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INVERSE TORSIONAL EIGENVALUE PROBLEMS

C. Willis
(Ph.D. Thesis)

June 1983

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INVERSE TORSIONAL EIGENVALUE PROBLEMS

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June 1983

Ph.D. Thesis

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INVERSE TORSIONAL EIGENVALUE PROBLEMS

CATHERINE WILLIS

ABSTRACT

We undertake a numerical and theoretical investigation of the inverse problem for the reconstruction of the density ρ and S-wave velocity β of the Earth from its torsional oscillations. We assume a spherically symmetric, non-rotating Earth which consists of a perfect elastic, isotropic material and transform the differential equation governing the torsional oscillations to a Sturm-Liouville problem.

We present a numerical method for determining ρ and β in the upper mantle when ρ and β are smooth functions of radius. The method is closely related to the theory by Hald which proves that ρ and β are uniquely determined in the upper mantle by their values in the lower mantle and the periods of the torsional oscillations for two angular orders. The method, based on the Rayleigh-Ritz method, solves iteratively for the coefficients of a generalized Fourier series for the potential. We reconstruct several earth models to 2% accuracy. However, the method is sensitive to error in the data. This is not true of the inversion for the density alone and suggests that the simultaneous inversion for the density and velocity from free oscillation data may be unstable.

The smoothness assumption is a serious limitation of our numerical method, since most earth models have a discontinuity at the crust and many have gradients with discontinuities in the upper mantle. We study the associated discontinuous Sturm-Liouville problem and prove that if the

eigenfunctions have two discontinuities and if the potential is known in half the interval then the potential in the whole interval is uniquely determined from one spectrum. We apply this theorem to the discontinuous earth model to prove that given ρ in the lower mantle and β in the mantle and crust, then the torsional spectra of one angular order uniquely determine ρ in the upper mantle. In addition, if β is known only in the lower mantle, then two torsional spectra uniquely determine both ρ and β in the upper mantle.

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INTRODUCTION

In this work we investigate a mathematical model of the Earth. We want to describe the properties which govern the behavior of waves traveling through the Earth. To determine a model describing the material inside the Earth we must make some simplifying assumptions. In this paper we assume that the earth is a spherically symmetric body and consists of an isotropic, perfectly elastic material. Thus we can describe the Earth by the elastic parameters: the density, the rigidity and the bulk modulus, or equivalently, the density and the velocities of the P- and S- waves.

The purpose of this work is twofold. In Part I we present a numerical method for determining the density ρ and the velocity β of the S- waves in the upper mantle from the periods of the torsional oscillations. Our method requires ρ and β in the lower mantle. The terms upper and lower mantle have precise definitions which are given in Section 1.

Under certain assumptions the velocities α and β of the P- and S- waves can be determined from travel times of the waves. In 1939, Jeffreys and independently Gutenberg and Richter inverted travel times to obtain a velocity model for the Earth. If the elastic material is adiabatic, homogeneous and devoid of phase change, then the density at a given radii depends only on the pressure from the material above. The distribution of the density can then be determined by a second order non-linear differential equation known as the Adams-Williamson equation. Bullen used this differential equation together with certain constraints such as the mass and the moment of inertia of the Earth to determine a density distribution for the mantle. The earth model which consists of Jeffreys velocities and Bullen's density distribution

has become known as Model A. Bullen first tried an earth model satisfying Adams- Williamson equation throughout the mantle. This resulted in a core which was too light. From travel time data, Bullen decided to not use the differential equation in the region now known as region C. In this region he described the density by quadratic polynomials. Many current and frequently quoted earth models have a density distribution which satisfies the Adams-Williamson equation below a depth of 1000 km.

In the past twenty years one has been able to observe the free oscillations of the Earth on long period seismographs. On the other hand, the periods of the free oscillations of a given earth model can be calculated. We can then compare the calculated periods with the observed values and then either discard the model or change it to fit the data better. Today most earth models are constructed by a variation of this technique. One drawback is that the final model depends upon the starting model. Also, the existence and position of discontinuities is usually built into the model. This leads to the question: What data is necessary to determine the elastic parameters of the Earth uniquely?

This question was considered by Backus and Gilbert [6]. Under certain assumptions, such as linearity, they stated that a finite amount of data cannot determine the Earth uniquely. They asserted, however, that a fixed, finite data set could determine the average value of a given parameter at a certain radius uniquely. They defined the resolving length for the average and developed an algorithm to determine a linear combination of the data which minimized the resolving length in some sense. When error in the data is considered, one gets a trade-off curve: statistical error in the average of the parameter versus resolution of the parameter. Trade-off curves have been used to study the resolution of model parameters. The calculations are

expensive in general. By assuming that the error in the data is normally distributed, Bolt and Uhrhammer simplified the calculation of the trade-off curve.. They used this technique to look at the resolving power for the density in Model Cal5I0, [9].

We will only study the density ρ and velocity β in the mantle. What data is needed to determine these parameters uniquely? In Part II we address this question. It was conjectured by Backus and Gilbert that the torsional modes for two angular orders could determine the density. This is not true in general. Anderssen and Chandler [4] have shown that earth models can have redundant torsional spectra; no new information is obtained from spectra with different angular orders. Hald has found two distinct earth models that have the same periods for the torsional modes. However, if the density and velocities are smooth functions of radius, then Hald [26] proved that the density in the upper mantle is uniquely determined by the torsional modes of one angular order, provided that the density is given in the lower mantle and that the S-wave velocity is known throughout the mantle. The torsional modes of two angular orders will in addition determine the S-wave velocity uniquely in the upper mantle. Hald has recently extended these uniqueness results to models which have one discontinuity in the mantle. See [28]. In this case the position of the discontinuity may be determined as well.

All published earth models have a crustal layer above the upper mantle. The density and velocities of the elastic waves are discontinuous at the boundary between the mantle and crust. This discontinuity is known as the Mohorovicic discontinuity. In addition, the elastic parameters of many earth models have discontinuous gradients in the upper mantle. For example Model 1066B by Gilbert and Dziewonski [23], Bullen's Model A and Model B1 by Jordan

and Anderson [34]. Thus we were motivated to obtain uniqueness results for earth models with two discontinuities.

In Part II we prove that if the density is known in the lower mantle and the velocity of the S-waves is given in the mantle and in the crust then the torsional spectrum of one angular order determines the density in the upper mantle uniquely. This result is a natural extension of the work by Hald [26], [28], and we follow his proof. The details become more complicated when an second discontinuity is allowed, but the techniques are still valid. However, we had to make the assumption that the product of the density and S-wave velocity can neither increase to more than twice its value nor decrease to less than half its magnitude at either discontinuity. This is satisfied by all earth models I know.

Our numerical method is closely related to the theory by Hald [26]. Thus we assume that there is no discontinuity in the upper mantle and that our models do not have a crustal layer. If we use data from a model which is not smooth, the algorithm will diverge or converge to an incorrect solution. Since all published earth models have a crust, this is a serious limitation of our method. We hope it can be remedied in the future.

In practice, the existence of discontinuities cannot be determined from free oscillation data. Thus Gilbert and Dziewonski [23] have constructed two models 1066A and 1066B which fit the data used in the inversions equally well. Both models have a crust, but the latter has two discontinuities in the mantle at depths of 421 and 671 km., while the former is continuous from the Mohorovicic discontinuity to the core.

The numerical method presented in this paper is based on three ingredients. Our differential equation comes from separating the variables in

the equation for the torsional modes of a spherical, symmetric, isotropic, elastic and non-rotating earth. By using the Liouville transformation we obtain an equation in Liouville normal form. We determine the potential of this equation by solving an inverse Sturm-Liouville problem by the technique of Hald [27]. This algorithm is based on Rayleigh's principle and computes the coefficients of a generalized Fourier series of the potential. The potentials corresponding to two angular orders are calculated and are then used to define the inverse Liouville transformation and to determine a differential equation for the velocity β. We later modify the method to allow for the mixing of data from different angular orders and to create a more stable algorithm.

We have tested our method on several earth models. The models were obtained by smoothing Model A. We are able to reconstruct the models to within 1% accuracy. Our method is therefore not as accurate as the method by Hald[27] in which only the density is computed. Moreover, our final solution is quite sensitive to error in the data. This suggests that the simultaneous inversion for both the density and S-velocity from the torsional modes may be unstable. I have not proved the convergence of the method. However, Hald [24] and Yen [51] proved the convergence of similar, simpler algorithms. In practice, our method converged in eight to ten iterations, and the final solution is independent of our initial guess.

In Section 1, Part I we present the differential equation for the torsional waves and describe the Liouville transformation. Section 2 contains an algorithm for calculating the eigenvalues of a Sturm-Liouville equation. This forward method is presented to provide motivation for and understanding of the inverse method. The basis functions used in our approximation to the eigenfunction are also given in Section 2. In Section 3 we explain the inverse

Sturm-Liouville method and describe the methods for obtaining ρ and β from the solutions of the previous method. The next two sections contain an explanation of the numerical implementation of the method and our numerical experiments and the results of these experiments. The effect of error in the data on the computed model is examined in the final section. We also present a modification of the inverse Sturm-Liouville method which increases the stability of the algorithm and results of experiments using the new algorithm.

In the first six sections of Part II we are concerned with the theory of Sturm-Liouville problems with two interior discontinuities. We show that if the potential is given over half the interval and if one boundary condition is known and the magnitude of the jumps in the eigenfunctions satisfy certain restrictions then the eigenvalues determine the potential and the other boundary condition uniquely. This generalizes a theorem by Hochstadt and Lieberman [30] and extends a theorem by Hald [28].

Our technique follows that of Hochstadt and Lieberman and Hald and is based on an integral representation of the eigenfunctions. The eigenvalues of a Sturm-Liouville problem with discontinuities may not have an asymptotic expansion. See [41] and [5]. However, by adapting the Cauchy integral technique we were able to study the Wronskian and determine the asymptotic distribution of the eigenvalues. The straightforward application of this technique required an assumption on the size of the discontinuity. The exact restriction is given in the statement of the Theorem 1, but the restriction will be satisfied if the ratio of the right and left hand limits of the eigenfunction is greater than $1/\sqrt{2}$ and less than $\sqrt{2}$. The range of allowable jumps decreases as the number of discontinuities increase. Thus this portion of the proof would need to be changed for the theory of inverse Sturm-Liouville problems with an arbi-

trary number of discontinuities.

In the final section of Part II we apply the theory for inverse inverse Sturm-Liouville problems to the inverse problem for the mantle.

PART ONE

1: The differential equation and Liouville normal form

Our differential equation comes from the equation which governs the torsional oscillations of a spherical, symmetric, elastic and non-rotating earth. After separation of variables, the equation for the radial dependence of the oscillations is

$$\mu \ddot{W}_l + (\dot{\mu}+2\mu/r)\dot{W}_l - (\dot{\mu}/r + l(l+1)\mu/r^2 - \omega^2\rho)W_l = 0 \qquad (1.1)$$

with the surface condition

$$\dot{W}_l(R) - W_l/R = 0.$$

The symbol $\dot{}$ denotes differentiation with respect to r. At the core-mantle boundary, the requirements of continuity of displacement and stress lead to the conditions

$$\mu[\dot{W}_l - W/r]_+ = \mu[\dot{W}_l - W/r]_-$$

$$W_{l_+} = W_{l_-}.$$

These same equations can be used at any spherical interface. See Lapwood [39], page 100. As the torsional waves do not pass through the core, the right hand side is zero for both equations. The function μ is the rigidity and is related to the density ρ and the S-wave velocity β by $\mu=\beta^2\rho$. The constant l arises from separation of variables and is known as the angular order of the oscillation. The eigenfrequency is denoted by ω^2.

We transform the differential equation (1.1) to Liouville normal form. This will allow us to exploit the asymptotic form of the eigenvalues and eigenfunctions in our inverse method. In this section we describe the method of

transformation. It involves a change of variables which maps the interval $[R_c, R]$ to the interval $[0,\pi]$. The transformed equation has a potential q which depends on the derivatives of the density and the velocity of the S-waves. Thus we assume that the density and velocity are smooth functions of radius. This restricts the method to earth models which have no discontinuities at the crust or in the mantle.

If $W_l = ru$ it follows from (1.1) that:

$$-(r^4\mu u')' + \frac{(l+2)(l-1)}{r^2} r^4 \mu u = \omega^2 r^4 \rho u \tag{1.2}$$

$$u'(R_c) = u'(R) = 0$$

We now use the Liouville transformation, see Jorgens [36], page 4.2. Let $z = R-r$ parameterize depth. We define $x = K^{-1}\int_0^z \beta^{-1}\, ds$ where K is equal to $\pi^{-1}\int_0^{R-R_c} \beta^{-1} ds$. Hence a vertical S-wave travels from the surface to the core in a time of $2\pi K$ seconds. The surface of the earth corresponds to $x=0$ and the core-mantle boundary to $x=\pi$. Since β is positive, x is a monotone increasing function of z. Thus there exists an inverse function φ such that $\varphi(x) = z$.

To rewrite equation (1.2) as a differential equation in x we set $f(x) = r^2\sqrt{\rho\beta}$ where $r = R-\varphi(x)$ and let $y(x) = f(x)u(r)$. Substituting these expressions into equation (1.2) we get the Liouville normal form:

$$-y'' + qy = \lambda y \tag{1.3}$$

$$y'(0)-hy(0) = y'(\pi)+Hy(\pi)=0.$$

The potential q is given by

$$q = v + (l+2)(l-1)w \tag{1.4}$$

where l is the angular order or spectrum number, $v = f''/f$ and $w = \beta^2/r^2$.

The constants h and $-H$ in the boundary conditions are the values of f'/f evaluated at 0 and π and the eigenvalue λ is given by $\omega^2 K^2$ where ω^2 is the eigenvalue in equation (1.2). The transformation is valid if ρ' and β' are continuous and piecewise differentiable on $[0,\pi]$.

The complete set of eigenvalues for equation (1.3) cannot determine the potential q uniquely. See Hald [27], Anderssen and Cleary [4]. However, if $q(x)$ is known in the interval $[\pi/2,\pi]$ and the constant H is given then q for $0 \le x \le \pi/2$ and h are uniquely determined by the eigenvalues, see Hald [26], Hochstadt and Lieberman [30]. We assume that the potential q is given in $[\pi/2,\pi]$ and that the constant H is known. This is the case if the density and velocity model are given in part of the mantle. We determine the lower mantle to be precisely that portion where the density and velocity are needed. Let r_0 be determined such that $\int_{R_c}^{r_0}\beta^{-1}\,dr = \int_{r_0}^{R}\beta^{-1}\,dr$. Then $x(r_0) = \pi/2$. We let the lower mantle be the part of the mantle which lies below r_0 and call the remaining part the upper mantle and crust. In practice r_0 lies at a depth of approximately 1300 km.

Suppose that we have found two distinct potentials q_{l_1} and q_{l_2} corresponding to angular orders l_1 and l_2. We can then determine the functions $v = f''/f$ and $w = \beta^2/r^2$ by

$$v = \frac{(l_2+2)(l_2-1)q_{l_1} - (l_1+2)(l_1-1)q_{l_2}}{(l_2+l_1+1)(l_2-l_1)} \tag{1.5}$$

and

$$w = \frac{q_{l_2} - q_{l_1}}{(l_2+l_1+1)(l_2-l_1)}\,. \tag{1.6}$$

We use these functions in the reconstruction of the density and velocity in the upper mantle. Thus the eigenvalues of two spectra determine our density and

S-velocity models.

The eigenvalues of the differential equation (1.3) have the asymptotic form

$$\lambda_j = j^2 + A + o(j^{-1}) \tag{1.7}$$

where the constant A is given by

$$A = \frac{\pi}{2}(h + H) + \frac{1}{\pi}\int_0^{\pi} q\,dx \quad . \tag{1.8}$$

See Borg [11], Hochstadt [29].

Given only a finite number of eigenvalues we need the asymptotic constant A as data for the inverse method. The eigenvalues of earth models with internal discontinuities may not have an asymptotic expansion. Our method cannot be applied to such models, see Anderssen and Cleary [5]. Since we assume that the density and the velocity of the S-waves are smooth functions, the eigenvalues of our models will have the asymptotic form (1.7).

2: The direct method and the comparison problem

In this section we present a numerical method for estimating the eigenvalues of a Sturm-Liouville equation. Although the method is not efficient, our inverse method is based on it and we think that the presentation of the direct method will clarify and motivate the inverse method.

The method is a version of the Rayleigh-Ritz method. First we form the Rayleigh quotient of the Sturm-Liouville equation and then we approximate the eigenfunctions by a linear combination of simpler functions. By substituting this approximation into the Rayleigh quotient we reduce the quotient to a quadratic form. The eigenvalues of the matrix for the quadratic form are close to the eigenvalues of the differential equation. Finally, we obtain a nonlinear equation which we solve by iteration for the eigenvalues of the matrix.

We now give our choice of basis functions for the expansion of the eigenfunction of equation (1.3). Consider the comparison problem

$$-y_j'' = \mu_j y_j \tag{2.1}$$
$$y_j'(0) - hy_j(0) = y_j'(\pi) + Hy_j(\pi) = 0$$

The eigenfunction u_j corresponding to the eigenvalue μ_j is given by

$$\begin{aligned} u_j &= \cos(\sqrt{\mu_j}x) + \frac{h}{\sqrt{\mu_j}}\sin(\sqrt{\mu_j}x) && \mu>0 \\ &= 1 + hx && \mu=0 \\ &= \cosh(\sqrt{|\mu_j|}x) + \frac{h}{\sqrt{|\mu_j|}}\sinh(\sqrt{|\mu_j|}x) && \mu<0 \end{aligned} \tag{2.2}$$

Let $\rho_j = \int_0^\pi u_j^2 dx$. Then $\sqrt{\rho_j}$ is the norm of u_j and $y_j = u_j/\sqrt{\rho_j}$ is the eigenfunction with norm 1. As equation (2.1) is a Sturm-Liouville problem the eigenfunctions are mutually orthogonal and complete in L^2, see John [35], page 220.

From (2.2) we see that the eigenvalues of equation (2.1) are the zeros of the function given by

$$-\omega(u) = (h + H)\cos(\sqrt{\mu}\pi) + (hH - \mu)\frac{\sin(\sqrt{\mu}\pi)}{\sqrt{\mu}} \qquad \mu>0 \qquad (2.3)$$

$$= h + H(1+h\pi) \qquad \mu=0$$

$$= (h + H)\cosh(\sqrt{|\mu|}\pi) + (hH + |\mu|)\frac{\sinh(\sqrt{|\mu|}\pi)}{\sqrt{|\mu|}} \qquad \mu<0.$$

Note that ω is the Wronskian of the differential equation (2.1). Since the differential equation is symmetric the roots will be real and simple. At most two eigenvalues may be negative.

An advantage of our choice of basis functions $\{y_j\}$ is that the asymptotic expansions of the eigenvalues $\{\mu_j\}$ for equation (2.1) and of the eigenvalues $\{\lambda_j\}$ for equation (2.2) are close. We have

$$\mu_j = j^2 + \tilde{A} + o(j^{-1})$$

where

$$\tilde{A} = \frac{2}{\pi}(h + H)$$

By comparing this result to equations (1.7) and (1.8) we see that if the average of our potential q is 0, then the two expansions have the same leading term.

To form the Rayleigh quotient we multiply equation (1.3) by the eigenfunction y and integrate. By using integration by parts and the boundary conditions we arrive at the Rayleigh quotient

$$\lambda = \frac{\int_0^\pi (y'^2 + qy^2)dx + Hy^2(\pi) + hy^2(0)}{\int_0^\pi y^2 dx} .$$

The stationary points of the quotient are the eigenvalues of the matrix A, Courant and Hilbert [18], page 402. Let y be approximated by $\sum_0^n c_j y_j$. We replace y by this expression in the Rayleigh quotient. Since the

eigenfunctions $\{y_j\}$ are orthogonal and normalized, the quotient reduces to

$$R[c] = \frac{\sum_{i,j} c_i c_j (\lambda_j \delta_{ij} + \int_0^\pi q y_i y_j \, dx)}{\sum_{i,j} c_i c_j \delta_{ij}}$$

where i and j are summed from 0 to n. Thus,

$$\lambda = R[c] = \frac{c^T A c}{c^T c}$$

with c the vector consisting of the coefficients in the expansion of y, and $A = (\alpha_{ij})$ the $(n+1)$x$(n+1)$ matrix with components

$$\alpha_{ij} = \lambda_j \delta_{ij} + \int_0^\pi q y_i y_j \, dx \quad . \tag{2.4}$$

$R[c]$ is the quadratic form associated with the minimum-maximum problem for the eigenvalues of the matrix A, Courant and Hilbert [18], page 31. It can be shown that the eigenvalues of the matrix A converge to the eigenvalues of the differential equation (1.3) as the dimension of the matrix increases.

We now describe a method to calculate the eigenvalues of A. We first introduce some notation. Let A_j be the $n x n$ matrix obtained from A by deleting the $(j+1)^{st}$ row and $(j+1)^{st}$ column. Recall that the first row of A has elements denoted by α_{0j} for $0 \le j \le n$. Let a_j be the vector consisting of the $(j+1)^{st}$ row of A with the diagonal element removed.

To present the idea we consider the case $j = 0$. We can write A as

$$A = \begin{bmatrix} \alpha_{00} & a_0^T \\ a_0 & A_0 \end{bmatrix}.$$

Let λ be an eigenvalue of A, but not of A_0, and let v be the corresponding eigenvector of the form $v^T = (1, z^T)$. Thus

$$(A - \lambda)v = 0.$$

is equivalent to the system

$$\alpha_{00} - \lambda + a_0{}^T z = 0$$

$$a_0 + (A_0 - \lambda)z = 0$$

By solving the second equation for z and substituting the solution into the first equation we arrive at the identity

$$\lambda = \alpha_{00} - a_0{}^T(A_0 - \lambda)^{-1}a_0.$$

Now let λ_j be an eigenvalue of matrix A, but not of the submatrix A_j. A similar argument produces the basic identity

$$\lambda_j = \alpha_{jj} - a_j{}^T(A_j - \lambda_j)^{-1}a_j. \tag{2.5}$$

This identity suggests an iterative method, namely

$$\lambda^{(n)} = \alpha_{jj} - a_j{}^T(A_j - \lambda^{(n-1)})^{-1}a_j. \tag{2.6}$$

We show that the method converges if the off diagonal terms are small in comparison to the gap between the eigenvalues. It is convenient to think of the matrix A as $A = D + E$ where $D = (\mu_j \delta_{ij})$ is a diagonal matrix and E has components $e_{ij} = \int_0^\pi q y_i y_j dx$. We can bound the 2-norm of E in terms of the L^2 norm of q. If the L^2 norm of q is small, then the eigenvalues of A will be close to the eigenvalues of the comparison problem (2.1) as well as to those of the original problem (1.3).

LEMMA 1: *Consider the eigenvalues* $\mu_0 < \cdots < \mu_n$ *and eigenfunctions* $y_0, \ldots, y_n$ *of the comparison problem (2.1). Let* $\gamma = \min_{k,l} |\mu_k - \mu_l|$ *with* $k \neq l$ *and let* $C_l = \max|y_l(x)|$ *for* x *in* $[0,\pi]$ *and* $l = 0, 1, \ldots, n$. *Suppose the potential* q *of the Sturm-Liouville problem satisfies* $\| q \| < \gamma/(4C)$. *If* $\lambda^{(0)}$ *is equal to be* μ_j, *then the sequence* $\lambda^{(n)}$ *defined by equation (2.6) converges to the eigenvalue* λ_j *of* A.

Remark: If $h = H = 0$, then $\gamma = 1$, and $C \leq \sqrt{2\pi^{-1}}$. The eigenvalues, and consequently γ and C depend continuously on h and H. In our experiment $|h|$, $|H| < 1$, $\gamma > 1$, and $C < 1$.

Proof: Let $\lambda^{(n)}$ be the result of n iterations steps. We use the basic identity (2.5) and the definition (2.6) to get

$$\lambda_j - \lambda^{(n)} = -a_j{}^T((A_j - \lambda_j)^{-1} - (A_j - \lambda^{(n-1)})^{-1})a_j$$

By factoring, we rewrite the right hand side as

$$a_j{}^T(A_j - \lambda_j)^{-1}(A_j - \lambda^{(n-1)})^{-1}(\lambda_j - \lambda^{(n-1)})a_j$$

Let $e_n = |\lambda_j - \lambda^{(n)}|$. We estimate and obtain the inequality

$$e_n \leq \|a_j\|^2 \|(A_j - \lambda_j)^{-1}\| \|(A_j - \lambda^{(n-1)})^{-1}\| e_{n-1} = M_n e_{n-1} \tag{2.7}$$

where the norms are the appropriate 2-norms. To prove the convergence of the method we use an induction argument to show that $M_n < \frac{1}{2}$ for all n.

We first need a bound for the norm of matrix E. Let k be fixed, $0 \leq k \leq n$, and set $c_l = \int_0^\pi q y_k y_l \, dx$. Thus $e_{kl} = c_l$ for $0 \leq l \leq n$. We can expand the function $q y_k$ in terms of the basis functions $\{y_l\}$, i.e. $q y_k = \sum_0^\infty c_l y_l$. By using Parseval's identity, we get the inequality $\sum_0^n e_{kl}{}^2 \leq \sum_0^\infty c_l{}^2 \leq \|q y_k\|^2 < \gamma/4$. This estimate is valid for all k. Therefore, the 2-norms of the row vectors of and thus the 2-norm of E are less than $\gamma/4$. It follows that a_j has norm less that $\gamma/4$. Since A_j is of the same form as A, we obtain the two bounds

$$|\nu_k(A_j) - \mu_k| < \gamma/4 \qquad |\lambda_j - \mu_j| < \gamma/4 \tag{2.8}$$

where $\nu_k(A_j)$ are the eigenvalues of A_j with $0 \leq k \leq n$ and $k \neq j$. By using these inequalities we can estimate the norms of $(A_j - \lambda_j)^{-1}$ and $(A_j - \lambda^{(n-1)})^{-1}$.

