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Equivariant Stable Homotopy Theory for Diagrams

A dissertation submitted in partial satisfaction of the

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by

Hannah Christine Housden

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ABSTRACT OF THE DISSERTATION

Equivariant Stable Homotopy Theory for Diagrams

by

Hannah Christine Housden

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2022

Professor Michael A. Hill, Chair

We begin with the observation that a group G is just a category with one object where every morphism is an isomorphism and that a G-space is just a functor out of G. The rest of the dissertation re-develops equivariant stable homotopy theory by replacing G with a (usually finite) category, D.

The first chapter considers the unstable case. Our main tool is that of an *orbit*, a generalization of subgroups of the form G/H. We show that several notions often framed in terms of subgroups $H \leq G$ can be rephrased purely in terms of orbits.

The second chapter explores the homotopical structure and homotopy invariants of D-spaces. Its final section gives a generalization of "Elmendorf's Theorem," which states that the category of D-spaces is Quillen equivalent to the category of functors out of the orbit category of D.

The third chapter considers the stable case. We build equivariant spectra as *D-spectral Mackey functors*, originally introduced in the group case by Guillou and May. We then construct examples including *suspension spectra* and *Eilenberg-MacLane spectra*. The penultimate section gives a generalization of *geometric fixed points*, and the final section gives specific computations of Mackey functors over the two-object category $\mathbb{J}$.

The dissertation of Hannah Christine Housden is approved.

Paul Balmer

Raphaël Roquier

Andrew J. Blumberg

Michael A. Hill, Committee Chair

University of California, Los Angeles

2022

To my late father, Chris, who was always proud of me,

and to my mother, Gayle, who still is,

I love you both.

Contents

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I would like to thank Jenny Harrison for advising me to prioritize these qualities when selecting an advisor. It was by far the most valuable piece of advice I received as an undergrad.

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Speaking of Andrew, I am grateful for his many conversations with me about what geometric fixed points ought to be and how to define them. This was a major goal for my project, and I'm glad to have accomplished it. In addition, his course notes from 2017[1] that were compiled by his students have been an excellent resource.

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Anna Marie Bohmann answered many questions I had about her paper with Angélica

[1] https://web.ma.utexas.edu/users/a.debray/lecture_notes/m392c_EHT_notes.pdf

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BIOGRAPHICAL SKETCH

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0 Introduction

This work seeks to generalize the objects of study in equivariant stable homotopy theory. As such, it seems prudent to provide a brief overview of what equivariant stable homotopy theory is and how a generalization might be achieved.

The main objects of (non-equivariant) stable homotopy theory are *spectra*, which can loosely be thought of as pointed spaces X where where iterated applications of the reduced suspension operation $\Sigma X \cong X \wedge S^1$ are invertible. Classically, these provide a framework for studying generalized cohomology theories on spaces, and there are several Quillen equivalent models of spectra.

On the other hand, equivariant (non-stable) homotopy theory studies spaces equipped with the continuous action of a (usually finite) group, G. Putting these together, equivariant stable homotopy theory studies *equivariant spectra*, objects equipped with some form of G-action where suspension is invertible. There are many competing models.

The central question in reconciling the equivariant and stable structures is: *what action do spheres have?* In his 4-page paper "Equivariant Stable Homotopy Theory" [15] that named the field, Graeme Segal restricted his attention to *representation spheres*, those created by the one-point compactification of real G-representations. This choice allows equivariant cohomology theories to be indexed by the ring of real representations of G, $RO(G)$.

The goal of this dissertation is to build equivariant stable homotopy theory where our spaces/spectra in question have the action of a category D, rather than a group G. Most of the non-stable machinery generalizes quickly, with much work being done in the 80s by Emmanuel Dror Farjoun and Alexander Zabrodsky in [4] and [5], William Dwyer and Daniel Kan in [6], and Elmendorf in [7]. This dissertation's main contribution to the non-stable story is "Elmendor's Theorem," which builds off of results from [6] and [7]:

Theorem 2.4.1. Let D be a small category, let $\mathcal{F}$ be some collection of orbits of D that contains all of the free orbits, and let $\mathcal{O}_{\mathcal{F}} \subseteq Top^D$ be the full subcategory spanned by $\mathcal{F}$. Then, $Top^{\mathcal{O}_{\mathcal{F}}^{op}}$ and Top^D are Quillen equivalent categories.

As for the stable story, we diverge from the common treatment using representation spheres. In general, to get a one-point compactification of a representation, the group G has to act via distance-preserving linear transformations. This is not an issue when working with a finite group, but the condition turns out to be rather restrictive when instead working over a category, D:

Theorem 3.0.5. Let D be a locally finite category. Then, D is a groupoid if and only if every objectwise-finite D-orbit embeds inside of an orthogonal D-representation.

("D-orbits" are the generalization of G-sets of the form G/H and are extensively discussed in Chapters 1 and 2.) This theorem tells us that orthogonal representations are very restrictive. Figuring out a "nice" less restrictive class of spheres will be a major undertaking and the subject of future work. For this dissertation, we focus on a model of equivariant spectra known as *spectral Mackey functors*, which avoid explicitly using spheres or representations. We'll be using a version developed by Bertrand Guillou and Peter May in [9] and enhanced by Anna Marie Bohmann and Angélica Osorno in [2]. Clark Barwick independently came up with an infinity-categorical version in [1].

In Chapter 3, we construct many examples of D-spectral Mackey functors, largely following [2]. Our most involved construction is that of *geometric fixed points*:

Theorem 3.7.3. Given a D-orbit O, there exists a *geometric O-fixed points functor*,

$$\Phi^O : Sp^{D\mathcal{B}^{op}} \to Sp^{(End(O)^{op}\mathcal{B})^{op}}$$

that preserves representable objects, is strong symmetric monoidal, and is the left adjoint in a Quillen adjunction.

1 Generalizing Equivariance

Equivariance is often encoded as a continuous group action on a topological space. But we can reformulate this via category theory: an equivalent definition of a group is that it is a (small) category with one object where every morphism is an isomorphism. The reader unfamiliar with this fact should check it for themselves.[2] In this context, a space with a G-action is just a functor $X : G \to Top$, where G is regarded as a category.

This definition would work just as well if G were any category, which leads us to:

Definition 1.0.1. Given a small[3] category D, a *D-space* is a functor $X : D \to Top$. A morphism of D-spaces $\alpha : X \to Y$ is a natural transformation.

We can apply this framework to generalizing other kinds of "objects with group action." For instance, a *D-set* is a functor $X : D \to Set$ and a *D-representation* is a functor $X : D \to Vect_k$ where $Vect_k$ is the category of vector spaces over a fixed field, k. As with D-spaces, morphisms are natural transformations.

Our main tool for understanding this more general notion of equivariance is via *D-orbits*, which generalize G-sets of the form G/H and are originally due to Emmanuel Dror Farjoun and Alexander Zabrodsky.

Definition 1.0.2. [4] Given a category D, we say that a D-space $X : D \to Top$ is an *orbit* if the colimit of X is terminal. (that is, a one-point space).

One crucial class of examples is:

Definition 1.0.3. [5, 2.2] Given a category D and object d, the *free orbit of d*, F^d, is the representable D-set (and discrete D-space) $D(d, -)$.

For any object $d \in D$, the free orbit $D(d, -)$ is indeed an orbit.

[2] HINT: The structure of a group involves an identity element and a binary operation known as "multiplication." These will be related to identity morphisms and composition, respectively.

[3] That is, a category with an actual set of objects where $Hom(X, Y)$ is also always a set.

Proof. For any morphism $f : d \to d'$ with source d, $id_d \circ f = f$. Thus, $D(d, f) : D(d, d) \to D(d, d')$ sends id_d to f, so id_d and f are glued together in $\mathrm{colim}(D(d, -))$. Because this holds for all f with source d, (that is, all elements of $\amalg_{d'} D(d, d')$) the colimit of $D(d, -)$ is a one-point set, meaning $D(d, -)$ is an orbit. □

When D is a group, the sole free orbit is isomorphic to $D/\{e\}$. As for the other orbits, one can indeed check that D being a group implies that any D-orbit is isomorphic to D/H for some subgroup H. However, things generally get much more exciting for non-group categories, even very simple ones. For instance:

Definition 1.0.4. $\mathbb{J} = s \xrightarrow{f} t$ is the category with two objects ('s' for 'source' and 't' for 'target') and one non-identity morphism $f : x \to y$.

Proposition 1.0.5. There is an equivalence of categories between the category of $\mathbb{J}$-orbits and Top.

Proof. By the usual description of a colimit in Top, the colimit of $X : \mathbb{J} \to Top$ is given as the quotient $(X_s \amalg X_t)/\sim$, where $\sim$ identifies each $x_s \in X_s$ with $f(x_s)$. Note that each $x_s \in X_s$ is identified to exactly one point in X_t because f is a function. Thus, it's not possible for two distinct points in X_t to become identified, so if X is an orbit, X_t must only have one point. But then all points in X_s are identified with the unique $x_t \in X_t$. Hence, an orbit in $\mathbb{J}$ is precisely a $\mathbb{J}$-space $X : \mathbb{J} \to Top$ such that X_t has a single point. From this we observe that any continuous map $\alpha_s : X_s \to Y_s$ will yield a commutative square

$$\begin{tikzcd} X_s \arrow[r, "\alpha_S"] \arrow[d, "X_f"'] & Y_s \arrow[d, "Y_f"'] \\ X_t \arrow[r, "\alpha_t"] & Y_t \end{tikzcd}$$

whenever Y is an orbit. In other words, the functor $Top^{\mathbb{J}} \to Top$ taking X to X_s is full, faithful, and essentially surjective, i.e., an equivalence. □

We'll use $\mathbb{J}$-orbits frequently as a source of examples, so it will be helpful to have simple notation for them:

Notation. For any $n \in \mathbb{N}$, $[n]$ is the $\mathbb{J}$-orbit $\{0, \ldots n-1\} \to \{pt\}$.

Observation 1.0.6. The free orbits of $\mathbb{J}$ are $[0] \cong F^t = \mathbb{J}(t, -)$ and $[1] \cong f^s = \mathbb{J}(s, -)$.

For a general small category D, we can still have a lot of orbits to work with. In most situations, we only need to consider the orbits that actually appear in our D-space X. It is helpful to have some vocabulary to describe these orbits.

Convention 1.0.7. For any topological space A, we will abuse notation and also refer to its corresponding constant D-space as A. (Thus, the statement $A_d = A$ is perfectly valid.)

Definition 1.0.8. [4, Section 2.2] Given a D-space X and a point x_d of X_d for some object $d \in D$, the *orbit of* x_d, O_{x_d}, is the D-space of points in X that get identified with x_d in $\operatorname{colim}(X)$. This can be identified with the following pullback: (The spaces on the bottom row are viewed as constant D-spaces, following Convention 1.0.7.)

$$\begin{tikzcd} O_{x_d} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d] \\ \{x_d\} \arrow[r, hook] & \operatorname{colim}(X) \end{tikzcd}$$

Orbits classify discrete D-spaces (that is, D-sets) as follows:

Proposition 1.0.9. Given a small category D, any D-set X can be decomposed as the coproduct of its (necessarily discrete) D-orbits. Furthermore, this decomposition is unique up to reordering and isomorphic replacement of the factors.

Proof. Given a point x_d of X, each point $x \in O_{x_d}$ is by definition sent to the same point in $\operatorname{colim}(X)$. Thus, each point of X is in precisely one orbit. This gives a disjoint union decomposition of X, which is precisely the claimed coproduct structure.

To see uniqueness, suppose X is isomorphic to both $\coprod_{i\in I} O_i$ and $\coprod_{j\in J} \tilde{O}_j$, where each O_i and $\tilde{O}_j$ is a D-orbit. The fact that these are both decompositions of X give us an isomorphism

$$f : \coprod_{i\in I} O_i \to \coprod_{j\in J} \tilde{O}_j.$$

Because isomorphisms preserve colimits, f induces an isomorphism of sets

$$g : \mathrm{colim}(\coprod_{i\in I} O_i) \to \mathrm{colim}(\coprod_{j\in J} \tilde{O}_j).$$

Since colimits commute with coproducts, g can instead be viewed as a bijective function

$$g : \coprod_{i\in I} \mathrm{colim}(O_I) \to \coprod_{j\in J} \mathrm{colim}(\tilde{O}_j).$$

But each $\mathrm{colim}(O_i)$ and $\mathrm{colim}(\tilde{O}_j)$ is a one-point set, so g corresponds to a bijection from I to J, which we will abusively call g. The fact that g sends i to $g(i)$ means that f sends points in O_i to points in $\tilde{O}_{g(i)}$. Because g is a bijection, *only* the points in O_i can be sent to $\tilde{O}_{g(i)}$. Thus, since f is an isomorphism, the restriction of f to $O_i \to \tilde{O}_{g(i)}$ must also be an isomorphism. This shows the desired uniqueness. □

1.1 Properties of Top^D

Let us now explore the categorical properties of Top^D:

Proposition 1.1.1. For any small category D, Top^D has all small limits and colimits.

Proof. This is immediate from the fact that Top has all small limits and colimits and the fact that limits and colimits in a functor category can be computed objectwise. □

From the objectwise computation of limits and colimits, we get:

Corollary 1.1.2. Given a small category D, finite category I, and a functor $A : I \to Top^D$

such that each $A_i(d)$ is a finite set, we have that each $\mathrm{colim}(A)_d$ is a finite set. In other words, the finite limit of objectwise-finite D-sets is an objectwise-finite D-set.

Next, we'd like see that Top^D is nicely enriched in Top. What follows is largely a recap of [4, Section 3].

Definition 1.1.3. For any D-spaces X and Y, we topologize $D(X,Y)$ with the subspace topology from its inclusion into $Top(\amalg_d X_d, \amalg_d Y_d)$.

Corollary 1.1.4. We can view Top^D as enriched in Top.

However, we can also view $D(X,Y)$ as a D-space:

Definition 1.1.5. [5, Proposition 2.17] For any D-spaces X and Y, we can view $D(X,Y)$ as a D-space where $D(X,Y)_d = D(X \times F^d, Y)$. For any morphism $f : d \to d'$ in D,

$$D(X,Y)_f : D(X \times F^d, Y) \to D(X \times F^{d'}, Y)$$

is induced by the natural map

$$D(f,-) : F^{d'} = D(d',-) \to D(d,-) = F^d.$$

This construction makes Top^D enriched in itself.

In Section 3.7, we'll use a third type of extra structure on $D(X,Y)$:

Definition 1.1.6. For any D-spaces X and Y, we can view $D(X,Y)$ as a being an $End(X)^{op}$-space, where $End(X) = D(X,X)$. The action is given by pre-composition.

WARNING. This does not specify an enrichment of Top^D because the action of $End(X)^{op}$ only exists on morphism spaces $D(X,Y)$ with source X. We don't have a general action on $D(X',Y)$ for $X' \neq X$.

Many times, we'll use the following notation for $D(X,Y)$:

Notation. For D-spaces X and Y, $Y^X := D(X,Y)$. Whether we wish for Y^X to be a space, D-space, or $End(X)^{op}$-space will depend on context.

Most commonly, we'll be applying this notation to the context of the representable functor $(-)^Y$. Let's list a few of this functor's properties:

Proposition 1.1.7. For any D-space X, the functor $(-)^X$ (valued in either Top, Top^D, or $Top^{End(X)^{op}}$) preserves limits.

Proof. For the functors valued in Top or Top^D, this is immediate from the fact that (enriched) representable functors preserve limits. For the functor valued in $Top^{End(X)^{op}}$, we recall that limits are computed objectwise and that $End(X)^{op}$ has just one object, O. Thus, the map

$$(\mathrm{Lim}_i X_i)^O \to \mathrm{Lim}_i X_i^O,$$

which is necessarily $End(X)^{op}$-equivariant, is also an isomorphism as a map of spaces. Since equivariant maps that have objectwise inverses are isomorphisms, we're done. $\square$

Proposition 1.1.8. For any D-orbit O, the functor $(-)^O$ (valued in either Top, Top^D or $Top^{End(X)^{op}}$) preserves pushouts and coproducts.

Proof. We begin with pushouts:

Consider any pushout $Y \sqcup_W Z$ of D-spaces. Because $\mathrm{colim}(O) = \{o\}$ is a one-point space, a map $f : O \to Y \sqcup_W Z$ is factored by a map $O \to Y$ or $O \to Z$ based on whether a given point $f(o)$ lands in $\mathrm{colim}(Y) \subseteq \mathrm{colim}(Y \sqcup_W Z)$ or in $\mathrm{colim}(Z) \subseteq \mathrm{colim}(Y \sqcup_W Z)$. If $f(o)$ lands in both $\mathrm{colim}(Y)$ and $\mathrm{colim}(Z)$, then $O \to Y$ and $O \to Z$ are both factored by a map $O \to X$. In other words, $(Y \sqcup_W Z)^O$ has the universal property of a pushout of $Y^O \leftarrow W^O \to Z^O$, so $(Y \sqcup_W Z)^O$ and $Y^O \sqcup_{W^O} X^O$ are naturally isomorphic as spaces. Hence, $(-)^O$ preserves pushouts, at least when it's valued in Top.

Similarly, for any coproduct $\amalg_{i \in I} Y_i$ of D-spaces, a map

$$O \to \amalg_{i \in I} Y_i$$

is factored based on which $\mathrm{colim}_{d \in D}(Y_i) \subseteq \mathrm{colim}_{d \in D}(\amalg_{i \in I} Y_i)$ that $f(o)$ lands in. Thus, $(\amalg_{i \in I} Y_i)^O$ has the universal property of $\amalg_{i \in I} Y_i^O$, at least when $(-)^O$ is valued in Top.

To get the version valued in Top^D, recall that X_d^O is the space $Top^D(F^d, X^O)$. Since F^d is an orbit, we conclude the pushouts and coproducts are preserved at each object. By the dual of the argument at the end of Proposition 1.1.7, this means the Top^D-valued functor preserves pushouts and coproducts. A similar argument shows that the $Top^{End(O)^{op}}$-valued functor also preserves pushouts and coproducts. □

We can say more about this enrichment if we take Top to be the category of compactly generated weak Hausdorff spaces. This condition implies that, for any spaces A, B, C, we have an isomorphism of spaces

$$Top(A \times B, C) \cong Top(A, Top(B, C)).$$

As long as we have this condition, we get the following:

Proposition 1.1.9. Top^D is a closed symmetric monoidal category, with monoidal structure given by objectwise Cartesian product.

