Title
Scattering resonances and the complex absorbing potential method

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Publication Date
2022

Peer reviewed|Thesis/dissertation

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by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

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Spring 2022

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Abstract

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Scattering resonances are the analogues of eigenvalues for problems on non-compact domains. The real part and imaginary part of the resonances capture the rates of oscillation and decay of the scattering waves. Hence the location of resonances reflects the long-time behavior of the waves on non-compact domains.

In this thesis we study a computational technique for scattering resonances, that is the method of *complex absorbing potentials* (CAP). We show that the CAP method for computing resonances applies to the case of scattering by exponentially decaying potentials. We also show that the CAP method is valid for an abstractly defined class of *black box* perturbations of dilation analytic second order differential operators which is close to the Laplacian near infinity. The black box formalism allows a unifying treatment of diverse problems ranging from obstacle scattering to scattering on finite volume surfaces without addressing the details of specific situations.

The black box scattering problem motivates us to study the boundary perturbations in obstacle scattering. We show that all resonances in obstacle scattering with Dirichlet boundary condition are generically simple in the class of obstacles with C^k (and C^∞) boundaries, $k \geq 2$. This generalizes the case of eigenvalues of second order elliptic operators on a compact domain that all eigenvalues are simple for a generic compact domain.

To my family.

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Acknowledgments

First and foremost, I would like to thank my advisor, Maciej Zworski, for all the encouragement and support over the years, for suggesting these projects, and for teaching me the techniques to approach the problems. It would be impossible for me to complete this thesis without his guidance. Partial support by the National Science Foundation grants DMS-1500852 and DMS-1901462 is also acknowledged.

I am very grateful to Semyon Dyatlov, Michael Christ, Daniel Tataru, Fraydoun Rezakhanlou, John Lott, Kiril Datchev, Jeffery Galkowski, Mengxuan Yang, and Jian Wang for teaching me different aspects of mathematics and for many helpful discussions during the last several years. I am also thankful to anonymous referees of my papers for many valuable suggestions.

Finally, I would like to express my gratitude to my family for their patience and encouragement, especially my wife for her constant understanding and support during the hard times of this pandemic.

Chapter 1

Introduction and statement of results

Scattering resonances are the analogues of eigenvalues for the wave equations on non-compact domains. Each resonance corresponds to a resonant wave. The local energy of the wave decays exponentially due to the fact that energy can escape to infinity. This is in contrast to the case of waves on compact domains where the energy is conserved. The rate of decay of the resonant wave is determined by the imaginary part of the resonance, while the frequency of the wave corresponds to the real part of the resonance. Hence the location of resonances is crucial in understanding the long time behavior of the wave on non-compact domain.

There are many tools that have been used for computation of scattering resonances: the method of complex scaling, which will be reviewed in §3.5, has been brought to computational chemistry by Reinhardt [44]; the method of perfect layer potentials (PML) has been used in computational physics – see Berenger [4]. In this chapter, we review another technique for computing resonances – the complex absorbing potential (CAP) method, first used in physical chemistry – see Seideman–Miller [46], Riss–Meyer [45], Mandelshtam–Taylor [36] for an early treatment and Jagau et al [29] for some recent developments. The CAP method is rougher but easier to implement, which provides an accurate approximation for the location of scattering resonances – see for instance Figure 1.2. We also review generic simplicity of scattering resonances, which means that resonances of higher multiplicity are very rare. This property is very useful when we deal with the difficulties caused by higher multiplicities of resonances. Then we give a brief introduction to the new results obtained by the author, and introduce some related open problems. In the last section of this chapter, we give the outline of this thesis.

1.1 The complex absorbing potential method

The complex absorbing potential method is based on replacing the Hamiltonian P by

$$P_\varepsilon := P - i\varepsilon x^2, \quad \varepsilon > 0.$$

For simplicity, throughout this thesis we shall write

$$x^2 := x_1^2 + \cdots + x_n^2, \quad x \in \mathbb{R}^n.$$

The potential x^2 is an example of a CAP and other potentials have also been used. The operator P_ε has discrete spectrum and as $\varepsilon \to 0+$ the eigenvalues converge to resonances of P uniformly on compact subsets of some neighborhood of the real axis. We refer to these limits as viscosity limits by analogy to the case of Pollicott–Ruelle resonances in Dyatlov–Zworski [13]. In that case, the analogue of P_ε is given by $X+\varepsilon\Delta$ where X (the analogue of our iP) is the generator of an Anosov flow on a compact manifold and Δ, the Laplace–Beltrami operator for some metric, is an analogue of our x^2 (using Fourier transform). This then corresponds to a standard "viscosity/stochastic" regularization.

We shall mention that fixed complex absorbing potentials have already been used in mathematical literature on scattering resonances. Stefanov [53] showed that semiclassical resonances close to the real axis can be well approximated using eigenvalues of the Hamiltonian modified by a complex absorbing potential. For applications of fixed complex absorbing potentials in generalized geometric settings see for instance Nonnenmacher–Zworski [39, 40] and Vasy [56].

Zworski [65] showed that scattering resonances of $-\Delta + V$, $V \in L^\infty_{\mathrm{comp}}$, are limits of eigenvalues of $-\Delta + V - i\varepsilon x^2$ as $\varepsilon \to 0+$, see Figure 1.1 and 1.2 for a numerical illustration. The author extends Zworski's result to dilation analytic potentials [63], exponentially decaying potentials [62] and black box Hamiltonians [61]. Kameoka [30] characterized the resonances of Stark Hamiltonians using the complex absorbing potential method. The analogous results were proved for kinetic Brownian motion by Drouot [11], for gradient flows by Dang–Rivière [8] (following earlier work of Frenkel–Losev–Nekrasov [14]), and for 0th order pseudodifferential operators, motivated by problems in fluid mechanics, by Galkowski–Zworski [17] while the dynamics of viscosity limits for 0th order pseudodifferential operators were studied by Wang [58]. A very different example is the Wigner–von Neumann-type Hamiltonian, for instance, $P = -\frac{d^2}{dx^2} + a\frac{\sin x}{x}$, in which case Kameoka and Nakamura [31] showed that the corresponding limits exist only away from a discrete set of thresholds.

We should point out that results like that of [13] and [11] are not valid for non-hyperbolic flows. A negative example is the geodesic flow on the torus, $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with the flat metric. The geodesic flow on $S^*\mathbb{T}^2 = \mathbb{S}^1_{x_1} \times \mathbb{S}^1_{x_2} \times \mathbb{S}^1_\theta$ is generated by

$$V = \cos\theta\, \partial_{x_1} + \sin\theta\, \partial_{x_2}.$$

Let $P_\varepsilon := V/i + i\varepsilon\Delta$, $\varepsilon > 0$, where Δ is the flat Laplacian, by the Fourier expansion in x, we have

$$\operatorname{Spec}(P_\varepsilon) = \bigcup_{n\in\mathbb{Z}^2} \operatorname{Spec}(P_\varepsilon(n)), \quad P_\varepsilon(n) := -i\varepsilon D_\theta^2 + |n|\cos\left(\theta - \tan^{-1}\frac{n_1}{n_2}\right) - i|n|^2\varepsilon.$$

Recalling the asymptotic behaviour of the spectrum of these Galtsev–Shafarevitch operators [18] as $\varepsilon \to 0+$, we obtain the accumulation points in the case of the generator of the geodesic flow on $\mathbb{T}^2$ regularized using the flat Laplacian:

$$-i[0,\infty) \cup \bigcup_{n\in\mathbb{Z}^2\setminus\{0\}} \{z : |\operatorname{Re} z| \le |n|,\ \operatorname{Im} z = -C|n| + C|\operatorname{Re} z|\},$$

which form a discrete set of lines – see [13, Figure 3.], where $C \approx 0.85$ is a special constant – see [18].

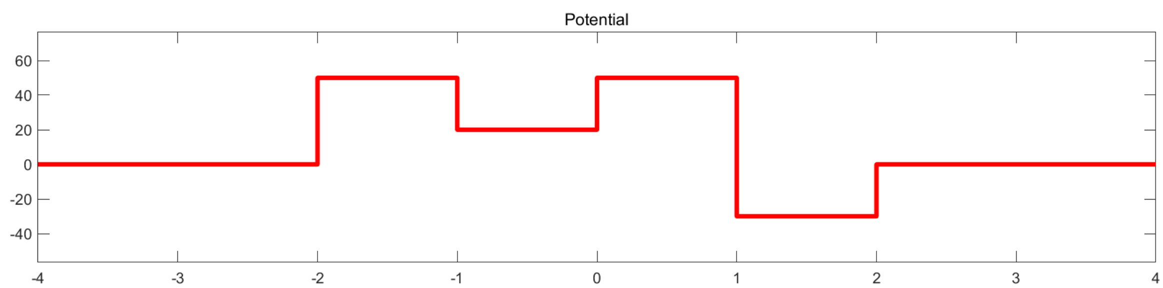


Figure 1.1: An example of $V \in L^\infty_{\text{comp}}(\mathbb{R})$ for illustrating the complex absorbing potential method for $P = -\frac{d^2}{dx^2} + V$.

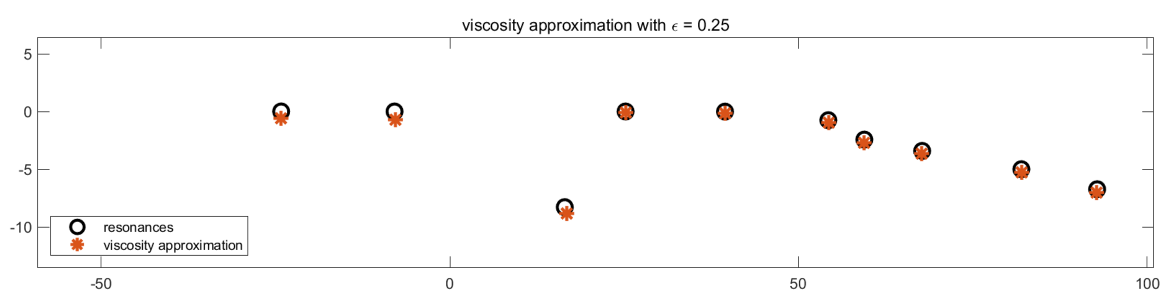


Figure 1.2: An illustration of the complex absorbing potential method in the case of a compactly supported potential shown on Figure 1.1. Resonances are computed using squarepot.m [5].

1.2 Generic simplicity of resonances

A property that holds on an intersection of open dense sets is called generic. In a famous paper Uhlenbeck [55] proved generic properties of eigenvalues and eigenfunctions of second order elliptic operators on a compact manifold. The analogous result was proved for eigenvalues of the Laplacian on a Riemannian cover by Zelditch [64].

In the case of resonances, Klopp and Zworski [33] showed that a generic potential perturbation in Euclidean scattering splits the multiplicities of all resonances. That means that for any r, there exists $\mathcal{V} \subset L^\infty(B(0,r);\mathbb{R})$, an intersection of open dense sets, such that all resonances of $-\Delta + V$, $V \in \mathcal{V}$ are simple (multiplicity $= 1$), and $L^\infty(B(0,r);\mathbb{R})$ can be replaced by other spaces of functions. Their argument in fact applies to a class of non-self-adjoint Fredholm operators on an abstract Hilbert space (see for instance §3.3 and §3.4). This result is very useful when we deal with the difficulties caused by multiplicities of resonances, as many statements about resonances are easy when there is no multiplicity but become more complicated in the case of higher multiplicity. Using Agmon's peturbation theory of resonances [1] instead of the exterior complex scaling used in [33], Borthwick and Perry [6] extended this result to scattering on asymptotically hyperbolic manifolds.

Apart from potential perturbations, I also mention that the evolution of eigenvalues of second order elliptic operators under boundary perturbations have been studied since Hadamard [24]. Henry [25] developed a general theory on perturbation of domains for second order elliptic operators. The author [60] studied resonances in obstacle scattering under generic boundary perturbations.

1.3 Statement of results

In this section, we introduce the main results in this thesis. To describe the convergence of the eigenvalues of P_ε as $\varepsilon \to 0+$, we adopt the Hausdorff metric, that is for two non-empty subsets A, B of $\mathbb{C}$,

$$d_{\mathrm{H}}(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\right\}.$$

1.3.1 CAP method for exponentially decaying potentials

We show that the CAP method for computing scattering resonances applies to the case of exponentially decaying potentials. We consider the following Schrödinger operator:

$$P = -\Delta + V \text{ acting on } L^2(\mathbb{R}^n), \quad |V(x)| \le Ce^{-2\gamma|x|} \text{ for some } C,\gamma > 0.$$

The exponentially weighted resolvent $\sqrt{V}(P-\lambda^2)^{-1}\sqrt{V}$ can be meromorphically continued to the strip $\operatorname{Im}\lambda > -\gamma$, see Froese [15], Gannot [19] and a review in §3.2.

Resonances of P, denoted by $\mathrm{Res}(P)$, are the poles in this meromorphic continuation. We show the following

Theorem 1. *The operator* $P_\varepsilon := P - i\varepsilon x^2$, $\varepsilon > 0$, *is a non-normal unbounded operator on* $L^2(\mathbb{R}^n)$ *with a discrete spectrum. Let*

$$\mathrm{Spec}(P_\varepsilon) \cap \mathbb{C} \setminus e^{-i\pi/4}[0,\infty) = \{\lambda_j(\varepsilon)^2\}_{j=1}^{\infty}, \quad -\pi/8 < \arg\lambda_j(\varepsilon) < 7\pi/8.$$

Then for any $0 < a' < a < b$ *and* $\gamma' < \gamma$ *such that the rectangle*

$$\Omega := (a', a) + i(-\gamma', b) \Subset \{\lambda \in \mathbb{C} : -\pi/8 < \arg\lambda < 7\pi/8\}. \tag{1.3.1}$$

we have

$$\lim_{\varepsilon\to 0+} \{\lambda_j(\varepsilon)\}_{j=1}^{\infty} \cap \Omega = \mathrm{Res}(P) \cap \Omega,$$

where the limit is taken with respect to the Hausdorff metric.

1.3.2 CAP method for black box scattering

We show that the CAP method also applies to an abstractly defined class of black box perturbations of the Laplacian in $\mathbb{R}^n$ which can be analytically extended from $\mathbb{R}^n$ to a conic neighborhood in $\mathbb{C}^n$ near infinity. The abstract setting of black box scattering was introduced by Sjöstrand and Zworski in [49]. The black box formalism allows a unifying treatment of diverse problems ranging from obstacle scattering to scattering on finite volume surfaces – see Examples 1–3 in §3.4, and we don't need address the details of specific situations.

The black box Hamiltonian P is assumed to act on a Hilbert space with an orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0)).$$

The orthogonal projections onto $\mathcal{H}_{R_0}$ (the *black box*) and $L^2(\mathbb{R}^n \setminus B(0,R_0))$ are denoted by $1_{B(0,R_0)}$ and $1_{\mathbb{R}^n\setminus B(0,R_0)}$ respectively. We refer the readers to §3.3 for a more detailed introduction of the black box formalism.

We do not assume P to be equal to $-\Delta$ near infinity as in [49]. Instead, we follow Sjöstrand [50] and assume that P is a dilation analytic perturbation of $-\Delta$ near infinity,

$$1_{\mathbb{R}^n\setminus B(0,R_0)} P u = \big(-\sum_{j,k=1}^{n} \partial_{x_j}(g^{jk}(x)\partial_{x_k}) + c(x)\big)(u|_{\mathbb{R}^n\setminus B(0,R_0)}), \quad \forall\, u \in \mathrm{Dom}(P),$$

where g^{jk}, $c \in \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R})$ with all derivatives bounded satisfying

$$g^{jk} = g^{kj},\ \forall\, j,k, \quad \Big|\sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k\Big| \geq C^{-1}|\xi|^2,\ \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k + c(x) \to \xi^2,\ |x| \to \infty,$$

$$\begin{gathered}\text{and there exist } \theta_0 \in [0, \pi/8],\ \delta > 0, \text{ and } R \geq R_0, \text{s.t. } g^{jk}, c \text{ extend analytically} \\ \text{to } \{s\omega : \omega \in \mathbb{C}^n,\ \operatorname{dist}(\omega, \mathbb{S}^{n-1}) < \delta,\ s \in \mathbb{C},\ |s| > R,\ \arg s \in (-\delta, \theta_0 + \delta)\} \\ \text{and the second half of (3.4.2) remains valid in this larger set.}\end{gathered}$$

The scattering resonances of P are defined by the method of complex scaling – see [49], [50] and a review in §3.5. We prove the following result:

Theorem 2. *The operator $P_\varepsilon := P - i\varepsilon(1-\chi(x))x^2$, $\varepsilon > 0$, where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 near $\overline{B(0, R_0)}$, is an unbounded operator on $\mathcal{H}$ with a discrete spectrum. Denote by $\operatorname{Res}(P)$ the set of resonances of P. Then, uniformly on any precompact open subset Ω of the sector $\{z \in \mathbb{C} \setminus \{0\} : -2\theta_0 < \arg z < 3\pi/2 + 2\theta_0\}$,*

$$\lim_{\varepsilon \to 0+} \operatorname{Spec}(P_\varepsilon) \cap \Omega = \operatorname{Res}(P) \cap \Omega.$$

1.3.3 Generic simplicity of resonances for obstacles

An obstacle scattering problem was used in the proof of Theorem 2, which motivates us to study the behavior of resonances in obstacle scattering under boundary perturbations. Suppose that $\mathcal{O} \subset \mathbb{R}^n$ is a bounded open set such that $\partial\mathcal{O}$ is a C^k ($k \geq 2$) hypersurface in $\mathbb{R}^n$. Let $\Delta_{\mathcal{O}}$ be the self-adjoint Dirichlet Laplacian on $\mathbb{R}^n \setminus \mathcal{O}$ with domain

$$\mathcal{D}(\Delta_{\mathcal{O}}) := H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H_0^1(\mathbb{R}^n \setminus \mathcal{O}). \tag{1.3.2}$$

The resolvent of $-\Delta_{\mathcal{O}}$,

$$R_{\mathcal{O}}(\lambda) := (-\Delta_{\mathcal{O}} - \lambda^2)^{-1} : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O}), \quad \operatorname{Im} \lambda > 0,$$

continues meromorphically as an operator from $L^2_{\mathrm{comp}}(\mathbb{R}^n \setminus \mathcal{O})$ to $L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{O})$ – see for instance Dyatlov–Zworski [12, §4.2], when n is odd the continuation is to $\lambda \in \mathbb{C}$ and when n is even to the logarithmic cover of $\mathbb{C} \setminus \{0\}$: $\Lambda = \exp^{-1}(\mathbb{C} \setminus \{0\})$. We denote the set of poles of $R_{\mathcal{O}}(\lambda)$ by $\operatorname{Res}(\mathcal{O})$, whose elements are called scattering resonances for the obstacle $\mathcal{O}$. For $\lambda \in \operatorname{Res}(\mathcal{O})$, its multiplicity is given by

$$m_{\mathcal{O}}(\lambda) := \operatorname{rank} \oint_\lambda R_{\mathcal{O}}(\zeta) d\zeta,$$

where the integral is over a circle containing no other pole of $R_{\mathcal{O}}(\zeta)$ than λ. A resonance $\lambda \in \operatorname{Res}(\mathcal{O})$ is called simple if $m_{\mathcal{O}}(\lambda) = 1$.

Consider a class of obstacles diffeomorphic to a fixed obstacle $\mathcal{O}_0$ (for example, $\mathcal{O}_0 = B_{\mathbb{R}^n}(0,1)$), that is,

$$X := \left\{ \Phi(\mathcal{O}_0) : \begin{array}{c} \Phi \in C^k(\mathbb{R}^n; \mathbb{R}^n) \text{ is a } C^k\text{-diffeomorphism, } \Phi(\partial\mathcal{O}) = \partial\,\Phi(\mathcal{O}) \\ \text{and } \Phi(x) = x,\ \forall\, |x| > R, \quad \text{for some } R > 0. \end{array} \right\}. \tag{1.3.3}$$

A topology can be introduced to this set using C^k norms of the diffeomorphisms, see Pereira [42]. We show the following result.

Theorem 3. *For any fixed obstacle $\mathcal{O}_0$ and the corresponding family X given by (1.3.3), there exists a generic set $\mathcal{X} \subset X$ such that for every $\mathcal{O} \in \mathcal{X}$, all resonances $\lambda \in \mathrm{Res}(\mathcal{O})$ are simple. By a generic set we mean an intersection of open dense sets.*

Remark. We should point out that an analogue of this result for Robin boundary condition (and in particular for the Neumann boundary condition) remains an open problem – see §1.4.4 for a detailed discussion.

1.4 Open problems

In this section we introduce some open problems.

1.4.1 CAP method on aymptotically hyperbolic manifolds

It remains an open problem whether the CAP method for finding resonances works for the hyperbolic space and more generally, asymptotically hyperbolic manifolds. The problem can be formulated as follows: let (M, g) be a complete Riemannian manifold of dimension $n+1$ with boundary ∂M given by $\{\rho = 0\}$ where $\rho : \overline{M} \to [0, \infty)$ is a $\mathcal{C}^\infty$ function such that $d\rho \neq 0$ on ∂M, and $\rho > 0$ on M. Suppose that the metric $\rho^2 g$ extends to a smooth Riemannian metric on $\overline{M}$ and that $|d\rho|_{\rho^2 g} = 1$ on ∂M. Let $\Delta_g \geq 0$ be the Laplace–Beltrami operator for the metric g. Since the spectrum is contained in $[0, \infty)$ the operator $\Delta_g - n^2/4 - \lambda^2$ is invertible on $L^2(M, d\,\mathrm{vol}_g)$ for $\mathrm{Im}\,\lambda > n/2$. Consider the resolvent

$$R(\lambda) := (\Delta_g - n^2/4 - \lambda^2)^{-1} : L^2(M, d\,\mathrm{vol}_g) \to H^2(M, d\,\mathrm{vol}_g), \quad \mathrm{Im}\,\lambda > n/2.$$

Let $\dot{\mathcal{C}}^\infty(M)$ denote functions which are extendable to smooth functions supported in $\overline{M}$. It follows from elliptic regularity that $R(\lambda) : \dot{\mathcal{C}}^\infty(M) \to \mathcal{C}^\infty(M)$, $\mathrm{Im}\,\lambda > n/2$. $R(\lambda) : \dot{\mathcal{C}}^\infty(M) \to \mathcal{C}^\infty(M)$ continues meromorphically from $\mathrm{Im}\,\lambda > n/2$ to $\mathbb{C}$ with poles of finite rank, see Mazzeo–Melrose [37], Guillarmou [21], Guillopé–Zworski [22], Vasy [57], [56] and [12, Chapter 5]. We denote the poles by $\mathrm{Res}(\Delta_g) = \{\lambda_j\}_{j=1}^\infty$. Does there exist a function f such that the operator $\Delta_g - n^2/4 - i\varepsilon f$, $\varepsilon > 0$, has discrete $L^2(M, d\,\mathrm{vol}_g)$ spectrum $\{\lambda_j(\varepsilon)^2\}$ and that

$$\lambda_j(\varepsilon) \to \lambda_j, \quad \text{as } \varepsilon \to 0+?$$

I would like to mention that for finite volume surface with cusp ends this holds with $f(x) = d(x, x_0)^2$ where d is the hyperbolic distance – see [61, Example 3], and some inconclusive and simple results are represented in §5.5.

1.4.2 Hyperbolic analogues of the complex harmonic oscillator

In [65], the complex harmonic oscillator $\Delta - i\varepsilon x^2$, $\varepsilon > 0$ plays an important role as it is an unbounded operator on $L^2(\mathbb{R}^n)$ with a discrete spectrum given by

$$\{e^{-i\pi/4}\sqrt{\varepsilon}(2|\alpha| + n) : \alpha \in \mathbb{N}_0^n\}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n,$$

see §5.1 for more details. It remains an open problem to find an analogue of this operator in the hyperbolic setting. That is, on a hyperbolic manifold (M, g), I wish to find a complex-valued function $f \in \mathcal{C}^\infty(M)$ such that $\Delta_g + f$ is an operator on $L^2(M, d\,\mathrm{vol}_g)$ with discrete spectrum. f should also be unbounded near infinity like the function $-i\varepsilon x^2$ in the Euclidean case, which will provide the compactness of the resolvent $(\Delta_g - n^2/4 + f - z)^{-1} : L^2(M, d\,\mathrm{vol}_g) \to L^2(M, d\,\mathrm{vol}_g)$. A candidate $f = \omega^2 \tanh^2 r$ where $\omega \in \mathbb{C}$, r is the hyperbolic radius, has been studied in [59] or §5.5, but in this case $\Delta_g + f$ (the "complex Higgs oscillator") has both eigenvalues and resonances.

1.4.3 More general CAPs

A natural question is whether we can use more general complex absorbing potentials than the quadratic potential x^2 for computing resonances. We might expect the complex absorbing potential to have some analyticity. The CAP $-i\varepsilon x^2$ has been most commonly used in the field since the CAP-regularized Laplacian, $-\Delta - i\varepsilon x^2$, is well understood – see §5.1 and it is convenient for numerical experiments.

1.4.4 Generic simplicity of resonances in Neumann obstacle scattering

The generic simplicity result for resonances in obstacle scattering with Robin boundary condition (and in particular the Neumann boundary condition) remains an open problem. The difficulty was overcome by Uhlenbeck [54] in the case of Neumann eigenvalue problem in a bounded domain Ω by using Transversality Theorem in infinite dimensions and then deriving a contradiction from the equation $\nabla_{\partial\Omega} u \cdot \nabla_{\partial\Omega} v = \lambda uv$ on $\partial\Omega$ where $\lambda > 0$, $u, v \in C^2(\partial\Omega; \mathbb{R})$ and $uv \neq 0$ on an open dense subset of $\partial\Omega$, see also [25, Example 6.4] for more details. In the case of obstacle scattering with Neumann boundary condition, this argument does not seem to apply for $\nabla_{\partial\Omega} u \cdot \nabla_{\partial\Omega} v = zuv$ when u, v are complex-valued and z is a complex resonance.

1.5 Outline

In chapter 2, we briefly review some basic notions and tools in semiclassical analysis and spectral theory. In §2.3 we introduce the analytic Fredholm theory, which is a standard result to show meromorphy of operators; In §2.4 we review the Gohberg–Sigal theory, which is the central tool for showing convergence of eigenvalues of P_ε in Theorem 1 and 2. The estimate of the decay of Green function in §2.5 is needed in the proof of Theorem 2. §2.6 is a preparation for the discussion in §5.5.

In Chapter 3, we first review various notions about scattering resonances in the simplest setting of 1D scattering by compactly supported potentials. In §3.2 we define the resonances for scattering by exponentially decaying potentials. In §3.3 and §3.4, we introduce the black box formalisms of scattering, which is used to define resonances in Theorem 3 and 2 respectively. In §3.5, we review the method of complex scaling, which is used to characterize resonances in black box scattering as eigenvalues. In §3.6, we adapt Agmon's theory to obstacle scattering, in which resonances are realized as eigenvalues of a operator on an abstractly constructed Banach space. In contrast to complex scaling, Agmon's method allows us to capture all resonances on the logarithmic plane in even dimensions.

In Chapter 4, we first study the deformation of obstacle and introduce the deformed Laplacian outside the obstacle with Dirichlet boundary condition. The variation of domain is transferred to the coefficients of the deformed operator. In 4.2, we apply Agmon's theory to the case of boundary perturbations. In §4.3 we present the proof of Theorem 3.

In Chapter 5, we study the Hamiltonians modified by CAPs. In §5.1 we review the properties of CAP-reularized free Laplacian on $\mathbb{R}^n$ (we call it Davies harmonic oscillator). In §5.2 we prove an estimate for the resolvent of Davies harmonic oscillator with exponential weights, which is the key estimate in the proof of Theorem 1. In §5.3 we study review the meromorphy of the resolvent of P_ε in Theorem 1. In §5.4 we study the CAP-regularized black box Hamiltonian $-P_\varepsilon$ appeared in Theorem 2, and its complex scaled version. In §5.5, we give a introduction to the complex Higgs oscillator and represent some numerical results about its spectrum.

In Chapter 6, We represent the proofs of Theorem 1 and Theorem 2. In §6.1 we complete the proof of Theorem 1 using the estimate obtained in §5.2. §6.2 – §6.6 are devoted to the proof of Theorem 2. In §6.2, an auxiliary obstacle is introduced to separate the abstract black box from the differential operator outside, we also study the interior and exterior reference operators associated to the obstacle and the black box Hamiltonian P. In §6.3, we introduce the Dirichlet-to-Neumann operator associated to the obstacle, we show that away from a discrete set depending on the obstacle, the resonances of P and the eigenvalues of P_ε can be characterized as the eigenvalues of the Dirichlet-to-Neumann operator. There is a special subset of the discrete set above, which consists of the embedded eigenvalues with compactly supported eigenfunctions of P, this set is discussed in §6.4. In §6.5 we show that the obstacle can be chosen so that the characterization of $\mathrm{Res}(P)$ and $\mathrm{Spec}(P_\varepsilon)$ through the Dirichlet-to-Neumann operator is valid in a neighborhood of $\mathrm{Res}(P)$. The proof of Theorem 2 is completed in §6.6 by obtaining further estimates on the Dirichlet-to-Neumann operators.

Chapter 2

Preliminaries

In this chapter we give a brief introduction to basic notions or tools in semiclassical analysis and spectral theory. We refer the reader to Zworski [66], Gohberg–Sigal [20], [12, Appendix C], and Shubin [47] for a complete treatment.

2.1 Semiclassical Quantization

2.1.1 Symbol classes

We first review the notion of symbols, which are smooth functions on $\mathbb{R}^{2n}$ that can be quantized to *semiclassical pseudodifferential operators*.

Definition 2.1.1. *A measurable function $m : \mathbb{R}^{2n} \to (0,\infty)$ is called an* order function *if there exist constants C and k such that for all $z, w \in \mathbb{R}^{2n}$,*

$$m(z) \le C\langle z-w\rangle^k m(w).$$

Examples. The most common examples of order functions are

$$m(z) \equiv 1, \qquad m(z) = \langle z\rangle = (1+|z|^2)^{1/2}.$$

We also check that if m_1, m_2 are order functions, so is $m_1 m_2$.

Definition 2.1.2. *Given an order function m on $\mathbb{R}^{2n}$, the corresponding symbol class is defined as follows. ($\mathbb{N}_0$ denotes the set of nonnegative integers.)*

$$S(m) := \{a \in \mathcal{C}^\infty(\mathbb{R}^{2n}) : \textit{for all } \alpha \in \mathbb{N}_0^{2n},\ \textit{there exists } C_\alpha \textit{ so that } |\partial^\alpha a| \le C_\alpha m\}.$$

Remarks. (i) Symbols $a(x,\xi)$ in $S(m)$ are allowed to depend on the small semiclassical parameter h, though our notation usually does not show this dependence.
(ii) If $a = a(x,\xi;h) \in S(m)$ depends on h, we require that the constants C_α in the definition be uniform for $0 < h < h_0$ for some $h_0 > 0$.

2.1.2 Asymptotic series

Next we review infinite sum of symbols.

Definition 2.1.3. *Let $a_j \in S(m)$ for $j = 0, 1, \dots$ We say that $a \in S(m)$ is asymptotic to the following series:*

$$a \sim \sum_{j=0}^{\infty} h^j a_j \quad \text{in } S(m),$$

if for every $N = 1, 2, \dots$

$$a - \sum_{j=0}^{N-1} h^j a_j = \mathcal{O}_{S(m)}(h^N),$$

which means that for each $\alpha \in \mathbb{N}_0^{2n}$,

$$\left| \partial^\alpha \left(a - \sum_{j=0}^{N-1} h^j a_j \right) \right| \le C_{N,\alpha} \, h^N m.$$

We call a_0 the principal symbol of a.

2.1.3 Quantization of symbol

Let $a \in S(m)$, we recall the Weyl quantization formula:

$$a^{\mathrm{w}}(x, hD) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n), \ \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n),$$

$$a^{\mathrm{w}}(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi \rangle} a(\tfrac{x+y}{2}, \xi) u(y) \, dy d\xi.$$

Here $\mathscr{S}$ denotes the space of Schwartz functions, and $\mathscr{S}'$ denotes the space of tempered distributions.

We recall the following properties:

Proposition 2.1.4. *Suppose that $a \in S(m_1)$ and $b \in S(m_2)$. Then*

$$a^{\mathrm{w}}(x, hD) \circ b^{\mathrm{w}}(x, hD) = (a \# b)^{\mathrm{w}}(x, hD),$$

for the symbol $a \# b \in S(m_1 m_2)$

$$a \# b(x, \xi) \sim \sum_{j=0}^{\infty} \frac{i^j h^j}{2^j j!} (\langle D_\xi, D_y \rangle - \langle D_x, D_\eta \rangle)^j (a(x, \xi) b(y, \eta)) \Big|_{\substack{y=x \\ \eta=\xi}},$$

where $D_{x_j} = i^{-1} \partial_{x_j}$.

Proposition 2.1.5. *If $a \in S(1)$ (we take the order function $m \equiv 1$), then*

$$a^{\mathrm{w}}(x, hD) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \|a^{\mathrm{w}}(x, hD)\|_{L^2 \to L^2} \le C \sum_{|\alpha| \le Mn} h^{|\alpha|/2} \sup_{\mathbb{R}^n} |\partial^\alpha a|,$$

where M is a universal constant.

2.1.4 Semiclassical ellipticity

Definition 2.1.6. *The symbol $a \in S(m)$ is called* elliptic *in $S(m)$ if there exists a constant $\gamma > 0$, independent of h, such that*

$$|a(x,\xi)| \geq \gamma\, m(x,\xi) \quad \text{for all } (x,\xi) \in \mathbb{R}^{2n}.$$

Proposition 2.1.7. *Suppose that $a \in S(m)$ and that a is elliptic in $S(m)$.*
(i) If $m \geq 1$, there exist $h_0, C > 0$ such that for all $0 < h < h_0$,

$$\|a^{\mathrm{w}}(x,hD)u\|_{L^2} \geq C\|u\|_{L^2}.$$

(ii) If $m = 1$, there exists $h_0 > 0$ such that for $0 < h < h_0$, $a^{\mathrm{w}}(x,hD)$ has a bounded inverse on $L^2(\mathbb{R}^n)$.

To end this section, we explore a concrete example of semiclassical elliptic operator: $-h^2\Delta + x^2 + 1$, which will be used in Chapter 5.

Example 2.1.8. ***(The quantum harmonic oscillator).*** *$H = -h^2\Delta + x^2$ is called* the quantum harmonic oscillator*, a closed densely defined operator on $L^2(\mathbb{R}^n)$, equipped with the domain*

$$\mathcal{D}(H) = \{u \in L^2(\mathbb{R}^n) : Hu \in L^2(\mathbb{R}^n)\}.$$

H has a discrete spectrum given by

$$\{(2|\alpha| + n)h : \alpha \in \mathbb{N}_0^n,\ |\alpha| = \alpha_1 + \cdots + \alpha_n\}.$$

Thus $-h^2\Delta + x^2 + 1 = H + 1$ is invertible. In the following we obtain a semiclassical asymptotic expansion for the inverse of $-h^2\Delta + x^2 + 1$.

Let $m = 1 + x^2 + \xi^2$ be an order function on $\mathbb{R}^{2n}$, we notice that m is also a symbol, $m \in S(m)$, and

$$m^{\mathrm{w}}(x,hD) = -h^2\Delta + x^2 + 1.$$

Similarly, $m^{-1} \in S(m^{-1})$, then we compute

$$(m\#m^{-1})(x,\xi) = 1 + h^2 b(x,\xi), \quad b(x,\xi) = \frac{2(x^2 - \xi^2)}{(1 + x^2 + \xi^2)^3} \in S(1).$$

Thus $m\#m^{-1}$ is elliptic in $S(1)$, by Proposition 2.1.7,

$$(m\#m^{-1})^{\mathrm{w}}(x,hD) = m^{\mathrm{w}}(x,hD)\,(m^{-1})^{\mathrm{w}}(x,hD)$$

has an inverse on L^2, and we have

$$m^{\mathrm{w}}(x,hD)\,(m^{-1})^{\mathrm{w}}(x,hD)[1 + h^2 b^{\mathrm{w}}(x,hD)]^{-1} = I_{L^2 \to L^2}.$$

It follows that

$$(m^{-1})^{\mathrm{w}}(x,hD)[1 + h^2 b^{\mathrm{w}}(x,hD)]^{-1} = m^{\mathrm{w}}(x,hD)^{-1} : L^2 \to \mathcal{D}(H).$$

Therefore, we obtain that $m^{\mathrm{w}}(x,hD)^{-1} = q^w(x,hD)$ with $q \in S(m^{-1})$ satisfying

$$q = m^{-1} + h^2 m^{-1} b + \mathcal{O}_{S(m^{-1})}(h^3).$$

2.2 Grushin problems

Suppose that

$$P : X_1 \to X_2, \quad R_+ : X_1 \to X_+, \quad R_- : X_- \to X_2,$$

are bounded operators on Banach spaces X_1, X_2, X_+, X_-.

Definition 2.2.1. *A Grushin problem is a system of equations as follows.*

$$\begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{2.2.1}$$

If (2.2.1) *is invertible, we say it is* well-posed, *and write its inverse as*

$$\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{2.2.2}$$

where $E : X_2 \to X_1$, $E_+ : X_+ \to X_1$, $E_- : X_2 \to X_-$, $E_{-+} : X_+ \to X_-$.

Remarks. Suppose that the Grushin problem (2.2.1) is well-posed. Then
(i) the operators R_+, E_- are surjective, the operators R_-, E_+ are injective;
(ii) the invertibility of P is equivalent to the invertibility of E_{-+}, and

$$P^{-1} = E - E_+ E_{-+}^{-1} E_-, \quad E_{-+}^{-1} = -R_+ P^{-1} R_-.$$

We next consider the perturbed Grushin problem. The following result is based on a Neumann series calculation.

Proposition 2.2.2. *Suppose that* (2.2.1) *is well-posed with the inverse* (2.2.2)*. Let* $A : X_1 \to X_2$ *be a bounded operator satisfying*

$$\|AE\|_{X_2 \to X_2} < 1 \ \text{ and } \ \|EA\|_{X_1 \to X_1} < 1.$$

Then

$$\begin{pmatrix} P + A & R_- \\ R_+ & 0 \end{pmatrix} \text{ is well-posed with the inverse } \begin{pmatrix} \tilde{E} & \tilde{E}_+ \\ \tilde{E}_- & \tilde{E}_{-+} \end{pmatrix},$$

where

$$\tilde{E}_{-+} = E_{-+} - \sum_{j=0}^{\infty} (-1)^j E_- A (EA)^j E_+.$$

2.3 Analytic Fredholm theory

Definition 2.3.1. *(i) A bounded linear operator $P : X_1 \to X_2$ is called a* Fredholm operator *if*

$$\ker P := \{u \in X_1 : Pu = 0\}, \text{ and } \operatorname{coker} P := X_2/PX_1,$$

are both finite dimensional.
(ii) The index of a Fredholm operator is

$$\operatorname{ind} P := \dim \ker P - \dim \operatorname{coker} P.$$

Grushin problems and Fredholm operators are connected by the following:

Proposition 2.3.2. *(i) Suppose that $P : X_1 \to X_2$ is a Fredholm operator. Then there exist finite dimensional spaces $X_\pm$ and operators $R_- : X_- \to X_2$, $R_+ : X_1 \to X_+$ such that the Grushin problem (2.2.1) is well-posed.*

(ii) Conversely, suppose that for some $X_\pm$ and $R_\pm$, the Grushin problem (2.2.1) is well-posed. Then $P : X_1 \to X_2$ is a Fredholm operator if and only if $E_{-+} : X_+ \to X_-$ is a Fredholm operator, and we have

$$\operatorname{ind} P = \operatorname{ind} E_{-+}.$$

Remarks. 1. The set of Fredholm operators is open in $\mathcal{L}(X_1, X_2)$, and the index is constant in each component of that set.
2. A bounded linear operator $P : X_1 \to X_2$ is a Fredholm operator if and only if there exists a bounded linear operator $E : X_2 \to X_1$ such that

$$PE = I_{X_2} + K_2, \quad EP = I_{X_1} + K_1, `$$

where $K_j : X_j \to X_j$ are compact operators.

Definition 2.3.3. *Suppose that $\Omega \subset \mathbb{C}$ is a connected open set, X and Y are Banach spaces. Then we say that $z \mapsto A(z) \in \mathcal{L}(X, Y)$ is analytic in Ω if for any $x \in X$ and $y^* \in Y^*$, $z \mapsto y^*(B(z)x)$ is an analytic function in Ω.*

Definition 2.3.4. *We say that $z \mapsto B(z)$ is a meromorphic family of operators (with poles of finite rank) in Ω if for any $z_0 \in \Omega$ there exist finite rank operators B_j, $1 \leq j \leq J$, and a family of operators $z \mapsto B_0(z)$, analytic near z_0, such that*

$$B(z) = B_0(z) + \frac{B_1}{z - z_0} + \cdots + \frac{B_J}{(z - z_0)^J}, \quad \text{near } z_0.$$

$B(z)$ is a meromorphic family of Fredholm operators if for every z_0, $B_0(z)$ is a Fredholm operator for z near z_0. Away from poles, $B_0(z) = B(z)$.

Theorem 4. ***(Analytic Fredholm Theory).*** *Suppose that Ω is a connected open subset of $\mathbb{C}$ and $\Omega \ni z \mapsto A(z)$ is an analytic family of Fredholm operators.*

If $A(z_0)$ is invertible for some $z_0 \in \Omega$, then $\Omega \ni z \mapsto A(z)^{-1}$ is a meromorphic family of operators with poles of finite rank.

2.4 Gohberg–Sigal theory

In this section we review some important properties of meromorphic families of Fredholm operators introduced in [20]. We start with the following factorization theorem.

Theorem 5. *Suppose that Ω is a connected open subset of $\mathbb{C}$ and that*

$$\lambda \mapsto A(\lambda), \quad \lambda \in \Omega,$$

is a meromorphic family of operators on a Banach space. If A_0 in the Laurent expansion

$$A(\lambda) = \sum_{j=-J}^{\infty} (\lambda - \mu)^j A_j, \quad \text{near } \mu$$

is of index 0, then there exist analytic families of operators $E(\lambda)$, $F(\lambda)$, invertible near μ, such that $A(\lambda)$ admits a factorization near μ:

$$A(\lambda) = E(\lambda)\Big(P_0 + \sum_{m=1}^{M}(\lambda-\mu)^{k_m} P_m\Big)F(\lambda), \quad k_m \in \mathbb{Z}\setminus\{0\}, \tag{2.4.1}$$

where P_m, $0 \le m \le M$ are mutually orthogonal projections, P_m, $m \ge 1$ are one dimensional, and $I - P_0$ is finite dimensional.

Moreover, $A(\lambda)^{-1}$ exists as a meromorphic family near μ if and only if $\sum_{m=0}^{M} P_m = I$, in which case

$$A(\lambda)^{-1} = F(\lambda)^{-1}\Big(P_0 + \sum_{m=1}^{M}(\lambda-\mu)^{-k_m} P_m\Big)E(\lambda)^{-1}.$$

Remark. This shows that if $A(\lambda_0)^{-1}$ exists for some $\lambda_0 \in \Omega$, by the connectedness, $A(\lambda)^{-1}$ is a meromorphic family of operators in Ω, which generalizes Theorem 4 to the case where $z \mapsto A(z)$ is meromorphic.

The factorization of $A(\lambda)$ allows one to define the null multiplicity at μ:

Definition 2.4.1. *In the notation of Theorem 5,*

$$N(A(\mu)) := \begin{cases} \displaystyle\sum_{k_m>0} k_m, & \text{if } \displaystyle\sum_{m=0}^{M} P_m = I, \\ \infty, & \text{otherwise.} \end{cases}$$

If $N(A(\mu)) < \infty$, then $A(\lambda)^{-1}$ is meromorphic and

$$N(A(\mu)^{-1}) = -\sum_{k_m<0} k_m.$$

We call $N(A(\mu)^{-1})$ the polar multiplicity of $A(\lambda)$ at μ.

Using Theorem 5, Gohberg and Sigal [20] obtained an operator generalization of the logarithmic residue theorem for meromorphic functions.