Let λ be any number. Then the eigenvalues of A_j can be bounded away from λ by

$$|\nu_k - \lambda| \geq |\mu_k - \mu_j| - |\mu_k - \nu_k| - |\lambda_j - \mu_j| - |\lambda_j - \lambda| \tag{2.9}$$
$$> \gamma - \gamma/4 - \gamma/4 - |\lambda_j - \lambda| = \gamma/2 - |\lambda_j - \lambda|.$$

The matrix $(A_j - \lambda I)^{-1}$ is symmetric with eigenvalues $(\nu_k - \lambda)^{-1}$. Hence, the matrix has norm $= \max_k (\nu_k - \lambda)^{-1} = (\min_k (\nu_k - \lambda))^{-1}$. By replacing λ with λ_j and μ_j in the inequalities (2.9) and (2.8), we see that $\|(A_j - \lambda_j)^{-1}\| < 2/\gamma$ and $\|(A_j - \mu_j)^{-1}\| < 4/\gamma$.

We chose $\lambda^{(0)}$ to be μ_j, hence the constant M_1 satisfies

$$M_1 = \|a_j\|^2 \|(A_j - \lambda_j)^{-1}\| \|(A_j - \lambda^{(0)})^{-1}\|$$
$$< (\gamma^2/16)(2/\gamma)(4/\gamma) = 1/2.$$

Suppose that $M_k < \frac{1}{2}$ for $k < n$. Since $e_{n-1} = M_{n-1} e_{n-2}$, this implies that $e_{n-1} \leq e_{n-2} \leq \ldots \leq e_0 < \gamma/4$. Therefore $|\nu_k - \lambda^{(n-1)}| > \gamma/2 - e_0 > \gamma/4$. Hence, $\|(A_j - \lambda^{(n-1)})^{-1}\| < 4/\gamma$. We use this and our previous bounds in the inequality (2.7) to obtain the desired result $M_n < \frac{1}{2}$.

3: The inverse method

In this section we present a numerical method for calculating the density profile and S-wave velocity in the upper mantle. We assume that the density and S-velocity are known in the lower mantle. The terms upper and lower mantle are defined in Section 1. We require as data a finite number of toroidal modes which correspond to two angular orders l_1 and l_2, and the leading term in the asymptotic expansions of the eigenvalues of these two spectra.

We first perform the Liouville transformation to the given data as described in Section 1. This transforms the equations for the toroidal modes with angular orders l_1 and l_2 to two Sturm-Liouville problems of the form (1.3) where the potentials $q_1(x)$ and $q_2(x)$ are known only in the interval $[\pi/2,\pi]$. We reconstruct the potentials $q_1(x)$ and $q_2(x)$ for $0 \le x \le \pi/2$ and from these functions determine the density and S-velocity in the upper mantle.

We now restrict our attention to a numerical method for the inverse problem for equation (1.3). In particular we show how we find the potential $q(x)$ in the interval $[0,\pi/2]$ and the constant h in the left boundary condition. The data is the potential q for $\pi/2 \le x \le \pi$, the constant H in the right boundary condition, the leading term in the asymptotic expansion of the eigenvalues and a finite set of eigenvalues of the differential equation.

The basic idea is to approximate the unknown potential by a finite linear combination of suitably chosen functions and then find the coefficients of the approximation. By equating the eigenvalues of the differential equation with the eigenvalues of the matrix A defined by equation (2.4) and then using the basic identity (2.5), we obtain a system of nonlinear equations which involve the coefficients of our approximation of the potential. We solve the nonlinear

equation by iteration. We make use of a result by Hald [27] to define the appropriate basis functions which together with the coefficients define our approximation of the potential.

We do not work directly with equation (2.2) but consider instead the problem

$$-y'' + (q - q_0)y = \lambda y$$
$$y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0$$

where $q_0 = \pi^{-1}\int_0^\pi q dx$ is the average of the potential. The modified potential has a smaller L^2 norm and its average is zero. The given eigenvalues must also be shifted by the amount q_0. The value q_0 is not known, but is computed during the iteration.

As in the direct method we use a linear combination of the eigenfunctions $\{y_j\}$ of the comparison problem (2.1) to approximate the eigenfunction of the unknown equation. It is convenient to write the normalized eigenfunction y_j as $u_j / \sqrt{\rho_j}$ where u_j is given by equation (2.2) and $\rho_j{}^2 = \int_0^\pi u_j^2 dx$ is the normalizing constant. The value of the constant h which appears in the left boundary condition of both the original equation and the comparison problem is not known, but is obtained by iteration. Hence the basis functions $\{y_j\}$ may change in each iteration. This complicates the method from a theoretical and computational point of view, but increases the rate of convergence.

In the following derivation of the nonlinear equations, we assume that h, q_0 and the matrix A are fixed. Let $\lambda_0 < \cdots < \lambda_m$ be the given eigenvalues of the differential equation and assume that $\lambda_j - q_0$ are eigenvalues of the matrix A, see equation (2.4). Then our basic identity (2.5) gives

$$\lambda_j - q_0 = \mu_j + \rho_j{}^{-1}\int_0^\pi (q - q_0)u_j{}^2 dx - a_j{}^T(A_j - (\lambda_j - q_0))^{-1}a_j \ .$$

Since the average of the potential $q-q_0$ is zero, we can split the integral into two parts and obtain

$$\lambda_j - q_0 = \mu_j + \rho_j^{-1}\int_0^{\pi/2}(q-q_0)(u_j^2-\tfrac{1}{2})dx + \int_{\pi/2}^{\pi}(q-q_0)(u_j^2-\tfrac{1}{2})dx$$
$$- a_j^T(A_j - (\lambda_j - q_0))^{-1}a_j \ .$$

By reordering the terms in this equation we get the identity

$$\rho_j^{-1}\int_0^{\pi/2}(q-q_0)(u_j^2-\tfrac{1}{2})dx = \tag{3.1}$$
$$\lambda_j - \mu_j - q_0 - \rho_j^{-1}\int_{\pi/2}^{\pi}(q-q_0)(u_j^2-\tfrac{1}{2})dx + a_j^T(A_j - (\lambda_j - q_0))^{-1}a_j.$$

We denote the left hand side of the equation by γ_j.

If the constants h and H in the boundary conditions vanish, the motivation for the method becomes more apparent. In this case $y_0 = \sqrt{1/\pi}$ and $y_j = \sqrt{2/\pi}\cos(jx)$ for $j>0$. Thus

$$\gamma_j = \frac{2}{\pi}\int_0^{\pi/2}(q-q_0)(\cos^2(jx)-\tfrac{1}{2})dx = \frac{1}{2\pi}\int_0^{\pi}(q(x/2)-q_0)\cos(jx)dx$$

and we note that $4\gamma_j$ are the Fourier coefficients of the cosine expansion of $q(x/2)-q_0$ for $0\le x\le\pi$. If the γ_j's are known we can expand $q(x)-q_0$ in the interval $[0,\pi/2]$ in terms of $u_j(2x)$ and get a truncated Fourier series as our approximation.

In the general case the functions $y_j^2-\tfrac{1}{2}$ are complete on the interval $[0,\pi/2]$, but not orthogonal. Thus it is more difficult to determine the basis functions for the expansion. Hald [27] has found the set of functions which are biorthogonal to the squares of the eigenfunctions and we use these functions in our expansion for the potential.

LEMMA 2: *Let u_j be the eigenfunctions of equation (2.1) with $u_j(0)=1$ and let ρ_j be the normalizing constants. Assume that $\sum_1^\infty|\gamma_j|$ is bounded. If q is*

integrable and

$$\gamma_j = \rho_j^{-1}\int_0^{\pi/2} q(x)(u_j^2-\tfrac{1}{2})dx \quad .$$

then

$$q(x)-q_0 = 4\sum_1^{\infty}\gamma_j w_j(2x)$$

almost everywhere where

$$w_j(x) = u_j(x)-he^{hx}\int_x^{\pi} e^{-ht}u_j(t)dt \quad .$$

For a proof see Hald [27].

We now describe one iteration of the inverse method. Assume that initial values of h and the coefficients $\gamma_0, \cdots, \gamma_m$ are given. For the first iteration we let $q_0 = A - 2\pi^{-1}(h + H)$ where A is the leading term in the asymptotic expansion of the eigenvalues. The eigenvalues and eigenfunctions of the comparison problem are computed for the current value of h. We then determine the matrix A and evaluate the right hand-side of equation (3.1) for each j, $0 \le j \le m$, to update the value of the coefficient γ_j. The new approximation to the potential q is then integrated and this gives the new value of q_0. Finally we let $h = (\pi/2)(A-q_0) - H$.

To solve the inverse problem for the mantle we must reconstruct two potentials with different angular orders. We get the constant h in the first inversion. The second inverse problem is slightly different. When h is known the average of the potential is determined by the asymptotic constant and equation (1.8). In this case we do not require the lowest eigenvalue, but set

$$\gamma_0 = \frac{\pi q_0 - 4\sum_0^m \int_0^{\pi}\gamma_j w_j\, dx}{\int_0^{\pi} w_0 dx}. \tag{3.2}$$

This guarantees that equation (1.8) is satisfied. The coefficients γ_j for $j \ge 1$ are

solved in the manner described above.

Let us assume that we have solved the inverse Sturm-Liouville problem, have obtained two potentials q_1 and q_2 with angular orders $l_2 > l_1$. We now complete the description of the inverse method for the mantle. By using equations (1.5) and (1.6) we determine $v = f''/f$ and $w = \beta^2/r$ from our approximations of q_1 and q_2. The density and S-velocity are found by solving two differential equations which involve v and w.

Consider the second order differential equation

$$f'' = vf. \tag{3.3}$$

The function $f = r^2\sqrt{\rho\beta}$ is determined by our data in the interval $[\pi/2, \pi]$. Since both f and f' are assumed continuous we can use the values of f and f' at $\pi/2$ as initial data for the differential equation (3.3) and get f for $0 \le x \le \pi/2$.

To determine the density in the upper mantle we must first determine the velocity structure. Let $\varphi(x) = R - r(x)$. It follows from our definition of x and the inverse function theorem that

$$\frac{d\varphi}{dx} = -\frac{dr}{dx} = K\beta .$$

Since $\beta = r\sqrt{w}$ and $r = R - \varphi$ we have the linear differential equation

$$\frac{d\varphi}{dx} = (R - \varphi)\sqrt{w} \tag{3.4}$$

$$\varphi(0) = 0$$

which we solve for $\varphi(x)$ for $0 \le x \le \pi/2$. We then set $\beta = (R - \varphi)\sqrt{w}$ and $\rho = f^2/(r^4\beta)$. This gives the density and velocity models as a function of x.

The final step of the method is to describe the density and velocity as functions of radius. The function φ obtained from solving equation (3.4)

defines the inverse transformation, i.e. $r(x) = R - \varphi(x)$. Recall from Section 1 that the upper mantle is defined by r, where $r_0 \leq r \leq R$. Since $r(0)=R$ and $r(\pi/2)=r_0$, we have the desired profiles given by $\rho(r(x))$ and $\beta(r(x))$ in the upper mantle.

4: Numerical Implementation

In our numerical experiments we test the method on published earth models. Typically these models give the density and the velocities of the compression and shear waves at certain radii throughout the earth. We use the values in the lower mantle and find the values in the upper mantle. Thus our data contains radius points $R_c = r_1 < \cdots < r_{n_0} = r_0$, the corresponding density values $\rho_1, \ldots, \rho_{n_0}$ and the S-velocities $\beta_1, \ldots, \beta_{n_0}$ and we find the density and velocity at the points $r_{n_0+1} < \cdots < r_n = R$ in the upper mantle. This is not the only approach to obtain the density in the lower mantle. In many earth models the Adams-Williamson equation, which relates the change of density to the velocities of the compression and shear waves, is satisfied below a depth of approximately 984 km. Hence if the velocity is given in the lower mantle one can solve the Adam-Williamson equation to obtain the density distribution there. See Bullen [13], pages 154, 162. We have not taken this approach, as our method is not applicable to real data at this time.

To implement the Liouville transformation we set $x_1 = \pi$ and $x_j = x_1 - I(j)$, where $I(j)$ is an approximation to $K^{-1}\int_{r_1}^{r_j} \beta^{-1} dr$. Then x_j corresponds to the mesh point r_j but is in terms of the variable x. For the numerical integration we use Simpson's rule for a non-equidistant mesh. Note that $\pi = x_1 > \cdots > x_{n_0} = \pi/2$. With this ordering the values of the arrays correspond, i.e. β_j corresponds to both $\beta(r_j)$ and $\beta(x_j)$. We have now transformed the problem to the interval $0 \le x \le \pi$ and will describe the numerical implementation of the inverse Sturm-Liouville method.

In the method we assume that the two potentials in equation (1.3) are known $(\pi/2, \pi)$. To find the potentials in $(\pi/2, \pi)$ we evaluate $v = f''/f$ and

$w=\beta^2/r^2$ and then use equation (1.4). We find the function values at the points $x_1, \cdots, x_{n_0}$. Both f and w come directly from the data; we set $f_j = r_j^2\sqrt{\rho_j\beta_j}$ and $w_j = \beta_j^2/r_j^2$. To approximate f'', we interpolate the function at the meshpoints $x_1, \cdots, x_{n_0}$ by a cubic spline and calculate the second derivative of the spline at these points. Since we test the method on known models, data is given throughout the mantle. Thus $f(x)$ is interpolated at points throughout $(0,\pi)$ and our spline is defined throughout this interval. We use the IMSL routine ICSSCU with the not-a-knot endpoint condition. This condition requires the continuity of the third derivatives at the second and next to last knot. If we instead use a natural spline f'' is zero at both 0 and π and this can create large errors in the potentials. In addition we use the spline to evaluate $H = -f'/f$ at $x=\pi$ and f and f' at $x=\pi/2$. The constant H is needed for the inverse Sturm-Liouville method and f and f' are used in the inverse Liouville transformation.

An essential step of the inverse method is the ability to calculate the eigenvalues of the comparison problem (2.1). The eigenvalues are the zeros of the Wronskian, see equation (2.3). To find the roots we must give upper and lower bounds for the eigenvalues such that each interval contains exactly one eigenvalue. There are five cases and the bounds depend on h and H, see Table 1 which is due to Hald [27]. If the eigenvalue is small in absolute value, a straight forward evaluation of the Wronskian becomes numerically unstable. We avoid this difficulty by approximating the function by the first two terms of its Taylor series expansion if $|\mu|<10^{-4}$. We use the algorithm ZERP, due to Kahan, to find the zeros of the Wronskian. The algorithm is based on the secant method. We do not solve directly for the eigenvalue, but instead find $k=\sqrt{|\mu|}$ and then determine the sign of μ from Table 4.1. Once the eigenvalues

are known, the eigenfunction u_j can be evaluated at any point by equation (2.2). When $|\mu_j|<10^{-4}$ we approximate the eigenfunction by the first two terms of its Taylor series.

	Case	μ_0	μ_1	μ_2	μ_3
1	$h+H>.01$ $h+H+\pi hH\geq 0$	$(-.01,1)$	$(1,4)$	$(4,9)$	$(3^2,4^2)$
2	$h+H>.01$ $h+H+\pi hH<0$	$(-\lvert hH\rvert,.01)$	$(1,4)$	$(4,9)$	$(3^2,4^2)$
3	$\lvert h+H\rvert\leq .01$	$(-A^2,.04)$	$\left(\frac{1}{4},\frac{9}{4}\right)$	$\left(\frac{9}{4},\frac{25}{4}\right)$	$\left((3-\frac{1}{2})^2,(3+\frac{1}{2})^2\right)$
4	$h+H<-.01$ $h+H+\pi hH<0$	$(-A^2,-B^2)$	$(-B^2,1)$	$(1,4)$	$(2^2,3^2)$
5	$h+H<-.01$ $h+H+\pi hH\geq 0$	$(-A^2,-B^2)$	$(-B^2,.01)$	$(1,4)$	$(2^2,3^2)$

Table 4.1. Bounds for the eigenvalues of equation (2.1):

$$A=1+|h+H|+|hH|,\ B=\max(|h|,|H|).$$

In the calculations of the matrix A and the coefficients γ_j many numerical integrations must be performed. Our calculation of the integrals differs from Hald's. We have two methods for approximating the integrals. If the integrand can be evaluated at any point, we take intervals of length $\pi/10$ and use Gaussian quadrature with five points in each subinterval. This gives high accuracy. In particular, we use this method for all integrals over the interval $(0,\pi/2)$ and when the basis function w_j must be integrated from 0 to π. When the integrand is given on a discrete set of points, we use Simpson's rule for a non-equidistant mesh. The method is exact for all polynomials of degree three or less. We use this method for the integrals over the interval $(\pi/2,\pi)$ where our data is given on a mesh. In all integrals involving the potential Hald inter-

polates the potential by a piecewise linear function and then integrates exactly. In many cases we integrate exactly. For example, we evaluate the normalizing constant $\rho_j^2 = \int_0^\pi u_j^2 dx$ and the basis function $w_j(x) = u_j(x) - e^{hx}\int_x^\pi e^{-ht} u_j(t)dt$ exactly. The calculation becomes unstable if the eigenvalue μ_j is small. When $|\mu_j| < 10^{-4}$, we approximate the integrand by its truncated Taylor series of degree three and integrate these exactly.

We find the term $a_j^T(A_j - (\lambda_j - q_0))^{-1}a_j$ by solving the equation $(A_j - (\lambda_j - q_0))^{-1}s = a_j$ by Gaussian elimination with partial pivoting. We then compute the inner product of a_j and s in the obvious way.

To prepare for the inverse Liouville transformation and to find the velocity, we solve the differential equation (3.3) for φ by the classical Runge-Kutta method. Our step size is $\pi/100$. Note that the function w can be evaluated at all points in $(0,\pi/2)$, see equation (1.6) and Lemma 2. The calculation gives us the function φ at the points $x = k\pi/100$ for $0 \le k \le 50$. From the relation $\beta = (R - \varphi)\sqrt{w}$, we find β on the this equidistant mesh.

We are interested in the density and velocity at the particular radii $r_{n_0+1}, \cdots, r_n$. We use the values of φ on the equidistant mesh and find the x_k such that $r_k = R - \varphi(x_k)$ by linear interpolation. This gives the non-equidistant mesh of x coordinates which corresponds to the given radii. Finally, we compute the velocity β at the desired points by linear interpolation.

Next we write the equation (3.3) as a first order system, solve it by a second order Runge-Kutta method and obtain f on the non-equidistant mesh. Here we divide the interval (x_k, x_{k+1}) into 10 intervals of equal length and solve the system in each subinterval. Finally, we get the density ρ at the desired radii by setting $\rho_j = f_j^2/(r_j^4\beta_j)$. This completes the reconstruction process for the density and S-velocity in the upper mantle.

5: Numerical Experiments

We reconstruct several earth models to test the accuracy and stability of our method. In this section we describe our experiments and discuss the results. There are many factors which affect the accuracy of our method. Perhaps the greatest of these is the smoothness of the model. If the density or the velocity have a quickly changing gradient then the potential in the transformed equation will have a peak with small oscillations nearby. In this case the potential cannot be well approximated by a linear combination of a small number of our basis functions. This creates error in the computed solution. We have observed that the accuracy increases if the number of given eigenvalues increases and decreases slightly as the spectrum number of the eigenvalues increases. The error is also increased if the two potentials are reconstructed from a different number of eigenvalues and the approximations for the potentials do not have the same number of terms.

Our first example is a model with constant density and constant velocity. We let ρ = 4 gm/cm^3, β = 6 km/sec, the radius of the earth R = 6371 km and the radius of the core R_c = 2898 km. Since ρ and β are constant we can differentiate $f = r^2\sqrt{\rho\beta}$ to find the potential q and the boundary conditions h and H explicitly. We find the asymptotic constants for spectra 1 and 2 by integrating the corresponding potentials exactly. To perform the inversion we use the torsional modes ${}_0T_1, \cdots, {}_7T_1$ and ${}_1T_2, \cdots, {}_7T_2$. These values were provided by Hald. In this and all inversions we take the matrix A to be $2m$x$2m$ where m eigenvalues are given for the first spectrum and $m-1$ are given for the second spectrum. In this case A is a 16x16 matrix. The results of inversions using 4, 6, and 8 eigenvalues from spectra 1 and 2 are given in Table 5.1. By using the velocity and the modes ${}_0T_2, \cdots, {}_7T_2$, Hald was able to recon-

struct ρ with a maximal error of .00003. Our error is greater. To study the error we ran the inversion using the exact values of β in the final calculation $\rho=f^2/r^4\beta$. The maximal error of ρ was reduced to .0006 when 15 eigenvalues were given. This indicates that the calculation of the velocity contributes more to the inaccuracy of the method than the calculation of the density.

Number of Eigenvalues	ρ	β
4	.078	.115
6	.049	.073
8	.038	.056

Table 5.1. Maximum error in the reconstruction of a constant earth: $\rho=4$, $\beta=6$.

For a more realistic example we use Model A as presented by Jeffreys [33], page 200. Model A consists of the velocities of the P- and S- waves by Jeffreys (1939) and the associated density distribution by Bullen (1940). Our data is at radius points r_j which lie between the core-mantle boundary and the Mohorovicic discontinuity. This discontinuity, whose position we denote by r_m, occurs at a radius of 6338 km. The radius of the core in Model A is .548 r_m, but the first data point in the mantle is at the radius .55r_m. We extend our model in a continuous manner to the crustal region and to the core-mantle boundary as explained below.

The Model A has a discontinuity in the gradients of the density and velocities at a depth of 413 km. To avoid a large peak in the potential we must smooth the model before attempting a reconstruction, see figure 5.1. The smoothing is done in two steps. First we approximate ρ and β at the given data points by cubic splines. We compute the least squares approximation by the

IMSL routine ICSFKU. The routine finds the spline which minimizes the normalized L^2 norm

$$\| f-g \|^2 = \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{x_n - x_0} \frac{(f(x_i)-g(x_i))^2+(f(x_{i-1})-g(x_{i-1}))^2}{2}.$$

After much experimentation, we chose the 8 knots used by Hald. They are given by r/r_m = 1, .944, .94, .936, .88, .835, .587 and .55. The maximal errors in the approximations were .005 for β and .004 for ρ while the normalized L^2 errors were both less than .003. To obtain a good approximation to the data several knots are needed close to the discontinuity which occurs near a depth of 413 km. By extrapolating the splines we get the values of ρ and β between the Mohorovicic discontinuity and the surface and near the core-mantle boundary.

To further smooth the model, we multiply the splines by triangular functions with base 2N and centered at the data point r_j. We then integrate over the base and use this averaged value as the function value at r_j. This is done for each r_j. When necessary, we evaluate the spline at radii $r>R$ or $r<R_c$. Table 5.2 and figure 5.2 show the change to Model A due to the smoothing.

Smoothing	N =	50	100	200	300
Normalized	ρ	.003	.005	.012	.020
L^2	β	.003	.006	.015	.027
Maximum	ρ	.014	.032	.065	.094
Norm	β	.017	.037	.074	.106

Table 5.2. Change to Model A due to smoothing.

We now determine the data for the inversion and attempt the inversion. We need the data in the lower mantle, however, the boundary of the lower mantle is not known. We find $K=\frac{1}{\pi}\int_{R_c}^{R}\beta^{-1}\,dr$ by integrating throughout the

mantle. To find r_0 where $r<r_0$ defines the lower mantle, we calculate $\int_{R_c}^{r_j}\beta^{-1}dr$ and use linear interpolation to determine r_0 where $r_{j-1}<r_0<r_j$.

We solve the direct problem for the torsional modes of the model to complete the data. To exclude systematic errors we calculate the eigenfrequencies by a method which is not based on the Rayleigh quotient. Specifically, we use the equation (1.1) for the torsional modes of the earth. We rewrite the equation as a 2 point boundary value problem for a system of first order differential equations by letting $y_1 = y$ be the eigenfunction and $y_2 = \mu(y_1 - y_1'/r)$. The boundary conditions are then $y_2(R_c) = y_2(R) = 0$. See Lapwood [39], page 101. We add the equations $y_3' = y_1^2$ and $y_4' = 0$ with the boundary conditions $y_3(R_c) = 0$ and $y_3(R) = (R - R_c)/2$. Thus y_4 is the eigenvalue $\lambda = K^2\omega^2$ and y_3 fixes the norm of the eigenfunction and completes our set of boundary conditions. See Keller [37], Chapter 3. We use the NAG routine D02RAF to solve the boundary value problem. The routine implements a method by V. Pereyra [45]. It uses a finite difference method with deferred correction and Newton iteration. We choose a mesh and provide an initial guess on this mesh. The mesh is then refined by D02RAF to improve the solution. In all our experiments the number of points remains below 350. The density and velocity are given by the spline interpolants and we calculate the rigidity $\mu = \beta^2\rho$ from these splines.

The routine failed to solve the problem as given and we found it necessary to use the continuation option on a sequence of three families of equations. The families depend on the parameter ε which is increased from 0 to 1 in steps of .1. The first family has the equation $y'' = \lambda y$ for $\varepsilon=0$ and the equation for the torsional modes of a constant earth with angular order l_1 for $\varepsilon=1$. The exact solution for $\varepsilon=0$ is a linear combination of $\sin nxl$ and $\cos nxl$ and is

given as the initial guess. For $\varepsilon=0$, the remaining families correspond to the final equation of the preceding family and the final solution of the preceding family is given as the initial guess. We continue from the constant earth to the given earth model for angular order l_1 and finally to the torsional modes of the given model with angular order l_2. This process is repeated for each wave number used in the inversion. Note that the initial guess determines the wave number of the eigenvalue. In all cases the routine estimated the absolute error in the computed eigenvalue to be less than 10^{-5}.

We obtain four different models by smoothing Model A with N=50, 100, 200 and 300 km. The corresponding eigenvalues are given in Table 5.3, together with the computed values of K, the asymptotic constants A_1 and A_2, the constants h and H, and the position of the lower mantle r_0. Note that the model with N=0 is obtained by interpolating the data by a cubic spline rather than by approximating the data by a spline. The interpolating spline was extrapolated to the surface and to the core-mantle boundary. We were unable to invert for this model using 15 eigenvalues. For each model we use the eigenvalues corresponding to the modes ${}_0T_1, \cdots, {}_7T_1$ and ${}_1T_2, \cdots, {}_7T_2$ in the inversion. The results of the inversions are shown in Table 5.4. Because the accuracy increase with the smoothness of the model, the reconstruction of the model with 200 km smoothing is closest to Model A.

