ϕ-twisted Dirac structures on M and groupoids over M equipped with a certain extra structure. Examples of over-presymplectic groupoids that cannot be reduced to presymplectic groupoids are given in Section 6.3.

(ii) Similarly, if (G, ω) is a ϕ-twisted presymplectic groupoid over M, then, again modulo global regularity issues, one can quotient it out by $\mathrm{Ker}(\omega)$ to obtain a ϕ-twisted symplectic groupoid over the ϕ-twisted Poisson manifold $M/\mathrm{Ker}(L)$. Details of this construction will be given elsewhere.

5. Reconstructing multiplicative forms

In this section we explain how multiplicative forms can be reconstructed from their infinitesimal counterpart. In particular, we complete the proof of Theorem 2.4.

Recall that (cf. Prop. 3.5) for $\omega \in \Omega^2(G)$ multiplicative and relatively ϕ-closed, the associated bundle map ρ^*_ω satisfies

$$\langle \rho^*_\omega(\beta), \rho(\alpha)\rangle = -\langle \rho^*_\omega(\alpha), \rho(\beta)\rangle \quad \text{for all } \alpha, \beta \in \Gamma(A), \tag{5.1}$$

$$\rho^*_\omega([\alpha, \beta]) = \mathscr{L}_\alpha\big(\rho^*_\omega(\beta)\big) - \mathscr{L}_\beta\big(\rho^*_\omega(\alpha)\big) - d\langle \rho^*_\omega(\beta), \rho(\alpha)\rangle + i_{\rho(\alpha)\wedge\rho(\beta)}(\phi). \tag{5.2}$$

We use the notation

$$(d_A\rho^*_\omega)(\alpha, \beta) := \rho^*_\omega([\alpha, \beta]) - \mathscr{L}_\alpha\big(\rho^*_\omega(\beta)\big) + \mathscr{L}_\beta\big(\rho^*_\omega(\alpha)\big) + d\langle \rho^*_\omega(\beta), \rho(\alpha)\rangle.$$

The rest of this section is devoted to the proof of the next result (see also [8]).

THEOREM 5.1

If G is an s-simply connected Lie groupoid, and $\phi \in \Omega^3(M)$ is closed, then the correspondence $\omega \mapsto \rho^*_\omega$ induces a bijection between the space of relatively ϕ-closed multiplicative 2-forms on G and bundle maps $\rho^* : A \longrightarrow T^*M$ satisfying conditions (5.1) and (5.2).

By Proposition 3.5(iii), we only have to check that the map $\omega \mapsto \rho^*_\omega$ is surjective; that is, given $\rho^* : A \longrightarrow T^*M$ satisfying (5.1) and (5.2), we must produce a relatively ϕ-closed multiplicative 2-form ω on G. By Corollary 4.8, if $\mathrm{Ker}(\rho^*) \cap \mathrm{Ker}(\rho) = \{0\}$, then the resulting 2-form makes G into a twisted presymplectic groupoid; note that if

L is a Dirac structure on M, then Theorem 5.1 (applied to $\rho^* = \mathrm{pr}_2 : L \longrightarrow T^*M$) and Corollary 4.8 imply the correspondence in Theorem 2.4.

Let us recall (see [13]) how G can be reconstructed from A. Let $I = [0, 1]$, and let $p : A \longrightarrow M$ be the natural projection. (We also denote other bundle projections by p whenever the context is clear.) Consider the Banach manifold $\tilde{P}(A)$ consisting of paths $a : I \longrightarrow A$ of class C^1, whose base path $\gamma = p \circ a : I \longrightarrow M$ is of class C^2, and consider the submanifold $P(A)$ defined by the equation $\rho \circ a = \frac{d}{dt}\gamma$ (i.e., a is an A-path). The manifold $P(A)$ comes endowed with an infinitesimal action of the infinite-dimensional Lie algebra $\mathfrak{g}$ consisting of time-dependent sections η_t ($t \in [0, 1]$)* of A, with $\eta_0 = \eta_1 = 0$. To define the Lie algebra map

$$\mathfrak{g} \ni \eta \mapsto X_\eta \in \mathscr{X}\big(P(A)\big)$$

describing the action, it is more convenient to introduce the flows of the vector fields X_η. One advantage of this approach is that X_η is defined on the entire $\tilde{P}(A)$. Given $a_0 \in P(A)$, we construct the flow $a_\epsilon = \Phi^\epsilon_{X_\eta}(a_0)$ in such a way that a_ϵ are paths above $\gamma_\epsilon(t) = \Phi^\epsilon_{\rho(\eta_t)}\gamma_0(t)$, where γ_0 is the base path of a_0, and $\Phi^\epsilon_{\rho(\eta_t)}$ is the flow of the vector field $\rho(\eta_t)$. We choose a time-dependent section ξ_0 of A with $\xi_0(t, \gamma_0(t)) = a_0(t)$, and we consider the (ϵ, t)-dependent section of A, $\xi = \xi(\epsilon, t)$, to be a solution of

$$\frac{d\xi}{d\epsilon} - \frac{d\eta}{dt} = [\xi, \eta], \quad \xi(0, t) = \xi_0(t). \tag{5.3}$$

Then $a_\epsilon(t) = \xi_\epsilon(t, \gamma_\epsilon(t))$. This defines the desired vector fields X_η, the action of $\mathfrak{g}$, and the foliation on $P(A)$. It is clear from the definition that

$$(dp)(X_\eta) = \rho(\eta) \circ p \qquad \text{and} \qquad p \circ \Phi^\epsilon_{X_\eta}(a) = \Phi^\epsilon_{\rho(\eta)} \circ p(a). \tag{5.4}$$

Now, $G(A) = P(A)/\sim$ is a topological groupoid for any A. The source (resp., target) map is obtained by taking the starting (resp., ending) point of the base paths, and the multiplication is defined by concatenation of paths. Moreover, G must be isomorphic to $G(A)$.

We construct forms on $G(A)$ by constructing forms on $P(A)$ which are basic with respect to the action (i.e., $\mathscr{L}_{X_\eta}\omega = 0$ and $i_{X_\eta}\omega = 0$ for all $\eta \in \mathfrak{g}$). First, the 3-form ϕ on M induces a 2-form on TM that, at $X \in T_xM$, is $i_X(p_M^*\phi_x)$, where $p_M : TM \longrightarrow M$ is the projection. We pull it back by ρ to A and lift it to $\tilde{P}(A)$ to get a 2-form ω_ϕ on $\tilde{P}(A)$:

$$\omega_{\phi,a}(V, W) = \int_0^1 \phi\big(\rho(a(t)), (dp)_{a(t)}(V(a(t))), (dp)_{a(t)}(W(a(t)))\big)\, dt.$$

In order to produce basic forms, we must understand the behavior of $\omega_{\phi,a}$ when we take derivatives along, or interior products by, vector fields of the form X_η.

*In this section, t denotes a "time" parameter.

LEMMA 5.2
For any closed 3*-form* ϕ *on* M*,*

$$\omega_\phi(X_{\eta,a}, X_a) = \int_0^1 \phi\big(\rho a(t), \rho\eta(t,\gamma(t)), (dp)_{a(t)}(X_a(t))\big)\, dt,$$
$$d\omega_\phi = \mathbf{t}^*\phi - \mathbf{s}^*\phi,$$

where $\eta \in \mathfrak{g}$*,* X_a *is a vector tangent to* $\tilde{P}(A)$ *at* $a \in P(A)$*, and* $\mathbf{s}, \mathbf{t} : \tilde{P}(A) \longrightarrow M$ *take the start/end points of the base path.*

Proof
The first formula is immediate from the definition of ω_ϕ and the first equation in (5.4), while the second formula follows from Stokes's theorem. □

Let us now consider the Liouville 1-form σ^c on T^*M, and the associated canonical symplectic form ω^c. Recall that, on T^*M,

$$\sigma^c_{\xi_x}(X_{\xi_x}) = \big\langle \xi_x, (dp)_{\xi_x}(X_{\xi_x})\big\rangle \quad (\xi_x \in T^*_x M, X_x \in T_{\xi_x}T^*M)$$

and $\omega^c = -d\sigma^c$. Using the map $\rho^* : A \longrightarrow T^*M$, we pull σ^c and ω^c back to A, and we denote by $\tilde{\sigma}$ and $\tilde{\omega}$ the resulting forms on $\tilde{P}(A)$. Hence $\tilde{\omega} = -d\tilde{\sigma}$, and

$$\tilde{\sigma}_a(X_a) = \int_0^1 \big\langle \rho^*(a(t)), (dp)_{a(t)}(X_a(t))\big\rangle\, dt.$$

LEMMA 5.3
If ρ^* *satisfies (5.1), then, for any* A*-path* a *and any vector field* X *on* $\tilde{P}(A)$*,*

$$i_{X_{\eta,a}}(\tilde{\sigma}) = -\int_0^1 \big\langle \rho^*\eta(t,\gamma(t)), \rho(a(t))\big\rangle\, dt,$$

and

$$\mathscr{L}_{X_\eta}(\tilde{\sigma})(X_a) = \int_0^1 \big\langle (d_A\rho^*)(a(t), \eta(t,\gamma(t))), (dp)(X_a(t))\big\rangle\, dt$$
$$- i_{X_a} d\bigg(\int_0^1 \Big\langle \rho^*\eta\big(t,\gamma(t)\big), \frac{d\gamma}{dt}\Big\rangle\, dt\bigg),$$

where the last term is the differential of the function $a \mapsto \int_0^1 \langle \rho^*\eta(t,\gamma(t)), d\gamma/dt\rangle\, dt$*,* γ *is the base path of* a*, and* $\eta \in \mathfrak{g}$*.*

Proof
For the first formula, we use the definition of $\tilde{\sigma}$,

$$\tilde{\sigma}(X_{\eta,a}) = \sigma^c_{\rho^*(a)}\big((d\rho^*)_a(X_{\eta,a})\big) = \int_0^1 \big\langle \rho^*(a), (dp)(X_{\eta,a})\big\rangle,$$

the first formula in (5.4), and (5.1). To prove the second formula, we use the definition of $\tilde{\sigma}$ and the second formula in (5.4) to rewrite the Lie derivative

$$\begin{aligned}\mathscr{L}_{X_\eta}(\tilde{\sigma})(X_a) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde{\sigma}(d\Phi^{\epsilon}_{X_\eta})_a(X_a)\\ &= \int_0^1 \frac{d}{d\epsilon}\Big|_{\epsilon=0}\big\langle \rho^*\xi_\epsilon(t,\gamma_\epsilon(t)), (d\Phi^{\epsilon}_{\rho(\eta_t)})_{\gamma(t)}(X^{'}(t))\big\rangle\, dt\end{aligned}$$

where $\gamma_\epsilon(t)$ and ξ_ϵ are as in the construction above of the vector fields X_η, and $X^{'}(t) = (dp)_{a(t)}(X_a(t))$. To compute this expression, we use a connection ∇ on M. The expression in the last integral has the following two terms:

$$\big\langle \rho^*\xi_\epsilon(t,\gamma_\epsilon(t)), \partial_\epsilon(d\Phi^{\epsilon}_{\rho(\eta_t)})_{\gamma(t)}(X^{'}(t))\big\rangle, \tag{5.5}$$

$$\big\langle \partial_\epsilon\rho^*\xi_\epsilon(t,\gamma_\epsilon(t)), X^{'}(t)\big\rangle, \tag{5.6}$$

where ∂_ϵ is the derivation of paths in TM and T^*M induced by the connection. On the other hand, for any vector fields V and W on M, $\partial_\epsilon(d\Phi^{\epsilon}_{W})_x(V_x) = \overline{\nabla}_{V_x}(W)$, where $\overline{\nabla}_V(W) = \nabla_W(V) + [V, W]$ is the conjugated connection. (This is a simple check in local coordinates.) Hence (5.5) equals

$$\big\langle \rho^*(\xi), [X^{'}, \rho(\eta)] + \nabla_{\rho(\eta)}(X^{'})\big\rangle_{\gamma(t)} = \big\langle \mathscr{L}_{\rho(\eta)}(\rho^*(\xi)) - \nabla_{\rho(\eta)}(\rho^*(\xi)), X^{'}(t)\big\rangle_{\gamma(t)},$$

where, in the last equation, we have made $X^{'}$ into a vector field extending $X^{'}(t)$ (for each fixed t). On the other hand, (5.6) equals

$$\Big\langle \nabla_{d\gamma/dt}\big(\rho^*(\xi)\big) + \frac{d\rho^*(\xi_\epsilon)}{d\epsilon}, X^{'}(t)\Big\rangle = \Big\langle \nabla_{\rho(\eta)}\big(\rho^*(\xi)\big) + \rho^*([\xi,\eta]) + \frac{d\rho^*(\eta_t)}{dt}, X^{'}(t)\Big\rangle$$