Proof. We just need to confirm that there is a natural isomorphism of *sets*

$$Top^D(X \times Y, Z) \cong Top^D(X, Top^D(Y, Z)).$$

Given a natural transformation

$$\alpha : X \times Y \to Z,$$

we get a natural transformation

$$\beta : X \to Top^D(Y, Z),$$

where

$$\beta_d : X_d \to Top^D(Y, Z)_d = Top^D(Y \times F^d, Z)$$

is defined by

$$[\beta_d(x_d)](y_{d'}, f) = \alpha(f(x_d), y_{d'}).$$

(Here, f is a generic element of $F^d(d')$, meaning it's a morphism $f : d \to d'$.) The fact that we're working with compactly generated weak Hausdorff spaces ensures that each β_d is continuous. We can recover α from β by setting

$$\alpha_d(x_d, y_d) = [\beta_d(x_d)](y_d, id_d).$$

Again, the fact that we're working with compactly generated weak Hausdorff spaces ensures that each α_d is continuous. $\square$

If we apply the same argument to pointed (compactly generated weak Hausdorff) spaces, using the isomorphism

$$Top_\bullet(D \wedge E, F) \cong Top_\bullet(D, Top(E, F)),$$

for any pointed spaces D, E, and F, we get:

Corollary 1.1.10. $Top_\bullet^D$ is closed symmetric monoidal category, with monoidal structure given by objectwise smash product.

1.2 Orbits vs. Subgroups

In the group case, keeping track of orbits of G is essentially the same task as keeping track of subgroups of G. One way of making this precise is the following:

Proposition 1.2.1. The category of G-orbits with G-equivariant maps, $Orb(G)$ is equivalent to the category of subgroups of G with inclusions and conjugations for morphisms.

For general categories D, the natural generalization of this proposition that uses "subcategory" instead of "subgroup" is extremely false. For $\mathbb{J} = s \xrightarrow{f} t$, we saw that the orbit category was equivalent to Top, but we can see there are only finitely many subcategories! In practice, this is because "subgroup" is the wrong notion to capture the equivariant structure, and our story can be explained purely in terms of orbits. Let's explore how to go about this; for the group case, this will involve translating notions that use the subgroup $H \leq G$ to instead use the G-orbit G/H. But first, we'll need a definition:

Definition 1.2.2. Given a D-set $T : D \to Set$, its *translation category*, $B_D(T)$, has objects given by elements of $\amalg_{d \in D} T_d$ and where morphism-sets are defined by

$$B_D(T)(a, b) = \{f \in D \mid T_f(a) = b\}$$

.

Proposition 1.2.3. This construction is functorial.

WARNING. We will have to be vigilant for abuse of notation. Given any morphism $f \in D$ with source d, 'f' will denote the corresponding morphism in $B_D(T)(a, T_{f(a)})$ for every $a \in T_d$. For this reason, morphisms in the translation category will generally have their source and target explicitly specified.

Example 1.2.4. When we pick D to be a group and T to be some orbit D/H, we get what is often called the *translation groupoid*. This translation groupoid, $B_D(D/H)$ is in fact equivalent (in the categorical sense) to H! (The reader unfamiliar with this fact should check it for themselves.[4]) Thus, the functor categories Top^H and $Top^{B_D(D/H)}$ are equivalent.

This allows us to talk about "restriction" in terms of orbits: while any G-space X has a "restricted" H action given by the inclusion $H \leq G$, X also gives rise to a $B_G(G/H)$-space that encodes the same data. This is the subject of the next section:

1.3 Restriction and its Adjoints

By the functoriality of $B_D(-)$, a map of D-sets $\tau : T_1 \to T_2$ induces a map

$$B_D(\tau) : B_D(T_1) \to B_D(T_2)$$

. On functor categories, this becomes:

Definition 1.3.1. Given a map of D-sets $\tau : T_1 \to T_2$, the *restriction functor of* τ,

$$Res^\tau : Top^{B_D(T_2)} \to Top^{B_D(T_2)},$$

is defined by precomposition with $B_D(\tau)$.

This generalizes what we see in the group case: If T_2 is the terminal orbit G/G and T_1 is some G/H, the map $\tau : G/H \to G/G$ corresponds to the inclusion $H \leq G$. Up to applying the equivalence $B_G(G/H) \simeq H$ and the isomorphism $B_G(G/G) \cong G$, Res^τ is precisely the functor $Top^G \to Top^H$ that takes a space with G-action and outputs the action given by elements of $H \leq G$.

The restriction functor gives us a nice way to convert from a $B_D(T_2)$-space to a $B_D(T_1)$-space. This is a contravariant construction. But can we give a covariant construction? The answer is yes, as we can functorially give a left adjoint and a right adjoint to Res^τ:

Definition 1.3.2. Let D be a small category and let $\tau : T_1 \to T_2$ be an equivariant map of

[4]HINT: What are the endomorphisms of a given object in $B_D(D/H)$?

D-sets. The *induction of* τ functor, $Ind^{\tau} : Top^{B_D(T_1)} \to Top^{B_D(T_2)}$, is given by:

- Given a $B_D(T_1)$-space X and object $t_2 \in B_D(T_2)$, we define

$$[Ind^{\tau}(X)]_{t_2} = \coprod_{t_1 \in \{t_1 | \tau(t_1) = t_2\}} X_{t_1}.$$

- Given a $B_D(T_1)$-space X and morphism $f : t_2 \to t_2' \in B_D(T_2)$, we define the map

$$[Ind^{\tau}(X)]_f : \coprod_{t_1 \in \{t_1 | \tau(t_1) = t_2\}} X_{t_1} \to \coprod_{t_1' \in \{t_1' | \tau(t_1') = t_2'\}} X_{t_1'}$$

  by mapping the factor indexed by t_1 to the factor $f(t_1)$ via the map $X_f : X_{t_1} \to X_{f(t_1)}$.

- Given a morphism of $B_D(T_1)$-spaces $\alpha : X \to Y$ and object $t_2 \in B_D(T_2)$,

$$[Ind^{\tau}(\alpha)]_{t_2} = \coprod_{t_1 \in \{t_1 | \tau(t_1) = t_2\}} \alpha_{t_1}.$$

Definition 1.3.3. Let D be a small category and let $\tau : T_1 \to T_2$ be an equivariant map of D-sets. The *coinduction of* τ functor, $Coind^{\tau} : Top^{B_D(T_1)} \to Top^{B_D(T_2)}$, is given by:

- Given a $B_D(T_1)$-space X and object $t_2 \in B_D(T_2)$, we define

$$[Coind^{\tau}(X)]_{t_2} = \prod_{t_1 \in \{t_1 | \tau(t_1) = t_2\}} X_{t_1}.$$

- Given a $B_D(T_1)$-space X and morphism $f : t_2 \to t_2' \in B_D(T_2)$, we define the map

$$[Coind^{\tau}(X)]_f : \prod_{t_1 \in \{t_1 | \tau(t_1) = t_2\}} X_{t_1} \to \prod_{t_1' \in \{t_1' | \tau(t_1') = t_2'\}} X_{t_1'}$$

  by mapping the factor indexed by t_1 to the factor $f(t_1)$ via the map $X_f : X_{t_1} \to X_{f(t_1)}$.

- Given a morphism of $B_D(T_1)$-spaces $\alpha : X \to Y$ and object $t_2 \in B_D(T_2)$,

$$[Coind^\tau(\alpha)]_{t_2} = \prod_{t_1 \in \{t_1 | \tau(t_1) = t_2\}} \alpha_{t_1}.$$

Proposition 1.3.4. Ind^τ and $Coind^\tau$ are functorial in τ.

Proof. This is immediate from the naturality of coproducts and products, respectively. $\square$

Theorem 1.3.5. Given any morphism $\tau : T_1 \to T_2$ of D-sets, Ind^τ is the left adjoint of Res^τ and $Coind^\tau$ is the right adjoint of Res^τ.[5] Pictorially:

$$\begin{tikzcd} Top^{B_D(T_2)} \arrow[d, "\dashv Res^\tau \dashv" description] \\ Top^{B_D(T_1)} \arrow[u, bend left=60, "Ind^\tau"] \arrow[u, bend right=60, "Coind^\tau"'] \end{tikzcd}$$

Proof. To show Ind^τ is the left adjoint to Res^τ, we will construct a natural isomorphism

$$Top^{B_D(T_2)}(Ind^\tau(X), Y) \cong Top^{B_D(T_1)}(X, Res^\tau(Y)).$$

By a dual argument, we will see that $Coind^\tau$ is the right adjoint to Res^τ.

Let X be a $B_D(T_1)$-space, let Y be a $B_D(T_2)$-space, and consider a $B_D(T_1)$-equivariant map $\alpha : X \to Res^\tau(Y)$. Thus, for every object $t_1 \in T_1$, we have a continuous map

$$\alpha_{t_1} : X_{t_1} \to [Res^\tau(Y)]_{(t_1)} = Y_{\tau(t_1)}$$

[5] "Induction" and "Coinduction" are terms from representation theory that predate category theory and the subsequent convention that the prefix "co-" to describe constructions that are left adjoints.

. By the universal property of coproducts, this is the same as having a continuous map

$$\beta_{t_1} : [Ind^\tau(X)]_{t_2} = \coprod_{t_1 \in \{t_1 | \tau(t_1) = t_2\}} X_{t_1} \to Y_{t_2}$$

for every object $t_2 \in T_2$. By the same universal property, asking that α be T_1-equivariant is equivalent to asking that β be T_2-equivariant. The naturality in both arguments follows immediately from the description of the isomorphism.

The dual argument for seeing that $Coind^\tau$ is a right adjoint is done by swapping Ind^τ for $Coind^\tau$, reversing the direction of α and β, and using products and their universal property instead of coproducts. $\square$

As is always the case for functors with adjoints on both sides, Res^τ preserves limits and colimits, and if we work with pointed spaces, there is a natural map from the left adjoint to the right adjoint. By the description of Ind^τ and $Coind^\tau$, this natural map just involves sending a coproduct of spaces to a product of those same spaces. If we were working in an additive category instead of Top and the indexing sets were finite, we would get that this natural map from $Ind^\tau(X)$ to $Coind^\tau(X)$ is an isomorphism.

1.4 Slice Categories

We have another way to view restriction and its adjoints, which is sometimes more convenient. This is done via *slice categories*:

Definition 1.4.1. Given a D-set T, the *slice category over* T, $Top^D_{/T}$, is the category whose objects are morphisms $\alpha : X \to T$ in Top^D and where a morphism between $\alpha : X \to T$ and $\beta : Y \to T$ is a map $\gamma : X \to Y$ such that the square

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & Y \arrow[dl, "\beta"] \\ & T & \end{tikzcd}$$

commutes.

In the group case, one classical result is:

Theorem 1.4.2. Given a subgroup $H \leq G$, the category of H-spaces, Top^H, is equivalent (in the categorical sense) to $Top^G_{/(G/H)}$.

Recall again that H is equivalent as a category to $B_G(G/H)$. Thus, Theorem 1.4.2 can be stated purely in terms of orbits, and it's this version of the statement that we can generalize:

Theorem 1.4.3. For any small category D and D-set T, there is a categorical equivalence $Top^{B_D(T)} \simeq Top^D_{/T}$.

Proof. We will construct a functor $F : Top^D_{/T} \to Top^{B_D(T)}$ that is full, faithful, and essentially surjective.

Let $\alpha : X \to T$ be an element of $Top^D_{/T}$, and define $F(\alpha)_t = \alpha_d^{-1}(\{t\})$ for any object $d \in D$ and point $t \in T_d$. Similarly, for any morphism $f : d \to d'$ in D such that $X_f(t) = u$, define $F(\alpha)_f : F(\alpha)_t \to F(\alpha)_u$ as X_f applied to $F(\alpha)_t = \alpha_d^{-1}(\{t\})$. To check that $F(\alpha)$ preserves composition, let $g : d' \to d''$ be a morphism in D such that $X_g(u) = v$. By definition, $F(\alpha)_{g \circ f}$ is $X_{g \circ f}$ applied to $F(\alpha)_t$. But since X is a functor, this is the same as applying X_f and then X_g. In other words, $F(\alpha)_{g \circ f} = F(\alpha)_g \circ F(\alpha)_f$.

Thus far, we have only defined F on objects. For any morphism from $\alpha : X \to T$ to $\beta : Y \to T$ given by $\gamma : X \to Y$, let $F(\gamma) : F(\alpha) \to F(\beta)$ be defined at each object $t \in B_D(T)$ as γ_d applied to $F(\alpha)_t$. Checking that this respects composition is similar to checking that $F(\alpha)$ preserves composition: Namely, let $\delta : Z \to T$ be a third object of $Top^D_{/T}$ and consider any map from β to δ given by $\epsilon : Y \to Z$. Then, by definition, $F(\epsilon \circ \gamma)$ is $(\epsilon \circ \gamma)_d$ applied to $F(\alpha)_t$. But this gives the same result as applying γ_d and then ϵ_d, completing the proof that F is indeed a functor.

To see that F is full, we need to show that any natural transformation $\zeta : F(\alpha) \to F(\beta)$ is of the form $F(\gamma)$ for some morphism in the slice category given by $\gamma : X \to Y$. Because α_d is a function, each $x \in X_d$ is in exactly one $\alpha_d^{-1}(\{t\}) = F(\alpha)_t$, where t ranges over elements of T_d. Moreover, since T_d is discrete, X_d is the disjoint union $\coprod_{t \in T_d} F(\alpha)_t$, so we simply define $\gamma_d = \coprod_{t \in T_d} \zeta_t$. By construction, $F(\gamma) = \zeta$, so F is full.

To see that F is faithful, observe that if ζ_1 and ζ_2 differ on some $x \in F(\alpha)_t$, then the corresponding γ_1 and γ_2 differ on x, this time regarded as an element of X_d.

Finally, we extract essential surjectivity from the observation that $X_d = \coprod_{t \in T_d} F(\alpha)_t$ and that the decomposition is functorial in d. □

So what about restriction? Recall that for any map of D-sets $\tau : T_1 \to T_2$, we have a restriction functor

$$Res^{\tau} : Top^{B_D(T_2)} \to Top^{B_D(T_1)}$$

in the contravariant direction, which has adjoints on both sides that thus go in the covariant direction. However, when we look for functors between $Top^D_{/T_1}$ and $Top^D_{/T_2}$, the most obvious natural map, post-composition by τ, gives a functor in the covariant direction. How does this compare to Ind^{τ} and $Coind^{\tau}$? It's Induction:

Proposition 1.4.4. There is a commutative square:

$$\begin{tikzcd} Top^D_{/T_1} \arrow[r, "\simeq"] \arrow[d, "\tau \circ -"'] & Top^{B_D(T_1)} \arrow[d, "Ind^{\tau}"] \\ Top^D_{/T_2} \arrow[r, "\simeq"] & Top^{B_D(T_2)} \end{tikzcd}$$

Proof. Recall that the equivalence $F_{T_1} : Top^D_{/T_1} \simeq Top^{B_D(T_1)}$ sends $\alpha : X \to T_1$ to the $B_D(T_1)$-space $F(\alpha)$ where $F(\alpha)_{t_1} = \alpha_d^{-1}(\{t_1\})$. (Here, $t_1 \in T_d$.) Similarly, $F_{T_2}(\tau \circ \alpha)$ is the $B_D(T_2)$-space defined by $F(\tau \circ \alpha)_{t_2} = (\tau \circ \alpha)^{-1}(\{t_2\})$. By observation, $(\tau \circ \alpha)^{-1}(\{t_2\})$ can be re-written as

$$\coprod_{t_1 \in \{t_1 | \tau(t_1) = t_2\}} \alpha_d^{-1}(\{t_1\}).$$

But this is precisely $Ind^{\tau}(F(\alpha)$!

Thus, the diagram above commutes at all objects $\alpha \in Top^D_{/T_1}$. A similar chase shows that, for any morphism $\gamma : \alpha \to \beta$ in $Top^D_{/T_1}$, $F(\tau \circ \alpha)$ and $Ind^{\tau}(F(\gamma))$ agree as maps of $B_D(T_2)$-spaces. Hence, we indeed have a commutative square of categories. $\square$

2 The Homotopy Theory of D-Spaces

2.1 Invariants

To use algebraic invariants in the equivariant context, we need to make choices about how they interact with the equivariant structure. For instance, the nth homotopy groups of a pointed space X, $\pi_n(X)$, is often viewed as consisting of the homotopy classes of pointed maps from S^n to X. If we want to do this equivariantly, X will have an action attached to it, and we'll need to decide what action S^n has. If we give S^n the constant "trivial" action, (that is, viewing S^n as a functor $D \to Top_\bullet$ that sends every object to the sphere S^n and every morphism to the identity on S^n) then we're extremely limited in the power of our invariants. For instance:

Example 2.1.1. Let C_2 denote the group of order 2, and let V be the m-dimensional orthogonal C_2-representation where the non-identity morphism acts via multiplication by -1. For any n, there is only one pointed C_2-equivariant map from S^n to S^V.

In other words, pointed C_2-equivariant maps from various S^n can't distinguish between the S^V above and a point! If we built a homotopy invariant out of the homotopy classes of such maps, we'd have a very weak invariant. Before discussing stronger invariants, let's first note that this phenomenon isn't unique to the group case:

Example 2.1.2. Let X be the $\mathbb{J}$-space where $X_s = \{pt\}$ and $X_t = S^m$. For any n, there is only one $\mathbb{J}$-equivariant map from S^n to X. (Here, we're following Convention 1.0.7 and treating S^n as a constant $\mathbb{J}$-space, which is the same as saying that S^n has the "trivial" action.)

In both cases, the issue was orbit types: an equivariant map can only send points of orbit type O_1 to points of orbit type O_2 if there's a D-equivariant map from O_1 to O_2. When we used S^n with trivial action, there was only one orbit type represented,[6] that orbit being

[6] This is true for all connected categories. In general, a constant D-space has orbit types precisely corresponding to the connected components of D.

C_2/C_2 in the first example and [1] in the second example. However, there were other orbit types present in the codomain, namely $C_2/\{id\}$ and [0], respectively.

We can capture the homotopical data for these other orbit types by replacing S^n with the "free" space $S^n \wedge O_+$. (O_+ is the pointed D-space obtained from O by adding a disjoint base point at every object) Aside from potentially the base points, every point in $S^n \wedge O_+$ has orbit type O. By Corollary 1.1.10, an equivariant map from $S^n \wedge O_+$ to X is equivalent to the data of an equivariant map from S^n to X^O.