Smoothing N =	0	50	100	200	300	300
Spectrum 1						
${}_0\lambda_1$	.0001	.0001	.0001	.0001	.0001	
${}_1\lambda_1$	1.3419	1.3417	1.3412	1.3400	1.3367	
${}_2\lambda_1$	4.2472	4.2481	4.2491	4.2524	4.2559	
${}_3\lambda_1$	9.0855	9.0867	9.0930	9.1160	9.1483	
${}_4\lambda_1$	16.2774	16.2732	16.2725	16.2710	16.2717	
${}_5\lambda_1$	25.4139	25.4066	25.3919	25.3459	25.3018	
${}_6\lambda_1$	36.3591	36.3510	36.3386	36.3013	36.2726	
${}_7\lambda_1$	49.2109	49.2145	49.2276	49.2557	49.2675	
A_1	.3137	.2786	.2777	.2763	.2753	
Spectrum 2						
${}_0\lambda_2$	.1302	.1301	.1301	.1276	.1275	
${}_1\lambda_2$	1.5331	1.5329	1.5324	1.5307	1.5276	
${}_2\lambda_2$	4.4169	4.4179	4.4190	4.4226	4.4267	
${}_3\lambda_2$	9.2582	9.2594	9.2656	9.2883	9.3203	
${}_4\lambda_2$	16.4482	16.4439	16.4429	16.4407	16.4404	
${}_5\lambda_2$	25.5800	25.5726	25.5578	25.5115	25.4673	
${}_6\lambda_2$	36.5233	36.5150	36.5028	36.4660	36.4322	
${}_7\lambda_2$	49.3744	49.3783	49.3917	49.4202	49.4322	
A_2	.4774	.4423	.4413	.4398	.4388	
h	.0631	−.0225	−.0232	−.0257	−.0300	
H	.5998	.5894	.5901	.5931	.5982	
K	148.5	148.4	148.4	148.5	148.5	
r_0	5095.0	5094.9	5094.8	5094.6	5094.1	

Table 5.3. Inversion data for the smoothed versions of Model A.

Smoothing	N =	50	100	200	300
Normalized	ρ	.067	.051	.017	.010
L^2	β	.083	.062	.019	.010
Maximum	ρ	.134	.102	.034	.041
Norm	β	.144	.108	.036	.055

Table 5.4. | Computed - Smoothed | in the reconstruction of the smoothed versions of Model A.

All remaining tests are done on our version of Model A smoothed with N=200 km. To illustrate the effect of increasing the number of eigenvalues we use eigenvalues associated with ${}_0T_1, \ldots, {}_7T_1$, and ${}_1T_3, \ldots, {}_7T_3$. We invert for the model using 4, 6 and 8 eigenvalues from spectrum 1 along with 3, 5 and 7 eigenvalues from spectrum 3. The results are shown in Table 5.5. The L^2 error steadily decreases, although the maximal error is greater for 6 eigenvalues than for 4.

Number of eigenvalues		4	6	8
Normalized	ρ	.042	.027	.017
L^2	β	.048	.035	.019
Maximum	ρ	.083	.105	.035
Norm	β	.093	.138	.037

Table 5.5. Errors in the reconstruction: spectra 1 and 3.

Next we compare the reconstructive powers of eigenvalues from spectra 1, 2, 3 and 4. Table 5.6 shows that the error increases slightly as the angular order increases. Recall that only one of the spectra determines the constant h in the left boundary condition. We will refer to the spectrum which determines h as the first spectrum in the inversion. Table 5.7 indicates that there

is an indication that a spectrum with large angular order cannot determine h as well as a spectrum with smaller angular order.

First Spectrum		1	1	1	4	2	3
Second Spectrum		2	3	4	1	3	2
Normalized	ρ	.017	.018	.018	.018	.018	.018
L^2	β	.019	.019	.019	.019	.019	.019
Maximum	ρ	.034	.035	.036	.041	.036	.034
Norm	β	.036	.037	.037	.036	.037	.036

Table 5.6. Errors in the reconstructions: 8 terms in the expansions for the potentials.

We will now look more closely at the error in the reconstructed potentials. We denote the potential associated with spectrum l by q_l and its reconstruction by $\tilde{q}_l$. The potential can be written as

$$q_l = v + (l+2)(l-1)w$$

where v and w are defined in Section 2. We note that the magnitude of w is less than that of the error in the reconstructed potentials, see figures 5.5 and 5.7. Yet we determine w from these reconstructed potentials. Moreover, a 10% error in the potentials is reduced to less than a 2% error in the reconstructed earth model.

Spectrum	1	2	3	4
h	-.02589	-.02584	-.02619	-.02644
Relative error	.06	.04	.18	.28

Table 5.7. The values of h computed with data from different spectra.

The coefficients in our expansion of the potentials are linear functionals. Hence the coefficient γ_j of the potential q_l can be written as

$$\gamma_j = \nu_j + (l+2)(l-1)\omega_j$$

where ν_j and ω_j are the corresponding coefficients of v and w in terms of our basis functions. Therefore we assume that the error in the reconstruction of the potential can be written as

$$q_l - \tilde{q}_l = e_v + (l+2)(l-1)e_w \tag{5.1}$$

where e_v and e_w are the errors in the reconstructions of v and w. The function $v = f''/f = q_1$ has a large peak, while w is smooth and slowly varying. One would expect to better approximate w than v by a finite expansion. Figure 5.7 shows that the errors in q_l are virtually identical for $l = 1, \ldots, 4$. If e_w were large, this would not be the case. We assume equation (5.1) and determine e_w and e_v from the various reconstructed potentials. In these calculations we use potentials reconstructed with spectrum 1 as the first spectrum. This guarantees that h and hence the basis functions are the same for all of the reconstructions. We let

$$e_w = \frac{(q_{l_2} - \tilde{q}_{l_2}) - (q_{l_1} - \tilde{q}_{l_1})}{(l_2 + l_1 + 1)(l_2 - l_1)} \tag{5.2}$$

$$e_v = \frac{(l_2+2)(l_2-1)(q_{l_1} - \tilde{q}_{l_1}) - (l_1+2)(l_1-1)(q_{l_2} - \tilde{q}_{l_2})}{(l_2 + l_1 + 1)(l_2 - l_1)} . \tag{5.3}$$

The results of the calculations with $l_1 = 1$ in (5.2) are shown in figures 5.8 and 5.9. We see that the errors are similar and we conclude that the various combinations of the spectra give the same coefficients.

Since we subtract the potentials from one another in the calculation of $w = (q_{l_2} - q_{l_1})/(l_2 + l_1 + 1)(l_2 - l_1)$, the error in v will cancel and we get a good approximation to w. This cancellation in the error is essential to the success of the method. In Section 7, we will see that the method is unstable if e_v is not independent of the spectrum.

To emphasize the importance of the cancellation of the error in v and show how the algorithm can break down, we invert for the Model A smoothed with N=200 km., using 8 coefficients in the expansion of q_1, but only 6 in the expansion of q_2. Specifically, we use the eigenvalues associated with ${}_0T_1, \cdots, {}_7T_1$ and ${}_1T_2, \cdots, {}_5T_2$. The reconstructed potentials and their errors are shown in figures 5.10 and 5.11. The differential equation for β involves the square root of w, but since $\tilde{q}_2 < \tilde{q}_1$ for some x, the inversion could not be completed to determine the earth model. We repeated this experiment but replaced the data from spectrum 2 with data from spectrum 4. The fourth potential is sufficiently large to keep the computed w positive, however the error was considerable. Thus our method works best if the number of coefficients in the expansions for the potentials are equal . This also suggests that information from spectra with angular orders that are far apart are better for the simultaneous inversion for ρ and β.

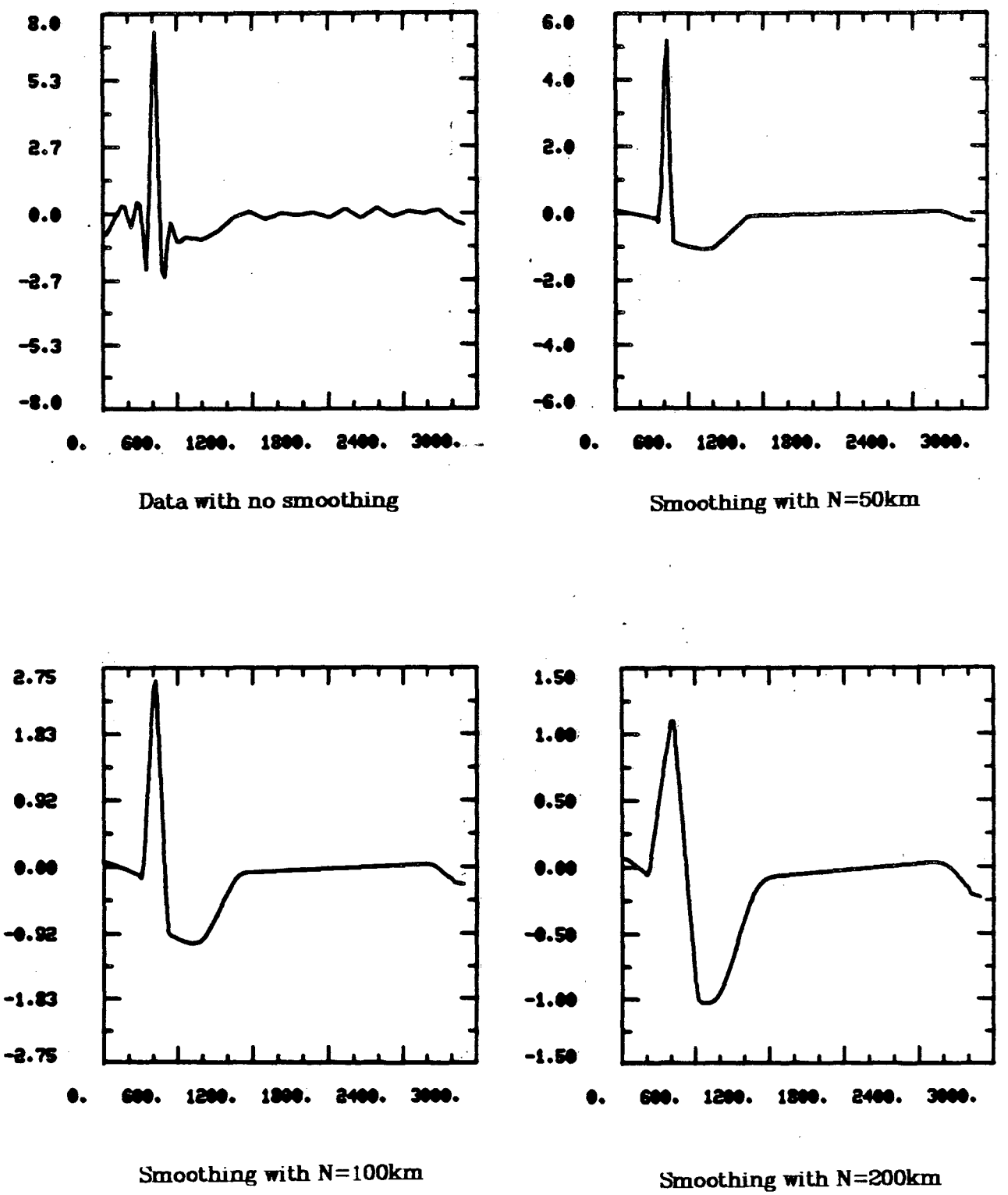


Figure 5.1. The potential as a function of depth for the smoothed versions of Model A. Note the difference in scaling.

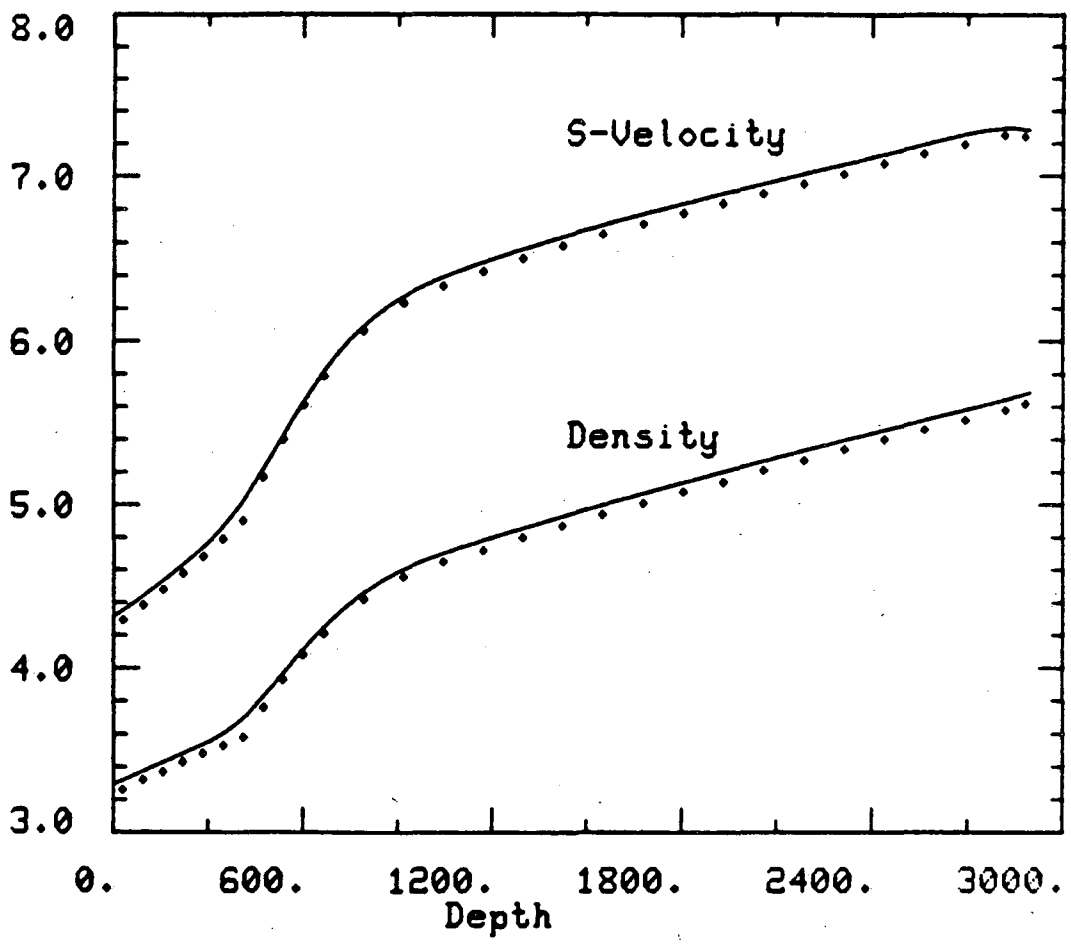


Figure 5.2. Solid Line : Density Change: N = 200 Km.

Dash : S-velocity change: N = 200 Km.

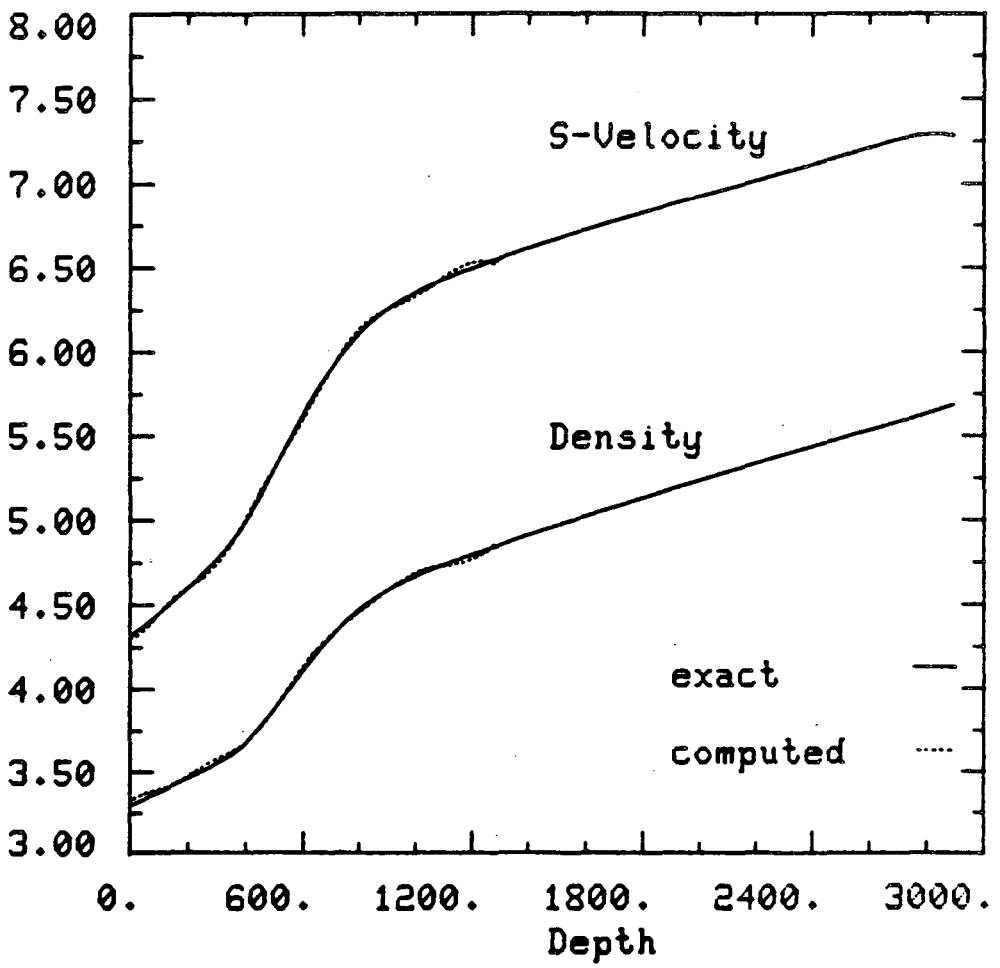


Figure 5.3. Model A: 200 Km smoothing. Reconstructed from spectra 1 and 3; 8 coefficients.

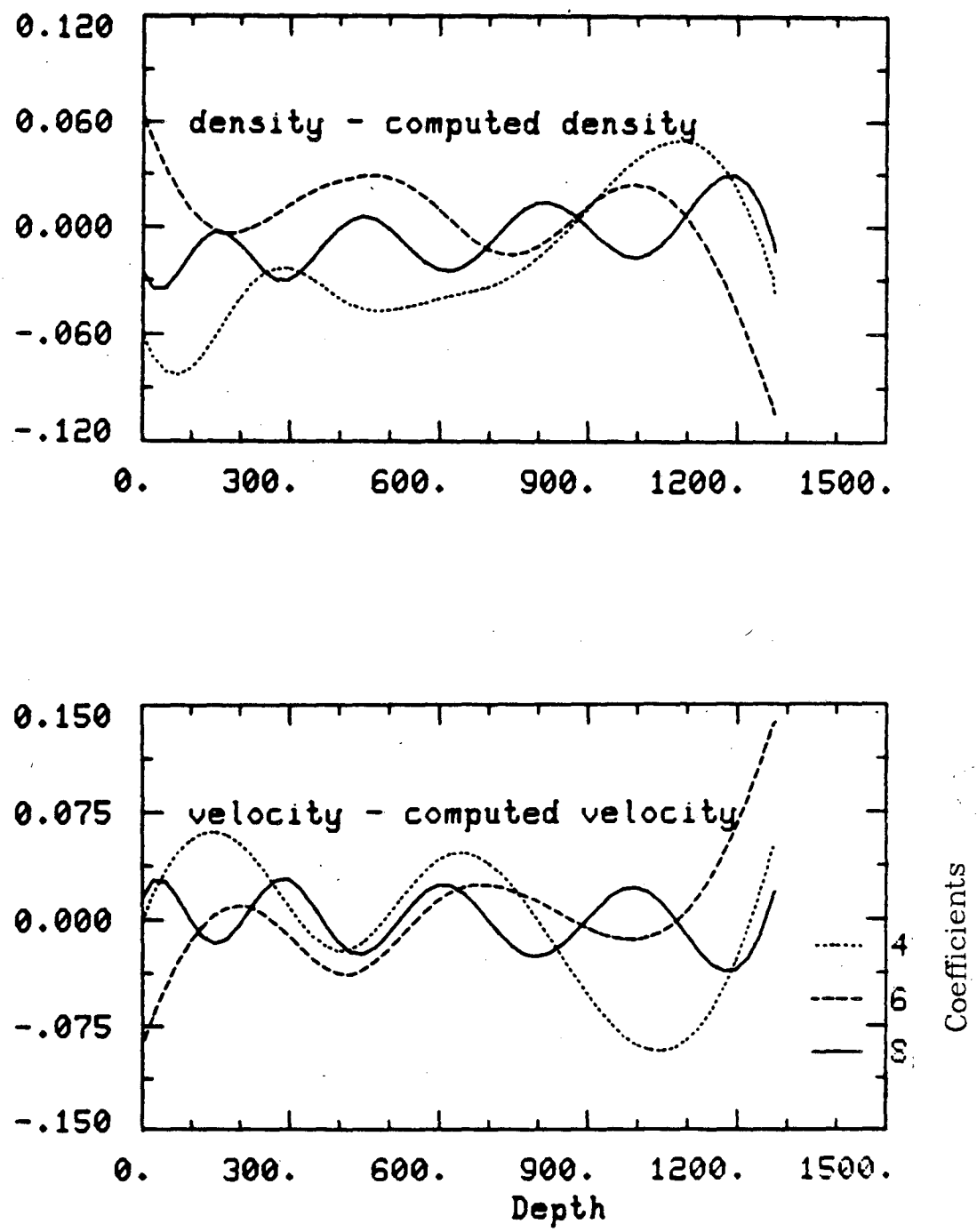


Figure 5.4. Errors in the computed model. Spectra 1 and 3; smoothed Model A, N = 200 Km.

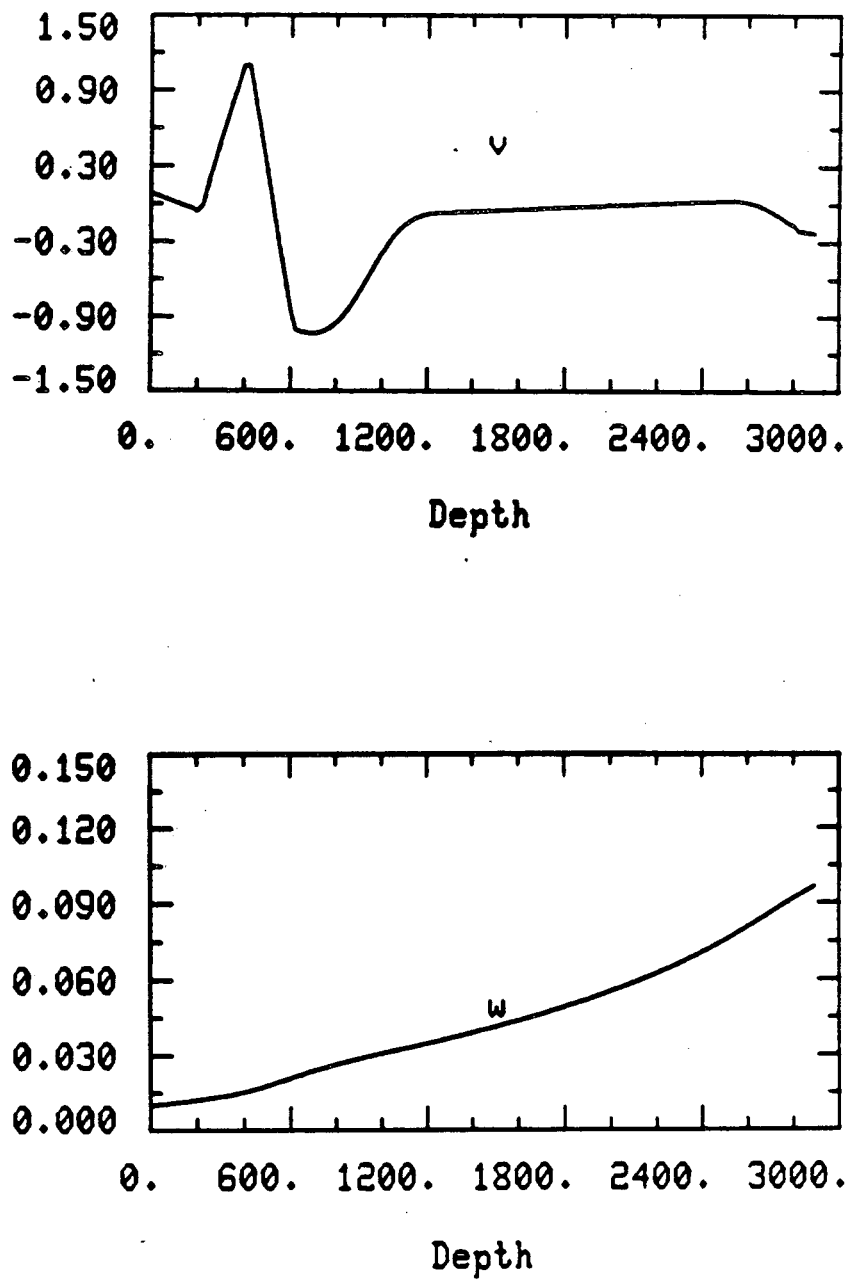


Figure 5.5. The functions v and w where the potential $q = v+(l+2)(l-1)w$; smoothed Model A, N = 200 Km.