(at the point $\gamma(t)$), where we have used the defining equation (5.3) for ξ. Adding the two expressions we obtained for (5.5) and (5.6) (at $\epsilon = 0$), we get

$$\begin{aligned}\mathscr{L}_{X_\eta}(\tilde{\sigma})(X_a) &= \int_0^1 \Big\langle \mathscr{L}_{\rho(\eta)}\big(\rho^*(\xi_0)\big) + \rho^*([\xi_0,\eta]) + \frac{d\rho^*(\eta_t)}{dt}, X^{'}(t)\Big\rangle_{\gamma(t)} dt\\ &= \int_0^1 \Big\langle (d_A\rho^*)(\xi_0,\eta) + i_{\rho(\xi_0)}\big(d\rho^*(\eta)\big) + \frac{d\rho^*(\eta_t)}{dt}, X^{'}(t)\Big\rangle dt,\end{aligned}$$

where, in the last equality, we used the definition of $d_A(\rho^*)$. At this point, the computation is transfered to M since the expression

$$\mathscr{L}_{X_\eta}(\tilde{\sigma})(X_a) - \int_0^1 \big\langle (d_A\rho^*)(a(t), \eta(t,\gamma(t))), (dp)_{a(t)}(X_a(t))\big\rangle\, dt$$

equals

$$\int_0^1 \Big\langle i_{d\gamma/dt}(du^t) + \frac{du^t}{dt}, X^{'}(t)\Big\rangle_{\gamma(t)} dt, \tag{5.7}$$

where $u^t = \rho^*(\eta^t)$. To finish the proof, we use the next lemma. □

LEMMA 5.4
For any path γ on M, any path $X^{'} : I \longrightarrow TM$ above γ, and any time-dependent 1-form u^t on M, we have

$$\mathscr{L}_{X'}\Big(\int_0^1 \Big\langle u\big(t,\gamma(t)\big), \frac{d\gamma}{dt}\Big\rangle dt\Big) + \int_0^1 \Big\langle i_{d\gamma/dt}(du^t) + \frac{du^t}{dt}, X^{'}(t)\Big\rangle_{\gamma(t)} dt = \big\langle u(t,\gamma(t)), X^{'}(t)\big\rangle\big|_0^1.$$

(The function on which $\mathscr{L}_{X'}$ acts is defined as in Lem. 5.3.)

Proof
We may assume that there is a vector field Z such that $Z(\gamma(t)) = X^{'}(t)$. (Otherwise, one just breaks γ into smaller paths; note that the formula to be proven is additive with respect to concatenation of paths.) We first compute the first integral using, as above, a connection ∇ and the formula $\partial_\epsilon(d\Phi^\epsilon_W)_x(V_x) = \overline{\nabla}_{V_x}(W)$:

$$\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0}&\int_0^1 \Big\langle u\big(t, \Phi^\epsilon_Z(\gamma(t))\big), \frac{d}{dt}\big(\Phi^\epsilon_Z(\gamma(t))\big)\Big\rangle dt \\
&= \int_0^1 \Big\langle u^t, \partial_\epsilon(d\Phi^\epsilon_Z)_{\gamma(t)}\Big(\frac{d\gamma}{dt}\Big)\Big\rangle dt + \Big\langle \partial_\epsilon\big(u(t,\Phi^\epsilon_Z(\gamma(t)))\big), \frac{d\gamma}{dt}\Big\rangle \\
&= \int_0^1 \Big(\big\langle u^t, \overline{\nabla}_{d\gamma/dt}(X^{'})\big\rangle + \Big\langle u^{t}, \frac{d\gamma}{dt}\Big\rangle\Big) dt.
\end{aligned}$$

Now, it is easy to see that the sum of the term in the last integral and the term appearing in the second integral in the statement is precisely

$$\big\langle u, \nabla_{d\gamma/dt}(X^{'})\big\rangle_{\gamma(t)} + \Big\langle \nabla_{d\gamma/dt}(u) + \frac{du^t}{dt}, X^{'}\Big\rangle_{\gamma(t)} = \frac{d}{dt}\big\langle u(t,\gamma(t)), X^{'}(t)\big\rangle. \qquad \square$$

Using Cartan's formula $\mathscr{L}_X = di_X + i_X d$, the next result follows directly from Lemma 5.2.

LEMMA 5.5
If ρ^* satisfies (5.1), then, for any A-path a and any vector field X on $\tilde{P}(A)$, we have

$$\begin{aligned}
i_{X_{\eta,a}}(\tilde{\omega})(X_a) = i_{X_a} d\Big(\int_0^1 \Big\langle \rho^*(\eta), \frac{d\gamma}{dt} - \rho(a)\Big\rangle dt\Big) \\
- \int_0^1 \big\langle (d_A\rho^*)(a,\eta), (dp)(X_a)\big\rangle dt.
\end{aligned}$$

We can now complete the proof of Theorem 5.1, that is, reconstruct the 2-form ω out of ρ^*. Let us assume that ρ^* satisfies both conditions (5.1) and (5.2), and put

$\tilde{\omega}_\phi = \tilde{\omega} + \omega_\phi$. Then the first equation in Lemma 5.2 and the equation in Lemma 5.5 show that $i_{X_\eta}(\tilde{\omega}_\phi) = 0$ on $P(A)$. On the other hand, the second equation in Lemma 5.2 and the fact that $d\mathbf{s}$ and $d\mathbf{t}$ vanish on X_η's (since η vanishes at end points) imply that

$$i_{X_\eta} d\tilde{\omega}_\phi = i_{X_\eta}(\mathbf{t}^*\phi - \mathbf{s}^*\phi) = 0.$$

Hence $\tilde{\omega}_\phi$ is basic, and it induces a 2-form ω_0 on $G(A)$. The multiplicativity of ω_0 follows from the additivity of integration. To compute the associated $\rho^*_{\omega_0}(\alpha)(X) = \omega(\alpha, X)$, one has to look at the identification of the Lie algebroid of $G(A)$ with A (see [13]). After straightforward computations, we find that

$$\rho^*_{\omega_0}(\alpha)(X) = \omega_{\mathrm{can}}\big((\rho(\alpha), \rho^*(\alpha)), (X, 0)\big).$$

Here, ω_{can} is the linear version of the symplectic form,

$$\omega_{\mathrm{can}}\big((X_1, \eta_1), (X_2, \eta_2)\big) = \eta_2(X_1) - \eta_1(X_2).$$

We see that $\rho^*_{\omega_0} = -\rho^*$. On the other hand, $d\omega_0 = t^*\phi - s^*\phi$, as it follows from the similar formula satisfied by ω_ϕ (cf. Lem. 5.2). Hence $\omega = -\omega_0$ will have the desired properties.

The construction of $\tilde{\omega}_\phi$ is inspired by the one in [8], which we recover when ρ^* is an isomorphism.

6. Examples

We discuss in this section some examples of multiplicative 2-forms, presymplectic groupoids, and their corresponding Dirac structures.

6.1. *Multiplicative 2-forms: First examples*

We now discuss some basic examples of multiplicative 2-forms on Lie groupoids.

Example 6.1 (*Lie groups*)
If H is a Lie group (so the base M is just the one-point space consisting of the identity in H), then the zero form is the only multiplicative form on H. This follows from Lemma 3.1 (part (ii) or part (iv)).

Example 6.2 (*Lie groupoids integrating tangent bundles*)
Let M be a ϕ-twisted presymplectic manifold. Hence $\phi \in \Omega^3(M)$ is closed, and M is equipped with a 2-form ω_M with $d\omega_M + \phi = 0$. Consider the pair groupoid $G = M \times M$ with the product $(x, y) \circ (y, z) = (x, z)$ (hence $s = \mathrm{pr}_2$, $t = \mathrm{pr}_1$). A simple computation shows that $M \times M$ equipped with the 2-form $\omega = \mathrm{pr}_1^*\,\omega_M - \mathrm{pr}_2^*\,\omega_M$ is a ϕ-twisted presymplectic groupoid, and that the ϕ-twisted Dirac structure induced on M (identified with the diagonal in $M \times M$) is just the one associated with ω_M. As usual,

one obtains the s-simply connected ϕ-twisted presymplectic groupoid corresponding to ω_M by pulling $\omega \in \Omega^2(M \times M)$ back to $\Pi(M)$, the fundamental groupoid of M, using the natural covering map $\Pi(M) \longrightarrow M \times M$ (which is also a groupoid morphism).

Example 6.3 (*Pullbacks*)
Let L be a ϕ-twisted Dirac structure on M, and let $(G(L), \omega_L)$ be a presymplectic groupoid integrating it. If $f : P \longrightarrow M$ is a submersion (we will see that weaker conditions are possible), we can form the *pullback* groupoid $f^*G(L) := P \times_M G(L) \times_M P$ consisting of triples (p, g, q) with $g : f(p) \leftarrow f(q)$, $s = \mathrm{pr}_3$, $t = \mathrm{pr}_1$, and $(p, g, q) \cdot (q, h, r) = (p, gh, r)$. It is simple to check that $\dim(f^*G(L)) = 2\dim(P)$, and that the form $\mathrm{pr}_2^* \omega_L \in \Omega^2(f^*G(L))$ is multiplicative, robust and relatively $f^*\phi$-closed. So $(f^*G(L), \mathrm{pr}_2^* \omega_L)$ is a $f^*\phi$-twisted presymplectic groupoid over P. Infinitesimally, it corresponds to the *pullback* Dirac structure (see, e.g., [5])

$$f^*L = \big\{(X, f^*(\xi)) : ((df)(X), \xi) \in L\big\}.$$

We remark that the construction of the pullbacks f^*L and $f^*G(L)$ is also possible in situations where f is not a submersion. In this case, $(f^*G(L), \mathrm{pr}_2^* \omega_L)$ is often just a (twisted) *over*-presymplectic groupoid, but not presymplectic. Such examples arise, for instance, when one considers inclusions of submanifolds (see Example 6.7).

Example 6.4 (*Multiplicative 2-forms of non-Dirac type*)
There are closed multiplicative 2-forms that are not of Dirac type. In order to provide an explicit example, we start with a general observation. Let G be a groupoid over M, and let θ be a closed 2-form on M.

CLAIM
If the multiplicative 2-form $\omega = t^*\theta - s^*\theta$ on G is of Dirac type, then

$$\mathrm{Im}(\rho_x) + \mathrm{Im}(\rho_x)^{\perp_\theta}$$

has constant dimension along points x in a fixed orbit of G. (Here "$\perp_\theta$" is the orthogonal with respect to θ.)

Proof
We prove the claim following the notation of Section 4.

If ω is of Dirac type, then the ranges of $t_*(L_{\omega,g})$ and $L_{M,t(g)}$ must coincide. Let us denote $s(g) = x$, $t(g) = y$. Since $\omega = t^*\theta - s^*\theta \in \Omega^2(G)$, we can write $\rho_\omega^*(\alpha) = i_{\rho(\alpha)}(\theta)$. Now the second formula in (4.5) implies that $\mathrm{Ker}(\omega) \cap T_yM = \mathrm{Im}(\rho_y)^{\perp_\theta}$ for all $y \in M$, and, using (4.2), we see that

$$\dim\big(\mathscr{R}(L_{M,y})\big) = \dim\big(\mathrm{Im}(\rho_y) + \mathrm{Im}(\rho_y)^{\perp_\theta}\big).$$

On the other hand, it is easy to see from the definition of $t_*(L_{\omega,g})$ that $\mathscr{R}(t_*(L_{\omega,g})) = (dt)_g(\mathrm{Ker}(dt_g)^{\perp})$ which, as shown in the proof of Claim 2 (Lem. 4.3), has dimension equal to $(dt)_x(\mathrm{Ker}(dt_x)^{\perp}) = (dt)_x(\mathrm{Ker}(ds_x) + \mathrm{Ker}(\omega_x))$. Hence

$$\dim\big(\mathscr{R}(t_*(L_{\omega,g}))\big) = \dim\big(\mathrm{Im}(\rho_x) + \mathrm{Ker}(\omega_x) \cap T_x M\big) = \dim\big(\mathrm{Im}(\rho_x) + \mathrm{Im}(\rho_x)^{\perp_\theta}\big),$$

and this proves the claim. □

To find our example, let v be a vector field on M, and let Φ_v denote its flow. The domain $G(v) \subset \mathbb{R} \times M$ of Φ_v is a groupoid over M (the source is the second projection, the target is Φ_v, and the multiplication is defined by $(t_1, y)(t_2, x) = (t_1 + t_2, x)$) integrating the action Lie algebroid defined by v (the underlying vector bundle is the trivial line bundle, the anchor is multiplication by v, and the bracket is $[f, g] = f v(g) - v(f) g$). On such a groupoid, the image of ρ is either zero- or one-dimensional, so $\mathrm{Im}(\rho_x) + \mathrm{Im}(\rho_x)^{\perp_\theta} = \mathrm{Im}(\rho_x)^{\perp_\theta}$ for any closed $\theta \in \Omega^2(M)$, and this need not be constant along orbits of v. For instance, one can take

$$M = \mathbb{R}^2, \qquad v = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad \theta = y\, dx\, dy.$$

Then the circle S^1 is an integral curve, and $\mathrm{Im}(\rho_x)^{\perp}$ is one-dimensional everywhere on S^1, except for the points $(1, 0)$ and $(-1, 0)$.