Thus, instead of having a single n-th homotopy group, we have one for each orbit. By the representability of homotopy groups, these can be arranged into a functor:

Definition 2.1.3. Given a pointed D-space X, its *n-th equivariant homotopy group functor* is a contravariant functor $\pi_n^*(X) : Orb_D^{op} \to Grp$ given by

$$\pi_n^O(X) = [S^n \wedge O_+, X]^D \cong [S^n, X^O]^D.$$

Let's explore this invariant with a few examples for $\mathbb{J}$.

Example 2.1.4. Let X be a constant $\mathbb{J}$-space. Then, $\pi_n^*(X)$ is the constant functor

$$\pi_n(X_t) = \pi_n(X_s).$$

Proof. Consider any morphism $\alpha : (S^n \wedge O_+) \to X$:

$$\begin{tikzcd}
(S^n \wedge O_+)_s \arrow[r, "\alpha_s"] \arrow[d, "(S^n \wedge O_+)_f"'] & X_s \arrow[d, "X_f"] \\
(S^n \wedge O_+)_t \arrow[r, "\alpha_t"] & X_t
\end{tikzcd}$$

Since X_f is an identity morphism, α_s is uniquely determined as the composite $\alpha_t \circ$

$(S^n \wedge O_+)_f$. Thus, the homotopy classes of $\mathbb{J}$-equivariant maps from $S^n \wedge O_+$ to X can be identified with the non-equivariant homotopy classes of maps from $(S^n \wedge O_+)_t$ to X_t. Recall that when O is a $\mathbb{J}$-orbit, O_t is a one-point space. Thus, $(S^n \wedge O_+)_t \cong S^n$, so $\pi_n^O(X) \cong \pi_n(X_t)$. Because this identification can be made compatibly for each orbit, we conclude that $\pi_n^*(X)$ is a constant functor. □

In the above example, we didn't get any interesting orbit data. This was just because X only had one orbit type. To see a more general behavior, let's revisit the case where $X_s = \{pt\}$ and $X_t = S^m$:

Example 2.1.5. Let X be the $\mathbb{J}$-space with $X_s = \{pt\}$ and $X_t = S^m$. Then, $\pi_n^{[0]}(X) = \pi_n(S^m)$ and $\pi_n^O(X) = 0$ for all other orbits O. (The structure maps are all uniquely determined.)

Before going on to further examples, let's introduce a lemma:

Lemma 2.1.6. For any $\mathbb{J}$-space X and any non-[0] $\mathbb{J}$-orbit O, consider the unique map $g : O \to [1]$. In this context, $\pi_n^g(X) : \pi_n^{[1]}(X) \to \pi_n^O(X)$ is a surjection.

Proof. When O is an orbit other than [0], $id_{[1]}$ factors as $[1] \to O \xrightarrow{g} [1]$, where the first map is arbitrary. The rest follows from the fact that π_n^* is a functor. □

Now, what about some other maps involving spheres? For instance:

Example 2.1.7. Let X be the $\mathbb{J}$-space with $X_s = S^1$ and $X_t = S^1$, but where $X_f = 2$ is the "double counter-clockwise winding" map. (If one views S^1 as the unit sphere in $\mathbb{C}$, this is the map given by $z \mapsto z^2$.) Then, $\pi_n^O(X) = \pi_n(S^1)$ for all orbits O. For any $\mathbb{J}$-equivariant map $g : O_1 \to O_2$, $\pi_n^g(X)$ is multiplication by 2 when O_1 is the orbit [0] and O_2 is a different orbit; otherwise, $\pi_n^g(X)$ is the identity map.

Proof. Note that $X_f = 2$ is a covering map of S^1 onto itself, and consider any morphism $\alpha : (S^n \wedge O_+) \to X$:

$$\begin{tikzcd}
(S^n \wedge O_+)_s \arrow[r, "\alpha_s"] \arrow[d, "(S^n \wedge O_+)_f"'] & S^1 \arrow[d, "2"] \\
(S^n \wedge O_+)_t \arrow[r, "\alpha_t"] & S^1
\end{tikzcd}$$

Because $X_f = 2$ is a covering, any pointed map into S^1 is automatically the unique pointed lift of its composition with $X_f = 2$. In particular, there must be only one lift of $\alpha_t \circ (S^n \wedge O_+)_f$. Let's compare α_s with another such lift:

When O is not the orbit $[0]$, there exists maps $h : O_t \to O_s$ such that $O_f \circ h = id_{O_t}$. Thus, we have a map $\tilde{\alpha}_t : (S^n \wedge O_+)_t \to S^1$ given by $\tilde{\alpha}_t = \alpha_s \circ (S^n \wedge h)$. However, we can consider the diagram

$$\begin{tikzcd}
(S^n \wedge O_+)_t \arrow[dr, "h"] \arrow[ddr, "id_{(S^n \wedge O_+)_t}"'] \arrow[drr, dashed, "\tilde{\alpha}_t"] & & \\
& (S^n \wedge O_+)_s \arrow[r, "\alpha_s"] \arrow[d, "(S^n \wedge O_+)_f"] & S^1 \arrow[d, "2"] \\
& (S^n \wedge O_+)_t \arrow[r, "\alpha_t"] & S^1
\end{tikzcd}$$

and compute that $2 \circ \tilde{\alpha}_t = 2 \circ \alpha_s \circ h = \alpha_t \circ (S^n \wedge O_+)_f \circ h = \alpha_t$.

Thus, $\tilde{\alpha}_t \circ (S^n \wedge O_+)_f$ is a pointed lift of $\alpha_t \circ (S^n \wedge O_+)_f$. By the uniqueness of pointed lifts, this means $\alpha_s = \tilde{\alpha}_t \circ (S^n \wedge O_+)_f$. In other words, we have a commutative diagram:

$$\begin{tikzcd}
(S^n \wedge O_+)_s \arrow[r, "\alpha_s"] \arrow[d, "(S^n \wedge O_+)_f"'] & S^1 \arrow[d, "2"] \\
(S^n \wedge O_+)_t \arrow[r, "\alpha_t"] \arrow[ur, dashed, "\tilde{\alpha}_t"] & S^1
\end{tikzcd}$$

Assuming $O \neq [0]$, observe that a $\mathbb{J}$-equivariant map from $(S^n \wedge O_+)$ to X thus uniquely determines a map from the constant $\mathbb{J}$-space $(S^n \wedge O_+)_t$ to X, and vice versa. We could repeat the same argument replacing $(S^n \wedge O_+)$ with $(S^n \wedge O_+) \times I$ to get a lifting of homotopies. We also note that the constant $\mathbb{J}$-space $(S^n \wedge O_+)_t$ is isomorphic to $(S^n \wedge [1])$. Thus, when $O \neq [0]$, we compute: $\pi_n^O(X) = [S^n \wedge O_+, X]^{\mathbb{J}} \cong [S^n \wedge [1], X]^{\mathbb{J}} = [S^n, X_s] = \pi_n(S^1)$. These isomorphisms specify that the structure maps $\pi_n^g(X)$ are isomorphisms for $g : O_1 \to O_2$ when neither O_1 nor O_2 are $[0]$.

When $O = [0]$, we compute

$$\pi_n^{[0]}(X) = [S^n \wedge [0]_+, X]^{\mathbb{J}} \cong [S^n, X^{[0]}] \cong [S^n, X_t] = [S^n, S^1] = \pi_n(S^1).$$

Now, we just need to determine the unresolved structure maps. Since $\pi_n^g(X)$ is a isomorphism when O_1 and O_2 are not $[0]$, we only need to consider the unique map $j : [0] \to [1]$. (All of the other unresolved maps are obtained by composing this map with some already-known isomorphism.)

Recall again that $[0]$ and $[1]$ are isomorphic to the free orbits $\mathbb{J}(t, -)$ and $\mathbb{J}(s, -)$, respectively. Under this identification, $j : [0] \to [1]$ becomes $\mathbb{J}(f, -)$. Hence the structure map from

$$\pi_n^{[1]}(X) = [S^n \wedge [1]_+, X]^{\mathbb{J}} \cong [S^n, X_s] = \pi_n(X_s)$$

to $\pi_n^{[1]}(X) = [S^n \wedge [0]_+, X]^{\mathbb{J}} \cong [S^n, X_t] = \pi_n(X_t) is given by \pi_n(X_f)$, which is multiplication by 2.

$\square$

Comparing this example with the one about constant $\mathbb{J}$-spaces shows why we needed to have structure maps: without the maps, the X above would have been indistinguishable from the constant $\mathbb{J}$-space S^1. For our last two examples, let's see how orbits other than $[0]$ and $[1]$ provide useful data:

Example 2.1.8. Let X be the pointed $\mathbb{J}$-space with $X_s = S^m$, $X_t = S^\infty = \bigcup_{n\in\mathbb{N}} S^n$, where X_f is the inclusion of S^m into S^∞. Then, $\pi_n^{[0]}(X) = 0$, while $\pi_n^O(X) = \pi_n(X_s)$ for all other obits. Given any $g : O_1 \to O_2$ where O_1 and O_2 are not $[0]$, $\pi_n^g(X)$ is the identity map.

Proof. For $O \neq [0]$, $(S^n \wedge O_+)_f$ and $(S^n \wedge O_+ \times I)_f$ are surjective, so any map (or homotopy of maps) from $S^n \wedge O_+$ to X is factored by the inclusion of the constant $\mathbb{J}$-space S^m into X. Thus, for $O \neq [0]$, (and the structure maps between such O) $\pi_n^O(X)$ agrees with $\pi_n^O(S^m) \cong \pi_n(S^m)$. □

We contrast this with the following:

Example 2.1.9. Let Y be the pointed $\mathbb{J}$-space with $Y_s = S^m$ and $Y_t = \{pt\}$. Then, $\pi_n^{[i]}(Y) \cong \prod_{z\in[i]_s} \pi_n(S^m)$, and $g : [j] \to [i]$ acts by sending $(z_1, \ldots, z_i)$ to $(z_{g(1)}, \ldots, z_{g(j)})$.

Proof. Because Y_t is terminal, the data of a map α from $S^n \wedge [i]_+$ to Y is the same as the data of $\alpha_s : (S^n \wedge [i]_+)_s \to Y_s$. Since $(S^n \wedge [i]_+)_s$ is homeomorphic to the i-fold wedge product $S^n \vee \cdots \vee S^n$, we have that

$$\pi_n^{[i]}(Y) \cong [S^n \vee \cdots \vee S^n, S^m] \cong \prod_{z\in[i]_s} \pi_n(S^m),$$

where the last isomorphism follows from the fact that $\vee$ is the coproduct in the category of pointed topological spaces with homotopy classes of maps. Our description of $\pi_n^g(Y)$ then follows from chasing through the two isomorphisms. □

In these last two examples, $\pi_n^{[0]}(X)$, $\pi_n^{[1]}(X)$, and $\pi_n^{[0]\to[1]}(X)$ agree with their counterparts for Y. Only by using the other orbits can we homotopically distinguish between X and Y. This is desirable because while $X_s \simeq Y_s$ and $X_t \simeq Y_t$ are homotopy equivalent *as spaces*, X and Y are not homotopic as $\mathbb{J}$-spaces.

Our homotopy group functors provide powerful algebraic invariants for Top^D. We will see later that they become even *more* structured in the stable case. In fact, this additional structure, of additional "transfer" maps in the covariant direction, is one of the defining features of the stable case.

2.2 D-CW-Complexes and D-cell complexes

As in the non-equivariant case, we have a notion of CW-complexes, objects that are completely described by homotopy group functors. The approach here largely follows the original exposition given by Dror Fajoun and Zabrodsky, with some more modern updates.

Definition 2.2.1. [4] Given a collection of orbits $\mathcal{F}$ of a small category D and a D-space X, a *relative D-CW structure of type $\mathcal{F}$ on X* is a sequence of D-spaces

$$X^{-1} \hookrightarrow X^0 \hookrightarrow X^1 \hookrightarrow \cdots \hookrightarrow X^n \hookrightarrow \cdots \hookrightarrow X$$

such that for each $i \geq 0$, X^i is obtained from X^{i-1} as a pushout

$$\begin{tikzcd} S^{i-1} \times A_i \arrow[r] \arrow[d, hook] & X^{i-1} \arrow[d, hook] \\ D^i \times A_i \arrow[r] & X^i \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where each A_i is a disjoint union of D-orbits in $\mathcal{F}$. If X^{-1} is the constant empty D-space, we drop the word relative and say that

$$X^0 \hookrightarrow X^1 \hookrightarrow \cdots \hookrightarrow X^n \hookrightarrow \cdots \hookrightarrow X$$

is a *D-CW structure on X*.

As in the non-equivariant case, a map $\alpha : X \to Y$ of D-CW-complexes is a D-homotopy equivalence if an only if all nth homotopy group functors (including $n = 0$) induce isomorphisms. This result is usually called "Whitehead's theorem" in the non-equivariant case and

"Bredon's theorem" in the group-equivariant case. We now present the case of a general small category, which was proven by Dror Farjoun and Zabrodsky.

Definition 2.2.2. [4] Let $\mathcal{F}$ be a collection of orbits of a small category D. We say a D-space X is *of type* $\mathcal{F}$ if

$$\mathcal{O}_X = \{O_x \mid x \in \mathrm{colim}(X)\} \subseteq \mathcal{F}.$$

Theorem 2.2.3. [4] Let D be a small category and let $\alpha : X \to Y$ be a D-equivariant map of D-CW-complexes of type $\mathcal{F}$. Then, α is a D-homotopy equivalence if and only if $\alpha^O : X^O \to Y^O$ is a homotopy equivalence of spaces for all $O \in \mathcal{F}$. (That is, that $\pi_n^O(\alpha) : \pi_n^O(X) \to \pi_n^O(Y)$ is a isomorphism for all $n \in \mathcal{N}$ and $O \in \mathcal{F}$.)

In the previous section, we saw that $X = (S^m \hookrightarrow S^\infty)$ and $Y = (S^m \to \{pt\})$ were not $\mathbb{J}$-homotopy equivalent. Since X is of type $\mathcal{O}_X = \{[0], [1]\}$ and Y is of type $\mathcal{O}_Y = \{Y\}$, (Y is an orbit!) we know that any $\alpha : X \to Y$ must have $\pi_n^O(\alpha)$ fail to be an isomorphism for some $n \in \mathbb{N}$ and $O \in \{[0], [1], Y\}$. Back then, we showed that $\pi_n^{[i]}(X)$ and $\pi_n^{[i]}(Y)$ were not isomorphic for $i \geq 2$ and any n where $\pi_n(S^m) \neq 0$. The theorem above says we could have simply checked $\pi_m^Y(X)$ and $\pi_m^Y(Y)$.

In general, once one has a D-CW structure on X, it's straightforward to know which orbits to check:

Proposition 2.2.4. If X has (non-relative) D-CW structure of type $\mathcal{F}$, then X is a D-space of type $\mathcal{F}$.

Proof. Let X have a non-relative D-CW structure of type $\mathcal{F}$, and consider any point $x_d \in X_d$ for any object $d \in D$. We wish to show that $O_{x_d} \in \mathcal{F}$. By construction, x_d is a point in the interior of $D^i \times A_i$ for exactly one $i \in \mathbb{N}$. By equivariance, each point in O_{x_d} must also be a point in the interior of $D^i \times A_i$. Thus, the orbit type of x_d in X must be the same as its orbit type in $D^i \times A_i$. Because the latter is an orbit type in $\mathcal{F}$ and x_d is a generic point, we're done. $\square$

Overall, D-CW-complexes are tame combinatorial objects that allow us to isolate certain homotopical behavior. However, sometimes we want to remove the restriction that higher-dimensional cells only attach onto lower-dimensional cells. When we get rid of this restriction, we get a more general notion of *D-cell complexes*. These play a central role in several model structures on Top^D, and we will use them heavily in the next two sections.

Definition 2.2.5. Given a collection of orbits $\mathcal{F}$ of a small category D and a D-space X, a *D-cell stucture of type $\mathcal{F}$ on X* is a (potentially transfinite) sequence of pushouts of the form:

$$\begin{tikzcd} S^{n-1} \times O_\mu \arrow[r] \arrow[d] & X_\mu \arrow[d] \\ D^n \times O_\mu \arrow[r] & X_{\mu+1} \end{tikzcd},$$

such that $\operatorname{colim}(X_\mu) = X$, where each $O_\mu \in \mathcal{F}$ and where $n \geq 0$ varies with respect to μ. (Here, the (-1)-sphere is the empty space.)

That is, there is an ordinal λ such that $X = \operatorname{colim}_{\nu \leq \lambda} X_\nu$. For successor ordinals $\lambda = \mu + 1$, X_λ is obtained by the above pushout. For limit ordinals λ, $X_\lambda = \operatorname{colim}_{\nu < \lambda} X_\nu$. If X_0 is the constant empty D-space, we drop the word relative and say X is a *D-cell complex of type $\mathcal{F}$*.

As with D-CW-complexes, D-cell complexes of type $\mathcal{F}$ are spaces of type $\mathcal{F}$:

Proposition 2.2.6. If X is a (non-relative) D-cell complex of type $\mathcal{F}$, then X is a D-space of type $\mathcal{F}$.

Proof. Take the proof of proposition 2.2.4 and replace "D-CW structure" with "D-cell structure." $\square$

Example 2.2.7. Any relative D-CW-complex is a relative D-cell complex.

Proof. Let X have a relative D-CW structure. By definition, each A_i involved in the construction is a disjoint union of orbits O_α. By the usual axioms of set theory, this collection of orbits can be well-ordered. We can then order the orbits of $A_0, A_1, \ldots$ lexicographically and get a new well-ordered set of all the orbits involved. This corresponds to some ordinal λ which we will now use for labeling. For any orbit O_μ, we build our attaching pushout as

$$\begin{tikzcd}
 & X^{n-1} \arrow[d, hook] \\
S^{n-1} \times O_\mu \arrow[ur] \arrow[r] \arrow[d] & X_\mu \arrow[d] \\
D^n \times O_\mu \arrow[r] & X_{\mu+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd},$$

where the inclusion $X^n \hookrightarrow X_\mu$ is guaranteed by our lexicographic ordering and where the map $S^{n-1} \times O_\mu \to X^n$ is given by the disjoint union decomposition of A_n (and corresponding decomposition of $S^{n-1} \times A_n$). X_0 is defined as X^{-1}. $\square$

From the last sentence of the proof, we are also able to conclude:

Example 2.2.8. Any D-CW-complex is a D-cell complex.