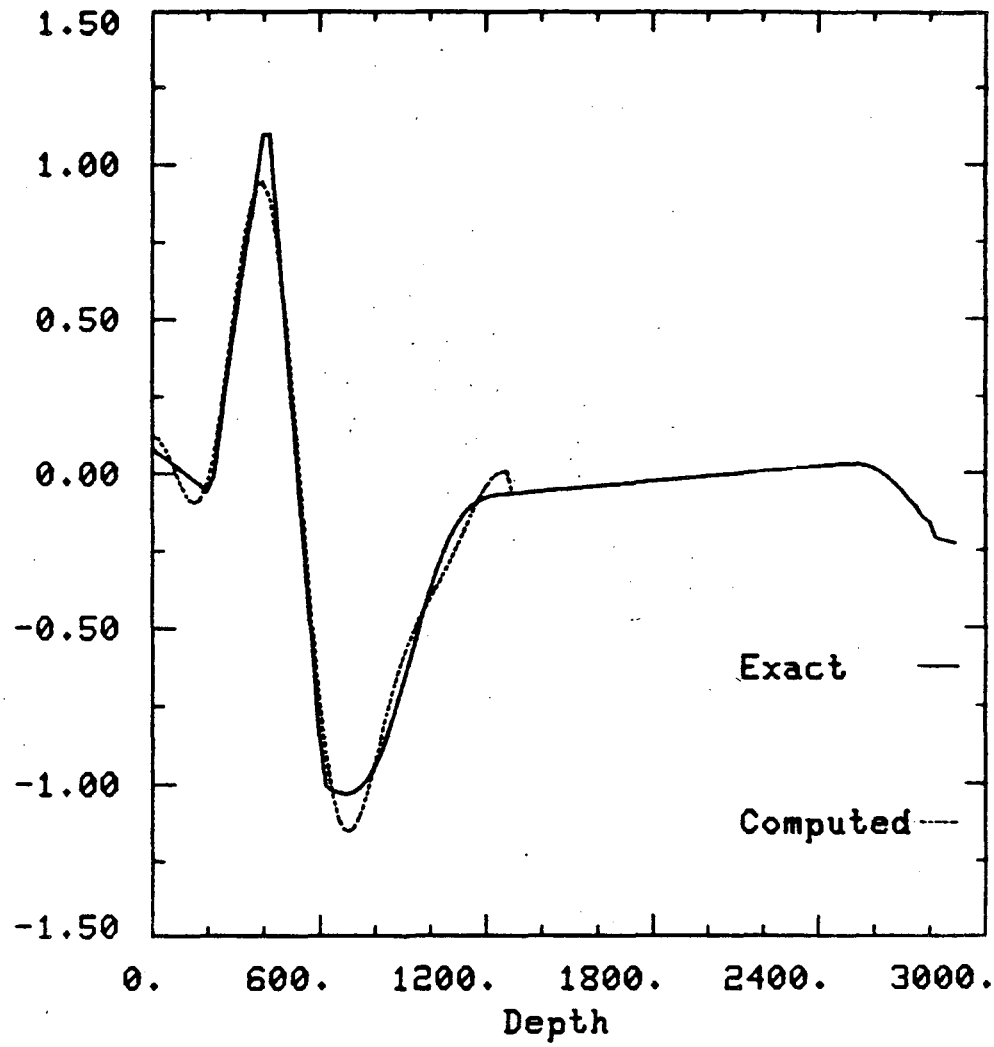


Figure 5.6. Potential q_1 for Model A with 200 Km smoothing; reconstructed with 8 coefficients.

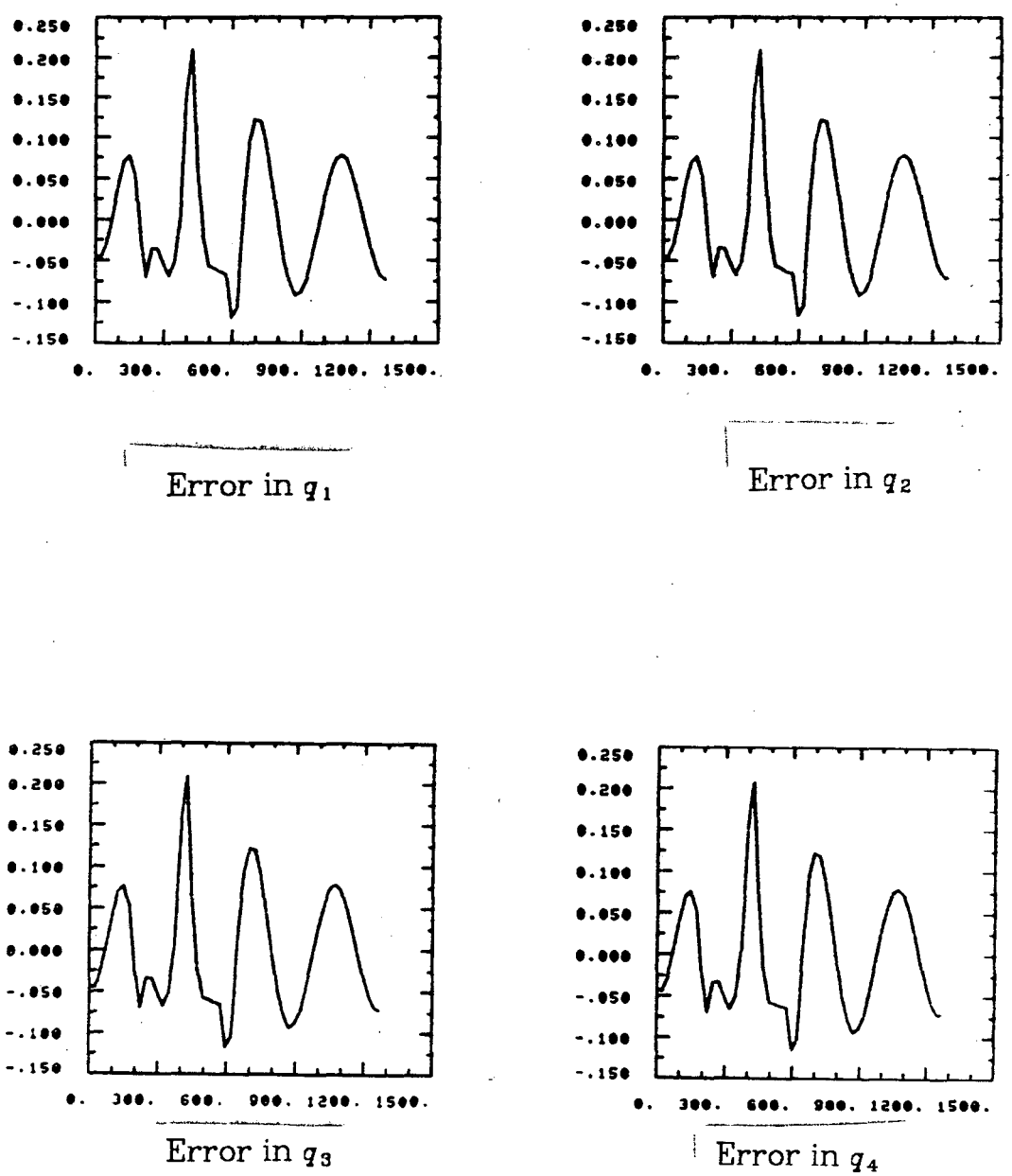


Figure 5.7. The potentials approximated with 8 terms in the expansion.

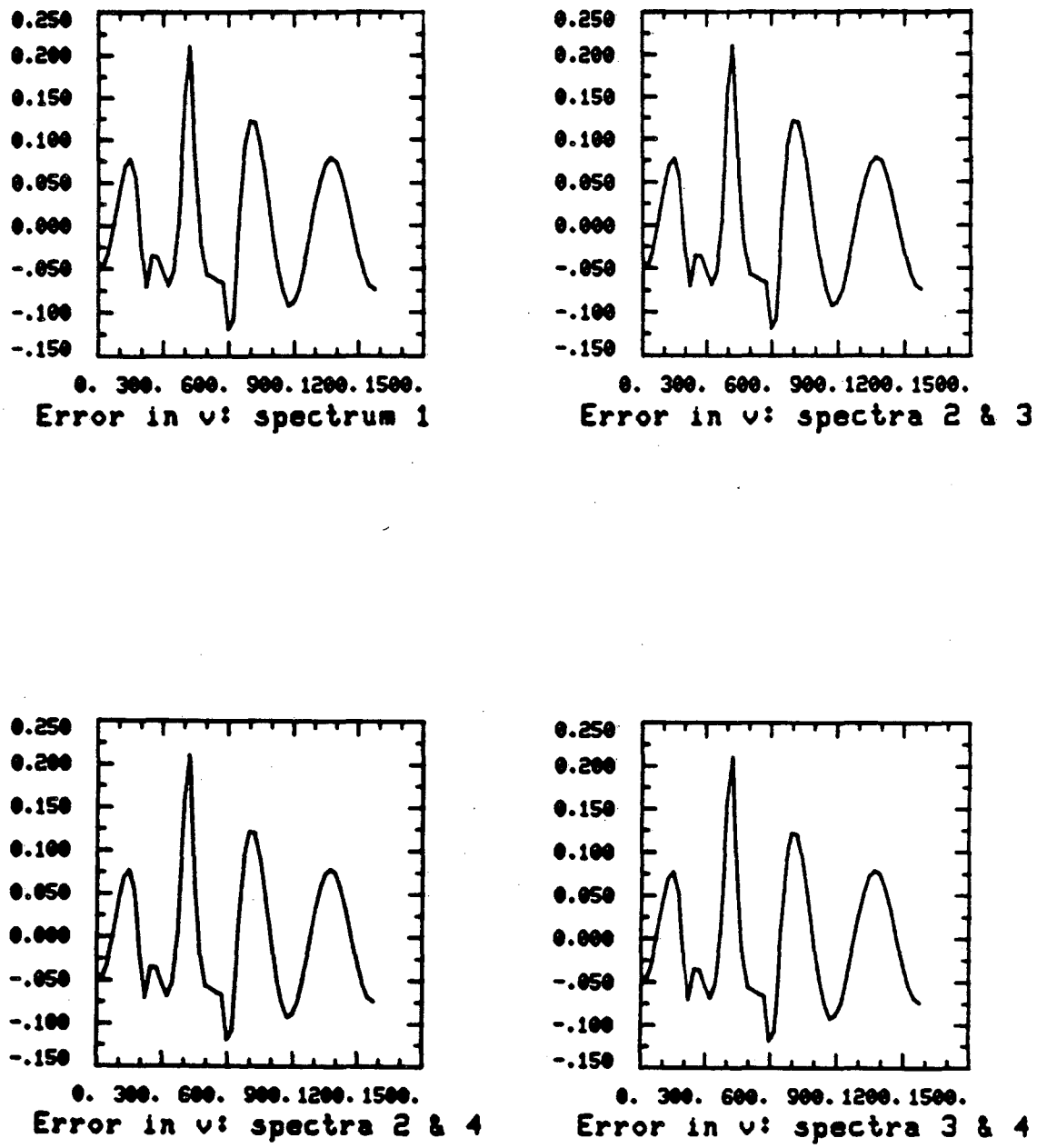


Figure 5.8. The estimated errors in the calculated v.

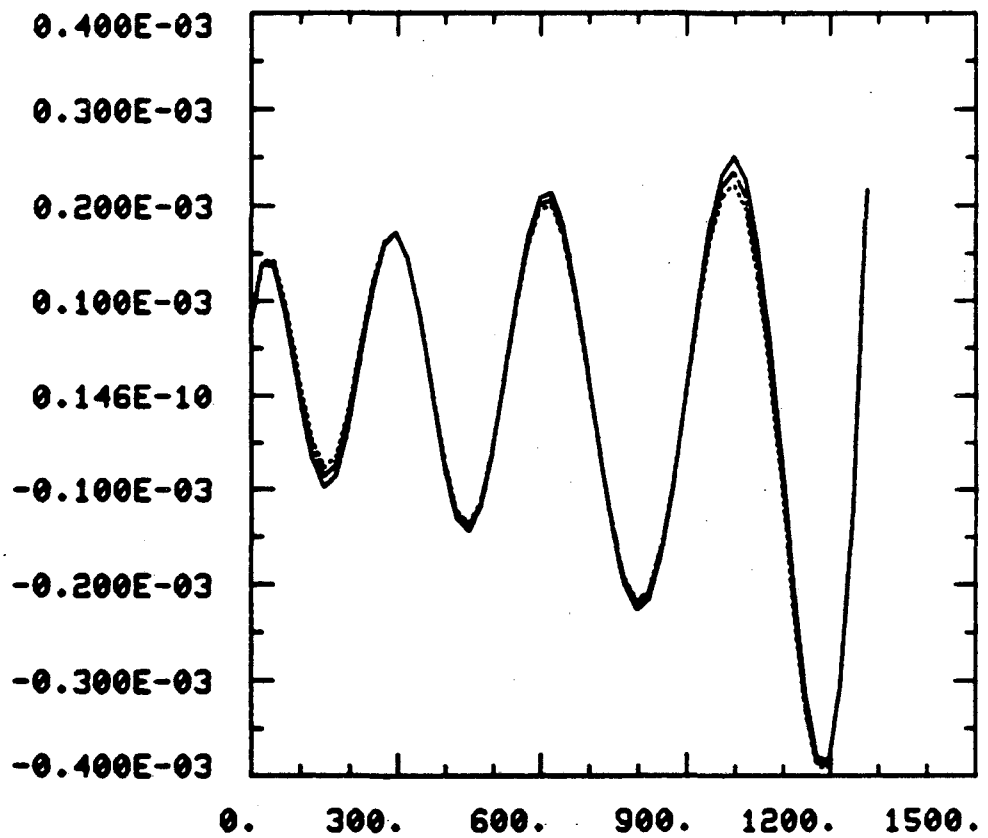
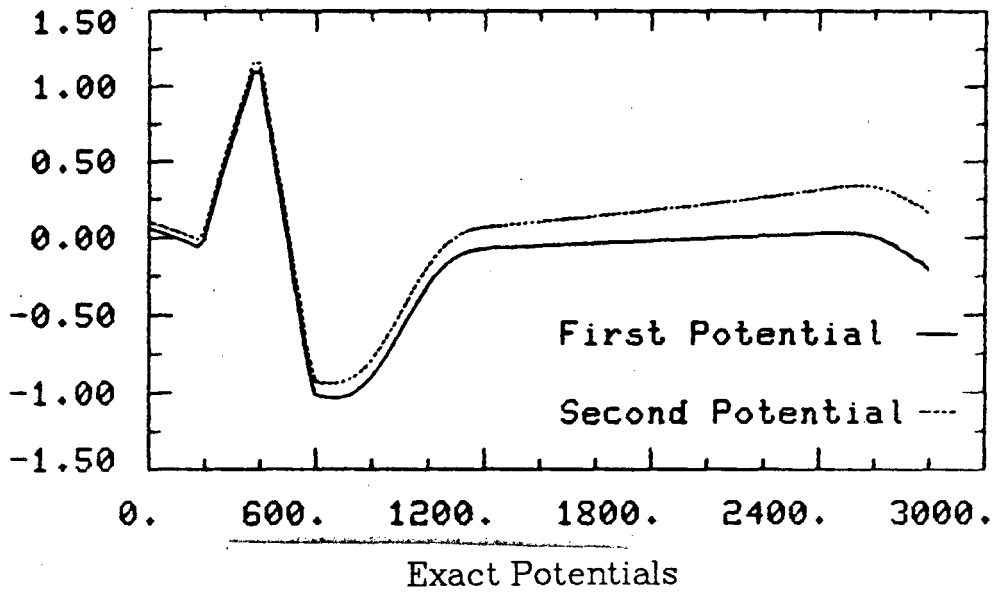


Figure 5.9. The estimated errors in the calculated w. Solid: spectra 1 and 2; dash: spectra 1 and 4; dot: spectra 1 and 3.



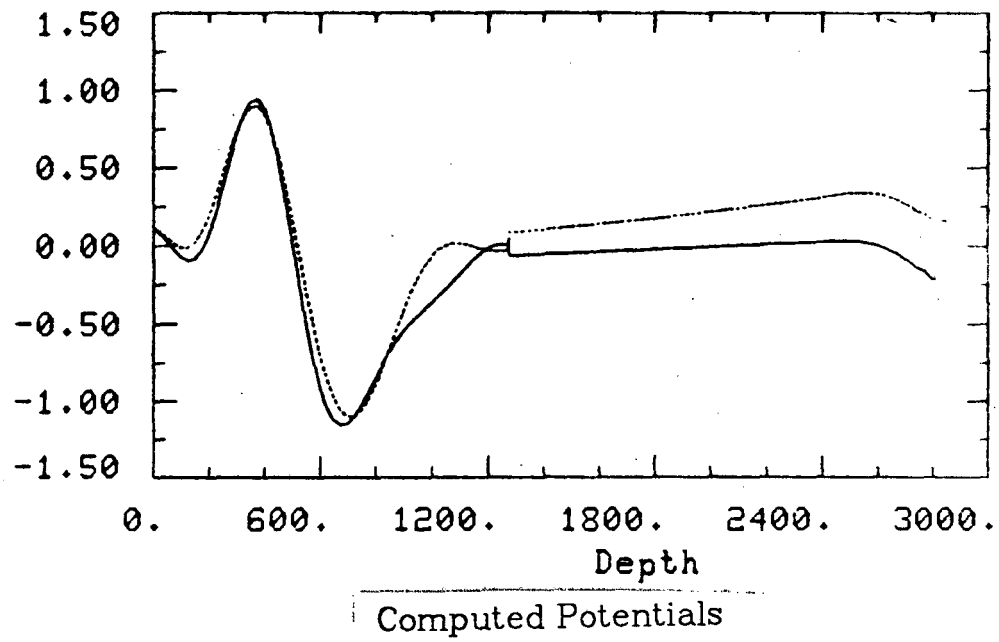
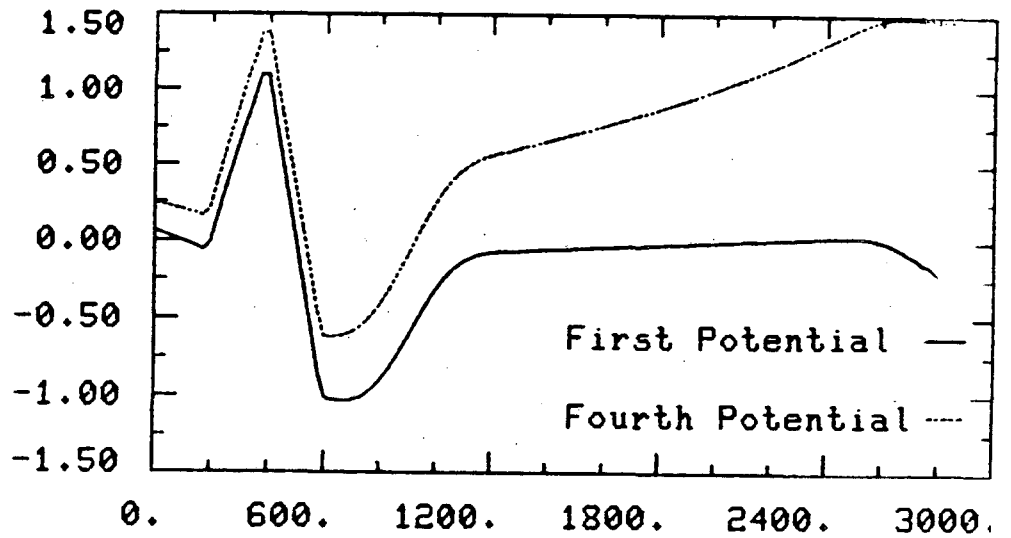
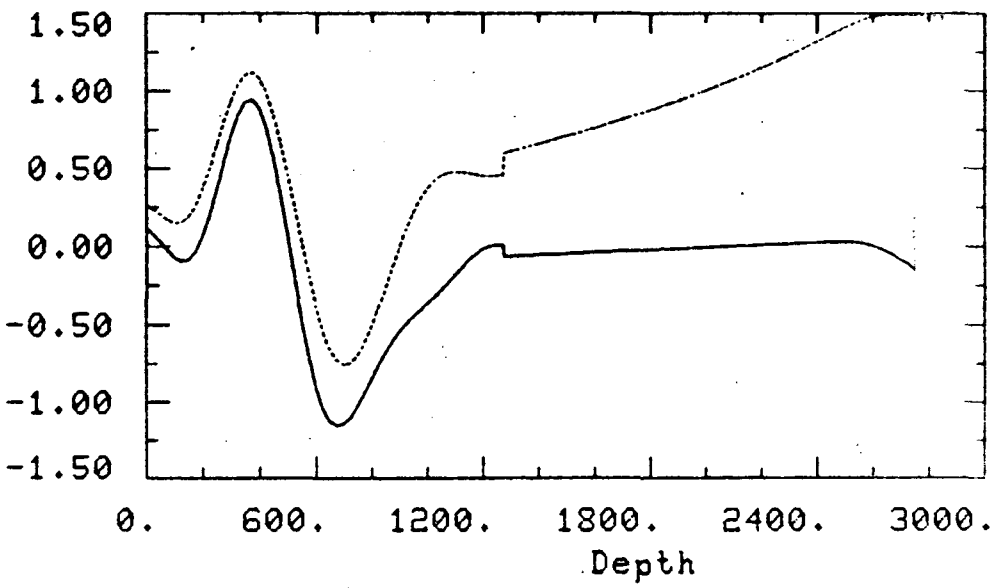


Figure 5.10. Note that the difference between the computed second and first potentials changes sign.



Exact Potentials



Computed Potentials

Figure 5.11. The fourth potential is greater than the first for both exact and computed potentials.

6: Error in the Data

In this section we discuss the effect of the errors in the eigenvalues on the reconstructed model. We show that the method applied to the inverse problem for the mantle is unstable for large perturbations. We then describe a modification which increases the stability. The scheme is motivated by the fact that the errors in the eigenvalues can be considered random. The new version of the method allows for the use of data from more than two spectra.

We invert for the model obtained by smoothing Model A with N=200 km, replacing one of the eigenvalues by a slightly perturbed value. In each of the inversions, we use spectrum 1 as the first spectrum in the inversion. The eigenvalue ${}_0\lambda_1 = 0$ is replaced by .01 and all other eigenvalues are changed by ±.1%. This corresponds to a perturbation of .05% in the associated torsional mode. The perturbations in spectrum 1 affect the value of h and hence the coefficients in the expansion for both potentials. Since the basis functions depend on h a direct comparison of the coefficients does not describe the change in the reconstruction. In Table 6.1 we show the differences between the models computed with one eigenvalue perturbed and the model computed with the original data. Only the results for changes in spectra 1 and 4 are given; changes in spectra 2 and 3 produced similar results. There is a marked increase in instability with the wavenumber of the perturbed eigenvalue. In the inversion for the density only, Hald noted stability and no apparent effect due to wavenumber. We believe that the instability is due to the relative sizes of the functions w and v and the errors in the potentials.

The final model depends on the asymptotic constant, which cannot be measured. We have tried to perturb these constants. In the first experiment A_1 was increased by .1 %. In the second A_3 was increased by .1 %. The normal-

ized L^2 error was .09 for β and .07 for ρ. This is close to the error resulting from an equivalent perturbation in a third eigenvalue. We observe that most of the change lies near the surface, see figures 6.2 and 6.3. This could indicate a relation between the asymptotic constants and the boundary condition at the surface.

Overtone of the perturbed eigenvalue			λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
	Maximum	ρ	.20	.03	.14	.40	.61	1.7
	Norm	β	.07	.06	.19	.48	.71	1.5
Spectrum	Maximum	ρ	5.9	1.0	4.3	12	17	52
1	Percent	β	1.6	1.5	4.4	11	16	35
	Normalized	ρ	.01	.00	.01	.05	.12	.42
	L^2	β	.01	.00	.02	.09	.30	.67
	Maximum	ρ		.13	.15	.52	.81	1.2
	Norm	β		.16	.20	.58	.84	1.2
Spectrum	Maximum	ρ		3.9	4.3	16	24	34
4	Percent	β		3.7	4.5	13	20	27
	Normalized	ρ		.01	.02	.07	.18	.43
	L^2	β		.01	.03	.11	.31	.71

Table 6.1. The changes to the computed model due to perturbations in the eigenvalues of spectra 1 and 4.

To partially explain the increase in the instability with wavenumber we examine the coefficients in the expansion of the reconstructed potentials. In Tables 6.2-6.5, we see that the dominant effect of a perturbation of $\delta\lambda$ in the jth eigenvalue is a corresponding perturbation of $\delta\lambda$ in the jth coefficient. Hence, the error in the potential will have the additional term $\delta\lambda w_j$ where w_j is the the jth basis function in our expansion. When the potentials are subtracted to determine w, this term will not cancel, (see equation (5.1)). As the

wavenumber of λ increases, the magnitude of $\delta\lambda$ also increases and hence the error increases. It can then dominate w. This loss of resolution of w causes error in the computed β. Since β is needed to compute ρ, the error is repeated in the density.

coeffi-cients	Unperturbed values	Perturbation of				
		${}_1\lambda_2$	${}_2\lambda_2$	${}_3\lambda_2$	${}_4\lambda_2$	${}_5\lambda_2$
$\delta\lambda$		15	44	−93	−164	255
γ_0	−.0243	3	−2	−3	−3	4
γ_1	.1004	17	−1	1	−2	3
γ_2	−.0053	3	43	5	4	−3
γ_3	−.1472	−1	1	−95	−3	8
γ_4	.0117	1	1	−3	168	7
γ_5	.0781	−0	1	−1	−3	255
γ_6	.0281	−0	0	−3	−2	−7
γ_7	−.0175	−0	−0	−1	3	4

Table 6.2. The perturbations of the coefficients of q_2 due to perturbations in the eigenvalues of Spectrum 2. All perturbations are multiplied by 10^4.

coefficients	Unperturbed values	Perturbation of $_1\lambda_3$	$_2\lambda_3$	$_3\lambda_3$	$_4\lambda_3$	$_5\lambda_3$
$\delta\lambda$		18	−47	−95	167	258
γ_0	−.0615	4	3	−4	−4	4
γ_1	.0923	22	5	−3	−6	6
γ_2	−.0033	−3	−46	10	7	−6
γ_3	−.1473	−2	−3	−98	−7	10
γ_4	.0119	2	2	−6	170	3
γ_5	.0776	0	−2	0	−0	258
γ_6	.0284	1	0	−3	−3	−3
γ_7	−.0179	0	0	−0	−3	−6

Table 6.3. The perturbations to the coefficients of q_3 due to perturbations in the eigenvalues of Spectrum 3. All perturbations are multiplied by 10^4.

coefficients	Unperturbed values	Perturbation of $_1\lambda_4$	$_2\lambda_4$	$_3\lambda_4$	$_4\lambda_4$	$_5\lambda_4$
$\delta\lambda$		−22	50	−99	−170	261
γ_0	−.1112	2	2	3	14	6
γ_1	.0814	−29	−13	−9	−12	−11
γ_2	−.0007	0	54	−15	−11	−9
γ_3	−.1474	3	6	−101	12	13
γ_4	.0122	−2	−3	−10	−173	−2
γ_5	.0769	0	2	1	−5	261
γ_6	.0287	0	0	−4	4	2
γ_7	−.0183	0	0	0	−4	−8

Table 6.4. The perturbations to the coefficients of q_4 due to perturbations in the eigenvalues of Spectrum 4. All perturbations are multiplied by 10^4.