Theorem 6. *Suppose that $\Omega \subset \mathbb{C}$ is open and connected, $A(\lambda)$ and $A(\lambda)^{-1}$, $\lambda \in \Omega$ are meromorphic families of Fredholm operators on a Banach space X. Then*

$$\oint_{\mu} \partial_\lambda A(\lambda) A(\lambda)^{-1}\, d\lambda \text{ is of finite rank and}$$

$$\frac{1}{2\pi i} \oint_{\mu} \partial_\lambda A(\lambda) A(\lambda)^{-1}\, d\lambda = N(A(\mu)) - N(A(\mu)^{-1}).$$

Here the integral is over a positively oriented circle enclosing μ and containing no poles other than possibly μ.

Another result is an operator generalization of Rouché's theorem:

Theorem 7. *Let Ω be a connected open subset of $\mathbb{C}$. Suppose that $A(\lambda)$ and $B(\lambda)$ satisfy the assumptions of Theorem 6 and that $U \Subset \Omega$ is a simply connected open set with a C^1 boundary on which $A(\lambda)$ and $B(\lambda)$ are both invertible. If*

$$\|A(\lambda)^{-1}(A(\lambda) - B(\lambda))\|_{X\to X} < 1, \quad \forall\, \lambda \in \partial U,$$

$$\text{then} \quad \sum_{\mu\in U} N(A(\mu)) - N(A(\mu)^{-1}) = \sum_{\mu \in U} N(B(\mu)) - N(B(\mu)^{-1}).$$

2.5 Decay of the Green function

Let M be a complete Riemannian manifold, $\dim M = n$ and $d : M \times M \to [0, \infty)$ be the Riemannian distance. Let $\exp_x : T_x M \to M$ be the usual exponential geodesic map. For $x \in M$, we define

$$r_x := \sup\{r : \exp_x|_{B(0,r)\subset T_x M} \text{ is a diffeomorphism}\}.$$

Then the injectivity radius of M is given by $r_{\text{inj}} = \inf_{x\in M} r_x$. The coordinates

$$B(0, r) \ni y \mapsto \exp_x(y) \in M$$

are called canonical coordinates on $U_{x,r} := \exp_x(B(0, r))$.

Definition 2.5.1. *We say that M is a* manifold of bounded geometry *if the following conditions hold:*
(i) $r_{inj} > 0$,
(ii) $|\nabla^k R| \le C_k$, $\forall\, k$, *i.e. all covariant derivatives of the curvature tensor are bounded.*

Examples. A trivial example of manifold of bounded geometry is the Euclidean space. Other examples include Lie groups and covering manifolds of compact manifolds.

Definition 2.5.2. *Suppose that $A : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is a differential operator of order m. We say A is $\mathcal{C}^\infty$-bounded if in any canonical coordinates*

$$A = \sum_{|\alpha|\leq m} a_\alpha(y)\partial_y^\alpha, \quad |\partial_y^\beta a_\alpha(y)| \leq C_{\alpha\beta}, \text{ for all } \beta \in \mathbb{N}_0^n.$$

Furthermore, A is called uniformly elliptic *if there exists constant $C > 0$ such that*

$$\Big|\sum_{|\alpha|=m} a_\alpha(y)\,\eta^\alpha\Big| \geq C|\eta|^m, \ \forall\,(y,\eta) \in T^*M.$$

We recall the following estimate from [47]:

Proposition 2.5.3. *Let M be a manifold of bounded geometry and $A : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ be a $\mathcal{C}^\infty$-bounded uniformly elliptic differential operator. Suppose that $\lambda \notin \operatorname{Spec}(A)$ (L^2 spectrum). Denote by $G(\lambda; x, y)$ the Schwartz kernel of the resolvent*

$$(A-\lambda)^{-1} : L^2(M) \to L^2(M).$$

Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ and $\alpha, \beta \in \mathbb{N}_0^n$, $\exists\, C_{\alpha\beta\delta} > 0$ s.t.

$$|\partial_x^\alpha \partial_y^\beta G(\lambda; x, y)| \leq C_{\alpha\beta\delta}\, e^{-\varepsilon d(x,y)} \quad \text{if } d(x,y) > \delta. \tag{2.5.1}$$

Sketch of proof. One crucial step is to construct a smooth substitute for the Riemannian distance d. The conditions in Definition 2.5.1 guarantee that for every $\rho > 0$ we can construct $\tilde{d} \in \mathcal{C}^\infty(M \times M; [0,\infty))$ satisfying

$$|\tilde{d}(x,y) - d(x,y)| < \rho, \quad \forall\, x, y \in M \tag{2.5.2}$$

and

$$\sup_{x,y\in M} |\partial_y^\alpha \tilde{d}(x,y)| < C_\alpha, \quad \forall\, \alpha \in \mathbb{N}_0^n, \ |\alpha| \geq 1, \tag{2.5.3}$$

where the derivative ∂_y^α is taken with respect to canonical coordinates. For a complete construction, we refer to Kordyukov [34] or [47, Appendix 1].

Next we introduce exponential weights $f_{\varepsilon,y} \in \mathcal{C}^\infty(M)$ for any $\varepsilon > 0$, $y \in M$:

$$f_{\varepsilon,y}(x) = e^{\varepsilon \tilde{d}(y,x)}, \quad (F_{\varepsilon,y}u)(x) := f_{\varepsilon,y}(x)u(x).$$

It follows from (2.5.3) that

$$A_{\varepsilon,y} := F_{\varepsilon,y} \circ A \circ F_{\varepsilon,y}^{-1} = A + \varepsilon B_{\varepsilon,y},$$

where $B_{\varepsilon,y}$ is a uniformly $\mathcal{C}^\infty$-bounded differential operator of order $m-1$ (defined by Definition 2.5.2 with uniform constants $C_{\alpha\beta}$ on M). Then for $\varepsilon > 0$ sufficiently small,

$$A_{\varepsilon,y} - \lambda \text{ is invertible and } (A_{\varepsilon,y} - \lambda)^{-1} : H^s(M) \to H^{s+m}(M),\ \forall\, s \in \mathbb{R},$$

where $H^s(M)$ are Sobolev spaces on manifold M. This is obtained by standard elliptic estimates and the invertibility of $A - \lambda$. We notice that

$$e^{\varepsilon \tilde{d}(x,y)} G(\lambda; x, y) = [F_{\varepsilon,y}(A-\lambda)^{-1}\delta_y](x) = [(A_{\varepsilon,y} - \lambda)^{-1}(f_{\varepsilon,y}\delta_y)](x)$$

where δ_y is the Dirac function supported at y. For every $\delta > 0$ and $x, y \in M$ with $d(x,y) > \delta$, one can apply the interior elliptic estimate to obtain that

$$\|f_{\varepsilon,y}(\cdot)G(\lambda; \cdot, y)\|_{H^s(B(x,\delta/2))} \le C_{\delta,s}, \quad \forall\, s \in \mathbb{R}.$$

The Sobolev embedding theorem then shows that

$$|\partial_x^\alpha G(\lambda; x, y)| \le C_{\alpha,\delta} e^{-\varepsilon \tilde{d}(x,y)} \quad \text{if } d(x,y) > \delta.$$

This and (2.5.2) imply (2.5.1) if $\beta = 0$; the case where $\beta \neq 0$ follows from the same arguments with A replaced by A^t (the transpose of A). □

Proposition 2.5.3 can be adapted to some manifolds that are not complete, for instance $M = \mathbb{R}^n \backslash K$, where K is a compact set with smooth boundary such that $\mathbb{R}^n \backslash K$ is connected. In this case for each $\rho > 0$ we first construct $\tilde{d} \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n; [0,\infty))$ as a smooth substitute of $d = d(x,y) = |x-y|$, then restrict $\tilde{d}$ to $\mathbb{R}^n \setminus K$, thus

$$\left|\tilde{d}(x,y) - |x-y|\right| < \rho, \quad \forall\, x, y \in \mathbb{R}^n \setminus K, \text{ and}$$

$$\sup_{x,y \in M} |\partial_y^\alpha \tilde{d}(x,y)| < C_\alpha, \quad \forall\, \alpha \in \mathbb{N}_0^n,\ |\alpha| \ge 1.$$

Therefore, we have

Corollary 2.5.4. *Let $A : \mathcal{C}^\infty(\mathbb{R}^n \setminus K) \to \mathcal{C}^\infty(\mathbb{R}^n \setminus K)$ be a $\mathcal{C}^\infty$-bounded, second order uniformly elliptic differential operator. A can be viewed as an operator:*

$$A : L^2(\mathbb{R}^n \setminus K) \to L^2(\mathbb{R}^n \setminus K) \text{ with domain } H^2(\mathbb{R}^n \setminus K) \cap H_0^1(\mathbb{R}^n \setminus K).$$

Suppose that $\lambda \notin \operatorname{Spec}(A)$ and let $G(\lambda; x, y)$ be the Schwartz kernel of $(A-\lambda)^{-1}$. Then there exists $\beta > 0$ such that for each $\delta > 0$, $\exists\, C_\delta > 0$ s.t.

$$|G(\lambda; x, y)| \le C_\delta e^{-\beta|x-y|} \quad \text{if } |x-y| > \delta.$$

2.6 Pöschl–Teller potentials

We recall the following definition from Pöschl–Teller [43]: the Pöschl–Teller potential is defined on $\mathbb{R}$ by

$$V_{\mu,\nu}(r) := \frac{\mu(\mu+1)}{\sinh^2 r} - \frac{\nu(\nu+1)}{\cosh^2 r}, \quad r \in \mathbb{R},$$

$V_{\mu,\nu}$ is a real potential if μ, ν are taken in $-1/2 + i(0,\infty) \cup [-1/2,\infty)$. In this section we will focus on the case in which $V_{\mu,\nu}$ is complex-valued and review some properties of the Hamiltonian $D_r^2 + V_{\mu,\nu}(r)$ on the half line $(0,\infty)_r$. The following result is based on the analysis of $D_r^2 + V_{\mu,\nu}(r)$ in Guillopé–Zworski [23, Appendix]:

Proposition 2.6.1. *The Schrödinger operator $D_r^2 + V_{\mu,\nu}$ (respectively $D_r^2 + V_{0,\nu}$) has $\mathbb{R}^+$ as continuous spectrum. The determinant of the scattering matrix for $D_r^2 + V_{\mu,\nu}$ is given by the reflection coefficient*

$$s^{PT}_{\mu,\nu}(k) = -\frac{\Gamma(ik)\Gamma((\mu+\nu-ik)/2+1)\Gamma((\mu-\nu-ik+1)/2)2^{-ik}}{\Gamma(-ik)\Gamma((\mu+\nu+ik)/2+1)\Gamma((\mu-\nu+ik+1)/2)2^{ik}}, \tag{2.6.1}$$

and for $D_r^2 + V_{0,\nu}$ given by

$$s^{PT}_{\nu}(k) = -\frac{\Gamma(ik)^2\Gamma(\nu-ik+1)\Gamma(-\nu-ik)}{\Gamma(-ik)^2\Gamma(\nu+ik+1)\Gamma(-\nu+ik)}. \tag{2.6.2}$$

The Schrödinger operator $D_r^2 + V_{\mu,\nu}$ (resp. $D_r^2 + V_{0,\nu}$) has non-empty discrete spectrum if and only if $\mathrm{Re}(\nu-\mu) > 1$ *(resp.* $\mathrm{Re}\,\nu > 0$*). The discrete spectrum is given by*

$$\sigma_d(D_r^2 + V_{\mu,\nu}) = \{-(\nu-\mu-1-2n)^2 : n \in \mathbb{N},\, 2n < \mathrm{Re}(\nu-\mu-1)\},$$
$$\sigma_d(D_r^2 + V_{0,\nu}) = \{-(\nu-n)^2 : n \in \mathbb{N},\, n < \mathrm{Re}\,\nu\}.$$

Proof. Through a conjugation by $\sinh^{\mu+1} r \cosh^{\nu+1} r$ and the change of variable $u = -\sinh^2 r$, the Schrödinger equation

$$D_r^2\psi + V_{\mu,\nu}\psi - k^2\psi = 0 \tag{2.6.3}$$

is reduced to the hypergeometric equation

$$\begin{aligned} u(1-u)F''(u) + [(\mu+3/2) - (\mu+\nu+3)u]F'(u) \\ - [((\mu+\nu+2)/2)^2 + (k/2)^2]F = 0. \end{aligned}$$

The Schrödinger equation (2.6.3) has the following independent solutions (if $\mu \neq -\frac{1}{2}$):

$$\begin{aligned} E_{\mu,\nu}(k)(r) =& \sinh^{1+\mu} r \cosh^{1+\nu} r \\ & \times {}_2F_1((\mu+\nu-ik+2)/2, (\mu+\nu+ik+2)/2, \mu+\frac{3}{2}; -\sinh^2 r), \end{aligned} \tag{2.6.4}$$

$$F_{\mu,\nu}(k)(r) = \sinh^{-\mu} r \cosh^{1+\nu} r \\ \times {}_2F_1((-\mu+\nu-ik+1)/2, (-\mu+\nu+ik+1)/2, \frac{1}{2}-\mu; -\sinh^2 r). \tag{2.6.5}$$

The asymptotic expansion of (2.6.4) at infinity is given, if ik is not an integer, by

$$\begin{aligned} E_{\mu,\nu}(k)(r) \approx & \frac{\Gamma(\mu+3/2)\Gamma(ik)}{\Gamma((\mu+\nu+ik+2)/2)\Gamma((\mu-\nu+ik+1)/2)} \coth^{\nu+1} r \sinh^{ik} r \\ & \times {}_2F_1((\mu+\nu-ik+1)/2, (-\mu+\nu-ik+1)/2, 1-ik; -\sinh^{-2} r) \\ & + \frac{\Gamma(\mu+3/2)\Gamma(-ik)}{\Gamma((\mu+\nu-ik+2)/2)\Gamma((\mu-\nu-ik+1)/2)} \coth^{\nu+1} r \sinh^{-ik} r \\ & \times {}_2F_1((\mu+\nu+ik+1)/2, (-\mu+\nu+ik+1)/2, 1+ik; -\sinh^{-2} r), \end{aligned} \tag{2.6.6}$$

recalling the definition of reflection coefficient for potential scattering (see for instance Dyatlov–Zworski [12, §2.4]), we obtain (2.6.1).

The potential $V_{0,\nu}$ is smooth on $\mathbb{R}$, the operator $D_r^2 + V_{0,\nu}$ can be decomposed as the sum of the Dirichlet (H_ν^D) and Neumann (H_ν^N) extensions of $D_r^2 + V_{0,\nu}$. The eigenfunctions of the spectral resolution of H_ν^N are the $F_{0,\nu}(k)$ from (2.6.5) and a similar asymptotic expansion at infinity to (2.6.6) gives the reflection coefficient $s(H_\nu^N)$. The scattering coefficient $s_\nu^{PT}(k)$ (2.6.2) is then the product $s_{0,\nu}^{PT}(k)s(H_\nu^N)(k)$.

The asymptotic properties of the eigenfunctions (2.6.4) and (2.6.5) determine the discrete spectra. □

Chapter 3

Review of scattering resonances

3.1 Scattering by compactly supported potentials

In this section we review briefly various notions of scattering theory in the simplest setting of scattering by compactly supported potentials in dimension one, we refer the reader to [12, Chapter 2] for a full introduction.

We consider the following operator:

$$P = D_x^2 + V(x), \quad D_x = i^{-1}\partial_x \quad \text{acting on } L^2(\mathbb{R}),$$

$$V \in L^\infty_{\text{comp}}(\mathbb{R};\mathbb{R}), \quad \operatorname{supp} V \subset [-R,R], \quad R > 0.$$

Solutions of the stationary equation

$$(P - \lambda^2)u = 0$$

admit the following decomposition outside the support of V:

$$u(x) = u_{\text{in}}(x) + u_{\text{out}}(x), \quad |x| > R,$$

where

$$u_{\text{in}}(x) = b_- e^{i\lambda x},\ x < -R; \quad u_{\text{in}}(x) = b_+ e^{-i\lambda x},\ x > R,$$

corresponds to incoming waves: $e^{i\lambda(x \pm t)}$, $x \gtrless 0$;

$$u_{\text{out}}(x) = a_- e^{-i\lambda x},\ x < -R; \quad u_{\text{out}}(x) = a_+ e^{i\lambda x},\ x > R,$$

corresponds to outgoing waves: $e^{i\lambda(x \mp t)}$, $x \gtrless 0$.

The scattering matrix is defined by

$$S(\lambda) : \begin{pmatrix} b_- \\ b_+ \end{pmatrix} \mapsto \begin{pmatrix} a_+ \\ a_- \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

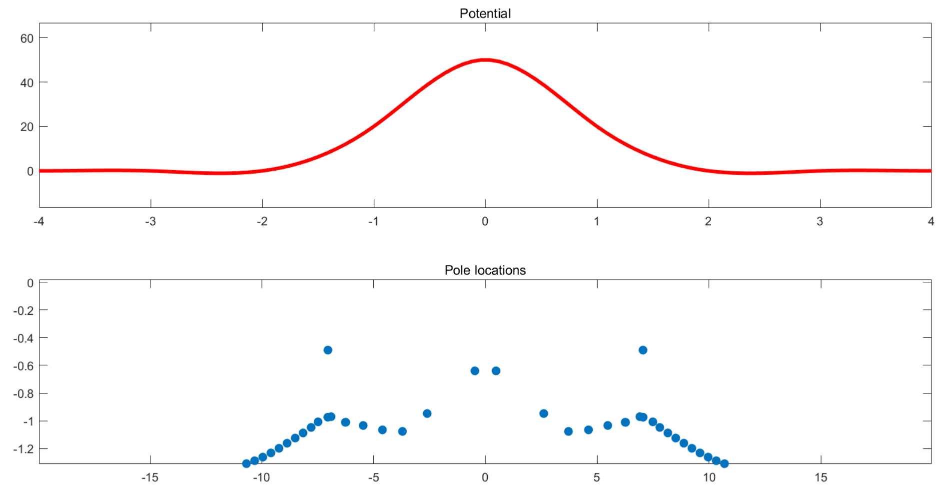


Figure 3.1: Resonances of scattering by a bump potential shown on the top. Resonances are computed using splinepot.m [5].

It is well known that $S(\lambda)$ can be meromorphically continued to $\lambda \in \mathbb{C}$ – see [12, §2.4]. The poles of the scattering matrix are called *scattering resonances.*

Another way to define the resonances is using the resolvent of $P = D_x^2 + V$

$$R_V(\lambda) := (D_x^2 + V - \lambda^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \operatorname{Im} \lambda > 0.$$

$R_V(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$ as a family of operators: $L^2_{\mathrm{comp}}(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R})$, see [12, Theorem 2.2]. The resonances are defined as the poles of $R_V(\lambda)$. The multiplicity of a resonance λ is given by

$$m(\lambda) := \operatorname{rank} \oint_\lambda R_V(\zeta)\, d\zeta,$$

where the integral is over a small circle enclosing no other poles than λ.

One important property of resonances is that they capture the frequencies and rates of decay of solutions of the wave equation

$$\begin{gathered}(\partial_t^2 + P)u = (\partial_t^2 - \partial_x^2 + V(x))u = 0, \quad t > 0,\ x \in \mathbb{R}, \\ u(0, x) = 0, \quad \partial_t u(0, x) = \varphi(x) \in L^2_{\mathrm{comp}}(\mathbb{R}).\end{gathered} \tag{3.1.1}$$

The solution of (3.1.1) admits a *resonance expansion* – see [12, Theorem 2.9]. For simplicity we assume all resonances λ_j of P are simple, i.e. $m(\lambda_j) = 1$ (this is true for a generic potential V by [33]), then for any $A > 0$, we have

$$u(t,x) = \sum_{\operatorname{Im}\lambda_j > -A} e^{-i\lambda_j t} u_j(x) + E_A(t),$$

where the sum is finite and u_j are called *resonant states* satisfying $(P - \lambda_j)u_j = 0$, which can be calculated using residues of $R_V(\lambda)$ at λ_j:

$$u_j(x) = -ie^{i\lambda_j t} \operatorname{Res}_{\lambda=\lambda_j}(e^{-i\lambda t} R_V(\lambda)\varphi).$$

The error term decays exponentially: for any L such that $\operatorname{supp}\varphi \subset (-L, L)$,

$$\|E_A(t)\|_{H^2([-L,L]_x)} \leq Ce^{-At}\|\varphi\|_{L^2}, \quad \text{for some } C = C(L, A).$$

3.2 Scattering by exponentially decaying potentials

In this section we consider the Schrödinger operator

$$P := -\Delta + V \quad \text{acting on } L^2(\mathbb{R}^n),$$

where $V \in L^\infty(\mathbb{R}^n; \mathbb{R})$ is exponentially decaying

$$|V(x)| \leq Ce^{-2\gamma|x|}, \text{ for some } C, \gamma > 0. \tag{3.2.1}$$

We denote the resolvent of P by

$$R_V(\lambda) := (P - \lambda^2)^{-1} = (-\Delta + V - \lambda^2)^{-1} : L^2 \to L^2, \quad \operatorname{Im}\lambda > 0.$$

To meromorphically continue $R_V(\lambda)$ to the lower half plane, we introduce the weighted resolvent:

$$\sqrt{V} R_V(\lambda) \sqrt{V} : L^2 \to L^2, \quad \operatorname{Im}\lambda > 0,$$

where we take the usual branch for the square root such that $\sqrt{-1} = +i$. Following Froese [15] and Gannot [19], we will show that the weighted resolvent can be meromorphically continued to the strip

$$\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > -\gamma\}. \tag{3.2.2}$$

Resonances of P are the poles in this meromorphic continuation. In fact, one can show that $\operatorname{Res}(P)$ are precisely equal to the poles of the scattering matrix, confirming that our definition is consistent with the standard definition. We refer the reader to [15] for a detailed presentation.

Now we introduce the meromorphic continuation of $\sqrt{V}R_V(\lambda)\sqrt{V}$ from the upper half plane to the strip (3.2.2) under the assumption (3.2.1). Denote by

$$R_0(\lambda) := (-\Delta - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \operatorname{Im}\lambda > 0$$

the free resolvent on $\mathbb{R}^n$. For $\operatorname{Im}\lambda > 0$, the resolvent equation

$$R_0(\lambda) - R_V(\lambda) - R_V(\lambda)VR_0(\lambda) = 0$$

implies

$$(I - \sqrt{V}R_V(\lambda)\sqrt{V})(I + \sqrt{V}R_0(\lambda)\sqrt{V}) = I.$$

We recall that

$$\|R_0(\lambda)\|_{L^2\to L^2} = \mathcal{O}(|\operatorname{Im}\lambda|^{-1}),$$

thus for $\operatorname{Im}\lambda$ sufficiently large, $I + \sqrt{V}R_0(\lambda)\sqrt{V}$ is invertible by a Neumann series argument. Therefore

$$I - \sqrt{V}R_V(\lambda)\sqrt{V} = (I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1}, \quad \operatorname{Im}\lambda \gg 1. \tag{3.2.3}$$

In the following lemma, we show that the right side of (3.2.3) admits a meromorphic continuation. Some bounds of the free resolvent with exponential weights are used in the proof, we refer the reader to [19] for a complete treatment.

Lemma 1. *For any $a > 0$ and $\gamma' < \gamma$,*

$$\lambda \mapsto (I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1}, \quad \operatorname{Re}\lambda > a,\ \operatorname{Im}\lambda > -\gamma',$$

is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ with poles of finite rank.

Proof. Choose $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ satisfying $\varphi(x) = |x|$ for large $|x|$, it is well known that for each $c > 0$, the weighted resolvent:

$$e^{-c\varphi}R_0(\lambda)e^{-c\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

extends analytically across $\operatorname{Re}\lambda > 0$ to the strip $\operatorname{Im}\lambda > -c$, see [19, §1] and references given there. Moreover, Gannot [19, §1] proved that for any $a, c, \varepsilon > 0$ and $\alpha \in \mathbb{N}^n$, $|\alpha| \le 2$ there exists $C_\alpha = C_\alpha(a, c, \varepsilon)$ such that

$$\|D^\alpha(e^{-c\varphi}R_0(\lambda)e^{-c\varphi})\|_{L^2\to L^2} \le C_\alpha|\lambda|^{|\alpha|-1}, \quad \text{for } \operatorname{Im}\lambda > -c+\varepsilon,\ \operatorname{Re}\lambda > a. \tag{3.2.4}$$

In particular, for $\operatorname{Re}\lambda > a$ and $\operatorname{Im}\lambda > -\gamma'$,

$$\lambda \mapsto e^{-\gamma'\varphi}R_0(\lambda)e^{-\gamma'\varphi}$$

is an analytic family of operators $L^2 \to H^2$. It follows from (3.2.1) that

$$\lim_{|x|\to\infty} |\sqrt{V(x)}e^{\gamma'\varphi(x)}| = 0$$

thus we have $\sqrt{V} e^{\gamma'\varphi} : H^2 \to L^2$ is compact. Hence,

$$\lambda \mapsto \sqrt{V} R_0(\lambda) \sqrt{V} = \sqrt{V} e^{\gamma'\varphi} (e^{-\gamma'\varphi} R_0(\lambda) e^{-\gamma'\varphi}) \sqrt{V} e^{\gamma'\varphi}$$

is an analytic family of compact operators $L^2 \to L^2$ for $\operatorname{Re}\lambda > a$, $\operatorname{Im}\lambda > -\gamma'$. Recalling that $I + \sqrt{V} R_0(\lambda) \sqrt{V}$ is invertible for $\operatorname{Im}\lambda \gg 1$, then by Theorem 4,

$$\lambda \mapsto (I + \sqrt{V} R_0(\lambda) \sqrt{V})^{-1} : L^2 \to L^2$$

is a meromorphic family of operators in the same range of λ. □

We shall identify the resonances λ_j, in the region Ω – see (1.3.1), with the poles of the meromorphic family of operators

$$\Omega \ni \lambda \mapsto (I + \sqrt{V} R_0(\lambda) \sqrt{V})^{-1} : L^2 \to L^2$$

with agreement of multiplicities. In view of Theorem 6, the multiplicity of resonance λ is given by

$$m(\lambda) := \frac{1}{2\pi i} \operatorname{tr} \oint_{\lambda} (I + \sqrt{V} R_0(\zeta) \sqrt{V})^{-1} \partial_\zeta (\sqrt{V} R_0(\zeta) \sqrt{V}) \, d\zeta, \tag{3.2.5}$$

where the integral is over a positively oriented circle enclosing λ and containing no poles other than λ.

3.3 Black box scattering

In this section we will follow [12, §4] to introduce a general class of compactly supported self-adjoint perturbations of the Laplacian on $\mathbb{R}^n$, which are called black box Hamiltonians.

Let $\mathcal{H}$ be a complex separable Hilbert space with an orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)), \tag{3.3.1}$$

where $B(x, R) = \{y \in \mathbb{R}^n : |x - y| < R\}$ and R_0 is fixed. The corresponding orthogonal projections will be denoted by

$$u \mapsto u|_{B(0,R_0)} \text{ and } u \mapsto u|_{\mathbb{R}^n \setminus B(0,R_0)},$$

or simply by the characteristic function 1_L of the corresponding set L.

We now consider an unbounded self-adjoint operator

$$P : \mathcal{H} \to \mathcal{H} \quad \text{with domain } \mathcal{D}(P). \tag{3.3.2}$$

Assume that

$$\mathcal{D}(P)|_{\mathbb{R}^n\setminus B(0,R_0)} \subset H^2(\mathbb{R}^n \setminus B(0,R_0)), \tag{3.3.3}$$

and conversely, $u \in \mathcal{D}(P)$ if $u \in H^2(\mathbb{R}^n \setminus B(0,R_0))$ and u vanishes near $B(0,R_0)$;

$$1_{B(0,R_0)}(P+i)^{-1} \text{ is compact.} \tag{3.3.4}$$

We also assume that,

$$1_{\mathbb{R}^n\setminus B(0,R_0)}Pu = -\Delta(u|_{\mathbb{R}^n\setminus B(0,R_0)}), \quad \text{for all } u \in \mathcal{D}(P). \tag{3.3.5}$$

It is well known that the resolvent $(P-\lambda^2)^{-1}$, $\operatorname{Im}\lambda > 0$, $\lambda^2 \notin \operatorname{Spec}_{\text{point}}(P)$, has a meromorphic continuation to $\mathbb{C}$ when n is odd; to $\Lambda = \exp^{-1}(\mathbb{C}\setminus\{0\})$ when n is even: as an operator $(P-\lambda^2)^{-1} : \mathcal{H}_{\text{comp}} \to \mathcal{D}_{\text{loc}}$, see [49, Theorem 1.1] and [12, Theorem 4.4]. However, we need to meromorphically continue $(P-\lambda^2)^{-1}$ as an operator between some Banach spaces to apply Agmon's method [1] and prove Theorem 3. For that we define a weighted subspace of $\mathcal{H}$ for any large constant $A > 0$,

$$\mathcal{H}_0^A := \mathcal{H}_{R_0} \oplus e^{-A|x|}L^2(\mathbb{R}^n \setminus B(0,R_0)), \tag{3.3.6}$$

and a larger space containing $\mathcal{H}$:

$$\mathcal{H}_1^A := \mathcal{H}_{R_0} \oplus e^{A|x|}L^2(\mathbb{R}^n \setminus B(0,R_0)). \tag{3.3.7}$$

The space $\mathcal{D}_1^A(P)$ is defined using (3.3.7),

$$\begin{aligned} \mathcal{D}_1^A(P) := \{u \in \mathcal{H}_1^A : \chi \in \mathcal{C}_c^\infty(\mathbb{R}^n),\ \chi|_{B(0,R_0)} \equiv 1 \\ \Rightarrow \chi u \in \mathcal{D}(P),\ \Delta((1-\chi)u) \in \mathcal{H}_1^A\}. \end{aligned} \tag{3.3.8}$$

We also denote the strips in $\mathbb{C}$ by

$$T_A := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > -A\}, \tag{3.3.9}$$

and a family of subsets of Λ by

$$\begin{aligned} S_m &:= \{\lambda \in \Lambda : -m\pi < \arg\lambda < m\pi\}, \quad m \in \mathbb{N}_+, \\ \Lambda_A &:= \{\lambda \in \Lambda : 0 < \arg\lambda < \pi\} \cup \{\lambda \in S_{\lfloor A \rfloor} : |\lambda| < A\}. \end{aligned} \tag{3.3.10}$$

We are now ready to state the main result of this section:

Proposition 3.3.1. *Suppose that P is a black box Hamiltonian. Then*

$$R(\lambda) := (P-\lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}(P) \quad \textit{is meromorphic for } \operatorname{Im}\lambda > 0. \tag{3.3.11}$$

Moreover, when n is odd, the resolvent extends to a meromorphic family

$$R(\lambda) : \mathcal{H}_0^A \to \mathcal{D}_1^A(P), \quad \lambda \in T_A. \tag{3.3.12}$$

When n is even (3.3.12) holds with T_A replaced by Λ_A.

The proof is the same as the one of [12, Theorem 4.4]. The only difference is that unlike the free resolvent $R_0(\lambda) := (-\Delta - \lambda^2)^{-1}$ mermorphically continued as an operator between $L^2_{\mathrm{comp}}(\mathbb{R}^n)$ and $H^2_{\mathrm{loc}}(\mathbb{R}^n)$ there, we have to show that

$$\begin{aligned} \lambda &\mapsto R_0(\lambda) : e^{-A|x|}L^2(\mathbb{R}^n) \to e^{A|x|}L^2(\mathbb{R}^n), \\ \lambda &\mapsto [\Delta, \chi]R_0(\lambda) : e^{-A|x|}L^2(\mathbb{R}^n) \to L^2_{\mathrm{comp}}(\mathbb{R}^n), \quad \forall \chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n), \end{aligned} \tag{3.3.13}$$

are meromorphic in $\lambda \in T_A$ when n is odd, $\lambda \in \Lambda_A$ when n is even.

Denote by $R_0(\lambda, x, y)$ the Schwartz kernel of the free resolvent $R_0(\lambda)$, which can be written in terms of the Hankel functions of the first kind:

$$R_0(\lambda, x, y) = c_n \lambda^{n-2}(\lambda|x-y|)^{-\frac{n-2}{2}} H^{(1)}_{\frac{n}{2}-1}(\lambda|x-y|). \tag{3.3.14}$$

We recall some well known facts about $R_0(\lambda, x, y)$ as follows, see for instance [12, §3.1] for a detailed account. When n is odd, (3.3.14) admits a finite expansion:

$$R_0(\lambda, x, y) = \lambda^{n-2} e^{i\lambda|x-y|} \sum_{j=\frac{n-1}{2}}^{n-2} \frac{c_{n,j}}{(\lambda|x-y|)^j}. \tag{3.3.15}$$

For $x \neq y$ this form extends meromorphically to $\lambda \in \mathbb{C}$. When n is even, using the relation:

$$R_0(e^{i\ell\pi}\lambda, x, y) - R_0(\lambda, x, y) = c_n \ell (-1)^{\frac{n-2}{2}(\ell+1)} \lambda^{\frac{n-2}{2}} |x-y|^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda|x-y|), \tag{3.3.16}$$

where $J_d(z)$ is the Bessel function, we see that $R_0(\lambda, x, y)$, $x \neq y$ extends to $\lambda \in \Lambda$. In view of (3.3.15), when n is odd we have the upper bounds for $\lambda \in \mathbb{C}$:

$$|R_0(\lambda, x, y)| \leq \begin{cases} C(\lambda)|x-y|^{2-n}, & |x-y| \leq |\lambda|^{-1}; \\ C(\lambda)e^{-\operatorname{Im}\lambda|x-y|}|\lambda|^{\frac{n-3}{2}}|x-y|^{\frac{1-n}{2}}, & |x-y| \geq |\lambda|^{-1}. \end{cases} \tag{3.3.17}$$

For n even, $n \neq 2$, the bounds (3.3.17) hold for $-\pi < \arg\lambda < 2\pi$. This follows from the asymptotics of $H^{(1)}_d(z)$, see also Galkowski–Smith [16] for more details. Using (3.3.16) and the following formulas about $J_d(z)$:

$$J_d(z) \sim \frac{1}{\Gamma(d+1)}\left(\frac{z}{2}\right)^d, \quad \text{as } z \to 0, \text{ when } d \in \mathbb{Z},$$

$$J_d(z) = \sqrt{\frac{2}{\pi z}}\left(\cos\left(z - \frac{d\pi}{2} - \frac{\pi}{4}\right) + e^{|\operatorname{Im} z|}\mathcal{O}\left(|z|^{-1}\right)\right), \quad \text{as } |z| \to \infty, \ |\arg z| < \pi.$$

we can extend (3.3.17) to any $\lambda \in \Lambda$, $\arg\lambda \leq -\pi$ or $\arg\lambda \geq 2\pi$:

$$|R_0(\lambda, x, y)| \leq \begin{cases} C(\lambda)\,|x-y|^{2-n}, & |x-y| \leq |\lambda|^{-1}; \\ C(\lambda)\,e^{|\operatorname{Im}\lambda||x-y|}|\lambda|^{\frac{n-3}{2}}|x-y|^{\frac{1-n}{2}}, & |x-y| \geq |\lambda|^{-1}. \end{cases} \tag{3.3.18}$$

In the case that $n = 2$, $|x-y|^{2-n}$ in (3.3.18) is replaced by $-\ln|x-y|$ when $|x-y| \leq |\lambda|^{-1}$. Now we can conclude from (3.3.17) and (3.3.18) that for any (except possible poles) $\lambda \in T_A$ when n is odd, $\lambda \in \Lambda_A$ when n is even,

$$\sup_x \int_{\mathbb{R}^n} e^{-A|x|-A|y|}|R_0(\lambda,x,y)|dy, \quad \sup_y \int_{\mathbb{R}^n} e^{-A|x|-A|y|}|R_0(\lambda,x,y)|dx < \infty.$$

Using the formula about derivatives of the Hankel functions

$$\frac{d}{dz}H_m^{(1)}(z) = H_{m-1}^{(1)}(z) - \frac{m}{z}H_m^{(1)}(z),$$

we can also conclude from the bounds (3.3.17) that

$$\sup_x \int_{\mathbb{R}^n} |[\Delta_x,\chi]R_0(\lambda,x,y)|e^{-A|y|}dy, \quad \sup_y \int_{\mathbb{R}^n} |[\Delta_x,\chi]R_0(\lambda,x,y)|e^{-A|y|}dx < \infty.$$

Hence (3.3.13) follows by the Schur test.

3.4 Long range perturbations of Laplacian

In this section, we introduce long range perturbations of $-\Delta$. We modify the black box formalism in §3.3 and follow [50] to assume that P is a dilation analytic perturbation of $-\Delta$ near infinity. The black box formalism allows an abstract treatment of diverse scattering problems without addressing the details of specific situations – see Examples 1–3 later in this section. We recall the setup as follows.

Let $\mathcal{H}$ be a complex separable Hilbert space with an orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0)),$$

where $B(x,R) = \{y \in \mathbb{R}^n : |x-y| < R\}$ and R_0 is fixed. We consider an unbounded self-adjoint operator P satisfying (3.3.2) – (3.3.4). We also assume that,

$$\begin{aligned} 1_{\mathbb{R}^n\setminus B(0,R_0)}Pu &= Q(u|_{\mathbb{R}^n\setminus B(0,R_0)}), \qquad \text{for all } u \in \mathcal{D}, \\ Q &= -\sum_{j,k=1}^n \partial_{x_j}(g^{jk}(x)\partial_{x_k}) + c(x), \quad g^{jk}, c \in \mathcal{C}_b^\infty(\mathbb{R}^n). \end{aligned} \tag{3.4.1}$$

Here $\mathcal{C}_b^\infty$ denotes the space of $\mathcal{C}^\infty$ functions with all derivatives bounded. Note that if $\psi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ is constant near $B(0,R_0)$, then there is a natural way to define the multiplication: $\mathcal{H} \ni u \mapsto \psi u \in \mathcal{H}$, and we have $\psi u \in \mathcal{D}$ if $u \in \mathcal{D}$.

We make the further assumptions on the coefficients of Q: g^{jk}, c are real-valued functions on $\mathbb{R}^n$ satisfying

$$\begin{gathered} g^{jk} = g^{kj},\ \forall\, j,k, \quad |\sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k| \geq C^{-1}|\xi|^2, \\ \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k + c(x) \to \xi^2,\ |x| \to \infty. \end{gathered} \tag{3.4.2}$$

We will use the method of complex scaling – see §3.5 to define the resonances of P. For that we follow [50] to make the following assumptions:

$$\begin{gathered} \text{There exist } \theta_0 \in [0, \pi/8],\ \delta > 0, \text{ and } R \geq R_0, \text{ such that} \\ \text{the coefficients } g^{jk}(x), c(x) \text{ of } Q \text{ extend analytically in } x \text{ to} \\ \{s\omega : \omega \in \mathbb{C}^n,\ \mathrm{dist}(\omega, \mathbb{S}^{n-1}) < \delta,\ s \in \mathbb{C},\ |s| > R,\ \arg s \in (-\delta, \theta_0 + \delta)\} \\ \text{and the second half of (3.4.2) remains valid in this larger set.} \end{gathered} \tag{3.4.3}$$

We define the resonances z_j of P in $\mathbb{C} \setminus e^{-2i\theta_0}[0, \infty)$ as the eigenvalues of P on a suitable contour in $\mathbb{C}^n$, this set consists of the negative eigenvalues of P plus a discrete set in the sector $\{z \in \mathbb{C} \setminus \{0\} : -2\theta_0 < \arg z \leq 0\}$, see [49] and §3.5.

Example 1. Obstacle scattering. Suppose that $\mathcal{O} \subset \overline{B(0, R_0)}$ is an open set such that $\partial\mathcal{O}$ is a smooth hypersurface in $\mathbb{R}^n$ and that $\mathbb{R}^n \setminus \mathcal{O}$ is connected. Let $\mathcal{H} = L^2(\mathbb{R}^n \setminus \mathcal{O})$, and $P = -\Delta|_{\mathbb{R}^n \setminus \mathcal{O}}$ on the exterior domain realized with any self-adjoint boundary conditions on $\partial\mathcal{O}$. For instance, the Dirichlet boundary condition

$$\mathcal{D} = \{u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) : u|_{\partial\mathcal{O}} = 0\}$$

or the Neumann/Robin boundary condition

$$\mathcal{D} = \{u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) : \partial_\nu u + \eta u|_{\partial\mathcal{O}} = 0\}$$

where ∂_ν is the normal derivative with respect to $\partial\mathcal{O}$ and η is a real-valued smooth function on $\partial\mathcal{O}$. Theorem 2 shows that the eigenvalues of $P - i\varepsilon x^2$ converge to the resonances of P (the irrelevance of the missing $i\varepsilon\chi(x)x^2$ term comes from continuity of resonances under compactly supported perturbations – see Stefanov [52]).

Example 2. Scattering on asymptotically Euclidean space. Let M be a real analytic manifold which is diffeomorphic to $\mathbb{R}^n$ near infinity and equipped with a real analytic metric g which is asymptotically Euclidean. More precisely, let $g_{ij} = \delta_{ij} + h_{ij}$ be the metric tensor then we assume that $h_{ij}(x)$ extend analytically in x to

$$\{s\omega : \omega \in \mathbb{C}^n,\ \mathrm{dist}(\omega, \mathbb{S}^{n-1}) < \delta,\ s \in \mathbb{C},\ |s| > R,\ \arg s \in (-\delta, \theta_0 + \delta)\}$$

for some $\theta_0 \in [0, \pi/8]$, $\delta > 0$, $R \geq R_0$, and that $h_{ij} \to 0$ in this larger set. We put $P = -\Delta_g$, the Laplace–Beltrami operator with respect to the metric g, then all the black box assumptions are satisfied. Suppose that $\chi \in \mathcal{C}_c^\infty(M; [0,1])$ is equal to 1 near some compact set K and that $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus \overline{B(0, R_0)}$. Then the operator $-\Delta_g - i\varepsilon(1-\chi(x))x^2$ has a discrete spectrum for $\varepsilon > 0$ and the eigenvalues converge to the resonances of $-\Delta_g$ uniformly on compact subsets of $-2\theta_0 < \arg z < 3\pi/2 + 2\theta_0$.

Example 3. Scattering on finite volume surfaces. This example was already discussed in [65] but we will provide a complete proof in the black box setting. Consider the modular surface

$$M = SL_2(\mathbb{Z})\backslash\mathbb{H}^2$$

(or any surfaces with cusps – see [12, §4.1, Example 3]) equipped with the Poincaré metric g and $\Delta_M \leq 0$ the Laplacian on M. We choose the fundamental domain of $SL_2(\mathbb{Z})$ to be

$$\{x + iy \in \mathbb{H}^2 : |x| \leq 1/2, x^2 + y^2 \geq 1\}$$

then Δ_M in the cusp $y > 1$ is given by $y^2(\partial_x^2 + \partial_y^2)$. Let $r = \log y$, $\theta = 2\pi x$, then M in (r, θ) coordinates admits the following decomposition:

$$M = M_0 \cup M_1, \ (M_1, g|_{M_1}) = ([0, \infty)_r \times \mathbb{S}^1_\theta, \, dr^2 + (2\pi)^{-2}e^{-2r}d\theta^2), \ \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

We recall the black box setup in this case from [12, §4.1, Example 3]. Let

$$\mathcal{H} = \mathcal{H}_0 \oplus L^2([0, \infty), dr), \quad \mathcal{H}_0 = L^2(M_0) \oplus \mathcal{H}_0^0,$$

where (with $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$)

$$\mathcal{H}_0^0 = \left\{ \{a_n(r)\}_{n \in \mathbb{Z}^*} : a_n \in L^2([0, \infty)), \sum_{n \in \mathbb{Z}^*} \int_0^\infty |a_n(r)|^2 dr < \infty \right\}.$$

We can identify $L^2(M)$ with $\mathcal{H}$ via the following isomorphism:

$$\iota : L^2(M) \ni u \mapsto \left(u|_{M_0}, \{e^{-r/2}u_n(r)\}_{n \in \mathbb{Z}^*}, e^{-r/2}u_0(r)\right) \in \mathcal{H},$$
$$u_n(r) := \frac{1}{2\pi} \int_{\mathbb{S}^1} u(r, \theta) e^{-in\theta} d\theta, \quad r > 0.$$

Then $P := -\Delta_M - 1/4$ is a black box Hamiltonian on $\mathcal{H}$ which equals $-\partial_r^2$ on $L^2([0, \infty), dr)$ – see [12, §4.1, Example 3]. In the language of Theorem 2 and in (x, y) coordinates

$$P_\varepsilon = -\Delta_M - 1/4 - i\varepsilon(1 - \chi(y))(\log y)^2 \Pi_0, \quad \Pi_0 u(x, y) := \int_{-1/2}^{1/2} u(x', y) \, dx'.$$

where $\chi \in \mathcal{C}_c^\infty([0,\infty))$, $\chi(y) \equiv 1$ for $y < 2$ and $\chi(y) \equiv 0$ for $y > 3$. The eigenvalues of P_ε converge to the resonances of P uniformly on compact subsets of $\arg z > -\pi/4$. Equivalently if we define

$$s(\varepsilon) \in \Sigma_\varepsilon \Longleftrightarrow s(\varepsilon)(1 - s(\varepsilon)) - 1/4 \in \mathrm{Spec}(P_\varepsilon),$$

then the limit points of Σ_ε, $\varepsilon \to 0+$, in $\mathrm{Re}\, s < 1/2$, $\arg(s - 1/2) \neq 11\pi/8$ are given by the nontrivial zeros of $\zeta(2s)$ where ζ is the Riemann zeta function – see [65, Example 2] and [12, §4.4 Example 3].

