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Noncommuting spherical coordinates

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Restricting the states of a charged particle to the lowest Landau level introduces a noncommutativity between Cartesian coordinate operators. This idea is extended to the motion of a charged particle on a sphere in the presence of a magnetic monopole. Restricting the dynamics to the lowest energy level results in noncommutativity for angular variables and to a definition of a noncommuting spherical product. The values of the commutators of various angular variables are not arbitrary but are restricted by the discrete magnitude of the magnetic monopole charge. An algebra, isomorphic to angular momentum, appears. This algebra is used to define a spherical star product. Solutions are obtained for dynamics in the presence of additional angular dependent potentials.

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Noncommutativity between operators corresponding to space coordinates on a plane can be brought about via two, not totally disconnected, procedures. In the first case we replace the ordinary product between two functions by the Moyal star product [1]

$$f(x) \star g(x) = \exp\left(i\frac{\theta^{ab}}{2}\partial_a^{(x)}\partial_b^{(y)}\right)f(x)g(y)|_{y=x}; \quad (1)$$

θ_{ab} is an antisymmetric tensor. The second approach consists of having a particle move on a plane in the presence of a very strong, constant magnetic field perpendicular to the plane. Letting the ratio of strength of the magnetic field to the mass of the particle approach infinity forces the system to lie in the lowest Landau level. Restricting the dynamics to this level permits us to treat one of the planar coordinates as a momentum conjugate to the other one and thus introduce a noncommutativity between coordinate variables [2–5]. In this work we extend this second approach to the motion of particles on a sphere, namely, to noncommutativity between angular variables. The idea of using magnetic monopoles to study the motion of charged particles on a sphere, in analogy to a uniform field for dynamics on a plane, goes back to investigations of the fractional quantum Hall effect [6] and was noted in the study of noncommuting variables in nonuniform magnetic fields [7]. For this purpose we consider a particle of charge e and mass μ moving on a sphere of radius r in the presence of a magnetic field due to a monopole of charge q/e; the Dirac quantization condition limits q to the values $n/2$ where n is an integer. In the northern patch, the one excluding the south pole, the Hamiltonian is [8]

$$H = \frac{1}{2\mu r^2}\left\{p_\theta^2 + \frac{[p_\phi - q(1-\cos\theta)]^2}{\sin^2\theta}\right\}. \quad (2)$$

The simple approach would be to consider the above Hamiltonian in the limit $\mu \to 0$ where we obtain the constraints $p_\theta = 0$ and $p_\phi = q(1-\cos\theta)$ which in turn would imply the commutator

$$[\cos\theta, \phi] = \frac{i}{q}. \quad (3)$$

In the Cartesian case the right-hand side of the above takes on any value inversely proportional to the strength of the applied magnetic field. In the present situation these values are restricted by the discrete possibilities of the magnetic monopole charge. For functions periodic in ϕ this may be rewritten as

$$[\cos\theta, e^{i\phi}] = -\frac{e^{i\phi}}{q}. \quad (4)$$

Multiplying both sides by $\sin\theta$ we obtain a commutator of variables well defined on a sphere

$$[\cos\theta, \sin\theta e^{i\phi}] = -\frac{\sin\theta e^{i\phi}}{q}. \quad (5)$$

We shall obtain a version of (5) in a more rigorous way by considering the algebra of spherical harmonics to be restricted to the lowest level of (2). Wu and Yang [8,9] studied this problem extensively and wave functions and their properties are discussed in these references. The eigenvalues of (2) are $E_{q;l,m} = [l(l+1) - q^2]/(2\mu r^2)$, with $l = |q|, |q|+1, |q|+2, \ldots,$ and $-l \le m \le l$; each level is $(2l+1)$ fold degenerate with eigenvalues being the monopole harmonics [10], $Y_{q;l,m}(\theta, \phi)$; up to a phase these harmonics are multiples of rotation matrix elements [9]. The lowest eigenvalue, $E_{q;q,m} = |q|/(2\mu r^2)$, is separated by $2(|q|+1)/(2\mu r^2)$ from the next level. Thus in the limit $\mu \to 0$ we may restrict the dynamics to the lowest level with states $|q; q, m\rangle$. As most expressions depend on $|q|$ we shall treat the case $q > 0$.

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To this end we define a spherical q product:

$$\langle q;q,m_2|(f(\theta,\phi)\cdot g(\theta,\phi))_q|q;q,m_1\rangle = \sum_m \langle q;q,m_2|f(\theta,\phi)|q;q,m\rangle \times\langle q;q,m|g(\theta,\phi)||q;q,m_1\rangle, \quad (6)$$

where

$$\langle q;q,m'|f(\theta,\phi)|q;q,m\rangle = \int Y^*_{q:q,m'}(\theta,\phi) \times f(\theta,\phi)Y_{q:q,m}(\theta,\phi)d\Omega. \quad (7)$$

Equation (5) suggests that we look at the matrix elements of $Y_{1,m}(\theta,\phi)$ in the level $l=q$. All such expressions may be found in [9].

$$\langle q;q,m_2|Y_{1,m}(\theta,\phi)|q;q,m_1\rangle = (-1)^{m_2+1-q}(2q+1) \times\sqrt{\frac{3}{4\pi}}\begin{pmatrix} q & 1 & q \\ -q & 0 & q\end{pmatrix} \times\begin{pmatrix} q & 1 & q \\ -m_2 & m & m_1\end{pmatrix}, \quad (8)$$

where the arrays are Wigner $3j$ symbols and $m=m_2-m_1$. Explicit expressions for these $3j$ symbols are readily available [11] yielding

$$\langle q;q,m_2|Y_{1,m}(\theta,\phi)|q;q,m_1\rangle = \frac{(-1)^{m+1}}{q+1}\sqrt{\frac{3}{4\pi}}\begin{cases}\sqrt{(q+m_2)(q-m_1)} & \text{for } m=1, \\ m_1 & \text{for } m=0, \\ -\sqrt{(q-m_2)(q+m_1)} & \text{for } m=-1.\end{cases} \quad (9)$$

