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The Greatest Mathematical Discovery?

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1 Introduction

Question: What mathematical discovery more than 1500 years ago:

- Is one of the greatest, if not the greatest, single discovery in the field of mathematics?
- Involved three subtle ideas that eluded the greatest minds of antiquity, even geniuses such as Archimedes?
- Was fiercely resisted in Europe for hundreds of years after its discovery?
- Even today, in historical treatments of mathematics, is often dismissed with scant mention, or else is ascribed to the wrong source?

Answer: Our modern system of positional decimal notation with zero, together with the basic arithmetic computational schemes, which were discovered in India about 500 CE.

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2 Why?

As the 19th century mathematician Pierre-Simon Laplace explained:

> *It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to all computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.* [4, pg. 527]

As Laplace noted, the scheme is anything but "trivial," since it eluded the best minds of the ancient world, even extraordinary geniuses such as Archimedes. Archimedes saw far beyond the mathematics of his time, even anticipating numerous key ideas of modern calculus and numerical analysis. He was also very skilled in applying mathematical principles to engineering and astronomy. Nonetheless he used the traditional Greek-Roman numeral system for performing calculations [9, 10]. It is worth noting that Archimedes' computation of π was a *tour de force* of numerical interval analysis performed without either positional notation or trigonometry [1, 9].

Perhaps one reason this discovery gets so little attention today is that it is very hard for us to appreciate the enormous difficulty of using Roman numerals, counting tables and abacuses. As Tobias Dantzig (father of George Dantzig, the inventor of linear programming) wrote,

> *Computations which a child can now perform required then the services of a specialist, and what is now only a matter of a few minutes [by hand] meant in the twelfth century days of elaborate work.* [3, pg. 27]

Michel de Montaigne, Mayor of Bordeaux and one of the most learned men of his day, confessed in 1588 (prior to the widespread adoption of decimal arithmetic in Europe) that in spite of his great education and erudition, "*I cannot yet cast account either with penne or Counters.*" That is, he could not do basic arithmetic [7, pg. 577]. In a similar vein, at about the same time a wealthy German merchant, consulting a scholar regarding which European university offered the best education for his son, was told the following:

> *If you only want him to be able to cope with addition and subtraction, then any French or German university will do. But if you are intent on your son going on to multiplication and division—assuming that he has sufficient gifts—then you will have to send him to Italy.* [7, pg. 577]

We observe in passing that Claude Shannon (1916–2001) constructed a mechanical calculator wryly called *Throback 1* at Bell Labs in 1953, which computed in Roman, so as to demonstrate that it was possible, if very difficult, to compute this way.

To our knowledge, the best source currently available on the history of our modern number system is by French scholar Georges Ifrah [7]. He chronicles in encyclopedic detail the rise of modern numeration from its roots in primitive hand counting and tally schemes, to the Babylonian, Egyptian, Greek, Roman, Mayan, Indian and Chinese systems, and finally to the eventual discovery of full positional decimal arithmetic with zero in India, and its belated, kicking-and-screaming adoption in the West. Ifrah describes the significance of this discovery in these terms:

> *Now that we can stand back from the story, the birth of our modern number-system seems a colossal event in the history of humanity, as momentous as the mastery of fire, the development of agriculture, or the invention of writing, of the wheel, or of the steam engine.* [7, pg. 346–347]

Indeed, the development of this system hinged on three key abstract (and certainly non-intuitive) principles [7, pg. 346]:

(a) The idea of attaching to each basic figure graphical signs which were removed from all intuitive associations, and did not visually evoke the units they represented;

(b) The idea of adopting the principle according to which the basic figures have a value which depends on the position they occupy in the representation of a number; and

(c) The idea of a fully operational zero, filling the empty spaces of missing units and at the same time having the meaning of a null number.

The Mayans came close—before 36 BCE they had devised a place-value system that included a zero. However, in their system successive positions represented the mixed sequence $(1, 20, 360, 7200, 144000, \cdots)$ rather than the purely vigesimal (base-20) sequence $(1, 20, 400, 8000, 160000, \cdots)$, which precluded the possibility that their numerals could be routinely used as part of an efficient system for computation.