3.5 The method of complex scaling

In this section we review the method of complex scaling. Complex scaling has been a standard technique in resonance theory since the works of Aguilar–Combes [2], Balslev–Combes [3] and Simon [48]. Here we follow rather closely the presentation in [50] since our assumptions on the operator P is weaker than [49].

A smooth submanifold $\Gamma \subset \mathbb{C}^n$ is said to be totally real if $T_x\Gamma \cap iT_x\Gamma = \{0\}$ for every $x \in \Gamma$, where we identify $T_x\Gamma$ with a real subspace of $T_x\mathbb{C}^n \simeq \mathbb{C}^n$. We say that Γ is maximally totally real if Γ is totally real and of maximal (real) dimension n, the natural example is $\Gamma = \mathbb{R}^n$. Let $\Gamma \subset \mathbb{C}^n$ be smooth and of real dimension n, then locally Γ can be represented using real coordinates: $\mathbb{R}^n \ni x \mapsto f(x) \in \Gamma$. Let $\tilde{f}$ be an almost analytic extension of f so that $\bar\partial\tilde{f}$ vanishes to infinite order on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$, then since $d\tilde{f}(x)$ is complex linear, $iT_{f(x)}\Gamma = d\tilde{f}(x)(iT_x\mathbb{R}^n)$. Hence Γ is totally real in a neighborhood of $f(x)$ if and only if $d\tilde{f}(x)$ is injective, i.e. $\det df(x) \neq 0$.

Let $\Omega \subset \mathbb{C}^n$ be an open neighborhood of Γ such that Γ is closed in Ω, and let

$$A(z, D_z) = \sum_{|\alpha| \leq m} a_\alpha(z) D_z^\alpha, \quad D_{z_j} := \frac{1}{i}\partial_{z_j}, \quad D_z^\alpha = D_{z_1}^{\alpha_1} \cdots D_{z_n}^{\alpha_n},$$

be a differential operator on Ω with holomorphic coefficients. Define $A_\Gamma : \mathcal{C}^\infty(\Gamma) \to \mathcal{C}^\infty(\Gamma)$ by

$$A_\Gamma u = (A\tilde{u})|_\Gamma, \tag{3.5.1}$$

where $\tilde{u}$ is an almost analytic extension of u, that is, a smooth extension of u to a neighborhood of Γ such that $\bar\partial\tilde{u}$ vanishes to infinite order on Γ. A_Γ is then a differential operator on Γ with smooth coefficients, and for the principal symbols we have

$$a_\Gamma = a|_{T^*\Gamma},$$

where a is the principal symbol of A.

We recall a deformation result from [49, Lemma 3.1]:

Lemma 2. *Suppose that $W \subset \mathbb{R}^n$ is open and that $F : [0,1] \times W \ni (s,x) \mapsto F(s,x) \in \mathbb{C}^n$, is a smooth proper map satisfying for all $s \in [0,1]$*

$$\det \partial_x F(s,x) \neq 0, \quad \textit{and } x \mapsto F(s,x) \textit{ is injective,}$$

and assume that $x \in W \setminus K \implies F(s,x) = F(0,x)$ for some compact $K \subset W$.

Let $A(z, D_z)$ be a differential operator with holomorphic coefficients defined in a neighborhood of $F([0,1] \times W)$ such that for $0 \leq s \leq 1$ and $\Gamma_s := F(\{s\} \times W)$, A_{Γ_s} is elliptic.

If $u_0 \in \mathcal{C}^\infty(\Gamma_0)$ and $A_{\Gamma_0} u_0$ extends to a holomorphic function in a neighborhood of $F([0,1] \times W)$, then the same holds for u_0.

The lemma will be applied to a family of deformations of $\mathbb{R}^n$ in $\mathbb{C}^n$. We aim to restrict the operators P_ε, $\varepsilon \geq 0$, to the corresponding totally real submanifolds. For given $\alpha_0 > 0$ and $R_1 > R_0$, we can construct a smooth function

$$[0, \theta_0] \times [0, \infty) \ni (\theta, t) \mapsto g_\theta(t) \in \mathbb{C},$$

injective for every θ, with the following properties:

1. $g_\theta(t) = t$ for $0 \leq t \leq R_1$,
2. $0 \leq \arg g_\theta(t) \leq \theta, \quad \partial_t g_\theta(t) \neq 0$,
3. $\arg g_\theta(t) \leq \arg \partial_t g_\theta(t) \leq \arg g_\theta(t) + \alpha_0$,
4. $g_\theta(t) = e^{i\theta} t$ for $t \geq T_0$, where T_0 depends only on α_0 and R_1.

We now define the totally real submanifolds, Γ_θ, as images of $\mathbb{R}^n$ under the maps

$$f_\theta : \mathbb{R}^n \ni x = t\omega \mapsto g_\theta(t)\omega \in \mathbb{C}^n, \ t = |x|.$$

Then a dilated operator $\mathcal{P}_\theta$ can be defined as follows. Let

$$\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0, R_0)), \tag{3.5.2}$$

where $B(0, R_0)$ denotes the real ball as before. If $\chi \in \mathcal{C}_c^\infty(B(0, R_1))$ is equal to 1 near $\overline{B(0, R_0)}$, we put

$$\mathcal{D}_\theta = \{u \in \mathcal{H}_\theta : \chi u \in \mathcal{D}, \ (1 - \chi)u \in H^2(\Gamma_\theta \setminus B(0, R_0))\}.$$

Let $\mathcal{P}_\theta$ be the unbounded operator $\mathcal{H}_\theta \to \mathcal{H}_\theta$ with domain $\mathcal{D}_\theta$, given by

$$\mathcal{P}_\theta u := P(\chi u) + Q_\theta((1 - \chi)u), \quad Q_\theta := -\sum_{j,k=1}^{n} (\partial_{z_j}(g^{jk}(z)\partial_{z_k}) + c(z))|_{\Gamma_\theta}. \tag{3.5.3}$$

These definitions do not depend on the choice of χ.

We recall some properties of the dilated Laplacian from [49, §3]. Let

$$\Delta_\theta := (\Delta_z)|_{\Gamma_\theta}, \quad x_\theta := z|_{\Gamma_\theta}.$$

Parametrizing Γ_θ by $[0,\infty)\times\mathbb{S}^{n-1} \ni (t,\omega) \mapsto g_\theta(t)\omega$, we obtain

$$-\Delta_\theta = (g_\theta'(t)^{-1}D_t)^2 - i(n-1)(g_\theta(t)g_\theta'(t))^{-1}D_t + g_\theta(t)^{-2}D_\omega^2, \tag{3.5.4}$$

where $D_t = -i\partial_t$ and $D_\omega^2 = -\Delta_{\mathbb{S}^{n-1}}$. If ω^{*2} denotes the principal symbol of D_ω^2 and we let τ be the dual variable of t, then the principal symbol of $-\Delta_\theta$ is

$$\sigma(-\Delta_\theta) = g_\theta'(t)^{-2}\tau^2 + g_\theta(t)^{-2}\omega^{*2},$$

so pointwise on Γ_θ, $-\Delta_\theta$ is elliptic and the principal symbol takes values in an angle of size $\le 2\alpha_0$, while globally, $\sigma(-\Delta_\theta)$ takes values in the sector $-2\theta - 2\alpha_0 \le \arg z \le 0$. The basic result based on ellipticity at infinity is

$$\begin{gathered}-2\theta+\delta < \arg z < 2\pi - 2\theta - \delta, \quad |z| > \delta \implies \\ (-\Delta_\theta - z)^{-1} = O_\delta(|z|^{\frac{j-2}{2}}) : L^2(\Gamma_\theta) \to H^j(\Gamma_\theta),\ j = 0,1,2.\end{gathered} \tag{3.5.5}$$

This follows from [49, Lemmas 3.2–3.5 and §4] applied with $P = -\Delta$.

$\mathcal{P}_\theta$, as a perturbation of $-\Delta_\theta$, is also elliptic – see [50, §5]. More precisely, choosing R_1 large enough, it follows from the assumptions (3.4.2) and (3.4.3) that

$$\begin{gathered}\text{In } \Gamma_\theta \setminus B(0,R_0),\ \mathcal{P}_\theta \text{ is an elliptic differential operator whose principal} \\ \text{symbol pointwise on } \Gamma_\theta \text{ takes its values in an angle of size} \le 3\alpha_0, \\ \text{and globally in a sector } -2\theta - 3\alpha_0 \le \arg z \le \alpha_0.\end{gathered} \tag{3.5.6}$$

$$\begin{gathered}\text{The coefficients of } \mathcal{P}_\theta - e^{-2i\theta}(-\Delta) \text{ tend to zero when } \Gamma_\theta \ni x \to \infty, \\ \text{where we identify } \Gamma_\theta \text{ and } \mathbb{R}^n, \text{ by means of } f_\theta.\end{gathered} \tag{3.5.7}$$

We recall some basic results about $\mathcal{P}_\theta$ from [50, §5]:

Lemma 3. *If $z \in \mathbb{C}\setminus\{0\}$, $\arg z \ne -2\theta$, then $\mathcal{P}_\theta - z : \mathcal{D}_\theta \to \mathcal{H}_\theta$ is a Fredholm operator of index 0. In particular the spectrum of $\mathcal{P}_\theta$ in $\mathbb{C}\setminus e^{-2i\theta}[0,\infty)$ is discrete.*

Proof. The first part of the lemma is the same as Lemma 7.3 in the lecture notes by Sjöstrand [51], the corresponding proof can be found there. It remains to show that $\mathcal{P}_\theta$ has a discrete spectrum in $\mathbb{C}\setminus e^{-2i\theta}[0,\infty)$. For that, let $z_0 = iL,\ L \ge 1$, we put

$$E(z_0) = \tilde{\chi}_1(P - z_0)^{-1}\chi_1 + (1-\chi_0)(-\Delta_\theta - z_0)^{-1}(1-\chi_1), \tag{3.5.8}$$

where $\chi_1 \in \mathcal{C}_c^\infty(B(0,R_1))$ is equal to 1 near $\operatorname{supp}\chi_0$ and $\chi_0 = 1$ on $B(0,R_1-\delta)$, for some $\delta > 0$ small. Then we have

$$(\mathcal{P}_\theta - z_0)E(z_0) = I + K(z_0) + K_1(z_0),$$

where

$$K(z_0) = [P,\tilde{\chi}_1](P-z_0)^{-1}\chi_1 + [\Delta_\theta,\chi_0](-\Delta_\theta - z_0)^{-1}(1-\chi_1),$$
$$K_1(z_0) = (\mathcal{P}_\theta - (-\Delta_\theta))(1-\chi_0)(-\Delta_\theta - z_0)^{-1}(1-\chi_1).$$

Choosing R_1 sufficiently large, we may assume by (3.5.5) and (3.5.7) that

$$\|K_1(z_0)\|_{\mathcal{H}_\theta\to\mathcal{H}_\theta} \leq 1/2, \quad \forall\, z_0 = iL,\ L \geq 1.$$

Then we get

$$(\mathcal{P}_\theta - z_0)E(z_0)(I + K_1(z_0))^{-1} = I + K(z_0)(I+K_1(z_0))^{-1}.$$

It follows from (3.5.5) that $K(iL) = O(L^{-1/2}) : \mathcal{H}_\theta \to \mathcal{H}_\theta$, then for $z_0 = iL,\ L \gg 1$,

$$\|K(z_0)(I+K_1(z_0))^{-1}\|_{\mathcal{H}_\theta\to\mathcal{H}_\theta} \leq 1/2,$$

thus $\mathcal{P}_\theta - z_0$ has a right inverse:

$$E(z_0)(I+K_1(z_0))^{-1}(I + K(z_0)(I+K_1(z_0))^{-1})^{-1},$$

which implies that $\mathcal{P}_\theta - z_0$ is surjective. Since $\mathcal{P}_\theta - z_0$ is a Fredholm operator of index 0, it must also be injective. Hence by the inverse mapping theorem, $\mathcal{P}_\theta - z_0$ is invertible and we have

$$(\mathcal{P}_\theta - z_0)^{-1} = E(z_0)(I+K_1(z_0))^{-1}(I + K(z_0)(I+K_1(z_0))^{-1})^{-1}. \tag{3.5.9}$$

Theorem 4 then shows that $\mathcal{P}_\theta$ has a discrete spectrum. □

Lemma 4. *Assume that $0 \leq \theta_1 < \theta_2 \leq \theta_0$ and let $z_0 \in \mathbb{C} \setminus e^{-2i[\theta_1,\theta_2]}[0,\infty)$. Then*

$$\dim\ker(\mathcal{P}_{\theta_1} - z_0) = \dim\ker(\mathcal{P}_{\theta_2} - z_0).$$

This is identical to [49, Lemma 3.4] and the proof is the same as there using Lemma 2.

Lemma 4 shows that the spectrum in the sector $-2\theta_0 < \arg z \leq 0$ is independent of θ in the following sense: We say that $z \in \mathbb{C}\setminus\{0\}$, $-2\theta_0 < \arg z \leq 0$ is a resonance for P if and only if $z \in \operatorname{Spec}(\mathcal{P}_\theta)$ with $-2\theta < \arg z \leq 0$ for some $\theta \in (0,\theta_0]$. For such a resonance $z_0 \in e^{-2i[0,\theta)}(0,\infty)$, the spectral projection

$$\Pi_\theta(z_0) = \frac{1}{2\pi i}\oint_{z_0}(z-\mathcal{P}_\theta)^{-1}dz, \tag{3.5.10}$$

where the integral is over a positively oriented circle enclosing z_0 and containing no resonances other than z_0, is of finite rank. The restriction of $\mathcal{P}_\theta - z_0$ to $\operatorname{Ran}\Pi_\theta(z_0)$ is nilpotent. If $\tilde{\theta} \in [0,\theta_0]$ is a second number with $z_0 \in e^{-2i[0,\tilde{\theta})}(0,\infty)$, then since Lemma 4 can be extended to $\dim\ker(\mathcal{P}_\theta - z_0)^k = \dim\ker(\mathcal{P}_{\tilde{\theta}} - z_0)^k$ for all k, $\Pi_\theta(z_0)$ and $\Pi_{\tilde{\theta}}(z_0)$ have the same rank, which by definition is the multiplicity of the resonance z_0:

$$m(z_0) := \operatorname{rank}\Pi_\theta(z_0), \quad -2\theta < \arg z_0 \leq 0. \tag{3.5.11}$$

3.6 Agmon's perturbation theory of resonances

In this section we adapt Agmon's perturbation theory of resonances [1] to study resonances in obstacle scattering, in which resonances are realized as eigenvalues of a non-self-adjoint operator on an abstractly constructed Banach space. We remark that the method of complex scaling is also capable of characterizing resonances in $\mathbb{C}$ in odd dimensions and resonances in the logarithmic plane Λ in even dimensions with small argument, but Agmon's method allows us to prove the generic simplicity of all resonances in the whole Λ in even dimensions.

Let $\mathcal{O} \subset \mathbb{R}^n$ be an obstacle and $\Delta_{\mathcal{O}}$ be the corresponding self-adjoint Dirichlet Laplacian on $\mathbb{R}^n \setminus \mathcal{O}$, see §1.3.3 for the definitions. We note that $-\Delta_{\mathcal{O}}$ is a black box Hamiltonian reviewed in §3.3, whose resolvent admits a meromorphic continuation by Proposition 3.3.1. More precisely, for any obstacle $\mathcal{O}$ and constant $A > 0$ let

$$B_0 := e^{-A|x|}L^2(\mathbb{R}^n \setminus \mathcal{O}), \quad B_1 := e^{A|x|}L^2(\mathbb{R}^n \setminus \mathcal{O}),$$

the resolvent of $-\Delta_{\mathcal{O}}$ extends to a meromorphic family

$$(-\Delta_{\mathcal{O}} - \lambda^2)^{-1} : B_0 \to \mathcal{D}_1(\mathcal{O}) \subset B_1, \quad \lambda \in T_A \text{ when } n \text{ odd}, \ \lambda \in \Lambda_A \text{ when } n \text{ even},$$

where $\mathcal{D}_1(\mathcal{O})$ is the same as (3.3.8) except that B_1 replaces $\mathcal{H}_1$ there:

$$\mathcal{D}_1(\mathcal{O}) = \{u \in B_1 \cap H^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{O}) : u|_{\partial\mathcal{O}} = 0, \ \Delta u \in B_1\}, \tag{3.6.1}$$

and T_A, Λ_A are given by (3.3.9), (3.3.10). The poles in this meromorphic continuation are called scattering resonances for the obstacle $\mathcal{O}$.

We recall the following facts from [12, Theorem 4.19] (for n odd) and Christiansen [7, §6] (for n even) that:

$$\begin{gathered} 0 \text{ is not a resonance when } n \text{ is odd;} \\ 0 \text{ is not a limit point of resonances when } n \text{ is even.} \end{gathered} \tag{3.6.2}$$

Therefore, resonances lie in $T_A \setminus i[0,\infty)$ when n is odd. We consider the map:

$$T_A \setminus i[0,\infty) \ni \lambda = re^{i\theta} \mapsto z = r^2 e^{2i\theta} = \lambda^2 \in \Lambda,$$

which is invertible. Throughout this section, we will replace parameter λ by z with $z = \lambda^2$ defined above. We introduce the image of $T_A \setminus i[0,\infty)$ or Λ_A under this map:

$$D_+ := \begin{cases} \{\lambda^2 \in \Lambda : \lambda \in T_A \setminus \{0\}, \ -\frac{3\pi}{2} < \arg\lambda < \frac{\pi}{2}\} & \text{when } n \text{ is odd;} \\ \{z : 0 < \arg z < 2\pi\} \cup \{z \in S_{2\lfloor A\rfloor} : |z| < A^2\} & \text{when } n \text{ is even.} \end{cases} \tag{3.6.3}$$

We write the resolvent of $\Delta_{\mathcal{O}}$ as follows:

$$\mathcal{R}(z) := (-\Delta_{\mathcal{O}} - z)^{-1} : B_0 \to B_1, \quad z \in D_+,$$

which is a meromorphic family by Proposition 3.3.1. We denote by $\mathrm{Res}(\mathcal{O})$, the poles of $\mathcal{R}(z)$, $z \in D_+$, which is also the image of the resonances under the map $\lambda \mapsto z = \lambda^2$.

We note that $-\Delta_{\mathcal{O}}$ as an operator acting on B_1 is closable. Denote by P_1 the closure of $-\Delta_{\mathcal{O}}$ in B_1, by (3.6.1) we have

$$P_1 = -\Delta : B_1 \to B_1 \quad \text{with domain } \mathcal{D}_1(\mathcal{O}). \tag{3.6.4}$$

Let us take $B = L^2(\mathbb{R}^n \setminus \mathcal{O})$, $D = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Then one can check that $P := -\Delta_{\mathcal{O}}$ satisfies the abstract hypotheses of Agmon's theory:

Hypothesis 3.6.1. *(i) P is a closed, densely defined operator acting in some Banach space B;*

(ii) The resolvent $(P - z)^{-1}$ is a meromorphic family of operators in $\mathcal{L}(B)$ for $z \in D$;

(iii) There are two reflexive Banach spaces B_0 and B_1 such that $B_0 \subset B \subset B_1$;

(iv) P as an operator on B_1 is closable, and denoting the closure of P in B_1 by P_1, the resolvent $(P_1 - z)^{-1}$ exists for some $z \in D$ as an operator in $\mathcal{L}(B_1)$;

(v) The resolvent $(P - z)^{-1}$ admits a meromorphic continuation from D to D_+ as an operator in $\mathcal{L}(B_0, B_1)$.

(iv) can be seen from the following calculation

$$e^{-A|x|}(-\Delta - z)e^{A|x|} = -\Delta - \frac{2Ax}{|x|} \cdot \nabla - \frac{(n-1)A}{|x|} - A^2 - z,$$

which is invertible as an operator from $\mathcal{D}(\Delta_{\mathcal{O}})$ to $L^2(\mathbb{R}^n \setminus \mathcal{O})$ for $z \in D$, $\operatorname{Im} z \gg A^2$.

Now we fix a resonance $z_0 \in \mathrm{Res}(\mathcal{O}) \subset D_+$, $z_0 \neq 0$ then choose Σ, a bounded domain containing z_0, with a C^1 boundary Γ, satisfying

$$(i)\ \overline{\Sigma} \subset D_+; \quad (ii)\ \Gamma \cap \mathrm{Res}(\mathcal{O}) = \emptyset; \quad (iii)\ \Sigma \cap D \neq \emptyset. \tag{3.6.5}$$

Having chosen Σ we denote by B_Γ, the subspace of B_1 consisting of elements f, admitting a representation of the form:

$$f = g + \int_\Gamma \mathcal{R}(\zeta)\varphi(\zeta)d\zeta, \quad g \in B_0,\ \varphi \in C(\Gamma; B_0), \tag{3.6.6}$$

We recall [1] that B_Γ is a Banach space with the norm

$$\|f\|_{B_\Gamma} := \inf_{g,\varphi}(\|g\|_{B_0} + \|\varphi\|_{C(\Gamma;B_0)}) \tag{3.6.7}$$

where the infimum is taken over all $g \in B_0$ and $\varphi \in C(\Gamma; B_0)$ which verify (3.6.6). Then $B_0 \subset B_\Gamma \subset B_1$ are continuous inclusions. Agmon [1] also introduced a linear operator $R_\Gamma(z)$ on B_Γ associated to any $z \in \Sigma \setminus \mathrm{Res}(\mathcal{O})$,

$$R_\Gamma(z)f := \mathcal{R}(z)g + \int_\Gamma (\zeta - z)^{-1}(\mathcal{R}(\zeta) - \mathcal{R}(z))\varphi(\zeta)d\zeta, \tag{3.6.8}$$

where $f \in B_\Gamma$ is given by (3.6.6). Under Hypothesis 3.6.1, Agmon [1] showed that $R_\Gamma(z)$ is a well-defined operator in $\mathcal{L}(B_\Gamma)$, which is actually the resolvent of an operator P_Γ acting on B_Γ:

$$R_\Gamma(z) = (P_\Gamma - z)^{-1} \quad \text{for} \quad z \in \Sigma \setminus \mathrm{Res}(\mathcal{O}), \tag{3.6.9}$$

where P_Γ is closed linear operator in B_Γ defined as follows:

$$\mathcal{D}(P_\Gamma) = \mathrm{Ran}\, R_\Gamma(w_0), \quad P_\Gamma u = w_0 u + f \tag{3.6.10}$$

for $u = R_\Gamma(w_0)f \in \mathcal{D}(P_\Gamma)$, $f \in B_\Gamma$. Here w_0 is a fixed point in $\Sigma \cap D$. Moreover, P_1 extends P_Γ in the sense that

$$\mathcal{D}(P_\Gamma) \subset \mathcal{D}_1(\mathcal{O}), \quad P_\Gamma u = P_1 u \quad \text{for } u \in \mathcal{D}(P_\Gamma), \tag{3.6.11}$$

where $\mathcal{D}(P_\Gamma) \subset \mathcal{D}_1(\mathcal{O})$ is continuous if they are equipped with the graph norms:

$$\|u\|_{\mathcal{D}(P_\Gamma)} := \|u\|_{B_\Gamma} + \|P_\Gamma u\|_{B_\Gamma}; \quad \|u\|_{\mathcal{D}_1(\mathcal{O})} := \|u\|_{B_1} + \|\Delta u\|_{B_1}.$$

We recall from [1] the following properties that relate P_Γ to $-\Delta_\mathcal{O}$:

Proposition 3.6.2. *P_Γ has a discrete spectrum in Σ, given by $\mathrm{Res}(\mathcal{O}) \cap \Sigma$. Furthermore, let $z_0 \in \mathrm{Res}(\mathcal{O}) \cap \Sigma$ be an eigenvalue of P_Γ, $\mathcal{E}_\Gamma(z_0)$ denote the generalized eigenspace of P_Γ at z_0, then*

$$\mathcal{E}_\Gamma(z_0) := \left(\oint_{z_0} (P_\Gamma - \zeta)^{-1} d\zeta \right)(B_\Gamma) = \left(\oint_{z_0} \mathcal{R}(\zeta) d\zeta \right)(B_0), \tag{3.6.12}$$

where the integral is over a circle containing no other resonance than z_0. In particular, the multiplicity of $z_0 \in \mathrm{Spec}(P_\Gamma)$ satisfies

$$m_\Gamma(z_0) := \dim \mathcal{E}_\Gamma(z_0) = m_\mathcal{O}(\lambda_0), \quad \text{with } z_0 = \lambda_0^2. \tag{3.6.13}$$

Let us turn to the perturbation theory for resonances. Let Ω be an open neighborhood of the origin in $\mathbb{C}$. We assume the following:

Hypothesis 3.6.3. *There exists a family of linear operators $V(t) : \mathcal{D}_1(\mathcal{O}) \to B_0$, $t \in \Omega$, with $V(0) = 0$, such that*

1. $\|V(t)u\|_{B_0} = \mathcal{O}(t)\|u\|_{D_1(\mathcal{O})}$, $\forall\, u \in D_1(\mathcal{O})$ *as* $\Omega \ni t \to 0$;

2. *$\Omega \ni t \mapsto V(t)u$ is an analytic B_0-valued function, for any $u \in D_1(\mathcal{O})$.*

Then we consider a family of operators on B_Γ:

$$P_\Gamma(t) = P_\Gamma + V(t), \quad \text{with domain } \mathcal{D}(P_\Gamma(t)) := \mathcal{D}(P_\Gamma). \tag{3.6.14}$$

Since P_Γ is closed, it follows from the bound in Hypothesis 3.6.3 and a well-known result by Kato [32] that $P_\Gamma(t)$ is also closed in B_Γ provided $|t|$ sufficiently small. Shrinking Ω if necessary, we assume from now on that $P_\Gamma(t)$ is a closed operator for all $t \in \Omega$, then we can apply analytic perturbation theory to the eigenvalues of $P_\Gamma(t)$ – see [32, Chapter VII, §1] for a full treatment.

Fixing a resonance $z_0 \in \mathrm{Res}(\mathcal{O}) \cap \Sigma$, we choose $\Sigma' \Subset \Sigma$ with $z_0 \in \Sigma'$, z_0 is also an eigenvalue of P_Γ by Proposition 3.6.2. We recall the following perturbation result from [1, Theorem 7.4]:

Proposition 3.6.4. *There exists an open neighborhood of $0 \in \mathbb{C}$, $\Omega_0 \subset \Omega$, such that*

(i) for each $t \in \Omega_0$, $P_\Gamma(t)$ has a discrete spectrum in Σ';

(ii) the spectrum of $P_\Gamma(t)$ depends analytically on t in the following sense: for each $t \in \Omega_0$, there exist a polynomial $q_t^\Gamma(z)$, of degree independent of t, with coefficients analytic in t, such that the zeros of $q_t^\Gamma(z)$ in Σ' coincide with the eigenvalues of $P_\Gamma(t)$ in Σ', with agreement of multiplicities.

Shrinking Ω_0 if necessary, we may assume by Hypothesis 3.6.3 that

$$\mathcal{R}(z,t) := (-\Delta_{\mathcal{O}} + V(t) - z)^{-1} : B_0 \to \mathcal{D}_1(\mathcal{O})$$

exists for $\operatorname{Im} z > c > 0$ for all $t \in \Omega_0$. It was shown in [1, Theorem 7.5] that for any $t \in \Omega_0$, $\mathcal{R}(z,t)$ admits a meromorphic continuation with poles of finite rank to $z \in \Sigma'$ given by

$$\mathcal{R}(z,t)f = (P_\Gamma(t) - z)^{-1} f, \quad \forall\, f \in B_0,\ z \in \Sigma' \setminus \operatorname{Spec}(P_\Gamma(t)).$$

The connection between the poles of $z \mapsto \mathcal{R}(z,t)$ and the eigenvalues of $P_\Gamma(t)$ was established in [1, Theorem 7.7], which shows that these two discrete sets are identical, more precisely, the multiplicity of z_t as an eigenvalue of $P_\Gamma(t)$ equals its rank as a pole of $\mathcal{R}(z,t)$. This correspondence and Proposition 3.6.4 yield the following perturbation result for resonances – see [1, Proposition 8.1]:

Proposition 3.6.5. *Suppose that the multiplicity of resonance z_0 equals m. Let $K \subset \Sigma'$ be any disc centered at z_0 containing no other resonances. Then there exists a neighborhood of $0 \in \mathbb{C}$, $\Omega_0' \subset \Omega_0$, such that for any $t \in \Omega_0'$,*

(i) The total rank of the poles of $\mathcal{R}(z,t)$ in K is equal to m.

(ii) *Denote by* $z_1(t), \ldots, z_m(t)$ *the poles of* $\mathcal{R}(z,t)$ *in* K*, each repeated with respect to its rank. Then* $z_j(t) \to z_0$ *as* $t \to 0$, $j = 1, \ldots, m$.

(iii) *The average* $\hat{z}(t) := m^{-1} \sum_{j=1}^{m} z_j(t)$ *is an analytic function of* t *in* Ω_0'.

Chapter 4

Boundary perturbation in obstacle scattering

4.1 Deformation of obstacle

In this section we study the deformation of obstacle and the corresponding deformed Dirichlet Laplacian. To describe the deformation of obstacle, we follow Pereira [42] and introduce a set of C^k-smooth mappings ($k \geq 2$) which deforms the obstacle $\mathcal{O}$:

$$\mathrm{Diff}(\mathcal{O}) := \left\{ \begin{matrix} \Phi \in C^k(\mathbb{R}^n; \mathbb{R}^n) \text{ is a } C^k\text{-diffeomorphism}: \ \Phi(\partial\mathcal{O}) = \partial\,\Phi(\mathcal{O}), \\ \text{and } \Phi(x) = x,\ \forall\,|x| > R, \quad \text{for some } R > 0. \end{matrix} \right\}. \tag{4.1.1}$$

For every $\Phi \in \mathrm{Diff}(\mathcal{O})$, $\Phi(\mathcal{O})$ is a deformed obstacle satisfying all the requirements in §1.3.3, thus we can define the Dirichlet Laplacian $\Delta_{\Phi(\mathcal{O})}$. We conjugate $\Delta_{\Phi(\mathcal{O})}$ by the pullback Φ^*. This will transform the deformed domain $\mathbb{R}^n \setminus \Phi(\mathcal{O})$ to the original one. As a result, the variation is transferred to the coefficients of the newly-defined differential operator. For $\mathcal{O}$, an obstacle, and $\Phi \in \mathrm{Diff}(\mathcal{O})$ given in (4.1.1), the pullback Φ^* is a bounded operator from $L^2(\mathbb{R}^n \setminus \Phi(\mathcal{O}))$ to $L^2(\mathbb{R}^n \setminus \mathcal{O})$, which is invertible with the inverse $(\Phi^{-1})^*$. In view of (1.3.2), the restricted map $\Phi^* : \mathcal{D}(\Delta_{\Phi(\mathcal{O})}) \to \mathcal{D}(\Delta_{\mathcal{O}})$ is also invertible with the inverse $(\Phi^{-1})^*$, since Φ^* preserves the Dirichlet boundary condition. Hence we can define the deformed operator $\Delta_{\mathcal{O}}^{\Phi}$ of $\Delta_{\mathcal{O}}$ associated to the deformation Φ:

$$\Delta_{\mathcal{O}}^{\Phi} := \Phi^* \Delta_{\Phi(\mathcal{O})} (\Phi^{-1})^* : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O}), \quad \text{with domain } \mathcal{D}(\Delta_{\mathcal{O}}). \tag{4.1.2}$$

Let $J_{\Phi}^{ij}(x)$ denote $[D\Phi(x)^{-1}]_{ij}$, by a direct calculation we have

$$\Phi^* \Delta (\Phi^{-1})^* = \sum_{ij\ell} J_{\Phi}^{i\ell} J_{\Phi}^{j\ell} \partial^2_{x_i x_j} + \sum_{ij\ell mq} (\partial^2_{x_i x_\ell} \Phi^m) J_{\Phi}^{jm} J_{\Phi}^{iq} J_{\Phi}^{\ell q} \partial_{x_j},$$

where $\Phi^m(x)$ is the m-th component of $\Phi(x) = (\Phi^1(x), \cdots, \Phi^n(x))$. Now we define

$$\begin{aligned} V &:= \Delta - \Phi^*\Delta(\Phi^{-1})^* = \sum_{i,j} a_{ij}(x)\partial^2_{x_i x_j} + \sum_j b_j(x)\partial_{x_j}, \\ \text{where} \quad a_{ij} &= \delta_{ij} - \sum_\ell J_\Phi^{i\ell} J_\Phi^{j\ell}, \quad b_j = -\sum_{i\ell mq} (\partial^2_{x_i x_\ell}\Phi^m) J_\Phi^{jm} J_\Phi^{iq} J_\Phi^{\ell q}, \end{aligned} \tag{4.1.3}$$

then by (4.1.1) we obtain that for all $1 \le i, j \le n$,

$$a_{ij} \in C_c^{k-1}(\mathbb{R}^n), \; b_j \in C_c^{k-2}(\mathbb{R}^n), \quad \|a_{ij}\|_\infty, \|b_j\|_\infty \le C\|\Phi - \mathrm{id}\,\|_{C^2}. \tag{4.1.4}$$

We note that $-\Delta_{\Phi(\mathcal{O})}$ is a self-adjoint black box Hamiltonian, whose resolvent admits a meromorphic continuation by Proposition 3.3.1. More precisely,

$$(-\Delta_{\Phi(\mathcal{O})} - \lambda^2)^{-1} : e^{-A|x|}L^2(\mathbb{R}^n \setminus \Phi(\mathcal{O})) \to \mathcal{D}_1(\Phi(\mathcal{O})),$$

is a meromorphic family of operators for $\lambda \in \mathbb{C}$ when n is odd, $\lambda \in \Lambda$ when n is even. Here $\mathcal{D}_1(\Phi(\mathcal{O}))$ is defined as in (3.6.1). Since Φ^* gives isomorphisms between

$$\mathcal{D}_1(\Phi(\mathcal{O})) \overset{\Phi^*}{\cong} \mathcal{D}_1(\mathcal{O}), \quad e^{-A|x|}L^2(\mathbb{R}^n \setminus \Phi(\mathcal{O})) \overset{\Phi^*}{\cong} e^{-A|x|}L^2(\mathbb{R}^n \setminus \mathcal{O})$$

respectively, it follows from (4.1.2) that the resolvent of $-\Delta_\mathcal{O}^\Phi$ also has a meromorphic continuation given by

$$(-\Delta_\mathcal{O}^\Phi - \lambda^2)^{-1} = \Phi^*(-\Delta_{\Phi(\mathcal{O})} - \lambda^2)^{-1}(\Phi^{-1})^* = \Phi^* R_{\Phi(\mathcal{O})}(\lambda)(\Phi^{-1})^*, \tag{4.1.5}$$

whose poles, denoted by $\mathrm{Res}(-\Delta_\mathcal{O}^\Phi)$, coincide, with agreement of multiplicities, with the resonances of $\Phi(\mathcal{O})$.

4.2 Agmon's theory and boundary perturbation

In this section we consider Agmon's theory for the deformed operators $-\Delta_\mathcal{O}^\Phi$. It follows from (4.1.2) and (4.1.3) that $-\Delta_\mathcal{O}^\Phi$ is also closable on B_1 with the closure

$$P_1^\Phi := \Phi^*(-\Delta)(\Phi^{-1})^* = -\Delta + V : B_1 \to B_1 \quad \text{with domain } \mathcal{D}_1(\mathcal{O}). \tag{4.2.1}$$

Thus $-\Delta_\mathcal{O}^\Phi$ also satisfies the abstract hypotheses of Agmon's theory reviewed in §3.6.

For a fixed domain Σ with boundary Γ satisfying (3.6.5), by Proposition 3.3.1 we can assume that

$$\sup_{\zeta \in \Gamma} \|\mathcal{R}(\zeta)\|_{\mathcal{L}(B_0, \mathcal{D}_1(\mathcal{O}))} < C_\Gamma \quad \text{for some constant } C_\Gamma > 0, \tag{4.2.2}$$

where $\mathcal{D}_1(\mathcal{O})$ is equipped with the graph norm:

$$\|u\|_{\mathcal{D}_1(\mathcal{O})} = \|u\|_{B_1} + \|\Delta u\|_{B_1}.$$

Since V defined in (4.1.3) is a second order differential operator with compactly supported coefficients, it can be viewed as an operator in $\mathcal{L}(\mathcal{D}_1(\mathcal{O}), B_0)$ satisfying

$$\|V\|_{\mathcal{L}(\mathcal{D}_1(\mathcal{O}),B_0)} \leq C\|\Phi - \mathrm{id}\,\|_{C^2}, \tag{4.2.3}$$

there exists $\delta_\Gamma > 0$ sufficiently small such that for all $\zeta \in \Gamma$,

$$\|\Phi - \mathrm{id}\,\|_{C^2} < \delta_\Gamma \implies \|V\mathcal{R}(\zeta)\|_{\mathcal{L}(B_0,B_0)} < 1/2, \tag{4.2.4}$$

which guarantees that $I+V\mathcal{R}(\zeta) : B_0 \to B_0$ is invertible by a Neumann series argument. Thus we have

$$\mathcal{R}_\Phi(\zeta) := (-\Delta_\mathcal{O}^\Phi - \zeta)^{-1} = \mathcal{R}(\zeta)(I + V\mathcal{R}(\zeta))^{-1}, \quad \zeta \in \Gamma, \tag{4.2.5}$$

which can be justified first for ζ near $\Gamma \cap \{z : 0 < \mathrm{Im}\, z < \pi\}$ and then by meromorphic continuation. In particular, $\Gamma \cap \mathrm{Res}(-\Delta_\mathcal{O}^\Phi) = \emptyset$. Hence for the same domain Σ with boundary Γ, we can define $B_{\Gamma,\Phi}$, $R_{\Gamma,\Phi}$ and $P_{\Gamma,\Phi}$ for the deformed operator $-\Delta_\mathcal{O}^\Phi$, as in (3.6.6), (3.6.8) and (3.6.10) with $\mathcal{R}(\zeta)$ replaced by $\mathcal{R}_\Phi(\zeta)$.