6.2. Examples related to Poisson manifolds

Example 6.5 (*Symplectic groupoids*)
Let (G, ω) be a presymplectic groupoid over M, and let L be the corresponding Dirac structure on M. Recall that L comes from a Poisson structure if and only if $\mathrm{Ker}(L) = \mathrm{Ker}(\omega) \cap TM = \{0\}$. But this condition is equivalent to $\mathrm{Ker}(\omega_x) = 0$ for all $x \in M$ (by Cor. 4.8(v) and Lem. 3.3), which is in turn equivalent to $\mathrm{Ker}(\omega_g) = 0$ for all $g \in G$ (by Lem. 4.2(iv)). Hence L comes from a Poisson structure if and only if ω is nondegenerate. We see in this way that our main result, restricted to Poisson structures, recovers the well-known correspondence between Poisson manifolds and symplectic groupoids. In the presence of a closed 3-form, we recover the twisted version of this correspondence, which was conjectured in [27] and proved in [8].

Example 6.6 (*Gauge transformations of Poisson manifolds*)
Following [5], we now explain how to produce twisted presymplectic groupoids out of symplectic groupoids through gauge transformations associated to 2-forms.

Let M be a smooth manifold equipped with a ϕ-twisted Poisson structure π. We denote the corresponding ϕ-twisted Dirac structure by $L_\pi = \mathrm{graph}(\widetilde{\pi})$. The gauge transformation of L_π associated to a 2-form B on M is given by

$$L_\pi \mapsto \tau_B(L_\pi) = \big\{\big(\widetilde{\pi}(\eta), \eta + \widetilde{B}(\widetilde{\pi}(\eta))\big),\ \eta \in T^*M\big\}.$$

As explained in [27], $\tau_B(L_\pi)$ is a $(\phi - dB)$-twisted Dirac structure that may fail to be Poisson.

Let (G, ω) be a ϕ-twisted symplectic groupoid integrating π. Since L_π and $\tau_B(L_\pi)$ have isomorphic Lie algebroids (see [27]), $\tau_B(L_\pi)$ is integrable as an algebroid to a groupoid isomorphic to G. The 2-form

$$\tau_B(\omega) := \omega + t^*B - s^*B \in \Omega^2(G) \tag{6.1}$$

is easily seen to be multiplicative, robust, and relatively $(\phi - dB)$-closed, and, as remarked in [5, Th. 2.16], it induces the Dirac structure $\tau_B(L_\pi)$ on M. So $(G, \tau_B(\omega))$ is the presymplectic groupoid corresponding to $\tau_B(L_\pi)$.

Our results also show that this is true more generally: if L is a (twisted) Dirac structure on M associated with a presymplectic groupoid $(G(L), \omega_L)$, then

$$\rho^*_{\tau_B(\omega_L)} = \rho^*_\omega + i_{\rho(\alpha)}B;$$

hence the image of $(\rho, \rho^*_{\tau_B(\omega_L)})$ is $\tau_B(L)$, and $(G(L), \tau_B(\omega_L))$ is a presymplectic groupoid integrating $\tau_B(L)$.

Example 6.7 (*Dirac submanifolds of Poisson manifolds*)
In this example we relate our results to those in [14, Sec. 9]. We describe how certain submanifolds of Dirac manifolds carrying an induced Dirac structure (such submanifolds of Poisson manifolds were studied in [14], [33]) give rise to over-presymplectic groupoids, whose reduction (in the sense of Rem. 4.9(i)) produces presymplectic groupoids of the submanifolds. For simplicity, we restrict the discussion to the untwisted case.

Let L_M be a Dirac structure on M, let $N \hookrightarrow M$ be a submanifold, and suppose that the pullback Dirac structure induced at each point by inclusion,

$$L_N := \left\{(X, \xi|_{TN}) : X \in T_xN, (X, \xi) \in L_M\right\} \subset TN \oplus T^*N,$$

is a smooth bundle; it is not difficult to check that L_N defines a Dirac structure on N. In the particular case of L_M coming from a Poisson structure π_M on M, and L_N coming from a Poisson structure π_N on N, N is called a *Poisson-Dirac submanifold** of (M, π_M) (see [14, Sec. 9]).

Let us consider the vector bundle

$$\mathfrak{g}_N(M) := TN^\circ \cap (L_M \cap T^*M) = \left(\mathscr{R}(L_M) + TN\right)^\circ$$

over N, which is, in fact, a bundle of Lie algebras, and let us assume that it has constant rank. (Here $^\circ$ denotes the annihilator.) In this case, the restriction of the

*A Poisson submanifold is a Poisson-Dirac submanifold for which the inclusion $(N, \pi_N) \hookrightarrow (M, \pi_M)$ is a Poisson map; this is equivalent to $\mathrm{Im}(\tilde{\pi}_N) = \mathrm{Im}(\tilde{\pi}_M)$.

Lie algebroid L_M to N (see [17]) is well defined and determines a Lie subalgebroid $L_N(M)$ whose underlying vector bundle is

$$\{(X,\xi) : X \in TN, (X,\xi) \in L_M\}.$$

This Lie algebroid fits into the following exact sequence of Lie algebroids:

$$0 \longrightarrow \mathfrak{g}_N(M) \longrightarrow L_N(M) \longrightarrow L_N \longrightarrow 0.$$

Let us assume that L_M is integrable, and let $(G(L_M), \omega_M)$ be the associated presymplectic groupoid. Then $L_N(M)$ is also integrable (as it sits inside L_M as a Lie subalgebroid), and the associated groupoid $G(L_N(M))$ is a subgroupoid of $G(L_M)$. Moreover, the restriction $\omega_{N,M}$ of ω_M to $G(L_N(M))$ makes $G(L_N(M))$ into an over-presymplectic groupoid over N, corresponding to the pullback Dirac structure L_N.

We observe, however, that the reduction procedure of Remark 4.9(i) produces a smooth presymplectic groupoid if and only if L_N is also integrable. In this case, the reduced groupoid will be precisely the pullback (see Exam. 6.3) presymplectic groupoid $(G(L_N), \omega_N)$ of L_N. Hence the presymplectic groupoids of a Dirac structure and of a Dirac submanifold are related by reduction of an intermediary over-presymplectic groupoid, as illustrated:

$$\begin{tikzcd}
\big(G(L_N(M)), \omega_{N,M}\big) \arrow[r, hook] \arrow[dr] & \big(G(L_M), \omega_M\big) \\
& \big(G(L_N), \omega_N\big)
\end{tikzcd}$$

In general, this quotient space may not be a manifold, but it is the same as the Weinstein groupoid introduced in [13] (see also [7]).

If L_M comes from a Poisson structure π_M, the discussion above shows that N is a Poisson-Dirac submanifold of (M, π_M) if and only if $G(L_N(M))$ is an over-symplectic groupoid, and the reduction procedure just described recovers [14, Prop. 9.13].

6.3. Over-presymplectic groupoids and singular presymplectic groupoids

In this subsection we discuss examples of over-presymplectic groupoids that cannot be reduced to presymplectic groupoids, as already mentioned in Example 6.7. In particular, we provide concrete examples of nonintegrable Poisson structures admitting over-symplectic groupoids inducing them.

Let us consider the following particular case of the construction in Example 6.7. Let (M, π) be a Poisson manifold with an s-simply connected symplectic groupoid (G, ω). Let C be a Casimir function on M, and let a be a regular value of C. Then the level manifold $N = C^{-1}(a) \hookrightarrow M$ is a Poisson submanifold of M. As we saw in Example 6.7, the restriction $G_{N,M} = t^*(N) = s^*(N)$ of G to N is an s-simply connected groupoid over N, and the pullback $\omega_{N,M}$ of ω to $G_{N,M}$ makes it into an over-symplectic groupoid over N, inducing on N its Poisson structure. Indeed, the kernel of $\omega_{N,M}$ is spanned by the Hamiltonian vector field of $t^*C = s^*C$, and since this vector field projects to zero on M, it is easy to check that condition (iv) of Proposition/Definition 4.5 is satisfied. To pass to a symplectic groupoid for N, we simply form the quotient of $G_{N,M}$ by the Hamiltonian flow.

Example 6.8 (*Over-symplectic groupoids of nonintegrable Poisson submanifolds*) To obtain an interesting class of examples of the above construction, take G to be the cotangent bundle $T^*H \cong H \ltimes \mathfrak{h}^*$ of a simply connected Lie group H, $M = \mathfrak{h}^*$, and take

$$C : \mathfrak{h}^* \to \mathbb{R}, \qquad u \mapsto \frac{1}{2}(u, u)_{\mathfrak{h}^*}$$

to be the kinetic energy function of a bi-invariant (possibly indefinite) metric $(\cdot, \cdot)_{\mathfrak{h}^*}$ on H. The Hamiltonian flow of $t^*(C)$ is then the geodesic flow on T^*H. If N is the unit "sphere" $\mathfrak{h}^*_{1/2} = C^{-1}(1/2)$, then the unit co"sphere" bundle $G_{N,M} = (T^*H)_{1/2}$ is an oversymplectic groupoid over it. The quotient Q of $(T^*H)_{1/2}$ by the geodesic flow is then the canonical symplectic groupoid for the Poisson manifold $\mathfrak{h}^*_{1/2}$. The elements of Q are geodesics in H considered as oriented submanifolds. One can view them as cosets of an open subset of the connected oriented one-dimensional subgroups of H. From this point of view, the groupoid structure of Q is just the one induced from the Baer groupoid (see [31]) consisting of all the cosets in H.

A specific case of the construction in the previous paragraph recovers the "pathology" of symplectic groupoids discussed in [7, Sec. 6]. We let H be the product of a "space manifold" $SU(2) = S^3$ with its usual Riemannian metric and a "time manifold" $\mathbb{R}$ carrying the negative of its usual metric. The unit "sphere" in $\mathfrak{h}^*$ for this Lorentz metric may then be identified with the product $S^2 \times \mathbb{R}$, with the Poisson structure for which the S^2-slice over each $\tau \in \mathbb{R}$ is a symplectic leaf with symplectic structure equal to $1 + \tau^2$ times the standard symplectic structure. The critical point of $1+\tau^2$ at $\tau = 0$ is responsible for a singularity in the space of unit-speed (equivalently, spacelike) geodesics. Most of these geodesics are diffeomorphic to $\mathbb{R}$, but the ones that are perpendicular to the τ-direction are circles. As a result, the 6-dimensional quotient groupoid, that is, the space of geodesics in $S^3 \times \mathbb{R}$, is a singular fiber bundle over $S^2 \times S^2$—the smooth 4-dimensional manifold of oriented geodesics* in S^3. The fiber, which is neither Hausdorff nor locally Euclidean, is the quotient space $\mathbb{R}^2/(x, y) \sim (x, x + y)$. It can be obtained from the standard cone in $\mathbb{R}^3$ by remov-

*The projection $T^*H \to T^*S^3$ is equivariant with respect to the geodesic flows.

ing the vertex and replacing it with a line, with a topology such that any sequence of points on the cone converging to where the vertex used to be now converges to every point on the line.

An alternative way of obtaining over-presymplectic groupoids (which may not be reducible) inducing a given Dirac structure L is by means of extensions of the Lie algebroid of L by 2-cocycles (see, e.g., [23]).

Let L be a Dirac structure on M, and let $u \in \Gamma(\Lambda^2 L^*)$ be a 2-cocycle of the Lie algebroid of L. Let us assume, for simplicity, that L is not twisted. Then there is an associated algebroid $L \ltimes_u \mathbb{R} = L \oplus \mathbb{R}$ which fits into an exact sequence

$$0 \to \mathbb{R} \to L \ltimes_u \mathbb{R} \to L \to 0. \tag{6.2}$$

The anchor of $L \ltimes_u \mathbb{R}$ is just $(X, a) \mapsto \rho(X)$, where ρ is the anchor of L, while the Lie bracket on $\Gamma(L)$ is

$$[(X, a), (Y, b)] = \big([X, Y], \mathscr{L}_X b - \mathscr{L}_Y a + u(X, Y)\big).$$

The interesting point of this construction is that $L \ltimes_u \mathbb{R}$ may be integrable even when L is not. Moreover, the associated groupoid $G(L \ltimes_u \mathbb{R})$ admits a canonical 1-form σ_u (whose construction is similar to the construction of the form σ in Prop. 8.1(iii)), and $(G(L \ltimes_u \mathbb{R}), d\sigma_u)$ is an over-presymplectic groupoid over M inducing L on the base. In the case of a Poisson manifold (M, π) ($L = L_\pi$), the groupoid corresponding to the extension of the Lie algebroid T^*M by $u = \pi$, together with the 1-form σ_u, is an example of a contact groupoid.