2.3 Model Structures on Top^D

We can now establish a model structure on Top^D from that on Top:

Definition 2.3.1. [14] The *classical model structure on Top* is given by:

- Weak equivalences are weak homotopy equivalences. (maps that induce isomorphisms for all π_n)
- Fibrations are "Serre fibrations."
- Cofibrations are retracts of relative cell complexes.

(As in any model category, a choice of two of {Fibrations, Cofibrations, Weak Equivalences} uniquely determines the third. It is thus a theorem that the weak equivalences and fibrations described above determine the cofibrations of the definition.)

By [10, Theorem 11.6.1] in Hirschhorn, this gives us a model structure on Top^C for any (possibly large) category, C. The model structure we're about to describe will most often be used on $C = Top^{\mathcal{O}_{\mathcal{F}}^{op}}$, where $\mathcal{O}_{\mathcal{F}}$ is the full subcategory of Top^D who objects are orbits $O \in \mathcal{F}$. We'll then use this to get a model structure on Top^D, which is why we're using a second letter.

Definition 2.3.2. Given a (possibly large) category C, the *projective model structure on* Top^C is given by:

- Weak equivalences $\beta : R \to S$ are such that β_X is a weak homotopy equivalence for each object $X \in C$.
- Fibrations $\beta : R \to S$ are such that β_X is a Serre fibration for each object $X \in C$.
- Cofibrations are retracts of relative C-cell complexes of type $Free$, the collection of free orbits of C.

We will contrast this with:

Definition 2.3.3. Let D be a small category and let $\mathcal{F}$ be some collection of D-orbits that contains all of the free orbits. Then, the *$\mathcal{F}$-model structure* on Top^D is given by:

- Weak equivalences $\alpha : X \to Y$ are such that $Top^D(O, \alpha) : Top^D(O, X) \to Top^D(O, Y)$ is a weak equivalence for each $O \in \mathcal{F}$.
- Fibrations $\alpha : X \to Y$ are such that $Top^D(O, \alpha) : Top^D(O, X) \to Top^D(O, Y)$ is a Serre fibration for each $O \in \mathcal{F}$.

The main theorem of the next section is there there is a Quillen equivalence: (Top^D has the $\mathcal{F}$-model structure and $Top^{\mathcal{O}_{\mathcal{F}}^{op}}$ has the projective model structure.)

$$\begin{tikzcd}
Top^D \arrow[d, bend left=60, "\Phi"] \arrow[d, phantom, "\dashv"] \\
Top^{\mathcal{O}_{\mathcal{F}}^{op}} \arrow[u, bend left=60, "K"]
\end{tikzcd}.$$

Using this will enable us to compute the cofibrations in the $\mathcal{F}$-model structure, which will be the subject of Proposition 2.4.6

2.4 Elmendorf's Theorem

In the group-equivariant case, there is a celebrated result known as "Elmendorf's theorem" that states that the homotopical data of a G-space is the same as that of an Orb_G^{op}-space. This is made precise via a Quillen equivalence. This theorem was originally proven by Anthony Elmendorf in [7] and reformulated using model categories by Robert Piacenza in [13].

Our main theorem of this section is that the model-theoretic "Elmendorf's theorem" also holds in the D-equivariant case for any small category D. Our approach follows a modern treatment of the group-equivariant case by Marc Stephan in [17], although there is a similar theorem for simplicial sets given by William Dwyer and Daniel Kan in [6] that uses a different notion of "orbit."

The reader less interested by the model-theoretic proof may wish to simply read the theorem statement and move on to the next section. Our main takeaway is that we can replace D-spaces with spaces indexed by some category of D-orbits; it's this orbit-centric approach that we will use later on to define the stable case.

Theorem 2.4.1. Let D be a small category, let $\mathcal{F}$ be some collection of orbits of D that contains all of the free orbits, and let $\mathcal{O}_{\mathcal{F}} \subseteq Top^D$ be the full subcategory spanned by $\mathcal{F}$.

Then, there is a Quillen equivalence

$$K : Top^{\mathcal{O}_{\mathcal{F}}^{op}} \simeq Top^{D} : \Phi,$$

where $Top^{\mathcal{O}_{\mathcal{F}}^{op}}$ has the projective model structure and Top^D has the $\mathcal{F}$-model structure.

Definition 2.4.2. The functors that comprise the Quillen equivalence are:

- $K : Top^{\mathcal{O}_{\mathcal{F}}^{op}} \to Top^D$ is defined by $K(R) = R \circ i$, where i is the inclusion of D into $\mathcal{O}_{\mathcal{F}}^{op}$.
- $\Phi : Top^D \to Top^{\mathcal{O}_{\mathcal{F}}^{op}}$ is defined by $\Phi(X)(O) = Top^D(O, X)$ for all $O \in \mathcal{F}$.

To prove Theorem 2.4.1, we will show first show that (K, Φ) is an adjunction, then show that (K, Φ) is a Quillen adjunction, and then finally show that (K, Φ) is in fact a Quillen equivalence.

Lemma 2.4.3. *K is a left inverse to Φ. That is, there is a natural isomorphism $K\Phi \cong id_{Top^D}$.*

Proof. Let X be a D-space. By definition, $K\Phi(X)$ is the D-space where

$$[K\Phi(X)]_d = Top^D(F^d, X).$$

But by the Yoneda lemma, $Top^D(F^d, X) \cong X_d$, and the naturality of the Yoneda lemma in the first argument gives us that that $K\Phi(X) \cong X$. The naturality of the Yoneda lemma in the second argument allows us to conclude that $K\Phi \cong id_{Top^D}$. □

Lemma 2.4.4. *(K, Φ) is an adjunction.*

Proof. We will construct a natural isomorphism

$$Top^D(K(R), X) \cong Top^{\mathcal{O}_{\mathcal{F}}^{op}}(R, \Phi(X)),$$

where R is an $\mathcal{O}_{\mathcal{F}}^{op}$-space and X is a D-space. In this context, given a D-equivariant map

$f : K(R) \to X$, its adjunct is the $\mathcal{O}_{\mathcal{F}}^{op}$-equivariant map $g : R \to \Phi(X)$ defined as follows:

For any $O \in \mathcal{F}$,

$$g(O) : R(O) \to [\Phi(X)](O) = Top^D(O, X)$$

is the continuous map that sends $r \in R(O)$ to the D-map $[g(O)](r) : O \to X$. $[g(O)](r)$ is defined by: for any $d \in D$ and $o_d \in O_d$,

$$([g(O)](r))_d(o_d) = f(R(o_d^*)(r)),$$

where $o_d^* : F^d \to O$ is the unique D-map that sends id_d to o_d.

For this construction to be valid, we need to check that $[g(O)](r)$ is D-equivariant and then that g is $\mathcal{O}_{\mathcal{F}}^{op}$-equivariant. (That is, $[g(O)](r)$ and g need to be natural transformations, so we need to check the commutative squares that natural transformations are required to satisfy.) We'll begin with $[g(O)](r)$:

Let $\alpha : d \to d'$ be a morphism in D. To see that the square

$$\begin{tikzcd} O_d \arrow[r, "([g(O)](r))_d"] \arrow[d, "O_\alpha"'] & X_d \arrow[d, "X_\alpha"] \\ O_{d'} \arrow[r, "([g(O)](r))_{d'}"'] & X_{d'} \end{tikzcd}$$

commutes, consider a generic element $o_d \in O_d$ and observe that:

1. $[O_\alpha(o_d)]^* = o_d^* \circ Top^D(\alpha, -)$ because the maps agree on $id_{d'}$ and are D-equivariant.

2. Applying R to both sides gives us $R([O_\alpha(o_d)]^*) = R(Top^D(\alpha, -)) \circ R(o_d^*)$. (R is contravariant!)

3. By definition of K, $K(R)_d = R(F^d)$, $K(R)_{d'} = R(F^{d'})$, and $K(R)_\alpha = R(Top^D(\alpha, -))$, so the previous line can be rephrased as $R([O_\alpha(o_d)]^*) = K(R)_\alpha \circ R(o_d^*)$.

4. Since $f : K(R) \to X$ is a D-equivariant map, $f_{d'} \circ K(R)_\alpha = X_\alpha \circ f_d$.

5. Thus, combining the previous three steps, we see that

$$f_{d'} \circ R([O_\alpha(h_d)]^*) = f_{d'} \circ R(Top^D(\alpha, -)) \circ R(o_d^*) = X_\alpha \circ f_d \circ R(o_d^*).$$

6. The above are all continuous maps from $R(O)$ to $X_{d'}$. Hence, for any $r \in R(O)$, $f_{d'} \circ R([O_\alpha(o_d)]^*)(r) = X_\alpha \circ f_d \circ R(o_d^*)(r)$ as elements of $X_{d'}$.

7. By definition of $g(O)(r)$, this shows that $[g(O)(r)]_{d'} \circ O_\alpha(o_d) = X_\alpha \circ [g(O)(r)]_d(o_d)$. Since o_d was arbitrary, our square commutes and we conclude that $g(O)(r)$ is indeed D-equivariant.

Now let's confirm that g is $\mathcal{O}_{\mathcal{F}}^{op}$-equivariant, which is to say that the square

$$\begin{tikzcd}
R(O) \arrow[r, "g(O)"] & \Phi(X)(O) = Top^D(O, X) \\
R(P) \arrow[u, "R(\sigma)"] \arrow[r, "g(P)"'] & \Phi(X)(P) = Top^D(P, X) \arrow[u, "Top^D(\sigma,X)=(-\circ\sigma)"']
\end{tikzcd}$$

commutes, where $\sigma : O \to P$ is any map of D-orbits. (Note the direction of the vertical arrows; R and $\Phi(X)$ are contravariant.)

Let r be a generic element of $R(P)$. We will see $g(O)(R(\sigma)(r))$ and $g(P)(r) \circ \sigma$ are the same element of $Top^D(O, X)$ by the following:

1. For any object $d \in D$ and point $o_d \in O_D$, we know that $\sigma \circ o_d^* = (\sigma(o_d))^*$ because they are both D-equivariant maps from F^d that agree on id_d.

2. Thus, applying R to both sides, we get that $R(o_d^*) \circ R(\sigma)$ and $R((\sigma(o_d))^*)$ are equal as functions from $R(P)$ to $R(F^d)$. Hence, for any $r \in R(P)$, $R(o_d^*) \circ R(\sigma)(r) = R((\sigma(o_d))^*)(r)$.

3. Since $R(F^d) = K(R)_d$, we can post-compose f to both sides and see that

$$f(R(o_d^*) \circ R(\sigma)(r)) = f(R((\sigma(o_d))^*)(r)).$$

4. But by definition, the left-hand side of this equation is $g(O)(R(\sigma)(r))(o_d)$, and the right-hand side is $g(P)(r) \circ \sigma(o_d)$. Because this holds for all possible r and o_d, g is indeed $\mathcal{O}_{\mathcal{F}}^{op}$-equivariant.

Having constructed the adjunct g, let us now show that the assignment of $f : K(R) \to X$ to $g : R \to \Phi(X)$ as described above yields a bijection $Top^D(K(R), X) \cong Top^{\mathcal{O}_{\mathcal{F}}^{op}}(R, \Phi(X))$. To see that the assignment is injective, observe that if $f_1, f_2 : K(R) \to X$ differ at $r \in K(R)_d\ R(F^d)$, then the corresponding g_1 and g_2 differ because $g_i(F^d)(r)(id_d) = f_i(r)$.

To show surjectivity, we will demonstrate that any $g : R \to \Phi(X)$ has an $f : K(R) \to X$ assigned to it, namely the one defined by $f(r) = ([g(F^d)](r))_d(id_d)$ for any $r \in R(F^d) = K(R)_d$.

If we were to continue with the notation just used, the rest of the proof would be quite cumbersome. It's time to simplify:

Notation. From now on, $([g(O)](r))_d(o_d)$ will be denoted $g(O)(r)(o_d)$. (In particular, the domain of $g(O)(r)$, treated as a continuous map, will be implicit from its argument.)

Resuming the proof of surjectivity, we first note that this f is indeed D-equivariant because the naturality of g in F^d gives that f is natural in d, and that f_d is continuous because $g(F^d)$ is. To finish proving surjectivity, we just need to show that f is actually assigned to g, which we do as follows:

1. The adjunct $\widehat{g}$ that f is assigned to is defined by

$$\widehat{g}(O)(r_O)(o_d) = f(R(o_d^*)(r_O))$$

for any $r_O \in R(O)$. However, we've defined f above such that

$$f(R(o_d^*)(r_O)) = (g(F^d)(R(o_d^*)(r_O))(id_d).$$

2. By naturality of g in O, we know that $\widehat{g}(O)(r_O) \circ o_d^*$ and $g(F^d) \circ (R(o_d^*)(r_O))$ are equal as elements of $Top^D(F^d, X)$. In particular, they agree on the evaluation of id_d, which means

$$\widehat{g}(O)(r_O)(o_d) = g(F^d)(R(o_d^*)(r_O))(id_d) = g(O)(r)(o_d).$$

Hence, $\widehat{g} = g$, so the generic assignment of f to g is surjective.

Finally, to complete the proof of the adjunction, we just need to show that the bijection (isomorphism of sets) $Top^D(K(R), X) \cong Top^{\mathcal{O}_{\mathcal{F}}^{op}}(R, \Phi(X))$ is natural in both X and R:

1. Let $\alpha : X \to Y$ be a map of D-spaces. To show naturality in X, we will check that the diagram

$$\begin{tikzcd} Top^D(K(R), X) \arrow[r, "\cong"] \arrow[d, "\alpha\circ-"'] & Top^{\mathcal{O}_{\mathcal{F}}^{op}}(R, \Phi(X)) \arrow[d, "\Phi(\alpha)\circ-"] \\ Top^D(K(R), Y) \arrow[r, "\cong"] & Top^{\mathcal{O}_{\mathcal{F}}^{op}}(R, \Phi(Y)) \end{tikzcd}$$

commutes. Consider any $f \in Top^D(K(R), X)$, which thus has adjunct $g \in Top^{\mathcal{O}_{\mathcal{F}}^{op}}(R, \Phi(X))$ defined by $g(O)(r)(h_d) = f(R(o_d^*)(r))$, for all $O \in \mathcal{O}_{\mathcal{F}}$, $r \in R(O)$, and $o_d \in O_d$. By definition of Φ, the composition $\Phi(\alpha) \circ g$ thus satisfies $\Phi(\alpha) \circ g(O)(r)(o_d) = \alpha \circ f(R(o_d^*)(r))$. But $\alpha \circ f(R(o_d^*)(r))$ is precisely the formula that defines that adjunct to $\alpha \circ f \in Top^D(K(R), Y)$. Hence, the diagram commutes, so $Top^D(K(R), X) \cong Top^{\mathcal{O}_{\mathcal{F}}^{op}}(R, \Phi(X))$ is natural in X.

2. Similarly, let $\gamma : R \to S$ be a map of $\mathcal{O}_{\mathcal{F}}^{op}$-spaces. To show naturality in R, we will

check that the diagram

$$\begin{tikzcd}
Top^D(K(R),X) \arrow[r, "\cong"] & Top^{\mathcal{O}_{\mathcal{F}}^{op}}(R,\Phi(X)) \\
Top^D(K(S),X) \arrow[r, "\cong"] \arrow[u, "-\circ K(\gamma)"] & Top^{\mathcal{O}_{\mathcal{F}}^{op}}(S,\Phi(X)) \arrow[u, "-\circ\gamma"']
\end{tikzcd}$$

commutes. (Note the direction of the vertical arrows.) We know that any $f \in Top^D(K(S),X)$ is assigned to the adjunct $g \in Top^{\mathcal{O}_{\mathcal{F}}^{op}}(S,\Phi(X))$ defined by $g(O)(s)(o_d) = f(S(o_d^*)(s))$ for all $O \in \mathcal{O}_{\mathcal{F}}$, $s \in S(O)$, and $o_d \in O_d$. The composition $g \circ \gamma$ then satisfies

$$(g \circ \gamma)(O)(r)(o_d) = g(O)(\gamma(r))(o_d) = f(S(o_d^*)(\gamma(s)))f(\gamma \circ R(o_d^*)(\gamma(s))).$$

(The first equation is the definition of $g \circ \gamma$, the second follows by plugging $\gamma(r)$ into the adjunct formula, and the third is given by the fact that $\gamma(R) = S$.) But $f(\gamma \circ R(o_d^*)(\gamma(s)))$ is precisely the formula that defines the adjunct of $f \circ K(\gamma)$. Hence, $Top^D(K(R),X) \cong Top^{\mathcal{O}_{\mathcal{F}}^{op}}(R,\Phi(X))$ is natural in R.

Thus, we've completed the proof of that (K,Φ) is an adjunction.

□

Lemma 2.4.5. (K,Φ) is a Quillen adjunction.

Proof. One of the equivalent conditions for an adjunction to be a Quillen adjunction is that the right adjoint, Φ, preserve fibrations and trivial fibrations. (that is, fibrations that are also weak equivalences) Recall from Definitions 2.3.3 and 2.3.2 that the model structures we're using are:

- The weak equivalences (or fibrations) in Top^D are maps $\beta : X \to Y$ such that $Top^D(O,\beta) : Top^D(O,X) \to Top^D(O,Y)$ is a weak equivalence (or fibration) in Top for all $O \in \mathcal{F}$.

- The weak equivalences (or fibrations) in $Top^{\mathcal{O}_{\mathcal{F}}^{op}}$ are maps $\gamma : R \to S$ such that $\gamma(O) : R(O) \to S(O)$ is a weak equivalence (or fibration) in Top for all $O \in \mathcal{F}$.

We observe from this description that, since $\Phi(X)(O) = Top^D(O, X)$, a D-equivariant map $\beta : X \to Y$ is a weak equivalence (resp. fibration) if and only if $\Phi(\beta) : \Phi(X) \to \Phi(Y)$ is an equivalence (resp. fibration). Hence, Φ preserves weak equivalences and fibrations, and thus also trivial fibrations. □

We're now able to describe the cofibrations in the $\mathcal{F}$-model structure:

Proposition 2.4.6. Any relative cell complex $\alpha : X_0 \to X$ of type $\mathcal{F}$ is a cofibration in Top^D under the $\mathcal{F}$-model structure.