Ifrah emphasizes more than once that the Indian discovery was by no means obvious or inevitable:

> *The measure of the genius of Indian civilization, to which we owe our modern system, is all the greater in that it was the only one in all history to have achieved this triumph. ... Some cultures succeeded, earlier than the Indian, in discovering one or at best two of the characteristics of this intellectual feat. But none of them managed to bring together into a complete and coherent system the necessary and sufficient conditions for a number-system with the same potential as our own.* [7, pg. 346]

It is astonishing how many years passed before this system finally gained full acceptance in the rest of the world. There are indications that Indian numerals reached southern Europe perhaps as early as 500 CE, but with Europe mired in the Dark Ages, few paid any attention. Similarly, there is mention in Sui Dynasty (581–618 CE) records of Chinese translations of the *Brahman Arithmetical Classic*, although sadly none of these copies seem to have survived [6].

The Indian system (also known as the Indo-Arabic system) was introduced to Europeans by Gerbert of Aurillac in the tenth century. He traveled to Spain to learn about the system first-hand from Arab scholars, prior to being named Pope Sylvester II in 999 CE. However, the system subsequently encountered stiff resistance, in part from accountants who did not want their craft rendered obsolete, to clerics who were aghast to hear that the Pope had traveled to Islamic lands to study the method. It was widely rumored that he was a sorcerer, and that he had sold his soul to Lucifer during his travels. This accusation persisted until 1648, when papal authorities reopened Sylvester's tomb to make sure that his body had not been infested by Satanic forces [7, pg. 579].

The Indo-Arabic system was reintroduced to Europe by Leonardo of Pisa, also known as Fibonacci, in his 1202 CE book *Liber Abaci.* However, usage of the system remained limited for many years, in part because the scheme continued to be considered "diabolical," due in part to the mistaken impression that it originated in the Arab world (in spite of Fibonacci's clear descriptions of the "nine Indian figures" plus zero) [7, pg. 361-362]. Indeed, our modern English word "cipher" or "cypher," which is derived from the Arabic *zephirum* for zero, and which alternately means "zero" or "secret code" in modern usage, is very likely a linguistic memory of the time when using decimal arithmetic was deemed evidence of dabbling in the occult, which was potentially punishable by death [7, pg. 588-589].

Decimal arithmetic began to be widely used by scientists beginning in the 1400s, and was employed, for instance, by Copernicus, Galileo, Kepler and Newton, but it was not universally used in European commerce until after the French Revolution in 1793 [7, pg. 590]. In limited defense of the Roman system, it is harder to alter Roman entries in an account book or the sum payable in a cheque, but this does not excuse the continuing practice of using Roman numerals and counting tables for arithmetic.

The Arabic world, by comparison, was much more accepting of the Indian system—in fact, as mentioned briefly above, the West owes its knowledge of the scheme to Arab scholars. One of the first to popularize the method was al-Khowarizmi, who in the ninth century wrote at length about the Indian place-value system and also described algebraic methods for the solution of quadratic equations. In 1424, Al-Kashi of Samarkand, "*who could calculate as eagles can fly*" computed 2π in *sexagecimal* (good to an equivalent of 16 decimal digits) using $3 \cdot 2^{28}$-gons and a base-60 variation of Indian positional arithmetic [1, Appendix on Arab Mathematics]:

$$2\pi \approx 6 + \frac{16}{60^1} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{01}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9}.$$

This is a personal favorite of ours: re-entering it on a computer centuries later and getting the predicted answer still produces goose-bumps.

So who exactly discovered the Indian system? Sadly, there is no record of the individual who first discovered the scheme, who, if known, would surely rank among the greatest mathematicians of all time. As Dantzig notes,

> *When viewed in this light, the achievement of the unknown Hindu who some time in the first centuries of our era discovered [positional decimal arithmetic] assumes the proportions of a world-event. Not only did this principle constitute a radical departure in method, but we know now that without it no progress in arithmetic was possible.* [3, pg. 29–30]