We now describe the modification of the method. Let γ_j^l be the jth coefficient of the potential q_l. Suppose we are given the coefficients $\gamma_j^{l_i}$, for $i=1, \cdots, m$ where $m \geq 2$. From these and the relation

$$\gamma_j^l = \nu_j + (l+2)(l-1)\omega_j \tag{6.1}$$

we attempt to determine the jth coefficients ν_j and ω_j of the functions v and w. We look for averages

$$\nu_j = a_1\gamma_j^{l_1} + \cdots + a_m\gamma_j^{l_m}$$

$$\omega_j = b_1\gamma_j^{l_1} + \cdots + b_m\gamma_j^{l_m}$$

where

$$a_1 + \cdots + a_m = 1 \qquad (l_1+2)(l_1-1)a_1 + \cdots + (l_m+2)(l_m-1)a_m = 0$$

and

$$(l_1+2)(l_1-1)b_1 + \cdots + (l_m+2)(l_m-1)b_m = 1 \qquad b_1 + \cdots + b_m = 0.$$

This is an underdetermined system for $m>2$.

Thus we choose the linear combination that will minimize the variance of the error in the coefficient. We assume that the errors in the eigenvalues are independent, random variables with mean 0 and variance σ_j^2 which is independent of angular order. We also assume that all error in the computed coefficients comes from error in the eigenvalues. Let e_j^l be the error in the eigenvalue with wavenumber j and angular order l. Since our results indicate that the error in γ_j^l will be close to e_j^l, we assume that the computed coefficient $\tilde{\gamma}_j^l = \gamma_j^l + e_j^l$. For any a_i the expected value of $\sum_1^m a_i(\gamma_j^{l_i}+e_j^{l_i})$ is $\sum_1^m a_i\gamma_j^{l_i}$ and the variance σ^2 is given by $\sum_1^m a_i^2\sigma_j^2 = \sigma_j^2\sum_1^m a_i^2$. Therefore, we choose the linear combination which minimizes $V = \sum a_i^2$. For each coefficient we have a minimization problem with two constraints. We use the homogeneous constraint to eliminate one of the variables and reduce the problem to a

minimization problem with one constraint. We then use Lagrange multipliers to solve for the a_i.

Note that any number of spectra can be used with this method. If we use m terms in the expansion approximating the potential, we must have eigenvalues from at least two spectra for the wavenumbers 2 through m and 1 fundamental eigenvalue. However, the spectra need not be represented in all of these wavenumbers.

In one iteration step we calculate the coefficients for each potential as described in Section 4. We determine only those coefficients for which the corresponding eigenvalue is given. The first spectrum determines h, which is then fixed for the remainder of the iteration step. Except for the first potential, the lowest coefficients are determined by equation (4.2). Once the coefficients for the represented potentials are calculated, we use our averaging method to determine the coefficients for w and v. These values are then used in relation (6.1) to determine the coefficients corresponding to missing eigenvalues and to redefine the calculated coefficients.

The first two tests of our modification use unperturbed eigenvalues as data. The data for Test 1 consists of the first eight eigenvalues of spectrum 1 and the first through seventh eigenvalues for spectra 2,3 and 4. Test 2 omits λ_2^2, λ_3^3, λ_5^1, λ_6^2, λ_6^3 and λ_7^1. The results are given in Table 6.6. Since the algorithm averages the information from the various spectra, we expect the error to lie in the range of that of the original algorithm. Comparison with Table 5.6 shows this is the case.

Test		1	2
Normalized	ρ	.017	.018
L^2	β	.019	.019
Maximum	ρ	.035	.035
Norm	β	.037	.037

Table 6.6. Errors in the reconstruction for Tests 1 and 2.

The next five experiments use data with added error. We determine the amount of the error by letting $\tilde{\lambda} = (1+z)\lambda$ where z is chosen randomly. We use the Fortran routine RAN( ). This returns a random variable from a uniform distribution of [0,1]. Tables 6.7 through 6.9 give the amount of the perturbations $z\lambda$. For test 3 we scale the random variable so that the error lies between −.1 and .1%. The modified algorithm converges, but the calculated model changes by as much as 50%. If the data from only two spectra is given, the computed w becomes negative and the inversion fails. With data from spectra 1,3 and 4, the final solution had slightly less error. This could be a result of the distribution of the error. Test 5 used data with error ranging from −.05 to .05 %. The results are somewhat better, but still large. For the last two tests, the error was kept between −.01 and .01%. Test 6 used a complete set of data from spectra 1,2,3 and 4 while Test 7 used data from spectra 1 and 4 only. There is an improvement with more data although the results from both inversions are decent. The non-linearity of the inversion is illustrated by the growth of error seen in these examples.

Test		3	4	5	6	7
Normalized	ρ	.874	.808	.513	.053	.064
L^2	β	.993	.939	.614	.071	.081
Maximum	ρ	2.89	2.55	2.43	.122	.138
Norm	β	2.09	1.94	1.80	.141	.181

Table 6.6. Errors in the reconstruction for the tests with error in the data.

The errors in the computed model for tests 3 and 4 are displayed in graphs 6.4 through 6.7. The error in the function v is similar for the two tests, while the error in w, ρ and β is much greater for Test 3 than for Test 6. This again emphasizes that the sensitivity of the algorithm lies in the calculation of w.

Overtone	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
Spectrum 1	0	−4	−31	−65	−80	173	170	−334
Spectrum 2	-	8	24	62	−88	−85	205	21
Spectrum 3	-	−16	−25	16	145	−164	328	−23
Spectrum 4	-	−6	21	−88	19	236	−292	−232

Table 6.7: Perturbations x 10^4 in the eigenvalues for Test 3.

Overtone	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
Spectrum 1	0	2	13	−23	−23	96	23	−133
Spectrum 2	-	−5	11	−7	43	88	159	−222
Spectrum 3	-	−7	4	20	−19	82	133	−136
Spectrum 4	-	6	−26	32	−53	−19	−167	−119

Table 6.8: Perturbations x 10^4 in the eigenvalues for Test 4.

Overtone	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
Spectrum 1	0	.3	−2.7	8.9	11.0	−13.7	−26.2	32.3
Spectrum 2	-	1.4	.2	1.9	−4.3	16.8	12.4	30.9
Spectrum 3	-	1.5	−.6	5.0	1.7	8.3	−23.9	2.4
Spectrum 4	-	−1.6	−4.9	−3.4	−7.7	13.9	−17.3	24.6

Table 6.9: Perturbations x 10^4 in the eigenvalues for Tests 6 and 7.

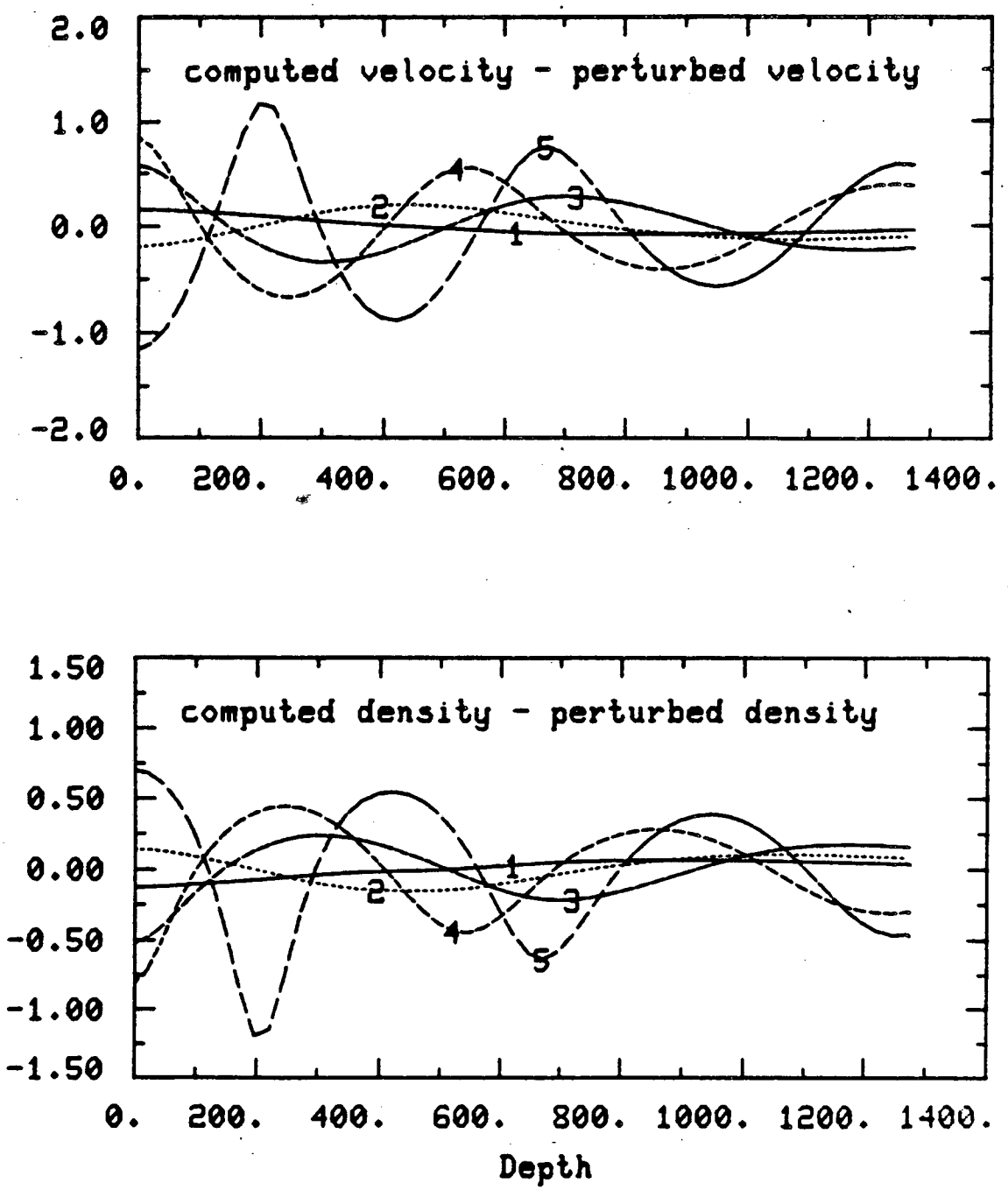


Figure 6.1. Changes to the final model resulting from a change in one eigenvalue. The results are for perturbations in spectrum 4, λ_n, $n = 1,\ldots,5$.

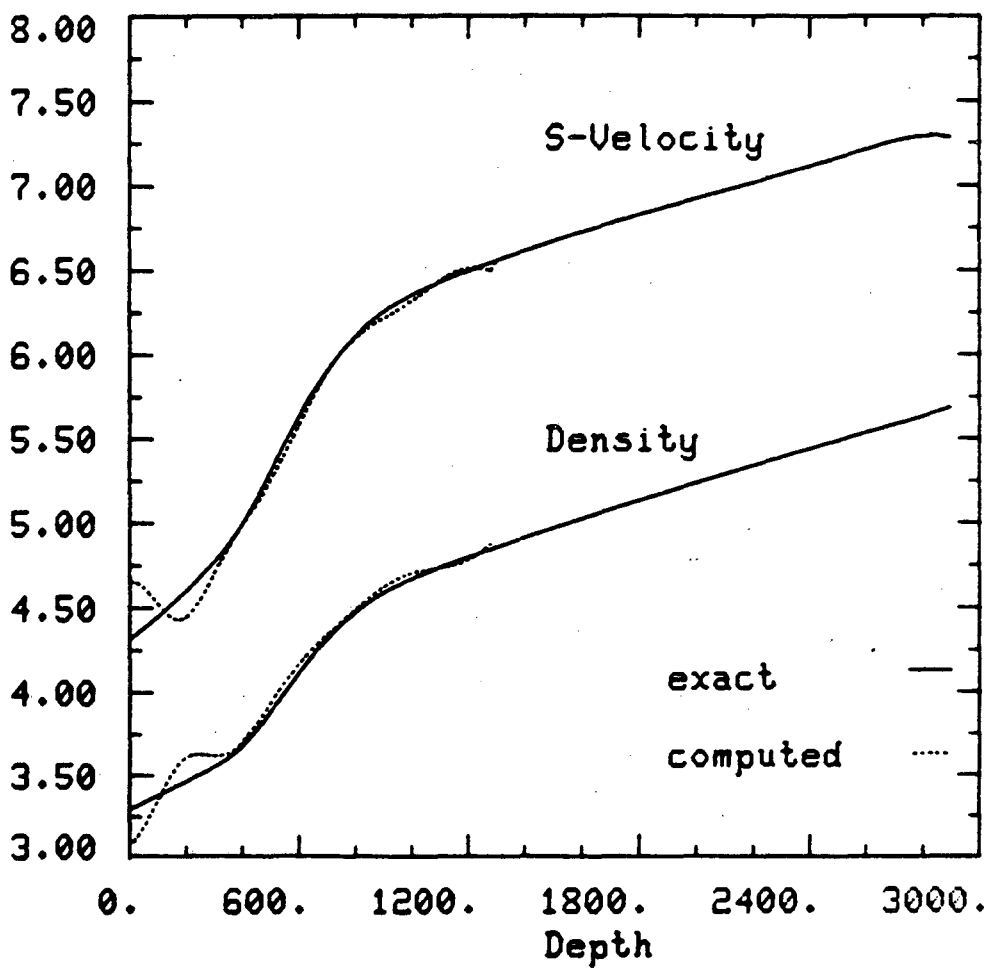


Figure 6.2. Change in the computed model resulting from a .1% increase in A_1.

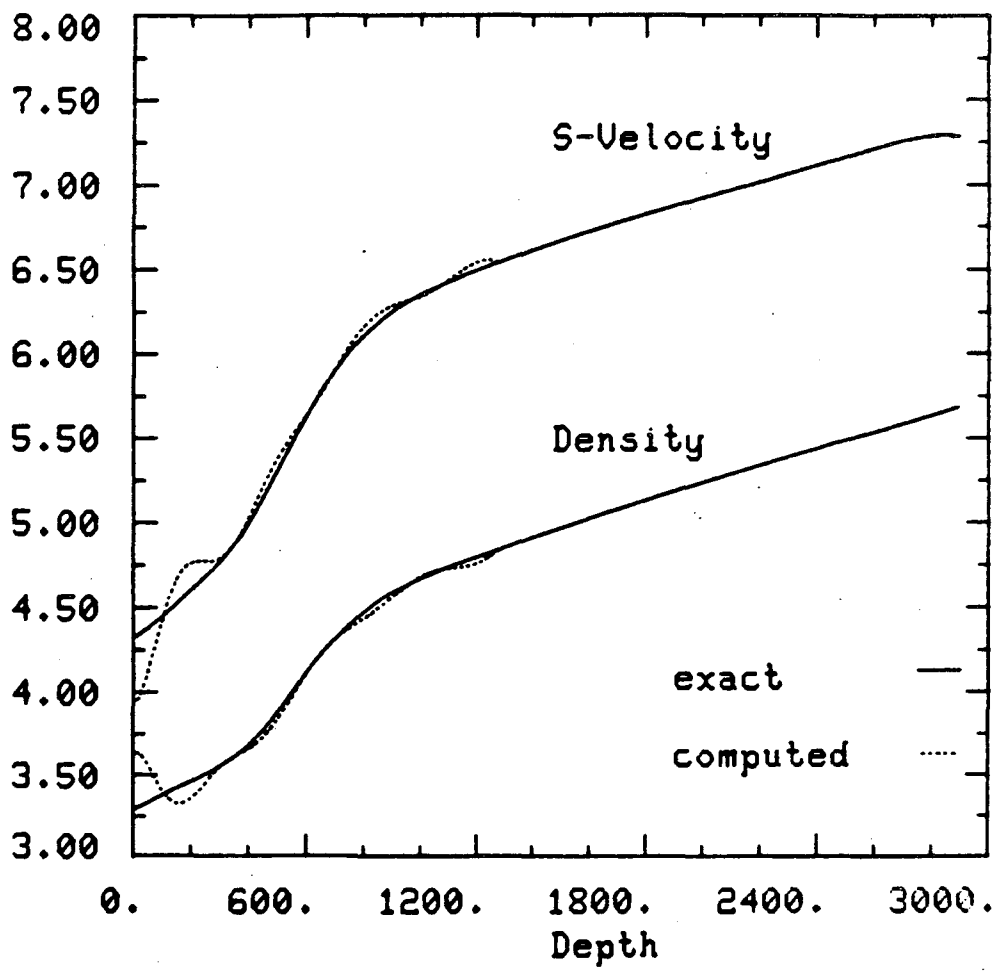


Figure 6.3. Change in the computed model resulting from a .1% increase in A_3.

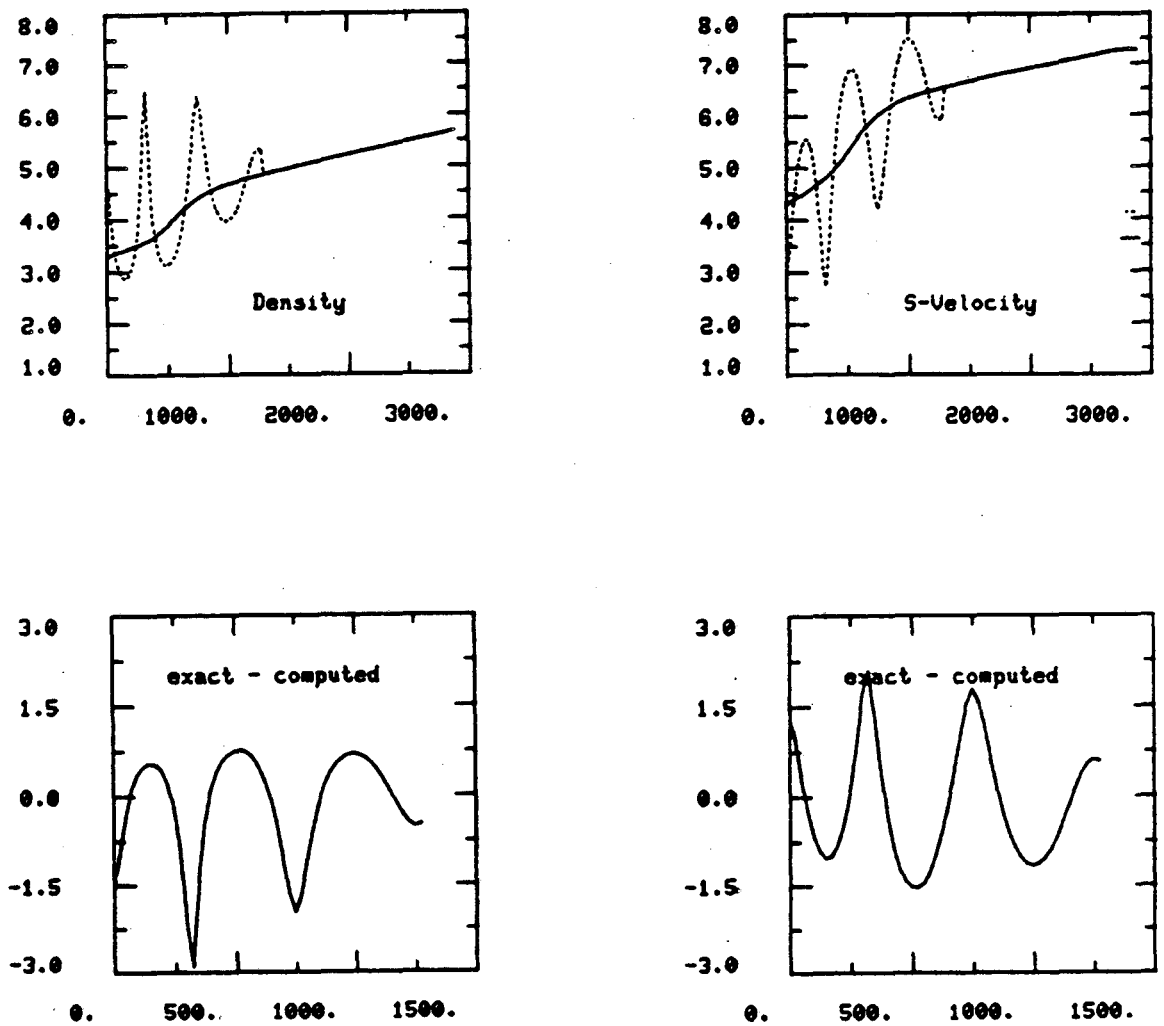


Figure 6.4. Error in Test 3.

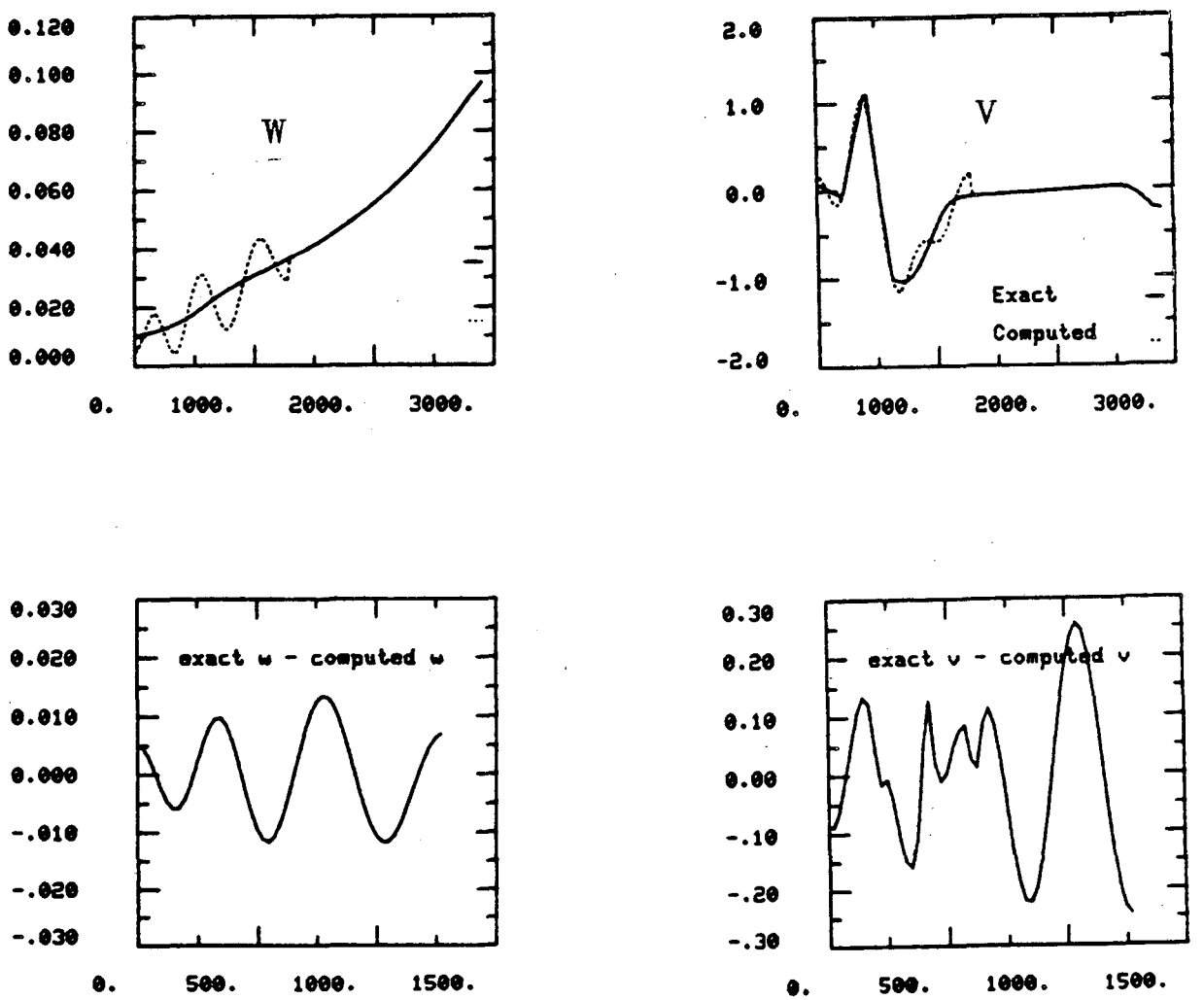


Figure 6.5. Error in Test 3.