Proof. The left adjoint in a Quillen adjunction preserves cofibrations. It also preserves pushouts and general colimits. Thus, any D-cell complex X of type $\mathcal{F}$ is the image under K of a $Top^{\mathcal{O}_{\mathcal{F}}^{op}}$-cell complex of type $Free$. (A D-orbit in $\mathcal{F}$ is a free $\mathcal{O}_{\mathcal{F}}^{op}$-orbit under the Yoneda embedding. Thus, a D-cell complex where $D^n \times O_\mu$ is attached at the μth stage is hit by a $Top^{\mathcal{O}_{\mathcal{F}}^{op}}$-cell complex where $Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O_\mu, -) \times D^n$ is attached at the μth stage.) □

We can now finish the proof of the theorem:

Theorem 2.4.7. (K, Φ) is a Quillen equivalence.

Proof. We need to show that, for any cofibrant $R \in Top^{\mathcal{O}_{\mathcal{F}}^{op}}$ and fibrant $X \in Top^D$, any $f : K(R) \to X$ is a weak equivalence if and only if its adjunct $g : R \to \Phi(X)$ is a weak equivalence. But by how we've defined our weak equivalences, f is a weak equivalence if and only if $\Phi(f) : \Phi K(R) \to \Phi(X)$ is. By the 2-of-3 property of weak equivalences and that fact that the unit of the adjunction at R, η_R, factors $\Phi(f)$ as $f \circ \eta_R$, it is sufficient (and necessary) to show that η_R is a weak equivalence for all cofibrant $R \in Top^{\mathcal{O}_{\mathcal{F}}^{op}}$. We will in fact show that η_R is an *isomorphism* for each cofibrant R!

Recall that cofibrations in the projective model structure on $Top^{\mathcal{O}_{\mathcal{F}}^{op}}$ are retracts of relative $Top^{\mathcal{O}_{\mathcal{F}}^{op}}$-cell complexes of type $Free$. Hence, every cofibrant R can be realized as a retract of some R', where R' is a transfinite composition of pushouts of the form

$$\begin{tikzcd} Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-) \times A \arrow[r] \arrow[d, "Id_{Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-)} \times c"'] & \widetilde{R}_{\mu} \arrow[d] \\ Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-) \times B \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \widetilde{R}_{\mu+1} \end{tikzcd} .$$

Since R is a retract of R', we can show η_R is an isomorphism by showing that $\eta_{R'}$ is an isomorphism. We will do this by transfinite induction:

Let λ be an ordinal such that there is a functor $\widetilde{R} : \lambda \to Top^{\mathcal{O}_{\mathcal{F}}^{op}}$ with colimit R' and such that for all successor ordinals $\mu + 1 < \lambda$, $\widetilde{R}_{\mu+1}$ is the pushout given above, where O is some orbit in $\mathcal{O}_{\mathcal{F}}$ and $c : A \to B$ is a generating cofibration in Top.

Initial Case: If λ is the initial ordinal, then the colimit of λ is the initial object of $Top^{\mathcal{O}_{\mathcal{F}}^{op}}$. In this case, $R'(O) = \{\}$ for all $O \in \mathcal{F}$. Hence, $K(R')_d = \{\}$ for all objects $d \in D$, and consequently $\Phi K(R')(O) = Top^D(O, K(R')) = \{\}$ as all orbits have at least one point and there are no continuous maps from a non-empty to the empty space. Thus, in this case, $\eta_{R'}$ is not just an isomorphism, but an equality.

Successor Case: If $\lambda = \mu + 1$ for some ordinal μ, we inductively assume that $\eta_{\widetilde{R}_{\mu}}$ is an isomorphism. To see that $\eta_{R'}$ is an isomorphism, we will check that $\Phi K(-)$ preserves pushouts and that $\eta_{Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-)\times A}$ is an isomorphism for all $O \in \mathcal{F}$ and $A \in Top$. For pushout-preservation, we first note that K automatically preserves pushouts by being a left adjoint. Since colimits (and limits) in a functor category are computed object-wise, showing that Φ preserves pushouts is equivalent to showing that $Top^D(O, Y \sqcup_X Z)$ is naturally isomorphic to $Top^D(O, Y) \sqcup_{Top^D(O,X)} Top^D(O, Z)$. But this is immediate from Proposition 1.1.8, since pushouts are colimits.

To complete the successor ordinal case, we just need to show that $\eta_{Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-)\times A}$ is an isomorphism for all $O \in \mathcal{F}$ and $A \in Top$. To see this, note that

$$K(Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-) \times A)_d = Top^D(O, F^d) \times A \cong O_d \times A.$$

The naturality of the Yoneda lemma in the second argument shows us that

$$K(Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-) \times A) \cong O \times A.$$

Applying Φ to both sides gives

$$\Phi K(Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-) \times A) \cong \Phi(O \times A).$$

Next, observe that, at any orbit $P \in \mathcal{F}$,

$$\Phi(O \times A)(P) = Top^D(P, O \times A) \cong Top^D(P, O) \times A = Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O, P) \times A.$$

(To see the natural isomorphism above, first observe that

$$Top^D(P, O \times A) \cong Top^D(P, O) \times Top^D(P, A)$$

because representable functors preserve limits. Then, simplify by noting that $Top^D(P, A) \cong A$ because $\mathrm{colim}(P) = \{p\}$ is a one-point space and every morphism in A is an identity morphism.) Since $Top^D(P, O) \times A$ is the same space as $Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-) \times A$ evaluated at P, we conclude that $\Phi K(Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-) \times A)$ is naturally isomorphic to $Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-) \times A$. By construction, the natural isomorphism described above is precisely $\eta_{Top^{\mathcal{O}_{\mathcal{F}}^{op}}(O,-)\times A}$, meaning the successor case is complete.

Limit Case: Let λ be a limit ordinal. We inductively assume that $\eta_{\widetilde{R}_\nu}$ is an isomorphism for all $\nu < \lambda$. Since λ is a limit ordinal, $R' = \mathrm{colim}_{\nu<\lambda}(\widetilde{R}_\nu)$. As before, showing that

$\eta_{R'} : \Phi K(R') \to R'$ is an isomorphism reduces to showing each $\eta_{R'}(O) : \Phi K(R')(O) \to R'(O)$ is an isomorphism of spaces. Note that each $K(\widetilde{R}_\nu)$ is a D-cell complex by construction, (specifically, it's only built out of cells of orbit types $O \in \mathcal{F}$) and $K(\widetilde{R}_\nu) \to K(\widetilde{R}_\xi)$ is a D-cellular inclusion for any $\nu < \xi < \lambda$. Thus, since K preserves colimits, we only need to check that $Top^D(O, -)$ preserves colimits indexed by ordinals where each map is a D-cellular inclusion of D-cell complexes. Combined with the inductive hypothesis, this will show that $\eta_{R'}(O) : \Phi K(R')(O) \to R'(O)$ is an isomorphism of spaces.

Let $f : O \to R' = \mathrm{colim}_{\nu<\lambda}(\widetilde{R}_\nu)$ be a D-equivariant map. We wish to find an ordinal $\nu < \lambda$ and map $g : O \to \widetilde{R}_\nu$ such that g factors f. Let $o_d \in O_d$ be a point of O, and set ν to be the smallest ordinal such that $f(o_d) \in (\widetilde{R}_\nu)_d$. Observe that when $f(o_d)$ was added to $\widetilde{R}_\nu$, it was added as the interior of some D-disk $Orb^{op}_{\mathcal{F}}(P, F^d) \times D^{n+1} \cong P \times D^{n+1}$. By equivariance and the colimit property of orbits, all points in $\widetilde{R}_\nu$ that are in the orbit of $f(o_d)$ must also have been in the interior of the new $(P \times D^{n+1})$-cell. Thus, all points in the orbit of $f(o_d)$ must lie in $\widetilde{R}_\nu$, which means that f is indeed factored by a map $g : O \to \widetilde{R}_\nu$. This completes the limit case and the proof.

$\square$

3 Stability

In the non-equivariant setting, the basic objects one works with are called spectra. There are many distinct models of spectra with various “nice” properties, but they all have the same homotopy category. This homotopy category has the property that arbitrary suspension (that is, smashing with an n-sphere for some n) is an invertible operation. Non-equivariantly, we work with a very tame class of spheres: Up to isomorphism, there's exactly one for each each natural number. But when we do things equivariantly, we have many more spheres to consider given the variety of possible actions. In practice, we don't need to consider *all* possible spheres, and since Graeme Segal's 1970 paper [15] that first established equivariant stable homotopy theory, one usually only considers *representation spheres*. This chapter will explore the usual equivariant practice, and how one might adapt it.

Definition 3.0.1. Given a small category D, an *orthogonal representation* of D is a functor $V : D \to \text{Orth}$, where Orth is the category whose objects are finite dimensional real inner product spaces and whose morphisms are distance-preserving linear maps, which are necessarily injective. We denote the category of orthogonal D-representations by $Orth^D$.

Definition 3.0.2. Given an orthogonal D-representation V, its *representation sphere*, S^V is the (pointed) D-space given by one-point compactification at each object, where any morphism $d_1 \xrightarrow{f} d_2$ has V_f extended to send the new point in d_1 to the new point in d_2.

One of our main uses for representations is each orbit can be nicely embedded inside of one:

Definition 3.0.3. Given a small category D and orbit O, the *real*[7] *representation spanned by* O,

$$\mathbb{R}(O) : D \to Vect_{\mathbb{R}},$$

is the real D-representation where:

[7]this definition generalizes to any field k, but we're only interested in the real case for its relation to orthogonality

- For each object $d \in D$, $\mathbb{R}(O)_d$ has basis O_d.
- For any morphism $d \to d'$ in D, $\mathbb{R}(O)_f$ is defined by linearity on the basis vectors.

Note that $\mathbb{R}(O)$ is distance-preserving (and thus an orthogonal D-representation under the usual inner product structure) precisely when each O_f is an injection. Since this is guaranteed when D is a groupoid, we see that every orbit of a groupoid embeds inside of an orthogonal D-representation, and hence a representation sphere. What's more, the ability to embed orbits inside of orthogonal representations essentially *determines* whether D is a groupoid:

Lemma 3.0.4. If D is a small category that is is not a groupoid, then there exists a D-orbit O and morphism $f : d \to d'$ such that O_f is not injective.

Proof. Since D is not a groupoid, there is at least one morphism in D that lacks a postcompositional inverse.[8] Call this $f : d \to d'$, and consider the D-set T in the following pushout:

$$\begin{tikzcd}
D(d', -) \arrow[r, "D(f,-)"] \arrow[d, "D(f,-)"'] & D(d, -) \arrow[d] \\
D(d, -) \arrow[r] & T \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Concretely, for any object $d'' \in D$, $T_{d''}$ is obtained from $D(d, d'')$ by adding a duplicate copy of any $g : d \to d''$ that cannot be factored as $g = h \circ f$ for some $h : d' \to d$. In particular, T_d has two copies of id_d because f lacks a postcompositional inverse. Similarly, $T_{d'}$ has only a single copy of f because $f = id_{d'} \circ f$. Thus, T_f is a non-injective map. If take O to be the orbit of $f \in T_{d'}$, (see Definition 1.0.8) then O_f is also a non-injective map. $\square$

Theorem 3.0.5. Let D be a locally finite category. Then, D is a groupoid if and only if every objectwise-finite D-orbit embeds inside of an orthogonal D-representation.

[8] If $f' \circ f = id_d$ and $f'' \circ f' = id_{d'}$, then $f'' = f'' \circ id_d = f'' \circ f' \circ f = id_{d'} \circ f = f$. Thus, if every morphism had a postcompositional inverse, each morphism would also have a precompositional inverse, making D a groupoid.

Proof. Assume D is a groupoid, and let consider $\mathbb{R}(O)$ for any objectwise-finite D-orbit O. Because every morphism in D is an isomorphism, each O_f is an isomorphism, so each $\mathbb{R}(O)_f$ is given by a bijection of basis vectors from the domain to the codomain. Thus, with the usual inner product structure on each $\mathbb{R}(O)_d$, each $\mathbb{R}(O)_f$ is distance preserving. Since O is objectwise-finite, each $\mathbb{R}(O)_d$ finite dimensional, meaning $\mathbb{R}(O)$ is an orthogonal D-representation. Hence, O embeds into an orthogonal representation.

Conversely, suppose D is not a groupoid and consider the orbit O constructed in Lemma 3.0.4. If D is locally finite, then T is automatically objectwise-finite, and thus so is O. Hence, O cannot embed inside of any orthogonal D-representation, as all maps in orthogonal D-representations are injective by virtue of being distance-preserving linear maps. □

This theorem makes orthogonal representations less useful to us in the non-groupoid case, because we're losing so much orbit data. For the category $\mathbb{J} = s \xrightarrow{f} t$, recall that a $\mathbb{J}$-orbit is any $\mathbb{J}$-space $X : \mathbb{J} \to Top$ such that X_t is a single point. Up to isomorphism, there are only two $\mathbb{J}$-orbits with injective maps: those where X_s is the empty space, and those where X_s has a single point. In other words, of the large category of $\mathbb{J}$ orbits, (recall it's equivalent to Top!) orthogonal representations only recover *two* of the orbit classes!

This is a major problem for addressing stability via representation spheres. Thankfully, there are other approaches, and these don't run into the same difficulties:

3.1 Spectral Mackey Functors

Spectral Mackey functors are a tool for describing equivariant stable homotopy theory. In the group case, they are originally due to Bert Guillou and Peter May in [9], although Clark Barwick independently developed an infinity-categorical approach in [1]. This section will adapt the theory to the case of small (usually locally finite) categories by following a more recent treatment by Anna Marie Bohmann and Angélica Osorno in [2] of the group case. We will define a *D-spectral Mackey functor* as a spectrally-enriched contravariant functor from the *spectrally-enriched Burnside category* to the category of spectra. We'll build the spectrally-enriched Burnside category in two stages: (Note: All of these "Burnside Categories" will have finite D-sets for objects; these are precisely the D-sets obtained by finite coproducts of finite D-orbits.)

1. First, we'll build a version of the Burnside category enriched in permutative categories. The objects of this category are finite D-sets, and the morphism-categories are built out of spans $X \leftarrow Z \rightarrow Y$ of finite D-sets.

2. Then, we'll apply a "K-theory functor" $\mathbb{K}$ on the morphism-categories to get morphism spectra. This gives the spectrally-enriched Burnside category.

The above process is made rigorous through the language of multicategories, for which we now give an overview. The reader who is only interested in the definitions of the various Burnside categories may wish to skip ahead to Definition 3.1.6.

Definition 3.1.1. A small[9] *multicategory* M is the data of:

- A set of *objects* $ob(M)$
- For any natural number k and objects $a_1, \ldots, a_k, b$, a *k-multimorphism set* $M(a_1, \ldots, a_k; b)$
- For $k = 1$ and $a_1 = b$, an *identity* element $id_b \in M(b; b)$
- For any natural numbers $k_1, \ldots k_n$ and objects

$$a_{1,1}, \ldots a_{1,k_1}, \ldots, a_{n,1}, \ldots a_{n,k_n}, b_1, \ldots, b_n, c,$$

  a *composition* function

$$M(b_1, \ldots, b_n; c) \times M(a_{1,1}, \ldots, a_{1,k_1}; b_1) \times \cdots \times M(a_{n,1}, \ldots, a_{n,k_1}; b_n) \to M(a_{1,1}, \ldots, a_{n,k}; c).$$

[9] As with ordinary categories, some set-theoretic care needs to be taken to define non-small multicategories, where the collection of objects and the collections of morphisms may not be sets. The reader aware of these subtle issues will note that the only multicategories considered in this paper are either small or constructible using a limited number of axioms of universes. (a.k.a. "Groethendieck Universes")

The composition function is subject to identity associativity diagrams. We omit them here, but they can be found in [12, Section 2.1] by Tom Leinster.

Definition 3.1.2. A *multifunctor* $F : M \to N$ between multicategories M and N is the data of:

- For any object $m \in M$, an object $F(m) \in N$
- For any objects $a_1, \ldots, a_k, b \in M$, a function

$$M(a_1, \ldots, a_k; b) \to N(F(a_1), \ldots, F(a_k); F(b))$$

  that respects composition.

Example 3.1.3. A monoidal category $(C, \otimes)$ gives rise to a multicategory C' with the same objects, where

$$C'(a_1, \ldots, a_k; b) \coloneqq C(a_1 \otimes \cdots \otimes a_n, b)$$

.

Definition 3.1.4. A *permutative category* is a symmetric monoidal category that is strictly associative and strictly unital.

Proposition 3.1.5. [8, Theorem 1.1, first part] There is a multicategory, $\mathbf{P}$, whose objects are small permutative categories and whose multimorphisms are given by multifunctors. (Here, each permutative category is viewed as a multicategory as in the example above.)

With the basics of multicategories now established, let's define the various Burnside categories, starting with one enriched in commutative groups:

Definition 3.1.6. Given a small category D, its *Burnside category*, $\mathcal{B}_D$ is the additive category where:

- The objects of $\mathcal{B}_D$ are finite D-sets.
- For any D-sets X and Y, $\mathcal{B}_D$ is the group completion of the commutative monoid of

isomorphism[10] classes of spans of D-sets $X \leftarrow Z \rightarrow Y$. The monoidal structure is given by disjoint union.

- Composition is given by pullback.

Now we give the $\mathbf{P}$-enriched version:

Definition 3.1.7. [11] Given a small category D, its $\mathbf{P}$-*enriched Burnside category*, $D\mathcal{E}$, is the $\mathbf{P}$-enriched category where:

- The objects of $D\mathcal{E}$ are finite D-sets.
- For any D-sets X and Y, with $X \neq Y$ $D\mathcal{E}(X, Y)$ is the permutative category whose objects are spans of finite D-sets from $X \leftarrow Z \rightarrow Y$ where the objects of Z are all of the form $\{0, \ldots, n-1\}$ for some $n \in \mathbb{N}$[12] and whose morphisms are isomorphisms of spans. The monoidal structure is given by disjoint union, which is strictly associative and strictly unital.
- For any D-set X, $D\mathcal{E}(X, X)$ is the permuative category created by taking the permutative category of spans from X to X and adding a new object I_X with a specified isomorphism $\xi_X : I_X \rightarrow (X \leftarrow X \rightarrow X)$. (The monoidal structure on $D\mathcal{E}(X, X)$ is given by having $I_X \otimes f = f \otimes I_X = f$ whenever f is a span with roof a non-empty D-set Z, and $I_X \otimes f = f \otimes I_X = I_X$ when the roof of f is the empty D-set.)
- Composition involving any I_X is defined by $I_X \circ f = f = f \circ I_Y$. This makes composition strictly unital.
- If neither f nor g are some I_X, composition is defined by pullback, where we take the

[10] A morphism of spans is a commutative diagram of the form

$$\begin{tikzcd} & W \arrow[dl] \arrow[d] \arrow[dr] & \\ X & Z \arrow[l] \arrow[r] & Y. \end{tikzcd}$$

[11] This is a modification of [2, Definition 7.2] via the "whiskering" described in [9, Section 5]. The version in [2] was NOT strictly associative.