Using (6), the q commutator is defined as

$$[f(\theta,\phi),g(\theta,\phi)]_q = (f(\theta,\phi)\cdot g(\theta,\phi))_q - (g(\theta,\phi) \cdot f(\theta,\phi))_q, \quad (10)$$

and we obtain

$$[Y_{1,0}(\theta,\phi),Y_{1,1}(\theta,\phi)]_q = -\frac{1}{q+1}\sqrt{\frac{3}{4\pi}}Y_{1,1}(\theta,\phi), \quad (11)$$

which agrees with (5) for large q. The q commutator of $Y_{1,1}$ with $Y_{1,-1}$ is

$$[Y_{1,1}(\theta,\phi),Y_{1,-1}(\theta,\phi)]_q = \frac{2}{q+1}\sqrt{\frac{3}{4\pi}}Y_{1,0}(\theta,\phi). \quad (12)$$

From (9) or from (11) and (12) we find that under the spherical q product the $Y_{1,m}$'s form an algebra isomorphic to angular momentum with

$$(q+1)\sqrt{\frac{4\pi}{3}}Y_{1,1}\leftrightarrow L_+, \qquad (q+1)\sqrt{\frac{4\pi}{3}}Y_{1,0}\leftrightarrow -L_z, \qquad (q+1)\sqrt{\frac{4\pi}{3}}Y_{1,-1}\leftrightarrow -L_-, \quad (13)$$

or equivalently,

$$-(q+1)\hat{\mathbf{r}}\leftrightarrow \mathbf{L}. \quad (14)$$

In addition to nontrivial commutation relations for angular position operators, we would like to obtain a definition of a star product for these variables. Such a star product will agree with the q product only for commutators but not for simple products [3,5]. We do require that a star product reduce to an ordinary one when multiplying commuting variables; the q product does not do that. For Cartesian coordinates the most direct way of obtaining a star product in (1), consistent with $[r_a,r_b]=i\theta_{ab}$, is through the Fourier transform. Namely,

$$e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{q}\cdot\mathbf{r}} = e^{(i/2)\theta^{ab}k_aq_b}e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}}. \quad (15)$$

For the angular case, we must modify the product of two spherical harmonics to allow for noncommuting angular variables. To this end we start with an unconventional expression for the coefficient of $Y_{L,M}$ in the expansion of the product of two spherical harmonic (usually written as a product of $3j$ symbols), $\int Y_{l_1,m_1}(\hat{\mathbf{r}})Y_{l_2,m_2}(\hat{\mathbf{r}})Y^*_{L,M}(\hat{\mathbf{r}})d\hat{\mathbf{r}}$. From the expansion of a plane wave in terms of spherical waves we find

$$Y_{l,m}(\hat{\mathbf{r}}) = \frac{i^{-l}}{4\pi j_l(kr)}\int e^{i\mathbf{k}\cdot\mathbf{x}}Y_{l,m}(\hat{\mathbf{k}})d\hat{\mathbf{r}}; \quad (16)$$

this expression is independent of the magnitudes of $\mathbf{k}$ and $\mathbf{r}$. The previously discussed expansion coefficient becomes

$$\int Y_{l_1,m_1}(\hat{\mathbf{r}})Y_{l_2,m_2}(\hat{\mathbf{r}})Y^*_{L,M}(\hat{\mathbf{r}})d\hat{\mathbf{r}} = \frac{i^{-l_1-l_2}}{j_{l_1}(kr)j_{l_2}(qr)}\int e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{q}\cdot\mathbf{r}}Y_{l_1,m_1}(\hat{\mathbf{k}})Y_{l_2,m_2}(\hat{\mathbf{q}})Y^*_{L,M}(\hat{\mathbf{r}})d\hat{\mathbf{r}}d\hat{\mathbf{k}}d\hat{\mathbf{q}}. \quad (17)$$

When the components of $\hat{\mathbf{r}}$ commute with each other the product of the two exponentials in the above integrals is treated normally. In the noncommuting situation we have to define such a product to be consistent with the commutators in (11) and (12). Using the correspondence in (14) we may make the replacement

$$\exp(i\mathbf{k}\cdot\mathbf{r}) \to \exp\left(-ir\frac{1}{q+1}\mathbf{k}\cdot\mathbf{L}\right) \tag{18}$$

with a similar expression for $\exp(i\mathbf{q}\cdot\mathbf{r})$. The product of the two exponentials is treated as a product of two rotations. The result is then inserted into (17) to obtain the desired coefficients. This time the result depends on the magnitudes k and r indicating that, as in the Cartesian case, different star products will result in the same star commutator. This procedure is a specific construction for the rotation group which agrees with a general study of such products using coherent state bases [12].

Following Peierls [13], who studied the problem of a charged particle that, in addition to the strong magnetic field, is acted on by some potential, we can add an angle dependent potential, $V(\theta, \phi)$, to the present problem. In general, the solution requires the diagonalization of a $(2q+1)\times(2q+1)$ matrix. In the simple case $V(\theta, \phi) = \lambda\cos\theta$ the eigenstates are still the $|q; q, m\rangle$'s and the corresponding energies are

$$E_{q;q,m} = q/(2\mu r^2) - (-1)^m\frac{\lambda m}{q+1}. \tag{19}$$

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