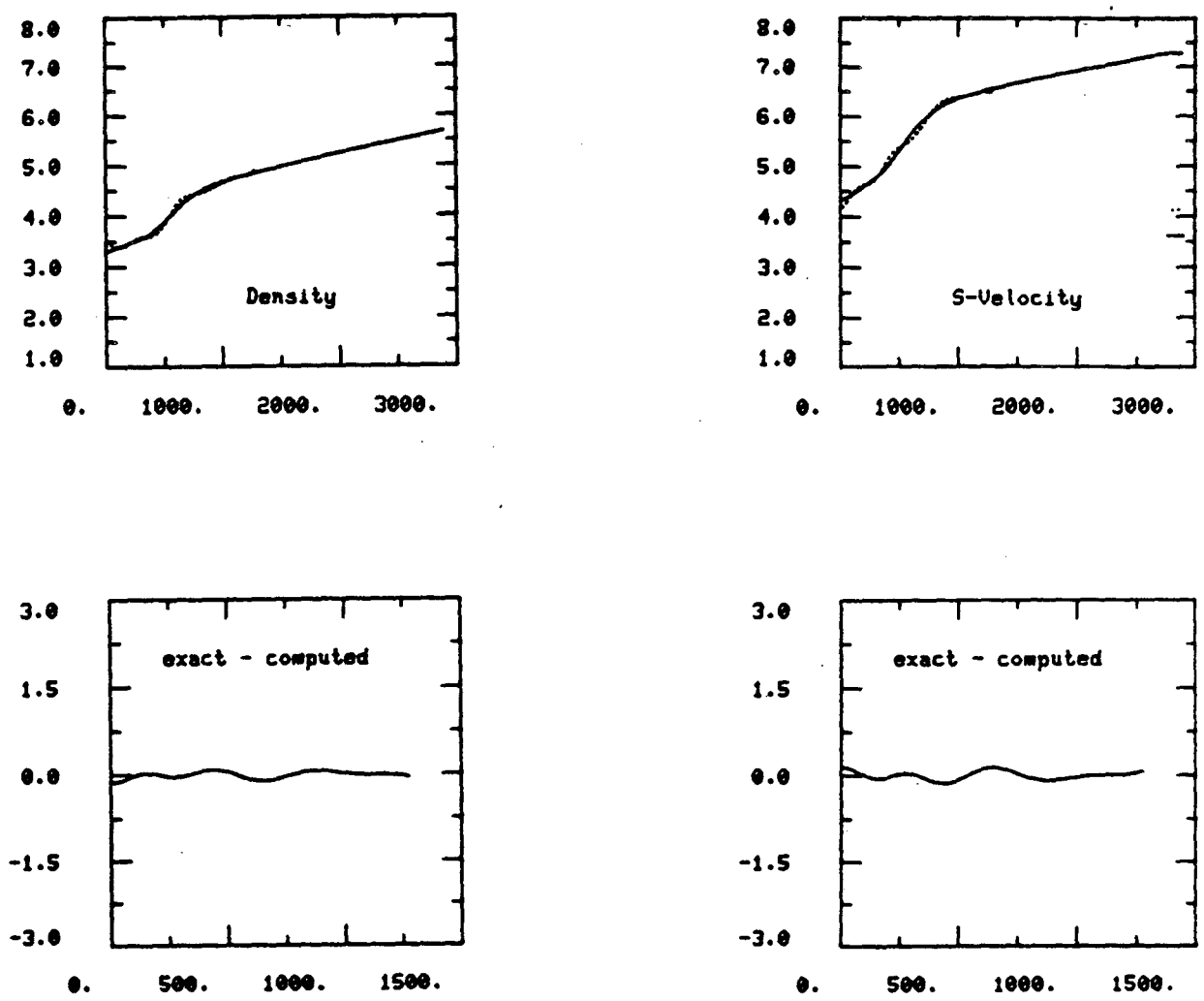


Figure 6.6. Error in Test 6.

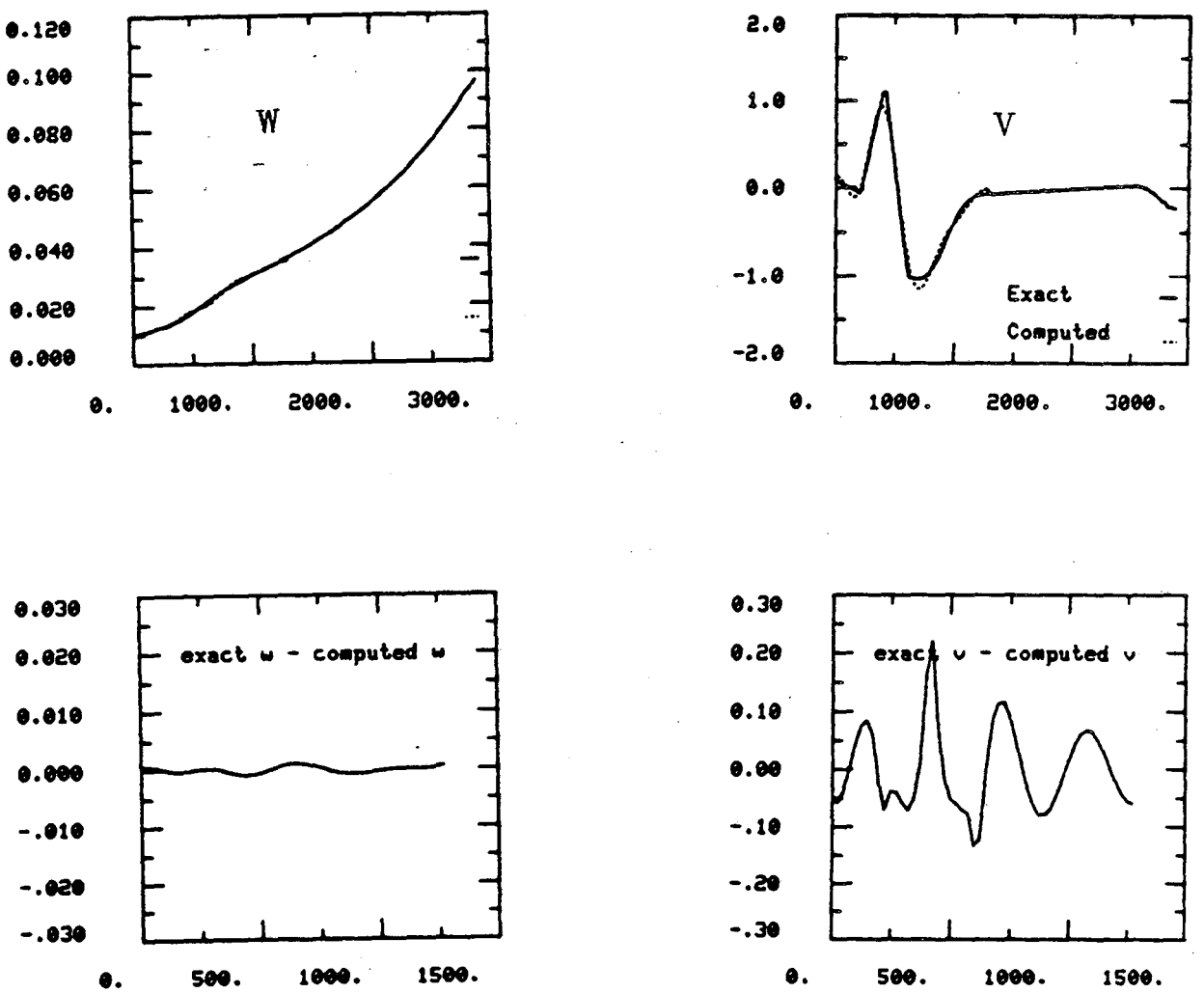


Figure 6.7. Error in Test 6.

PART TWO

1: Main Result

In this section we present our main result for the inverse Sturm-Liouville theory. The proof of Theorem 1 is based on the asymptotic form of the eigenvalues and eigenfunctions. We must extend the theory of a Sturm-Liouville problem with no discontinuities to include problems with two interior discontinuities. We find an integral equation and use this equation to develop the theory. Our techniques are standard.

THEOREM 1. *Consider the eigenvalue problem*

$$-u''+q(x)u=\lambda u \tag{1.1}$$

on the interval $0<x<\pi$ *and with the boundary conditions*

$$u'(0)-hu(0)=u'(\pi)+Hu(\pi)=0 \tag{1.2}$$

and with jump conditions

$$u(d_1+)=a_1u(d_1-),\quad u'(d_1+)=a_1^{-1}u'(d_1-)+b_1u(d_1-) \tag{1.3}$$

$$u(d_2+)=a_2u(d_2-),\quad u'(d_2+)=a_2^{-1}u'(d_2-)+b_2u(d_2-) \tag{1.4}$$

where q *is an integrable function,* $0<d_1<d_2<\pi/2$, $a_1, a_2>0$, $|a_1-1|+|b_1|>0$ *and* $|a_2-1|+|b_2|>0$. *Let* $\lambda_0, \lambda_1, \cdots$ *be the eigenvalues. Consider the eigenvalue problem with* $a_1, b_1, d_1, a_2, b_2, d_2, h, H, \lambda$ *and* q *replaced by* $\tilde{a}_1, \tilde{b}_1, \tilde{d}_1, \tilde{a}_2, \tilde{b}_2, \tilde{d}_2, \tilde{h}, \tilde{\lambda}$ *and* $\tilde{q}$. *If* $\lambda_j=\tilde{\lambda}_j$ *for* $j\geq 0$, $H=\tilde{H}$ *and* $q=\tilde{q}$ *almost everywhere in* $(\pi/2,\pi)$ *then* $a_1=\tilde{a}_1$, $b_1=\tilde{b}_1$, $d_1=\tilde{d}_1$, $a_2=\tilde{a}_2$, $b_2=\tilde{b}_2$, $d_2=\tilde{d}_2$, $h=\tilde{h}$ *and* $q=\tilde{q}$ *almost everywhere.*

Proof: Consider the solution u of equations (1.1), (1.3) and (1.4) that has the initial values $u=1$ and $u'=h$ at $x=0$. We will find an integral equation that u satisfies and will then study the integral equation to prove the theorem.

Let $u=u_1$ for $0\le x<d_1$, u_2 for $d_1<x<d_2$ and $u=u_3$ for $d_2<x\le\pi$. Note that u is not defined at d_1 and d_2 but u_1 , u_2 and u_3 are. It is well known that u_1 satisfies a Volterra integral equation of the second kind. We find the integral equation by letting $u=u_1$ in equation (1.1), multiplying by a function $G(x,t)$ and then integrating with respect to t from 0 to x. After integrating by parts twice and using the initial conditions we arrive at

$$hG(x,0)-G_t(x,0)$$
$$-u_1'(x)G(x,x)+u_1(x)G_t(x,x)-\int_0^x G_{tt}u_1dt+\int_0^x Gqu_1dt = \int_0^x \lambda Gu_1dt.$$

By solving

$$-G_{tt} = \lambda G$$
$$G(x,x) = 0 \qquad G_t(x,x) = -1.$$

we obtain the Volterra integral equation for u_1 :

$$u_1(x) = g_1(x) + \int_0^x G_{11}(x,t)q(t)u_1(t)dt \tag{1.5}$$

where $0\le x\le d_1$, $G_{11} = G$ and $g_1(x)=hG(x,0)-G_t(x,0)$ is given below. We use the same technique on the intervals (d_1,x) and (d_2,x) to get the results for u_2 and u_3, but replace the initial conditions for u with the jump conditions at d_1 and d_2. We have

$$u_2(x) = g_2(x) + \int_0^{d_1} G_{21}(x,t)q(t)u_1(t)dt + \int_{d_1}^x G_{22}(x,t)q(t)u_2(t)dt \tag{1.6}$$
$$(d_1<x<d_2)$$

$$u_3(x) = g_3(x) + \int_0^{d_1} G_{31}(x,t)q(t)u_1(t)dt \tag{1.7}$$

$$+ \int_{d_1}^{d_2} G_{32}(x,t)q(t)u_2(t)dt + \int_{d_2}^{x} G_{33}(x,t)q(t)u_3(t)dt$$

$$(d_2 < x < \pi)$$

where

$$g_1(x) = \cos kx + \frac{h}{k}\sin kx \tag{1.8}$$

$$\begin{aligned} g_2(x) = {} & a_1 \left[\cos kd_1 + \frac{h}{k}\sin kd_1\right]\cos k(x-d_1) \\ & + a_1^{-1}\left[-\sin kd_1 + \frac{h}{k}\cos kd_1\right]\sin k(x-d_1) \\ & + \frac{b_1}{k}\left[\cos kd_1 + \frac{h}{k}\sin kd_1\right]\sin k(x-d_1) \end{aligned} \tag{1.9}$$

$$\begin{aligned} g_3(x) = {} & a_2 \left[a_1\cos k(d_2-d_1)\cos kd_1 - a_1^{-1}\sin k(d_2-d_1)\sin kd_1\right. \\ & + \frac{b_1}{k}\sin k(d_2-d_1)\cos kd_1 + \frac{h}{k}\{a_1^{-1}\sin k(d_2-d_1)\cos kd_1 \\ & + a_1\cos k(d_2-d_1)\sin kd_1 + \frac{b_1}{k}\sin k(d_2-d_1)\sin kd_1\}\left.\right]\cos k(x-d_2) \\ & + a_2^{-1}\left[-a_1\sin k(d_2-d_1)\cos kd_1 - a_1^{-1}\cos k(d_2-d_1)\sin kd_1\right. \\ & + \frac{b_1}{k}\cos k(d_2-d_1)\cos kd_1 + \frac{h}{k}\{a_1^{-1}\cos k(d_2-d_1)\cos kd_1 \\ & - a_1\sin k(d_2-d_1)\sin kd_1 + \frac{b_1}{k}\cos k(d_2-d_1)\sin kd_1\}\left.\right]\sin k(x-d_2) \\ & + \frac{b_2}{k}\left[a_1\cos k(d_2-d_1)\cos kd_1 - a_1^{-1}\sin k(d_2-d_1)\sin kd_1\right. \\ & + \frac{b_1}{k}\sin k(d_2-d_1)\cos kd_1 + \frac{h}{k}\{a_1^{-1}\sin k(d_2-d_1)\cos kd_1 \\ & + a_1\cos k(d_2-d_1)\sin kd_1 + \frac{b_1}{k}\sin k(d_2-d_1)\sin kd_1\}\left.\right]\sin k(x-d_2) \end{aligned} \tag{1.10}$$

and

$$G_{11}(x,t) = G_{22}(x,t) = G_{33}(x,t) = \frac{\sin k(x-t)}{k} \tag{1.11}$$

$$G_{21}(x,t) = \frac{1}{k}(a_1 \sin k(d_1-t)\cos k(x-d_1) \tag{1.12}$$

$$+ a_1^{-1}\cos k(d_1-t)\sin k(x-d_1) + \frac{b_1}{k}\sin k(d_1-t)\sin k(x-d_1))$$

$$G_{31}(x,t) = \frac{1}{k}(a_2[a_1\cos k(d_2-d_1)\sin k(d_1-t) \tag{1.13}$$

$$+ a_1^{-1}\sin k(d_2-d_1)\cos k(d_1-t)$$

$$+ \frac{b_1}{k}\sin k(d_2-d_1)\sin k(d_1-t)]\cos k(x-d_2)$$

$$+ a_2^{-1}[-a_1\sin k(d_2-d_1)\sin k(d_1-t) + a_1^{-1}\cos k(d_2-d_1)\cos k(d_1-t)$$

$$+ \frac{b_1}{k}\cos k(d_2-d_1)\sin k(d_1-t)]\sin k(x-d_2)$$

$$+ \frac{b_2}{k}[a_1\cos k(d_2-d_1)\sin k(d_1-t) + a_1^{-1}\sin k(d_2-d_1)\cos k(d_1-t)$$

$$+ \frac{b_1}{k}\sin k(d_2-d_1)\sin k(d_1-t)]\sin k(x-d_2))$$

$$G_{32}(x,t) = \frac{1}{k}(a_2\sin k(d_2-t)\cos k(x-d_2) + a_2^{-1}\cos k(d_2-t)\sin k(x-d_2) \tag{1.14}$$

$$+ \frac{b_2}{k}\sin k(d_2-t)\sin k(x-d_2))$$

Thus we can write

$$u(x) = g(x) + \int_0^{\pi} G(x,t)q(t)u(t)dt \tag{1.15}$$

for $0 \le x \le \pi$, $x \ne d_1, d_2$. Here $g(x)$ satisfies equations (1.1), (1.3) and (1.4) with $q=0$. The kernel G is discontinuous at $x=d_1$ and d_2, satisfies the jump conditions in x and is integrable in t. Any solution of (1.15) will also satisfy equations (1.1), (1.3) and (1.4).

Our proof depends on estimations of the solution u on each interval. We will use the following lemma to bound u_1, u_2 and u_3.

LEMMA 1. *Consider the integral equation*

$$u(x)-\int_a^x K(x,t)q(t)u(t)dt=f(x)$$

where f and K are continuous and q is integrable. This equation has a unique solution u which is continuous and satisfies

$$|u(x)|\leq M(x)e^{L(x)\rho(x)}$$

where $M(x)=\max_{a\leq t\leq x}|f(t)|$, $L(x)=\max_{a\leq t\leq x}|K(x,t)|$ *and* $\rho(x)=\int_a^x|q(t)|dt$.

Remark: The proof of Lemma 1 uses the method of successive approximations. Let $u_0=f$ and $u_{n+1}=f+\int_a^x Kqu_n\,dt$. Then $\sum_0^\infty u_n$ is shown to converge uniformly to a solution u of the integral equation. Therefore if f and K are analytic functions of λ, u_n and hence u will be analytic in λ. For details of the proof see Hald [28]. This lemma can be used to prove the existence and uniqueness of the solution to the integral equation (1.15).

In the next lemma we find upper bounds for u_1, u_2 and u_3.

LEMMA 2: *Let u_1, u_2 and u_3 be the solutions of equations (1.5), (1.6) and (1.7). Let* $\sqrt{\lambda}=\sigma+i\tau$, $c=\max(|b_1|,\ |b_2|,\ |h|,\ \int_0^\pi|q(t)|dt\ )$ *and* $A_1=a_1+a_1^{-1}$ *and* $A_2=a_2+a_2^{-1}$. *Then u_1, u_2 and u_3 are entire functions of λ of order ½ and*

$$|u_1(x,\lambda)|\leq(1+c\pi)e^{cx+|\tau|x} \qquad (0\leq x\leq d_1) \qquad (1.16)$$

$$|u_2(x,\lambda)|\leq A_1(1+c\pi)^3e^{cx+|\tau|x} \qquad (d_1\leq x\leq d_2) \qquad (1.17)$$

$$|u_3(x,\lambda)|\leq A_1A_2(1+c\pi)^5e^{cx+|\tau|x} \qquad (d_2\leq x\leq\pi)\ . \qquad (1.18)$$

Proof : The bounds (1.16) and (1.17) have been established by Hald [28]. Let $k=\sqrt{\lambda}=\sigma+i\tau$ and $\nu\geq|\tau|$. To prove equation (1.18) we follow the technique of Hald and write equation (1.7) as

$$\begin{aligned} e^{-\nu x}u_3 = e^{-\nu x}g_3(x) &+ \int_0^{d_1} e^{-\nu(x-t)}G_{31}(x,t)q(t)e^{-\nu t}u_1(t)dt \\ &+ \int_{d_1}^{d_2} e^{-\nu(x-t)}G_{32}(x,t)q(t)e^{-\nu t}u_2(t)dt \\ &+ \int_{d_2}^{x} e^{-\nu(x-t)}G_{33}(x,t)q(t)e^{-\nu t}u_3(t)dt \quad . \end{aligned}$$

This is an integral equation for $e^{-\nu x}u_3(x)$ which we rewrite in the notation of Lemma 1 as

$$e^{-\nu x}u_3(x) = f(x) + \int_{d_2}^{x} K(x,t)q(t)e^{-\nu t}u_3(t)dt$$

where $K \sim=\sim e^{-\nu(x-t)}G_{33}$ and

$$\begin{aligned} f(x) = e^{-\nu x}g_3(x) &+ \int_0^{d_1} e^{-\nu(x-t)}G_{31}(x,t)q(t)e^{-\nu t}u_1(t)dt \\ &+ \int_{d_1}^{d_2} e^{-\nu(x-t)}G_{32}(x,t)q(t)e^{-\nu t}u_2(t)dt \end{aligned}$$

and $K = G_{33}$. We will bound f and K and apply Lemma 1 to obtain (1.17). We make repeated use of the inequalities

$$|\cos kx|,\ |\sin kx|,\ \left|\frac{\sin kx}{kx}\right| \leq e^{\nu x}$$

which hold if $x\geq 0$ and $\nu\geq|\mathrm{Im}k|$.

We begin our estimation of f by writing $e^{-\nu x}$ as $e^{-\nu(d_2-d_1)}e^{-d_1}e^{-\nu(x-d_2)}$. From equation (1.10) follows

$$\begin{aligned} e^{-\nu x}\ |g_3(x)| = a_2\,[a_1 &+ a_1^{-1} + c(d_2-d_1) \\ &+ c\{a_1^{-1}(d_2-d_1) + a_1d_1 + c(d_2-d_1)d_1\}\,] \\ &+ a_2^{-1}\,[a_1 + a_1^{-1} + c(x-d_2) + c(x-d_2)\{a_1^{-1} + a_1 + cd_1\}\,] \\ &+ c(x-d_2)\,[a_1 + a_1^{-1} \\ &+ c(d_2-d_1) + c\{a_1d_1 + a_1^{-1}(d_2-d_1) + c(d_2-d_1)d_1\}] \end{aligned}$$

We factor, recall that $A_i \geq 2$ and estimate upwards to get

$$e^{-\nu x}\ |g_3(x)| \leq A_1 A_2 (1 + c\pi)^3 \quad .$$

Next we consider the first integral appearing in f. We have

$$e^{-\nu(x-t)}\ G_{31}(x,t) = e^{-\nu(d_2-d_1)} e^{-\nu(d_1-t)} e^{-\nu(x-d_2)} G_{31}(x,t) \quad .$$

Thus from (1.13) follows

$$\begin{aligned} e^{-\nu(x-t)}\ |G_{31}(x,t)| \leq\ & a_2\,[a_1(d_1-t) + a_1^{-1}(d_2-d_1) + c\ (d_2-d_1)(d_1-t)\,] \\ & + a_2^{-1}\ [a_1(d_2-d_1) + a_1^{-1}(x-d_2) + c\,(d_1-t)(x-d_2)\,] \\ & + c\,(x-d_2)\,[a_1(d_1-t) + a_1^{-1}(d_2-d_1) + c\,(d_2-d_1)(d_1-t)\,] \\ \leq\ & A_1 A_2 \pi (1 + c\pi)^2 \quad . \end{aligned}$$

Similarly we let $e^{-\nu(x-t)} G_{32}(x,t) = e^{-\nu(d_2-t)} e^{-\nu(t-x)}$ and use equation (1.14) to obtain

$$e^{-\nu(x-t)}\ |G_{32}(x,t)| \leq A_2 \pi (1 + c\pi) \quad .$$

We can use our bounds for u_1 and u_2 to conclude

$$\begin{aligned} |f\,| \leq\ & A_1 A_2 (1 + c\pi)^3 + A_1 A_2 \pi (1 + c\pi)^2\ \left|\int_0^{d_1} q e^{-\nu t} u_1(t)\,dt\,\right| \\ & + A_2 \pi (1 + c\pi) \left|\int_{d_1}^{d_2} q e^{-\nu t} u_2(t)\,dt\,\right| \\ \leq\ & A_1 A_2\ ((1 + c\pi)^3 + c\pi(1 + c\pi)^3 e^{cd_1} + c\pi(1 + c\pi)^4 e^{cd_2} \\ \leq\ & A_1 A_2 (1 + c\pi)^5 e^{cd_2}. \end{aligned}$$

Finally $e^{-\nu(x-t)}\ |G_{33}(x,t)| \leq x - d_2$ for $d_2 < t < x$. Thus equation (1.18) follows from Lemma 1 where we let $\nu = |\tau|$ to obtain the best bound. Both f and G_{33} are analytic in λ. Hence from our remark after Lemma 1 and our bound we get that u_3 is analytic in λ of order ½. This concludes the proof of Lemma 2.

We investigate the asymptotic behavior of the solution $u(x,\lambda)$. Here we will be concerned with the first order terms of g_1, g_2 and g_3. We denote these by φ_1, φ_2 and φ_3. By using the product formulas for basic trigonometric functions we get

$$\varphi_1(x) = \cos kx \tag{1.19}$$

$$\varphi_2(x) = \frac{A_1}{2}[\cos kx + \alpha_1 \cos k(x-2d_1)] \tag{1.20}$$

$$\varphi_3(x) = \frac{A_1 A_2}{4}[\cos kx + \alpha_1 \cos k(x-2d_1) \tag{1.21}$$

$$+ \alpha_2 \cos k(x-2d_2) + \alpha_1\alpha_2 \cos k(x-2d_2+2d_1)]$$

where $\alpha_1 = a_1 - a_1^{-1} / a_1 + a_1^{-1}$ and $\alpha_2 = a_2 - a_2^{-1} / a_2 + a_2^{-1}$. We have $|\alpha_i| < 1$. Let $\varphi = \varphi_1$ for $0 \le x < d_1$, φ_2 for $d_1 < x < d_2$ and φ_3 for $d_2 < x < \pi$. Note that φ_i is defined on the closed interval although φ is not. We have φ is a solution of equations (1.1), (1.3) and (1.4) if $h = b_1 = b_2 = 0$ and $q = 0$.

LEMMA 3: *Let u_1, u_2, and u_3 be the solutions of the integral equations (1.5)-(1.7). Let $k = \sqrt{\lambda} = \sigma + i\tau$ and $c = \max(|b_1|, |b_2|, |h|, \int_0^\pi |q|\,dt)$. If $|k| \ge 3c$, then*

$$|u_1(x)| \le 2e^{|\tau|x} \tag{1.22}$$

$$|u_1(x) - \varphi_1(x)| \le \frac{3c}{|k|} e^{|\tau|x} \tag{1.23}$$

$$(0 \le x \le d_1)$$

$$|u_2(x)| \le 3A_1 e^{|\tau|x} \tag{1.24}$$

$$|u_2(x) - \varphi_2(x)| \le \frac{5c}{|k|} A_1 e^{|\tau|x} \tag{1.25}$$

$$|u_2'(x) - \varphi_2'(x)| \le 5cA_1 e^{|\tau|x} \tag{1.26}$$

$$(d_1 \le x \le d_2)$$

$$|u_3(x)| \le 11A_1A_2e^{|\tau|x} \tag{1.27}$$

$$|u_3(x)-\varphi_3(x)| \le \frac{28c}{|k|}A_1A_2e^{|\tau|x} \tag{1.28}$$

$$|u_3'(x)-\varphi_3'(x)| \le 24cA_1A_2e^{|\tau|x} \tag{1.29}$$

$$(d_2 \le x \le \pi)$$

where the φ_i are defined by equations (1.19)-(1.21).

Remark: If $c = 0$, $\varphi_i = u_i$ and we set $\frac{c}{|k|} = 0$. This lemma is an extension of a lemma by Hald. The estimates involving u_1 and u_2 come directly from his results. Thus it is only necessary to prove the last three bounds. The proof is based on the Volterra integral equation for the eigenfunctions. This approach was previously used by Borg [11] and Hald [28].