[12] This specific choice is made to create a version of Cartesian product that is *strictly* associative.

following choice of pullback to make composition unique: The pullback of $X \to A \leftarrow Y$ is the *unique* pullback Z whose objects are of the form $0, \dots, n-1$ and such that Z is order-isomorphic to the lexicographic ordering of the usual description $X \times_A Y$.

Now we can apply the functor $\mathbb{K}$ to the morphism-categories to get the spectrally-enriched Burnside category. The functor $\mathbb{K}$ was originally given by Segal in [16], and his construction is recounted in our appendix. However, to use $\mathbb{K}$ in the context of multicategories, we'll need a slightly different version described by Elmendorf and Mandell:

Theorem 3.1.8. [8, Theorem 1.1, second part] There is a multifunctor $\mathbb{K}$ from $\mathbf{P}$ to the category of symmetric spectra that is weakly equivalent to the definition of K-theory given by Segal.

The reader is encouraged to treat this K-theory machinery as a black box.

The spectrally-enriched Burnside category is then obtained by applying $\mathbb{K}$ to the morphism spaces:

Definition 3.1.9. Given a small category D, its *spectrally-enriched Burnside category*, $D\mathcal{B}$, is the spectrally enriched category where:

- The objects of $D\mathcal{B}$ are finite D-sets
- For any D orbits, $D\mathcal{B}(X, Y) = \mathbb{K}(D\mathcal{E}(X, Y))$

Definition 3.1.10. Given a small category D, *D-spectral Mackey functor* is a spectrally-enriched contravariant functor $\underline{M} : D\mathcal{B}^{op} \to Sp$.

This is a specific instance of the following general fact: (with $F = \mathbb{K}$)

Proposition 3.1.11. [2, Proposition 2.11] Any multifunctor $F : M \to N$ induces a 2-functor

$$F_\bullet : M\text{-}Cat \to N\text{-}Cat,$$

where *M-Cat* and *N-Cat* are the categories of M-enriched and N-enriched categories respectively.[13] For any M-enriched category D, $F_\bullet(D)$ agrees with $F(D)$ on objects, and for any objects X, Y in D, $(F_\bullet(D))(X,Y)$ is defined as $F(D(X,Y))$, which is F applied to the M-enriched morphism space $D(X,Y)$.

We conclude this section with the following fact relating our various Burnside categories:

Theorem 3.1.12. [2, Paragraph preceding Proposition 6.5] There is a commutative diagram

$$\begin{tikzcd}
D\mathcal{E} \arrow[r, "\mathbb{K}_\bullet"] \arrow[dr, "Q"'] & D\mathcal{B} \arrow[d, "(\pi_0)_\bullet"] \\
 & \mathcal{B}_D
\end{tikzcd}$$

Here, $Q : D\mathcal{E} \to \mathcal{B}_D$ is the functor that is the identity on objects and that sends a span $X \leftarrow Z \rightarrow Y$ to its isomorphism class.

3.2 Model Structures on $Sp^{D\mathcal{B}^{op}}$

Now that we've defined D-spectral Mackey functors, we can discuss the mode structure on $Sp^{D\mathcal{B}^{op}}$. When we proved Elmendorf's theorem, we equipped each $Top^{\mathcal{O}_{\mathcal{F}}^{op}}$ with the projective model structure. Here, we're viewing $D\mathcal{B}$ as the stable analog of $\mathcal{O}_{\mathcal{F}}$, (both encode the data of what happens at orbits) so it makes sense to again use the projective model structure. In the finite group case, this is the model structure Guillou and May used to construct a zig-zag of Quillen equivalences between different models of equivariant spectra.

Definition 3.2.1. The projective model structure on $Sp^{D\mathcal{B}}$ is given by:

- A morphism $\gamma : \underline{M}_1 \to \underline{M}_2$ in $Sp^{D\mathcal{B}^{op}}$ is a weak equivalence (resp. fibration) if $\gamma(O) : \underline{M}_1(O) \to \underline{M}_2(O)$ is a weak equivalence (resp. fibration) for each $O \in D\mathcal{B}$.
- A morphism $\gamma : \underline{M}_1 \to \underline{M}_2$ in $Sp^{D\mathcal{B}^{op}}$ is a cofibration if it has the left lifting property with respect to all trivial fibrations.

[13] As usual, we have to be careful working with proper classes. In practice, we'll usually have M-*Cat* and N-*Cat* consist of categories constructed assuming a single axiom of universes, which means mean M-*Cat* and N-*Cat* themselves need only two axioms of universes.

Proposition 3.2.2. The projective model structure exists and is cofibrantly generated.

Proof. This follows immediately from [10, Theorem 11.6.1] and the fact that the usual model structure on Sp is cofibrantly generated. $\square$

3.3 Monoidal Structures on $Sp^{D\mathcal{B}^{op}}$

We have to be careful about what monoidal structure we're working with. One could use the *external* monoidal structure given by the smash product of spectra, $\wedge$. (That is, given two D-spectral Mackey functors $\underline{M}_1$ and $\underline{M}_2$, $\underline{M}_1 \wedge \underline{M}_2$ is defined by $(\underline{M}_1 \wedge \underline{M}_2)(X) = \underline{M}_1(X) \wedge \underline{M}_2(X)$.) However, we'll use a different model, known as *Day convolution*, for two reasons:

1. Day convolution is the usual monoidal structure on non-spectral Mackey functors.
2. In [9] by Guillou and May, Day convolution is expected to agree in the finite group case with the monoidal structure of other models of equivariant spectra. (That is, the zig-zag of Quillen equivalences between them should all be monoidal.)

Definition 3.3.1. Given D-spectral Mackey functors $\underline{M}_1$ and $\underline{M}_2$, their *Day convolution*, $\underline{M}_1 \otimes \underline{M}_2$ is defined as the enriched left Kan extension of $\underline{M}_1 \wedge \underline{M}_2$ along $- \times -$. Diagrammatically:

$$\begin{tikzcd}
 & D\mathcal{B}^{op} \arrow[dd, "\underline{M}_1 \otimes \underline{M}_2"] \\
D\mathcal{B}^{op} \times D\mathcal{B}^{op} \arrow[ur, "- \times -"] \arrow[dr, "\underline{M}_1 \wedge \underline{M}_2"'] & \Uparrow \eta_{\underline{M}_1 \wedge \underline{M}_2} \\
 & Sp
\end{tikzcd}$$

Proposition 3.3.2. If D has a finite number of objects, $(Sp^{D\mathcal{B}^{op}}, \otimes, D\mathcal{B}(-, \{pt\})$ defines a monoidal structure on $Sp^{D\mathcal{B}^{op}}$, where $\{pt\}$ is the terminal D-set.

Proof. This follows immediately from [3, Theorem 3.3] by Brian Day. $\square$

3.4 Invariants of D-Spectral Mackey Functors

Spectra have natural invariants, their (stable) homotopy groups. When working equivariantly over a (usually finite) group G, this construction is naturally seen to be a G-Mackey Functor, a functor $\mathcal{B}_G^{op} \to CommGrp$, from the Burnside category to the category of commutative groups. In this section, we will see that one can apply the same procedure to D-spectral Mackey functors to produce what we will call D-Mackey functors.

By Proposition 3.1.11, the functor $\pi_0 : Sp \to CommGrp$ induces 2-functor

$$(\pi_0)_\bullet : Sp\text{-}Cat \to CommGrp\text{-}Cat$$

. Thus, for any D-spectral Mackey functor $X : D\mathcal{B}^{op} \to Sp$, we get a new functor

$$(\pi_0)_\bullet(X) : (\pi_0)_\bullet(D\mathcal{B}^{op}) \to (\pi_0)_\bullet(Sp)$$

. We make sense of this with the following facts:

Proposition 3.4.1. $(\pi_0)_\bullet(Sp)$ is equivalent to $HoSpec$, the homotopy category of spectra.

Proposition 3.4.2. $(\pi_0)_\bullet(D\mathcal{B}^{op}) \cong \mathcal{B}_D^{op}$.

Thus, $(\pi_0)_\bullet(X)$ can be viewed as a contravariant functor from the D-Burnside category to $HoSp$. Thus, any functor out of $HoSp$ produces an invariant on D-spectral Mackey functors. We are most interested in functors to $CommGrp$, such as those given by π_n. In the group case these form well-studied objects called Mackey Functors, which we can now generalize:

Definition 3.4.3. Given a small category D, a *D-Mackey Functor* $\underline{M}$ is an additive functor $\underline{M} : \mathcal{B}_D^{op} \to CommGrp$

Example 3.4.4. For any D-spectral Mackey functor X and $n \in \mathbb{N}$, there is a D-Mackey Functor $\underline{\pi_n(X)} : \mathcal{B}_D^{op} \to CommGrp$ given by $\pi_n \circ (\pi_0)_\bullet(X)$.

Mackey Functors provide a connection between equivariant stable homotopy theory and representation theory:

Definition 3.4.5. Given a small category D and a field k, the *representation ring of D over k, $R_k(D)$*, is the ring whose elements are formal differences $[V]-[W]$ of isomorphism classes of finite dimensional D-representations. (That is, $V, W : D \to FinVect_k$.) Addition is given by direct sum on each factor, and multiplication is given by tensor product. (with distribution over the formal difference) This can alternatively be viewed as the group completion of the semiring of D-representations over k, with addition and multiplication given by direct sum and tensor product, respectively.

Example 3.4.6. Let D be a small category and let k be a field. There is a *representation ring D-Mackey functor* defined by $\underline{M}(T) = R_k(B_D(T))$. For $\tau : T_1 \to T_2$, restriction maps are induced by $Res^\tau : FinVect_k^{B_D(T_2)} \to FinVect_k^{B_D(T_1)}$, and transfer maps are induced by the corresponding induction/coinduction. (Because $FinVect_K$ is an additive category, induction and coinduction agree!)

3.5 Eilenberg-MacLane Spectra

One classical construction in algebraic topology is that of *Eilenberg-MacLane spaces* $K(G, n)$ for any commutative group, G. These are spaces with the special property that $\pi_n(K(G, n)) \cong G$, and $\pi_i(K(G, n)) = 0$ for $i \neq n$. These naturally fit into an *Eilenberg-MacLane spectrum*, HG whose nth space is $K(G, n)$. The 0-th (stable) homotopy group of this spectrum will be G, and we could obtain spectra whose nth stable homotopy group is G by suspending HG.

So far, that's a purely non-equivariant story: there may be some groups involved, but they're not acting on our spaces. To include equivariance, we need to consider different invariants, as we did in section 2.1. In the non-stable case, these will take the form of contravariant functors out of the orbit category; in the stable case, we will consider Mackey functors. For now, we will consider the stable case. Thankfully, this has already been well-studied in the group case, and there is a relatively straightforward way to construct equivariant Eilenberg-MacLane spectra out of spectral Mackey functors (as opposed to other models of equivariant spectra). We again follow the approach given by Bohmann and Osorno in Section 8 of [2].

We will make heavy use of $Perm$, the $\mathbf{P}$-enriched category of (small) permutative categories:

Definition 3.5.1. [8] $Perm$ is the $\mathbf{P}$-enriched category of small permutative categories, where morphisms are strictly unital lax symmetric monoidal functors. For any two permutative categories A, B, the permutative structure on $Perm(A, B)$ is given by:

- For any $F, G \in Perm(A, B)$, $[Perm(A, B)](F, G)$ is the set of monoidal natural transformations from F to G.
- For any $F, G \in Perm(A, B)$, the monoidal structure is given by $(F \oplus G)(a) = F(a) \oplus G(a)$, where a is either an object or morphism in A. This is symmetric and strictly associative/unital because B is symmetric and strictly associative/unital.
- The "lax" structure maps $\delta_{F\oplus G}$ on $F \oplus G$ are given as the composite[14]

$$\begin{tikzcd}
 & F(a)\oplus F(a')\oplus G(a)\oplus G(a') \arrow[dr, "\delta_F\oplus\delta_G"] & \\
F(a)\oplus G(a)\oplus F(a')\oplus G(a') \arrow[ur, "Id_{F(a)}\oplus\gamma\oplus Id_{G(a')}"] \arrow[rr, "\delta_{F\oplus G}"'] & & G(a\oplus a').
\end{tikzcd}$$

Now, given a D-Mackey functor $\underline{M} : \mathcal{B}_D^{op} \to CommGrp$, we wish to produce a D-spectral Mackey Functor $H\underline{M} : D\mathcal{B}^{op} \to Sp$ such that

[14]We implicitly use the strict associativity of B multiple times.

$$\pi_0(H\underline{M}) \cong \begin{cases} \underline{M} & \text{if } n = 0 \\ \underline{0} & \text{if } n \neq 0 \end{cases}.$$

We will do this in three stages, first producing a **P**-enriched functor $\widetilde{\underline{M}} : D\mathcal{E}^{op} \to Perm$, then applying $\mathbb{K}_\bullet$, and finally obtaining $H\underline{M}$ by post-composing with a spectrally-enriched functor

$$\Phi : \mathbb{K}_\bullet(Perm) \to Sp.$$

At the end, we will check that $\pi_0(H\underline{M})$ is indeed $\underline{M}$.

Our first lemma is generalization of [2, Lemma 8.2].

Lemma 3.5.2. Any D-Mackey functor $\underline{M} : \mathcal{B}_D^{op} \to CommGrp$ determines a **P**-enriched functor $\widetilde{\underline{M}} : D\mathcal{E} \to Perm$.

Proof. Recall that $D\mathcal{E}$ and $\mathcal{B}_D$ have the same objects, finite D-sets. Also recall that for any finite D sets X, Y, the morphism-**P**-category $D\mathcal{E}(X, Y)$ is the category whose objects are spans $X \leftarrow Z \to Y$ from X to Y and whose morphisms are isomorphisms of spans, while the morphism-group $\mathcal{B}_D(X, Y)$ is group completion applied to the monoid of isomorphism classes of spans. Thus, there is a natural additive "quotient" functor $Q : D\mathcal{E}^{op} \to \mathcal{B}_D^{op}$; this is just the opposite of the functor Q from Theorem 3.1.12. Observe that Q becomes a **P**-enriched functor when we view each morphism-group of $\mathcal{B}_D^{op}$ as a the permutative category whose objects are the elements of the morphism-group and where the monoidal structure comes from the group multiplication. By the same logic, we can regard each morphism-group in $CommGrp$ as a permutative category, in fact a subcategory of $Perm$. This description of **P**-enrichment for $\mathcal{B}_D^{op}$ and $CommGrp$ turns the additivity of $\underline{M}$ into **P**-enrichment. Thus, we have a composition

$$\widetilde{\underline{M}} : D\mathcal{E}^{op} \xrightarrow{Q} \mathcal{B}_D^{op} \xrightarrow{\underline{M}} CommGrp \hookrightarrow Perm$$

of $\mathbf{P}$-enriched functors, as desired. $\square$

Lemma 3.5.3. Any $\mathbf{P}$-enriched functor $\widetilde{\underline{M}} : D\mathcal{E}^{op} \to Perm$ determines a spectrally-enriched functor $\widehat{\underline{M}} : D\mathcal{B}^{op} \to \mathbb{K}_\bullet(Perm)$.

Proof. This is a specific instance of proposition 3.1.11 with $F = \mathbb{K}$. (Recall that $D\mathcal{B}^{op}$ is defined as $\mathbb{K}_\bullet(D\mathcal{E}^{op})$.) $\square$

Lemma 3.5.4. [2, Theorem 6.2] There is a spectrally-enriched functor $\Phi : \mathbb{K}_\bullet(Perm) \to Sp$ such that for any permutative category C, $\Phi(C) = \mathbb{K}(C)$.

From here, we obtain $H\underline{M}$ as the composite $\Phi \circ \widehat{\underline{M}}$, and we just need to check its homotopy groups. In the group case, this is [2, Theorem 8.1].

Theorem 3.5.5. For any D-Mackey functor $\underline{M}$, $\pi_0(H\underline{M}) \cong M$, and $\pi_n(H\underline{M}) = \underline{0}$ for $n \neq 0$. ($\underline{0}$ is the constantly 0 Mackey functor.)

Proof. We will first show that $\pi_n(H\underline{M}) = \underline{0}$ for $n \neq 0$. By our construction of $H\underline{M}$ and the definition of $\underline{\pi_n}$, $\underline{\pi_n(H\underline{M})}$ is the composite

$$\mathcal{B}_D = (\pi_0)_\bullet\mathbb{K}_\bullet(D\mathcal{E}^{op}) \xrightarrow{(\pi_0)_\bullet\mathbb{K}_\bullet(\widetilde{\underline{M}})} (\pi_0)_\bullet\mathbb{K}_\bullet(Perm) \xrightarrow{(\pi_0)_\bullet\Phi} HoSpec \xrightarrow{\pi_n} CommGrp.$$

Since $(\pi_0)_\bullet$ and $\mathbb{K}_\bullet$ preserve objects, we see that for any finite D-set X, $\underline{\pi_n(H\underline{M})}(X)$ is $\pi_n(\Phi(\widetilde{\underline{M}}(X)))$. By construction, $\widetilde{\underline{M}}(X)$ is the commutative group $\underline{M}(X)$, viewed as a permutative category. By definition, $\Phi(\widetilde{\underline{M}}(X)) = \mathbb{K}(\widetilde{\underline{M}}(X))$, which K-theory tells us the (non-equivariant) Eilenberg-MacLane spectrum $H(\underline{M}(X))$.

Hence, by the defining property of (non-equivariant) Eilenberg-MacLane spectra,

$$\underline{\pi_n(H\underline{M})}(X) = \pi_n(H(\underline{M}(X)))$$

is 0 when $n \neq 0$. Furthermore, when $n = 0$, we see that $\underline{\pi_n(H\underline{M})}(X) = \pi_n(H(\underline{M}(X))) =$

$\underline{M}(X)$. Thus, to complete the proof, we just need to show that the structure morphisms of $\underline{\pi_n(H\underline{M})}$ and $\underline{M}$ agree.