Now we explore the relationships between $B_{\Gamma,\Phi}$, $R_{\Gamma,\Phi}$, $P_{\Gamma,\Phi}$ and B_Γ, R_Γ, P_Γ. Assuming that $\|\Phi - \mathrm{id}\,\|_{C^2} < \delta_\Gamma$, by (4.2.5) we have for any $f \in B_\Gamma$,

$$f = g + \int_\Gamma \mathcal{R}(\zeta)\varphi(\zeta)d\zeta = g + \int_\Gamma \mathcal{R}_\Phi(\zeta)(I + V\mathcal{R}(\zeta))\varphi(\zeta)d\zeta.$$

Since $(I + V\mathcal{R}(\zeta))\varphi(\zeta) \in C(\Gamma; B_0)$, $f \in B_{\Gamma,\Phi}$ thus we have $B_\Gamma \subset B_{\Gamma,\Phi}$. Furthermore, (4.2.4) implies that

$$\|g\|_{B_0} + \|(I + V\mathcal{R}(\zeta))\varphi(\zeta)\|_{C(\Gamma;B_0)} \leq \frac{3}{2}(\|g\|_{B_0} + \|\varphi\|_{C(\Gamma;B_0)}),$$

by taking the infimum as in (3.6.7), we obtain that $\|f\|_{B_{\Gamma,\Phi}} \leq 3/2\,\|f\|_{B_\Gamma}$. Similarly, for $f \in B_{\Gamma,\Phi}$ we have

$$f = g + \int_\Gamma \mathcal{R}_\Phi(\zeta)\varphi(\zeta)d\zeta = g + \int_\Gamma \mathcal{R}(\zeta)(I + V\mathcal{R}(\zeta))^{-1}\varphi(\zeta)d\zeta \in B_\Gamma,$$

and again by (4.2.4) we can deduce that $\|f\|_{B_\Gamma} \leq 2\,\|f\|_{B_{\Gamma,\Phi}}$. Therefore,

$$B_{\Gamma,\Phi} = B_\Gamma,\ \|\cdot\|_{B_{\Gamma,\Phi}} \text{ and } \|\cdot\|_{B_\Gamma} \text{ are equivalent}, \quad \text{if } \|\Phi - \mathrm{id}\,\|_{C^2} < \delta_\Gamma. \tag{4.2.6}$$

Henceforth, we identify $B_{\Gamma,\Phi}$ with B_Γ. Suppose that $f = g + \int_\Gamma \mathcal{R}_\Phi(\zeta)\varphi(\zeta)d\zeta \in B_\Gamma$, then for w_0 chosen in (3.6.10), in view of (3.6.8) and (4.2.5) we have

$$\begin{aligned} R_{\Gamma,\Phi}(w_0)f &= \mathcal{R}_\Phi(w_0)g + \int_\Gamma (\zeta - w_0)^{-1}(\mathcal{R}_\Phi(\zeta) - \mathcal{R}_\Phi(w_0))\varphi(\zeta)d\zeta \\ &= \mathcal{R}(w_0)g_1 + \int_\Gamma (\zeta - w_0)^{-1}(\mathcal{R}(\zeta) - \mathcal{R}(w_0))\varphi_1(\zeta)d\zeta, \end{aligned}$$

where $\varphi_1(\zeta) := (I + V\mathcal{R}(\zeta))^{-1}\varphi(\zeta) \in C(\Gamma; B_0)$ and

$$g_1 := (I + V\mathcal{R}(w_0))^{-1}g + \int_\Gamma \frac{(I + V\mathcal{R}(\zeta))^{-1} - (I + V\mathcal{R}(w_0))^{-1}}{\zeta - w_0}\varphi(\zeta)d\zeta \in B_0.$$

Thus $R_{\Gamma,\Phi}(w_0)f = R_\Gamma(w_0)f_1$ for $f_1 := g_1 + \int_\Gamma \mathcal{R}(\zeta)\varphi_1(\zeta)d\zeta \in B_\Gamma$, which implies that $\operatorname{Ran} R_{\Gamma,\Phi}(w_0) \subset \operatorname{Ran} R_\Gamma(w_0)$. We can also derive that $\operatorname{Ran} R_\Gamma(w_0) \subset \operatorname{Ran} R_{\Gamma,\Phi}(w_0)$ by similar arguments. Therefore, recalling (3.6.10) we obtain that

$$\mathcal{D}(P_{\Gamma,\Phi}) := \operatorname{Ran} R_{\Gamma,\Phi}(w_0) = \operatorname{Ran} R_\Gamma(w_0) = \mathcal{D}(P_\Gamma).$$

We recall [1] that P_1^Φ extends $P_{\Gamma,\Phi}$ as in (3.6.11), then for any $u \in \mathcal{D}(P_\Gamma)$, (4.2.1) and (3.6.11) imply that

$$P_{\Gamma,\Phi}u = P_1^\Phi u = P_1 u + Vu = P_\Gamma u + Vu \tag{4.2.7}$$

Hence $P_{\Gamma,\Phi}$ and P_Γ are related as follows

$$P_{\Gamma,\Phi} = P_\Gamma + V : B_\Gamma \to B_\Gamma \quad \text{with domain } \mathcal{D}(P_\Gamma). \tag{4.2.8}$$

Now we substitute P_Γ by $P_{\Gamma,\Phi}$ in Proposition 3.6.2 and recall (4.1.5) to conclude:

Proposition 4.2.1. *Let Σ with boundary Γ be chosen as in (3.6.5) and suppose that $\Phi \in \operatorname{Diff}(\mathcal{O})$ satisfies $\|\Phi - \mathrm{id}\,\|_{C^2} < \delta_\Gamma$ for some $\delta_\Gamma > 0$ in (4.2.4), then $P_{\Gamma,\Phi}$ has a discrete spectrum in Σ, given by $\operatorname{Res}(\Phi(\mathcal{O})) \cap \Sigma$.*

Furthermore, let $z \in \operatorname{Res}(\Phi(\mathcal{O})) \cap \Sigma$ be an eigenvalue of $P_{\Gamma,\Phi}$, denote by $\mathcal{E}_{\Gamma,\Phi}(z)$ the generalized eigenspace of $P_{\Gamma,\Phi}$ at z, then

$$\mathcal{E}_{\Gamma,\Phi}(z) := \left(\oint_z (P_{\Gamma,\Phi} - \zeta)^{-1} d\zeta\right)(B_\Gamma) = \left(\oint_z \mathcal{R}_\Phi(\zeta)d\zeta\right)(B_0) \tag{4.2.9}$$

where the integral is over a circle containing no other resonance than z. In particular, the multiplicity of $z \in \operatorname{Spec} P_{\Gamma,\Phi}$ satisfies

$$m_{\Gamma,\Phi}(z) := \dim \mathcal{E}_{\Gamma,\Phi}(z) = m_{\Phi(\mathcal{O})}(\lambda), \quad \text{with } z = \lambda^2. \tag{4.2.10}$$

4.3 Generic simplicity of resonances in obstacle scattering

We will follow the strategy of [33] and [6] in the case of potential perturbations to prove Theorem 3. However we have to overcome the additional difficulties produced by boundary perturbations using the results obtained in §4.1 and §4.2. For simplicity we identify $\mathbb{C} \setminus i[0, \infty)$ with $\{\lambda \in \Lambda : -3\pi/2 < \arg \lambda < \pi/2\}$ when n is odd. Let X be the class of obstacles diffeomorphic to a fixed obstacle $\mathcal{O}_0$ – see (1.3.3), that is for some $k \geq 2$,

$$X := \{\Phi(\mathcal{O}_0) : \Phi \in \mathrm{Diff}(\mathcal{O}_0)\},$$

with $\mathrm{Diff}(\mathcal{O}_0)$ defined by (4.1.1). We introduce a topology in this set by defining a sub-basis of the neighborhoods of any $\mathcal{O} \in X$ by

$$\mathcal{V}_\varepsilon(\mathcal{O}) := \{\Phi(\mathcal{O}) : \Phi \in \mathrm{Diff}(\mathcal{O}),\ \|\Phi - \mathrm{id}\,\|_{C^k} < \varepsilon \text{ with } \varepsilon \text{ sufficiently small}\}.$$

For any $\theta_1, \theta_2 \in \mathbb{R}$ and $r > 1$, we define

$$\begin{aligned} S^r_{\theta_1,\theta_2} &:= \{\lambda \in \Lambda : \theta_1 < \arg \lambda < \theta_2,\ 1/r < |\lambda| < r\}, \\ E^r_{\theta_1,\theta_2} &:= \{\mathcal{O} \in X : m_{\mathcal{O}}(\lambda) \leq 1,\ \forall\, \lambda \in S^r_{\theta_1,\theta_2}\}. \end{aligned} \tag{4.3.1}$$

To prove Theorem 3 it suffices to show that for each θ_1, θ_2 and r, $E^r_{\theta_1,\theta_2}$ is open and dense in X, since we can then obtain the generic set $\mathcal{X}$ by taking

$$\mathcal{X} := \bigcap_{m=1}^{\infty} \bigcap_{N=1}^{\infty} E^N_{-m\pi,m\pi} \text{ when } n \text{ is even}; \quad \mathcal{X} := \bigcap_{N=1}^{\infty} E^N_{-\frac{3\pi}{2},\frac{\pi}{2}} \text{ when } n \text{ is odd.}$$

We proceed the proof of Theorem 3 in steps:

Proof of Theorem 3. Step 1. As in §3.6, for $\mathcal{O} \in X$ we write $\mathrm{Res}(\mathcal{O})$ for the image of resonances under the map $\lambda \mapsto z = \lambda^2$, and for any z the multiplicity is given by $m_{\mathcal{O}}(z) := m_{\mathcal{O}}(\lambda)$ provided $z = \lambda^2$. Then

$$E^r_{\theta_1,\theta_2} = \{\mathcal{O} \in X : m_{\mathcal{O}}(z) \leq 1,\ \forall\, z \in S^{r^2}_{2\theta_1,2\theta_2}\}.$$

In view of (3.6.3), we can choose $A > 0$ large enough, such that $S^{r^2}_{2\theta_1,2\theta_2} \Subset D_+$ (when n is odd, we only need to check for $\theta_1 = -3\pi/2$, $\theta_2 = \pi/2$). Suppose that there is exactly one resonance z_0 in $B(z_0, 2\delta) \subset S^{r^2}_{2\theta_1,2\theta_2}$, where $B(z_0, r)$ denotes the disc in $\mathbb{C}$ centered at z_0 with radius r. For $\Omega := B(z_0, \delta)$ we then define

$$\Pi_{\mathcal{O}}(\Omega) := -\frac{1}{2\pi i} \int_{\partial\Omega} (-\Delta_{\mathcal{O}} - \zeta)^{-1} d\zeta, \quad m_{\mathcal{O}}(\Omega) := \mathrm{rank}\, \Pi_{\mathcal{O}}(\Omega). \tag{4.3.2}$$

Now we choose a bounded domain Σ containing $B(z_0, 2\delta)$ with boundary Γ satisfying (3.6.5). We also assume that $\Sigma \Subset S^{r^2}_{2\theta_1,2\theta_2}$. By Proposition 3.6.2, elements in $\mathrm{Res}(\mathcal{O})$ coincide with the eigenvalues of P_Γ in Σ. In view of (3.6.13), we have the relationship:

$$\Pi_\Gamma(\Omega) := -\frac{1}{2\pi i}\int_{\partial\Omega}(P_\Gamma-\zeta)^{-1}d\zeta, \text{ then } m_\Gamma(\Omega) := \operatorname{rank}\Pi_\Gamma(\Omega) = m_{\mathcal{O}}(\Omega). \tag{4.3.3}$$

Let $\mathcal{U}_\varepsilon(\mathcal{O})$ be a set of deformations defined for small $\varepsilon > 0$,

$$\mathcal{U}_\varepsilon(\mathcal{O}) := \{\Phi \in \mathrm{Diff}(\mathcal{O}) : \|\Phi - \mathrm{id}\,\|_{C^k} < \varepsilon\}.$$

Assuming that $\varepsilon < \delta_\Gamma$ for constant δ_Γ given in (4.2.4), then for every $\Phi \in \mathcal{U}_\varepsilon(\mathcal{O})$ Proposition 4.2.1 implies that

$$\Pi_{\Gamma,\Phi}(\Omega) := -\frac{1}{2\pi i}\int_{\partial\Omega}(P_{\Gamma,\Phi}-\zeta)^{-1}d\zeta, \; m_{\Gamma,\Phi}(\Omega) := \operatorname{rank}\Pi_{\Gamma,\Phi}(\Omega) = m_{\Phi(\mathcal{O})}(\Omega). \tag{4.3.4}$$

We recall (4.2.8) that $P_{\Gamma,\Phi} = P_\Gamma + V$ with V defined in (4.1.3), then by (4.2.3) we obtain that if ε is sufficiently small, then for $\zeta \in \partial\Omega$ and $\Phi \in \mathcal{U}_\varepsilon(\mathcal{O})$,

$$(P_{\Gamma,\Phi}-\zeta)^{-1} = (P_\Gamma-\zeta)^{-1}(I + V(P_\Gamma-\zeta)^{-1})^{-1}$$

and $\sup_{\zeta\in\partial\Omega}\|(P_{\Gamma,\Phi}-\zeta)^{-1} - (P_\Gamma-\zeta)^{-1}\|_{B_\Gamma\to B_\Gamma} < C(\Omega)\varepsilon$. Then we can derive that $\Pi_\Gamma(\Omega)$ and $\Pi_{\Gamma,\Phi}(\Omega)$ have the same rank for any $\Phi \in \mathcal{U}_\varepsilon(\mathcal{O})$ if ε is sufficiently small. We restate this as follows:

$$m_{\Phi(\mathcal{O})}(\Omega) \text{ is constant for } \Phi \in \mathcal{U}_\varepsilon(\mathcal{O}) \text{ if } \varepsilon \text{ is sufficiently small.} \tag{4.3.5}$$

Hence $\mathcal{O} \in E^r_\theta$ implies that $\{\Phi(\mathcal{O}) : \Phi \in \mathcal{U}_\varepsilon(\mathcal{O})\} \subset E^r_\theta$ for some ε sufficiently small, in other words, E^r_θ is an open subset of X.

Step 2. It remains to show that E^r_θ is dense in X, which is equivalent to:

$$\forall\,\mathcal{O} \in X \text{ and } \varepsilon > 0, \quad \exists\,\Phi \in \mathcal{U}_\varepsilon(\mathcal{O}) \text{ such that } \Phi(\mathcal{O}) \in E^r_\theta. \tag{4.3.6}$$

Since the number of resonances for the obstacle $\mathcal{O}$ in $S^r_{\theta_1,\theta_2}$ is finite, it is enough to prove a local statement as it can be applied successively to obtain (4.3.6) (once a resonance is simple it stays simple under small deformations due to (4.3.5)). We will define Ω for any given $\mathcal{O}$ and $z_0 \in \mathrm{Res}(\mathcal{O})$ as in Step 1, thus to obtain (4.3.6) it suffices to prove that for

$$\forall\,\mathcal{O} \in X,\; z_0 \in \mathrm{Res}(\mathcal{O}) \text{ and } \varepsilon > 0, \quad \exists\,\Phi \in \mathcal{U}_\varepsilon(\mathcal{O}) \text{ s.t. } m_{\Phi(\mathcal{O})}(z) \le 1,\; \forall\, z \in \Omega. \tag{4.3.7}$$

To establish (4.3.7) we proceed by induction. We note that for each $\mathcal{O} \in X$, $z_0 \in \mathrm{Res}(\mathcal{O})$, one of the following cases has to occur:

$$\forall\,\varepsilon > 0, \quad \exists\,\Phi \in \mathcal{U}_\varepsilon(\mathcal{O}) \text{ s.t. } 1 \le m_{\Phi(\mathcal{O})}(z) < m_{\Phi(\mathcal{O})}(\Omega), \quad \forall\, z \in \Omega, \tag{4.3.8}$$

or

$$\exists\,\varepsilon>0,\ \text{s.t.}\quad \forall\,\Phi\in\mathcal{U}_\varepsilon(\mathcal{O}),\ \exists z=z(\Phi)\in\Omega,\ m_{\Phi(\mathcal{O})}(z)=m_{\Phi(\mathcal{O})}(\Omega)>1. \tag{4.3.9}$$

The first possibility means that by applying an arbitrarily small deformation Φ to $\mathcal{O}$ we can obtain at least two distinct resonances for $\Phi(\mathcal{O})$ in Ω. The second possibility means that under any small deformations the maximal multiplicity persists.

Step 3. Assuming (4.3.8) we can prove (4.3.7) by induction on $m_{\mathcal{O}}(z_0)$. If $m_{\mathcal{O}}(z_0)=1$ there is nothing to prove. Assuming that we proved (4.3.7) in the case $m_{\mathcal{O}}(z_0)<M$, we now assume that $m_{\mathcal{O}}(z_0)=M$. We note that for any $\Phi_1\in\operatorname{Diff}(\mathcal{O})$ and $\Phi_2\in\operatorname{Diff}(\Phi_1(\mathcal{O}))$, there exists $C=C(k,n)$ such that

$$\|\Phi_2\circ\Phi_1-\operatorname{id}\|_{C^k}\le C\big(\|\Phi_2-\operatorname{id}\|_{C^k}+\|\Phi_1-\operatorname{id}\|_{C^k}\big).$$

In view of (4.3.8) we can find $\Phi_0\in\operatorname{Diff}(\mathcal{O})$ with $\|\Phi_0-\operatorname{id}\|_{C^k}<\varepsilon/(2C)^M$ such that $m_{\Phi_0(\mathcal{O})}(\Omega)=m_{\mathcal{O}}(\Omega)$ (using (4.3.5)) and that all resonances in Ω, denoted by $z_1,\cdots,z_\ell$, satisfy $m_{\Phi_0(\mathcal{O})}(z_j)<M$. We now find r_j such that

$$B(z_j,2r_j)\subset\Omega,\ \{z_j\}=B(z_j,2r_j)\cap\operatorname{Res}(\Phi_0(\mathcal{O})),\quad B(z_j,2r_j)\cap B(z_i,2r_i)=\emptyset.$$

We put $\Omega_j:=B(z_j,r_j)$ and apply (4.3.7) successively to $\Phi_{j-1}\circ\cdots\circ\Phi_0(\mathcal{O})$, $j=1,\dots,\ell$ with $\|\Phi_j-\operatorname{id}\|_{C^k}<\varepsilon/(2C)^{\ell+1-j}$ (by (4.3.5) we can assume that Φ_j is sufficiently close to the identity map such that resonances in $\Omega_0,\cdots,\Omega_{j-1}$ that are already simple stay simple while total multiplicities in $\Omega_{j+1},\cdots,\Omega_\ell$ are invariant). Then we obtain the desired $\Phi=\Phi_\ell\circ\cdots\circ\Phi_0\in\mathcal{U}_\varepsilon(\mathcal{O})$ since (note that $\ell<M$)

$$\|\Phi_\ell\circ\cdots\circ\Phi_0-\operatorname{id}\|_{C^k}<\sum_{j=1}^{\ell}C^{\ell+1-j}\frac{\varepsilon}{(2C)^{\ell+1-j}}+C^\ell\frac{\varepsilon}{(2C)^M}\le\varepsilon.$$

Step 4. It remains to show (4.3.9) is impossible. For that, we shall argue by contradiction. Suppose that there exist an obstacle $\mathcal{O}\in X$ and a resonance $z_0\in\Omega$ with some disc $\Omega=B(z_0,r)$ containing no other resonances, such that (4.3.9) holds. In fact we may assume further that $\mathcal{O}$ has $\mathcal{C}^\infty$-boundary since we can deform $\mathcal{O}$ to a smooth obstacle $\tilde{\mathcal{O}}$ through some $\tilde\Phi\in\operatorname{Diff}(\mathcal{O})$ with $\|\tilde\Phi-\operatorname{id}\|_{C^k}\ll\varepsilon$, decreasing ε if necessary, then (4.3.9) still holds with $\tilde{\mathcal{O}}$ and $\tilde z_0=z(\tilde\Phi)$ replacing $\mathcal{O}$ and z_0. Hence we assume in the following that $\mathcal{O}$ is a smooth obstacle.

Let $M=m_{\mathcal{O}}(\Omega)$. Suppose that Σ and Γ are chosen as in Step 1. Using (4.3.3) and (4.3.4) we obtain an equivalent statement to (4.3.9):

$$\exists\,\varepsilon>0,\ \text{s.t.}\quad \forall\,\Phi\in\mathcal{U}_\varepsilon(\mathcal{O}),\ \exists\,z=z(\Phi)\in\Omega,\ m_{\Gamma,\Phi}(z)=m_{\Gamma,\Phi}(\Omega)>1. \tag{4.3.10}$$

For $\Phi\in\mathcal{U}_\varepsilon(\mathcal{O})$, we define

$$q(\Phi):=\min\{q\in\mathbb{N}:(P_{\Gamma,\Phi}-z(\Phi))^q\Pi_{\Gamma,\Phi}(\Omega)=0\},$$

then $1 \leq q(\Phi) \leq M$. It follows from (4.2.8) and (4.1.3) that if $\|\Phi_j - \Phi\|_{C^{2M}} \to 0$ and $(P_{\Gamma,\Phi_j} - z(\Phi_j))^q \Pi_{\Gamma,\Phi_j}(\Omega) = 0$, then $(P_{\Gamma,\Phi} - z(\Phi))^q \Pi_{\Gamma,\Phi}(\Omega) = 0$. We now define

$$q_0 := \max\{q(\Phi) : \Phi \in \mathcal{U}_{\varepsilon/2}(\mathcal{O})\},$$

and assume that the maximum is attained at Φ_0 i.e. $q(\Phi_0) = q_0$, then there exists $\varepsilon' > 0$ such that

$$\|\Phi - \Phi_0\|_{C^{2M}} < \varepsilon' \implies q(\Phi) = q_0.$$

Therefore, we can choose a $\tilde{\Phi}_0 \in \mathrm{Diff}(\mathcal{O})$ that is in $\mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\tilde{\Phi}_0 - \Phi_0\| \ll \varepsilon'$. Replacing $\mathcal{O}$ in (4.3.10) by $\tilde{\Phi}_0(\mathcal{O})$ and decreasing ε such that $\varepsilon \ll \varepsilon'$, we assume in the following that

$$\begin{aligned} &\forall\, \Phi \in \mathrm{Diff}(\mathcal{O}),\ \|\Phi - \mathrm{id}\,\|_{C^{2M}} < \varepsilon,\ \exists\, z(\Phi) \text{ and } 1 \leq q_0 \leq M \text{ such that} \\ &m_{\Gamma,\Phi}(z(\Phi)) = \mathrm{rank}\,\Pi_{\Gamma,\Phi}(\Omega) = M > 1, \\ &(P_{\Gamma,\Phi} - z(\Phi))^{q_0}\Pi_{\Gamma,\Phi}(\Omega) = 0, \quad (P_{\Gamma,\Phi} - z(\Phi))^{q_0-1}\Pi_{\Gamma,\Phi}(\Omega) \neq 0. \end{aligned} \tag{4.3.11}$$

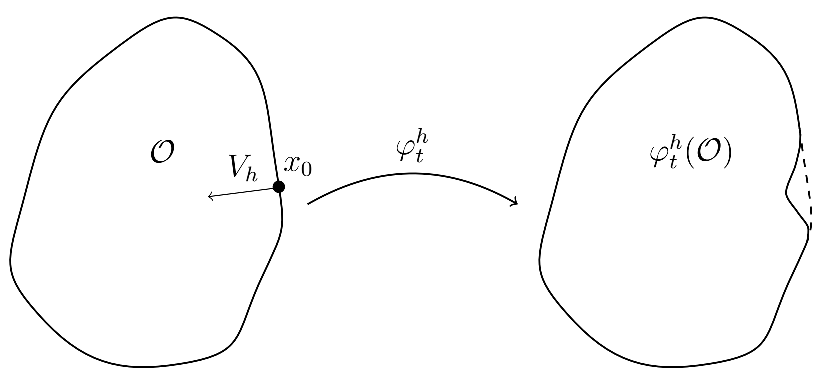


Figure 4.1: Deformation φ_t^h in $\mathrm{Diff}(\mathcal{O})$ acting near a fixed point on $\partial\mathcal{O}$, which is used in Step 5 of the proof of Theorem 3.

Step 5. Before proving (4.3.11) is impossible we introduce a family of deformations in $\mathrm{Diff}(\mathcal{O})$ acting near a point on $\partial\mathcal{O}$. For any fixed $x_0 \in \partial\mathcal{O}$, we consider the normal coordinates near x_0, that is there is some $U = B_{\mathbb{R}^n}(x_0, 2r_0)$ such that for each $x \in U$ there exist unique $(x', x_n) \in \partial\mathcal{O} \times \mathbb{R}$ with $x = x' + x_n\nu(x')$, where $\nu(x')$ is the normal vector at x' pointing to the interior of $\mathcal{O}$. Let $\rho \in \mathcal{C}_c^\infty(\mathbb{R}; [0,1])$ be a bump function such that $\rho(0) = 1$ and $\mathrm{supp}\,\rho \subset (-r_0, r_0)$. Fixing $h_0 > 0$ small, we choose a family of functions $\chi_h \in \mathcal{C}^\infty(\partial\mathcal{O}; [0,\infty))$ depending continuously in $h \in (0, h_0]$ such that

$$\int_{\partial\mathcal{O}} \chi_h(x')dS(x') = 1, \quad \mathrm{supp}\,\chi_h \subset B_{\partial\mathcal{O}}(x_0, h) \subset U, \quad \forall\, h \in (0, h_0], \tag{4.3.12}$$

where $B_{\partial\mathcal{O}}(x_0, h)$ is a geodesic ball on $\partial\mathcal{O}$ with center x_0 and radius h. For each $h \in (0, h_0]$, we construct a smooth vector field $V_h \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ as follows

$$\begin{gathered} V_h(x) = \chi_h(x')\rho(x_n)\nu(x'), \text{ for } x = x' + x_n\nu(x') \in U, \\ \text{and } V_h(x) = 0 \text{ for all } x \in \mathbb{R}^n \setminus U. \end{gathered} \tag{4.3.13}$$

Then we introduce a family of smooth deformations produced by V_h:

$$\varphi_t^h \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n), \quad \varphi_t^h(x) := x + tV_h(x). \tag{4.3.14}$$

It follows from (4.3.13) that for every $h \in (0, h_0]$ there is $t_0 = t_0(h) \ll 1$ such that

$$\forall\, t \in (-t_0, t_0), \quad \varphi_t^h \in \mathrm{Diff}(\mathcal{O}), \ \|\varphi_t^h - \mathrm{id}\,\|_{C^{2M}} < \varepsilon.$$

Step 6. To show that (4.3.11) is impossible we first assume the case $q_0 > 1$. We recall (3.6.12) that $\Pi_\Gamma(\Omega)(B_\Gamma) = \Pi_\Gamma(\Omega)(B_0)$, let

$$\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}}) := \{f \in \mathcal{C}_c^\infty(\mathbb{R}^n) : \operatorname{supp} f \subset \mathbb{R}^n \setminus \overline{\mathcal{O}}\}$$

then $\operatorname{Ran}\Pi_\Gamma(\Omega) = \Pi_\Gamma(\Omega)(\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}}))$ since $\Pi_\Gamma(\Omega)$ is finite rank and $\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$ is dense in B_0. Thus by (4.3.11) we can find $w \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$ such that

$$u := (P_\Gamma - z_0)^{q_0 - 1}\Pi_\Gamma(\Omega)w \neq 0, \quad \text{here } z_0 = z(\mathrm{id}). \tag{4.3.15}$$

Fixing $x_0 \in \partial\mathcal{O}$ and $h \in (0, h_0]$, we define φ_h^t as in Step 5 and write $\Phi_t := \varphi_h^t$, $t \in (-t_0, t_0)$. If we set

$$u(t) := ({\Phi_t}^{-1})^* v(t), \ v(t) := (P_{\Gamma,\Phi_t} - z(t))^{q_0-1}\Pi_{\Gamma,\Phi_t}(\Omega)w, \ z(t) := z(\Phi_t). \tag{4.3.16}$$

Then by (4.3.11) we have for any

$$\forall\, t \in (-t_0, t_0), \quad m_{\Gamma,\Phi_t}(z(t)) = \operatorname{rank}\Pi_{\Gamma,\Phi_t}(\Omega) = M, \ (P_{\Gamma,\Phi_t} - z(t))v(t) = 0. \tag{4.3.17}$$

Recalling (4.2.1) and (4.2.7), we obtain the equation for $u(t)$:

$$(-\Delta - z(t))u(t) = 0 \quad \text{on } \mathbb{R}^n \setminus \Phi_t(\mathcal{O}), \tag{4.3.18}$$

in the sense of L^2_{loc} functions.

We next aim to show that $z(t)$ is differentiable at 0. For that we extend (4.3.14) to $\Phi_t \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^n)$, $t \in \mathbb{C}$:

$$\Phi_t(x) := x + tV_h(x), \quad t \in \mathbb{C}$$

We set $t_1 = t_1(h)$ sufficiently small such that for all $|t| < t_1$ and $x \in \mathbb{R}^n$, $D\Phi_t(x) = I + tDV_h(x)$ is invertible. Denoting by $J_{\Phi_t}^{ij}(x) = [D\Phi_t(x)^{-1}]_{ij}$, $\Phi_t^m(x)$ the m-th component of $\Phi_t(x)$, we replace Φ by Φ_t in (4.1.3) to define

$$\mathcal{V}(t) := \sum_{i,j} a_{ij}(t,x)\partial^2_{x_i x_j} + \sum_j b_j(t,x)\partial_{x_j},$$

$$\text{where} \quad a_{ij} = \delta_{ij} - \sum_{\ell} J_{\Phi_t}^{i\ell} J_{\Phi_t}^{j\ell}, \quad b_j = -\sum_{i\ell mq} (\partial^2_{x_i x_\ell} \Phi_t^m) J_{\Phi_t}^{jm} J_{\Phi_t}^{iq} J_{\Phi_t}^{\ell q}.$$

Repeating the calculation that yields (4.1.4), we also have for some $C = C(h) > 0$,

$$a_{ij}(t,\cdot),\ b_j(t,\cdot) \in \mathcal{C}_c^\infty(\mathbb{R}^n_x), \quad \|a_{ij}(t,\cdot)\|_\infty,\ \|b_j(t,\cdot)\|_\infty < C|t|.$$

It follows that $\mathcal{V}(t)$, $|t| < t_1$ satisfies Hypothesis 3.6.3. For $t \in \mathbb{C}$, $|t| < t_1$, we follow (3.6.14) to define

$$P_\Gamma(t) = P_\Gamma + \mathcal{V}(t), \quad \text{with domain } \mathcal{D}(P_\Gamma(t)) := \mathcal{D}(P_\Gamma).$$

Recalling Propositions 3.6.4 and 3.6.5, decreasing t_0 if necessary, for any $t \in \mathbb{C}$, $|t| < t_0$, $P_\Gamma(t)$ has a discrete spectrum in some neighborhood K of z_0 and the total multiplicity of the eigenvalues of $P_\Gamma(t)$ in K equals M. Moreover, if we denote by $z_1(t), \dots, z_M(t)$ the eigenvalues of $P_\Gamma(t)$ in K, repeated with multiplicity, then $\hat{z}(t) = M^{-1}\sum_{j=1}^M z_j(t)$ is an analytic function in $t \in \mathbb{C}$, $|t| < t_0$. On the other hand, if we consider real t, $t \in (-t_0, t_0)$, then (4.1.3) and (4.2.8) imply that

$$P_\Gamma(t) = P_\Gamma + \mathcal{V}(t) = P_{\Gamma,\Phi_t}, \quad -t_0 < t < t_0.$$

It follows from (4.3.17) that for $t \in (-t_0, t_0)$ the eigenvalues of $P_\Gamma(t)$ near z_0 don't split, i.e. $z_j(t) = z(t)$, $j = 1, \dots, M$. Thus $z(t) = \hat{z}(t)$ when t is real, $t \in (-t_0, t_0)$. The analyticity of $\hat{z}(t)$ gives the smoothness of $z(t)$ on $(-t_0, t_0)$. As a consequence, $u(t)$ and $v(t)$ defined in (4.3.16) also depend smoothly on $t \in (-t_0, t_0)$.

Since $\Phi_t(\mathcal{O}) \subset \mathcal{O}$ for $t \geq 0$, we can restrict (4.3.18) to the region $\mathbb{R}^n \setminus \mathcal{O}$ then differentiate the equation in t, by taking $t = 0$, we obtain that

$$(-\Delta - z_0)\partial_t u(0,x) = z'(0)u(x) \quad \text{on } \mathbb{R}^n \setminus \mathcal{O}. \tag{4.3.19}$$

We recall (4.3.16) that $u(t,x) = v(t, \Phi_t^{-1}(x))$, using $u(0,x) = v(0,x) = u(x)$ and (4.3.14) we can calculate the derivative in t:

$$\partial_t u(0,x) = \partial_t v(t, \Phi_t^{-1}(x))|_{t=0} = \partial_t v(0,x) - \partial_x u \cdot V_h(x).$$

In view of (3.6.12) and (4.3.15), $u \in \mathcal{E}_\Gamma(z_0)$ is a resonant state of $-\Delta_{\mathcal{O}}$ at z_0, thus we recall [12, Theorem 4.7] that $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \mathcal{O})$. Then by (4.3.13) we conclude that

$$\begin{aligned} &(-\Delta - z_0)(\partial_t v(0,x) - f) = z'(0)u(x) \quad \text{on } \mathbb{R}^n \setminus \mathcal{O}, \\ &f := \partial_x u \cdot V_h(x) \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \mathcal{O}), \quad f|_{\partial\mathcal{O}} = \chi_h \partial_\nu u. \end{aligned} \tag{4.3.20}$$

It follows from $v(t,x) \in \mathcal{D}(P_\Gamma)$, $t \in (-t_0, t_0)$ that $\partial_t v(0,x) \in D(P_\Gamma)$, thus the first equation in (4.3.20) reduces to

$$(P_\Gamma - z_0)\partial_t v(0,x) = (-\Delta - z_0)f + z'(0)u \quad \text{on } \mathbb{R}^n \setminus \mathcal{O}. \tag{4.3.21}$$

We introduce the bilinear form on $B_0 \times B_1$ (no complex conjugation),

$$\langle u, v\rangle := \int_{\mathbb{R}^n\setminus\mathcal{O}} uv\, dx, \quad u \in B_1,\ v \in B_0.$$

We now apply the projection Π_Γ (omitting Ω) to both sides of (4.3.21), pair with $(P_\Gamma - z_0)^{q_0-1}w \in B_0$ (since $w \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$), use the fact that $(P_\Gamma - z_0)\Pi_\Gamma g = \Pi_\Gamma(P_\Gamma - z_0)g$, $\forall\, g \in \mathcal{D}(P_\Gamma)$ to obtain that

$$\begin{aligned}&\langle (P_\Gamma - z_0)\Pi_\Gamma \partial_t v(0,x), (P_\Gamma - z_0)^{q_0-1}w\rangle\\ &= \langle \Pi_\Gamma(-\Delta - z_0)f, (P_\Gamma - z_0)^{q_0-1}w\rangle + z'(0)\langle u, (P_\Gamma - z_0)^{q_0-1}w\rangle.\end{aligned}$$

By Green's formula, $\langle P_\Gamma g_1, g_2\rangle = \langle g_1, P_\Gamma g_2\rangle$ for any $g_1 \in \mathcal{D}(P_\Gamma)$, $g_2 \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$. It then follows from (4.3.11) and (4.3.15) that

$$\langle (P_\Gamma - z_0)\Pi_\Gamma \partial_t v(0,x), (P_\Gamma - z_0)^{q_0-1}w\rangle = \langle (P_\Gamma - z_0)^{q_0}\Pi_\Gamma \partial_t v(0,x), w\rangle = 0,$$

and that

$$\langle u, (P_\Gamma - z_0)^{q_0-1}w\rangle = \langle (P_\Gamma - z_0)u, (P_\Gamma - z_0)^{q_0-2}w\rangle = 0.$$

Since $\langle \Pi_\Gamma f_1, f_2\rangle = \langle \Pi_\Gamma f_2, f_1\rangle$ for any $f_1, f_2 \in B_0$, we conclude that

$$\begin{aligned}0 &= \langle \Pi_\Gamma(-\Delta - z_0)f, (P_\Gamma - z_0)^{q_0-1}w\rangle\\ &= \langle (-\Delta - z_0)f, (P_\Gamma - z_0)^{q_0-1}\Pi_\Gamma w\rangle = \langle (-\Delta - z_0)f, u\rangle.\end{aligned}$$

Now we apply Green's formula and recall (4.3.20) and $u_{\partial\mathcal{O}} = 0$ to obtain

$$0 = \langle (-\Delta - z_0)f, u\rangle = \int_{\partial\mathcal{O}} f\, \partial_\nu u\, dS = \int_{\partial\mathcal{O}} \chi_h(x')(\partial_\nu u(x'))^2 dS(x').$$

Since the above equation holds for any $h \in (0, h_0]$, (u is independent of x_0 and h) sending h to $0+$, by (4.3.12) we can derive that $\partial_\nu u(x_0) = 0$. We note that $x_0 \in \partial\mathcal{O}$ can be chosen arbitrarily, thus $\partial_\nu u|_{\partial\mathcal{O}} \equiv 0$. However, it follows from (4.3.11) and (4.3.15) that $u \in \mathcal{D}_1(\mathcal{O})$ satisfying $(-\Delta - z_0)u = 0$ on $\mathbb{R}^n \setminus \mathcal{O}$. Extending u into $\mathcal{O}$ by $u|_\mathcal{O} = 0$, it then follows from (4.3.18) and the boundary values $u|_{\partial\mathcal{O}} = 0$, $\partial_\nu u|_{\partial\mathcal{O}} = 0$ that $u \in H^1_{\rm loc}(\mathbb{R}^n)$ is a weak solution of $(-\Delta - z_0)u = 0$ on $\mathbb{R}^n$. The unique continuation property of second order elliptic differential equations shows that $u \equiv 0$, which contradicts (4.3.15).

Step 7. It remains to consider the case $q_0 = 1$ in (4.3.11). Let $\{w_j\}_{j=1}^M$ be a set of vectors in $\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$ such that $\{\Pi_\Gamma w_j\}_{j=1}^M$ is a basis for $\operatorname{Ran}\Pi_\Gamma$. Since Π_Γ is symmetric with respect to the bilinear form $\langle \cdot, \cdot\rangle$ on $B_0 \times B_0$, the matrix A, $A_{ij} := \langle \Pi_\Gamma w_i, w_j\rangle$ is a complex symmetric matrix. To see A is nondegenerate, we suppose that

$$\exists\, x \in \mathbb{C}^M, \quad \langle \Pi_\Gamma w_i, \sum_j x_j w_j\rangle = 0,\ i = 1, \cdots, M.$$

Since $\{\Pi_\Gamma w_i\}_{i=1}^M$ spans $\operatorname{Ran}\Pi_\Gamma$, we have $\langle \Pi_\Gamma w, \sum x_j w_j\rangle = 0$ for all $w \in B_0$, which implies that $\langle \sum x_j \Pi_\Gamma w_j, w\rangle = 0$, $\forall\, w \in B_0$. Hence $\sum x_j \Pi_\Gamma w_j = 0 \Rightarrow x = 0$. We apply the Takagi factorization to the matrix A to obtain that

$$A = U^T \operatorname{Diag}(r_1, \cdots, r_M)U, \text{ where } U \text{ is unitary, } r_j^2 \text{ are the eigenvalues of } AA^*.$$

We remark that U^T is the real transpose. Then we can write $A = B^T B$, B nondegenerate due to the nondegeneracy of A. Transforming $\{w_j\}_{j=1}^M$ by the matrix B and putting $u_j := \Pi_\Gamma w_j$, we may assume now that

$$\operatorname{Ran}\Pi_\Gamma = \operatorname{span}\{u_j\}_{j=1}^M, \quad \langle u_j, w_i\rangle = \delta_{ij}.$$

For any fixed $x_0 \in \partial\mathcal{O}$ and $h \in (0, h_0]$, we define the evolution of each u_j as in (4.3.16):

$$u_j(t) := (\Phi_t^{-1})^* v_j(t), \quad v_j(t) := \Pi_{\Gamma,\Phi_t}(\Omega) w_j, \;\; z(t) := z(\Phi_t). \tag{4.3.22}$$

We note that (4.3.21) still holds with $\partial_t v(0, x), u, f$ replaced by $\partial_t v_j(0, x), u_j$ and f_j defined as in (4.3.20). The same arguments as in Step 6 show that

$$\langle (P_\Gamma - z_0)\Pi_\Gamma v_j'(0), w_i\rangle = \langle \Pi_\Gamma(-\Delta - z_0) f_j, w_i\rangle + z'(0)\langle u_j, w_i\rangle.$$

Since $(P_\Gamma - z_0)\Pi_\Gamma = 0$ by (4.3.11) with $q_0 = 1$, it then follows that

$$\langle (-\Delta - z_0) f_j, u_i\rangle = -z'(0)\delta_{ij}.$$

We apply Green's formula with boundary value of f_j like (4.3.20) to obtain that

$$-z'(0)\delta_{ij} = \langle (-\Delta - z_0)u_i, f_j\rangle + \int_{\partial\mathcal{O}} f_j \partial_\nu u_i \, dS = \int_{\partial\mathcal{O}} \chi_h (\partial_\nu u_i)(\partial_\nu u_j)\, dS.$$

Since $M \geq 2$, for any $x_0 \in \partial\mathcal{O}$ and $h \in (0, h_0]$ we have

$$\int_{\partial\mathcal{O}} \chi_h (\partial_\nu u_1)^2 dS = \int_{\partial\mathcal{O}} \chi_h (\partial_\nu u_2)^2 dS, \quad \int_{\partial\mathcal{O}} \chi_h \partial_\nu u_1 \partial_\nu u_2 \, dS = 0.$$

Sending $h \to 0+$, it follows from (4.3.12) that

$$(\partial_\nu u_1(x_0))^2 = (\partial_\nu u_2(x_0))^2, \quad \partial_\nu u_1(x_0)\partial_\nu u_2(x_0) = 0,$$

thus $\partial_\nu u_1(x_0) = \partial_\nu u_2(x_0) = 0$. Since $x_0 \in \partial\mathcal{O}$ is arbitrary, $\partial_\nu u_1 \equiv 0$. Hence the same arguments as in the end of Step 6 show that $u_1 \equiv 0$, which gives a contradiction. □

Chapter 5

The CAP-regularized operator

In this chapter we study the Hamiltonians modified by the complex absorbing potantials.

5.1 The Davies harmonic oscillator

The operator

$$H_{\varepsilon,\theta} := -e^{-2i\theta}\Delta - i\varepsilon e^{2i\theta}x^2, \quad \varepsilon > 0,\ 0 \leq \theta \leq \pi/8,$$

was used by Davies [9] to illustrate properties of non-normal differential operators. Some known facts about $H_{\varepsilon,\theta}$ and its resolvent have been reviewed in [65], we recall those facts here:

Proposition 5.1.1. *$H_{\varepsilon,\theta}$ is a closed densely defined operator on $L^2(\mathbb{R}^n)$ equipped with the domain $H^2(\mathbb{R}^n) \cap \langle x\rangle^{-2}L^2(\mathbb{R}^n)$. The spectrum is given by*

$$\operatorname{Spec}(H_{\varepsilon,\theta}) = \{e^{-i\pi/4}\sqrt{\varepsilon}(2|\alpha| + n) : \alpha \in \mathbb{N}_0^n\}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n. \tag{5.1.1}$$

If $\Omega \Subset \{z : -\pi/2 + 2\theta < \arg z < -2\theta\} \setminus e^{-i\pi/4}[0,\infty)$, then there exist constants $C_1 = C_1(\Omega)$ and $C_2 = C_2(\Omega)$ such that

$$C_1 e^{C_1\varepsilon^{-\frac{1}{2}}} \leq \|(H_{\varepsilon,\theta} - z)^{-1}\|_{L^2\to L^2} \leq C_2 e^{C_2\varepsilon^{-\frac{1}{2}}}, \quad z \in \Omega. \tag{5.1.2}$$

In addition for any $\delta > 0$ we have uniformly in $\varepsilon > 0$,

$$\begin{gathered}(H_{\varepsilon,\theta} - z)^{-1} = O_\delta(|z|^{\frac{j-2}{2}}) : L^2(\mathbb{R}^n) \to H^j(\mathbb{R}^n), \quad j = 0,1,2, \\ \text{for } -2\theta + \delta < \arg z < 3\pi/2 + 2\theta - \delta, \quad |z| > \delta.\end{gathered} \tag{5.1.3}$$

Proof. By rescaling $y = \sqrt{\varepsilon}x$, $H_{\varepsilon,\theta}$ can be viewed as a semiclassical Weyl quantization of a complex-valued quadratic form, with $h = \sqrt{\varepsilon}$,

$$H_{\varepsilon,\theta} = q^{\mathrm{w}}(y, hD), \quad q : \mathbb{R}^n_y \times \mathbb{R}^n_\eta \to \mathbb{C},\ (y,\eta) \mapsto e^{-2i\theta}\eta^2 - ie^{2i\theta}y^2,$$

which shall be viewed as a closed densely defined operator on $L^2(\mathbb{R}^n)$ equipped with the domain $\mathcal{D}(H_{\varepsilon,\theta}) := \{u \in L^2(\mathbb{R}^n) : H_{\varepsilon,\theta}u \in L^2(\mathbb{R}^n)\}$. For the analysis of the spectrum for general quadratic operators see Hitrik–Sjöstrand–Viola [27] and references given there, in particular we obtain (5.1.1).

Then we show $\mathcal{D}(H_{\varepsilon,\theta}) = H^2(\mathbb{R}^n) \cap \langle x\rangle^{-2}L^2(\mathbb{R}^n)$. For $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, let $f = H_{\varepsilon,\theta}u$, we integrate by parts to obtain:

$$\langle f, u\rangle_{L^2} = e^{-2i\theta}\|Du\|_{L^2}^2 - i\varepsilon e^{2i\theta}\|xu\|_{L^2}^2, \tag{5.1.4}$$

where $\langle f, u\rangle_{L^2} = \int_{\mathbb{R}^n} f(x)\overline{u(x)}dx$. We also have

$$\begin{aligned}\|f\|_{L^2}^2 &= \|\Delta u\|_{L^2}^2 + \varepsilon^2\|x^2u\|_{L^2}^2 - i\varepsilon e^{-4i\theta}\langle \Delta u, x^2u\rangle_{L^2} + i\varepsilon e^{4i\theta}\langle x^2u, \Delta u\rangle_{L^2}\\ &= \|\Delta u\|_{L^2}^2 + \varepsilon^2\|x^2u\|_{L^2}^2 + 2\varepsilon\sin 4\theta\|xDu\|_{L^2}^2 + 4\varepsilon\,\mathrm{Im}(e^{4i\theta}\langle xu, Du\rangle_{L^2}).\end{aligned} \tag{5.1.5}$$

Thus we can conclude a priori estimate from (5.1.4), (5.1.5), that is

$$\|u\|_{H^2} + \varepsilon\|x^2u\|_{L^2} \le C(\|u\|_{L^2} + \|H_{\varepsilon,\theta}u\|_{L^2}), \quad \forall\, u \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{5.1.6}$$

It then follows that

$$u \in L^2 \text{ and } H_{\varepsilon,\theta}u \in L^2 \implies u \in H^2 \text{ and } x^2u \in L^2.$$

In other words, $\mathcal{D}(H_{\varepsilon,\theta}) = H^2(\mathbb{R}^n) \cap \langle x\rangle^{-2}L^2(\mathbb{R}^n)$.

The lower bound in (5.1.2) follows from general arguments for operators with analytic coefficients – see Dencker–Sjöstrand–Zworski [10]. The upper bound in (5.1.2) is obtained and shown to be sharp in [27].

To obtain the bounds (5.1.3), we recall (5.1.4) to write

$$\langle (H_{\varepsilon,\theta} - z)u, u\rangle_{L^2} = e^{-2i\theta}\|Du\|_{L^2}^2 - i\varepsilon e^{2i\theta}\|xu\|_{L^2}^2 - z\|u\|_{L^2}^2.$$

Since $-2\theta + \delta < \arg z < 3\pi/2 + 2\theta - \delta$, $|z| > \delta$, we have

$$|z|\|u\|_{L^2}^2 \le C_\delta|\langle (H_{\varepsilon,\theta} - z)u, u\rangle_{L^2}| \implies \|u\|_{L^2} \le C_\delta|z|^{-1}\|H_{\varepsilon,\theta} - z)u\|_{L^2}, \tag{5.1.7}$$

which proves (5.1.3) for $j = 0$. Combining (5.1.6) we conclude that

$$\|u\|_{H^2} \le C(\|u\|_{L^2} + |z|\|u\|_{L^2} + \|(H_{\varepsilon,\theta} - z)u\|_{L^2}) \le C_\delta\|(H_{\varepsilon,\theta} - z)u\|_{L^2},$$

which proves (5.1.3) for $j = 2$, then the case $j = 1$ follows by interpolation. □

5.2 An estimate of the weighted resolvent

In this section we show how exponential weights dramatically improve the estimate (5.1.2) for the resolvent:

$$(-\Delta - i\varepsilon x^2 - \lambda^2)^{-1}, \quad \lambda \in \Omega$$

for Ω defined in (1.3.1). This will be crucial in the proof of Theorem 1.

Using the Fourier transform $\mathcal{F}u(\xi) = \hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} u(x)\, dx$, we have

$$-\Delta_x - i\varepsilon x^2 = \mathcal{F}^{-1}(\xi^2 + i\varepsilon\Delta_\xi)\mathcal{F}.$$

Inspired by [31] and the earlier work by Nakamura [38], first we introduce a family of spectral deformations in the Fourier space as follows.