The very earliest document clearly exhibiting familiarity with the decimal system is the Indian astronomical work *Lokavibhaga* ("Parts of the Universe"). Here, for example, the number 13,107,200,000 is written as

```
panchabhyah khalu shunyebhyah param dve sapta chambaram
ekam trini cha rupam cha
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("five voids, then two and seven, the sky, one and three and the form"), i.e., 0 0 0 0 0 2 7 0 1 3 1, which, when written in reverse order, is 13,107,200,000. One section of this same work gives detailed astronomical observations that confirm to modern scholars that this was written on the date it claimed to be written: 25 August 458 CE (Julian calendar). As Ifrah points out, this information not only allows us to date the document with precision, but it also proves its authenticity. Methods for computation were not explicitly mentioned in this work, although the frequent appearance of large numbers suggests that advanced arithmetic schemes were being used.

Fifty-two years later, in 510 CE, the Indian mathematician Aryabhata explicitly described schemes for various arithmetic operations, even including square roots and cube roots, which schemes likely were known earlier than this date. Aryabhata's actual algorithm for computing square roots, as described in greater detail in a 628 CE manuscript by a faithful disciple named Bhaskara I, is presented in Figure 1 (based on [7, pg. 497–498]). Additionally, Aryabhata gave a decimal value of $\pi = 3.1416$ [7, pg. 465].

From these and other sources there can be no doubt that our modern system of arithmetic—differing only in variations on the symbols used for the digits and minor details of computational schemes—originated in India at least by 510 CE and quite possibly by 458 CE.

3 Modern History

It is disappointing that this seminal development in the history of mathematics is given such little attention in modern published histories. For example, in one popular work on the history of mathematics, although the author describes Arab and Chinese mathematics in significant detail, he mentions the discovery of positional decimal arithmetic in India only in one two-sentence passage [2, pg. 253]. Another popular history of mathematics mentions the discovery of the "Hindu-Arabic Numeral System," but says only that

> *Positional value and a zero must have been introduced in India sometime before A.D. 800, because the Persian mathematician al-Khowarizmi describes such a completed Hindu system in a book of A.D. 825.* [5, pg. 23]

The Aryabhata-Bhaskara I square root algorithm (510–628 CE): Start with the highest-order digit of the input integer (if it has an odd number of digits) or with the two highest-order digits (if it has an even number of digits). Write, as the first digit of the result, the square root of the largest perfect square less than or equal to this one- or two-digit integer; in the next row of the tableau, subtract that perfect square from the one- or two-digit integer. Then repeat the following two-step scheme until the input digits are exhausted: (i) in the tableau, bring down and append the next input digit to the remainder; divide the new remainder by *twice* the current running square root; append the quotient digit (ignoring fraction if any) to the end of the running square root; subtract the quotient digit times the divisor from the remainder; (ii) bring down and append the next input digit to the remainder; subtract the *square* of the quotient digit produced in (i); if this subtraction result would be negative, backtrack and reduce the quotient digit obtained in (i) by one. Note that if the final remainder in the tableau is nonzero, then by continuing the procedure, the fractional digits of the square root can be generated one by one.

For example, to compute the square root of 45468049 (= 6743):

Tableau								**Result**				**Notes**
4	5	4	6	8	0	4	9	6				$\lfloor\sqrt{45}\rfloor = 6$
3	6											$6^2 = 36$
	9	4						6	7			$\lfloor 94/(2\cdot 6)\rfloor = 7$
	8	4										$7\cdot(2\cdot 6) = 84$
	1	0	6									
		4	9									$7^2 = 49$
		5	7	8				6	7	4		$\lfloor 578/(2\cdot 67)\rfloor = 4$
		5	3	6								$4\cdot(2\cdot 67) = 536$
			4	2	0							
				1	6							$4^2 = 16$
			4	0	4	4		6	7	4	3	$\lfloor 4044/(2\cdot 674)\rfloor = 3$
			4	0	4	4						$3\cdot(2\cdot 674) = 4044$
						0	9					
							9					$3^2 = 9$
							0					Finished; result is exact.

Figure 1: The Aryabhata-Bhaskara I scheme for computing square roots.

A third historical work briefly mentions this discovery, but cites a 662 CE Indian manuscript as the earliest known source [8, pg. 221]. A fourth reference states that the combination of decimal and positional arithmetic "*appears in China and then in India*" [12, pg. 67]. None of these authors devotes more than a few sentences to the subject, and, more importantly, none suggests that this discovery is regarded as particularly significant.

We entirely agree with Laplace, Dantzig and Ifrah that this discovery is of the first magnitude. The stark fact that the system is now taught (and mastered) in grade schools worldwide, and is implemented (in binary) in every computer ever manufactured, should not detract from its historical significance. To the contrary, these same facts emphasize the enormous advance that this system represented over earlier systems, both in simplicity and efficiency, as well as the huge importance of this discovery in modern life.

Perhaps some day we will finally learn the identity of this mysterious Indian mathematician. If we do, we surely must accord him or her the same accolades that we have granted to Archimedes, Newton, Gauss and Ramanujan.

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