Consider any span $X \leftarrow Z \rightarrow Y$ of finite D-sets, and let $f : X \rightarrow Y$ denote the corresponding morphism in $D\mathcal{E}^{op}$. Let us now examine what $(\pi_0)_\bullet\Phi$ does to the morphisms of spectra of form $(\pi_0)_\bullet\mathbb{K}_\bullet(\widetilde{\underline{M}})(f)$. K-theory tells us that for any permutative category P, $\pi_0\mathbb{K}(P)$ is the group completion of the commutative monoid of connected components of P. But $\widetilde{\underline{M}}$ is obtained by postcomposing the inclusion $CommGrp \hookrightarrow Perm$, so every morphism in $\widetilde{\underline{M}}$ is an identity map. Thus, by applying $\mathbb{K}$ then π_0, we get that $(\pi_0)_\bullet\mathbb{K}_\bullet(\widetilde{\underline{M}})(f) = \underline{M}(Q(f))$. (as morphisms of commutative groups)

From there, we apply $(\pi_0)_\bullet\Phi$, which we know from two paragraphs above gives us a map of (non-equivariant) Eilenberg-MacLane spectra $(\pi_0)_\bullet\Phi(\underline{M}(Q(f))) : H\underline{M}(Y) \rightarrow H\underline{M}(X)$ in the homotopy category of spectra. Proposition 6.5 in Bohmann–Osorno tells us that this map is precisely the homotopy class of $\mathbb{K}(\underline{M}(f)$. However, because this is a map of Eilenberg-MacLane spectra in the homotopy category, it is determined precisely by $\pi_0(\mathbb{K}(\underline{M}(f)))$. Furthemore, Corollary 6.6 of Bohmann–Osorno tells us that this is $\underline{M}(f)$. Hence, $\pi_0(H\underline{M}) \cong M$, which completes the proof.

□

3.6 Suspension Spectra

We have yet another way to produce examples of D-spectral Mackey functors, which is via *suspension spectra*. In the non-equivariant case, this takes a space X and produces a spectrum whose nth space is the nth suspension of X, hence the name. These are built similarly in some other models of equivariant spectra. However, for D-spectral Mackey functors, these will actually be the representable D-spectral Mackey functors, at least when X is a finite D-set.

We have two choices for constructing these representable functors, depending on whether we want to use representable functors directly from the spectrally-enriched Burnside category, $D\mathcal{B}$, or if we instead want to use the $\mathbf{P}$-enriched Burnside category, $D\mathcal{E}$, and apply $\mathbb{K}$. It turns out that these choices are the same:

Definition 3.6.1. Given a finite D-set X, its *suspension spectrum*, $\Sigma_D^\infty(X_+) : D\mathcal{B}^{op} \to Sp$, is the spectrally-enriched functor defined by

$$\Sigma_D^\infty(X_+) := D\mathcal{B}(-, X).$$

Definition 3.6.2. Givens a finite D-set X, the functor $S_X : D\mathcal{E}^{op} \to Perm$ is defined by

$$S_X = D\mathcal{E}(-, X)$$

.

Proposition 3.6.3. $\Sigma_D^\infty(X_+) \cong \Phi\mathbb{K}_\bullet(S_X)$.

Proof. By definition of $D\mathcal{B}(X, Y)$ as $\mathbb{K}(D\mathcal{E}(X, Y))$, the two functors agree on objects. (Recall that $\mathbb{K}_\bullet$ does not change objects.) Thus, we only have to check that they agree on morphisms.

To check that the morphism-spectra agree, we need to see that the diagram

$$\begin{tikzcd}
\mathbb{K}D\mathcal{E}(Y,Z) \arrow[r, "\mathbb{K}D\mathcal{E}(-,X)"] \arrow[d, "\mathbb{K}_\bullet(S_X)"'] & Spec(\mathbb{K}D\mathcal{E}(Z,X), \mathbb{K}D\mathcal{E}(Y,X)) \\
\mathbb{K}Perm(D\mathcal{E}(Z,X), D\mathcal{E}(Y,X)) \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

commutes. (Note that we've replaced each instance of $D\mathcal{B}$ by $\mathbb{K}D\mathcal{E}$.) But by the adjunction condition defining Φ, this is the same as saying the diagram

$$\begin{tikzcd}
\mathbb{K}D\mathcal{E}(Y,Z)\wedge\mathbb{K}D\mathcal{E}(Z,X) \arrow[rr] \arrow[d, "\mathbb{K}_\bullet(S_X)\wedge id_{\mathbb{K}D\mathcal{E}(Z,X)}"'] & & \mathbb{K}D\mathcal{E}(Y,X) \\
\mathbb{K}Perm(D\mathcal{E}(Z,X),D\mathcal{E}(Y,X))\wedge\mathbb{K}D\mathcal{E}(Z,X) \arrow[urr, "\mathbb{K}(ev)"'] & &
\end{tikzcd}$$

commutes, where the top arrow is (spectrally enriched) composition. By observation, this is just $\mathbb{K}: Perm \to Sp$ applied to the diagram

$$\begin{tikzcd}
(D\mathcal{E}(Y,Z),D\mathcal{E}(Z,X)) \arrow[rr] \arrow[d, "((S_X)\text{,}id_{D\mathcal{E}(Z,X)})"'] & & D\mathcal{E}(Y,X) \\
(Perm(D\mathcal{E}(Z,X),D\mathcal{E}(Y,X)),D\mathcal{E}(Z,X)) \arrow[urr, "ev"'] & &
\end{tikzcd}$$

where the top arrow is now **P**-enriched composition. (We're writing the monoidal product in $Perm$ as an ordered pair.) But this final diagram commutes by definition of ev, so the previous two diagrams must also commute. Hence, $\Sigma_D^\infty(X_+)$ and $\Phi\mathbb{K}_\bullet(S_X)$ agree on morphisms, which is the final step in showing they are isomorphic. $\square$

3.7 Fixed Points and Geometric Fixed Points

In the non-stable case (that is, considering D-spaces), we had a notion of fixed points. Namely for any D-orbit O and D-space X, the *O-fixed points of X*, X^O, was defined as $Top^D(O,X)$. These essentially keep track of the points of X that are of orbit type O. (However, X^O also detect any O'-fixed points if there's a map $O \to O'$ of D-orbits. Having maps from O to other orbits is extremely common, which is partly why we usually consider X^O for multiple values of O.) This function-space approach generalizes to D-spectral Mackey Functors:

Definition 3.7.1. Given a D-orbit O and D-spectral Mackey functor $\underline{M} \in Sp^{D\mathcal{B}^{op}}$, the *categorical O-fixed points of $\underline{M}$*, $\underline{M}^O$, is the spectrum $Sp^{D\mathcal{B}^{op}}(\Sigma_D^\infty(O_+), \underline{M})$.

Unfortunately, this notion of fixed points doesn't have some nice properties we might like. In the non-stable case, our fixed point functors $(-)^O$ preserved the monoidal structure on Top^D. (That is, $(X \times Y)^O \cong X^O \times Y^O$ and $(-)^O$ applied to the constant D-space $\{pt\}$ yields the non-equivariant space $\{pt\}$. Both of these follow from Top-enrichment of Top^D and the fact that the monoidal operation and monoidal unit are constructed via limits.) This isn't usually the case when we're working with the categorical fixed O-fixed points of D-spectral Mackey functors. Since the categorical O-fixed points are representable, the enriched Yoneda lemma tells us that, for any D-spectral Mackey functor $\underline{M}$,

$$\underline{M}^O = Sp^{D\mathcal{B}^{op}}(\underline{M}, \Sigma_D^\infty(O_+)) = Sp^{D\mathcal{B}^{op}}(\underline{M}, D\mathcal{B}(O, -)) \cong \underline{M}(O).$$

From this we can see that $(-)^O$ often fails to preserve the monoidal unit, and thus fails to be any kind of monoidal functor:

Proposition 3.7.2. Given a finite category D and finite orbit O, the O-categorical fixed points of the unit of $Sp^{D\mathcal{B}^{op}}$ are

$$D\mathcal{B}(O, \{pt\}) \cong \mathbb{K}(D\mathcal{E}(O, \{pt\})).$$

Proof. By Proposition 3.3.2, the unit of $Sp^{D\mathcal{B}^{op}}$ is the representable functor $D\mathcal{B}(-, \{pt\})$. The O-fixed points are then computed by the enriched Yoneda lemma. □

We'll need a different type of fixed point to regain some of these nicer properties we saw in the non-stable case:

Theorem 3.7.3. Given a D-orbit O, there exists a *geometric O-fixed points functor*,

$$\Phi^O : Sp^{D\mathcal{B}^{op}} \to Sp^{(End(O)^{op}\mathcal{B})^{op}}$$

with the following properties:

- For any finite D-space X, $\Phi^O(\Sigma^\infty_D(X_+)) \cong \Sigma^\infty_{End(O)^{op}}(X^O_+)$.
- Φ^O is a strong monoidal functor.
- Φ^O is the left adjoint in a Quillen adjunction.

To prove this, we'll need to actually construct such a functor Φ^O. We will build it as the left adjoint to "geometric inflation," Infl^O, which is constructed using the ordinary fixed-point functor $(-)^O : Top^D \to Top^{End(O)^{op}}$. (Recall that every morphism space $X^O = Top^D(O, X)$ is an $End(O)^{op}$-space via the action of precomposition.) These constructions go as follows:[15]

First, as a consequence of Propositions 1.1.7 and 1.1.8, we obtain:

Corollary 3.7.4. For any D-orbit O, the functor $(-)^O : Top^D \to Top^{End(O)^{op}}$ preserves coproducts and pullbacks.

This allows us to prove:

Proposition 3.7.5. Given a small category D and D-orbit O, $(-)^O$ induces a $\mathbf{P}$-enriched functor $(-)^O_{\mathbf{P}} : D\mathcal{E} \to End(O)^{op}\mathcal{E}$ on the respective $\mathbf{P}$-enriched Burnside categories.

Proof. Recall that the permutative structure on $D\mathcal{E}(X, Y)$ had spans for objects, isomorphisms of spans for morphisms, and a monoidal structure given by the coproduct, which is disjoint union. Composition was given by a strictly unital and associative model of pullback. Because $(-)^O$ is a functor, it automatically preserved spans and isomorphisms of spans. By Proposition 3.7.4, we know that $(-)^O$ also preserves coproducts and pullbacks. In the proof of Proposition 1.1.8, we implicitly confirmed that $(-)^O$ strictly preserves disjoint unions. (That is, $(X \sqcup Y)^O = X^O \sqcup Y^O$. This is an equality, not just an isomorphism.) Furthermore, we can see that $(-)^O$ preserves the pullback choices of Definition 3.1.7 because all functors

[15]The author would like to thank Bert Guillou for suggesting this approach; the group case should appear in an updated version of [9] currently awaiting publication.

preserve identities and because

$$Top^D(O, X \times_Z Y) = Top^D(O, X) \times_{Top^D(O,Z)} Top^D(O, Y)$$

when $X \times_Z Y$ and $Top^D(O, X) \times_{Top^D(O,Z)} Top^D(O, Y)$ are taken to denote the usual subsets of the corresponding cartesian products. $\square$

From $(-)^O_{\mathbf{P}}$, we can apply $\mathbb{K}$ to a functor on the spectrally-enriched Burnside categories:

Definition 3.7.6. Given a small category D and D-orbit O,

$$Fix^O : D\mathcal{B} \to End(O)^{op}\mathcal{B}$$

is the spectrally-enriched functor given by $Fix^O = \mathbb{K}_\bullet(-)^O_{\mathbf{P}}$.

This allows us to define Infl^O, which will be the right adjoint to Φ^O:

Definition 3.7.7. Given a small category D and D-orbit O, $\text{Infl}^O : Sp^{End(O)^{op}\mathcal{B}^{op}} \to Sp^{D\mathcal{B}^{op}}$ is pre-composition by Fix^O.

From here, we just need to confirm that Infl^O actually has a left adjoint, and that its left adjoint satisfies the properties of Definition 3.7.3.

Definition 3.7.8. Given a D-spectral Mackey functor $\underline{M}$, $\Phi^O(\underline{M})$ is the enriched left Kan extension of $\underline{M}$ along Fix^O. This means we have a diagram,

$$\begin{tikzcd}
 & End(O)^{op}\mathcal{B}^{op} \arrow[dd, dashed, "\Phi^O(\underline{M})"] \\
D\mathcal{B}^{op} \arrow[ur, "Fix^O"] \arrow[dr, "\underline{M}"'] \arrow[r, phantom, "\Uparrow \eta_{\underline{M}}"] & \\
 & Sp,
\end{tikzcd}$$

where $\eta_{\underline{M}}$ is a natural transformation $\eta_{\underline{M}} : X \Rightarrow \Phi^O(\underline{M}) \circ Fix^O$. This diagram is natural in the sense that given any other $\underline{N} : End(O)\mathcal{B}^{op} \to Sp$ and natural transformation $\theta : \underline{M} \Rightarrow$

$\underline{N} \circ Fix^O$, there is a unique natural transformation $\sigma : \Phi^O(X) \Rightarrow \underline{N}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \Phi^O(\underline{M}) \circ Fix^O \arrow[dd, Rightarrow, "\sigma"] \\
\underline{M} \arrow[ur, Rightarrow, "\eta_{\underline{M}}"] \arrow[dr, Rightarrow, "\theta"'] & \\
 & \underline{N} \circ Fix^O
\end{tikzcd}$$

Proposition 3.7.9. Φ^O defines a functor, which is a left adjoint to Infl^O

Proof. See [11, Theorem 4.50] □

With the existence of Φ^O now established, let's prove it has the desired properties:

Proposition 3.7.10. Given a small category D, a finite D-orbit O and a finite D-set X, $\Phi^O(\Sigma_D^\infty(X_+)) \cong \Sigma^\infty_{End(O)^{op}}(X_+^O)$.

Proof. Recall that, by definition, $\Sigma_D^\infty(X_+)$ is the representable spectrally-enriched functor $D\mathcal{B}(-, X)$. We wish to compute the geometric fixed points of this spectral Mackey functor. By definition, $\Phi^O(-)$ is enriched left Kan extension along Fix^O. We then apply the (enriched) Yoneda lemma twice and see:

$$\begin{aligned}
Sp^{D\mathcal{B}^{op}}(D\mathcal{B}(-, X), (\underline{N} \circ Fix^O)) &\cong (\underline{N} \circ Fix^O)(X) \\
&= \underline{N}(X^O) \\
&\cong Sp^{End(O)^{op}\mathcal{B}^{op}}(End(O)^{op}\mathcal{B}(-, X^O), \underline{N}).
\end{aligned}$$

Thus, $\Sigma^\infty_{End(O)^{op}}(X_+^O) = End(O)^{op}\mathcal{B}(-, X^O)$ satisfies the universal property of being a left Kan extension of $\Sigma_D^\infty(X_+) = D\mathcal{B}(-, X)$ along Fix^O, which is exactly what we wanted to prove. □

The reader may notice that the argument above only used the facts that $\Sigma_D^\infty(X_+)$ is representable and that $\Phi^O(-)$ is left Kan extension. In general, enriched left Kan extensions preserve representable functors; this appears as part of [11, Theorem 4.6].

Next, we'll show that Φ^O is a strong monoidal functor, recalling by Definition 3.3.1 that the monoidal structures on $Sp^{D\mathcal{B}^{op}}$ and $Sp^{End(O)^{op}\mathcal{B}^{op}}$ are given by Day convolution.

Proposition 3.7.11. Φ^O is strong monoidal with respect to $\otimes$.

Proof. Since $(-)^O$ is a representable functor, it commutes with $\times$, at least up to isomorphism. Thus, the following diagram commutes up to isomorphism.

$$\begin{tikzcd}
D\mathcal{B}^{op} \times D\mathcal{B}^{op} \arrow[r, "Fix^O \times Fix^O"] \arrow[d, "- \times -"'] & End(O)^{op}\mathcal{B}^{op} \times End(O)^{op}\mathcal{B}^{op} \arrow[d, "- \times -"] \\
D\mathcal{B}^{op} \arrow[r, "Fix^O"'] & End(O)^{op}\mathcal{B}^{op}
\end{tikzcd}$$

By iterating enriched left Kan extensions of the D-spectral Mackey functor $\underline{M}_1 \wedge \underline{M}_2$, we get a diagram

$$\begin{tikzcd}
D\mathcal{B}^{op} \times D\mathcal{B}^{op} \arrow[rr, "Fix^O \times Fix^O"] \arrow[dd, "- \times -"'] \arrow[dr, "\underline{M}_1 \wedge \underline{M}_2"'] & & End(O)^{op}\mathcal{B}^{op} \times End(O)^{op}\mathcal{B}^{op} \arrow[dd, "- \times -"] \arrow[dl, "\Phi^O(\underline{M}_1) \wedge \Phi^O(\underline{M}_2)"] \\
& Sp & \\
D\mathcal{B}^{op} \arrow[rr, "Fix^O"'] \arrow[ur, "\underline{M}_1 \otimes \underline{M}_2"] & & End(O)^{op}\mathcal{B}^{op} \arrow[ul]
\end{tikzcd}$$

where we only assume that the outer square commutes up to isomorphism. Note that we've left the bottom-right arrow unlabeled. That's because the right triangle suggests it should be $\Phi^O(\underline{M}_1) \otimes \Phi^O(\underline{M}_2)$, while the bottom triangle suggests it should be $\Phi^O(\underline{M}_1 \otimes \underline{M}_2)$. On the other hand, because left Kan extensions preserve left Kan extensions, this arrow should also be the left Kan extension of $\underline{M}_1 \wedge \underline{M}_2$ along $Fix^O \circ (- \times -) = (- \times -) \circ (Fix^O \times Fix^O)$. But by the preservation left Kan extensions under left Kan extensions, all three of these answers are isomorphic. In other words, $\Phi^O(\underline{M}_1) \otimes \Phi^O(\underline{M}_2) \cong \Phi^O(\underline{M}_1 \otimes \underline{M}_2)$. This isomorphism is natural because left Kan extension preservation is, which means that Φ^O is strong monoidal. □

Finally, we'll show that Φ^O is a left Quillen adjoint. Recall from Definition 3.2.1 that our categories of spectral Mackey functors have the projective model structure.