Example 6.9 (*Contact groupoids*)
Contact groupoids are groupoids associated with Jacobi manifolds, as we now briefly explain (see [15] and references therein for details).

A *Jacobi manifold* is a manifold equipped with a bivector Λ and a vector field E satisfying $[\Lambda, \Lambda] = 2\Lambda \wedge E$ and $[\Lambda, E] = 0$. A particular example is given by Poisson manifolds (M, π), in which case, $\Lambda = \pi$ and $E = 0$; more generally, if $g \in C^\infty(M)$, then $g\pi$ may fail to be Poisson, but $\Lambda = g\pi$ and $E = X_g$ define a Jacobi structure on M.

Any Jacobi manifold (M, Λ, E) determines a Lie algebroid structure on $T^*M \oplus \mathbb{R}$, which in the case of a Poisson manifold (M, π) is precisely $T^*M \ltimes_\pi \mathbb{R}$ defined in (6.2). Just as Poisson structures correspond to symplectic groupoids (or Dirac structures correspond to presymplectic groupoids), Jacobi manifolds are associated with *contact groupoids*. These are groupoids G endowed with a contact 1-form σ and a smooth function $f \in C^\infty(G)$ such that σ is f-multiplicative:

$$m^*\sigma = \mathrm{pr}_2^* f \cdot \mathrm{pr}_1^* \sigma + \mathrm{pr}_2^* \sigma.$$

(Here $\mathrm{pr}_j : G \times G \to G$ is the natural projection onto the jth factor.) When M is a Poisson manifold, the function f is the constant function 1, so σ is multiplicative. One can therefore associate two groupoids to a Poisson manifold (M, π): its symplectic groupoid (G_s, ω) and the contact groupoid (G_c, σ) obtained by regarding (M, π) as a Jacobi manifold. Since σ is multiplicative, so is the 2-form $d\sigma$, and one can check that $(G_c, d\sigma)$ is an over-symplectic groupoid over M inducing on M its Poisson structure. As a result, one can see (G_s, ω) as a reduction of $(G_c, d\sigma)$.

So, whenever a Poisson manifold (M, π) is integrable as a Jacobi manifold, it automatically admits an over-symplectic groupoid inducing π. As explained in [15], (M, π) can be a nonintegrable Poisson manifold and yet be integrable as a Jacobi manifold (and vice versa). For example, the co“sphere” bundle $(T^*H)_{1/2}$ in Example 6.8 is actually a contact groupoid with contact form given by the restriction of the canonical 1-form on T^*H. For more concrete examples and a detailed comparison of the obstructions to Poisson and Jacobi integrability, we refer the reader to [15].

6.4. Lie group actions and equivariant cohomology

Let H be a connected Lie group acting on a manifold M. We consider the *action groupoid* $H \ltimes M$ over M, with

$$s(g, x) = x, \qquad t(g, x) = gx, \quad \forall (g, x) \in H \times M,$$

and multiplication of composable pairs given by

$$m\big((g_1, x_1), (g_2, x_2)\big) = (g_1 g_2, x_2).$$

It was pointed out in [3] that the space of twisted multiplicative 2-forms on $H \ltimes M$ is closely related to the equivariant cohomology of M in degree three. Our main results provide the following description of this relationship at the infinitesimal level.

The Lie algebroid of $H \ltimes M$ (the *action Lie algebroid* $\mathfrak{h} \ltimes M$) is the trivial bundle $\mathfrak{h}_M := \mathfrak{h} \times M \to M$; the anchor is defined by the infinitesimal action, and the bracket is uniquely determined by the Leibniz rule and the Lie bracket on $\mathfrak{h}$. Following Theorem 2.5, the infinitesimal counterpart of twisted multiplicative 2-forms on $H \ltimes M$ are pairs (ρ^*, ϕ), where ϕ is a closed 3-form on M, and $\rho^* : \mathfrak{h}_M \longrightarrow T^*M$ is a bundle map satisfying conditions (5.1) and (5.2). We denote by $\omega_{\rho^*,\phi} \in \Omega^2(H \ltimes M)$ the relatively ϕ-closed, multiplicative 2-form associated with (ρ^*, ϕ).

When H is a compact Lie group, the equivariant cohomology of M can be computed by Cartan's complex of equivariant differential forms on M, denoted $\Omega^*_H(M)$ (see, e.g., [16]). This complex consists of H-invariant, $\Omega^*(M)$-valued polynomials on $\mathfrak{h}$, with degree twice the polynomial degree plus the form degree:

$$\Omega^k_H(M) = \Big(\bigoplus_{2i+j=k} S^i(\mathfrak{h}^*) \otimes \Omega^j(M) \Big)^H .$$

The differential is $d_H := d_1 - d_2$, where $d_1(P)(v) = d(P(v))$, $d_2(P)(v) = i_v(P(v))$; if α is an $\Omega^*(M)$-valued polynomial on $\mathfrak{h}$, invariance means that

$$g^*\big(\alpha(\mathrm{Ad}_g(v))\big) = \alpha(v) \tag{6.3}$$

for all $v \in \mathfrak{h}$ and $g \in H$.

Note that equivariantly closed 3-forms can be written as

$$\rho^* + \phi \in \Omega^3_H(M),$$

where $\phi \in \Omega^3(M)$ is closed, and $\rho^* \in \mathfrak{h}^* \otimes \Omega^1(M)$, which are both invariant (as in (6.3)) and satisfy

$$\begin{cases} i_v(\rho^*(v)) = 0, \\ i_v(\phi) - d(\rho^*(v)) = 0 \end{cases}$$

for all $v \in \mathfrak{h}$. Infinitesimally, the invariance condition for ρ^* reads

$$\rho^*([v, w]) = \mathscr{L}_v\big(\rho^*(w)\big)$$

for all $v, w \in \mathfrak{h}$. Using this equation, one can easily check that conditions (5.1) and (5.2) are satisfied, and hence there is a corresponding multiplicative 2-form $\omega_{\rho^*,\phi}$. Assuming that $\rho^* + \phi \in \Omega^3_H(M)$, we now describe how one can obtain a simple explicit formula for $\omega_{\rho^*,\phi}$ just using general properties of multiplicative forms.

Let pr_g and pr_x be the natural projections of $H \times M$ onto H and M, respectively, and let $\lambda, \overline{\lambda}$ denote the left- and right-invariant Maurer-Cartan forms on H $\big($i.e., $\lambda_g(V) = (dL_{g^{-1}})_g(V)$, $\overline{\lambda}_g(V) = (dR_{g^{-1}})_g(V)\big)$.

PROPOSITION 6.10

Suppose that ρ^* and ϕ satisfy conditions (5.1) and (5.2), and let $\omega = \omega_{\rho^*,\phi} \in \Omega^2(H \ltimes M)$ be the corresponding 2-form. The following are equivalent:

(i) $\rho^* + \phi \in \Omega^3_H(M)$;

(ii) *the restriction of ω to all slices $\{g\} \times M$ vanishes for all $g \in H$;*

(iii) *ω is given by the formula*

$$\omega_{g,x} = \Big\langle \rho^*_x \,\mathrm{pr}^*_g \,\lambda, \mathrm{pr}^*_x + \frac{1}{2}\rho_x \,\mathrm{pr}^*_g \,\lambda \Big\rangle$$

or, explicitly,

$$\begin{aligned}\omega_{g,x}\big((V, X), (V', X')\big) = {}& \big\langle \rho^*_x(\lambda_g(V)), \rho_x(\lambda_g(V'))\big\rangle \\ &+ \big\langle \rho^*_x(\lambda_g(V)), X'\big\rangle - \big\langle \rho^*_x(\lambda_g(V')), X\big\rangle. \end{aligned} \tag{6.4}$$

Proof
We first observe a few facts. The defining formula for ρ^* implies that

$$\omega\big((v,0),(0,X)\big)=\big\langle\rho^*(v),X\big\rangle \tag{6.5}$$

for $v\in\mathfrak{h}$, $X\in TM$. Using (3.4), we can write

$$\begin{aligned}\omega_{g,x}\big((V_g,0),(V'_g,X'_x)\big)&=\omega_{gx}\big(((dR_g^{-1})_g(V_g),0),dt_{g,x}(V'_g,X'_x)\big)\\&=\omega_{gx}\big(((dR_g^{-1})_g(V_g),0),dt_{gx}((dR_g^{-1})_gV'_g,gX'_x)\big)\\&=\big\langle\rho^*_{gx}(dR_g^{-1})_g(V_g),\rho_{gx}((dR_g^{-1})_g(V'_g))+gX'_x\big\rangle\end{aligned}$$

for all V_g, $V'_g\in T_gH$, $X'_x\in T_xM$. (Here gX denotes the infinitesimal action of H on TM.) Hence, for general pairs $((V_g,X_x),(V'_g,X'_x))$, we have

$$\begin{aligned}\omega_{g,x}\big((V_g,X_x),(V'_g,X'_x)\big)&=\omega^0_{g,x}\big((V_g,X_x),(V'_g,X'_x)\big)\\&\quad+\omega_{g,x}\big((0_g,X_x),(0_g,X'_x)\big),\end{aligned} \tag{6.6}$$

where

$$\begin{aligned}\omega^0_{g,x}\big((V_g,X_x),(V'_g,X'_x)\big)&=\big\langle\rho^*_{gx}dR_{g^{-1}}(V_g),\rho_{gx}dR_{g^{-1}}(V'_g)\big\rangle\\&\quad+\big\langle\rho^*_{gx}dR_{g^{-1}}(V_g),gX'_x\big\rangle-\big\langle\rho^*_{gx}dR_{g^{-1}}(V'_g),gX_x\big\rangle.\end{aligned} \tag{6.7}$$

Let $\omega^1=\omega-\omega^0$. We make two simple remarks. First, ω^1 encodes precisely the restrictions of ω to the slices $\{g\}\times M$ (see (6.6)). Second, if ρ^* is invariant, then (6.7) coincides with formula (6.4) in the statement. Hence it suffices to show that $\omega=\omega^0$ if and only if (ρ^*,ϕ) is an equivariant form. One possible route to prove that is as follows. One can show that ω^1 (or, equivalently, ω^0) is multiplicative if and only if ρ^* is invariant, and ω^1 is closed if and only if $i_v(\phi)-d(\rho^*(v))=0$. Since $\rho^*_{\omega^1}=0$, the uniqueness of Corollary 3.4 implies the proposition. We present an alternative argument instead.

Let us rewrite $\omega^1=\omega-\omega^0$ as $\langle c(g),X_x\wedge X'_x\rangle$, defining a smooth function $c\in C^\infty(H;\Omega^2(M))$. The multiplicativity of ω, applied on vectors $((0,X),(0,X'))$, reads

$$\omega_{hg,x}\big((0,X_x),(0,X'_x)\big)=\omega_{h,gx}\big((0,gX_x),(0,gX'_x)\big)+\omega_{g,x}\big((0,X_x),(0,X'_x)\big), \tag{6.8}$$

which precisely means that

$$c(hg)=g^*c(h)+c(g),\quad\forall\, g,h,\in H \tag{6.9}$$

(i.e., c is an $\Omega^2(M)$-valued 1-cocycle on H). Note that, in order to prove that $c=0$, it suffices to show that $\mathscr{L}_v(c)=0$ for all $v\in\mathfrak{h}$. Indeed, by differentiating (6.9), we obtain $\mathscr{L}_{V_g}(c) = 0$ for all $V_g = dR_g(v) \in T_g H$ and all $g \in H$, and c must be constant. Since, again by (6.9), $c(1)$ is clearly zero, c must vanish (see also Rem. 6.11).