For any fixed Ω given in (1.3.1), we choose $\rho \in \mathcal{C}^\infty([0,\infty);\mathbb{R})$ with $\rho \equiv 0$ near 0 and $\rho(t) \equiv 1$ for $t \gg 1$ such that

$$0 \le \rho'(t) < \gamma^{-1} \tan\frac{\pi}{8}, \ \forall\, t \ge 0; \quad \Omega \Subset \{x + iy : x > 0,\ y > -\gamma\rho(x)\}, \tag{5.2.1}$$

and define the map

$$\psi : \mathbb{R}^n \to \mathbb{R}^n, \quad \psi(\xi) = |\xi|^{-1}\rho(|\xi|)\,\xi, \tag{5.2.2}$$

then ψ is smooth with the Jacobian:

$$D\psi(\xi) = |\xi|^{-1}\rho(|\xi|)I + (|\xi|^{-2}\rho'(|\xi|) - |\xi|^{-3}\rho(|\xi|))\,\xi\cdot\xi^T. \tag{5.2.3}$$

Let A be an orthogonal matrix with n-th column $|\xi|^{-1}\xi$, then we have

$$A^T D\psi(\xi)\, A = \mathrm{diag}[\,|\xi|^{-1}\rho(|\xi|), \cdots, |\xi|^{-1}\rho(|\xi|),\ \rho'(|\xi|)\,]. \tag{5.2.4}$$

For $\theta \in \mathbb{R}$, we consider a family of deformations:

$$\varphi_\theta(\xi) = \xi + \theta\psi(\xi), \tag{5.2.5}$$

and the corresponding unitary operators $U_\theta,\ \theta \in \mathbb{R}$ defined by

$$U_\theta u(\xi) := (\det D\varphi_\theta(\xi))^{\frac{1}{2}} u(\varphi_\theta(\xi)). \tag{5.2.6}$$

Using (5.2.4), we can compute $\det D\varphi_\theta(\xi)$ explicitly, i.e.

$$J_\theta(\xi) \equiv \det D\varphi_\theta(\xi) = \det(I + \theta D\psi(\xi)) = (1 + \theta\rho'(|\xi|)\,)\,(1 + \theta|\xi|^{-1}\rho(|\xi|)\,)^{n-1}, \tag{5.2.7}$$

then by (5.2.1), U_θ is invertible as $\det D\varphi_\theta(\xi) \ne 0$ for $\theta \in \mathbb{R}$, $|\theta| < \gamma$, the inverse is given by

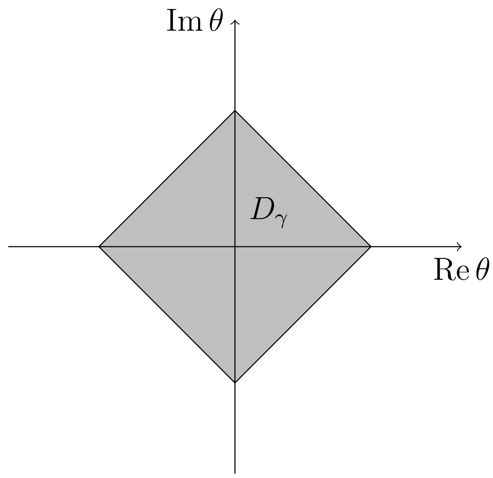
$$U_\theta^{-1} v(\xi) = (\det D\varphi_\theta(\varphi_\theta^{-1}(\xi)))^{-\frac{1}{2}} v(\varphi_\theta^{-1}(\xi)). \tag{5.2.8}$$

Now we consider the deformed operators of $\xi^2 + i\varepsilon\Delta_\xi$:

$$\begin{aligned} Q_{\varepsilon,\theta} &:= U_\theta(\xi^2 + i\varepsilon\Delta_\xi)U_\theta^{-1} \\ &= \varphi_\theta(\xi)^2 - i\varepsilon J_\theta(\xi)^{-\frac{1}{2}} D_{\xi_l} J^{lj}(\xi) J_\theta(\xi) J^{kj}(\xi) D_{\xi_k} J_\theta(\xi)^{-\frac{1}{2}} \end{aligned} \tag{5.2.9}$$

where $D_{\xi_k} = -i\partial_{\xi_k}$, $J_\theta(\xi) = \det D\varphi_\theta(\xi)$, $J^{lj}(\xi) = [D\varphi_\theta(\xi)^{-1}]_{jl}$. To extend $Q_{\varepsilon,\theta}$ to $\theta \in \mathbb{C}$, we define

$$D_\gamma := \{\theta \in \mathbb{C} : |\operatorname{Re}\theta| + |\operatorname{Im}\theta| < \gamma\}. \tag{5.2.10}$$



In view of (5.2.1) and (5.2.7), $D\varphi_\theta^{-1}$ and $\det D\varphi_\theta$ extend analytically to $\theta \in D_\gamma$. , we obtain that $Q_{\varepsilon,\theta}$, given by the second equation in (5.2.9), extends analytically to $\theta \in D_\gamma$.

Then we introduce some preliminary results about the spectrum of $Q_{\varepsilon,\theta}$:

Proposition 5.2.1. *There exists constant $\varepsilon_0 = \varepsilon_0(\Omega, \gamma)$ such that for all $0 < \varepsilon < \varepsilon_0$ and $\theta \in D_\gamma$,*

$$\operatorname{Spec}(Q_{\varepsilon,\theta}) \cap \{z \in \mathbb{C} : |z| > 1, \pi/2 < \arg z < \pi\} = \emptyset.$$

Proof. We note that for $\theta \in D_\gamma$, by (5.2.1),

$$1 - \tan\frac{\pi}{8} < 1 - |\theta||\rho'(t)| \le |1 + \theta\rho'(t)| \le 1 + |\theta||\rho'(t)| < 1 + \tan\frac{\pi}{8}, \quad \forall\, t \ge 0.$$

Thus, (5.2.7) implies that $C^{-1} < |J_\theta(\xi)| < C$ for some constant $C > 0$. Since

$$[D\varphi_\theta(\xi)]_{jl} = \left(1 + \theta\frac{\rho(|\xi|)}{|\xi|}\right)\delta_{jl} + \frac{\theta|\xi|\rho'(|\xi|) - \theta\rho(|\xi|)}{|\xi|^3}\xi_j\xi_l$$

by (5.2.3), and $\rho' \in \mathcal{C}_c^\infty((0,\infty))$, together with (5.2.7), we conclude that

$$J_\theta, J_\theta^{-1}, J^{lj} \in \mathcal{C}_b^\infty(\mathbb{R}^n),\ 1 \le j, l \le n. \tag{5.2.11}$$

Here $\mathcal{C}_b^\infty(\mathbb{R}^n) := \{u \in \mathcal{C}^\infty(\mathbb{R}^n) : |\partial^\alpha u| \le C_\alpha \text{ for all } \alpha \in \mathbb{N}_0^n\}$. Hence we have

$$Q_{\varepsilon,\theta} = \varphi_\theta(\xi)^2 - i\varepsilon J^{kj}(\xi)J^{lj}(\xi)D_{\xi_k}D_{\xi_l} + \varepsilon a_j(\xi)D_{\xi_j} + \varepsilon b(\xi), \tag{5.2.12}$$

where $a_j, b \in \mathcal{C}_b^\infty(\mathbb{R}^n)$. Let $h = \sqrt{\varepsilon}$, then $Q_{\varepsilon,\theta} = q_\theta^{\mathrm{w}}(\xi, hD_\xi)$ is a semiclassical differential operator with the full symbol:

$$q_\theta(\xi, \xi^*; h) = \varphi_\theta(\xi)^2 - i(D\varphi_\theta(\xi)^{-2}\xi^*)\cdot\xi^* + ha_j(\xi)\xi_j^* + h^2 b(\xi), \tag{5.2.13}$$

where (ξ, ξ^*) are coordinates of $\mathrm{T}^*\mathbb{R}^n$,

$$D\varphi_\theta(\xi)^{-2} = (D\varphi_\theta(\xi)^{-1})^T(D\varphi_\theta(\xi)^{-1})$$

as $D\varphi_\theta(\xi)$ is a symmetric matrix. Choose $m(\xi,\xi^*) = 1 + \xi^2 + \xi^{*2}$ as an order function, we recall the symbol class $S(m)$:

$$S(m) := \{a \in \mathcal{C}^\infty : |\partial^\alpha a| \le C_\alpha m \quad \text{for } \forall\, \alpha \in \mathbb{N}_0^{2n}\}. \tag{5.2.14}$$

Then by (5.2.1), (5.2.5) and (5.2.11), we have $q_\theta \in S(m)$. In view of §2.1.4 and Example 2.1.8, we aim to show that there exists constant $h_0 > 0$ such that for $h < h_0$,

$$q_\theta - z \text{ is elliptic in } S(m) \text{ for } |z| > 1,\ \pi/2 < \arg z < \pi.$$

Using (5.2.2) we calculate:

$$\varphi_\theta(\xi)^2 = (\xi + \theta\psi(\xi))\cdot(\xi + \theta\psi(\xi)) = (|\xi| + \theta\rho(|\xi|))^2. \tag{5.2.15}$$

Then for $\theta \in D_\gamma$, by (5.2.1), we have

$$-\pi/4 < \arg\varphi_\theta(\xi)^2 < \pi/4, \quad |\varphi_\theta(\xi)^2| > \left(1 - \tan\frac{\pi}{8}\right)^2 |\xi|^2. \tag{5.2.16}$$

To obtain similar bounds for the argument and modulus of $(D\varphi_\theta(\xi)^{-2}\xi^*)\cdot\xi^*$, we recall (5.2.4) to compute

$$(D\varphi_\theta^{-2}\xi^*)\cdot\xi^* = (1 + \theta\rho(|\xi|)|\xi|^{-1})^{-2}(\eta_1^{*2} + \cdots + {\eta_{n-1}^*}^2) + (1 + \theta\rho'(|\xi|))^{-2}\eta_n^{*2}, \tag{5.2.17}$$

where $\eta^* = A^T\xi^* \in \mathbb{R}^n$ with the same orthogonal matrix A as in (5.2.4). By (5.2.1), for $\theta \in D_\gamma$, we have

$$\pm\operatorname{Im}\theta \ge 0 \implies 0 \le \pm\arg(1 + \theta\rho(|\xi|)|\xi|^{-1}),\ \pm\arg(1 + \theta\rho'(|\xi|)) < \pi/8,$$

Hence, for all $\theta \in D_\gamma$,

$$\pm\operatorname{Im}\theta \ge 0 \implies 0 \le \mp\arg\left(D\varphi_\theta^{-2}\xi^*\right)\cdot\xi^* < \pi/4, \tag{5.2.18}$$

and by applying the following basic inequality with (5.2.1) to (5.2.17),

$$|r_1 e^{i\theta_1} + r_2 e^{i\theta_2}|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) \geq \frac{1 - |\cos(\theta_1 - \theta_2)|}{2}(r_1 + r_2)^2, \quad (5.2.19)$$

we also obtain that for all $\theta \in D_\gamma$,

$$|(D\varphi_\theta^{-2}\xi^*) \cdot \xi^*| \geq C|\eta^*|^2 = C|\xi^*|^2. \quad (5.2.20)$$

Since $\arg(\varphi_\theta(\xi)^2 - z) \in (-\pi/2, \pi/4)$ for $\pi/2 < \arg z < \pi$ and $\arg -i(D\varphi_\theta^{-2}\xi^*) \cdot \xi^* \in (-3\pi/4, -\pi/4)$ by (5.2.18), using (5.2.19) together with (5.2.16) and (5.2.20), we have

$$\begin{aligned} |\varphi_\theta(\xi)^2 - z - i(D\varphi_\theta^{-2}\xi^*) \cdot \xi^*| &\geq C|\varphi_\theta(\xi)^2 - z| + C| - i(D\varphi_\theta^{-2}\xi^*) \cdot \xi^*| \\ &\geq C|\varphi_\theta(\xi)^2| + C|z| + C|\xi^*|^2 \\ &\geq C(1 + |\xi|^2 + |\xi^*|^2) = Cm. \end{aligned} \quad (5.2.21)$$

Then by (5.2.13), we conclude that there exists $h_0 > 0$ such that for all $0 < h < h_0$, $|z| > 1$, $\pi/2 < \arg z < \pi$, we have $|q_\theta - z| \geq Cm$.

Recalling Example 2.1.8 that the principal symbol of $m^{\mathrm{w}}(\xi, hD)^{-1}$ is m^{-1}, we have

$$(q_\theta^{\mathrm{w}}(\xi, hD) - z)\, m^{\mathrm{w}}(\xi, hD)^{-1} = a_z^{\mathrm{w}}(\xi, hD), \quad a_z \text{ is elliptic in } S(1).$$

By Proposition 2.1.7, $a_z^{\mathrm{w}}(\xi, hD)$ has an inverse on L^2, thus for ε small enough,

$$(Q_{\varepsilon,\theta} - z)^{-1} = m^{\mathrm{w}}(\xi, hD)^{-1} a_z^{\mathrm{w}}(\xi, hD)^{-1}, \quad h = \sqrt{\varepsilon},$$

which completes the proof. □

Proposition 5.2.2. *For any $\beta \in (\gamma', \gamma)$ satisfying*

$$\Omega \Subset \{x + iy : x > 0,\ y > -\beta\rho(x)\}, \quad (5.2.22)$$

there exists $\varepsilon_0 = \varepsilon_0(\Omega, \gamma, \beta)$ such that for all $0 < \varepsilon < \varepsilon_0$,

$$\mathrm{Spec}(Q_{\varepsilon,-i\beta}) \cap \{\lambda^2 : \lambda \in \Omega\} = \emptyset.$$

Proof. Let $m = 1 + \xi^2 + \xi^{*2}$, as in the proof of Proposition 5.2.1, it suffices to show that there exists $h_0 = h_0(\Omega, \gamma, \beta)$ such that for $0 < h < h_0$,

$$q_{-i\beta}(\xi, \xi^*; h) - \lambda^2 \text{ is elliptic in } S(m) \text{ for } \lambda \in \Omega.$$

For a numerical illustration, see Figure 5.1.

Recalling (5.2.18) that

$$\arg -i(D\varphi_{-i\beta}^{-2}\xi^*) \cdot \xi^* \in [-\pi/2, -\pi/4),$$

in order to apply (5.2.19), we claim that for all $\lambda \in \Omega$, $\xi \in \mathbb{R}^n$,

$$\exists\, \delta > 0 \text{ s.t. } \arg(\varphi_{-i\beta}(\xi)^2 - \lambda^2) \le \pi/2 - \delta \text{ or } \ge 3\pi/4 + \delta. \tag{5.2.23}$$

In view of (5.2.15) We have for $|\xi| \gg 1$,

$$\varphi_{-i\beta}(\xi)^2 = (|\xi| - i\beta)^2 \implies \arg(\varphi_{-i\beta}(\xi)^2 - \lambda^2) \in (-\pi/4, 0),$$

in other words, there exists some large R such that (5.2.23) holds for $|\xi| > R$ with $\delta = \pi/2$. It remains to show that (5.2.23) holds for all $|\xi| \le R$ and $\lambda \in \Omega$. We argue by contradiction: if it does not hold, there must exist $\lambda \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$ such that $\arg(\varphi_{-i\beta}(\xi)^2 - \lambda^2) \in [\pi/2, 3\pi/4]$, i.e.

$$0 \le -\operatorname{Re}\left(\left(|\xi| - i\beta\rho(|\xi|)\right)^2 - \lambda^2\right) \le \operatorname{Im}\left(\left(|\xi| - i\beta\rho(|\xi|)\right)^2 - \lambda^2\right),$$

which immediately implies $\operatorname{Im}\lambda \le 0$. Let $t = |\xi|$ and write $\lambda = x - iy$, then we have

$$x^2 - y^2 - t^2 + \beta^2\rho(t)^2 \le 2xy - 2\beta t\rho(t) \tag{5.2.24}$$

$$\beta t\rho(t) \le xy \tag{5.2.25}$$

Since $x > 0$ and $0 \le y < \beta\rho(x)$ by (5.2.22), then (5.2.24) implies that

$$x^2 - 2\beta x\rho(x) - \beta^2\rho(x)^2 < t^2 - 2\beta t\rho(t) - \beta^2\rho(t)^2.$$

Let $S(x) = x^2 - 2\beta x\rho(x) - \beta^2\rho(x)^2$, by (5.2.1),

$$\begin{aligned} S'(x) &= 2x\left(1 - \beta\frac{\rho(x)}{x} - \beta\rho'(x) - \beta\frac{\rho(x)}{x}\cdot\beta\rho'(x)\right) \\ &> 2x\left(1 - 2\tan\frac{\pi}{8} - \tan^2\frac{\pi}{8}\right) = 0, \end{aligned}$$

thus $S(x) < S(t) \implies x < t$. Recalling that ρ is non-decreasing, we have $\beta t\rho(t) \ge \beta x\rho(x) > xy$, which contradicts (5.2.25). Hence (5.2.23) holds, using (5.2.19) and (5.2.20), we obtain that

$$|\varphi_{-i\beta}(\xi)^2 - \lambda^2 - i(D\varphi_{-i\beta}^{-2}\xi^*)\cdot\xi^*| \ge C(\delta)\left(|(|\xi| - i\beta\rho(|\xi|))^2 - \lambda^2| + |\xi^*|^2\right).$$

Since for $|\xi| \gg 1$,

$$|(|\xi| - i\beta\rho(|\xi|))^2 - \lambda^2| = |(|\xi| - i\beta)^2 - \lambda^2| \ge |\xi|^2 - \beta^2 - |\lambda|^2,$$

there exists $R = R(\Omega, \beta) > 0$ such that $|(|\xi| - i\beta\rho(|\xi|))^2 - \lambda^2| \ge (1 + |\xi|^2)/2$ whenever $|\xi| > R$. We also note that, by (5.2.22),

$$\operatorname{dist}\left(\{t - i\beta\rho(t) : t \ge 0\}, \pm\Omega\right) \ge C = C(\Omega, \gamma, \beta) > 0,$$

thus for $|\xi| \le R$,

$$|(|\xi| - i\beta\rho(|\xi|))^2 - \lambda^2| \ge C^2 \ge C^2(1+R^2)^{-1}(1+|\xi|^2).$$

Hence for some constant $C > 0$ determined by Ω, γ, β,

$$|\varphi_{-i\beta}(\xi)^2 - \lambda^2 - i(D\varphi_{-i\beta}^{-2}\xi^*)\cdot\xi^*| \ge C(1+|\xi|^2+|\xi^*|^2).$$

Then by (5.2.13), we conclude that there exist $h_0 = h_0(\Omega,\gamma,\beta)$ and $C = C(\Omega,\gamma,\beta) > 0$ such that

$$\text{for all } 0 < h < h_0,\ \lambda \in \Omega, \quad |q_{-i\beta}(\xi,\xi^*;h) - \lambda^2| \ge Cm. \tag{5.2.26}$$

Let $b = m/(q_{-i\beta} - \lambda^2) \in S(1)$, then there exists $r \in S(1)$ such that

$$(q^{\mathrm{w}}_{-i\beta}(\xi, hD) - \lambda^2)m^{\mathrm{w}}(\xi,hD)^{-1}b^{\mathrm{w}}(\xi,hD) = I + hr^{\mathrm{w}}(\xi,hD).$$

We may assume that $h_0(\Omega,\gamma,\beta)\|r^{\mathrm{w}}\|_{L^2\to L^2} < 1/2$, then for all $0 < h < h_0$,

$$(q^{\mathrm{w}}_{-i\beta}(\xi,hD) - \lambda^2)^{-1} = m^{\mathrm{w}}(\xi,hD)^{-1}b^{\mathrm{w}}(\xi,hD)(I + hr^{\mathrm{w}}(\xi,hD))^{-1},$$

which completes the proof. □

Now we state the main result of this section:

Lemma 5. *For any $0 < a' < a < b$ and $\gamma' < \gamma$ such that the rectangle*

$$\Omega := (a', a) + i(-\gamma', b) \Subset \{\lambda \in \mathbb{C} : -\pi/8 < \arg\lambda < 7\pi/8\},$$

there exist constant $C = C(\Omega,\gamma) > 0$ and $\varepsilon_0 = \varepsilon_0(\Omega,\gamma) > 0$ such that uniformly for $0 < \varepsilon < \varepsilon_0$,

$$\|e^{-\gamma|x|}(-\Delta - i\varepsilon x^2 - \lambda^2)^{-1}e^{-\gamma|x|}\|_{L^2\to L^2} \le C, \quad \forall\,\lambda\in\Omega.$$

Proof. We consider the matrix element

$$B^{\varepsilon}_{f,g}(\lambda) := \langle e^{-\gamma|x|}(-\Delta - i\varepsilon x^2 - \lambda^2)^{-1}e^{-\gamma|x|}f, g\rangle_{L^2_x}, \quad \text{for } f,g \in L^2(\mathbb{R}^n),$$

where $\langle u, v\rangle_{L^2_x} = \int_{\mathbb{R}^n} u\bar{v}\,dx$ is the standard L^2 inner product. It suffices to show that there exist C, ε_0 such that uniformly for $0 < \varepsilon < \varepsilon_0$,

$$|B^{\varepsilon}_{f,g}(\lambda)| \le C\|f\|_{L^2}\|g\|_{L^2}, \quad \text{for all } f, g \in L^2,\ \lambda\in\Omega. \tag{5.2.27}$$

Recalling (5.1.1), both $-\Delta_x - i\varepsilon x^2 - \lambda^2$ and $\xi^2 + i\varepsilon\Delta_\xi - \lambda^2$ are invertible for $\lambda \in \Omega$. Then we have

$$\begin{aligned} B^{\varepsilon}_{f,g}(\lambda) &= \langle\,(-\Delta_x - i\varepsilon x^2 - \lambda^2)^{-1}e^{-\gamma|x|}f,\, e^{-\gamma|x|}g\rangle_{L^2_x} \\ &= \langle \mathcal{F}^{-1}(\xi^2 + i\varepsilon\Delta_\xi - \lambda^2)^{-1}\mathcal{F}e^{-\gamma|x|}f,\, e^{-\gamma|x|}g\rangle_{L^2_x} \\ &= \langle\,(\xi^2 + i\varepsilon\Delta_\xi - \lambda^2)^{-1}\mathcal{F}(e^{-\gamma|x|}f)(\xi),\, \mathcal{F}(e^{-\gamma|x|}g)(\xi)\rangle_{L^2_\xi}. \end{aligned} \tag{5.2.28}$$

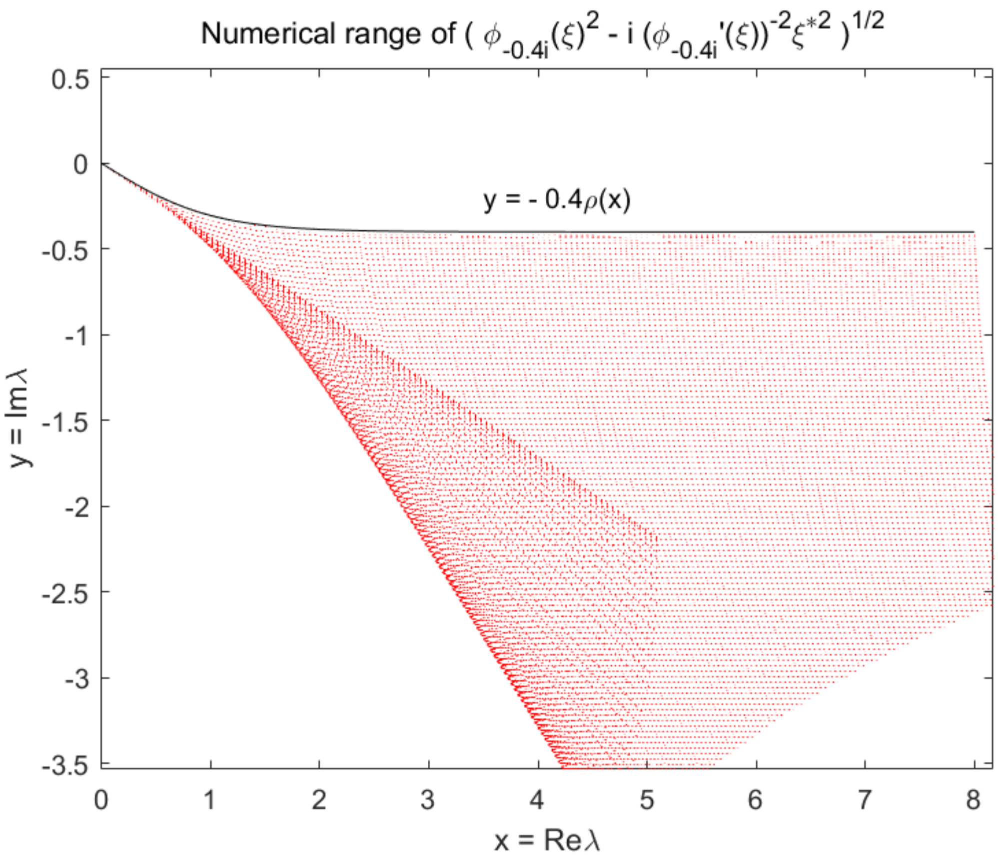


Figure 5.1: An illustration of the ellipticity of the deformed operator in the case of dim $= 1$, $\beta = 0.4$, which shows that the numerical range of the principal symbol of $ih^2\Delta_\xi + |\xi|^2$ is compressed to avoid the region $\{\lambda^2 : \lambda \in \Omega\}$. We choose $\rho(\cdot) = 0.4\tanh(\cdot)$ for calculation.

Let $F_\gamma(\xi) := \mathcal{F}(e^{-\gamma|x|}f)(\xi)$ and $G_\gamma(\xi) := \mathcal{F}(e^{-\gamma|x|}g)(\xi)$, recalling the formula

$$\mathcal{F}(e^{-|x|})(\xi) = c_n(1+\xi^2)^{-\frac{n+1}{2}}, \quad c_n = (2\pi)^{\frac{n}{2}}\Gamma((n+1)/2)\pi^{-\frac{n+1}{2}},$$

then $F_\gamma = K_\gamma * \hat{f}$ and $G_\gamma = K_\gamma * \hat{g}$, where $K_\gamma(\xi) = c_n\gamma\,(\gamma^2+\xi^2)^{-\frac{n+1}{2}}$.

First we consider, for $\theta \in \mathbb{R}$, $|\theta| < \gamma$ and U_θ defined by (5.2.6), the integral kernel of the map $U_\theta \circ (K_\gamma * \,)$:

$$K(\xi, \eta; \theta) := (\det D\varphi_\theta(\xi))^{\frac{1}{2}} K_\gamma(\varphi_\theta(\xi) - \eta), \quad \xi, \eta \in \mathbb{R}^n.$$

We claim that $K(\xi, \eta; \theta)$ has an analytic extension to $\theta \in D_\gamma$. Since K_γ extends analytically to the strip $\{\xi \in \mathbb{C}^n : |\operatorname{Im}\xi| < \gamma\}$, it suffices to show that $|\operatorname{Im}(\varphi_\theta(\xi) - \eta)| =$

$|\operatorname{Im}\theta\psi(\xi)| < \gamma$, which is a direct consequence of $\theta \in D_\gamma$ and $|\psi(\xi)| \le 1$ by (5.2.2). Then for $\theta \in D_\gamma$, using (5.2.1) and (5.2.7), we can estimate $K(\xi,\eta;\theta)$ as follows:

$$\begin{aligned}|K(\xi,\eta;\theta)| &\le C\gamma\,|\gamma^2 + (\xi+\theta\psi(\xi)-\eta)^2|^{-\frac{n+1}{2}}\\ &\le C\gamma\,|\gamma^2 - |\operatorname{Im}\theta|^2|\psi(\xi)|^2 + (\xi-\eta+\operatorname{Re}\theta\psi(\xi))^2|^{-\frac{n+1}{2}}\\ &\le C\gamma\,(\gamma^2 - |\operatorname{Im}\theta|^2 + (|\xi-\eta| - |\operatorname{Re}\theta|)^2)^{-\frac{n+1}{2}}\end{aligned}$$

thus

$$\begin{aligned}&\max\{\sup_{\xi\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(\xi,\eta;\theta)|d\eta,\ \sup_{\eta\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(\xi,\eta;\theta)|d\xi\,\}\\ &\le C\gamma\int_{x\in\mathbb{R}^n}(\gamma^2 - |\operatorname{Im}\theta|^2 + (|x| - |\operatorname{Re}\theta|)^2)^{-\frac{n+1}{2}}dx \le C(\gamma,\theta).\end{aligned} \tag{5.2.29}$$

Hence, by Schur's criterion, $U_\theta \circ (K_\gamma * \,)$, first defined for $\theta \in D_\gamma \cap \mathbb{R}$, with the integral kernel $K(\xi,\eta;\theta)$, extends to $\theta \in D_\gamma$ as an analytic family of operators $L^2 \to L^2$. In particular,

$$D_\gamma \ni \theta \mapsto U_\theta F_\gamma = U_\theta(K_\gamma * \hat{f}) \text{ and } U_\theta G_\gamma = U_\theta(K_\gamma * \hat{g}),$$

are two analytic families of functions in $L^2(\mathbb{R}^n)$.

Now we define

$$B^\varepsilon_{f,g}(\lambda;\theta) = \langle\,(Q_{\varepsilon,\theta} - \lambda^2)^{-1}U_\theta F_\gamma, U_{\bar\theta}G_\gamma\rangle$$

for $\theta \in D_\gamma$, with $Q_{\varepsilon,\theta}$ given by (5.2.9), where we write $U_{\bar\theta}G_\gamma$ instead of $U_\theta G_\gamma$. Then by Proposition 5.2.1, there exists $\varepsilon_0 = \varepsilon_0(\Omega,\gamma)$ such that for all $0 < \varepsilon < \varepsilon_0$, and $|\lambda| > 1$ with $\pi/4 < \arg\lambda < \pi/2$,

$$D_\gamma \ni \theta \mapsto B^\varepsilon_{f,g}(\lambda;\theta) \text{ is analytic.}$$

However, for $\theta \in D_\gamma \cap \mathbb{R}$, since U_θ is unitary, by (5.2.28) we have

$$\begin{aligned}B^\varepsilon_{f,g}(\lambda;\theta) &= \langle U_\theta(\xi^2 + i\varepsilon\Delta_\xi - \lambda^2)^{-1}U_\theta^{-1}U_\theta F_\gamma,\, U_\theta G_\gamma\rangle\\ &= \langle U_\theta(\xi^2 + i\varepsilon\Delta_\xi - \lambda^2)^{-1}F_\gamma,\, U_\theta G_\gamma\rangle\\ &= \langle\,(\xi^2 + i\varepsilon\Delta_\xi - \lambda^2)^{-1}F_\gamma,\, G_\gamma\rangle = B^\varepsilon_{f,g}(\lambda).\end{aligned}$$

Thus by analyticity, $B^\varepsilon_{f,g}(\lambda;\theta) \equiv B^\varepsilon_{f,g}(\lambda)$, $\forall\,\theta \in D_\gamma$ whenever $|\lambda| > 1$, $\pi/4 < \arg\lambda < \pi/2$. In particular, for fixed $\beta \in (\gamma',\gamma)$ satisfying (5.2.22),

$$B^\varepsilon_{f,g}(\lambda) = B^\varepsilon_{f,g}(\lambda;-i\beta) \text{ whenever } |\lambda| > 1,\ \pi/4 < \arg\lambda < \pi/2.$$

In view of Proposition 5.2.2 and (5.1.1), both $B^\varepsilon_{f,g}(\lambda)$ and $B^\varepsilon_{f,g}(\lambda;-i\beta)$ are analytic in Ω. Without loss of generality, we may assume that $a > 1$ in (1.3.1), then

$$\Omega \cap \{\lambda : |\lambda| > 1, \pi/4 < \arg\lambda < \pi/2\} \ne \emptyset,$$

where $B^\varepsilon_{f,g}(\lambda)$ and $B^\varepsilon_{f,g}(\lambda;-i\beta)$ coincide. Hence by analyticity, we conclude that for each $0<\varepsilon<\varepsilon_0$,

$$B^\varepsilon_{f,g}(\lambda)=B^\varepsilon_{f,g}(\lambda;-i\beta)=\langle\,(Q_{\varepsilon,-i\beta}-\lambda^2)^{-1}U_{-i\beta}F_\gamma,\,U_{i\beta}G_\gamma\,\rangle,\quad\forall\,\lambda\in\Omega. \tag{5.2.30}$$

By the elliptic theory of semiclassical differential operators introduced in §2.1, (5.2.26) implies that there exists $\varepsilon_0=\varepsilon_0(\Omega,\gamma,\beta)$ such that for all $0<\varepsilon<\varepsilon_0$,

$$\|(Q_{\varepsilon,-i\beta}-\lambda^2)^{-1}\|_{L^2\to L^2}\le C(\Omega,\gamma,\beta),\quad\forall\,\lambda\in\Omega. \tag{5.2.31}$$

Recalling (5.2.29), by Schur's criterion, we obtain that

$$\begin{aligned}\|U_{-i\beta}F_\gamma\|_{L^2}=\|U_{-i\beta}\circ(K_\gamma*\hat f)\|_{L^2}\le C(\gamma,\beta)\|\hat f\|_{L^2}=C(\gamma,\beta)\|f\|_{L^2}\\ \|U_{i\beta}G_\gamma\|_{L^2}=\|U_{i\beta}\circ(K_\gamma*\hat g)\|_{L^2}\le C(\gamma,\beta)\|\hat g\|_{L^2}=C(\gamma,\beta)\|g\|_{L^2}\end{aligned} \tag{5.2.32}$$

Combining (5.2.30), (5.2.31) and (5.2.32), also noticing that β can be determined by Ω,γ, we obtain (5.2.27) with $C=C(\Omega,\gamma)$, which completes the proof. □

5.3 The regularized operator with exponentially decaying potential

In this section we study the regularized operator

$$P_\varepsilon=-\Delta+V-i\varepsilon x^2,$$

where $|V(x)|\le Ce^{-2\gamma|x|}$ for some constants $C,\gamma>0$.

We will review the meromorphy of the resolvent

$$R_{V,\varepsilon}(\lambda):=(P_\varepsilon-\lambda^2)^{-1},\quad\varepsilon>0,$$

in a similar form to the meromorphic continuation of $\sqrt{V}R_V(\lambda)\sqrt{V}$ given by (3.2.3).

First we write $R_\varepsilon(\lambda):=(-\Delta-i\varepsilon x^2-\lambda^2)^{-1}$ then

$$(P_\varepsilon-\lambda^2)R_\varepsilon(\lambda)=I+VR_\varepsilon(\lambda),\quad-\pi/8<\arg\lambda<7\pi/8. \tag{5.3.1}$$

Since $R_\varepsilon(\lambda):L^2\to H^2$ is analytic in $\{\lambda:-\pi/8<\arg\lambda<7\pi/8\}$, see (5.1.1), $V:H^2\to L^2$ is compact by (3.2.1), we have

$$\lambda\mapsto VR_\varepsilon(\lambda):L^2\to L^2,\quad-\pi/8<\arg\lambda<7\pi/8,$$

is an analytic family of compact operators. Using (5.1.3) (with $\theta=0$), $I+VR_\varepsilon(\lambda)$ is invertible for $\pi/4<\arg\lambda<\pi/2$, $|\lambda|\gg1$. By Theorem 4,

$$\lambda\mapsto(I+VR_\varepsilon(\lambda))^{-1}:L^2\to L^2,\quad-\pi/8<\arg\lambda<7\pi/8,$$

is a meromorphic family of operators. Using (5.3.1), we conclude that

$$R_{V,\varepsilon}(\lambda) = R_\varepsilon(\lambda)(I + VR_\varepsilon(\lambda))^{-1}$$

is meromorphic for $-\pi/8 < \arg\lambda < 7\pi/8$ (in fact $R_{V,\varepsilon}(\lambda)$ is meromorphic for $\lambda \in \mathbb{C}$ by Theorem 5). The poles of $R_{V,\varepsilon}(\lambda)$ are $\{\lambda_j(\varepsilon)\}_{j=1}^\infty$ where

$$\operatorname{Spec}(P_\varepsilon) \cap \mathbb{C} \setminus e^{-i\pi/4}[0,\infty) = \{\lambda_j(\varepsilon)^2\}_{j=1}^\infty, \quad -\pi/8 < \arg\lambda_j(\varepsilon) < 7\pi/8. \tag{5.3.2}$$

Then we have

Lemma 6. *For each $\varepsilon > 0$,*

$$\lambda \mapsto (I + \sqrt{V}R_\varepsilon(\lambda)\sqrt{V})^{-1}, \quad -\pi/8 < \arg\lambda < 7\pi/8,$$

is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ with poles of finite rank. Moreover,

$$m_\varepsilon(\lambda) := \frac{1}{2\pi i}\operatorname{tr}\oint_\lambda (I + \sqrt{V}R_\varepsilon(\zeta)\sqrt{V})^{-1}\partial_\zeta(\sqrt{V}R_\varepsilon(\zeta)\sqrt{V})\,d\zeta, \tag{5.3.3}$$

where the integral is over a positively oriented circle enclosing λ and containing no poles other than possibly λ, satisfies

$$m_\varepsilon(\lambda) = \frac{1}{2\pi i}\operatorname{tr}\oint_\lambda (\zeta^2 - P_\varepsilon)^{-1}2\zeta\,d\zeta. \tag{5.3.4}$$

Remark. The multiplicity of an eigenvalue λ^2 of P_ε can be defined by the right side of (5.3.4), thus Lemma 6 implies that the poles of $(I + \sqrt{V}R_\varepsilon(\lambda)\sqrt{V})^{-1}$ coincide with $\{\lambda_j(\varepsilon)\}_{j=1}^\infty$ given in Theorem 1, with agreement of multiplicities.

Proof. Following the above argument, it easy to see that $\lambda \mapsto \sqrt{V}R_\varepsilon(\lambda)\sqrt{V}$ is an analytic family of compact operators for $-\pi/8 < \arg\lambda < 7\pi/8$. Then

$$\lambda \mapsto (I + \sqrt{V}R_\varepsilon(\lambda)\sqrt{V})^{-1}, \quad -\pi/8 < \arg\lambda < 7\pi/8,$$

is a meromorphic family of operators, since $I + \sqrt{V}R_\varepsilon(\lambda)\sqrt{V}$ is invertible for $\pi/4 < \arg\lambda < \pi/2$, $|\lambda| \gg 1$ by (5.1.3). In this range of λ, $I + VR_\varepsilon(\lambda)$ is also invertible by the Neumann series argument, thus we have

$$\begin{aligned}
(P_\varepsilon - \lambda^2)^{-1} &= R_\varepsilon(\lambda)(I + VR_\varepsilon(\lambda))^{-1} \\
&= R_\varepsilon(\lambda)\sum_{j=0}^\infty (-1)^j (VR_\varepsilon(\lambda))^j \\
&= R_\varepsilon(\lambda)(I - \sqrt{V}\sum_{j=0}^\infty (-1)^j(\sqrt{V}R_\varepsilon(\lambda)\sqrt{V})^j\,\sqrt{V}R_\varepsilon(\lambda)) \\
&= R_\varepsilon(\lambda)[\,I - \sqrt{V}(I + \sqrt{V}R_\varepsilon(\lambda)\sqrt{V})^{-1}\sqrt{V}R_\varepsilon(\lambda)\,].
\end{aligned} \tag{5.3.5}$$

Since both sides of (5.3.5) are meromorphic for $-\pi/8 < \arg\lambda < 7\pi/8$, by meromorphy, we conclude that (5.3.5) holds for all $-\pi/8 < \arg\lambda < 7\pi/8$, as an identity between meromorphic families of operators.

To obtain the multiplicity formula, we fix any λ with $-\pi/8 < \arg\lambda < 7\pi/8$, then there exists a neighborhood $\lambda \in U$ in this half plane and finite rank operators A_j, $1 \le j \le J$ such that

$$(I + \sqrt{V} R_\varepsilon(\zeta)\sqrt{V})^{-1} - \sum_{j=1}^{J} \frac{A_j}{(\zeta-\lambda)^j} \text{ is analytic in } \zeta \in U.$$

Let $\mathcal{C}_\lambda \subset U$ be a positively oriented circle enclosing λ and containing no poles of $(I+\sqrt{V}R_\varepsilon(\zeta)\sqrt{V})^{-1}$ other than possibly λ, thus it also contains no poles of $(\zeta^2 - P_\varepsilon)^{-1}$ other than possibly λ as a consequence of (5.3.5). On the one hand, we can compute

$$\begin{aligned} m_\varepsilon(\lambda) &= \frac{1}{2\pi i}\operatorname{tr}\int_{\mathcal{C}_\lambda} (I + \sqrt{V}R_\varepsilon(\zeta)\sqrt{V})^{-1}\sqrt{V}R_\varepsilon(\zeta)^2\sqrt{V}\,2\zeta d\zeta \\ &= \frac{1}{2\pi i}\operatorname{tr}\int_{\mathcal{C}_\lambda}\sum_{j=1}^{J}\frac{A_j\sqrt{V}R_\varepsilon(\zeta)^2 2\zeta\sqrt{V}}{(\zeta-\lambda)^j}d\zeta \\ &= \sum_{j=1}^{J}\sum_{k=0}^{j-1}\frac{1}{k!(j-1-k)!}\operatorname{tr} A_j\sqrt{V}\,\partial_\zeta^k R_\varepsilon(\zeta)\,\partial_\zeta^{j-1-k}(R_\varepsilon(\zeta)2\zeta)\sqrt{V}. \end{aligned} \tag{5.3.6}$$

On the other hand, by (5.3.5), we have

$$\begin{aligned} &\frac{1}{2\pi i}\operatorname{tr}\oint_\lambda (\zeta^2 - P_\varepsilon))^{-1}2\zeta d\zeta \\ =& \frac{1}{2\pi i}\operatorname{tr}\int_{\mathcal{C}_\lambda}\sum_{j=1}^{J}\frac{R_\varepsilon(\zeta)2\zeta\sqrt{V}A_j\sqrt{V}R_\varepsilon(\zeta)}{(\zeta-\lambda)^j}d\zeta \\ =& \sum_{j=1}^{J}\sum_{k=0}^{j-1}\frac{1}{k!(j-1-k)!}\operatorname{tr}\partial_\zeta^{j-1-k}(R_\varepsilon(\zeta)2\zeta)\sqrt{V}A_j\sqrt{V}\,\partial_\zeta^k R_\varepsilon(\zeta). \end{aligned} \tag{5.3.7}$$

Now we compare (5.3.6) and (5.3.7), since each A_j has finite rank, we can apply cyclicity of the trace to obtain the multiplicity formula (5.3.4). □

5.4 The regularized black box Hamiltonian and its analytic distortion

In this section we study the CAP-regularized black box Hamiltonian. We take P to be a long range perturbation of $-\Delta$ introduced in §3.4, that is

$$P : \mathcal{H} \to \mathcal{H} \quad \text{with domain } \mathcal{D}(P),$$

which satisfies (3.3.2) – (3.3.4) and (3.4.1) – (3.4.3). Let

$$P_\varepsilon = P - i\varepsilon(1-\chi(x))x^2, \quad \varepsilon > 0, \tag{5.4.1}$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 near $\overline{B(0,R_0)}$, $x^2 = x_1^2 + \cdots + x_n^2$. We show that P_ε is an unbounded operator on $\mathcal{H}$ with a discrete spectrum and that the spectrum of P_ε is invariant under complex scaling.

Choosing R_1 such that $\operatorname{supp}\chi \subset B(0,R_1)$ when we construct the complex contours Γ_θ in §3.5, the CAP $-i\varepsilon(1-\chi(x))x^2$ can be analytically extended to Γ_θ, thus it defines a multiplication on the following subspace of $\mathcal{H}_\theta$ (with $\mathcal{H}_\theta$ given by (3.5.2)):

$$\widehat{\mathcal{H}}_\theta := \mathcal{H}_{R_0} \oplus |x_\theta|^{-2}L^2(\Gamma_\theta \setminus B(0,R_0)),$$

where $x_\theta := f_\theta(x)$ denotes the parametrization of Γ_θ.

We now introduce the analytic distortion of P_ε on Γ_θ, $\theta \in [0,\theta_0)$:

$$\mathcal{P}_{\varepsilon,\theta} := \mathcal{P}_\theta - i\varepsilon(1-\chi(x_\theta))x_\theta^2, \quad \text{with the domain } \widehat{\mathcal{D}}_\theta := \mathcal{D}_\theta \cap \widehat{\mathcal{H}}_\theta. \tag{5.4.2}$$

It follows from (3.5.7) that $\mathcal{P}_{\varepsilon,\theta}$ near infinity is close to the Davies harmonic oscillator introduced in §5.1,

$$H_{\varepsilon,\theta} = -e^{-2i\theta}\Delta - i\varepsilon e^{2i\theta}x^2.$$

Using Proposition 5.1.1 we show that $\mathcal{P}_{\varepsilon,\theta}$ is a Fredholm operator for $z \notin e^{-i\pi/4}[0,\infty)$.