Proof: Let $\nu=|\tau|$ where $k=\sigma+i\tau$. Recall that φ_3 is defined to consist of the first order terms of g_3 and is a solution of (1.1), (1.3) and (1.4) with $c = 0$. Thus from Lemma 2 we have

$$|\varphi_3(x)| \le A_1A_2e^{|\tau|x}.$$

We use Lemma 1 to estimate $|u_3-\varphi_3|$ as we did in the proof of Lemma 2. The inhomogeneous term of the integral equation for $e^{-\nu x}(u_3-\varphi_3)$ is given by

$$\begin{aligned} e^{-\nu x}(g_3-\varphi_3) &+ \int_0^{d_1} e^{-\nu(x-t)}G_{31}(x,t)q(t)e^{-\nu t}u_1(t)dt \\ &+ \int_{d_1}^{d_2} e^{-\nu(x-t)}G_{32}(x,t)q(t)e^{-\nu t}u_2(t)dt \\ &+ \int_{d_2}^{x} e^{-\nu(x-t)}G_{33}(x,t)q(t)e^{-\nu t}\varphi_3(t)dt \ . \end{aligned}$$

By definition

$$e^{-\nu x}|g_3-\varphi_3| = e^{-\nu x}$$

$$|\frac{a_2}{k}[b_1\cos kd_1\sin k(d_2-d_1) + h\ \{a_1^{-1}\cos kd_1\sin k(d_2-d_1)$$

$$+ a_1 \sin kd_1 \cos k(d_2-d_1) + \frac{b_1}{k} \sin kd_1 \sin k(d_2-d_1)\}]\cos k(x-d_2)$$

$$+ \frac{a_2^{-1}}{k}[b_1 \cos kd_1 \cos k(d_2-d_1) + h\{a_1^{-1} \cos kd_1 \cos k(d_2-d_1)$$

$$- a_1 \sin kd_1 \sin k(d_2-d_1) + \frac{b_1}{k} \sin kd_1 \cos k(d_2-d_1)\}]\sin k(x-d_2)$$

$$+ \frac{b_2}{k}[a_1 \cos kd_1 \cos k(d_2-d_1) - a_1^{-1} \sin kd_1 \sin k(d_2-d_1)$$

$$+ \frac{b_1}{k} \cos kd_1 \sin k(d_2-d_1) + \frac{h}{k}\{a_1^{-1} \cos kd_1 \sin k(d_2-d_1)$$

$$+ a_1 \sin kd_1 \cos k(d_2-d_1) + \frac{b_1}{k} \sin kd_1 \sin k(d_2-d_1)\}]\sin k(x-d_2)|$$

We write $e^{-\nu t}$ as $e^{-\nu d_1} e^{-\nu(d_2-d_1)} e^{-\nu(x-d_2)}$ and conclude

$$e^{-\nu x} \; |g_3-\varphi_3| \le \frac{1}{|k|}(a_2 [c + c\{A_1+1/3\}] \tag{1.30}$$

$$+ \; a_2^{-1} [c + c\{A_1+1/3\}] + c \; [A_1+1/3+1/3\{A_1+1/3\}])$$

$$\le \frac{3cA_1A_2}{|k|}$$

where we have used $A_1, A_2 \ge 2$ and $\frac{c}{|k|} < 1/3$. To arrive at an upper bound for $|u_3-\varphi_3|$ which is proportional to $\frac{1}{|k|}$ we must redo some of the estimates in the proof of Lemma 2. We expand $e^{-\nu t}$ as before, let $|k| > 3c$ and use equation (1.13) to get

$$e^{-\nu(x-t)} \; |G_{31}(x,t)| \le \frac{1}{|k|}\left\{A_2[A_1+\frac{1}{3}] + \frac{1}{3}[A_1+\frac{1}{3}]\right\}$$

$$\le \frac{2A_1A_2}{|k|} .$$

From equation (1.22) we have

$$e^{-\nu x}\ \left|\int_0^{d_1} G_{31}(x,t)q(t)u_1(t)dt\right| \le \frac{4cA_1A_2}{|k|}$$

We use equation (1.14) and our previous methods to conclude

$$e^{-\nu(x-t)}\ |G_{32}(x,t)| \le \frac{2A_2}{|k|}\ .$$

Then

$$e^{-\nu x}\ \left|\int_{d_1}^{d_2} G_{32}(x,t)q(t)u_2(t)dt\right| \le \frac{6cA_1A_2}{|k|}$$

Finally we have

$$\begin{aligned} e^{-\nu x}\ \left|\int_{d_2}^{x} G_{33}(x,t)q(t)\varphi_3(t)dt\right| &\le \frac{1}{|k|}\int_{d_2}^{x} |q(t)e^{-\nu t}\ \varphi_3(t)|dt \\ &\le \frac{cA_1A_2}{|k|} \end{aligned}$$

Thus in the notation of Lemma 1 the inhomogeneous term in our integral equation satisfies

$$\begin{aligned} |f(x)| &\le \frac{cA_1A_2}{|k|}(\ 3+4+6+1\ ) \\ &\le \frac{14cA_1A_2}{|k|} \end{aligned}$$

As the kernel is $e^{-\nu(x-t)}\dfrac{\sin k(x-t)}{k}$ we have $L(x) \le \dfrac{1}{|k|}$. We apply Lemma 1 and use that $e^{1/3} < 2$ to conclude

$$|u_3(x)-\varphi_3(x)| \le \frac{28cA_1A_2}{|k|}e^{\nu x}$$

Combining the bounds for φ_3 and $|u_3-\varphi_3|$ and gives

$$|u_3(x)| \le 11A_1A_2e^{|\tau|x}$$

Now consider

$$
\begin{aligned}
|u_3'(x) - \varphi_3'(x)| \le\ & |g_3'(x) - \varphi_3'(x)| \\
& + \left|\int_0^{d_1} \frac{\partial G_{31}(x,t)}{\partial x} q(t) u_1(t)\,dt\right| \\
& + \left|\int_{d_1}^{d_2} \frac{\partial G_{32}(x,t)}{\partial x} q(t) u_2(t)\,dt\right| \\
& + \left|\int_{d_2}^{x} \frac{\partial G_{33}(x,t)}{\partial x} q(t) u_3(t)\,dt\right|
\end{aligned}
$$

Taking the derivative of $g_3 - \varphi_3$ has only the effect of multiplying by k, multiplying terms by ± 1 and changing sines and cosines, thus we inspect the definition of $g_3 - \varphi_3$ and use (1.30) to conclude that

$$|g_3'(x) - \varphi_3'(x)| \le 3A_1A_2ce^{|\tau|x}$$

Similarly the bounds for the partial derivatives of the kernels G_{31} and G_{32} differ by a factor of $|k|$ from the bounds for the kernels. Thus we have the first three terms of $|u_3' - \varphi_3'|$ bounded by

$$
\begin{aligned}
& cA_1A_2(3 + 4 + 6)e^{|\tau|x} \\
& \le 13cA_1A_2e^{|\tau|x}
\end{aligned}
$$

Finally $e^{-\nu(x-t)}\left|\frac{\partial G_{33}(x,t)}{\partial x}\right| = |\cos k(x-t)e^{-\nu(x-t)}| \le 1$. Hence from our bound for $|u_3|$ we have that

$$|u_3'(x) - \varphi_3'(x)| \le 24cA_1A_2e^{|\tau|x}$$

This completes the proof of Lemma 3.

We now investigate the distribution of the eigenvalues for the problem defined by equations (1.1)-(1.4). Our technique follows that of Titchmarsh [49]. I have been unable to obtain results without restricting the size of α_1 and α_2. The restriction on α_1 and α_2 is needed only in the proof of the next lemma. In a straightforward application of this technique the bounds on the sizes of the

jumps decrease proportionally to the number of discontinuities. Thus I do not see how to apply this method to eigenproblems with an arbitrary number of discontinuities.

Let $u(x,\lambda)$ be the solution of equation (1.1) which satisfies (1.3) and (1.4) and has the values $u = 1$ and $u' = h$ at $x=0$. Then λ is an eigenvalue if $u' + Hu = 0$ at $x = \pi$. Let $k = \sqrt{\lambda}$ and let R_n be the rectangle in the k plane with vertices $\pm\nu + i\cdot 0$ and $\pm\nu + i\nu$, where $\nu = n + \frac{1}{2}$, see Titchmarsh [49], page 13. Let Γ_n be the contour in the λ plane that corresponds to the points of R_n which lie in the upper half plane.

LEMMA 4: *Let u be the solution of (1.1), (1.3) and (1.4) with $u = 1$ and $u' = h$ at $x = 0$ and let $\omega(\lambda) = -u'(\pi) - Hu(\pi)$. Then ω is an entire function of λ of order ½ and its roots $\lambda_0 < \lambda_1 < \ldots\ldots$ are real and simple. Let $\sqrt{\lambda} = k = \sigma + i\tau$. Let $\alpha = |\alpha_1| + |\alpha_2| + |\alpha_1\alpha_2|$ and let $c = \max(|b_1|, |b_2|, |h|, |H|, \int_0^\pi |q|\,dt)$. If $\alpha < 1$ and*

$$n > \max\left(\frac{1+\alpha}{1-\alpha}, \frac{840c}{1-\alpha}\right)$$

then

$$|\omega| \geq \frac{\sqrt{\lambda}A_1A_2}{24} e^{|\tau|\pi}(1-\alpha) \tag{1.31}$$

for all points λ in the contour Γ_n.

Remark: If $a_1, a_2 < \sqrt{2}$ then $\alpha<1$.

Proof:

The jump conditions (1.3) and (1.4) at d_1 and d_2 ensure that

$$\int_0^\pi (-u'' + qu)v\, dx = \int_0^\pi (-v'' + qv)u\, dx$$

for all u and v which satisfy the boundary conditions and the jump conditions. Thus the usual arguments for a symmetric operator will show that the eigenvalues λ must be real, see Titchmarsh [49]. page 8. Moreover, Titchmarsh's proof can be used to show that the eigenvalues are simple, [49] page 4.

Consider the leading term ω_0 of ω. We obtain ω_0 by letting $b_1 = b_2 = h = H = 0$ and $q \equiv 0$. Thus

$$\omega_0(k^2) = -\varphi_3'(\pi) = \tag{1.32}$$

$$k\frac{A_1A_2}{4}(\sin k\pi + \alpha_1 \sin k(\pi-2d_1)$$

$$+ \alpha_2 \sin k(\pi-2d_2) + \alpha_1\alpha_2 \sin k(\pi-2d_2+2d_1))$$

Then ω_0 vanishes for $\lambda=0$. We show that the remaining zeros of ω_0 are positive. Assume $\lambda<0$ is a root and $k = i\tau$. As $\sin(i\tau y) = \sinh(\tau y)$ and the sinh function is strictly increasing we have

$$|\alpha_1 \sinh\tau(\pi-2d_1) + \alpha_2 \sinh\tau(\pi-2d_2) + \alpha_1\alpha_2 \sinh|\tau(\pi-2d_2+2d_1)$$
$$\leq \alpha \sinh(\tau\pi) < \sinh(\tau\pi)$$

for τ positive. Note that we have used $d_1, d_2 < \frac{\pi}{2}$. As

$$\omega_0 = \frac{kA_1A_2}{4}\begin{Bmatrix}(-1)^{n+1} + \delta_1 & \text{for } k = n - \frac{1}{2} \\ (-1)^n + \delta_2 & \text{for } k = n + \frac{1}{2}\end{Bmatrix}$$

where $|\delta_i| < 1$, there is a root of w_0 in each interval $n - \frac{1}{2} \leq k \leq n + \frac{1}{2}$.

To investigate the distribution of the eigenvalues we estimate ω_0 on Γ_n. We will show that $|\omega_0| > |\omega - \omega_0|$ on Γ_n. We can then apply Rouche's theorem to

show that the two functions have the same number of zeros in the interior of Γ_n. Since the assumptions of Lemma 4 implies that $c/|k| < 1/3$ it immediately follows from Lemma 3 that

$$|\omega - \omega_0| = |-u_3'(\pi) + \varphi_3'(\pi)| + |-Hu_3(\pi)|$$
$$\leq 35cA_1A_2\, e^{|\tau|\pi}$$

Let $k = \pm\nu + i\tau$ where $\nu = n + \frac{1}{2}$. Then

$$|w_0| \geq \frac{|k|A_1A_2}{4}[\cosh\tau\pi - |\alpha_1|\cosh\tau(\pi-2d_1)$$
$$- |\alpha_2|\cosh\tau(\pi-d_2) - |\alpha_1\alpha_2|\cosh\tau(\pi-2d_2+2d_1)]$$
$$\geq \frac{|k|A_1A_2}{8}e^{|\tau|\pi}(1-\alpha)$$

Now let $k = \sigma+i\nu=\sigma+i(n+\frac{1}{2})$. One can show that

$$\omega_0 \geq \frac{|k|A_1A_2}{4}[\sinh\nu\pi - \alpha\cosh\nu\pi]$$
$$\geq \frac{|k|A_1A_2}{8}e^{\nu\pi}(1-\alpha)(1-\frac{1+\alpha}{1-\alpha}e^{-2\nu\pi})$$
$$\geq \frac{|k|A_1A_2}{8}e^{\nu\pi}(1-\alpha)(1-e^{-\pi})$$

as $n > \frac{1+\alpha}{1-\alpha}$. But $1-e^{-\pi}>1/3 + 8/13$ and so by combining the above two estimates we get

$$|\omega_0| \geq \frac{A_1A_2|k|}{8}e^{|\tau|\pi}(1-\alpha)(1-e^{-\pi})$$
$$\geq \frac{A_1A_2|k|}{8}e^{|\tau|\pi}(1-\alpha)(\frac{1}{3}+\frac{8}{13})$$

on the three sides of R_n.

Now

$$|\omega - \omega_0| \leq 35cA_1A_2e^{|\tau|\pi}$$

$$< \frac{A_1 A_2 |k|}{24}(1-\alpha)e^{|\tau|\pi}$$

$$\leq |\omega_0|$$

Thus we apply Rouche's Theorem to conclude that both ω_0 and ω have at least $n+1$ roots in the interior of Γ_n.

Finally we combine our estimates for $|\omega - \omega_0|$ and $|\omega_0|$ to obtain equation (1.31). We have

$$|\omega| \geq |\omega_0| - |\omega - \omega_0|$$

$$\geq \frac{A_1 A_2 |k|}{13} e^{|\tau|\pi}(1-\alpha)$$

for n large. This completes the proof of Lemma 4.

2: Integral representation of the eigenfunctions

In this section we show that a solution of equations (1.1), (1.3) and (1.4) can be written as the sum of a trigonometric function and an integral. The importance of this representation is that it restricts the dependence of the eigenfunctions on the eigenvalue to the leading function and a cosine factor in the integrand. The trigonometric function can be found explicitly and can be used to approximate the asymptotic behavior of the solution. The kernel of the integral depends on the parameters a_i, a_i^{-1}, b_i , h and the potential q. There are two cases for its definition: $2d_1 \le d_2$ and $2d_1 > d_2$. The kernel will be defined by a different formula in each of the regions shown in Figures 2.1 and 2.2.

LEMMA 5: *Let u be the solution of equation (1.1), (1.3) and (1.4) that satisfies the initial conditions $u = 1$ and $u' = h$ at $x = 0$. Let $\lambda = k^2$ and $\varphi = \varphi_1$ for $0 \le x \le d_1$, φ_2 for $d_1 < x < d_2$ and φ_3 for $d_2 < x \le \pi$. Then there exists a bounded function $K(x,t)$ such that*

$$u(x,k^2) = \varphi(x,k^2) + \int_0^x K(x,t)\cos kt\,dt$$

for $0 \le x \le \pi$; $x \ne d_1, d_2$. Set $K(x,t) = 0$ for $t < 0$ or $t > x$.

Remark: This lemma is a generalization of the Povzner-Levitan representation of the eigenfunctions of a Sturm-Liouville problem. The complexity of the kernel increases as the number of discontinuities in the eigenproblem increases. In his study of eigenproblems with one discontinuity, Hald explicitly found the Fourier transform of the terms of $u-\varphi$ containing the factor $1/k$. We follow his technique but do not carry out the calculations to obtain this explicit formula for the problem with two discontinuities. The exact form

is not necessary for our purposes. We need only that the kernel is bounded. The kernel K will be continuous in the regions shown in Fig. 2.1 or Fig. 2.2.

Proof: From Lemmas 2 and 3 we see that the function $u - \varphi$ satisfies the conditions of the Paley-Wiener theorem. See Rudin [47] page 407. As u and φ are even in k and real for real values of k, we have

$$u(x,k^2) - \varphi(x,k^2) = \int_0^x K(x,t)\cos ktdt$$

where

$$K(x,t) = \frac{2}{\pi}\int_0^\infty (u - \varphi)(x,k^2)\cos ktdk$$

and $K(t)$ is square integrable on $(0,x)$. We now show that the kernel is bounded in x and t.

Hald has shown the result for $0 \le x < d_2$ [28]. Therefore we consider only the regions where $d_2 < x \le \pi$. Thus $\varphi=\varphi_3$ and $u=u_3$. Our proof involves two steps. We write

$$u(x,k^2) - \varphi(x,k^2) = E_1(x,k^2) + E_2(x,k^2)$$

where E_1 consists of all terms with the coefficient $1/k$ and E_2 is $O(\frac{1}{k^2})$. Then $K = \frac{2}{\pi}\int_0^\infty (E_1 + E_2)\cos ktdt$. We show by two separate arguments that $K_1 = \frac{2}{\pi}\int_0^\infty E_1\cos ktdt$ and $K_2 = \frac{2}{\pi}\int_0^\infty E_2\cos ktdt$ are bounded. It then follows that K is bounded.

First we consider E_1. From the definitions of φ_3 and u_3 we see that

$$E_1(x) = a_2[\frac{b_1}{k}\sin k(d_2-d_1)\cos kd_1 + \frac{h}{k}\{a_1^{-1}\sin k(d_2-d_1)\cos kd_1$$

$$+ a_1 \cos k(d_2-d_1)\sin kd_1\}]\cos k(x-d_2)$$

$$+a_2^{-1}[\frac{b_1}{k}\cos k(d_2-d_1)\cos kd_1 + \frac{h}{k}\{a_1^{-1}\cos k(d_2-d_1)\cos kd_1$$

$$- a_1 \sin k(d_2-d_1)\sin kd_1\}]\sin k(x-d_2)$$

$$+ \frac{b_2}{k}[a_1\cos k(d_2-d_1)\cos kd_1 - a_1^{-1}\sin k(d_2-d_1)\sin kd_1]\sin k(x-d_2)$$

$$+ \frac{1}{k}\int_0^{d_1}(a_2[a_1\cos k(d_2-d_1)\sin k(d_1-t)$$

$$+ a_1^{-1}\sin k(d_2-d_1)\cos k(d_1-t)]\cos k(x-d_2)$$

$$+ a_2^{-1}[-a_1\sin k(d_2-d_1)\sin k(d_1-t)$$

$$+ a_1^{-1}\cos k(d_2-d_1)\cos k(d_1-t))\sin k(x-d_2))q(t)\varphi_1(t)dt$$

$$+ \frac{1}{k}\int_{d_1}^{d_2}(a_2\cos k(x-d_2)\sin k(d_2-t)$$

$$+ a_2^{-1}\sin k(x-d_2)\cos k(d_2-t))q(t)\varphi_2(t)dt$$

$$+ \frac{1}{k}\int_{d_2}^{x}\sin k(x-t)q(t)\varphi_3(t)dt \quad .$$

The trigonometric identities allow us to write E_1 as

$$C_1\frac{\sin kx}{k} + C_2\frac{\sin k(x-2d_1)}{k} + C_3\frac{\sin k(x-2d_2)}{k}$$

$$+ C_4\frac{\sin k(x-2d_2+2d_1)}{k} + C_5\int_0^{d_1}\frac{\sin k(x-2s)}{k}q(s)ds$$

$$+ C_6\int_0^{d_1}\frac{\sin k(x-2d_1+2s)}{k}q(s)ds + C_7\int_0^{d_1}\frac{\sin k(x-2d_2+2s)}{k}q(s)ds$$

$$+ C_8\int_0^{d_1}\frac{\sin k(x-2d_2+2d_1-2s)}{k}q(s)ds + C_9\int_{d_1}^{d_2}\frac{\sin k(x-2s)}{k}q(s)ds$$

$$+ C_{10}\int_{d_1}^{d_2}\frac{\sin k(x-2s+2d_1)}{k}q(s)ds + C_{11}\int_{d_1}^{d_2}\frac{\sin k(x-2d_2+2s)}{k}q(s)ds$$

$$+ C_{12}\int_{d_1}^{d_2} \frac{\sin k(x-2d_2+2s-2d_1)}{k} q(s)ds + C_{13}\int_{d_2}^{x} \frac{\sin k(x-2s)}{k} q(s)ds$$

$$+ C_{14}\int_{d_2}^{x} \frac{\sin k(x-2s+2d_1)}{k} q(s)ds + C_{15}\int_{d_2}^{x} \frac{\sin k(x-2s+2d_2)}{k} q(s)ds$$

$$+ C_{16}\int_{d_2}^{x} \frac{\sin k(x-2d_2+2d_1-2s)}{k} q(s)ds \quad .$$

The coefficients C_i are products and sums of trigonometric functions, the parameters a_i, b_i, d_i, h and integrals of the form $\int_{s_1}^{s_2} q(s)ds$ where $0 \le s_1 < s_2 \le x$. Thus the coefficients are finite.

The first four terms of E_1 are of the form $C\frac{\sin k(x-b)}{k}$ where $b \ge 0$. We can write this as $C\int_0^{x-b} \cos ktdt$. Recall that $d_2 < x \le \pi$. By inspection we see that $b \le 2x$. Thus $-x \le x-b \le x$. If $x-b > 0$ we let

$$C\int_0^{x-b} \cos ktdt = C\int_0^{x} \chi_{[0,x-b]} \cos ktdt$$

where χ is the characteristic function. If $x-b<0$ we reflect about the t-axis to write

$$C\int_0^{x-b} \cos ktdt = -C\int_0^{b-x} \cos ktdt = \int_0^{x} -\chi_{[0,b-x]} \cos ktdt \quad .$$

By following this procedure we can explicitly find the contributions to K from the first four terms of E_1. It is clear that these contributions depend on the coefficients C_i and are bounded.

Next we consider the last twelve terms of E_1. To present the idea we examine the sixth term. We write

$$I_6 = \int_0^{d_1} \frac{\sin k(x-2d_1+2s)}{k} q(s)ds = \int_0^{d_1} q(s) \int_0^{x-2d_1+2s} \cos ktdt \quad .$$

We now change the order of integration to get

$$I_6 = \int_{x-2d_1}^{x} \cos kt \int_0^{(-x+t+2d_1)/2} q(s)dsdt \quad .$$

If $x-2d_1 \geq 0$ we write

$$I_6 = \int_0^x \cos kt \chi_{[x-2d_1,x]} \int_0^{(-x+t+2d_1)/2} q(s)\,ds\,dt \quad .$$

Otherwise we split the integral and reflect about the t-axis to obtain

$$I_6 = \int_0^x \cos kt - \chi_{[0,2d_1-x]} \int_0^{\frac{-x-t+2d_1}{2}} q(s)\,ds\,dt + \int_0^x \cos kt \int_0^{\frac{-x+t+2d_1}{2}} q(s)\,ds\,dt \quad .$$

In general the last twelve terms of E_1 have the form

$$I = \int_{s_1}^{s_2} \frac{\sin k(x-2s-b)}{k} q(s)\,ds$$

or

$$I = \int_{s_1}^{s_2} \frac{\sin k(x+2s-b)}{k} q(s)\,ds \quad .$$

We can see by inspection that $-x \leq x \pm 2s - b \leq x$. Thus the region of integration for each of the integrals is contained in the rectangle $-x \leq t \leq x$ and $0 \leq s \leq x$. We can therefore use the same procedure used for I_6 to obtain the explicit contribution to the kernel K from these terms. These contributions will depend on the coefficients C_i and integrals of the potential q over subintervals of $(0,\pi)$. As q is integrable, the kernel K_1 is bounded.

Finally we consider E_2. Let $k>0$. Note that in the equations (1.7), (1.11), (1.13) and (1.14) each factor k^{-n} is multiplied by $\sin k\beta_1 \cdots \sin k\beta_n$ where β_i depends on x, d_1 and d_2. Therefore each term of u_3 and φ_3 has finite limit as k tends to 0. The value of the limit depends on the parameters a_i, b_i and h. Thus E_2 is bounded as k tends to 0 and E_2 is continuous in x for $d_2 \leq x \leq \pi$ and we have $|E_2| \leq M_1$ for some constant M_1 if $d_2 \leq x \leq \pi$ and $0 \leq k \leq 1$. We see by inspection that E_2 consists of products and sums of trigonometric functions, the parameters of the differential equation and integrals of q, all divided by k^2. Hence $|E_2| \leq \frac{C}{k^2}$ for some constant C. We have

$$
\begin{aligned}
|K_2(x,t)| &= \left|\frac{2}{\pi}\int_0^\infty E_2(x,k^2)\cos kt\,dk\right| \\
&\le \left|\frac{2}{\pi}\int_0^1 E_2(x,k^2)\cos kt\,dk\right| + \left|\frac{2}{\pi}\int_1^\infty E_2(x,k^2)\cos kt\,dk\right| \\
&\le M_1 + \int_0^1 \frac{C}{k^2}dk \\
&\le M
\end{aligned}
$$

for some constant M. Therefore K_2 is bounded in x and t. This completes the proof of Lemma 5.