We now claim that

$$\mathscr{L}_v(c) = d_H(\rho^* + \phi)(v) = d\big(\rho^*(v)\big) - i_v(\phi) \tag{6.10}$$

for all $v \in \mathfrak{h}$. In order to prove (6.10), let V be a vector field on H extending v, and let X and X' be vector fields on M. We evaluate $d\omega = s^*\phi - t^*\phi$ on $(V, 0)$, $(0, X)$, $(0, X')$:

$$\begin{aligned} d\omega\big((V,0),(0,X),(0,X')\big) &= \mathscr{L}_{(V,0)}\big(\omega((0,X),(0,X'))\big) \\ &\quad - \mathscr{L}_{(0,X)}\big(\omega((V,0),(0,X'))\big) \\ &\quad + \mathscr{L}_{(0,X')}\big(\omega((V,0),(0,X))\big) + \omega\big((V,0),(0,[X,X'])\big) \\ &= -\phi\big(\rho(\bar{\lambda}(V)), X, X'\big), \end{aligned}$$

where $\bar{\lambda}(V)_g = (dR_g^{-1})_g(V_g)$. Using (6.5) and evaluating the previous formula at $g = 1 \in H$, we find

$$\begin{aligned} \mathscr{L}_v(c)(X, X') &= \mathscr{L}_X\big(\rho^*(v)(X')\big) \\ &\quad - \mathscr{L}_{X'}\big(\rho^*(v)(X)\big) - \rho^*(v)([X,Y]) - \phi\big(\rho(v), X, Y\big), \end{aligned}$$

which is just (6.10). This proves the proposition. □

Remark 6.11

We observe that the proof of Proposition 6.10 indicates how to express $\omega_{\rho^*,\phi}$ for general pairs (ρ^*, ϕ) (i.e., which satisfy only the conditions (5.1), (5.2)). More precisely, the cocycle condition (6.9) for c is equivalent to saying that $g \mapsto (g, c(g))$ is a group homomorphism from H into the group $H \ltimes \Omega^2(M)$ defined by $(h,a)(g,b) = (hg, g^*a + b)$. The proof of Proposition 6.10 then shows that the induced Lie algebra map $\mathfrak{h} \longrightarrow \mathfrak{h} \ltimes \Omega^2(M)$ is

$$v \mapsto \big(v, d(\rho^*(v)) - i_v(\phi)\big). \tag{6.11}$$

So, if H is simply connected, the Lie algebra cocycle (6.11) integrates uniquely to a group cocycle c, and ω is given by (6.7) plus $c(g)(X_x, Y_x)$.

Remark 6.12

In general, there is a natural map

$$H^*\big(\Omega_H^*(M)\big) \longrightarrow H_H^*(M)$$

from the cohomology of the Cartan complex into the equivariant cohomology of M, which is an isomorphism if H is compact (see [2]). The equivariant cohomology groups can be obtained from a double complex $\Omega^p(H^q \times M)$, with de Rham differential increasing the degree p, and a group-cohomology differential increasing q (see, e.g., [4]). Our result gives both an explicit description of this map in degree three,

$$\rho^* + \phi \mapsto \omega_{\rho^*,\phi} + \phi,$$

as well as an interpretation of this map in terms of multiplicative forms.

If $\rho^* + \phi \in \Omega^3_H(M)$ satisfies the nondegeneracy condition $\dim(\mathrm{Ker}(\rho) \cap \mathrm{Ker}(\rho^*)) = \dim(H) - \dim(M)$, then $(H \ltimes M, \omega_{\rho^*,\phi})$ becomes an over-presymplectic groupoid, which is presymplectic if $\dim(M) = \dim(H)$. The associated ϕ-twisted Dirac structure can be described directly as

$$L = \{(\rho(v), \rho^*(v)) : v \in \mathfrak{h}\} \subset TM \oplus T^*M.$$

A simple example is $M = \mathfrak{h}^*$ with the coadjoint action of H, $\phi = 0$, and ρ^* given by $\rho^*_\xi(v) = v$. The associated groupoid is $H \ltimes \mathfrak{h}^* \cong T^*H$ with the canonical symplectic form. A more interesting example is the AMM groupoid of Section 7.2.

7. Presymplectic realizations of Dirac structures

Let (M, π) be a Poisson manifold. Recall that a *symplectic realization* of M is a Poisson map from a symplectic manifold (P, η) to M (see, e.g., [6]). The following important property of symplectic realizations brings them close to the theory of Hamiltonian actions: any symplectic realization $\mu : P \to M$ induces a canonical action of the Lie algebroid T^*M, induced by π, on P by assigning to each $\alpha \in \Omega^1(M)$ the vector field $X \in \mathscr{X}(P)$ defined by

$$i_X \eta = \mu^* \alpha.$$

When μ is complete (i.e., the Hamiltonian vector field $X_{\mu^* f}$ is complete whenever $f \in C^\infty(M)$ has compact support), M is integrable (see [14]) and this action extends to a symplectic action of G, the s-simply connected symplectic groupoid of M (see [9], [14]) with moment map μ (see [24]). In this way, we get a natural correspondence between symplectic actions of G and complete symplectic realizations of M. In particular, if $M = \mathfrak{h}^*$ is the dual of a Lie algebra, the action of the associated symplectic groupoid $T^*H = H \ltimes \mathfrak{h}^*$ factors through an H-action, and complete symplectic realizations of $\mathfrak{h}^*$ become Hamiltonian H-spaces. In this section, we extend this picture to twisted Dirac manifolds.

7.1. Presymplectic realizations

We recall the definition introduced in Section 2.3.

Definition 7.1
A *presymplectic realization* of a ϕ-twisted Dirac manifold (M, L) is a Dirac map $\mu : (P, \eta) \longrightarrow (M, L)$, where η is a $\mu^*\phi$-closed 2-form (i.e., $d\eta + \mu^*\phi = 0$), such that $\mathrm{Ker}(d\mu) \cap \mathrm{Ker}(\eta) = \{0\}$.

The following results explain this definition.

LEMMA 7.2
Let (M, L) be a ϕ-twisted Dirac manifold, let $\mu : P \longrightarrow M$ be a smooth map, and let P be equipped with a 2-form η satisfying $d\eta + \mu^*\phi = 0$. The following are equivalent:
(i) *the map μ is a presymplectic realization of L;*
(ii) *for all $p \in P$, $(w, \xi) \in L_{\mu(p)}$, there exists a unique $X \in T_pP$ satisfying the equations*
$$\begin{cases} w = d\mu(X), \\ \mu^*(\xi) = i_X(\eta); \end{cases}$$
(iii) *the map μ is Dirac, and $d\mu$ maps* $\mathrm{Ker}(\eta)$ *isomorphically onto* $\mathrm{Ker}(L)$.

Proof
Note that μ being a Dirac map is equivalent to the equations in (ii) having a solution for X. The uniqueness of the solutions is equivalent to $\mathrm{Ker}(d\mu) \cap \mathrm{Ker}(\eta) = \{0\}$, so (i) and (ii) are equivalent. Note that $\mathrm{Ker}(L) = \{w = d\mu(X) \mid i_X\eta = 0\} = d\mu(\mathrm{Ker}(\eta))$, so $d\mu : \mathrm{Ker}(\eta) \to \mathrm{Ker}(L)$ is an isomorphism if and only if $\mathrm{Ker}(d\mu) \cap \mathrm{Ker}(\eta) = \{0\}$. Hence all the conditions are equivalent. □

Note that if the conditions in Lemma 7.2 hold, then Lemma 7.2(ii) defines a map $\rho_P : L_{\mu(p)} \longrightarrow T_pP$, $(w, \xi) \mapsto X$. A direct computation shows that
(1) the induced map $\rho_P : \Gamma(L) \longrightarrow \mathscr{X}(P)$ is a map of Lie algebras $\big(\Gamma(L)$ is equipped with twisted Courant bracket$\big)$,
(2) $d\mu(\rho_P(l)) = \rho(l)$ for all $l \in L$,

which precisely means that ρ_P is an infinitesimal action of the Lie algebroid L on P (and this is what we were after!).

COROLLARY 7.3
Any presymplectic realization $\mu : (P, \eta) \longrightarrow (M, L)$ of a ϕ-twisted Dirac structure is canonically equipped with an infinitesimal action of the Lie algebroid of L.

We call a presymplectic realization $\mu : P \to M$ *complete* if $\rho_P(l)$ is a complete vector field whenever $l \in \Gamma(L)$ has compact support. As with symplectic realizations of Poisson manifolds, a complete realization defines a complete Lie algebroid action (see [25]), which can be integrated to a global action of the groupoid $G(L)$ associated with L on P: indeed, an algebroid action of L defines a map $\nabla : \Gamma(L) \otimes C^\infty(P) \longrightarrow C^\infty(P)$ which behaves like a flat (L-)connection; then parallel transport defines the desired action of $G(L)$ on P (see [14, pp. 26–27] for details). For the integration of general Lie algebroid actions, see [25].

For complete symplectic realizations of Poisson manifolds, the induced action of the symplectic groupoid is *symplectic* (see [24]). The following property generalizes this fact. Let (M, L) be a ϕ-twisted Dirac structure, and let $(G(L), \omega_L)$ be the associated groupoid.

COROLLARY 7.4
If the realization $\mu : P \to M$ is complete (e.g., if P is compact), then there is an induced action of $G(L)$ on P, $m_P : G(L) \times_M P \longrightarrow P$. Moreover, if $G(L)$ is smooth (i.e., if L is integrable), then the action is smooth and

$$m_P^* \eta = \mathrm{pr}_G^* \, \omega_L + \mathrm{pr}_P^* \, \eta \tag{7.1}$$

(where $\mathrm{pr}_G : G(L) \times_M P \longrightarrow G(L)$ and $\mathrm{pr}_P : G(L) \times_M P \longrightarrow P$ are the natural projections).

Proof
In order to check (7.1), note that the 2-forms $\omega_1 := m_P^* \eta$ and $\omega_2 := \mathrm{pr}_G^* \, \omega_L + \mathrm{pr}_P^* \, \eta$ are both multiplicative in the semidirect product groupoid $G(L) \ltimes P$. A direct computation shows that $\rho^*_{\omega_1} = \rho^*_{\omega_2}$, so it follows from Theorem 5.1 that the forms must coincide. □

Remark 7.5
By the same arguments as in [14, Th. 8.2], the existence of a complete presymplectic realization $\mu : P \to M$ which is a surjective submersion implies the integrability of L. Note also that such realizations μ can be used to compute $(G(L), \omega_L)$ (though we do not know how to use this to give a direct proof of the integrability of L). First, we note that the groupoid $G(L) \ltimes P$ over P is isomorphic to the monodromy groupoid $G(\mathrm{Im}(\rho_P))$ of the (regular) foliation $\mathrm{Im}(\rho_P)$, so that $G(L)$ is a quotient of $G(\mathrm{Im}(\rho_P))$. Second, (7.1) says that the form $t^*\eta - s^*\eta$ on $G(\mathrm{Im}(\rho_P))$ descends to ω_L on $G(L)$.

7.2. Realizations of Cartan-Dirac structures and quasi-Hamiltonian spaces
As observed in [27, Exam. 4.2], any Lie group with a bi-invariant metric carries a ϕ-twisted Dirac structure, where ϕ is the associated bi-invariant Cartan form. We call it a *Cartan-Dirac structure*. In this section, we discuss presymplectic realizations and groupoids of Cartan-Dirac structures. We recover, in this framework, quasi-

Hamiltonian spaces (see [1]) and the AMM groupoid of [3], proving the following result.

THEOREM 7.6

Let H be a connected Lie group, and let $(\cdot,\cdot)_{\mathfrak{h}}$ be an invariant inner product on its Lie algebra $\mathfrak{h}$. Let L denote the associated Cartan-Dirac structure on H. Then

(i) *there is a one-to-one correspondence between presymplectic realizations of (H, L) and quasi-Hamiltonian $\mathfrak{h}$-spaces (which are infinitesimal versions of quasi-Hamiltonian H-spaces introduced in [1]);*

(ii) *the AMM groupoid of [3] is a presymplectic groupoid inducing the Cartan-Dirac structure L on H.*

We remark that the positivity of the invariant inner product does not play a role in Theorem 7.6. Before we prove this theorem, let us recall some definitions and fix our notation.