Lemma 7. *If $z \in \mathbb{C}\setminus\{0\}$, $\arg z \neq -\pi/4$, then for each $\varepsilon > 0$ and $0 \le \theta < \theta_0$, $\mathcal{P}_{\varepsilon,\theta} - z : \widehat{\mathcal{D}}_\theta \to \mathcal{H}_\theta$ is a Fredholm operator of index 0. In particular the spectrum of $\mathcal{P}_{\varepsilon,\theta}$ in $\mathbb{C}\setminus e^{-i\pi/4}[0,\infty)$ is discrete.*

Proof. We choose $\chi_j \in \mathcal{C}_c^\infty(\Gamma_\theta)$, $j = 0,1,2,3$, such that $\chi_j = 1$ near $\operatorname{supp}\chi_{j-1}$ and that $\chi_0(g_\theta(t)\omega) = 1$ for any $t \le T_0$, thus $1-\chi_j$ are supported in the region where $\Gamma_\theta \ni x_\theta = e^{i\theta}x$, $x \in \mathbb{R}^n$. Lemma 5.1.1 then shows that if $\arg z \neq -\pi/4$,

$$(1-\chi_0)(H_{\varepsilon,\theta}-z)^{-1}(1-\chi_1) : \mathcal{H}_\theta \to \widehat{\mathcal{D}}_\theta.$$

Now we fix $z \in \mathbb{C}\setminus\{0\}$ with $\arg z \neq -\pi/4$. Using (3.5.7) we may assume that $\operatorname{supp}\chi_0$ is large enough so that

$$\|(\mathcal{P}_{\varepsilon,\theta} - H_{\varepsilon,\theta})(1-\chi_0)(H_{\varepsilon,\theta}-z)^{-1}(1-\chi_1)\|_{\mathcal{H}_\theta\to\mathcal{H}_\theta} \le 1/2.$$

Then we choose $z_0 = iL$, $L \gg 1$ using (3.5.9) such that

$$\varepsilon\|(\chi_3-\chi)x_\theta^2(\mathcal{P}_\theta - z_0)^{-1}\|_{\mathcal{H}_\theta\to\mathcal{H}_\theta} \le 1/2,$$

thus $I - i\varepsilon(\chi_3-\chi)x_\theta^2(\mathcal{P}_\theta-z_0)^{-1}$ is invertible and

$$(\mathcal{P}_\theta - i\varepsilon(\chi_3-\chi)x_\theta^2 - z_0)^{-1} = (\mathcal{P}_\theta - z_0)^{-1}(I - i\varepsilon(\chi_3-\chi)x_\theta^2(\mathcal{P}_\theta-z_0)^{-1})^{-1} \tag{5.4.3}$$

exists. We put

$$E(z) = \chi_2(\mathcal{P}_\theta - i\varepsilon(\chi_3 - \chi)x_\theta^2 - z_0)^{-1}\chi_1 + (1-\chi_0)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_1).$$

Then we get

$$(\mathcal{P}_{\varepsilon,\theta} - z)E(z) = I + K(z) + K_1(z),$$

where

$$\begin{aligned} K(z) &= ((z_0 - z)\chi_2 + [\mathcal{P}_\theta, \chi_2])(\mathcal{P}_\theta - i\varepsilon(\chi_3 - \chi)x_\theta^2 - z_0)^{-1}\chi_1 \\ &\quad + [e^{-2i\theta}\Delta, \chi_0](H_{\varepsilon,\theta} - z)^{-1}(1-\chi_1) \\ K_1(z) &= (\mathcal{P}_{\varepsilon,\theta} - H_{\varepsilon,\theta})(1-\chi_0)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_1). \end{aligned}$$

Recalling that $\|K_1(z)\|_{\mathcal{H}_\theta \to \mathcal{H}_\theta} \leq 1/2$, we obtain that $I + K_1(z)$ is invertible, thus

$$(\mathcal{P}_{\varepsilon,\theta} - z)E(z)(I + K_1(z))^{-1} = I + K(z)(I + K_1(z))^{-1}.$$

Since $(\mathcal{P}_\theta - z_0)^{-1} : \mathcal{H}_\theta \to \mathcal{D}_\theta$, we conclude that $K(z)$ is compact: $\mathcal{H}_\theta \to \mathcal{H}_\theta$. Hence $E(z)(I + K_1(z))$ is an approximate right inverse of $\mathcal{P}_{\varepsilon,\theta} - z$.

As an approximate left inverse, we put

$$F(z) = \chi_1(\mathcal{P}_\theta - i\varepsilon(\chi_3 - \chi)x_\theta^2 - z_0)^{-1}\chi_2 + (1-\chi_1)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_0).$$

Then

$$F(z)(\mathcal{P}_{\varepsilon,\theta} - z) = I + L(z) + L_1(z),$$

where

$$\begin{aligned} L(z) &= \chi_1(\mathcal{P}_\theta - i\varepsilon(\chi_3 - \chi)x_\theta^2 - z_0)^{-1}((z_0 - z)\chi_2 - [\mathcal{P}_\theta, \chi_2]) \\ &\quad - (1-\chi_1)(H_{\varepsilon,\theta} - z)^{-1}[e^{-2i\theta}\Delta, \chi_0] \\ L_1(z) &= (1-\chi_1)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_0)(\mathcal{P}_{\varepsilon,\theta} - H_{\varepsilon,\theta}). \end{aligned}$$

We may assume again by (3.5.7) that $\|L_1(z)\|_{\widehat{\mathcal{D}}_\theta \to \widehat{\mathcal{D}}_\theta} \leq 1/2$, then

$$(I + L_1(z))^{-1}F(z)(\mathcal{P}_{\varepsilon,\theta} - z) = I + (I + L_1(z))^{-1}L(z).$$

Using (3.3.3), we see that $[e^{-2i\theta}\Delta, \chi_0]$ is compact: $\widehat{\mathcal{D}}_\theta \to \mathcal{H}_\theta$, thus $L(z)$ is compact: $\widehat{\mathcal{D}}_\theta \to \widehat{\mathcal{D}}_\theta$, $(I + L_1(z))^{-1}F(z)$ is an approximate left inverse.

In view of the remarks after Proposition 2.3.2, $\mathcal{P}_{\varepsilon,\theta} - z : \widehat{\mathcal{D}}_\theta \to \mathcal{H}_\theta$ is a Fredholm operator. This operator depends continuously on (θ, z), thus the index is constant under deformation in (θ, z). Deforming z into i and θ down to 0, we see that the index of $\mathcal{P}_{\varepsilon,\theta} - z$ is equal to the index of $P_\varepsilon - i : \widehat{\mathcal{D}} \to \mathcal{H}$ (where we omit the subscript 0). Repeating the arguments above, we can also show that for every $\gamma \in [0, \pi/2]$,

$$P + e^{-i\gamma}\varepsilon(1 - \chi(x))x^2 - i : \widehat{\mathcal{D}} \to \mathcal{H} \text{ is a Fredholm operator.}$$

Deforming γ from $\pi/2$ (that is for P_ε) to 0, it follows that the index of $P_\varepsilon - i$ is equal to the index of $P + \varepsilon(1-\chi(x))x^2 - i$, which is 0 since $P + \varepsilon(1-\chi(x))x^2 : \widehat{\mathcal{D}} \to \mathcal{H}$ is self-adjoint, see [27, §1]. Hence we conclude that $\mathcal{P}_{\varepsilon,\theta} - z$ is of index 0.

It remains to show that $\mathcal{P}_{\varepsilon,\theta}$ has a discrete spectrum in $\mathbb{C} \setminus e^{-i\pi/4}[0,\infty)$. Recalling first (5.4.3) and then (3.5.8), (3.5.9), $\|K(z_0)\|_{\mathcal{H}_\theta \to \mathcal{H}_\theta}$ can be controlled by

$$\|(-\Delta_\theta - z_0)^{-1}\|_{L^2(\mathbb{R}^n)\to H^1(\mathbb{R}^n)}, \quad \|(H_{\varepsilon,\theta} - z_0)^{-1}\|_{L^2(\mathbb{R}^n)\to H^1(\mathbb{R}^n)},$$
$$\text{and} \quad \|(1-\chi_0)(P - z_0)^{-1}\|_{\mathcal{H}\to H^1(\mathbb{R}^n)}.$$

It then follows from (3.5.5) and (5.1.3) that $K(iL) = O(L^{-1/2}) : \mathcal{H}_\theta \to \mathcal{H}_\theta$. Hence for $z_0 = iL$, $L \gg 1$, $I + K(z_0)(I + K_1(z_0))^{-1}$ is invertible and we have

$$(\mathcal{P}_{\varepsilon,\theta} - z_0)E(z_0)(I + K_1(z_0))^{-1}(I + K(z_0)(I + K_1(z_0))^{-1})^{-1} = I,$$

which implies that $\mathcal{P}_{\varepsilon,\theta} - z_0$ is surjective. Since $\mathcal{P}_{\varepsilon,\theta} - z_0$ is a Fredholm operator of index 0, it must also be injective. Thus $\mathcal{P}_{\varepsilon,\theta} - z_0$ is invertible by the inverse mapping theorem. Theorem 4 then shows that $\mathcal{P}_{\varepsilon,\theta}$ has a discrete spectrum. □

Lemma 8. *For each $0 \le \theta < \theta_0$ and $\varepsilon > 0$, let $\psi \in \mathcal{C}_c^\infty(B(0,R_1);[0,1])$ be equal to 1 near $\overline{B(0,R_0)}$ so that ψ is a function on Γ_θ and defines a multiplication on $\mathcal{H}_\theta$. Then we have, meromorphically in the region $-\pi/4 < \arg z < 7\pi/4$,*

$$\psi(P_\varepsilon - z)^{-1}\psi = \psi(\mathcal{P}_{\varepsilon,\theta} - z)^{-1}\psi. \tag{5.4.4}$$

Proof. We modify the proof of [65, Lemma 2]. It is sufficient to show that for $0 \le \theta_1 < \theta_2 < \theta_0$, $|\theta_1 - \theta_2| \ll 1$,

$$\psi(\mathcal{P}_{\varepsilon,\theta_1} - z)^{-1}\psi = \psi(\mathcal{P}_{\varepsilon,\theta_2} - z)^{-1}\psi. \tag{5.4.5}$$

It is also enough to establish this for $z \in e^{i(-2\theta_1+\pi/2)}(1,\infty)$ as then the result follows by analytic continuation. For that we show that for any

$$f \in \mathcal{H}_{R_0} \oplus L^2(B(0,R_1)\setminus B(0,R_0)) \subset \mathcal{H}_{\theta_j}, \quad j = 1,2,$$

there exists U holomorphic in a neighborhood $\Omega_{\theta_1,\theta_2}$ of

$$\bigcup_{\theta_1\le\theta\le\theta_2} (\Gamma_\theta \setminus B(0,R_0)) \subset \mathbb{C}^n$$

such that

$$U|_{\Gamma_{\theta_j}}(x) = [(\mathcal{P}_{\varepsilon,\theta_j} - z)^{-1}\psi f](x), \quad \forall\, x \in \Gamma_{\theta_j} \setminus B(0,R_0). \tag{5.4.6}$$

To show the existence of U such that (5.4.6) holds we apply Lemma 2 to a modified family of deformations, which is obtained as follows. Let $\rho \in \mathcal{C}_c^\infty((1,6);[0,1])$ be equal to 1 near $[2,4]$, and put for $T \ge 1$,

$$g_{\theta_1,\theta_2,T}(t) := g_{\theta_1}(t) + \rho(t/T)(g_{\theta_2}(t) - g_{\theta_1}(t)),$$
$$\Gamma_{\theta_1,\theta_2,T} := \{g_{\theta_1,\theta_2,T}(t)\omega : t \in [0,\infty),\ \omega \in \mathbb{S}^{n-1}\} \subset \mathbb{C}^n.$$

We can apply Lemma 2 to the family of totally real submanifolds interpolating between Γ_{θ_1} and $\Gamma_{\theta_1,\theta_2,T}$, $[0,1] \ni s \mapsto \Gamma_{\theta_1,(1-s)\theta_1+s\theta_2,T}$. It follows that there exists a holomorphic function U^T defined in a neighborhood of the union of these submanifolds which restricts to $u_1 := (\mathcal{P}_{\varepsilon,\theta_1} - z)^{-1}\psi f \in \mathcal{H}_{\theta_1}$. Varying T we obtain a family of functions agreeing on the intersections of their domains and that gives a holomorphic function U defined in the neighborhood $\Omega_{\theta_1,\theta_2}$.

It remains to show that U restricts to $u_2 \in \mathcal{H}_{\theta_2}$ (the equation $(\mathcal{P}_{\varepsilon,\theta_2} - z)u_2 = \psi f$ is automatically satisfied). For T large we put

$$\Omega_1(T) = \{z \in \mathbb{C}^n : T \leq |z| \leq 6T\} \cap \Gamma_{\theta_1,\theta_2,T} \supset \Gamma_{\theta_1,\theta_2,T} \setminus \Gamma_{\theta_1},$$
$$\Omega_2(T) = \{z \in \mathbb{C}^n : T/2 \leq |z| \leq 8T\} \cap \Gamma_{\theta_1,\theta_2,T}, \quad \Omega_2(T) \setminus \Omega_1(T) \subset e^{i\theta_1}\mathbb{R}^n,$$

and choose $\chi_T \in \mathcal{C}^\infty(\Omega_2(T);[0,1])$ such that $\chi_T = 1$ on $\Omega_1(T)$ with derivative bounds independent of T. We recall the following estimate from the proof of [65, Lemma 3]: for $u \in \mathcal{C}^\infty(\Gamma_{\theta_1,\theta_2,T})$, $\tau > 1$,

$$|\langle(-\Delta|_{\Gamma_{\theta_1,\theta_2,T}} - i\varepsilon(x|_{\Gamma_{\theta_1,\theta_2,T}})^2 - ie^{-2i\theta_1}\tau)u, u\rangle| \geq (\|u\|_{L^2}^2 + \|Du\|_{L^2}^2)/C,$$

with $C > 0$ independent of τ, T, here $\langle\cdot,\cdot\rangle$ is the L^2 inner product on $\Gamma_{\theta_1,\theta_2,T}$. Writing

$$\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T} := P|_{\Gamma_{\theta_1,\theta_2,T}} - i\varepsilon(x|_{\Gamma_{\theta_1,\theta_2,T}})^2,$$

it then follows from (3.4.1) that

$$\langle(\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T} - (-\Delta|_{\Gamma_{\theta_1,\theta_2,T}} - i\varepsilon(x|_{\Gamma_{\theta_1,\theta_2,T}})^2))u, u\rangle = \int_{\Gamma_{\theta_1,\theta_2,T}} (g^{jk} - \delta^{jk})\partial_k u \partial_j \bar{u} + c|u|^2.$$

In view of (3.4.2) and (3.4.3), we obtain that for T sufficiently large,

$$|\langle(\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T} - ie^{-2i\theta_1}\tau)\chi_T U, \chi_T U\rangle| \geq (\|\chi_T U\|_{L^2}^2 + \|D(\chi_T U)\|_{L^2}^2)/C,$$

thus $\|\chi_T U\|_{L^2} \leq C\|(\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T} - ie^{-2i\theta_1}\tau)\chi_T U\|_{L^2}$. We note that

$$(\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T} - ie^{-2i\theta_1}\tau)U^T = 0 \implies (\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T} - ie^{-2i\theta_1}\tau)\chi_T U = [\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T}, \chi_T]U,$$

which is supported on $\Omega_2(T) \setminus \Omega_1(T) \subset \Gamma_{\theta_1}$. Hence,

$$\|1_{2T\leq|z|\leq 4T}\, u_2\|_{L^2(\Gamma_{\theta_2})}^2 \leq C\|[\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T}, \chi_T]U\|_{L^2}^2 \leq C\|1_{T/2\leq|z|\leq 8T}\, u_1\|_{H^1(\Gamma_{\theta_1})}^2.$$

We now take $T = 2^j$ and sum over j, it follows that $u_2 \in \mathcal{H}_{\theta_2}$. □

Lemma 9. *For $0 \leq \theta < \theta_0$, $\varepsilon > 0$, the spectrum of $\mathcal{P}_{\varepsilon,\theta}$ is independent of θ. More precisely, for any $z_0 \in \mathbb{C} \setminus e^{-i\pi/4}[0,\infty)$ we have*

$$m_{\varepsilon,\theta}(z_0) := \operatorname{rank} \oint_{z_0} (\mathcal{P}_{\varepsilon,\theta} - z)^{-1} dz = \operatorname{rank} \oint_{z_0} (P_\varepsilon - z)^{-1}\, dz, \tag{5.4.7}$$

where the integral is over a positively oriented circle enclosing z_0 and containing no poles other than possibly z_0.

Proof. Lemma 7 shows that

$$\Pi_{\varepsilon,\theta}(z_0) := -\frac{1}{2\pi i}\oint_{z_0}(\mathcal{P}_{\varepsilon,\theta} - z)^{-1}dz, \tag{5.4.8}$$

is a finite rank projection which maps $\mathcal{H}_\theta$ to the generalized eigenspace of $\mathcal{P}_{\varepsilon,\theta}$ at z_0. In view of Lemma 8, it suffices to show that for each $0 \le \theta < \theta_0$,

$$\operatorname{rank}\Pi_{\varepsilon,\theta}(z_0) = \operatorname{rank}\psi\Pi_{\varepsilon,\theta}(z_0)\psi.$$

First we show that $\operatorname{rank}\Pi_{\varepsilon,\theta}(z_0) = \operatorname{rank}\Pi_{\varepsilon,\theta}(z_0)\psi$, which is equivalent to show that $\operatorname{rank}\psi\Pi_{\varepsilon,\theta}(z_0)^* = \operatorname{rank}\Pi_{\varepsilon,\theta}(z_0)^*$, since the adjoint of a finite rank operator is of finite rank with the same rank. For that we shall argue by contradiction. Suppose that $\operatorname{rank}\psi\Pi_{\varepsilon,\theta}(z_0)^* < \operatorname{rank}\Pi_{\varepsilon,\theta}(z_0)^*$, there would exist $0 \neq \tilde{v} \in \operatorname{Ran}\Pi_{\varepsilon,\theta}(z_0)^*$ satisfying $\psi\tilde{v} = 0$. Note that $\Pi_{\varepsilon,\theta}(z_0)^*$ is also a projection of the form (5.4.8) except that $\mathcal{P}^*_{\varepsilon,\theta}$ and $\bar{z}_0$ replace $\mathcal{P}_{\varepsilon,\theta}$ and z_0 there, we may assume

$$(\mathcal{P}^*_{\varepsilon,\theta} - \bar{z}_0)^k\tilde{v} = 0, \quad \tilde{u} := (\mathcal{P}^*_{\varepsilon,\theta} - \bar{z}_0)^{k-1}\tilde{v} \neq 0, \quad \text{for some } k \ge 1.$$

But that would mean that $\tilde{u}$ can be identified with an element of $H^2(\Gamma_\theta)$ satisfying

$$(Q^*_{\varepsilon,\theta} - \bar{z}_0)\tilde{u} = 0, \quad \tilde{u}|_{B(0,R_0)} \equiv 0, \quad Q_{\varepsilon,\theta} := Q_\theta - i\varepsilon(1-\chi(x_\theta))x_\theta^2.$$

Since $Q^*_{\varepsilon,\theta}$ is elliptic, unique continuation results for second order elliptic differential equations – see Hörmander [28, Chapter 17] show that $\tilde{u} \equiv 0$, thus a contradiction.

It remains to show that $\operatorname{rank}\psi\Pi_{\varepsilon,\theta}(z_0)\psi = \operatorname{rank}\Pi_{\varepsilon,\theta}(z_0)\psi$. Otherwise there would exist solutions $v \in \widehat{\mathcal{D}}_\theta$ to $(\mathcal{P}_{\varepsilon,\theta} - z_0)^\ell v = 0$, $u := (\mathcal{P}_{\varepsilon,\theta} - z_0)^{\ell-1}v \neq 0$ with $\psi v = 0$. It follows that u can be identified with an element of $H^2(\Gamma_\theta)$ satisfying

$$(Q_{\varepsilon,\theta} - z_0)u = 0, \quad u|_{B(0,R_0)} \equiv 0.$$

Again by the unique continuation results for second order elliptic differential equations, we obtain that $u \equiv 0$, thus a contradiction. □

The next lemma shows that the spectrum of $\mathcal{P}_{\varepsilon,\theta}$ must stay close to the spectrum of $\mathcal{P}_\theta$ when ε is sufficiently small:

Lemma 10. *Suppose that $0 \le \theta < \theta_0$ and that $\Omega \Subset \{z : -2\theta < \arg z < 3\pi/2 + 2\theta\}$ is disjoint with $\operatorname{Spec}(\mathcal{P}_\theta)$, then there exist $\varepsilon_0 = \varepsilon_0(\Omega)$ and $C = C(\Omega)$ such that, uniformly in $0 < \varepsilon < \varepsilon_0$, $\operatorname{Spec}(\mathcal{P}_{\varepsilon,\theta}) \cap \Omega = \emptyset$ and*

$$\|(\mathcal{P}_{\varepsilon,\theta} - z)^{-1}\|_{\mathcal{H}_\theta \to \mathcal{D}_\theta} \le C, \quad z \in \Omega.$$

Proof. We follow closely the proof of [65, Lemma 5] except that $\mathcal{P}_\theta$ replaces $-\Delta_\theta$ there. Let $\chi_j \in \mathcal{C}_c^\infty([0,\infty);[0,1])$ be equal to 1 on $[0,R_0]$ and satisfy $\chi_j = 1$ near $\operatorname{supp}\chi_{j-1}$, $j=1,2$. Parametrizing Γ_θ by $f_\theta : [0,\infty)\times\mathbb{S}^{n-1} \ni (t,\omega) \mapsto g_\theta(t)\omega \in \Gamma_\theta$, we define functions $\chi_j^h \in \mathcal{C}_c^\infty(\Gamma_\theta)$ by

$$\chi_j^h(g_\theta(t)\omega) := \chi_j(th), \quad 0 < h \le 1.$$

For $z \in \Omega$ we put

$$E^h_{\varepsilon,\theta}(z) := \chi_2^h(\mathcal{P}_\theta - z)^{-1}\chi_1^h + (1-\chi_0^h)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_1^h),$$

so that $(\mathcal{P}_{\varepsilon,\theta} - z)E^h_{\varepsilon,\theta}(z) = I + K^h_{\varepsilon,\theta}(z)$, where

$$\begin{aligned} K^h_{\varepsilon,\theta}(z) := & - i\varepsilon(1-\chi)x_\theta^2\chi_2^h(\mathcal{P}_\theta - z)^{-1}\chi_1^h + [\mathcal{P}_\theta, \chi_2^h](\mathcal{P}_\theta - z)^{-1}\chi_1^h \\ & + (\mathcal{P}_{\varepsilon,\theta} - H_{\varepsilon,\theta})(1-\chi_0^h)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_1^h) \\ & - [\mathcal{P}_\theta, \chi_0^h](1-\chi_0^h)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_1^h). \end{aligned}$$

Using (3.5.7) and (5.1.3) we see that for h small enough,

$$\|(\mathcal{P}_{\varepsilon,\theta} - H_{\varepsilon,\theta})(1-\chi_0^h)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_1^h)\|_{L^2(\Gamma_\theta)\to L^2(\Gamma_\theta)} < 1/4.$$

Noticing that

$$\|[Q_\theta, \chi_j^h]\|_{H^1(\Gamma_\theta)\to L^2(\Gamma_\theta)} = O(h) \text{ and } \|x_\theta^2\chi_2^h\|_{L^2(\Gamma_\theta)\to L^2(\Gamma_\theta)} = O(h^{-2}),$$

we can first choose h sufficiently small then take $\varepsilon_0 = \varepsilon_0(h,\Omega)$ small enough so that for all $\varepsilon < \varepsilon_0(h,\Omega)$ and $z \in \Omega$, $\|K^h_{\varepsilon,\theta}(z)\|_{\mathcal{H}_\theta\to\mathcal{H}_\theta} < 1/2$, thus $I + K^h_{\varepsilon,\theta}(z)$ has a uniformly bounded inverse and

$$(\mathcal{P}_{\varepsilon,\theta} - z)^{-1} = E^h_{\varepsilon,\theta}(z)(I + K^h_{\varepsilon,\theta}(z))^{-1}.$$

It follows from (5.1.3) that there exists $C = C(\Omega)$ independent of ε such that for $z \in \Omega$, $\|E^h_{\varepsilon,\theta}(z)\|_{\mathcal{H}_\theta\to\mathcal{D}_\theta} \le C$, which completes the proof. □

5.5 Complex Higgs oscillators

The Higgs oscillator [26] (see also Pallares-Rivera–Kirchbach [41]) is considered as an analogue of the quantum harmonic oscillator on the hyperbolic plane. In this section we discuss its complex version, in analogy to the complex harmonic oscillator in the Euclidean space studied by, among others, Davies [9].

As discussed in §1.4.2, for a hyperbolic manifold (M,g), we aim to find a complex-valued function $f \in \mathcal{C}^\infty(M)$ such that $\Delta_g + f$ is an operator on $L^2(M, d\,\mathrm{vol}_g)$ with discrete spectrum. Ideally, we should also require f to be unbounded near infinity like the function $-i\varepsilon x^2$ in the Euclidean case, which would provide the compactness of the resolvent $(\Delta_g - n^2/4 + f - z)^{-1} : L^2(M, d\,\mathrm{vol}_g) \to L^2(M, d\,\mathrm{vol}_g)$. However, it is hard to find a function f satisfying all the requirements above. We will explore a candidate $f = \omega^2 \tanh^2 r$ where $\omega \in \mathbb{C}$, r is the hyperbolic radius. The operator $\Delta_g + \omega^2 \tanh^2 r$ is called the Higgs oscillator in the hyperbolic space, whose spectrum and resonances can be explicitly computed, see [41] for more details. The drawback of this candidate is the boundedness of f thus we lose the compacteness of the resolvent $(\Delta_g - n^2/4 + f - z)^{-1}$. We remark that it is still an open problem to find an ideal analogue of the complex harmonic oscillator in the hyperbolic setting. We hope that the following introduction could popularize this natural problem.

5.5.1 Complex Higgs Oscillator on the Hyperbolic Plane

We consider the hyperbolic plane

$$\mathbb{H} := \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

with the Poincaré metric $y^{-2}(dx^2 + dy^2)$. Instead of coordinates (x, y), we will use the geodesic normal coordinates for hyperbolic metrics. These are coordinates (r, φ) for which the r-coordinate curves are unit speed geodesics and the φ-coordinate curves are geodesic circles. The Laplacian is given by

$$\Delta_{\mathbb{H}^2} = y^2(D_x^2 + D_y^2) = D_r^2 - i \coth r D_r + \sinh^{-2} r D_\varphi^2,$$

where $D_x = i^{-1}\partial_x$. $\Delta_{\mathbb{H}}$ is through conjugation by $\sinh^{1/2} r$, equivalent to

$$D_r^2 + \sinh^{-2} r(D_\varphi^2 - 1/4) + 1/4.$$

Now we define the complex version of Higgs Oscillator by

$$\Delta_{\mathbb{H}^2} + \omega^2 \tanh^2 r, \quad \omega \in \mathbb{C},$$

which is through the same conjugation as above, equivalent to the operator

$$D_r^2 + \frac{D_\varphi^2 - 1/4}{\sinh^2 r} - \frac{\omega^2}{\cosh^2 r} + \omega^2 + \frac{1}{4}$$

on $L^2((0, \infty)_r \times S^1_\varphi, dr d\varphi)$. We can expand this in terms of the eigenfunctions on S^1_φ to obtain

$$\bigoplus_{m \in \mathbb{Z}} D_r^2 + \frac{m^2 - 1/4}{\sinh^2 r} - \frac{\omega^2}{\cosh^2 r} + \omega^2 + \frac{1}{4}.$$

This decomposition leads to the one-dimensional Schrödinger operator with Pöschl–Teller potential – see §2.6, as follows.

$$D_r^2 + V_{\mu,\nu}, \quad \mu = |m| - 1/2, \ \nu = \sqrt{\omega^2 + 1/4} - 1/2.$$

It follows from Proposition 2.6.1 that

$$\operatorname{Spec}(D_r^2 + V_{\mu,\nu}) = \{(\nu - \mu - 1 - 2n)^2 : n \in \mathbb{N},\, 2n < \operatorname{Re}(\nu - \mu - 1)\}.$$

Hence we obtain the eigenvalues of $\Delta_{\mathbb{H}^2} + \omega^2 \tanh^2 r$:

$$\left\{\omega^2 + \frac{1}{4} - \Big(\sqrt{\omega^2 + \frac{1}{4}} - m - 1 - 2n\Big)^2 : m, n \in \mathbb{N},\, 2n < \operatorname{Re}\sqrt{\omega^2 + \frac{1}{4}} - m - 1\right\}.$$

The scattering matrix (2.6.1) gives the resonances of $\Delta_{\mathbb{H}^2} + \omega^2 \tanh^2 r$:

$$\left\{\omega^2 + \frac{1}{4} - \Big(\sqrt{\omega^2 + \frac{1}{4}} - m - 1 - 2n\Big)^2 : m, n \in \mathbb{N}\right\}.$$

5.5.2 Complex Higgs Oscillator with an Eckart barrier

We consider the one-dimensional Eckart barrier $V = \alpha \cosh^{-2} r$, $\alpha > 0$. The complex Higgs oscillator with an Eckart barrier is given by

$$D_r^2 + V + \omega^2 \tanh^2 r = D_r^2 + (\alpha - \omega^2)\cosh^{-2} r + \omega^2, \quad \omega \in \mathbb{C}.$$

This can be viewed as a Schrödinger operator with Pöschl–Teller potential

$$D_r^2 + V_{0,\nu}, \quad \nu = \sqrt{\omega^2 - \alpha + \frac{1}{4}} - \frac{1}{2}.$$

It follows from Proposition 2.6.1 that the discrete spectrum of the complex Higgs oscillator with a Eckart barrier is given by

$$\left\{\omega^2 - \Big(\sqrt{\omega^2 - \alpha + \frac{1}{4}} - \frac{1}{2} - n\Big)^2 : n \in \mathbb{N},\, n < \operatorname{Re}\sqrt{\omega^2 + \frac{1}{4}} - \frac{1}{2}\right\}.$$

And the scattering matrix (2.6.2) gives the resonances in this case:

$$\left\{\omega^2 - \Big(\sqrt{\omega^2 - \alpha + \frac{1}{4}} - \frac{1}{2} - n\Big)^2 : n \in \mathbb{N}\right\}.$$

5.5.3 Complex Higgs Oscillator on hyperbolic half-cylinder

Another interesting example is the hyperbolic half-cylinder

$$Y_{0l} \simeq (0, \infty)_r \times (\mathbb{R}/l\mathbb{Z})_\theta,$$

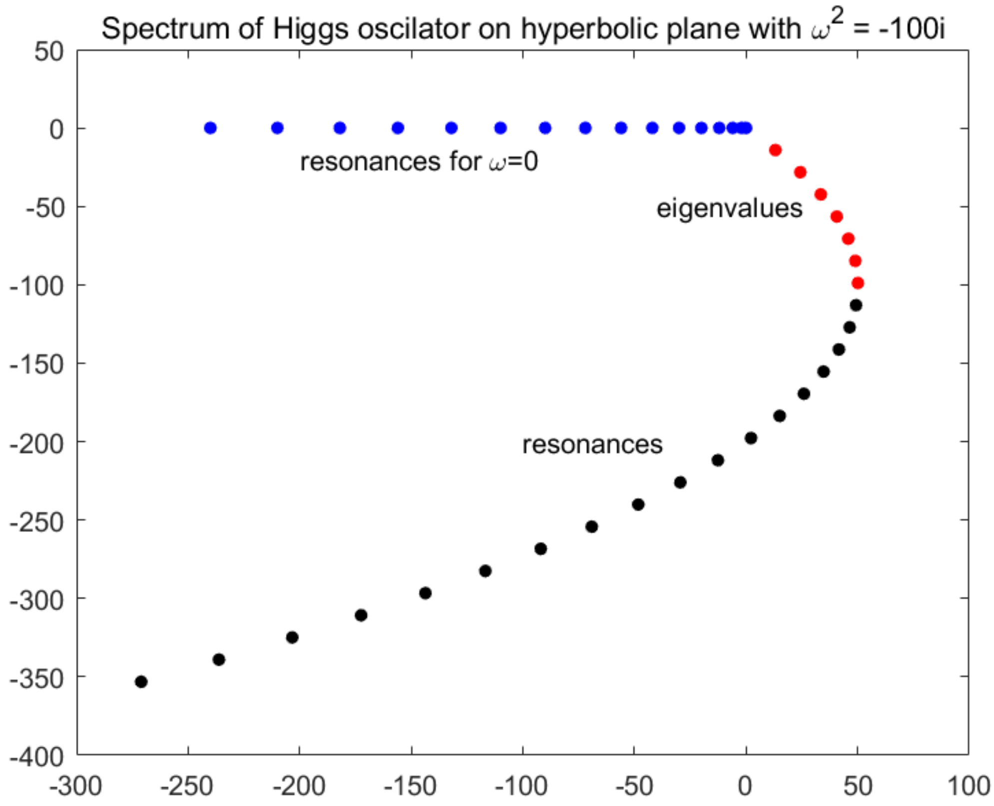


Figure 5.2: The spectrum of the complex Higgs osicllator on the hyperbolic plane. The red dots are the eigenvalues of $\Delta_{\mathbb{H}^2} + \omega^2 \tanh^2 r$ with $\omega^2 = -100i$ while the black dots are the resonances. We also plot the resonances of $\Delta_{\mathbb{H}^2}$, which are the blue dots on the real axis. This shows the deformation of resonances.

with the metric

$$g = dr^2 + \cosh^2 r d\theta^2.$$

The Laplacian with Dirichlet boundary condition on $\{r = 0\}$ is given by

$$\Delta_{Y_{0l}} = D_r^2 - i \tanh r D_r + \cosh^{-2} \Delta_{\mathbb{R}/l\mathbb{Z}},$$

which is, through a conjugation by $\cosh^{1/2} r$, equivalent to the operator

$$D_r^2 + \frac{\Delta_{\mathbb{R}/l\mathbb{Z}} + 1/4}{\cosh^2 r} + \frac{1}{4}.$$

Using the same conjugation, the complex Higgs oscillator

$$\Delta_{Y_{0l}} + \omega^2 \tanh^2 r, \quad \omega \in \mathbb{C},$$

is equivalent to the operator

$$D_r^2 - \frac{\omega^2 - \Delta_{\mathbb{R}/l\mathbb{Z}} - 1/4}{\cosh^2 r} + \omega^2 + \frac{1}{4},$$

which admits the following expansion:

$$\bigoplus_{m\in\mathbb{Z}} D_r^2 - \frac{\omega^2 - (2\pi m/l)^2 - 1/4}{\cosh^2 r} + \omega^2 + \frac{1}{4}.$$

The corresponding one-dimensional one-dimensional Schrödinger operator is $D_r^2 + V_{0,\nu}$ on $(0,\infty)$ with Dirichlet boundary condition, where we put $\nu = \sqrt{\omega^2 - (2\pi m/l)^2} - 1/2$. Hence by Proposition 2.6.1 the discrete specrtum of $\Delta_{Y_{0l}} + \omega^2 \tanh^2 r$ is

$$\{\omega^2 + 1/4 - \left(\sqrt{\omega^2 - (2\pi m/l)^2} - 2n - 3/2\right)^2 : m \in \mathbb{Z},$$
$$n \in \mathbb{N},\ 2n < \operatorname{Re}\sqrt{\omega^2 - (2\pi m/l)^2} - 3/2\},$$

while the analysis of (2.6.1) gives the resonances:

$$\{\omega^2 + 1/4 - \left(\sqrt{\omega^2 - (2\pi m/l)^2} - 2n - 3/2\right)^2 : m \in \mathbb{Z}, n \in \mathbb{N}\}.$$

Remark. The explicit formulae and the figures show that the resonances in all cases are deformed and some do become eigenvalues. However, in this setting we cannot obtain the original resonances for $\omega = 0$ by taking limits of these eigenvalues as $\omega \to 0$, as one would want for the CAP method. In fact, for $\omega \in \mathbb{C}$ with small modulus there are no eigenvalues for the complex Higgs oscillators at all.

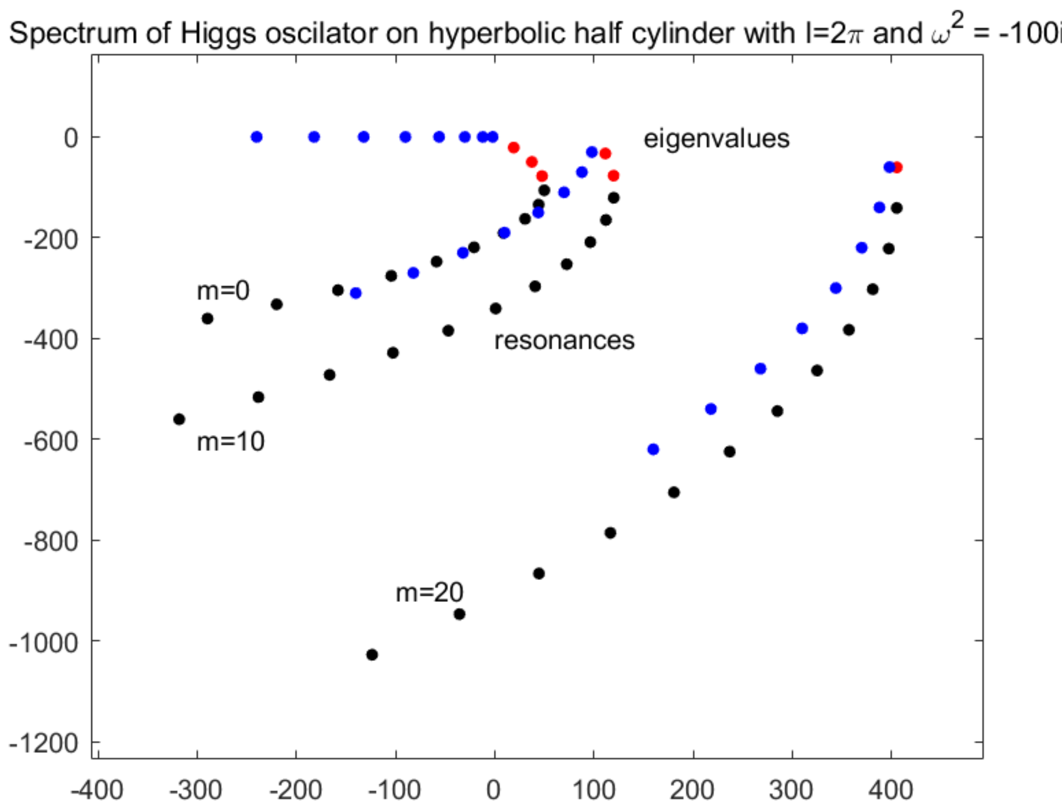


Figure 5.3: The spectrum of the Higgs osicllator on hyperbolic half-cylinder with parameter $l = 2\pi$ and $\omega^2 = -100i$. We only plot the spectrum with respect to the Fourier modes $m = 0, 10, 20$ for illustration. Here the red dots are eigenvalues and the black dots are resonances. We also plot resonances for $\omega = 0$ (blue dots) with respect to the same Fourier modes to show the deformation of resonances.

Chapter 6

Resonances as viscosity limits

6.1 The CAP method for exponentially decaying potentials

In this section, we prove Theorem 1. The proof is mainly based on Lemma 1, Lemma 6, with an application of Theorem 7. We first state a more precise version of Theorem 1 involving the multiplicities given in (3.2.5) and (5.3.3) as follows:

Theorem 8. *For any Ω given in (1.3.1), there exists $\delta_0 = \delta_0(\Omega)$ satisfying the following: for any $0 < \delta < \delta_0$, there exists $\varepsilon_\delta > 0$ such that for any $\lambda \in \Omega$ with $m(\lambda) > 0$,*

$$\#\{\lambda_j(\varepsilon)\}_{j=1}^{\infty} \cap B(\lambda, \delta) = m(\lambda), \quad \textit{for all } 0 < \varepsilon < \varepsilon_\delta,$$

where $\{\lambda_j(\varepsilon)\}_{j=1}^{\infty}$ given in (5.3.2) is counted with multiplicity, $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\}$.

Proof. In view of Lemma 1, the poles of $(I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1}$ are isolated in the region $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0, \operatorname{Im}\lambda > -\gamma\}$, thus there are finitely many $\lambda \in \Omega$ with $m(\lambda) > 0$, denoted by $\lambda_1, \dots, \lambda_J$. We choose $\delta_0 > 0$ such that $B(\lambda_j, \delta_0)$, $j = 1, \dots, J$ are disjoint discs in Ω, then for any fixed $0 < \delta < \delta_0$ and each $\lambda \in \Omega$ with $m(\lambda) > 0$, we have

$$\|(I + \sqrt{V}R_0(\zeta)\sqrt{V})^{-1}\|_{L^2 \to L^2} < C(\delta), \quad \forall\, \zeta \in \partial B(\lambda, \delta),$$

for some constant $C(\delta) > 0$.