Proposition 3.7.12. $(\Phi^O, \mathrm{Infl}^O)$ is a Quillen adjunction

Proof. By Proposition 3.7.9, we know $(\Phi^O, \mathrm{Infl}^O)$ is an adjunction. Since Infl^O is defined as pre-composition with Fix^O, it automatically preserves the fibrations and weak equivalences (and hence trivial fibrations) of the projective model structure. Thus, $(\Phi^O, \mathrm{Infl}^O)$ is a Quillen adjunction. □

This final piece establishes Φ^O as a homotopically-meaningful functor. In particular, Φ^O preserves weak equivalences between cofibrant objects.

3.8 $\mathbb{J}$-Mackey Functors

Much of what follows will be aided by the following computation:

Lemma 3.8.1. The pullback of $\mathbb{J}$-orbits is a $\mathbb{J}$-orbit.

Proof. Consider a pullback diagram in $Top^{\mathbb{J}}$, where X, Y, and Z are $\mathbb{J}$-orbits:

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

To see that the pullback W is a $\mathbb{J}$-orbit, we need only show that W_t is a one-point space. Because $Top^{\mathbb{J}}$ is a functor category and Top has all small limits, small limits can be computed object-wise. That is, we have a pullback square:

$$\begin{tikzcd}
W_t \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & Y_t \arrow[d] \\
X_t \arrow[r] & Z_t
\end{tikzcd}$$

Because X, Y, and Z are orbits, X_t, Y_t, and Z_t are all one-point spaces. It follows immediately that W_t is a one-point space, and we're done. □

Note that this is a rare property: it's not shared by any groups with more than one morphism, for instance. Since a D-orbit is just a D-space with a one-point colimit, asking for pullbacks to preserve D-orbits is asking for a large class of D-shaped colimits in Top to commute with pullback. Regardless, in $\mathbb{J}$, (or any other category D where pullback preserves D-orbits) we have the following computational simplification of $\mathbb{J}$-Mackey functors:

Proposition 3.8.2. *A functor $\widehat{M} : \widehat{FinOrb}(\mathbb{J})^{op} \to CommGrp$ uniquely extends to a $\mathbb{J}$-Mackey functor $\underline{M} : \mathcal{B}_{\mathbb{J}}^{op} \to CommGrp$.*

Proof. By the above lemma, $FinOrb(\mathbb{J})$ has pullbacks, so we can define its category of spans, $\widehat{FinOrb}(\mathbb{J})$. Then, by orbit decomposition on the "roof" (the middle term) of each span, any functor

$$\widehat{M} : \widehat{FinOrb}(\mathbb{J})^{op} \to CommGrp$$

uniquely determines a $CommMonoid$-enriched functor

$$\widehat{M}' : (\widehat{FinOrb}(\mathbb{J})')^{op} \to CommGrp$$

where $\widehat{FinOrb}(\mathbb{J})'$ is the category whose objects are finite $\mathbb{J}$-orbits and whose morphisms are equivalence classes of spans $X \leftarrow Z \to Y$ where Z a finite $\mathbb{J}$-set. (X, Y are still $\mathbb{J}$-orbits, and composition is done by pullback. The $CommMonoid$-enrichment of $\widehat{FinOrb}(\mathbb{J})'$ is given by disjoint union.) We can then group-complete each morphism-monoid to get a

$CommGrp$-enriched category $(\widehat{FinOrb}(\mathbb{J})'')^{op}$ and uniquely determine a $CommGrp$-enriched functor

$$\widehat{M}'' : \widehat{FinOrb}(\mathbb{J})'' \to CommGrp.$$

By applying orbit decomposition to the outer terms of each span, any $CommGrp$-enriched functor of the form above uniquely determines an an additive functor

$$\underline{M} : \mathcal{B}_{\mathbb{J}}^{op} \to CommGrp.$$

(that is, a Mackey functor) $\square$

Let's turn our attention then to $\widehat{FinOrb}(\mathbb{J})$. This category admits a relatively simple description:

Proposition 3.8.3. $\widehat{FinOrb}(\mathbb{J})$ is (categorically) equivalent to the category $Mat_{\mathbb{N}}$, whose objects are elements $n \in \mathbb{N}$ and where $Mat_{\mathbb{N}}(n, m)$ is the set of $\mathbb{N}$-valued matrices of size $(m \times n)$.[16]

Proof. From our description of $\mathbb{J}$-orbits, we know that $FinOrb(\mathbb{J})$ is equivalent to the full subcategory of Set whose objects are of the form $\tilde{n} = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. (If $n = 0$, $\tilde{n} = \{\}$.) We'll call this category $Set_{\mathbb{N}}$, and it gives a choice of a skeleton for $FinOrb(\mathbb{J})$. We get an isomorphism of categories $F : \widehat{Set_{\mathbb{N}}} \to Mat_{\mathbb{N}}$ by setting $F(\tilde{n}) = n$ and having $F(\tilde{n} \xleftarrow{a} \tilde{p} \xrightarrow{b} \tilde{m})$ be the $\mathbb{N}$-valued matrix whose (i, j)-th entry is the number of elements in the set

$$\{x \in \tilde{p} \mid a(x) = j, b(x) = i\}.$$

F^{-1} is defined by $F^{-1}(n) = \tilde{n}$ and $F^{-1}(A)$ is the equivalence class of the span $\tilde{n} \xleftarrow{a} \tilde{p} \xrightarrow{b} \tilde{m}$, where p is the sum of the entries and A and where a and b are constructed such that the first $A_{1,1}$ elements of $\tilde{p}$ satisfy $a(x) = 1, b(x) = 1$, the next $A_{1,2}$-many elements satisfy

[16]The swap here of m and n corresponds to the fact that one tends to write matrix multiplication in the same order as function composition.

$a(x) = 2, b(x) = 1$, proceeding so on lexicographically. This completes the equivalence $\widehat{FinOrb}(\mathbb{J}) \simeq Mat_{\mathbb{N}}$. $\square$

Now we can explore some examples:

Example 3.8.4. Consider the representable functor $Mat_{\mathbb{N}}(-, n) : Mat_{\mathbb{N}}^{op} \to Set$. To be able to extend this to a $\mathbb{J}$-Mackey functor, we need $Mat_{\mathbb{N}}(m, n)$ to somehow be a commutative group. One way to do this would be to instead use $\mathbb{Z}$-valued matrices, where the *CommGrp* structure is given by by component-wise addition. (In other words, we added a monoid structure to each $Mat_{\mathbb{N}}(m, n)$ and then group-completed.) This gives a *CommGrp*-enriched functor that takes values on each orbit of the form $[m]$. The induced $\mathbb{J}$-Mackey functor $\underline{M}$ is then obtained by imposing additivity.

When $n = 1$, this becomes:

Example 3.8.5. There is a $\mathbb{J}$-Mackey functor $\underline{M}$ such that

$$\underline{M}([n]) = \mathbb{Z}^n,$$

where each span $[m] \leftarrow X \rightarrow [n]$ of finite $\mathbb{J}$-sets corresponds to left multiplication by a $\mathbb{Z}$-valued matrix.

The above examples do *not* give a representable $\mathbb{J}$-Mackey functor. This is because our method of turning a set into a commutative group involved some choices of multiplicative structure. This is not the universal way to turn a set into a commutative group. Representable $\mathbb{J}$-Mackey functors are instead obtained by replacing $Mat_{\mathbb{N}}(m, n)$ with its free commutative group:

Example 3.8.6. Consider the representable functor $Mat_{\mathbb{N}}(-, n) : Mat_{\mathbb{N}}^{op} \to Set$, and then post-compose the free commutative group functor $\mathbb{Z}[-] : Set \to CommGrp$ to get a *CommGrp*-enriched functor. This composition extends (in the same way we extended $\widehat{M}''$ to $\underline{M}$ in Proposition 3.8.2) to the representable $\mathbb{J}$-Mackey Functor $\mathcal{B}_{\mathbb{J}}(-, n)$.

A Appendices

A.1 The $\mathbb{K}$-Theory Machine

The following is a condensed version of Graeme Segal's "Categories and Cohomology Theories" and shows how to turn a permutative category into a spectrum. This builds a functor $\mathbb{K}$ that turns a permutative category into a spectrum. This version of $\mathbb{K}$ is weakly equivalent to the multifunctor described in section 3.1, which is a modification due to Anthony Elemendorf and Mike Mandell in [8]. We will construct $\mathbb{K}$ in 3 stages:

1. To each small permutative category A, functorially generate a "Γ-category" $\widetilde{A}$.
2. To each Γ-category $\widetilde{A}$, functorially generate a "Γ-space" $|\widetilde{A}|$.
3. To each Γ-space $|\widetilde{A}|$, functorially generate a spectrum $\mathbb{K}(A)$.

As will be explained, Γ-categories and Γ-spaces are contravariant functors from Γ to Cat or Top respectively that satisfy two additional properties. Namely:

Definition A.1.1. Γ is the category whose objects are finite sets and whose morphism sets $\Gamma(S,T)$ consist of functions $\theta : S \to 2^T$ such that $\theta(x)$ and $\theta(y)$ are disjoint sets whenever $x \neq y$.[17]

For the following, we will let $\mathbf{n} \in \Gamma$ be the finite set $\mathbf{n} = \{0, \ldots, n-1\}$.

Definition A.1.2. [16, Definition 2.1] A Γ-category C is a functor $C : \Gamma^{op} \to Cat$ such that:

1. $C(\mathbf{0})$ is categorically equivalent to the terminal category.
2. For each $n > 0$,[18] the functor $C(\mathbf{n}) \to C(\mathbf{1}) \times \cdots \times C(\mathbf{1})$ induced by the maps $i_k : \mathbf{1} \to \mathbf{n}$ with $\{0\} \mapsto \{k\}$ is a categorical equivalence.

Definition A.1.3. [16, Definition 1.2] A Γ-space X is a functor $X : \Gamma^{op} \to Top$ such that:

[17] For an alternative characterization, note that Γ is categorically equivalent to the opposite of the category of pointed finite sets.

[18] The following condition also describes the $n = 0$ case, but we stated that case separately because of its simple description.

1. $X(\mathbf{0})$ is homotopy equivalent to the terminal space. (That is, $X(\mathbf{0})$ is contractible.)

2. For each $n > 0$,[19] the map $X(\mathbf{n}) \to X(\mathbf{1}) \times \cdots \times X(\mathbf{1})$ induced by the maps $i_k : \mathbf{1} \to \mathbf{n}$ with $\{0\} \mapsto \{k\}$ is a homotopy equivalence.

Now, we can do our 3-stage construction.

Definition A.1.4. For any small permutative (or just symmetric monoidal) category A, let $\widetilde{A}$ be the Γ-category where:

- $\widetilde{A}(S)$ is the category whose objects are contravariant functors from 2^S (viewed as a poset) to A that send $\sqcup$ to $\otimes$ and whose morphisms are isomorphisms of functors.
- For any structure map $\theta \in \Gamma(S, T)$, $\widetilde{A}(\theta)$ is precomposition with θ_*, where $\theta_* : 2^S \to 2^T$ is defined by $\theta_*(U) = \theta(U)$ for all $U \subseteq S$.

(Note that we're following the convention that functor $F : C^{op} \to D^{op}$ is given the same name as its corresponding functor $F : C \to D$.)

Proposition A.1.5. The construction of $\widetilde{A}$ is functorial.

Proof. We can post-compose with any strict monoidal functor $F : A \to B$ to get a map $\widetilde{F} : \widetilde{A}(S) \to \widetilde{B}(S)$ of Γ-categories. To see that that the square

$$\begin{tikzcd}
\widetilde{A}(S) \arrow[r, "\widetilde{F}_S"] & \widetilde{B}(S) \\
\widetilde{A}(T) \arrow[u, "\widetilde{A}(\theta)"] \arrow[r, "\widetilde{F}_T"'] & \widetilde{B}(T) \arrow[u, "\widetilde{B}(\theta)"']
\end{tikzcd}$$

commutes, recall that $\widetilde{A}(\theta)$ and $\widetilde{B}(\theta)$ are just pre-composition with $\theta_*$and that $\widetilde{F}$ is poscomposition with F. Thus, the square commutes because functor composition is associative. By inspection, we see that if F is the identity functor $id_A : A \to A$, then $\widetilde{F}$ is $Id_{\widetilde{A}}$ and that $\widetilde{G \circ F} = \widetilde{G} \circ \widetilde{F}$ for any $G : B \to C$. Hence, $\widetilde{(-)}$ is indeed a functor. $\square$

[19] As before, this description also covers the $n = 0$ case.

From $\widetilde{A}$, we can produce a Γ-space $|\widetilde{A}|$ via taking its nerve and then geometric realization. That is, given a Γ-category

$$\Gamma^{op} \xrightarrow{\widetilde{A}} Cat,$$

$|\widetilde{A}|$ is obtained as the composite

$$\Gamma^{op} \xrightarrow{\widetilde{A}} Cat \xrightarrow{N} SSet \xrightarrow{|-|} Top,$$

where $N : Cat \to SSet$ is the usual nerve construction and $|-| : SSet \to Top$ is geometric realization.

For our last step, we will realize $\mathbb{K}(A)$ as the spectrum $\mathbb{B}|\widetilde{A}|$ whose n-th space is $\mathbf{B}^n|\widetilde{A}|(\{pt\})$. This requires some unpacking:

Definition A.1.6. Given a Γ-space X and finite set S, X^S is the Γ-space given by the composite

$$\Gamma^{op} \xrightarrow{(S, id_\Gamma)} \Gamma^{op} \times \Gamma^{op} \xrightarrow{- \times -} \Gamma^{op} \xrightarrow{X} Top,$$

where S denotes the corresponding constant functor $S : \Gamma \to \Gamma$ and $(- \times -)$ is the product bifunctor. From this description, we have $X^S(T) = X(S \times T)$.

Note that the assignment $S \to X^S$ is functorial in S. This is because the functor $\Gamma^{op} \times -$ is the left adjoint to $Cat(\Gamma^{op}, -)$. (Cat is a closed monoidal category.) Thus, specifying the functor $X \circ (- \times -)$ is equivalent to specifying a functor $\Gamma^{op} \to Cat(\Gamma^{op}, Top)$, and this is by definition precisely $S \mapsto X^S$. Now, we want to turn the Γ-(Γ-space) X^-into a Γ-space by post-composing with some functor $Top^{\Gamma^{op}} \to Top$. But what is that functor?

Proposition A.1.7. Let Δ be the category whose objects are finite sets of the form $\mathbf{n}_\Delta = \{0, \ldots, n\}$ and whose morphism are order-preserving functions. (This is the category used to define simplicial sets.) Then, Δ faithfully embeds into Γ where a morphism $f : \mathbf{n}_\Delta \to \mathbf{m}_\Delta$ is sent to the map $\theta : \mathbf{n} \to \mathbf{m}$[20] with $\theta(i) = \{j | f(i-2) < k \leq f(i-1)\}$.

[20]Beware the off-by-one! $\mathbf{n}_\Delta \in \Delta$ is a set with $n+1$ elements in Δ, but $\mathbf{n} \in \Gamma$ is a set with n elements.

Hence, any Γ-space gives rise to a simplicial space by pre-composition with the embedding $\Delta^{op} \to \Gamma^{op}$. From here, we can turn our simplicial space into a space via geometric realization. In other words:

Definition A.1.8. Given a Γ-space X, $\mathbf{B}X$ is the Γ-space obtained as the composite

$$\Gamma \xrightarrow{X^{-}} Top^{\Gamma^{op}} \xrightarrow{-\circ\Delta} Top^{\Delta^{op}} \xrightarrow{|-|} Top.$$

In other words, the S-th space of $\mathbf{B}X$ is the geometric realization of X^S, viewed as a simplicial set.

We will construct a spectrum $\mathbb{B}X$ by iterating the functor $\mathbf{B}$, but first we need a proposition:

Proposition A.1.9. [16] Given a Γ-space X, its realization $|X|$ has a "1-skeleton" subspace given by its 0- and 1-simplices that is naturally homotopy equivalent to the suspension of $X(\mathbf{1})$. This homotopy equivalence defines a map

$$\Sigma X(\mathbf{1}) \to |X| \cong \mathbf{B}X(\mathbf{1}).$$

Proof. Let $|X_{\leq \mathbf{1}}|$ denote the subspace of $|X|$ given by the 0- and 1-simplices of X. Since $X(\mathbf{0})$ is contractible, $|X_{\leq \mathbf{1}}|$ is homotopy equivalent to the quotient $|X_{\leq \mathbf{1}}|/\sim$ formed by collapsing the 0-cells. Explicitly, $|X_{\leq \mathbf{1}}|/\sim$ is the space $\{(x,t)|x \in X(\mathbf{1}), t \in [0,1]\}/\sim'$, where $\sim'$ is the equivalence relation that identifies all (x,t) such that x is degenerate and/or $t \in \{0,1\}$. This receives a map (which is a homotopy equivalence) from the suspension of $X(\mathbf{1})$ by collapsing the degenerate 1-cells mapping the interval $[-1,1]$ onto $[0,1]$.

Thus, the homotopy equivalence $\Sigma X(\mathbf{1}) \xrightarrow{\sim} |X_{\leq \mathbf{1}}|$ can be realized as the composite

$$\Sigma X(\mathbf{1}) \xrightarrow{\sim} |X_{\leq \mathbf{1}}|/\sim \xrightarrow{\sim} |X_{\leq \mathbf{1}}|,$$

where the second morphism is a homotopy inverse to the quotient map.

Hence, we get a map $(\mathbf{1}) \to |X|$ via post-composition with the inclusion $|X_{\leq \mathbf{1}}| \hookrightarrow |X|$. The isomorphism $|X| \cong \mathbf{B}X(\mathbf{1})$ follow from the fact that $X \cong X^{\mathbf{1}}$ and that $\mathbf{B}X(\mathbf{1})$ is just $|X^1|$. This completes the proof. $\square$

Proposition A.1.10. [16, Proposition 1.4] For $n \geq 1$, the adjoint structure map $\mathbf{B}^n X(\mathbf{1}) \to \Omega \mathbf{B}^{n+1} X(\mathbf{1})$ is a homotopy equivalence.

Definition A.1.11. Given a Γ-space X, $\mathbb{B}X$ is the spectrum whose n-th space is $\mathbf{B}^n X(\mathbf{1})$ and whose structure maps $\Sigma \mathbf{B}^n X(\mathbf{1}) \to \mathbf{B}^{n+1} X(\mathbf{1})$ are given by the proposition above.

Finally, we can define the functor $\mathbb{K}$:

Definition A.1.12. Given a small permutative category A, $\mathbb{K}(A)$ is the spectrum $\mathbb{B}|\widetilde{A}|$.

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