A *quasi-Hamiltonian H-space* (see [1]) is a manifold P endowed with a smooth action of H, an invariant 2-form $\eta \in \Omega^2(P)$, and an equivariant map $\mu : P \longrightarrow H$ (the moment map), such that

(i) the differential of η is given by

$$d\eta = -\mu^*\phi;$$

(ii) the map μ satisfies

$$i_{\rho_P(v)}(\eta) = \frac{1}{2}\mu^*(\lambda + \bar{\lambda}, v)_{\mathfrak{h}};$$

(iii) at each $p \in P$, the kernel of η_p is given by

$$\mathrm{Ker}(\eta_p) = \big\{\rho_{P,p}(v) : v \in \mathrm{Ker}(\mathrm{Ad}_{\mu(p)} + 1)\big\}.$$

Here $\rho_P : \mathfrak{h} \longrightarrow TP$ is the induced infinitesimal action of $\mathfrak{h}$ on P, λ (resp., $\bar{\lambda}$) is the left- (resp., right-)invariant Maurer-Cartan form on H, and $\phi \in \Omega^3(H)$ is the bi-invariant Cartan form

$$\phi = \frac{1}{12}(\lambda, [\lambda, \lambda])_{\mathfrak{h}} = \frac{1}{12}(\bar{\lambda}, [\bar{\lambda}, \bar{\lambda}])_{\mathfrak{h}}.$$

On the Lie algebra, we have $\phi(u, v, w) = (1/2)(u, [v, w])_{\mathfrak{h}}$. The equivariance of μ is with respect to the action of H on itself by conjugation. Infinitesimally, equivariance becomes

$$(d\mu)_p\big(\rho_P(v)\big) = \rho_H(v) \tag{7.2}$$

for all $v \in \mathfrak{h}$, where $\rho_H : \mathfrak{h} \longrightarrow TH$ is the infinitesimal conjugation action. (Explicitly, $\rho_H(v) = v_r - v_l$, where v_l and v_r are the vector fields obtained from $v \in \mathfrak{h}$ by left and right translations.)

Definition 7.7
A *quasi-Hamiltonian* $\mathfrak{h}$*-space* is a manifold P carrying an $\mathfrak{h}$-action $\rho_P : \mathfrak{h} \longrightarrow TP$, together with an $\mathfrak{h}$-invariant 2-form $\eta \in \Omega^2(P)$ and an equivariant map $\mu : P \to H$ (as in (7.2)), satisfying conditions (1), (2), and (3).

Conditions (1), (2), and (3) in the definition of quasi-Hamiltonian spaces strongly resemble the conditions we used to define presymplectic realizations (see Lem. 7.2(ii)). In order to find the underlying Dirac structure L on H making quasi-Hamiltonian $\mathfrak{h}$-spaces into presymplectic realizations, recall that an $\mathfrak{h}$-action on P, together with an equivariant map $\mu : P \to H$, is equivalent to an action of the action Lie algebroid $\mathfrak{h} \ltimes H$ with moment μ (see [24]). Hence the Lie algebroid of L is isomorphic to $\mathfrak{h} \ltimes H$ with anchor ρ_H; in other words, there is a map $\rho^* : \mathfrak{h} \to T^*H$ such that $(\rho_H, \rho^*) : \mathfrak{h} \ltimes H \to L$ is an isomorphism. To find ρ^*, we compare condition (2) for quasi-Hamiltonian spaces and the second equation in Lemma 7.2(ii), and we obtain

$$\rho^*(v) = \frac{1}{2}(\lambda + \bar{\lambda}, v)_{\mathfrak{h}} = \frac{1}{2}(v_r + v_l),$$

where in the last equality we use the metric to identify T^*H with TH. More explicitly, the Dirac structure we obtain on H is

$$L = \left\{ \left(v_r - v_l, \frac{1}{2}(v_r + v_l) \right) : \ v \in \mathfrak{h} \right\} \subset TH \oplus TH,$$

which is precisely the ϕ-twisted Dirac structure discussed in [27, Exam. 4.2]. We call L the *Cartan-Dirac structure* on H associated with $(\cdot, \cdot)_{\mathfrak{h}}$.

We now proceed to the following proof.

Proof of Theorem 7.6
Suppose that $\mu : P \to H$ is a presymplectic realization of the Cartan-Dirac structure L on H. Let ρ_P^L be the induced infinitesimal action of L on P with moment μ. Since the Lie algebroid of L is isomorphic to $\mathfrak{h} \ltimes H$, it immediately follows that ρ_P^L determines an $\mathfrak{h}$-action ρ_P on P, for which μ is $\mathfrak{h}$-equivariant. Explicitly,

$$\rho_P(v) = \rho_P^L\left(v_r - v_l, \frac{1}{2}(\lambda + \bar{\lambda}, v)_{\mathfrak{h}} \right), \quad v \in \mathfrak{h}. \tag{7.3}$$

Since $d\eta + \mu^*\phi = 0$, (1) in Definition 7.7 holds. Condition (2) is just the second equation in Lemma 7.2(ii). Since $d\mu : \mathrm{Ker}(\eta) \to \mathrm{Ker}(L)$ is an isomorphism, and $\mathrm{Ker}(L_g) = \{\rho_H(v) : v \in \mathrm{Ker}(\mathrm{Ad}_g + 1)\}$, the equivariance of μ, $d\mu(\rho_P(v)) = \rho_H(v)$ implies condition (3). Finally, we note that η is $\mathfrak{h}$-invariant:

$$\mathscr{L}_{\rho_P(v)}(\eta) = di_{\rho_P(v)}\eta + i_{\rho_P(v)}\, d\eta = \frac{1}{2}d\left(\mu^*(\lambda + \bar{\lambda}, v)_{\mathfrak{h}}\right) - i_{\rho_P(v)}\mu^*\phi = 0,$$

where the last equality follows from the Maurer-Cartan equations for λ and $\bar{\lambda}$. So P is a quasi-Hamiltonian $\mathfrak{h}$-space.

Conversely, if P is a quasi-Hamiltonian $\mathfrak{h}$-space, then $d\eta+\mu^*\phi = 0$, and we must check that condition (ii) in Lemma 7.2 holds. If $(w, \xi) = (v_r - v_l, (1/2)(\lambda+\bar{\lambda}, v)_{\mathfrak{h}}) \in L$, then $w = \rho_H(v) = d\mu(\rho_P(v))$, so the first equation in Lemma 7.2(ii) has a solution $X = \rho_P(v)$. The second equation in Lemma 7.2(ii) is just (2) in Definition 7.7. The uniqueness of this solution follows from (3): if $i_X\eta = 0$, then $X = \rho_P(v)$ for some v with $v_l + v_r = 0$; since $d\mu(X) = \rho_H(v) = v_r - v_l = 0$, we must have $v = 0$. So $\mu : P \to H$ is a presymplectic realization. This proves part (i) of the theorem.

In order to prove part (ii), we describe the presymplectic groupoids associated with the Cartan-Dirac structure L. Since L, as a Lie algebroid, is isomorphic to the action Lie algebroid $\mathfrak{h} \ltimes H$, the action groupoid $H \ltimes H$ (with action given by conjugation, $g \cdot x = gxg^{-1}$) integrates it. $\big($On $H \ltimes H$, $s(g,x) = x$, $t(g,x) = gxg^{-1}$, $(g_1, x_1)\cdot(g_2, x_2) = (g_1g_2, x_2)$.$\big)$ We are exactly in the situation of Example 6.4. As observed in [1], $\rho^* + \phi \in \Omega^3_H(H)$. (In particular, it follows from this observation that L is indeed a ϕ-twisted Dirac structure.) Applying Proposition 6.10, we immediately obtain the formula for the multiplicative 2-form on $H \times H$ corresponding to L:

$$\omega_{(g,x)} = \frac{1}{2}\big((\mathrm{Ad}_x\, p_g^*\lambda, p_g^*\lambda)_{\mathfrak{h}} + (p_g^*\lambda, p_x^*(\lambda + \bar{\lambda}))_{\mathfrak{h}}\big).$$

As in Proposition 6.10, p_g and p_x denote the projections onto the first and second components of $H \times H$. This is precisely the 2-form in the "double" $D(H) = H \times H$ introduced in [1]; the groupoid $(H \ltimes H, \omega)$ also appears in [3], where it is called the *AMM groupoid*. So the AMM groupoid is a presymplectic groupoid associated with the Cartan-Dirac structure on H, though it is not necessarily s-simply connected. If H is simply connected, then $(G(L), \omega_L) = (H \ltimes H, \omega)$, while, in general, one must pull back ω to $\tilde{H} \ltimes H$, where $\tilde{H}$ is the universal cover of H. □

Finally, if the infinitesimal action can be integrated to a global action, then the notion of quasi-Hamiltonian H-space coincides with that of quasi-Hamiltonian $\mathfrak{h}$-space. For instance, if H is simply connected, there is a one-to-one correspondence between quasi-Hamiltonian H-spaces and complete presymplectic realizations of L (which, in turn, are equivalent to quasi-Hamiltonian $\mathfrak{h}$-spaces for which the $\mathfrak{h}$-action is by complete vector fields). In particular, we have the following.

COROLLARY 7.8

If H is simply connected, then there is a one-to-one correspondence between compact quasi-Hamiltonian H-spaces and compact presymplectic realizations of the Cartan-Dirac structure L on H.

8. Multiplicative 2-forms, foliations, and regular Dirac structures

In this section we explain connections between our results and some aspects of foliation theory: we see that multiplicative 2-forms on monodromy groupoids of foliations are directly related to foliated cohomology, and they are relevant for the explicit description of presymplectic groupoids associated to *regular* Dirac structures.

8.1. Foliations

Let us recall some basic facts of foliations theory. The reader is referred to [19] and references therein for details.

By the Frobenius theorem, a foliation on M can be viewed as a subbundle $\mathscr{F}$ of TM (of vectors tangent to the leaves) for which $[\mathscr{F}, \mathscr{F}] \subset \mathscr{F}$; alternatively, foliations are the same thing as algebroids with injective anchor map. The *monodromy groupoid* of $\mathscr{F}$ consists of leafwise homotopy classes of leafwise paths in M (i.e., each s-fiber $s^{-1}(x)$ is the universal cover of the leaf through x, constructed with x as base point). This groupoid is the same as the one described in Section 5; that is, it is the unique s-simply connected Lie groupoid integrating $\mathscr{F}$ viewed as an algebroid. We denote it by $G(\mathscr{F})$. The space of foliated forms on M, $\mathbf{\Omega}^\bullet(\mathscr{F}) = \Gamma\big(\bigwedge^\bullet \mathscr{F}^*\big)$, carries a foliated de Rham operator

$$\begin{aligned} d_{\mathscr{F}}\omega(X_1, \ldots, X_{p+1}) &= \sum_i (-1)^i \mathscr{L}_{X_i}\big(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})\big) \\ &\quad + \sum_{i<j} (-1)^{i+j-1} \omega\big([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}\big), \end{aligned} \tag{8.1}$$

and we denote by $H^\bullet(\mathscr{F})$ the resulting cohomology (which is just the cohomology of $\mathscr{F}$ as an algebroid). One defines, in a similar way, the foliated cohomology $H^\bullet(\mathscr{F}; E)$ with coefficients in a foliated bundle E, that is, a bundle E over M endowed with a flat $\mathscr{F}$-connection $\nabla : \Gamma(\mathscr{F}) \times \Gamma(E) \to \Gamma(E)$. The corresponding complex is now $\mathbf{\Omega}^\bullet(\mathscr{F}; E) = \Gamma(\bigwedge^\bullet \mathscr{F}^* \otimes E)$, and the differential is given just as in (8.1) with $\mathscr{L}_X$ replaced by ∇_X. The basic example of a foliated bundle is the normal bundle $\nu = TM/\mathscr{F}$ with $\mathscr{F}$-connection given by the well-known Bott connection, $\nabla : \Gamma(\mathscr{F}) \times \Gamma(\nu) \to \Gamma(\nu)$,

$$\nabla_V \overline{X} = \overline{[V, X]},$$

where $X \mapsto \overline{X}$ is the projection from TM onto ν. As usual, the Bott connection induces connections on the dual ν^* and on the associated tensor bundles.

Here we deal with the cohomology spaces $H^\bullet(\mathscr{F})$ and $H^\bullet(\mathscr{F}; \nu^*)$ (in degrees one and two). These spaces are related by a transversal de Rham operator

$$d_\nu : H^\bullet(\mathscr{F}) \longrightarrow H^\bullet(\mathscr{F}; \nu^*) \tag{8.2}$$

(which can be extended to higher exterior powers of ν^*). Let us give a direct description of this map in degree two since higher degrees can be treated analogously. Given a class $[\theta] \in H^2(\mathscr{F})$ represented by a foliated 2-form θ, let $\widetilde{\theta}$ be a 2-form on M with $\theta = \widetilde{\theta}|_{\mathscr{F}}$. Since $d\widetilde{\theta}|_{\mathscr{F}} = 0$, it follows that the map $\Gamma\big(\bigwedge^2 \mathscr{F}\big) \to \Gamma(\nu^*)$, defined by

$$(V, W) \mapsto d\widetilde{\theta}(V, W, -),$$

gives a closed foliated 2-form with coefficients in ν^*; we set $d_\nu([\theta])$ to be its class in $H^2(\mathscr{F}; \nu^*)$.

8.2. Multiplicative 2-forms on monodromy groupoids

In this example we relate the space of closed multiplicative 2-forms on monodromy groupoids to cohomology spaces that are well known in foliation theory.

Let $\mathrm{Mult}^2(G)$ denote the space of closed multiplicative 2-forms on a Lie groupoid G.