In order to apply the Theorem 7, we need to estimate :

$$\|\sqrt{V}R_\varepsilon(\zeta)\sqrt{V} - \sqrt{V}R_0(\zeta)\sqrt{V}\|_{L^2 \to L^2}, \quad \text{for any } \zeta \in \Omega.$$

1. Choose $\chi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ satisfying $\chi \equiv 1$ in $B_{\mathbb{R}^n}(0,1)$ and $\operatorname{supp}\chi \subset B_{\mathbb{R}^n}(0,2)$, here $B_{\mathbb{R}^n}(0,r) := \{x \in \mathbb{R}^n : |x| < r\}$, we define $\chi_R(x) = \chi(R^{-1}x)$ and write

$$\begin{aligned} &\sqrt{V}R_\varepsilon(\zeta)\sqrt{V} - \sqrt{V}R_0(\zeta)\sqrt{V} \\ &= (\sqrt{V}R_\varepsilon(\zeta)\sqrt{V} - \chi_R\sqrt{V}R_\varepsilon(\zeta)\chi_R\sqrt{V}) \\ &\quad + \sqrt{V}\chi_R(R_\varepsilon(\zeta) - R_0(\zeta))\chi_R\sqrt{V} \\ &\quad - (\sqrt{V}R_0(\zeta)\sqrt{V} - \chi_R\sqrt{V}R_0(\zeta)\chi_R\sqrt{V}). \end{aligned} \tag{6.1.1}$$

2. The first term can be written as

$$(1-\chi_R)\sqrt{V}R_\varepsilon(\zeta)\sqrt{V} + \chi_R\sqrt{V}R_\varepsilon(\zeta)(1-\chi_R)\sqrt{V}.$$

Let $\tilde{\gamma} = (\gamma + \gamma')/2$, then

$$(1-\chi_R)\sqrt{V}R_\varepsilon(\zeta)\sqrt{V} = (1-\chi_R)\sqrt{V}e^{\tilde{\gamma}|x|}(e^{-\tilde{\gamma}|x|}R_\varepsilon(\zeta)e^{-\tilde{\gamma}|x|})\sqrt{V}e^{\tilde{\gamma}|x|},$$

(3.2.1) implies that $|\sqrt{V(x)}e^{\tilde{\gamma}|x|}| \leq Ce^{(\tilde{\gamma}-\gamma)|x|} = Ce^{-(\gamma-\gamma')|x|/2}$. By Lemma 5, there exists $\varepsilon_0 = \varepsilon_0(\Omega,\gamma)$ such that

$$\|e^{-\tilde{\gamma}|x|}R_\varepsilon(\zeta))e^{-\tilde{\gamma}|x|}\|_{L^2\to L^2} \leq C(\Omega,\tilde{\gamma}), \quad \text{for any } 0<\varepsilon<\varepsilon_0.$$

We conclude that

$$\|(1-\chi_R)\sqrt{V}R_\varepsilon(\zeta)\sqrt{V}\|_{L^2\to L^2} \leq C(\Omega,\gamma)e^{-(\gamma-\gamma')R/2}, \quad \text{for any } 0<\varepsilon<\varepsilon_0.$$

Similarly, we can bound $\|\chi_R\sqrt{V}R_\varepsilon(\zeta)(1-\chi_R)\sqrt{V}\|_{L^2\to L^2}$ by the right side above. Hence there exists $C = C(\Omega,\gamma)$ such that for any $0<\varepsilon<\varepsilon_0$,

$$\|\sqrt{V}R_\varepsilon(\zeta)\sqrt{V} - \chi_R\sqrt{V}R_\varepsilon(\zeta)\chi_R\sqrt{V}\|_{L^2\to L^2} \leq Ce^{-(\gamma-\gamma')R/2}, \quad \forall\,\zeta\in\Omega. \tag{6.1.2}$$

3. We can estimate the third term in (6.1.1) by a similar argument. It follows from (3.2.4) that

$$\|e^{-\tilde{\gamma}|x|}R_0(\zeta)e^{-\tilde{\gamma}|x|}\|_{L^2\to L^2} \leq C(\Omega,\gamma), \quad \forall\,\zeta\in\Omega.$$

Hence, arguing as above, we obtain that

$$\|\sqrt{V}R_0(\zeta)\sqrt{V} - \chi_R\sqrt{V}R_0(\zeta)\chi_R\sqrt{V}\|_{L^2\to L^2} \leq Ce^{-(\gamma-\gamma')R/2}, \quad \forall\,\zeta\in\Omega. \tag{6.1.3}$$

4. We note that

$$\chi_R(R_\varepsilon(\zeta) - R_0(\zeta))\chi_R = i\varepsilon\,\chi_R(-\Delta - i\varepsilon x^2 - \zeta^2)^{-1}x^2(\Delta-\zeta^2)^{-1}\chi_R,$$

and recall [65] that there exists $C = C(\Omega,\chi_R)$ (independent of ε) such that

$$\|\chi_R(-\Delta - i\varepsilon x^2 - \zeta^2)^{-1}x^2(\Delta-\zeta^2)^{-1}\chi_R\|_{L^2\to L^2} \leq C, \quad \forall\,\zeta\in\Omega,\, \varepsilon>0,$$

which is proved using the method of complex scaling – see §3.5. Hence

$$\|\sqrt{V}\chi_R(R_\varepsilon(\zeta)-R_0(\zeta))\chi_R\sqrt{V}\|_{L^2\to L^2}\le C(\Omega,\chi_R)\,\varepsilon,\quad \forall\,\zeta\in\Omega,\ \varepsilon>0. \tag{6.1.4}$$

By (6.1.2) and (6.1.3), we can first fix R sufficiently large such that

$$\|\sqrt{V}R_\varepsilon(\zeta)\sqrt{V}-\chi_R\sqrt{V}R_\varepsilon(\zeta)\chi_R\sqrt{V}\|_{L^2\to L^2}\le 1/(3C(\delta)),\quad \forall\,\zeta\in\Omega,\ 0\le\varepsilon<\varepsilon_0.$$

Then by (6.1.4), there exists $\varepsilon_\delta>0$ such that for all $0<\varepsilon<\varepsilon_\delta$,

$$\|\sqrt{V}\chi_R(R_\varepsilon(\zeta)-R_0(\zeta))\chi_R\sqrt{V}\|_{L^2\to L^2}\le 1/(3C(\delta)),\quad \forall\,\zeta\in\Omega.$$

We may assume that $\varepsilon_\delta<\varepsilon_0$, thus by (6.1.1), we conclude that for each $0<\varepsilon<\varepsilon_\delta$,

$$\|(I+\sqrt{V}R_0(\zeta)\sqrt{V})^{-1}(\,I+\sqrt{V}R_\varepsilon(\zeta)\sqrt{V}-(I+\sqrt{V}R_0(\zeta)\sqrt{V})\,)\|_{L^2\to L^2}<1,$$

on $\partial B(\lambda,\delta)$.

Now we apply Theorem 7 to obtain that

$$\begin{aligned} m(\lambda)&=\frac{1}{2\pi i}\operatorname{tr}\int_{\partial B(\lambda,\delta)}(I+\sqrt{V}R_0(\zeta)\sqrt{V})^{-1}\partial_\zeta(\sqrt{V}R_0(\zeta)\sqrt{V})\,d\zeta\\ &=\frac{1}{2\pi i}\operatorname{tr}\int_{\partial B(\lambda,\delta)}(I+\sqrt{V}R_\varepsilon(\zeta)\sqrt{V})^{-1}\partial_\zeta(\sqrt{V}R_\varepsilon(\zeta)\sqrt{V})\,d\zeta, \end{aligned}$$

for each $0<\varepsilon<\varepsilon_\delta$. Denote by $\lambda_1(\varepsilon),\dots,\lambda_K(\varepsilon)$ the distinct poles of

$$B(\lambda,\delta)\ni\zeta\mapsto(I+\sqrt{V}R_\varepsilon(\zeta)\sqrt{V})^{-1},$$

then we have

$$\begin{aligned} m(\lambda)&=\sum_{k=1}^{K}\frac{1}{2\pi i}\operatorname{tr}\oint_{\lambda_k(\varepsilon)}(I+\sqrt{V}R_\varepsilon(\zeta)\sqrt{V})^{-1}\partial_\zeta(\sqrt{V}R_\varepsilon(\zeta)\sqrt{V})\,d\zeta\\ &=\sum_{k=1}^{K}m_\varepsilon(\lambda_k(\varepsilon)). \end{aligned}$$

Therefore, with Lemma 6 and (5.3.4), we obtain that

$$\#\,\{\lambda_j(\varepsilon)\}_{j=1}^{\infty}\cap B(\lambda,\delta)=m(\lambda),\quad \forall\,0<\varepsilon<\varepsilon_\delta,$$

which completes the proof. □

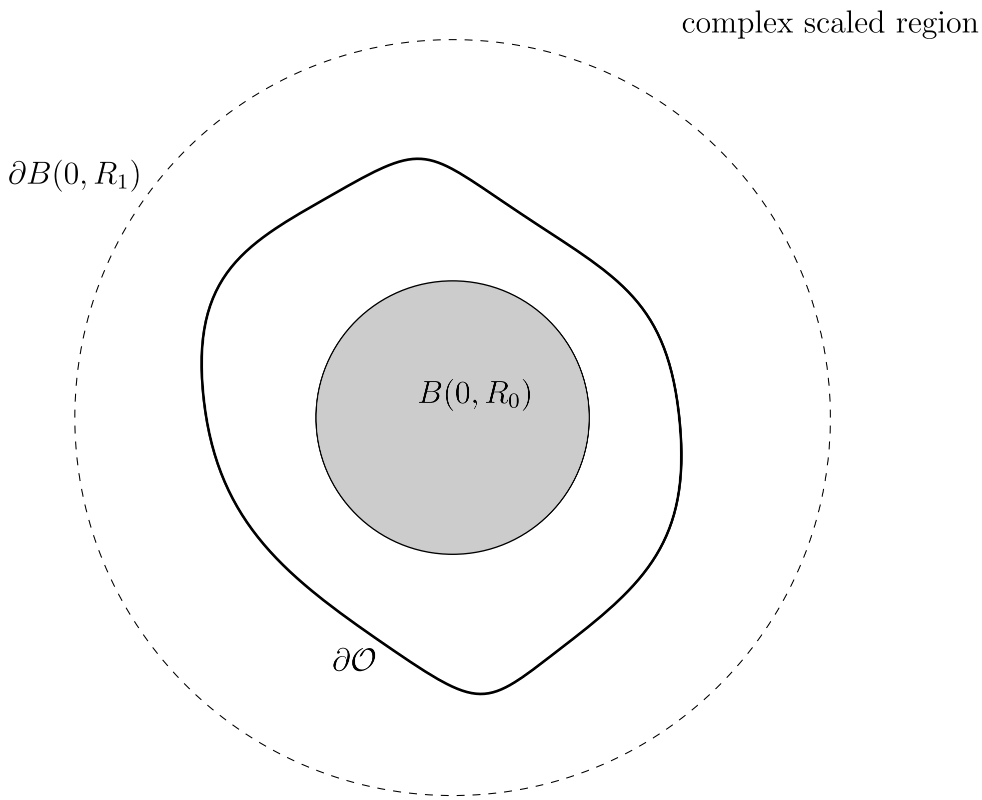


Figure 6.1: An auxiliary obstacle separating the black box from the differential operator outside.

6.2 An auxiliary obstacle problem

To prove Theorem 2 (the black box case) we cannot use the strategy of [65] which covers the case $P = -\Delta + V$, $V \in L^\infty_{\mathrm{comp}}$. Instead we introduce an auxiliary obstacle to separate the abstract black box from the differential operator outside. By an obstacle we mean a bounded open set $\mathcal{O}$ in $\mathbb{R}^n$ with smooth boundary, that is $\partial\mathcal{O}$ is a $\mathcal{C}^\infty$-hypersurface in $\mathbb{R}^n$. We shall assume that $\overline{B(0,R_0)} \subset \mathcal{O} \subset B(0,R_1)$, where $B(0,R_0)$ denotes the black box and $B(0,R_1)$ lies in the flat region of the complex contour Γ_θ – see §3.5. We also assume that the cutoff function χ in (5.4.1) be equal to 1 near $\overline{\mathcal{O}}$.

We first introduce a *reference operator* $P^{\mathcal{O}}$ associated with the obstacle $\mathcal{O}$. In the notation of (3.3.1), we put

$$\mathcal{H}^{\mathcal{O}} := \mathcal{H}_{R_0} \oplus L^2(\mathcal{O} \setminus B(0,R_0)). \tag{6.2.1}$$

The corresponding orthogonal projections are denoted by

$$u \mapsto 1_{B(0,R_0)}\, u = u|_{B(0,R_0)}, \quad u \mapsto 1_{\mathcal{O}\setminus B(0,R_0)}\, u = u|_{\mathcal{O}\setminus B(0,R_0)}.$$

Let P be a long range perturbation of $-\Delta$ introduced in §3.4, that is

$$P : \mathcal{H} \to \mathcal{H} \quad \text{with domain } \mathcal{D}(P),$$

which satisfies (3.3.2) – (3.3.4) and (3.4.1) – (3.4.3). Then we define

$$\begin{aligned} \mathcal{D}^{\mathcal{O}} := \{u \in \mathcal{H}^{\mathcal{O}} : \psi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathcal{O}),\ \psi = 1 \text{ near } \overline{B(0,R_0)} \Rightarrow \\ \psi u \in \mathcal{D}(P),\ (1-\psi)u \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})\} \end{aligned} \tag{6.2.2}$$

and, for any ψ with the property (6.2.2),

$$\begin{aligned} P^{\mathcal{O}} : \mathcal{D}^{\mathcal{O}} \to \mathcal{H}^{\mathcal{O}}, \\ P^{\mathcal{O}}u := P(\psi u) + Q((1-\psi)u). \end{aligned} \tag{6.2.3}$$

It follows from assumptions (3.3.3), (3.4.1) that these definitions are independent of the choice of the cutoff function ψ.

We recall some basic properties of the reference operator from [49, §7]:

Proposition 6.2.1. *Suppose that $\mathcal{O} \subset \mathbb{R}^n$ is an open set containing $\overline{B(0,R_0)}$ such that $\partial\mathcal{O}$ is a smooth hypersurface in $\mathbb{R}^n$. Let $P^{\mathcal{O}}$ be the reference operator defined in* (6.2.3)*. Then, with $\mathcal{H}^{\mathcal{O}}$ given by* (6.2.1),

$$P^{\mathcal{O}} : \mathcal{H}^{\mathcal{O}} \to \mathcal{H}^{\mathcal{O}},$$

is a self-adjoint operator with domain $\mathcal{D}^{\mathcal{O}}$ defined in (6.2.2)*. Furthermore, the resolvent $(P^{\mathcal{O}} + i)^{-1}$ is compact and thus $P^{\mathcal{O}}$ has discrete spectrum which is contained in $\mathbb{R}$.*

For the proof we refer to [12, Lemma 4.11] and we remark that the arguments there is still valid if we replace the assumption there: $P = -\Delta$ in $\mathbb{R}^n \setminus B(0,R_0)$, by the assumption (3.4.1).

Let $\mathcal{P}_\theta$ be the complex scaled P defined in (3.5.3). In parallel with the reference operator, we now introduce the restriction of $\mathcal{P}_\theta$ to $\Gamma \setminus \mathcal{O}$ with Dirichlet boundary condition. Outside the black box, P_θ is equal to a differential operator – see (3.5.3),

$$Q_\theta = -\sum_{j,k=1}^{n} (\partial_{z_j}(g^{jk}(z)\partial_{z_k}) + c(z))|_{\Gamma_\theta}.$$

We define

$$\begin{aligned} Q_\theta^{\mathcal{O}} : L^2(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O}),\ \text{with } \mathcal{D}(Q_\theta^{\mathcal{O}}) = H^2(\Gamma_\theta \setminus \mathcal{O}) \cap H_0^1(\Gamma_\theta \setminus \mathcal{O}), \\ Q_\theta^{\mathcal{O}}u := Q_\theta u, \quad \forall\, u \in \mathcal{D}(Q_\theta^{\mathcal{O}}). \end{aligned} \tag{6.2.4}$$

And the CAP-regularized version is defined by

$$Q_{\varepsilon,\theta}^{\mathcal{O}} : L^2(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O}), \text{ with } \mathcal{D}(Q_{\varepsilon,\theta}^{\mathcal{O}}) = \mathcal{D}(Q_\theta^{\mathcal{O}}) \cap |x_\theta|^{-2} L^2(\Gamma_\theta \setminus \mathcal{O}),$$
$$Q_{\varepsilon,\theta}^{\mathcal{O}} u := Q_\theta^{\mathcal{O}} u - i\varepsilon(1-\chi)x_\theta^2 u, \quad \forall\, u \in \mathcal{D}(Q_{\varepsilon,\theta}^{\mathcal{O}}). \tag{6.2.5}$$

We remark that $Q_\theta^{\mathcal{O}}$ can be viewed as $Q_0^{\mathcal{O}}$ ($\theta = 0$) being complex scaled, while $Q_0^{\mathcal{O}}$ being a long range perturbation of $-\Delta$ introduced in §3.4. Similarly, $Q_{\varepsilon,\theta}^{\mathcal{O}}$ is the complex scaled version of $Q_{\varepsilon,0}^{\mathcal{O}}$, which is $Q_0^{\mathcal{O}}$ being CAP-regularized – see §5.4. Therefore, we conclude from Lemma 3 and Lemma 7 that

$$Q_{\varepsilon,\theta}^{\mathcal{O}} : L^2(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O}), \quad \varepsilon \geq 0,$$

has a discrete spectrum in the range $\{z \in \mathbb{C} : -2\theta < \arg z < 3\pi/2 + 2\theta\}$.

6.3 Dirichlet-to-Neumann operator

In this section we use a different method from [65] and §5.3 to characterize the eigenvalues of $\mathcal{P}_{\varepsilon,\theta}$, $\varepsilon \geq 0$. We introduce the Dirichlet-to-Neumann operator associated with the obstacle $\mathcal{O}$ – see Figure 6.1. For that let $\nu(x)$ be the Euclidean normal vector of $\partial\mathcal{O}$ at x pointing into $\mathcal{O}$, we put

$$\nu_g(x) := (g^{jk}(x))_{n\times n} \cdot \nu(x), \quad x \in \partial\mathcal{O}, \tag{6.3.1}$$

where g^{jk} are the coefficients of Q satisfying (3.4.1) – (3.4.3).

First we introduce the interior Dirichlet-to-Neumann operator of P:

$$\mathcal{N}_P^{\text{in}}(z)\varphi := \frac{\partial u}{\partial \nu_g}, \quad \text{where } u \text{ solves } \begin{array}{c} (P-z)u = 0 \text{ in } \mathcal{O} \\ u = \varphi \text{ on } \partial\mathcal{O} \end{array}. \tag{6.3.2}$$

The boundary value problem in (6.3.2) has a unique solution if z is not an eigenvalue of $P^{\mathcal{O}}$ introduced in §6.2. More precisely, if we set $E^{\text{in}} : H^{3/2}(\partial\mathcal{O}) \to H^2(\mathcal{O})$ as a linear bounded extension operator such that

$$E^{\text{in}}\varphi|_{\partial\mathcal{O}} = \varphi \text{ and } \operatorname{supp} E^{\text{in}}\varphi \subset \overline{\mathcal{O}} \setminus B(0, R_0), \quad \forall\, \varphi \in H^{3/2}(\partial\mathcal{O}),$$

then for $z \notin \operatorname{Spec}(P^{\mathcal{O}})$, the unique solution to (6.3.2) can be written as

$$u = E^{\text{in}}\varphi - (P^{\mathcal{O}} - z)^{-1}(Q - z)E^{\text{in}}\varphi.$$

It follows that

$$\mathcal{N}_P^{\text{in}}(z)\varphi = \partial_{\nu_g}(E^{\text{in}}\varphi - (P^{\mathcal{O}} - z)^{-1}(Q - z)E^{\text{in}}\varphi), \tag{6.3.3}$$

thus $z \mapsto \mathcal{N}_P^{\text{in}}(z) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$ is a meromorphic family of operators on $\mathbb{C}$ with poles contained in $\operatorname{Spec}(P^{\mathcal{O}})$.

Similarly, we can define the exterior Dirichlet-to-Neumann operator of $\mathcal{P}_{\varepsilon,\theta}$ for every $0 \le \theta < \theta_0$ and $\varepsilon \ge 0$:

$$\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z)\varphi := \frac{\partial u}{\partial \nu_g}, \quad \text{where } u \text{ solves } \begin{array}{c} (Q_{\varepsilon,\theta} - z)u = 0 \text{ in } \Gamma_\theta \setminus \mathcal{O} \\ u = \varphi \text{ on } \partial\mathcal{O} \end{array}. \tag{6.3.4}$$

The existence and uniqueness of the solution to the exterior boundary value problem in (6.3.4) follows if z is not an eigenvalue of $Q^{\mathcal{O}}_{\varepsilon,\theta}$ defined in (6.2.5) with $-2\theta < \arg z < 3\pi/2 + 2\theta$. In fact let $E^{\text{out}} : H^{3/2}(\partial\mathcal{O}) \to H^2(\Gamma_\theta \setminus \mathcal{O})$ be a linear bounded extension operator such that

$$E^{\text{out}}\varphi|_{\partial\mathcal{O}} = \varphi \text{ and } \operatorname{supp} E^{\text{out}}\varphi \Subset \Gamma_\theta \setminus \mathcal{O}, \quad \forall\, \varphi \in H^{3/2}(\partial\mathcal{O}).$$

Then we have a more explicit expression for $\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z)$:

$$\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z)\varphi = \partial_{\nu_g}(E^{\text{out}}\varphi - (Q^{\mathcal{O}}_{\varepsilon,\theta} - z)^{-1}(Q_{\varepsilon,\theta} - z)E^{\text{out}}\varphi). \tag{6.3.5}$$

Hence $z \mapsto \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$ is a meromorphic family of operators in the region $-2\theta < \arg z < 3\pi/2 + 2\theta$, with poles contained in $\operatorname{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta})$.

Now we put

$$\mathcal{N}_{\varepsilon,\theta}(z) := \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z) - \mathcal{N}^{\text{in}}_{P}(z). \tag{6.3.6}$$

The following lemma shows that $\mathcal{N}_{\varepsilon,\theta}(z)$ is a Fredholm operator for suitable z.

Lemma 11. *Suppose that $0 \le \theta < \theta_0$, $\varepsilon \ge 0$ and that $-2\theta < \arg z < 3\pi/2 + 2\theta$ with $z \notin \operatorname{Spec}(P^{\mathcal{O}}) \cup \operatorname{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta})$, then $\mathcal{N}_{\varepsilon,\theta}(z) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$ is a Fredholm operator of index 0.*

Proof. Let $Q^{\mathcal{O}}_{\text{in}}$ and $\mathcal{N}^{\text{in}}_{Q}(z)$ be the the reference operator and the interior Dirichlet-to-Neumann operator associated with Q, defined as in (6.2.3) and (6.3.2) respectively except that Q replaces P there. Suppose that

$$z_0 \notin \operatorname{Spec}(Q^{\mathcal{O}}_{\text{in}}) \cup \operatorname{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta}) \cup \operatorname{Spec}(Q_{\varepsilon,\theta}). \tag{6.3.7}$$

Then $\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0)$ and $\mathcal{N}^{\text{in}}_{Q}(z_0)$ are well-defined. We claim that

$$\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0) - \mathcal{N}^{\text{in}}_{Q}(z_0) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O}) \quad \text{is invertible.} \tag{6.3.8}$$

To show injectivity, we argue by contradiction. Suppose that $0 \ne \varphi \in H^{3/2}(\partial\mathcal{O})$ satisfies $\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0)\varphi = \mathcal{N}^{\text{in}}_{Q}(z_0)\varphi$, it follows from (6.3.2) and (6.3.4) that

$$\exists\, u_1 \in H^2(\mathcal{O}) \text{ and } u_2 \in H^2(\Gamma_\theta \setminus \mathcal{O}) \ (|x_\theta|^2 u_2 \in L^2(\Gamma_\theta \setminus \mathcal{O}) \text{ when } \varepsilon > 0) \text{ such that}$$

$$\begin{array}{c} (Q - z_0)u_1 = 0 \text{ in } \mathcal{O} \\ u_1 = \varphi \text{ on } \partial\mathcal{O} \end{array} \quad \text{and} \quad \begin{array}{c} (Q_{\varepsilon,\theta} - z_0)u_2 = 0 \text{ in } \Gamma_\theta \setminus \mathcal{O} \\ u_2 = \varphi \text{ on } \partial\mathcal{O} \end{array}, \tag{6.3.9}$$

and that $\partial_{\nu_g} u_1 = \partial_{\nu_g} u_2$. Let

$$u = 1_{\mathcal{O}}\, u_1 + 1_{\Gamma_\theta \setminus \mathcal{O}}\, u_2,$$

we aim to show that $u \in H^2(\Gamma_\theta)$. For that it suffices to show the regularity of u near $\partial\mathcal{O}$. For any $x_0 \in \partial\mathcal{O}$, we choose $B_{x_0} := B(x_0, r) \subset B(0, R_1)$ such that $Q_{\varepsilon,\theta} = Q$ in B_{x_0} and put $v \in \mathcal{C}_{\mathrm{c}}^\infty(B_{x_0})$. Then we integrate by parts to obtain:

$$\begin{aligned}
&\int_{B_{x_0}} \left(\sum_{j,k=1}^n g^{jk} \partial_{x_k} u\, \partial_{x_j} v + cuv \right) dx \\
&= \int_{B_{x_0} \cap \mathcal{O}} \left(\sum_{j,k=1}^n g^{jk} \partial_{x_k} u_1 \partial_{x_j} v + c u_1 v \right) dx + \int_{B_{x_0} \setminus \mathcal{O}} \left(\sum_{j,k=1}^n g^{jk} \partial_{x_k} u_2 \partial_{x_j} v + c u_2 v \right) dx \\
&= \int_{B_{x_0} \cap \mathcal{O}} v\, Q u_1 dx - \int_{\partial\mathcal{O} \cap B_{x_0}} v\, \partial_{\nu_g} u_1 dS(x) + \int_{B_{x_0} \setminus \mathcal{O}} v\, Q u_2 dx + \int_{\partial\mathcal{O} \cap B_{x_0}} v\, \partial_{\nu_g} u_1 dS(x) \\
&= \int_{B_{x_0} \cap \mathcal{O}} z_0 u_1 v\, dx + \int_{B_{x_0} \setminus \mathcal{O}} z_0 u_2 v\, dx = \int_{B_{x_0}} z_0 u v\, dx.
\end{aligned}$$

Hence u is a weak solution of $(Q - z_0)u = 0$ in B_{x_0}, the regularity results for second order elliptic differential equations show that u is H^2 near x_0, thus $u \in H^2(\Gamma_\theta)$. It then follows from (6.3.9) that u solves the equation $(Q_{\varepsilon,\theta} - z_0)u = 0$, thus $z_0 \in \mathrm{Spec}(Q_{\varepsilon,\theta})$, which contradicts (6.3.7).

To show surjectivity, we first choose a linear bounded operator $L_g : H^{1/2}(\partial\mathcal{O}) \to H^2(\mathcal{O})$ satisfying the following:

$$\begin{aligned}
L_g \tilde{\varphi} := v, \quad &\text{where } v \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) \text{ satisfies} \\
\operatorname{supp} v \subset \overline{\mathcal{O}} \setminus B(0, R_0) &\text{ and } \partial_{\nu_g} v = \tilde{\varphi}, \quad \tilde{\varphi} \in H^{1/2}(\partial\mathcal{O}).
\end{aligned} \tag{6.3.10}$$

For any $\tilde{\varphi} \in H^{1/2}(\partial\mathcal{O})$, let $v := L_g \tilde{\varphi}$, $f := (Q_{\mathrm{in}}^{\mathcal{O}} - z_0)v \in L^2(\mathcal{O})$. By (6.3.7) we can define

$$u := (Q_{\varepsilon,\theta} - z_0)^{-1} \imath f \quad \text{and} \quad \varphi := u|_{\partial\mathcal{O}} \in H^{3/2}(\mathcal{O}),$$

where $\imath : L^2(\mathcal{O}) \hookrightarrow L^2(\Gamma_\theta)$ denotes the extension by zero. Then

$$u_1 := 1_{\mathcal{O}} u - v \quad \text{satisfies} \quad \begin{matrix} (Q - z_0)u_1 = 0 \text{ in } \mathcal{O} \\ u_1 = \varphi \text{ on } \partial\mathcal{O} \end{matrix};$$

$$u_2 := 1_{\Gamma_\theta \setminus \mathcal{O}}\, u \quad \text{satisfies} \quad \begin{matrix} (Q_{\varepsilon,\theta} - z_0)u_2 = 0 \text{ in } \Gamma_\theta \setminus \mathcal{O} \\ u_2 = \varphi \text{ on } \partial\mathcal{O} \end{matrix}.$$

Hence we have

$$\mathcal{N}_{\varepsilon,\theta}^{\mathrm{out}}(z_0)\varphi - \mathcal{N}_Q^{\mathrm{in}}(z_0)\varphi = \partial_{\nu_g} 1_{\Gamma_\theta \setminus \mathcal{O}}\, u - \partial_{\nu_g}(1_{\mathcal{O}} u - v) = \partial_{\nu_g} v = \tilde{\varphi}.$$

After proving (6.3.8), we now show that $\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z) - \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0)$ and $\mathcal{N}^{\text{in}}_{P}(z) - \mathcal{N}^{\text{in}}_{Q}(z_0)$ are compact: $H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$. Using (6.3.5) we have for any $\varphi \in H^{3/2}(\mathcal{O})$,

$$\begin{aligned}&\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z)\varphi - \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0)\varphi \\ &= \partial_{\nu_g}((Q^{\mathcal{O}}_{\varepsilon,\theta} - z_0)^{-1}(Q_{\varepsilon,\theta} - z_0) - (Q^{\mathcal{O}}_{\varepsilon,\theta} - z)^{-1}(Q_{\varepsilon,\theta} - z))E^{\text{out}}\varphi \\ &= (z - z_0)\partial_{\nu_g}(Q^{\mathcal{O}}_{\varepsilon,\theta} - z_0)^{-1}(I - (Q^{\mathcal{O}}_{\varepsilon,\theta} - z)^{-1}(Q_{\varepsilon,\theta} - z))E^{\text{out}}\varphi \in H^{5/2}(\partial\mathcal{O}),\end{aligned}$$

thus $\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z) - \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0) : H^{3/2}(\partial\mathcal{O}) \to H^{5/2}(\partial\mathcal{O}) \subset H^{1/2}(\partial\mathcal{O})$ is compact since the last inclusion map is compact. It remains to show that

$$\mathcal{N}^{\text{in}}_{P}(z) - \mathcal{N}^{\text{in}}_{Q}(z_0) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O}) \text{ is compact.}$$

For that let $\psi \in \mathcal{C}^\infty_{\text{c}}(\mathcal{O})$ be equal to 1 near $\overline{B(0,R_0)}$, $\varphi \in H^{1/2}(\mathcal{O})$, there exist u and v satisfying:

$$\begin{matrix}(P - z)u = 0 \text{ in } \mathcal{O} \\ u = \varphi \text{ on } \partial\mathcal{O}\end{matrix} \quad \text{and} \quad \begin{matrix}(Q - z_0)v = 0 \text{ in } \mathcal{O} \\ v = \varphi \text{ on } \partial\mathcal{O}\end{matrix},$$

recalling (6.2.2) that $(1-\psi)u \in H^2(\mathcal{O})$, thus we have

$$(\mathcal{N}^{\text{in}}_{P}(z) - \mathcal{N}^{\text{in}}_{Q}(z_0))\varphi = \partial_{\nu_g}((1-\psi)u - (1-\psi)v).$$

Using (3.4.1) we can show that $(1-\psi)u - (1-\psi)v \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$ satisfies:

$$\begin{aligned}Q((1-\psi)u - (1-\psi)v) &= (1-\psi)Pu - [P,\psi]u - (1-\psi)Qv + [Q,\psi]v \\ &= z(1-\psi)u - z_0(1-\psi)v - [P,\psi]u + [Q,\psi]v \in H^1(\mathcal{O}),\end{aligned}$$

then we conclude from the regularity results for second order elliptic differential equations that $(1-\psi)u - (1-\psi)v \in H^3(\mathcal{O})$, thus $(\mathcal{N}^{\text{in}}_{P}(z) - \mathcal{N}^{\text{in}}_{Q}(z_0))\varphi \in H^{3/2}(\partial\mathcal{O})$. Therefore, $\mathcal{N}^{\text{in}}_{P}(z) - \mathcal{N}^{\text{in}}_{Q}(z_0) : H^{3/2}(\partial\mathcal{O}) \to H^{3/2}(\partial\mathcal{O}) \subset H^{1/2}(\partial\mathcal{O})$ is compact.

So far we have shown that there exists a compact operator

$$K(z) := \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z) - \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0) - (\mathcal{N}^{\text{in}}_{P}(z) - \mathcal{N}^{\text{in}}_{Q}(z_0)) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O}),$$

such that $\mathcal{N}_{\varepsilon,\theta}(z) = \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0) - \mathcal{N}^{\text{in}}_{Q}(z_0) + K(z)$. Using (6.3.8) we can write

$$\mathcal{N}_{\varepsilon,\theta}(z) = (\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0) - \mathcal{N}^{\text{in}}_{Q}(z_0))(I + (\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0) - \mathcal{N}^{\text{in}}_{Q}(z_0))^{-1}K(z)),$$

it is a product of an invertible operator and a Fredholm operator of index 0, thus $\mathcal{N}_{\varepsilon,\theta}(z)$ is also a Fredholm operator of index 0. □

Remark. The compactness of $\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z) - \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0)$ and $\mathcal{N}^{\text{in}}_{P}(z) - \mathcal{N}^{\text{in}}_{Q}(z_0)$ can also be proved using the facts that the principal symbols of $\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z)$ and $\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0)$ are identical, same for $\mathcal{N}^{\text{in}}_{P}(z)$ and $\mathcal{N}^{\text{in}}_{Q}(z_0)$ – see for instance Lee–Uhlmann [35] for a detailed account.

In order to work on a single Hilbert space, we introduce

$$\widehat{\mathcal{N}}_{\varepsilon,\theta}(z) := \langle D_{\partial\mathcal{O}}\rangle^{-1}\mathcal{N}_{\varepsilon,\theta}(z) : H^{3/2}(\partial\mathcal{O}) \to H^{3/2}(\partial\mathcal{O}), \tag{6.3.11}$$

where $\langle D_{\partial\mathcal{O}}\rangle = (1-\Delta_{\partial\mathcal{O}})^{1/2}$ is the standard isomorphism between Sobolev spaces $H^s(\partial\mathcal{O})$ and $H^{s-1}(\partial\mathcal{O})$. The following lemma shows that the eigenvalues of $\mathcal{P}_{\varepsilon,\theta}$, $\varepsilon \geq 0$, can be characterized as the poles of $z \mapsto \widehat{\mathcal{N}}_{\varepsilon,\theta}(z)^{-1}$, with agreement of multiplicities.

Lemma 12. *Suppose that* $0 \leq \theta < \theta_0$, $\varepsilon \geq 0$ *and that*

$$\Omega \Subset \{z : -2\theta < \arg z < 3\pi/2 + 2\theta\} \text{ is disjoint from } \operatorname{Spec}(P^{\mathcal{O}}) \cup \operatorname{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta}).$$

Then

$$z \mapsto \widehat{\mathcal{N}}_{\varepsilon,\theta}(z)^{-1}, \quad z \in \Omega,$$

is a meromorphic family of operators on $H^{3/2}(\partial\mathcal{O})$ *with poles of finite rank. Moreover,*

$$n_{\varepsilon,\theta}(z) := \frac{1}{2\pi i}\operatorname{tr}\oint_z \widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1}\partial_w\widehat{\mathcal{N}}_{\varepsilon,\theta}(w)\,dw = m_{\varepsilon,\theta}(z), \tag{6.3.12}$$

where the integral is over a positively oriented circle enclosing z *and containing no poles other than possibly* z *and* $m_{\varepsilon,\theta}(z)$ *is given by* (5.4.7) *(and by* (3.5.11) *when* $\varepsilon = 0$*).*

Proof. 1. Suppose that $z \in \Omega$ is an eigenvalue of $\mathcal{P}_{\varepsilon,\theta}$, we choose $u \in \ker(\mathcal{P}_{\varepsilon,\theta} - z)$ and let $\varphi = u|_{\partial\mathcal{O}}$, then by (6.3.2) and (6.3.4),

$$\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z)\varphi - \mathcal{N}^{\text{in}}_{P}(z)\varphi = \partial_{\nu_g}(1_{\Gamma_\theta\setminus\mathcal{O}}\,u) - \partial_{\nu_g}(1_{\mathcal{O}}\,u) = 0.$$

We note that $\varphi \neq 0$ otherwise $1_{\mathcal{O}}\,u \in \mathcal{D}(P^{\mathcal{O}})$, $(P^{\mathcal{O}} - z)1_{\mathcal{O}}u = 0$, implies that $z \in \operatorname{Spec}(P^{\mathcal{O}})$. Thus $0 \neq \varphi \in \ker\widehat{\mathcal{N}}_{\varepsilon,\theta}(z)$.

On the other hand, suppose that $0 \neq \varphi \in \ker\widehat{\mathcal{N}}_{\varepsilon,\theta}(z)$, the same arguments as in the proof of Lemma 11 show that $z \in \operatorname{Spec}(\mathcal{P}_{\varepsilon,\theta})$. Hence

$$z \in \operatorname{Spec}(\mathcal{P}_{\varepsilon,\theta}) \Longleftrightarrow \ker\widehat{\mathcal{N}}_{\varepsilon,\theta}(z) \neq \{0\}, \tag{6.3.13}$$

and we conclude from Lemma 11 that $\widehat{\mathcal{N}}_{\varepsilon,\theta}(z)$ is invertible for $z \in \Omega \setminus \operatorname{Spec}(\mathcal{P}_{\varepsilon,\theta})$. Theorem (4) then shows that $\Omega \ni z \mapsto \widehat{\mathcal{N}}_{\varepsilon,\theta}(z)^{-1}$ is a meromorphic family of operators on $H^{3/2}(\partial\mathcal{O})$ with poles of finite rank.

2. Since (6.3.13) proves (6.3.12) in the case $m_{\varepsilon,\theta}(z) = 0$, we now assume that $m_{\varepsilon,\theta}(z) = M \geq 1$, and that $\mathcal{P}_{\varepsilon,\theta}$ has exactly one eigenvalue z in

$$D(z, 2r) := \{\zeta \in \mathbb{C}, |\zeta - z| < 2r\}.$$

We note that z is not a compactly supported embedded eigenvalue of P, by which we mean an eigenvalue admitting a compactly supported eigenfunction – see (6.4.1). This is because if $(P-z)u = 0$ for some $0 \neq u \in \mathcal{D}_{\mathrm{comp}}$, then u must vanish identically outside $B(0, R_0)$ by unique continuation results for second order elliptic differential equations, thus $u \in \mathcal{D}^{\mathcal{O}}$. It follows that $z \in \mathrm{Spec}(P^{\mathcal{O}})$ which contradicts our assumption on Ω. Then we claim that for any $\delta > 0$ there exists $V \in \mathcal{C}^\infty(\mathcal{O} \setminus B(0, R_0); \mathbb{R})$ with $\|V\|_\infty < \delta$ such that

$$\mathrm{rank} \int_{\partial D(z,r)} (\mathcal{P}_{\varepsilon,\theta} + V - w)^{-1} dw = M,$$

and that all the eigenvalues of $\mathcal{P}_{\varepsilon,\theta}+V$ in $D(z, r)$ are of multiplicity 1. This follows from the results of [33] (see also [12, Theorem 4.39]) and we omit the proof here. Replacing P by $P + V$ in (6.3.2), we can define $\widehat{\mathcal{N}}^V_{\varepsilon,\theta}$ for $\mathcal{P}_{\varepsilon,\theta} + V$ as in (6.3.6) and (6.3.11). Note that $\widehat{\mathcal{N}}_{\varepsilon,\theta}$ has no kernel except at z in $D(z, 2r)$ by (6.3.13), using (6.3.3) we can choose δ small enough such that for $\|V\|_\infty < \delta$,

$$\|\widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1}(\widehat{\mathcal{N}}_{\varepsilon,\theta}(w) - \widehat{\mathcal{N}}^V_{\varepsilon,\theta}(w))\|_{H^{3/2}(\mathcal{O})\to H^{3/2}(\mathcal{O})} < 1, \quad \forall w \in \partial D(z, r).$$

It then follows from Theorem 7 that

$$\frac{1}{2\pi i}\,\mathrm{tr} \int_{\partial D(z,r)} \mathcal{N}^V_{\varepsilon,\theta}(w)^{-1}\partial_w \mathcal{N}^V_{\varepsilon,\theta}(w)\, dw = n_{\varepsilon,\theta}(z).$$

Hence it is enough to prove (6.3.12) in the case $m_{\varepsilon,\theta}(z) = 1$ with $\mathcal{P}_{\varepsilon,\theta}$ replaced by $\mathcal{P}_{\varepsilon,\theta} + V$.

3. Now we assume that $m_{\varepsilon,\theta}(z) = 1$. In view of (6.3.13), $\widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1}$ has a pole at z, it remains to show that z has polar multiplicity 1 – see Theorem 6. For any w near z and $\tilde{\varphi} \in H^{1/2}(\partial\mathcal{O})$, we recall (6.3.10) that $L_g\tilde{\varphi} \in \mathcal{D}^{\mathcal{O}}$, then $(P^{\mathcal{O}} - w)L_g\tilde{\varphi} \in \mathcal{H}^{\mathcal{O}}$. Now we put

$$u := (\mathcal{P}_{\varepsilon,\theta} - w)^{-1}\imath(P^{\mathcal{O}} - w)L_g\tilde{\varphi}, \quad \varphi := u|_{\partial\mathcal{O}},$$

where $\imath : \mathcal{H}^{\mathcal{O}} \hookrightarrow \mathcal{H}_\theta$ is the extension by zero. Following the arguments in the proof of Lemma 11 while P replacing Q there, we can show that $\mathcal{N}_{\varepsilon,\theta}(w)\varphi = \tilde{\varphi}$, thus

$$\widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1}\tilde{\varphi} = ((\mathcal{P}_{\varepsilon,\theta} - w)^{-1}\imath(P^{\mathcal{O}} - w)L_g(\langle D_{\partial\mathcal{O}}\rangle\tilde{\varphi}))|_{\partial\mathcal{O}}, \quad \forall\tilde{\varphi} \in H^{3/2}(\partial\mathcal{O}).$$

We note that z is a pole of $w \mapsto (\mathcal{P}_{\varepsilon,\theta}-w)^{-1}$ with polar multiplicity 1 due to $m_{\varepsilon,\theta}(z) = 1$. It follows from the expression above that $n_{\varepsilon,\theta}(z) \leq 1$, but (6.3.13) and $m_{\varepsilon,\theta}(z) = 1$ imply that $n_{\varepsilon,\theta}(z) > 0$, thus $n_{\varepsilon,\theta}(z) = 1$. □

6.4 Compactly supported embedded eigenvalues

It has been shown that by introducing an auxiliary obstacle $\mathcal{O}$, we can characterize the eigenvalues of $\mathcal{P}_{\varepsilon,\theta}$, $\varepsilon \geq 0$, as the poles of $z \mapsto \widehat{\mathcal{N}}_{\varepsilon,\theta}(z)^{-1}$. The drawback of this characterization is that $\widehat{\mathcal{N}}_{\varepsilon,\theta}(z)$ can only be defined away from $\mathrm{Spec}(P^{\mathcal{O}})$ and

$\mathrm{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta})$. Therefore, we wish that we can always choose the obstacle $\mathcal{O}$ carefully according to the location of resonances of P, so that $\mathrm{Spec}(P^{\mathcal{O}})$ and $\mathrm{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta})$ stay away from the resonances. This is almost true except the fact that there is a special subset of resonances lying in $\mathrm{Spec}(P^{\mathcal{O}})$ for any $\mathcal{O}$ containing $B(0,R_0)$, that is $\mathrm{Spec}_{\mathrm{comp}}(P)$ – the *compactly supported embedded eigenvalues* of P, defined by

$$\mathrm{Spec}_{\mathrm{comp}}(P) := \{\lambda \in \mathbb{C} : \exists\, 0 \neq u \in \mathcal{D}_{\mathrm{comp}} \text{ such that } (P-\lambda)u = 0\}, \tag{6.4.1}$$

where $\mathcal{D}_{\mathrm{comp}} := \{u \in \mathcal{D}(P) : u|_{\mathbb{R}^n\setminus B(0,R_0)} \in H^2_{\mathrm{comp}}(\mathbb{R}^n \setminus B(0,R_0))\}$.

Remark. Since $Q(u|_{\mathbb{R}^n\setminus B(0,R_0)}) = 0$, by the unique continuation results for second order elliptic differential equations, u in (6.4.1) must vanish on $\mathbb{R}^n \setminus B(0,R_0)$, thus $u \in \mathcal{D}^{\mathcal{O}}$ for any $\mathcal{O}$ containing $\overline{B(0,R_0)}$, which implies that $\mathrm{Spec}_{\mathrm{comp}}(P) \subset \mathrm{Spec}(P^{\mathcal{O}})$.