PROPOSITION 8.1

Let $\mathscr{F}$ be a foliation on M, and let $G = G(\mathscr{F})$ be the monodromy groupoid of $\mathscr{F}$. Then

(i) *any $\omega \in \mathrm{Mult}^2(G)$ induces a cohomology class $c(\omega) \in H^2(\mathscr{F})$;*

(ii) *a foliated cohomology class $c \in H^2(\mathscr{F})$ is of type $c(\omega)$ if and only if $d_\nu(c) = 0$;*

(iii) *$c(\omega) = 0$ if and only if ω is multiplicatively exact; that is, $\omega = d\sigma$ with $\sigma \in \Omega^1(G)$ multiplicative.*

Before we prove this proposition, let us recall a more conceptual way to describe the operator (8.2). There is a spectral sequence associated to the foliation $\mathscr{F}$ (see, e.g., [19]), converging to $H^\bullet(M)$, with

$$E_1^{p,q} = H^p(\mathscr{F}; \Lambda^q \nu^*), \tag{8.3}$$

so that d_ν is just the boundary map $d_1^{p,q} : E_1^{p,q} \longrightarrow E_1^{p,q+1}$. The spectral sequence is associated to the filtration $F_p\Omega^\bullet(M)$ of $\Omega^\bullet(M)$ with

$$F_q\Omega^n(M) = \big\{\eta \in \Omega^n(M) : i_{V_1} \cdots i_{V_{n-q+1}} \eta = 0 \text{ for all } V_i \in \Gamma(\mathscr{F})\big\}$$

$\big(F_0\Omega^\bullet(M) = \Omega^\bullet(M)$, and $F_q\Omega^n(M) = 0$ for $q > n\big)$. It is easy to see that

$$E_0^{p,q} = F_q\Omega^{p+q}(M)/F_{q+1}\Omega^{p+q+1}(M) \cong \Omega^p(\mathscr{F}; \Lambda^q \nu^*)$$

and that the boundary $d_0^{p,q}$ is precisely the leafwise de Rham operator $d_{\mathscr{F}}$. This shows that the E_1-terms are indeed given by (8.3), and a standard computation shows that $d_1^{p,q}$ has the explicit description mentioned above.

Proof
Note that we have an isomorphism

$$\frac{F_0\Omega^2(M)}{F_2\Omega^2(M)} \xrightarrow{\sim} \left\{\rho^* : \mathscr{F} \longrightarrow T^*M : \rho^* \text{ satisfies (5.1)}\right\}$$

sending $[\eta]$ to ρ^*, defined by $\langle \rho^*(V), X\rangle = \eta(V, X)$. Moreover, the closedness of $[\eta]$ in the complex $F_0\Omega^\bullet(M)/F_2\Omega^\bullet(M)$ corresponds to (5.2) for ρ^*. On the other hand, ρ^* corresponds to an exact $[\eta]$ if and only if $\langle \rho^*(V), X\rangle = d\sigma(V, X)$ for some $\sigma \in F_0\Omega^1(M)/F_2\Omega^1(M) = \Omega^1(M)$. But then the closed multiplicative 2-form ω associated with ρ^* is $\omega_0 = d(t^*\sigma - s^*\sigma)$. (Note that ω_0 is multiplicative and closed, and it is easy to see that $\rho^*_{\omega_0} = \rho^*$.) As a result, we get an isomorphism

$$H^2\Big(\frac{F_0\Omega^\bullet(M)}{F_2\Omega^\bullet(M)}\Big) \xrightarrow{\sim} \frac{\mathrm{Mult}^2(G)}{\{d(t^*\sigma - s^*\sigma) : \sigma \in \Omega^1(M)\}}.$$

Now, using the short exact sequence of complexes

$$0 \longrightarrow \frac{F_1\Omega^\bullet(M)}{F_2\Omega^\bullet(M)} \longrightarrow \frac{F_0\Omega^\bullet(M)}{F_2\Omega^\bullet(M)} \longrightarrow \frac{F_0\Omega^\bullet(M)}{F_1\Omega^\bullet(M)} \longrightarrow 0,$$

we get an exact sequence in cohomology,

$$\begin{aligned} H^1(\mathscr{F}) \xrightarrow{d_\nu} H^1(\mathscr{F}; \nu^*) \longrightarrow \frac{\mathrm{Mult}^2(G(\mathscr{F}))}{\{d(t^*\sigma - s^*\sigma) : \sigma \in \Omega^1(M)\}} \\ \xrightarrow{c} H^2(\mathscr{F}) \xrightarrow{d_\nu} H^2(\mathscr{F}; \nu^*). \end{aligned} \tag{8.4}$$

This immediately implies statements (i) and (ii). Note that the map c in (8.4) is given by $c(\omega) = [c_\omega]$, where $c_\omega \in \Omega^2(\mathscr{F})$ is defined by $c_\omega(V, W) = \langle \rho^*_\omega(V), W\rangle$.

We now prove (iii). The fact that $c_\omega = 0$ for multiplicative 2-forms of type $\omega = d\sigma$, with σ multiplicative, follows by showing that the restriction of σ to $\mathscr{F}$ (a foliated 1-form) gives, after differentiation, precisely the foliated 2-form c_ω induced by ω. This can be checked by an argument similar to the one used to prove the formula of Proposition 3.5(ii) (but the argument is simpler, following from remark (2) and an analogue of remark (3) in that proof). For the converse, we fix ω with $[c_\omega] = 0$. From the exact sequence (8.4), we may assume that $c_\omega = 0$. But then Lemma 5.3 shows that the 1-form $\tilde{\sigma} = -\rho^*\sigma^c$ is basic, so it descends to a multiplicative 1-form σ on $G(\mathscr{F}) = G$. Since ω is induced by $\tilde{\omega} = d\tilde{\sigma}$, we conclude that $\omega = d\sigma$. □

8.3. Dirac structures associated to foliations

Any regular foliation $\mathscr{F}$ on M defines a Dirac structure $L_{\mathscr{F}}$ whose presymplectic leaves are precisely the leaves of $\mathscr{F}$, with the zero form. In other words,

$$L_{\mathscr{F}} = \mathscr{F} \oplus \nu^* \subset TM \oplus T^*M.$$

The Dirac structure $L_{\mathscr{F}}$ is always integrable, and we now describe the associated presymplectic groupoid. As above, we denote by $G(\mathscr{F})$ the monodromy groupoid of $\mathscr{F}$, and we denote by ν the normal bundle. The parallel transport with respect to the Bott connection ∇ (see Sec. 8.1) is well defined along leafwise paths (because ∇ is an $\mathscr{F}$-connection), and it is invariant under leafwise homotopy (because ∇ is flat). Hence it defines an action of $G(\mathscr{F})$ on ν, which we dualize to an action on ν^*. We form the semidirect product groupoid

$$G(\mathscr{F}) \ltimes \nu^*$$

consisting of pairs (g, v) with $v \in \nu^*_{s(g)}$, source and target maps induced by those of $G(\mathscr{F})$, and multiplication

$$(g, v)(h, w) = (gh, h^{-1}v + w).$$

Restricting the canonical symplectic form ω_{can} on T^*M to ν^*, its pullback

$$\omega_{\mathscr{F}} = \mathrm{pr}_2^*\, \omega_{\mathrm{can}}$$

by the second projection is a multiplicative 2-form on $G(\mathscr{F}) \ltimes \nu^*$. It is not difficult to check the following.

LEMMA 8.2
$(G(\mathscr{F}) \ltimes \nu^*, \omega_{\mathscr{F}})$ is the presymplectic groupoid associated with $L_{\mathscr{F}}$.

A slight "twist" of this result yields more examples of multiplicative 2-forms which are not of Dirac type.

Let us consider a closed 3-form ϕ on M with the property that

$$i_V i_W \phi = 0, \quad \forall\, V, W \in \Gamma(\mathscr{F}).$$

Using the filtration of Section 8.1, this condition means that $\phi \in F_2\Omega^3(M)$. By Theorem 5.1 applied with $\rho^* = 0$, there exists a unique multiplicative 2-form ω_ϕ on $G(\mathscr{F})$ such that

$$d\omega_\phi = s^*\phi - t^*\phi, \quad \omega_{\phi,x} = 0,\ \forall\, x \in M.$$

PROPOSITION 8.3
The following are equivalent:

(i) $\omega_\phi = 0$;

(ii) ω_ϕ *is of Dirac type;*

(iii) $\phi \in F_3\Omega^3(M)$ *(or, equivalently, ϕ is basic).*

Proof
By Lemma 3.1, both $\mathrm{Ker}(ds)$ and $\mathrm{Ker}(dt)$ sit inside $\mathrm{Ker}(\omega_\phi)$ at all points $g \in G(\mathscr{F})$. This implies that $\mathrm{Ker}(ds)_g^\perp = T_g G(\mathscr{F})$ and $\mathrm{Ker}(dt)_g + \mathrm{Ker}(\omega_{\phi,g}) = \mathrm{Ker}(\omega_{\phi,g})$, and the equivalence of (ii) and (i) follows from Lemma 4.2.

Next, note that, while (i) is equivalent to $s^*\phi - t^*\phi = 0$ at all $g \in G(\mathscr{F})$, (iii) is equivalent to the same condition at all $x \in M$ (by Cor. 3.4). Since $s^*\phi - t^*\phi$ is a multiplicative 3-form that has zero differential, the equivalence of (i) and (iii) follows from a degree three version of Corollary 3.4 (proven in the same way). □

Let us point out that, although ω_ϕ is not of Dirac type in general, this form is still relevant for the construction of forms of Dirac type and of presymplectic groupoids. In order to see that, note that the Dirac structure $L_{\mathscr{F}}$ is a ϕ-twisted Dirac structure since ϕ vanishes along the leaves of the foliation. By the properties of ω_ϕ in the proof above, adding ω_ϕ does not affect the isotropy bundle (see Def. 4.6) of a closed 2-form. Thus we get the following.

COROLLARY 8.4
Viewing $L_{\mathscr{F}}$ as a ϕ-twisted Dirac structure, the associated ϕ-twisted presymplectic groupoid is $G(\mathscr{F}) \ltimes \nu^*$ with the 2-form $\omega_{\mathscr{F}} + \mathrm{pr}_1^* \omega_\phi$.

8.4. Presymplectic groupoids of regular Dirac structures
We call a Dirac structure *regular* if its presymplectic leaves have constant dimension. To begin, we restrict ourselves to the untwisted case with $\phi = 0$. If L is regular, then it determines

(1) a regular foliation $\mathscr{F}$ (whose leaves are the presymplectic leaves of L),
(2) a closed foliated 2-form $\theta \in \Omega^2(\mathscr{F})$ (defined by the leafwise presymplectic forms of L).

Conversely, we can recover L from $\mathscr{F}$ and θ:

$$L = \big\{(X, \xi) : X \in \mathscr{F}, \xi|_{\mathscr{F}} = i_X(\theta)\big\}.$$

In this section, we discuss examples of regular Dirac structures for which $G(L)$ admits a simplified description in terms of this data, $\mathscr{F}$ and θ. (Note that the case $\theta = 0$ has been treated in Sec. 8.3.)

A simplified description of $G(L)$ depends on the *classifying class* of L, denoted by $c(L)$, that we now discuss. Using the transversal de Rham operator d_ν defined in (8.2), $c(L)$ is defined as

$$c(L) = d_\nu(\theta) \in H^2(\mathscr{F}; \nu^*).$$

While θ carries all the information of L as a Dirac structure, $c(L)$ characterizes L as a Lie algebroid. As suggested by the exact sequence

$$0 \longrightarrow \nu^* \longrightarrow L \longrightarrow \mathscr{F} \longrightarrow 0,$$

the relationship between L and $c(L)$ is the same as the one of extensions and 2-cocycles, as briefly discussed in Section 6.3. Let us make it more explicit. First of all, any closed $u \in \Omega^2(\mathscr{F}; \nu^*)$ defines an algebroid $\mathscr{F} \ltimes_u \nu^*$ with underlying vector bundle $\mathscr{F} \oplus \nu^*$, projection on the first factor as anchor map, and bracket

$$[(X, v), (Y, w)] = \big([X, Y], \nabla_X(w) - \nabla_Y(v) + u(X, Y)\big).$$

When $u = 0$, we simplify the notation to $\mathscr{F} \ltimes \nu^*$. (Note that this is the Lie algebroid underlying the Dirac structure $L_{\mathscr{F}}$ of the Sec. 8.3.) The isomorphism class of the Lie algebroid $\mathscr{F} \ltimes_u \nu^*$ depends only on the cohomology class of u: if $u' = u + dv$ with $v \in \Omega^1(\mathscr{F}; \nu^*)$, then

$$(X, \xi) \mapsto \big(X, \xi + v(X)\big)$$

is an isomorphism between $\mathscr{F} \ltimes_{u'} \nu^*$ and $\mathscr{F} \ltimes_u \nu^*$.

In order to see that $c(L)$ is the class corresponding to L, we choose a linear splitting σ of the map $L \longrightarrow \mathscr{F}$. On one hand,

$$u_\sigma(X, Y) = [\sigma(X), \sigma(Y)] - \sigma([X, Y]) \tag{8.5}$$

is a representative of $d_\nu(\theta)$ (this follows from the explicit description of d_ν given in Sec. 8.2); on the other hand, σ induces a linear isomorphism $L \cong \mathscr{F} \oplus \nu^*$ which maps the brackets on L into the brackets of $\mathscr{F} \ltimes_u \nu^*$.

Example 8.5 (*The case* $c(L) = 0$)
We now describe the presymplectic groupoid of L when $c(L) = 0$. This case is closely related to our discussion in Section 8.2, which we now extend. Any multiplicative 2-form ω on the monodromy groupoid $G(\mathscr{F})$ defines a foliated form $c_\omega = \omega|_{\mathscr{F}}$ (where we view $\mathscr{F} \subset TM \subset TG(\mathscr{F})$), whose cohomology class is precisely the $c(\omega)$ defined in Section 8.2. In particular, we have an induced regular Dirac structure L (namely, the one defined by $\mathscr{F}$ and c_ω). In this case, we say that L *comes from* ω, and we write $L = L(\omega)$.

COROLLARY 8.6
For a regular Dirac structure L on M, the following are equivalent:

(i) $c(L) = 0$;

(ii) *L comes from a closed multiplicative 2-form on the monodromy groupoid of $\mathscr{F}$;*

(iii) *the underlying algebroid of* L *is isomorphic to* $\mathscr{F} \ltimes \nu^*$.

In this case, L *is integrable. Moreover, if one chooses* ω *as in (ii) (in which case,* $L = L(\omega)$*), then*

$$\big(G(L), \omega_L\big) \cong \big(G(\mathscr{F}) \ltimes \nu^*, \omega_{\mathscr{F}} - \mathrm{pr}_1^* \omega\big).$$

(Here $G(\mathscr{F}) \ltimes \nu^*$ *and* $\omega_{\mathscr{F}} = \mathrm{pr}_2^* \omega_{\mathrm{can}}$ *are as in Sec. 8.3.)*

Proof

Using the map $\rho_\omega^* : \mathscr{F} \longrightarrow T^*M$ induced from ω, we have

$$L \cong \mathscr{F} \ltimes \nu^*, \qquad (v, \xi) \mapsto \big(v, \xi - \rho_\omega^*(v)\big),$$

which is an isomorphism of Lie algebroids. Hence $G(L) \cong G(\mathscr{F}) \ltimes \nu^*$.

To find the 2-form ω_L on $G(\mathscr{F}) \ltimes \nu^*$, we look at its infinitesimal counterpart $\rho^* : \mathscr{F} \ltimes \nu^* \longrightarrow T^*M$. This is obtained by transporting $\mathrm{pr}_2 : L \longrightarrow T^*M$ (which defines ω_L on $G(L)$) by the isomorphism above. Hence $\rho^*(v, \xi) = \xi - \rho_\omega^*(v)$. Now, $\rho_0^*(v, \xi) = \xi$ is precisely the infinitesimal counterpart of the multiplicative 2-form $\omega_{\mathscr{F}}$, while $\rho_1^*(v, \xi) = \rho_\omega^*(v)$ comes from the multiplicative form ω on $G(\mathscr{F})$ and the projection on the first factor. Hence the form induced by ρ^* is $\omega_{\mathscr{F}} - \mathrm{pr}_1^* \omega$. □

Another case in which we can make $G(L)$ more explicit is when $c(L)$ is integrable as a foliated cohomology class; as we will see, this is similar to van Est's approach to Lie's third theorem for Lie algebras (see [12] and references therein).

Example 8.7 (*The case of integrable* $c(L)$)

Recall that, if a Lie groupoid G acts on a vector bundle E, we can define differentiable cohomology groups $H^*_{\mathrm{diff}}(G; E)$, and the van Est map maps these cohomology groups into Lie algebroid cohomology with coefficients in E. We refer the reader to [32] and [12] for a general discussion. Here we deal only with the van Est map for $G(\mathscr{F})$, with coefficients in ν^*, and in degree two:

$$\Phi : H^2_{\mathrm{diff}}\big(G(\mathscr{F}); \nu^*\big) \longrightarrow H^2(\mathscr{F}; \nu^*).$$

We now recall its definition. A differentiable 2-cocycle on $G(\mathscr{F})$ with coefficients in ν^* is a smooth function c which associates to any composable pair (g, h) an element $c(g, h) \in \nu^*_{t(g)}$, which vanishes whenever g or h is a unit. We say that c is closed if

$$gc(h, k) - c(gh, k) + c(g, hk) - c(g, h) = 0$$

for all triples (g, h, k) of composable arrows in $G(\mathscr{F})$. Two cocycles c and c' are said to be cohomologous if their difference is of type $(g, h) \mapsto gd(h) - d(gh) + d(g)$ for some section $d \in \Gamma(G; t^*\nu^*)$. This defines $H^2_{\mathrm{diff}}(G(\mathscr{F}))$.

Any closed c defines a foliated form $\Phi(c) \in \Omega^2(\mathscr{F}; \nu^*)$: roughly speaking, $\Phi(c)$ is obtained from c by taking derivatives along leafwise vector fields (for the precise formulas, see [12], [32]). For our purpose, it is useful to give a more abstract description of $\Phi(c)$ using extensions. Analogously to Lie algebroid extensions by algebroid 2-cocycles, differentiable 2-cocycles induce groupoid structures on $G(\mathscr{F}) \times \nu^*$ with the multiplication extending the one in $G(\mathscr{F}) \ltimes \nu^*$:

$$(g, v)(h, w) = \big(gh, h^{-1}v + w + (gh)^{-1}c(g, h)\big).$$

The resulting groupoid is denoted by $G(\mathscr{F}) \ltimes_c \nu^*$. Still, as in the infinitesimal case, the isomorphism class of $G(\mathscr{F}) \ltimes_c \nu^*$ depends only on the cohomology class of c, and this groupoid fits into an exact sequence of groupoids

$$1 \longrightarrow \nu^* \longrightarrow G(\mathscr{F}) \ltimes_c \nu^* \longrightarrow G(\mathscr{F}) \longrightarrow 1. \tag{8.6}$$

Passing to Lie algebroids, this induces an extension of $\mathscr{F}$ by ν^*, and hence a cohomology class in $H^2(\mathscr{F}; \nu^*)$. This defines $\Phi([c])$ and determines Φ at the cohomology level. Note that this cohomology class has a canonical representative, and that defines $\Phi(c)$, that is, the map Φ at the chain level. The exact sequence (8.6) has a canonical splitting, which induces a linear splitting σ at the algebroid level; the associated foliated form (8.5) defines $\Phi(c)$.

COROLLARY 8.8
If the characteristic class $c(L)$ comes from a differentiable cocycle c (i.e., if $c(L)$ is integrable), then L is integrable and $G(L) \cong G(\mathscr{F}) \ltimes_c \nu^*$.

At first sight, this corollary is just the definition of the integrability of $c(L)$. This is due to our definition of Φ in terms of extensions. However, [12] gives us a precise description of when $c(L)$ is integrable, related to the monodromy groups of L. In this way the result becomes meaningful.

More precisely, by [12] we know that $u \in H^2(\mathscr{F}; \nu^*)$ is integrable if and only if all its leafwise periods vanish. This means that, for each leaf S and any 2-sphere γ in S, $\int_\gamma u|_S = 0$. On the other hand, by the very definition of the monodromy groups $N_x(L)$ (see [13]), and by the description of $c(L)$ above in terms of a splitting σ, we have

$$N_x(L) = \Big\{ \int_\gamma c(L)|_S : \gamma \in \pi_2(S, x) \Big\},$$

where S is the leaf through x. As in [14], these groups can be interpreted (or defined) as the groups defined by the variations of the presymplectic areas. And, still completely analogous to the Poisson case treated in [14], we state the conclusion without further details.

COROLLARY 8.9
The following are equivalent:
(i) $c(L)$ *is integrable;*
(ii) *all the leafwise periods of* $c(L)$ *vanish;*
(iii) *all the monodromy groups* $N_x(L)$ *vanish.*

The discussion of general regular Dirac structures (i.e., when $c(L)$ is not necessarily integrable) can be treated, again, exactly as in the Poisson case (see [14]).

So far, we have dealt with the case of $\phi = 0$. However, much of the discussion in this section extends to ϕ-twisted regular Dirac structures. For instance, one should use the ϕ-twisted Courant bracket in the construction of the foliated form (8.5). This defines the classifying class of L, $c(L) \in H^2(\mathscr{F}; \nu^*)$, and its integrals over leafwise 2-loops define the monodromy groups $N_x(L)$.

We finish the section with remarks on the particular twisted case discussed in Section 8.3.

Example 8.10 (*Twisted regular Dirac structures*)
Let $\phi \in F_1\Omega^3(M)$; that is, suppose that

$$i_U i_V i_W \phi = 0, \quad \forall\, U, V, W \in \mathscr{F}.$$

Let L be a regular Dirac structure L (with presymplectic foliation $\mathscr{F}$). Then L is automatically a ϕ-twisted regular presymplectic Dirac structure. To distinguish these two structures, we write (L, ϕ) for L viewed as a twisted Dirac structure. We also write $c(L, \phi)$ for its class, $G(L, \phi)$ for the associated groupoid, and so on.

As in Section 8.3, if $\phi \in F_2\Omega^3(M)$, then $G(L, \phi) = G(L)$; moreover, if $\phi \in F_3\Omega^3(M)$, then the corresponding multiplicative 2-forms coincide.

Let us now consider the class induced by ϕ, $\bar{\phi} \in F_1\Omega^3(M)/F_2\Omega^3(M) \cong \Omega^2(\mathscr{F}; \nu^*)$, that is,

$$\bar{\phi}(V, W) = i_V i_W(\phi).$$

Note that, if $\bar{\phi} = d\psi$ for some $\psi \in \Omega^1(\mathscr{F}; \nu^*)$, then Theorem 5.1 applies to $G(\mathscr{F})$, with the twist by ϕ, and to $\rho^* : \mathscr{F} \longrightarrow T^*M$, which is just ψ viewed as a bundle map. In this way we get an induced ϕ-twisted multiplicative 2-form

$$\omega_\psi \in \Omega^2\big(G(\mathscr{F})\big).$$

Let us denote by $\mathrm{pr} : G(L) \longrightarrow G(\mathscr{F})$ the projection induced by the first projection $\mathrm{pr}_1 : L \longrightarrow \mathscr{F}$.

COROLLARY 8.11
If $[\bar{\phi}] \in H^2(\mathscr{F}; \nu^*)$ *vanishes, then* $G(L, \phi) = G(L)$.

More precisely, any choice of $\psi \in \Omega^1(\mathscr{F}; \nu^*)$ *such that* $\bar{\phi} = d\psi$ *induces an isomorphism* $G(L, \phi) \cong G(L)$ *which maps the multiplicative* 2*-form on* $G(L, \phi)$ *to the form* $\omega_L + \mathrm{pr}^* \omega_\psi$.

Proof
First of all, by the description of the classifying classes in terms of splittings σ and of formula (8.5), it immediately follows that the forms u_σ representing $c(L)$, and $u_{\sigma,\phi}$ representing $c(L, \phi)$, satisfy $u_{\sigma,\phi} = u_\sigma + \bar{\phi}$. In particular,

$$c(L, \phi) = c(L) + [\bar{\phi}].$$

Hence, by the classifying properties of $c(L)$, the first assertion follows.

The second assertion follows from the general properties mentioned above: σ induces algebroid isomorphisms $L \cong \mathscr{F} \ltimes_{u_\sigma} \nu^*$, $(L, \phi) \cong \mathscr{F} \ltimes_{u_{\sigma,\phi}} \nu^*$, while ψ induces an isomorphism between the algebroids associated to the two cocycles. Actually, the resulting isomorphism does not depend on σ, and it is just $(X, \xi) \mapsto (X, \bar{\psi}(X) + \xi)$. It is now clear that the infinitesimal counterpart of the twisted multiplicative form is, on the untwisted L, just the sum of $\mathrm{pr}_2 : L \longrightarrow T^*M$ (defining ω_L) and the composition of $\mathrm{pr}_1 : L \longrightarrow \mathscr{F}$ with $\bar{\phi}$ (which is the infinitesimal counterpart of $\mathrm{pr}^* \omega_\psi$). This concludes the proof. □

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