Now we introduce a strategy to overcome the difficulty caused by $\mathrm{Spec}_{\mathrm{comp}}(P)$. Suppose that $\Omega \Subset \{z :\ -2\theta < \arg z < 3\pi/2 + 2\theta\}$. We define

$$V_0 := \{u \in \mathcal{D}_{\mathrm{comp}} : (P-z)u = 0 \text{ for some } z \in \Omega\}, \tag{6.4.2}$$

V_0 consists of compactly supported eigenfunctions corresponding to eigenvalues in Ω, thus V_0 is finite dimensional. By the remark above, V_0 is a subspace of $\mathcal{H}_{R_0}$ given in (3.3.1), $\mathcal{H}$ admits the following orthogonal decomposition:

$$\mathcal{H} = V_0 \oplus \tilde{\mathcal{H}}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0)). \tag{6.4.3}$$

Let $\Pi_0 : \mathcal{H} \to V_0$ be the orthogonal projection, Π_0 is also a spectral projection for P, thus $P\Pi_0 = \Pi_0 P$. If we replace $\mathcal{H}$ by

$$\tilde{\mathcal{H}} := \tilde{\mathcal{H}}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0)),$$

and define

$$\tilde{P} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}, \text{ with domain } \tilde{\mathcal{D}} := (I-\Pi_0)\mathcal{D}, \quad \tilde{P}u = (I-\Pi_0)Pu,$$

then we have $(\tilde{P}+i)^{-1} = (I-\Pi_0)(P+i)^{-1}(I-\Pi_0)$ and the assumptions (3.3.3), (3.3.4) and (3.4.1) are still satisfied. We remark that

$$\mathrm{Spec}_{\mathrm{comp}}(\tilde{P}) \cap \Omega = \emptyset.$$

Let $\tilde{\mathcal{P}}_\theta$ be the complex deformed $\tilde{P}$ on the contour Γ_θ, with

$$\tilde{\mathcal{H}}_\theta := \tilde{\mathcal{H}}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0,R_0)),$$

then for any $z \notin \mathrm{Spec}(\mathcal{P}_\theta)$,

$$(\tilde{\mathcal{P}}_\theta - z)^{-1} = (I-\Pi_0)(\mathcal{P}_\theta - z)^{-1} : \tilde{\mathcal{H}}_\theta \to \tilde{\mathcal{H}}_\theta. \tag{6.4.4}$$

In addition, let $\tilde{P}_\varepsilon := \tilde{P} - i\varepsilon(1-\chi(x))x^2$, then

$$(\tilde{P}_\varepsilon - z)^{-1} = (I-\Pi_0)(P_\varepsilon - z)^{-1} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}. \tag{6.4.5}$$

6.5 Eigenvalues and obstacle deformation

In this section we show that the eigenvalues of $Q_\theta^{\mathcal{O}}$ and $P^{\mathcal{O}}$ other than $\overline{\mathrm{Spec}_{\mathrm{comp}}(P)}$ can be moved by deforming the obstacle $\mathcal{O}$ while we always assume that $\overline{B(0,R_0)} \subset \mathcal{O} \subset B(0,R_1)$. To describe the deformations of obstacles, we modify (4.1.1) to define

$$\mathrm{Diff}(\mathcal{O}) := \left\{ \begin{matrix} \Phi \in \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ is a diffeomorphism : } \Phi(\partial\mathcal{O}) = \partial\,\Phi(\mathcal{O}), \\ \Phi(x) = x, \quad \text{for all } |x| \le R_0 \text{ or } |x| \ge R_1. \end{matrix} \right\} \tag{6.5.1}$$

We note that $\Phi \in \mathrm{Diff}(\mathcal{O})$ only deforms the region $\{x \in \mathbb{R}^n : R_0 < |x| < R_1\}$, then it also defines a diffeomorphism of Γ_θ, $0 \le \theta < \theta_0$. The pullback Φ^* gives an isomorphism between $L^2(\Gamma_\theta \setminus \Phi(\mathcal{O}))$ and $L^2(\Gamma_\theta \setminus \mathcal{O})$, which also restricts to an isomorphism between $\mathcal{D}(Q_\theta^{\Phi(\mathcal{O})})$ and $\mathcal{D}(Q_\theta^{\mathcal{O}})$ given in (6.2.4) since it preserves the Dirichlet boundary condition. Hence we can define

$$Q_{\theta,\Phi}^{\mathcal{O}} := \Phi^* Q_\theta^{\Phi(\mathcal{O})}(\Phi^*)^{-1}, \quad \text{with } \mathcal{D}(Q_{\theta,\Phi}^{\mathcal{O}}) = \mathcal{D}(Q_\theta^{\mathcal{O}}), \tag{6.5.2}$$

which is considered as the deformed operator of $Q_\theta^{\mathcal{O}}$ under the deformation Φ. The Fredholm properties of $Q_\theta^{\Phi(\mathcal{O})} - z$ immediately show that $Q_{\theta,\Phi}^{\mathcal{O}} - z$ is a Fredholm operator of index 0 for $-2\theta < \arg z < 3\pi/2 + 2\theta$, and (6.5.2) implies that the spectrum of $Q_{\theta,\Phi}^{\mathcal{O}}$ in this region is identical to the spectrum of $Q_\theta^{\Phi(\mathcal{O})}$. Moreover, $Q_{\theta,\Phi}^{\mathcal{O}}$ can be viewed as a restriction of $Q_{\theta,\Phi} := \Phi^* Q_\theta (\Phi^*)^{-1}$ to $\Gamma_\theta \setminus \mathcal{O}$ with Dirichlet boundary condition. A direct calculation shows that

$$A_\Phi := \Phi^* Q_\theta (\Phi^*)^{-1} - Q_\theta = \Phi^* Q (\Phi^*)^{-1} - Q = \sum_{|\alpha| \le 2} a_\alpha(x) \partial_x^\alpha, \tag{6.5.3}$$

where the coefficients a_α are supported in $B(0,R_1) \setminus \overline{B(0,R_0)} \subset \Gamma_\theta$. We note that $\|a_\alpha\|_\infty \le C \|\Phi - \mathrm{id}\,\|_{C^2}$, thus $A_\Phi = \mathcal{O}(\|\Phi - \mathrm{id}\,\|_{C^2}) : H^2(\Gamma_\theta) \to L^2(\Gamma_\theta)$.

Now we show that $\mathrm{Spec}(Q_\theta^{\mathcal{O}})$ can be moved by deforming the obstacle:

Lemma 13. *Suppose that the obstacle $\mathcal{O} \subset B(0,R_1)$ contains $\overline{B(0,R_0)}$ and that $-2\theta < \arg z_0 < 3\pi/2 + 2\theta$, then for any $\delta > 0$ there exists $\Phi \in \mathrm{Diff}(\mathcal{O})$ with $\|\Phi - \mathrm{id}\,\|_{C^2} < \delta$ such that $z_0 \notin \mathrm{Spec}(Q_\theta^{\Phi(\mathcal{O})})$.*

Proof. We may assume that $z_0 \in \mathrm{Spec}(Q_\theta^{\mathcal{O}})$, otherwise we can take $\Phi = \mathrm{id}$. Suppose that $Q_\theta^{\mathcal{O}}$ has exactly one eigenvalue in $D(z_0, 2r)$. For $D := D(z_0, r)$ we define

$$\Pi_{\mathcal{O}}(D) := -\frac{1}{2\pi i} \int_{\partial D} (Q_\theta^{\mathcal{O}} - \zeta)^{-1} d\zeta, \quad m_{\mathcal{O}}(D) := \mathrm{rank}\,\Pi_{\mathcal{O}}(D), \tag{6.5.4}$$

then $m_{\mathcal{O}}(D) = m_{\mathcal{O}}(z_0)$, where $m_{\mathcal{O}}(z_0)$ denotes the multiplicity of $z_0 \in \mathrm{Spec}(Q_\theta^{\mathcal{O}})$.

For $\delta > 0$ small, we put

$$\mathcal{U}_\delta(\mathcal{O}) := \{\Phi \in \mathrm{Diff}(\mathcal{O}) : \|\Phi - \mathrm{id}\,\|_{C^2(\mathbb{R}^n \setminus \mathcal{O})} < \delta\}.$$

It follows from (6.5.3) that $Q_{\theta,\Phi}^{\mathcal{O}} - Q_\theta^{\mathcal{O}} = O(\|\Phi - \mathrm{id}\,\|_{C^2}) : H^2(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O})$, thus for $\Phi \in \mathcal{U}_\delta(\mathcal{O})$ with δ sufficiently small,

$$(Q_{\theta,\Phi}^{\mathcal{O}} - \zeta)^{-1} = (Q_\theta^{\mathcal{O}} - \zeta)^{-1}(I + (Q_{\theta,\Phi}^{\mathcal{O}} - Q_\theta^{\mathcal{O}})(Q_\theta^{\mathcal{O}} - \zeta)^{-1})^{-1}, \quad \zeta \in \partial D,$$

exists and $\sup_{\zeta \in \partial D} \|(Q_{\theta,\Phi}^{\mathcal{O}} - \zeta)^{-1} - (Q_\theta^{\mathcal{O}} - \zeta)^{-1}\|_{L^2(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O})} < C(\Omega)\delta$. We define

$$\Pi_\Phi(D) := -\frac{1}{2\pi i}\int_{\partial D} (Q_{\theta,\Phi}^{\mathcal{O}} - \zeta)^{-1} d\zeta, \quad m_\Phi(D) := \mathrm{rank}\, \Pi_\Phi(D), \tag{6.5.5}$$

then $\Pi_\Phi(D)$ and $\Pi_{\mathcal{O}}(D)$ have the same rank for any $\Phi \in \mathcal{U}_\delta(\mathcal{O})$ if δ is sufficiently small. Since $m_\Phi(D) = m_{\Phi(\mathcal{O})}(D)$ by (6.5.2), we conclude that

$$m_{\Phi(\mathcal{O})}(D) \text{ is constant for } \Phi \in \mathcal{U}_\delta(\mathcal{O}) \text{ if } \delta \text{ is sufficiently small.} \tag{6.5.6}$$

We note that for every $\mathcal{O}$ and z_0, one of the following cases has to occur:

$$\forall\, \delta > 0, \quad \exists\, \Phi \in \mathcal{U}_\delta(\mathcal{O}) \text{ such that } m_{\Phi(\mathcal{O})}(z_0) < m_{\Phi(\mathcal{O})}(D), \tag{6.5.7}$$

or

$$\exists\, \delta > 0, \text{ such that } \quad \forall\, \Phi \in \mathcal{U}_\delta(\mathcal{O}),\ m_{\Phi(\mathcal{O})}(z_0) = m_{\Phi(\mathcal{O})}(D). \tag{6.5.8}$$

The first possibility means that by deforming $\mathcal{O}$ under an arbitrarily small Φ, we can obtain at least one eigenvalue of $Q_\theta^{\Phi(\mathcal{O})}$ other than z_0. The second possibility means that under any small deformation Φ, z_0 is the only eigenvalue of $Q_\theta^{\Phi(\mathcal{O})}$ in D and the maximal multiplicity persists.

Assuming (6.5.7) we can prove the lemma by induction on $m_{\mathcal{O}}(z_0)$. If $m_{\mathcal{O}}(z_0) = 1$, (6.5.6) shows that $m_{\Phi(\mathcal{O})}(D) = 1$ for $\Phi \in \mathcal{U}_\delta(\mathcal{O})$ with δ small. It then follows from (6.5.7) that we can find $\Phi \in \mathcal{U}_\delta(\mathcal{O})$ such that $m_{\Phi(\mathcal{O})}(z_0) < 1$, i.e. $z_0 \notin \mathrm{Spec}(Q_\theta^{\Phi(\mathcal{O})})$. Assuming that we proved the lemma in the case $m_{\mathcal{O}}(z_0) < M$, we now assume that $m_{\mathcal{O}}(z_0) = M$. We note that for any $\Phi_1 \in \mathrm{Diff}(\mathcal{O})$ and $\Phi_2 \in \mathrm{Diff}(\Phi_1(\mathcal{O}))$,

$$\|\Phi_2 \circ \Phi_1 - \mathrm{id}\,\|_{C^2} \le C(\|\Phi_1 - \mathrm{id}\,\|_{C^2} + \|\Phi_2 - \mathrm{id}\,\|_{C^2}),$$

where C is a constant depending only on the dimension n. For any $\delta > 0$, (6.5.7) implies that we can find $\Phi_1 \in \mathrm{Diff}(\mathcal{O})$ with $\|\Phi_1 - \mathrm{id}\,\|_{C^2} < \delta/2C$ such that $m_{\Phi_1(\mathcal{O})}(z_0) < M$. It then follows from our induction hypothesis that there exists $\Phi_2 \in \mathrm{Diff}(\Phi_1(\mathcal{O}))$ with $\|\Phi_2 - \mathrm{id}\,\|_{C^2} < \delta/2C$ such that $z_0 \notin \mathrm{Spec}(Q_\theta^{\Phi_2(\Phi_1(\mathcal{O}))})$. We now take $\Phi = \Phi_2 \circ \Phi_1$, then $\Phi \in \mathcal{U}_\delta(\mathcal{O})$ and $z_0 \notin \mathrm{Spec}(Q_\theta^{(\Phi(\mathcal{O})})$.

It remains to show that (6.5.8) is impossible. For that, we shall argue by contradiction, assume that $m_{\mathcal{O}}(D) = M$ and that (6.5.8) holds. For $\Phi \in \mathcal{U}_\delta(\mathcal{O})$, we define

$$k(\Phi) := \min\{k : (Q^{\mathcal{O}}_{\theta,\Phi} - z_0)^k \Pi_\Phi(D) = 0\},$$

then $1 \leq k(\Phi) \leq M$. It follows from (6.5.2) and (6.5.5) that if $\|\Phi_j - \Phi\|_{C^{2M}} \to 0$ and $(Q^{\mathcal{O}}_{\theta,\Phi_j} - z_0)^k \Pi_{\Phi_j}(D) = 0$, then $(Q^{\mathcal{O}}_{\theta,\Phi} - z_0)^k \Pi_\Phi(D) = 0$. We now put

$$k_0 := \max\{k(\Phi) : \Phi \in \mathcal{U}_{\delta/2}(\mathcal{O})\},$$

and assume that the maximum is attained at $\Phi_0 \in \mathcal{U}_{\delta/2}(\mathcal{O})$ i.e. $k(\Phi_0) = k_0$, then there exists $\delta' > 0$ such that $\|\Phi - \Phi_0\|_{C^{2M}} < \delta' \Rightarrow k(\Phi) = k_0$. Henceforth, we can replace our original obstacle $\mathcal{O}$ with $\Phi_0(\mathcal{O})$, decrease δ and then assume by (6.5.8) that

$$\begin{gathered}(Q^{\mathcal{O}}_{\theta,\Phi} - z_0)^{k_0} \Pi_\Phi(D) = 0, \quad (Q^{\mathcal{O}}_{\theta,\Phi} - z_0)^{k_0 - 1} \Pi_\Phi(D) \neq 0, \\ m_\Phi(z_0) = \operatorname{rank} \Pi_\Phi(D) = M, \quad \forall\, \Phi \in \operatorname{Diff}(\mathcal{O}),\ \|\Phi - \operatorname{id}\|_{C^{2M}} < \delta.\end{gathered} \tag{6.5.9}$$

Before proving that (6.5.9) is impossible we introduce a family of deformations in $\operatorname{Diff}(\mathcal{O})$ acting near a fixed point on $\partial\mathcal{O}$. For any fixed $x_0 \in \partial\mathcal{O}$ and some $h_0 > 0$ small we can choose a family of functions $\chi_h \in \mathcal{C}^\infty(\partial\mathcal{O}; [0,\infty))$ depending continuously in $h \in (0, h_0]$ with

$$\int_{\partial\mathcal{O}} \chi_h(x) dS(x) = 1, \quad \operatorname{supp} \chi_h \subset B_{\partial\mathcal{O}}(x_0, h), \quad \forall\, h \in (0, h_0], \tag{6.5.10}$$

where $B_{\partial\mathcal{O}}(x_0, h)$ denotes the geodesic ball on $\partial\mathcal{O}$ with center x_0 and radius h. For each $h \in (0, h_0]$, we construct a smooth vector field $V_h \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n; \mathbb{R}^n)$ with some small constant $\delta_h = \mathcal{O}(h^{2M+n-1})$ such that

$$\begin{gathered}V_h(x) = \delta_h \chi_h(x) \nu_g(x),\ \forall x \in \partial\mathcal{O}, \quad \|V_h\|_{C^{2M}} < \varepsilon/2, \\ \operatorname{supp} V_h \subset B_{\mathbb{R}^n}(x_0, Ch) \text{ for some } C > 0,\end{gathered} \tag{6.5.11}$$

where $\nu_g(x)$ is defined by (6.3.1). Let $\varphi^t_h : \mathbb{R}^n \to \mathbb{R}^n$ be the flow generated by the vector field V_h. It follows from (6.5.11) that for every $h \in (0, h_0]$ there exists $t_0 > 0$ such that

$$\varphi^t_h \in \operatorname{Diff}(\mathcal{O}), \quad \|\varphi^t_h - \operatorname{id}\|_{C^{2M}} < \delta, \quad \forall\, t \in (-t_0, t_0).$$

Assuming (6.5.9) we can find $w \in L^2(\Gamma_\theta \backslash \mathcal{O})$ so that $u := (Q^{\mathcal{O}}_\theta - z_0)^{k_0-1} \Pi_{\mathcal{O}}(D) w \neq 0$. For any fixed $x_0 \in \partial\mathcal{O}$ and $h \in (0, h_0]$, we take $\Phi_t := \varphi^t_h$, $t \in (-t_0, t_0)$ and put

$$u(t) := (\Phi_t{}^{-1})^* v(t), \quad v(t) := (Q^{\mathcal{O}}_{\theta,\Phi_t} - z_0)^{k_0-1} \Pi_{\Phi_t}(D) w.$$

In view of (6.5.2), $(Q^{\mathcal{O}}_{\theta,\Phi_t} - z_0) v(t) = 0$ implies that

$$(Q_\theta - z_0) u(t) = 0 \quad \text{in } \Gamma_\theta \setminus \Phi_t(\mathcal{O}). \tag{6.5.12}$$

Since $\Phi_t(\mathcal{O}) \subset \mathcal{O}$ for $t \geq 0$, we can restrict (6.5.12) to the region $\Gamma_\theta \backslash \mathcal{O}$ then differentiate it in t, by taking $t = 0$, we obtain that

$$(Q_\theta - z_0)u'(0) = 0 \quad \text{in } \Gamma_\theta \setminus \mathcal{O}. \tag{6.5.13}$$

Recalling that $u(t,x) = v(t, \varphi_h^{-t} x)$ and $u(0) = v(0) = u$, we conclude from the flow equation that $u'(0) = v'(0) - \partial_x u \cdot V_h$, thus by (6.5.11) we have

$$u'(0) = -\delta_h \chi_h(x) \partial_{\nu_g} u, \quad \text{on } \partial\mathcal{O}. \tag{6.5.14}$$

We now multiply (6.5.13) by u then integrate it on $\Gamma_\theta \setminus \mathcal{O}$, then

$$\begin{aligned} 0 &= \int_{\Gamma_\theta \backslash \mathcal{O}} u\,(Q_\theta - z_0)u'(0) \\ &= \int_{\Gamma_\theta \backslash \mathcal{O}} u'(0)\,(Q_\theta - z_0)u + \int_{\Gamma_\theta \backslash \mathcal{O}} \sum_{j,k} \partial_j (u'(0) g^{jk} \partial_k u - u g^{jk} \partial_k u'(0)) \\ &= \int_{\partial\mathcal{O}} (u'(0)\,\partial_{\nu_g} u - u\,\partial_{\nu_g} u'(0))\,dS. \end{aligned} \tag{6.5.15}$$

It then follows from $u|_{\partial\mathcal{O}} = 0$ and (6.5.14) that

$$0 = \int_{\partial\mathcal{O}} \chi_h(x) (\partial_{\nu_g} u(x))^2 dS(x),$$

sending $h \to 0+$, we conclude from (6.5.10) that $\partial_{\nu_g} u(x_0) = 0$. We note that $x_0 \in \partial\mathcal{O}$ can be chosen arbitrarily, thus $\partial_{\nu_g} u|_{\partial\mathcal{O}} \equiv 0$. Putting $\tilde{u} := 1_{\mathcal{O}} \cdot 0 + 1_{\Gamma_\theta \backslash \mathcal{O}} \cdot u$, the same arguments as in the proof of Lemma 11 show that $\tilde{u} \in H^2(\Gamma_\theta)$ and $(Q_\theta - z_0)\tilde{u} = 0$ on Γ_θ. But unique continuation results for second order elliptic differential equations show that $\tilde{u} \equiv 0$, thus a contradiction. □

Now we consider the behavior of $\mathrm{Spec}(P^{\mathcal{O}})$ under the deformations of $\mathcal{O}$. In the notation of §6.2, for $\Phi \in \mathrm{Diff}(\mathcal{O})$, the pullback Φ^* gives an isomorphism between $\mathcal{H}^{\Phi(\mathcal{O})}$ and $\mathcal{H}^{\mathcal{O}}$, which also restricts to an isomorphism between $\mathcal{D}^{\Phi(\mathcal{O})}$ and $\mathcal{D}^{\mathcal{O}}$. Like (6.5.2) we define the deformed operator of $P^{\mathcal{O}}$ associate with Φ:

$$P_\Phi^{\mathcal{O}} := \Phi^* P^{\Phi(\mathcal{O})} (\Phi^*)^{-1}, \quad \text{with domain } \mathcal{D}^{\mathcal{O}}. \tag{6.5.16}$$

Since $(P^{\Phi(\mathcal{O})} + i)^{-1}$ is compact by Proposition 6.2.1, the same holds for $P_\Phi^{\mathcal{O}}$, it follows that $P_\Phi^{\mathcal{O}}$ has a discrete spectrum. Moreover, $\mathrm{Spec}(P_\Phi^{\mathcal{O}})$ must be identical to $\mathrm{Spec}(P^{\Phi(\mathcal{O})})$, which lies in $\mathbb{R}$ due to the self-adjointness of $P^{\Phi(\mathcal{O})}$.

The next lemma shows that any eigenvalue of $P^{\mathcal{O}}$ other than those compactly supported embedded eigenvalues of P can still be moved by deforming the obstacle:

Lemma 14. *Suppose that the obstacle* $\mathcal{O} \subset B(0, R_1)$ *contains* $\overline{B(0, R_0)}$ *and* $z_0 \in \mathrm{Spec}(P^{\mathcal{O}}) \setminus \mathrm{Spec}_{\mathrm{comp}}(P)$*, then for any* $\delta > 0$ *there exists* $\Phi \in \mathrm{Diff}(\mathcal{O})$ *with* $\|\Phi - \mathrm{id}\,\|_{C^2} < \delta$ *such that* $z_0 \notin \mathrm{Spec}(P^{\Phi(\mathcal{O})})$*.*

Proof. The proof is similar to Lemma 13 except that we need a different approach from (6.5.15) since the integration by parts is not available inside the black box. Suppose that $z_0 \in \mathrm{Spec}(P^{\mathcal{O}})$ with multiplicity $m^P_{\mathcal{O}}(z_0)$ and that $P^{\mathcal{O}}$ has exactly one eigenvalue in $D(z_0, 2r)$. For $D := D(z_0, r)$ we put

$$\Pi^P_{\mathcal{O}}(D) := -\frac{1}{2\pi i}\int_{\partial D}(P^{\mathcal{O}} - \zeta)^{-1} d\zeta, \quad m^P_{\mathcal{O}}(D) := \mathrm{rank}\,\Pi^P_{\mathcal{O}}(D).$$

Using (6.2.3) and (6.5.3) we can deduce that $\partial D \ni \zeta \mapsto (P^{\mathcal{O}}_{\Phi} - \zeta)^{-1}$ exists for $\Phi \in \mathcal{U}_\delta(\mathcal{O})$ with δ small enough, then we define

$$\Pi^P_{\Phi}(D) := -\frac{1}{2\pi i}\int_{\partial D}(P^{\mathcal{O}}_{\Phi} - \zeta)^{-1} d\zeta, \quad m^P_{\Phi}(D) := \mathrm{rank}\,\Pi^P_{\Phi}(D) = m^P_{\Phi(\mathcal{O})}(D).$$

We remark that $m^P_{\mathcal{O}}(D)$ is also invariant under small deformations of obstacles:

$$m^P_{\Phi(\mathcal{O})}(D) \text{ is constant for } \Phi \in \mathcal{U}_\delta(\mathcal{O}) \text{ if } \delta \text{ is sufficiently small.} \tag{6.5.17}$$

In view of the proof of Lemma 13, it is enough to exclude the following case:

$$\exists\, \delta > 0, \text{ such that } \quad \forall\, \Phi \in \mathcal{U}_\delta(\mathcal{O}),\ m^P_{\Phi(\mathcal{O})}(z_0) = m^P_{\Phi(\mathcal{O})}(D). \tag{6.5.18}$$

Again we argue by contradiction, assume that (6.5.18) holds and $m^P_{\mathcal{O}}(D) = M \geq 1$. We remark that unlike the proof of Lemma 13, the self-adjointness of $P^{\Phi(\mathcal{O})}$ implies that $(P^{\Phi(\mathcal{O})} - z_0)\Pi^P_{\Phi(\mathcal{O})}(D) = 0$ thus $(P^{\mathcal{O}}_{\Phi} - z_0)\Pi^P_{\Phi}(D) = 0$ for any $\Phi \in \mathcal{U}_\delta(\mathcal{O})$. We now choose $w \in \mathcal{H}^{\mathcal{O}}$ such that $u := \Pi^P_{\mathcal{O}}(D)w \neq 0$. For any fixed $x_0 \in \partial\mathcal{O}$ and $h \in (0, h_0]$, we set $\Phi_t := \varphi^t_h$ where φ^t_h is the flow generated by V_h given in (6.5.11), there exists $t_0 > 0$ such that $\Phi_t \in \mathcal{U}_\delta(\mathcal{O})$ for all $-t_0 < t < t_0$. Let

$$v(t) := \Pi^P_{\Phi_t}(D)w \in \mathcal{D}^{\mathcal{O}}, \quad u(t) := (\Phi_t^{-1})^* v(t),$$

we have $(P^{\mathcal{O}}_{\Phi_t} - z_0)v(t) = 0$, thus $(P^{\Phi_t(\mathcal{O})} - z_0)u(t) = 0$. Recalling (6.2.3) we obtain that for some $\psi \in \mathcal{C}^\infty_c(\mathcal{O})$, $\psi = 1$ near $\overline{B(0, R_0)}$ and $t_0 > 0$ small enough,

$$\forall\, t \in (-t_0, t_0), \quad P(\psi u(t)) + Q((1 - \psi)u(t)) - z_0 u(t) = 0 \quad \text{in } \Phi_t(\mathcal{O}). \tag{6.5.19}$$

Since $\Phi_t(\mathcal{O}) \supset \mathcal{O}$ for $t \leq 0$, we can restrict (6.5.19) to $\mathcal{O}$ and differentiate it in t, by taking $t = 0$, we have

$$P(\psi u'(0)) + Q((1 - \psi)u'(0)) - z_0 u'(0) = 0 \quad \text{in } \mathcal{O}. \tag{6.5.20}$$

Next we compute the inner product of the left hand side and u on the Hilbert space $\mathcal{H}^{\mathcal{O}}$ defined by (6.2.1). For that, choose $\psi_j \in \mathcal{C}_c^\infty(\mathcal{O})$, $\psi_j = 1$ near $\overline{B(0, R_0)}$, so that

$$\psi_1 = 1 \text{ near } \operatorname{supp}\psi, \quad \psi = 1 \text{ near } \operatorname{supp}\psi_2. \tag{6.5.21}$$

Then we have, using the self-adjointness of P,

$$\langle P(\psi u'(0)), u\rangle_{\mathcal{H}^{\mathcal{O}}} = \langle P(\psi u'(0)), \psi_1 u\rangle_{\mathcal{H}} = \langle \psi u'(0), P(\psi_1 u)\rangle_{\mathcal{H}},$$

and

$$\langle Q((1-\psi)u'(0)), u\rangle_{\mathcal{H}^{\mathcal{O}}} = \langle Q((1-\psi)u'(0)), (1-\psi_2)u\rangle_{L^2(\mathcal{O})}.$$

Recalling (6.5.14), integration by parts as in (6.5.15) shows that

$$\begin{aligned}
&\langle Q((1-\psi)u'(0)), (1-\psi_2)u\rangle_{L^2(\mathcal{O})} - \langle (1-\psi)u'(0), Q((1-\psi_2)u)\rangle_{L^2(\mathcal{O})} \\
&= \int_{\mathcal{O}} \sum_{j,k} \partial_j((1-\psi)u'(0)g^{jk}\partial_k((1-\psi_2)\bar{u}) - (1-\psi_2)\bar{u}g^{jk}\partial_k((1-\psi)u'(0))) \\
&= \int_{\partial\mathcal{O}} -u'(0)\partial_{\nu_g}\bar{u} + \bar{u}\partial_{\nu_g}u'(0) = \int_{\partial\mathcal{O}} \delta_h\chi_h|\partial_{\nu_g}u|^2.
\end{aligned}$$

It follows from (6.2.3) and (6.5.21) that

$$\langle \psi u'(0), P(\psi_1 u)\rangle_{\mathcal{H}} = \langle u'(0), \psi(P^{\mathcal{O}}u - Q((1-\psi_1)u))\rangle_{\mathcal{H}^{\mathcal{O}}} = \langle u'(0), \psi P^{\mathcal{O}}u\rangle_{\mathcal{H}^{\mathcal{O}}};$$

and that

$$\begin{aligned}
\langle (1-\psi)u'(0), Q((1-\psi_2)u)\rangle_{L^2(\mathcal{O})} &= \langle u'(0), (1-\psi)(P^{\mathcal{O}}u - P(\psi_2 u))\rangle_{\mathcal{H}^{\mathcal{O}}} \\
&= \langle u'(0), (1-\psi)P^{\mathcal{O}}u\rangle_{\mathcal{H}^{\mathcal{O}}}.
\end{aligned}$$

We now conclude from (6.5.20) and all the calculation above that

$$0 = \langle u'(0), (P^{\mathcal{O}} - z_0)u\rangle_{\mathcal{H}^{\mathcal{O}}} + \int_{\partial\mathcal{O}} \delta_h\chi_h|\partial_{\nu_g}u|^2 = \int_{\partial\mathcal{O}} \delta_h\chi_h|\partial_{\nu_g}u|^2.$$

It follows that $\partial_{\nu_g}u(x_0) = 0$. Since $x_0 \in \partial\mathcal{O}$ can be chosen arbitrarily, we obtain that $\partial_{\nu_g}u|_{\partial\mathcal{O}} \equiv 0$. Putting $\tilde{u} := 1_{\mathcal{O}}u + 1_{\mathbb{R}^n\setminus\mathcal{O}}\cdot 0$, the same arguments as in the proof of Lemma 11 show that $\tilde{u} \in \mathcal{D}$ and $(P - z_0)\tilde{u} = 0$, which would imply that $z_0 \in \operatorname{Spec}_{\text{comp}}(P)$, a contradiction. □

6.6 The CAP method for black box scattering

Before proving the convergence of eigenvalues of P_ε to resonances as $\varepsilon \to 0+$, we recall the decay of the Green function of $Q_\theta^{\mathcal{O}}$ off the diagonal $\{(x,x) : x \in \Gamma_\theta \setminus \mathcal{O}\}$.

Lemma 15. *Suppose that the obstacle $\mathcal{O} \subset B(0,R_1)$ contains $\overline{B(0,R_0)}$ and that $z_0 \notin \operatorname{Spec}(Q_\theta^{\mathcal{O}})$ with $-2\theta < \arg z_0 < 3\pi/2 + 2\theta$. The Schwartz kernel of the resolvent $(Q_\theta^{\mathcal{O}} - z_0)^{-1} : L^2(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O})$ is denoted by $G(z_0; x_\theta, y_\theta)$, where $x_\theta = f_\theta(x)$ is the parametrization on Γ_θ. Then there exists $\beta > 0$ such that for every $\delta > 0$ there exists $C_\delta > 0$ such that*

$$|G(z_0; f_\theta(x), f_\theta(y))| \le C_\delta \, e^{-\beta|x-y|} \quad \textit{if} \quad |x-y| > \delta.$$

Proof. Identifying Γ_θ and $\mathbb{R}^n$ by means of f_θ, the pullback f_θ^* gives an isomorphism between $L^2(\Gamma_\theta \setminus \mathcal{O})$ and $L^2(\mathbb{R}^n \setminus \mathcal{O})$ since there exists $C > 0$ such that

$$C^{-1} < |\det df_\theta(x)| = |x|^{1-n} |g_\theta(|x|)|^{n-1} |g_\theta'(|x|)| < C, \quad \text{for all } x.$$

Let $\tilde{Q}_\theta^{\mathcal{O}} := f_\theta^* Q_\theta^{\mathcal{O}} (f_\theta^*)^{-1} : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O})$ then $\tilde{Q}_\theta^{\mathcal{O}}$ is uniformly elliptic – see Definition 2.5.2, and equipped with the domain $H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H_0^1(\mathbb{R}^n \setminus \mathcal{O})$. Moreover, $(\tilde{Q}_\theta^{\mathcal{O}} - z_0)^{-1}$ exists and we denote its Schwartz kernel by $\tilde{G}(z_0; x, y)$, $x, y \in \mathbb{R}^n \setminus \mathcal{O}$, i.e. $\tilde{G}(z_0; x, y) = [(\tilde{Q}_\theta^{\mathcal{O}} - z_0)^{-1} \delta_y(\cdot)](x)$ where δ_y is the Dirac function supported at y.

Corollary 2.5.4 shows that there exists $\beta > 0$ such that for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$|\tilde{G}(z_0; x, y)| \le C_\delta \, e^{-\beta|x-y|} \quad \text{if} \quad |x-y| > \delta.$$

Using $(\tilde{Q}_\theta^{\mathcal{O}} - z_0)^{-1} = f_\theta^* (Q_\theta^{\mathcal{O}} - z_0)^{-1} (f_\theta^*)^{-1}$ we obtain that

$$G(z_0; f_\theta(x), f_\theta(y)) = (\det df_\theta(y))^{-1} \tilde{G}(z_0; x, y), \quad x, y \in \mathbb{R}^n \setminus \mathcal{O},$$

the desired estimate of $G(z_0; x_\theta, y_\theta)$ then follows from the estimate of $\tilde{G}(z_0; x, y)$. □

Now we state a more precise version of Theorem 2:

Theorem 9. *Suppose that $\Omega \Subset \{z : -2\theta_0 < \arg z < 3\pi/2 + 2\theta_0\}$. Then there exists $\delta_0 = \delta_0(\Omega) > 0$ such that $\forall\, 0 < \delta < \delta_0$, $\exists\, \varepsilon_\delta > 0$ such that*

$$0 < \varepsilon < \varepsilon_\delta \implies \operatorname{Spec}(P_\varepsilon) \cap \Omega_\delta \subset \bigcup_{j=1}^{J} D(z_j, \delta), \tag{6.6.1}$$

where $\Omega_\delta := \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \delta\}$ and $z_1, \cdots, z_J$ are the resonances of P in Ω. Furthermore, for each resonance z_j with the multiplicity $m(z_j)$ given by (3.5.11),

$$\#\, \operatorname{Spec}(P_\varepsilon) \cap D(z_j, \delta) = m(z_j), \quad \forall\, 0 < \varepsilon < \varepsilon_\delta, \tag{6.6.2}$$

where the eigenvalue in $\operatorname{Spec}(P_\varepsilon)$ is counted with multiplicity defined in (5.4.7).

Proof. First we put $\delta_0 = \frac{1}{2}\min_{1\leq j\leq J}\operatorname{dist}(z_j, \partial\Omega)$ and fix $\theta \in [0, \theta_0)$ such that $\Omega \Subset \{z : -2\theta < \arg z < 3\pi/2 + 2\theta\}$. To prove (6.6.1) we argue by contradiction. Suppose that there exist some $\delta < \delta_0$ and a sequence $\varepsilon_k \to 0+$ such that

$$\exists\, z_k \in \operatorname{Spec}(P_{\varepsilon_k}) \cap \Omega_\delta \setminus \bigcup_{j=1}^{J} D(z_j, \delta), \quad k = 1, 2, \cdots$$

Then there exists a subsequence $z_{n_k} \to z_0$, as $k \to \infty$, for some $z_0 \in \overline{\Omega_\delta} \setminus \bigcup_{j=1}^{J} D(z_j, \delta)$. Since $z_0 \in \Omega$, we see that z_0 is not a resonance, thus $\mathcal{P}_\theta - z_0$ is invertible by definition. We may assume that $D(z_0, r)$ is disjoint with $\operatorname{Spec}(\mathcal{P}_\theta)$ for some $r > 0$, it then follows from Lemma 10 that $\operatorname{Spec}(\mathcal{P}_{\varepsilon,\theta}) \cap D(z_0, r) = \emptyset$ for ε small enough. However, Lemma 9 shows that

$$z_{n_k} \in \operatorname{Spec}(\mathcal{P}_{\varepsilon_{n_k},\theta}), \quad \text{while } z_{n_k} \to z_0 \text{ as } \varepsilon_{n_k} \to 0+$$

which gives a contradiction.

It remains to prove (6.6.2). For each resonance z_j, let

$$V_j := \{u \in \mathcal{D}_{\text{comp}} : (P - z_j)u = 0\},$$

then we have for V_0 defined by (6.4.2),

$$V_0 := V_1 \oplus \cdots \oplus V_J.$$

Let $\tilde{P}$ and $\tilde{\mathcal{P}}_\theta$ be defined as in §6.4. Recalling (3.5.10) and (3.5.11), it follows from (6.4.4) that

$$m(z_j) = \operatorname{rank} \oint_{z_j} (z - \tilde{\mathcal{P}}_\theta)^{-1} dz + \dim V_j.$$

Note that $V_j \neq \{0\}$ implies that $z_j \in \operatorname{Spec}(P_\varepsilon)$ for every $\varepsilon > 0$. For $\tilde{P}_\varepsilon$ defined in §6.4, (6.4.5) implies that

$$\#\operatorname{Spec}(P_\varepsilon) \cap D(z_j, \delta) = \#\operatorname{Spec}(\tilde{P}_\varepsilon) \cap D(z_j, \delta) + \dim V_j, \quad \forall\, \varepsilon > 0,$$

while both sides are counted with multiplicities. Hence it is enough to establish (6.6.2) for $\tilde{P}$. In other words, it suffices to prove (6.6.2) in the case that P has no compactly supported embedded eigenvalues in Ω.

Now we assume that $\operatorname{Spec}_{\text{comp}}(P) \cap \Omega = \emptyset$. Lemma 13 and Lemma 14 show that there exists an obstacle $\mathcal{O} \subset B(0, R_1)$ containing $\overline{B(0, R_0)}$ such that χ in (5.4.1) is equal to 1 near $\mathcal{O}$ and that $z_j \notin \operatorname{Spec}(P^{\mathcal{O}}) \cup \operatorname{Spec}(Q_\theta^{\mathcal{O}})$, $j = 1, \cdots, J$. Then we can decrease δ_0 such that $\operatorname{Spec}(P^{\mathcal{O}})$ and $\operatorname{Spec}(Q_\theta^{\mathcal{O}})$ are disjoint with $\bigcup_{j=1}^{J} D(z_j, 2\delta_0)$. For each $\delta \in (0, \delta_0)$, we can also decrease ε_δ in (6.6.1) such that

$$\forall\, 0 \leq \varepsilon < \varepsilon_\delta, \quad \bigcup_{j=1}^{J} D(z_j, 2\delta) \cap \operatorname{Spec}(Q_{\varepsilon,\theta}^{\mathcal{O}}) = \emptyset.$$

This follows from Lemma 10 applied with $\mathcal{P}_\theta = Q_\theta^{\mathcal{O}}$ and $\Omega = \bigcup_{j=1}^J D(z_j, 2\delta)$. Hence the Dirichlet-to-Neumann operators $\widehat{\mathcal{N}}_{\varepsilon,\theta}(z)$, $0 \le \varepsilon < \varepsilon_\delta$ introduced in §6.3, are well-defined for $z \in \bigcup_{j=1}^J D(z_j, 2\delta)$. In view of (6.6.1), Lemma 9 and Lemma 12 imply that

$$\partial D(z_j, \delta) \ni w \mapsto \widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1} \text{ exists,}$$

and that for all $0 < \varepsilon < \varepsilon_\delta$, $j = 1, \cdots, J$,

$$\# \operatorname{Spec}(P_\varepsilon) \cap D(z_j, \delta) = \frac{1}{2\pi i} \operatorname{tr} \int_{\partial D(z_j,\delta)} \widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1} \partial_w \widehat{\mathcal{N}}_{\varepsilon,\theta}(w) dw. \tag{6.6.3}$$

In order to apply Theorem 7, we need the estimate:

$$\forall\, 0 < \varepsilon < \varepsilon_\delta, \quad \|\widehat{\mathcal{N}}_{\varepsilon,\theta}(w) - \widehat{\mathcal{N}}_\theta(w)\|_{H^{3/2}(\partial\mathcal{O}) \to H^{3/2}(\partial\mathcal{O})} < 1, \quad w \in \partial D(z_j, \delta), \tag{6.6.4}$$

here we write $\widehat{\mathcal{N}}_\theta(\cdot) = \widehat{\mathcal{N}}_{0,\theta}(\cdot)$ for simplicity. To obtain this estimate, we first choose E^{out} in (6.3.5) such that

$$\chi = 1 \text{ near } \operatorname{supp} E^{\text{out}}\varphi, \ \forall\, \varphi \in H^{3/2}(\partial\mathcal{O}),$$

then (6.3.5) reduces to

$$\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z)\varphi = \partial_{\nu_g}(E^{\text{out}}\varphi - (Q^{\mathcal{O}}_{\varepsilon,\theta} - z)^{-1}(Q - z)E^{\text{out}}\varphi).$$

Therefore,

$$(\widehat{\mathcal{N}}_{\varepsilon,\theta}(w) - \widehat{\mathcal{N}}_\theta(w))\varphi = \langle D_{\partial\mathcal{O}}\rangle^{-1}\partial_{\nu_g}((Q^{\mathcal{O}}_\theta - w)^{-1} - (Q^{\mathcal{O}}_{\varepsilon,\theta} - w)^{-1})(Q - w)E^{\text{out}}\varphi.$$

Choosing $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\Gamma_\theta \setminus \mathcal{O})$ such that $\psi = 1$ near $\operatorname{supp} E^{\text{out}}\varphi$, $\forall \varphi \in H^{3/2}(\partial\mathcal{O})$ and that $\chi = 1$ near $\operatorname{supp}\psi$, (6.6.4) then follows from the following estimate: for $w \in \partial D(z_j, \delta)$,

$$((Q^{\mathcal{O}}_\theta - w)^{-1} - (Q^{\mathcal{O}}_{\varepsilon,\theta} - w)^{-1})\psi = O_\delta(\varepsilon) : L^2(\Gamma_\theta \setminus \mathcal{O}) \to H^2(\Gamma_\theta \setminus \mathcal{O}). \tag{6.6.5}$$

To obtain (6.6.5), we denote the Schwartz kernel of the operator $(1-\chi)x_\theta^2(Q^{\mathcal{O}}_{\varepsilon,\theta} - w)^{-1}\psi$ by $K(w; x_\theta, y_\theta)$. In the notation of Lemma 15, we have

$$K(w; f_\theta(x), f_\theta(y)) = (1 - \chi(x)) f_\theta(x)^2 G(w; f_\theta(x), f_\theta(y))\psi(y).$$

It follows from Lemma 15 that there exists $\beta_\delta > 0$ such that for all $w \in \partial D(z_j, \delta)$, $j = 1, \cdots, J$, $|K(w; f_\theta(x), f_\theta(y))| \le C|x|^2 e^{-\beta_\delta|x-y|}\psi(y)$, thus

$$\sup_{x_\theta} \int_{\Gamma_\theta\setminus\mathcal{O}} |K(w; x_\theta, y_\theta)||dy_\theta| \le C_\delta, \quad \sup_{y_\theta} \int_{\Gamma_\theta\setminus\mathcal{O}} |K(w; x_\theta, y_\theta)||dx_\theta| \le C_\delta.$$

The Schur test shows that $(1-\chi)x_\theta^2(Q^{\mathcal{O}}_{\varepsilon,\theta} - w)^{-1}\psi = O_\delta(1) : L^2(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O})$. Hence we can write

$$((Q^{\mathcal{O}}_\theta - w)^{-1} - (Q^{\mathcal{O}}_{\varepsilon,\theta} - w)^{-1})\psi = -i\varepsilon(Q^{\mathcal{O}}_{\varepsilon,\theta} - w)^{-1}(1-\chi)x_\theta^2(Q^{\mathcal{O}}_\theta - w)^{-1}\psi.$$

It remains to show that for $\varepsilon_\delta > 0$ small enough,

$$(Q^{\mathcal{O}}_{\varepsilon,\theta} - w)^{-1} = O_\delta(1) : L^2(\Gamma_\theta \setminus \mathcal{O}) \to H^2(\Gamma_\theta \setminus \mathcal{O}), \quad w \in \bigcup_{j=1}^{J} \partial D(z_j, \delta),\ 0 < \varepsilon < \varepsilon_\delta.$$

This follows from Lemma 10 with $\mathcal{P}_\theta = Q^{\mathcal{O}}_\theta$ and $\Omega = \bigcup_{j=1}^{J} \partial D(z_j, \delta)$. Using (6.6.5) we can decrease ε_δ such that (6.6.4) holds for $j = 1, \cdots, J$. Now we apply the Gohberg–Sigal–Rouché theorem to conclude that for all $0 < \varepsilon < \varepsilon_\delta$ and $j = 1, \cdots, J$,

$$\frac{1}{2\pi i} \operatorname{tr} \int_{\partial D(z_j,\delta)} \widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1} \partial_w \widehat{\mathcal{N}}_{\varepsilon,\theta}(w) dw = \frac{1}{2\pi i} \operatorname{tr} \int_{\partial D(z_j,\delta)} \widehat{\mathcal{N}}_{\theta}(w)^{-1} \partial_w \widehat{\mathcal{N}}_{\theta}(w) dw.$$

Finally, using Lemma 12, (6.6.3) and the equation above, we obtain (6.6.2). □

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