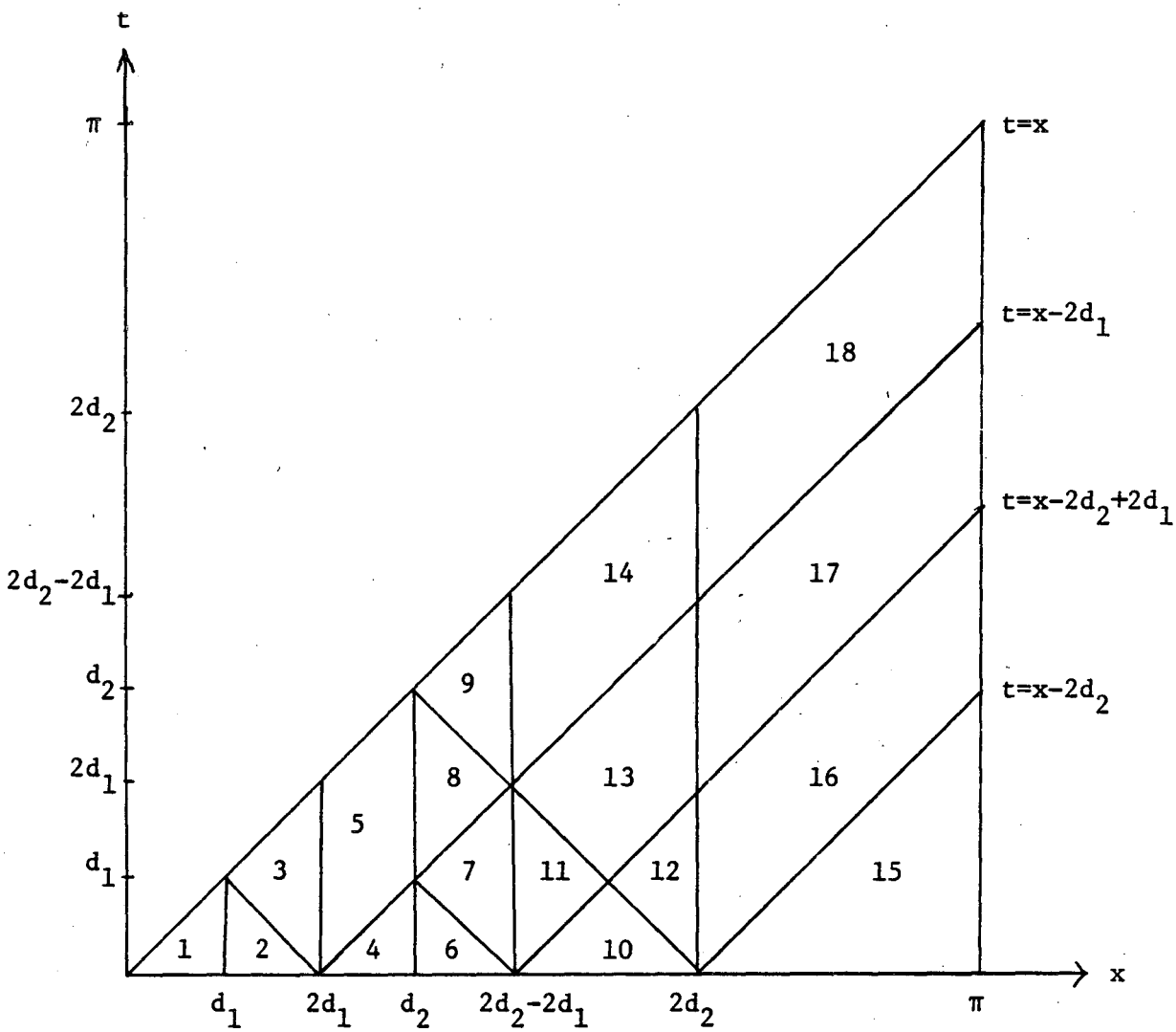


Figure 2.1. Case 1: $2d_1 \leq d_2$

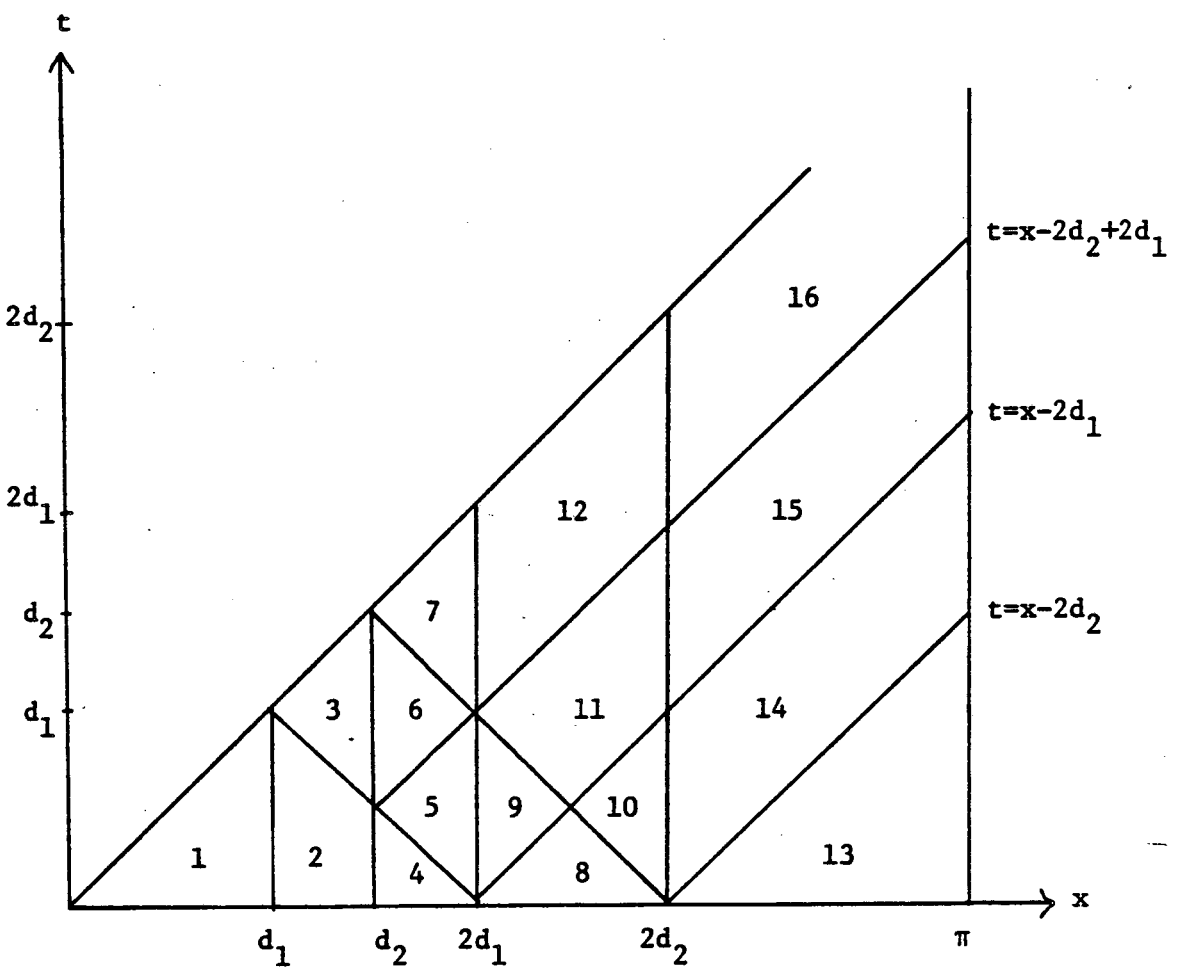


Figure 2.2. Case 2: $2d_1>d_2$

3: Uniqueness result for the positions and sizes of the discontinuities

In this section we examine the Wronskians of two different eigenvalue problems which have the same eigenvalues. By studying the asymptotic behavior of the difference of the Wronskians we show that the Wronskians are equal and that a_1, a_2, d_1 and d_2 are uniquely determined.

LEMMA 6: *The constants a_1, d_1, a_2, and d_2 are uniquely determined by the eigenvalues if* $0 < d_1 < d_2 < \pi/2$, $|a_i - 1| + |b_i| > 0$ *and* $\alpha = |\alpha_1| + |\alpha_2| + |\alpha_1\alpha_2| < 1$.

Proof: From Lemma 5 we have that the Wronskian $\omega(\lambda)$ is an entire function of order $1/2$. Thus it follows from Hadamard's theorem, [48] pages 249 and 250, that $\omega(\lambda) = \omega(0)\pi(1-\frac{\lambda}{\lambda_n})$ if all eigenvalues are different from zero. Here $\Pi(1-\frac{\lambda}{\lambda_n})$ is the canonical product of the genus zero formed by the eigenvalues. Let ω and $\tilde{\omega}$ be the Wronskians for two different eigenvalue problems which have the same eigenvalues. Then $\omega(\lambda) = C\tilde{\omega}(\lambda)$ for some $C \neq 0$ and $\omega_0 - C\tilde{\omega}_0 = C(\tilde{\omega} - \tilde{\omega}_0) - (\omega - \omega_0)$. From equation (1.32) follows

$$\begin{aligned}\frac{k}{4}[&(A_1A_2 - C\tilde{A}_1\tilde{A}_2)\sin k\pi + A_1A_2\alpha_1\sin k(\pi-2d_1) - C\tilde{A}_1\tilde{A}_2\tilde{\alpha}_1\sin k(\pi-2\tilde{d}_1) \\ &+ A_1A_2\alpha_2\sin k(\pi-2d_2) - C\tilde{A}_1\tilde{A}_2\tilde{\alpha}_2\sin k(\pi-2\tilde{d}_2) \\ &+ A_1A_2\alpha_1\alpha_2\sin k(\pi-2d_2+2d_1) - C\tilde{A}_1\tilde{A}_2\tilde{\alpha}_1\tilde{\alpha}_2\sin k(\pi-2\tilde{d}_2+2\tilde{d}_1]\end{aligned} \tag{3.1}$$

$$= C(\tilde{\omega} - \tilde{\omega}_0) - (\omega - \omega_0) .$$

Let $\bar{c} = \max(c, \tilde{c})$ where c and $\tilde{c}$ are defined as in Lemmas 2 and 3. We multiply equation (3.1) by $T^{-2}\sin k\pi$ and integrate with respect to k from $3\bar{c}$ to T. Since

$\omega_0 = -\varphi_3'(\pi)$ it follows from Lemma 3 that

$$|C(\tilde{\omega} - \tilde{\omega_0}) - (\omega - \omega_0)| \leq 48\bar{c}A_1A_2$$

for $k > 3\bar{c}$. Thus the integration yields

$$\frac{A_1A_2 - C\tilde{A_1}\tilde{A_2}}{4}[\frac{1}{4} + O(\frac{1}{T})] + O(\frac{1}{T}) = O(\frac{1}{T})$$

where we have used d_1, $d_2 < \pi/2$. We let T tend to infinity and conclude that $A_1A_2 = C\tilde{A_1}\tilde{A_2}$.

We now show that $\alpha_1 = \tilde{\alpha_1}$ and $\alpha_2 = \tilde{\alpha_2}$. There are two cases requiring different proofs. In the first case $d_1 = \tilde{d_1}$ and $d_2 = \tilde{d_2}$. The second case is characterized by $d_1 \neq \tilde{d_1}$ or $d_2 \neq \tilde{d_2}$. The proof is straightforward in the first case. We will show that the latter assumption leads to a contradiction of the hypotheses of the lemma.

First let $d_1 = \tilde{d_1}$ and $d_2 = \tilde{d_2}$ and $2d_1 \neq d_2$. Since $A_1A_2 = C\tilde{A_1}\tilde{A_2}$ it follows from equation (3.1) that

$$k\frac{A_1A_2}{4}[(\alpha_1 - \tilde{\alpha_1})\sin k(\pi - 2d_1) + (\alpha_2 - \tilde{\alpha_2})\sin k(\pi - 2d_2) \tag{3.2}$$

$$+ (\alpha_1\alpha_2 - \tilde{\alpha_1}\tilde{\alpha_2})\sin k(\pi - 2d_2 + 2d_1)$$

$$= C(\tilde{\omega} - \tilde{\omega_0}) - (\omega - \omega_0)$$

We multiply both sides of this equation by $T^{-2}\sin k(\pi - 2d_1)$ and integrate from $3\bar{c}$ to T to get

$$(\alpha_1 - \tilde{\alpha_1})[\frac{1}{4} + O(\frac{1}{T})] + O(\frac{1}{T}) = O(\frac{1}{T}) .$$

We let T tend to infinity and obtain $\alpha_1 = \tilde{\alpha_1}$. As $\alpha = \dfrac{a - a^{-1}}{a + a^{-1}}$ is strictly increasing for positive a and α_1, $\tilde{\alpha_1} \geq 0$, we find that $\alpha_1 = \tilde{\alpha_1}$. Similarly we multiply the

equation by $T^{-2}\sin k(\pi-2d_2)$ and integrate to conclude $a_2 = \tilde{a}_2$.

Now let $2d_1 = d_2$. Again we multiply both sides of equation (3.2) by $T^{-2}\sin k(\pi - 2d_1)$ and follow the same procedure to obtain $\alpha_1 - \tilde{\alpha}_1 + \alpha_1\alpha_2 - \tilde{\alpha}_1\tilde{\alpha}_2 = 0$. On the other hand, by multiplying with $T^{-2}\sin k(\pi - 2d_2)$ we get $\alpha_2 - \tilde{\alpha}_2 = 0$. We combine the two equations to obtain $(\alpha_1-\tilde{\alpha}_1)(1+\alpha_2) = 0$. As $|\alpha_2| < 1$ we have $\alpha_1 = \tilde{\alpha}_1$ and $\alpha_2 = \tilde{\alpha}_2$.

Now let $d_1 \neq \tilde{d}_1$ or $d_2 \neq \tilde{d}_2$. This part of the proof is more complicated than the first. We show that if either of the above assumptions are made then the problem degenerates into an eigenvalue problem with one or no discontinuities. Note that it is possible to have $d_1 = \tilde{d}_2$ or $\tilde{d}_2 = \tilde{d}_1$, but since $d_1 < d_2$ both equations cannot be true. We assume $d_1 \neq \tilde{d}_2$. There are 30 subcases which must be considered. We use only the above technique of multiplying by a sine function times T^{-2}, integrating with respect to k and letting T tend to infinity. It is convenient to recall that equation (3.1) can be written as

$$
\begin{aligned}
k\frac{A_1A_2}{4}[\ &\alpha_1\sin k(\pi-2d_1) - \tilde{\alpha}_1\sin k(\pi-2\tilde{d}_1) \\
&+ [\alpha_2\sin k(\pi-2d_2) - \tilde{\alpha}_2\sin k(\pi-2\tilde{d}_2)] \\
&+ \alpha_1\alpha_2\sin k(\pi-2d_2+2d_1) - \tilde{\alpha}_1\tilde{\alpha}_2\sin k(\pi-2\tilde{d}_2+2\tilde{d}_1) \\
= C\,&(\tilde{\omega}-\tilde{\omega}_0) - (\omega-\omega_0)
\end{aligned}
\tag{3.3}
$$

We consider the following cases.

CASE 1 $\quad d_1 = \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 \quad 2d_1 = d_2 \quad 2d_1 \neq \tilde{d}_2 \quad d_2 = \tilde{d}_2-\tilde{d}_1 \quad \tilde{d}_2 \neq d_2-d_1$

CASE 2 $d_1 = \tilde{d}_1$ $d_2 \neq \tilde{d}_2$ $2d_1 = d_2$ $2d_1 \neq \tilde{d}_2$ $d_2 \neq \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 = d_2 - d_1$

CASE 3 $d_1 = \tilde{d}_1$ $d_2 \neq \tilde{d}_2$ $2d_1 \neq d_2$ $2d_1 = \tilde{d}_2$ $d_2 = \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$

CASE 4 $d_1 = \tilde{d}_1$ $d_2 \neq \tilde{d}_2$ $2d_1 \neq d_2$ $2d_1 = \tilde{d}_2$ $d_2 \neq \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 = d_2 - d_1$

CASE 5 $d_1 = \tilde{d}_1$ $d_2 \neq \tilde{d}_2$ $2d_1 \neq d_2$ $2d_1 \neq \tilde{d}_2$ $d_2 \neq \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$

CASE 6 $d_1 = \tilde{d}_1$ $d_2 \neq \tilde{d}_2$ $2d_1 \neq d_2$ $2d_1 = \tilde{d}_2$ $d_2 = \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$

CASE 7 $d_1 = \tilde{d}_1$ $d_2 \neq \tilde{d}_2$ $2d_1 \neq d_2$ $2d_1 \neq \tilde{d}_2$ $d_2 \neq \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 = d_2 - d_1$

CASE 8 $d_1 \neq \tilde{d}_1$ $d_2 = \tilde{d}_2$ $2d_1 = d_2$ $2d_1 \neq \tilde{d}_2$ $d_2 \neq \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$

CASE 9 $d_1 \neq \tilde{d}_1$ $d_2 = \tilde{d}_2$ $2d_1 \neq d_2$ $2d_1 = \tilde{d}_2$ $d_2 \neq \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$

CASE 10 $d_1 \neq \tilde{d}_1$ $d_2 = \tilde{d}_2$ $2d_1 \neq d_2$ $2d_1 = \tilde{d}_2$ $d_2 \neq \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$

CASE 11 $d_2 = \tilde{d}_1$ $d_2 = \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$ $d_1 = d_2 - d_1$ $\tilde{d}_1 \neq d_2 - d_1$
$\tilde{d}_1 = \tilde{d}_2 - \tilde{d}_1$ $d_1 \neq \tilde{d}_2 - \tilde{d}_1$

CASE 12 $d_2 = \tilde{d}_1$ $d_2 \neq \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$ $d_1 = d_2 - d_1$ $\tilde{d}_1 \neq d_2 - d_1$
$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1$ $d_1 \neq \tilde{d}_2 - \tilde{d}_1$

CASE 13 $d_2 = \tilde{d}_1$ $d_2 \neq \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$ $d_1 \neq d_2 - d_1$ $\tilde{d}_1 \neq d_2 - d_1$
$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1$ $d_1 = \tilde{d}_2 - \tilde{d}_1$

CASE 14 $d_2 = \tilde{d}_1$ $d_2 \neq \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$ $d_1 \neq d_2 - d_1$ $\tilde{d}_1 \neq d_2 - d_1$
$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1$ $d_1 \neq \tilde{d}_2 - \tilde{d}_1$

CASE 15 $d_2 = \tilde{d}_1$ $d_2 = \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$ $d_1 = d_2 - d_1$ $\tilde{d}_1 \neq d_2 - d_1$
$\tilde{d}_1 = \tilde{d}_2 - \tilde{d}_1$ $d_1 \neq \tilde{d}_2 - \tilde{d}_1$

CASE 16 $d_2 = \tilde{d}_1$ $d_2 = \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$ $d_1 \neq d_2 - d_1$ $\tilde{d}_1 \neq d_2 - d_1$
$\tilde{d}_1 = \tilde{d}_2 - \tilde{d}_1$ $d_1 \neq \tilde{d}_2 - \tilde{d}_1$

In the next 13 cases $d_1 \neq \tilde{d}_1$ and $d_2 \neq \tilde{d}_2$.

CASE 17 $d_2 \neq \tilde{d}_1$ $d_2 = \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$ $d_1 = d_2 - d_1$ $\tilde{d}_1 \neq d_2 - d_1$
$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1$ $d_1 \neq \tilde{d}_2 - \tilde{d}_1$

CASE 18 $d_2 \neq \tilde{d}_1$ $d_2 = \tilde{d}_2 - \tilde{d}_1$ $\tilde{d}_2 \neq d_2 - d_1$ $d_1 \neq d_2 - d_1$ $\tilde{d}_1 = d_2 - d_1$

$$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1 \quad d_1 \neq \tilde{d}_2 - \tilde{d}_1$$

CASE 19 $d_2 \neq \tilde{d}_1 \quad d_2 = \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 \neq d_2 - d_1 \quad d_1 \neq d_2 - d_1 \quad \tilde{d}_1 \neq d_2 - d_1$

$$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1 \quad d_1 \neq \tilde{d}_2 - \tilde{d}_1$$

CASE 20 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 = d_2 - d_1 \quad d_1 \neq d_2 - d_1 \quad \tilde{d}_1 \neq d_2 - d_1$

$$\tilde{d}_1 = \tilde{d}_2 - \tilde{d}_1 \quad d_1 \neq \tilde{d}_2 - \tilde{d}_1$$

CASE 21 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 = d_2 - d_1 \quad d_1 \neq d_2 - d_1 \quad \tilde{d}_1 \neq d_2 - d_1$

$$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1 \quad d_1 = \tilde{d}_2 - \tilde{d}_1$$

CASE 22 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 = d_2 - d_1 \quad d_1 \neq d_2 - d_1 \quad \tilde{d}_1 \neq d_2 - d_1$

$$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1 \quad d_1 \neq \tilde{d}_2 - \tilde{d}_1$$

CASE 23 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 \neq d_2 - d_1 \quad d_1 \neq d_2 - d_1 \quad \tilde{d}_1 \neq d_2 - d_1$

$$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1 \quad d_1 \neq \tilde{d}_2 - \tilde{d}_1$$

CASE 24 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 \neq d_2 - d_1 \quad d_1 = d_2 - d_1 \quad \tilde{d}_1 \neq d_2 - d_1$

$$\tilde{d}_1 = \tilde{d}_2 - \tilde{d}_1 \quad d_1 \neq \tilde{d}_2 - \tilde{d}_1$$

CASE 25 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 \neq d_2 - d_1 \quad d_1 = d_2 - d_1 \quad \tilde{d}_1 \neq d_2 - d_1$

$$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1 \quad d_1 = \tilde{d}_2 - \tilde{d}_1$$

CASE 26 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 \neq d_2 - d_1 \quad d_1 = d_2 - d_1 \quad \tilde{d}_1 \neq d_2 - d_1$

$$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1 \quad d_1 \neq \tilde{d}_2 - \tilde{d}_1$$

CASE 27 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 \neq d_2 - d_1 \quad d_1 \neq d_2 - d_1 \quad \tilde{d}_1 = d_2 - d_1$

$$\tilde{d}_1 \neq \tilde{d}_2 - \tilde{d}_1 \quad d_1 = \tilde{d}_2 - \tilde{d}_1$$

CASE 28 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 \neq d_2 - d_1 \quad d_1 \neq d_2 - d_1 \quad \tilde{d}_1 = d_2 - d_1$

$$\tilde{d}_1 = \tilde{d}_2 - \tilde{d}_1 \quad d_1 \neq \tilde{d}_2 - \tilde{d}_1$$

CASE 29 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 \neq d_2 - d_1 \quad d_1 \neq d_2 - d_1 \quad \tilde{d}_1 \neq d_2 - d_1$

$$\tilde{d}_1 = \tilde{d}_2 - \tilde{d}_1 \quad d_1 \neq \tilde{d}_2 - \tilde{d}_1$$

CASE 30 $d_2 \neq \tilde{d}_1 \quad d_2 \neq \tilde{d}_2 - \tilde{d}_1 \quad \tilde{d}_2 \neq d_2 - d_1 \quad d_1 \neq d_2 - d_1 \quad \tilde{d}_1 = d_2 - d_1$

$$\tilde{d}_1 = \tilde{d}_2 - \tilde{d}_1 \quad d_1 \neq \tilde{d}_2 - \tilde{d}_1$$

For each case we use our technique of multiplying equation (3.) by T^{-2} times a

sine function, integrating and letting T tend to infinity. The appropriate functions and conclusions are given below.

CASE	MULTIPLY (3.3) BY	CONCLUDE
1	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1-\tilde{\alpha}_1+\alpha_1\alpha_2=0$
	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2-\tilde{\alpha}_1\tilde{\alpha}_2=0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$-\tilde{\alpha}_2=0$
2	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1-\tilde{\alpha}_1-\alpha_1\alpha_2=0$
	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2=0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$-\tilde{\alpha}_2+\alpha_1\alpha_2=0$
3	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1-\tilde{\alpha}_1-\tilde{\alpha}_1\tilde{\alpha}_2=0$
	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2-\tilde{\alpha}_1\tilde{\alpha}_2=0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$-\tilde{\alpha}_2=0$
4	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1-\tilde{\alpha}_1-\tilde{\alpha}_1\tilde{\alpha}_2=0$
	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2=0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$-\tilde{\alpha}_2+\alpha_1\alpha_2=0$
5	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1-\tilde{\alpha}_1=0$
	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2=0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$-\tilde{\alpha}_2=0$
6	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1-\tilde{\alpha}_1=0$
	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2-\tilde{\alpha}_1\tilde{\alpha}_2=0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2=0$
7	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1-\tilde{\alpha}_1=0$
	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2=0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$-\tilde{\alpha}_2+\alpha_1\alpha_2=0$
8	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2-\tilde{\alpha}_2=0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1(1+\alpha_2)=0$
	$T^{-2}\sin k(\pi-2\tilde{d}_1)$	$-\tilde{\alpha}_1=0$
9	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2-\tilde{\alpha}_2=0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1=0$

	$T^{-2}\sin k(\pi-2\tilde{d}_1)$	$-\tilde{\alpha}_1(1+\tilde{\alpha}_2) = 0$
10	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2-\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_1)$	$-\tilde{\alpha}_1 = 0$
11	$T^{-2}\sin k(\pi-2d_2)$	$-\tilde{\alpha}_1+\alpha_2-\tilde{\alpha}_1\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1(1-\alpha_2) = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$-\tilde{\alpha}_2 = 0$
12	$T^{-2}\sin k(\pi-2d_2)$	$-\tilde{\alpha}_1+\alpha_2-\tilde{\alpha}_1\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1(1-\alpha_2) = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$-\tilde{\alpha}_2 = 0$
13	$T^{-2}\sin k(\pi-2d_2)$	$-\tilde{\alpha}_1+\alpha_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1-\tilde{\alpha}_1\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2 = 0$
14	$T^{-2}\sin k(\pi-2d_2)$	$-\tilde{\alpha}_1+\alpha_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2 = 0$
15	$T^{-2}\sin k(\pi-2d_2)$	$-\tilde{\alpha}_1+\alpha_2-\tilde{\alpha}_1\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1(1-\alpha_2) = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2 = 0$
16	$T^{-2}\sin k(\pi-2d_2)$	$-\tilde{\alpha}_1+\alpha_2-\tilde{\alpha}_1\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2 = 0$
17	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2-\tilde{\alpha}_1\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1(1+\alpha_2) = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_1)$	$\tilde{\alpha}_1 = 0$
18	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2-\tilde{\alpha}_1\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_1)$	$\tilde{\alpha}_1-\alpha_1\alpha_2 = 0$
19	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2-\tilde{\alpha}_1\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_1)$	$\tilde{\alpha}_1 = 0$

20	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2-\alpha_1\alpha_2 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_1)$	$\tilde{\alpha}_1-\tilde{\alpha}_1\tilde{\alpha}_2 = 0$
21	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1-\tilde{\alpha}_1\tilde{\alpha}_2 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2-\alpha_1\alpha_2 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_1)$	$\tilde{\alpha}_1 = 0$
22	$T^{-2}\sin k(\pi-2d_2)$	$\alpha_2 = 0$
	$T^{-2}\sin k(\pi-2d_1)$	$\alpha_1 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2-\alpha_1\alpha_2 = 0$
	$T^{-2}\sin k(\pi-2\tilde{d}_1)$	$\tilde{\alpha}_1 = 0$
23-30	$\sin k(\pi-2d_2)$	$\alpha_2 = 0$
	$\sin k(\pi-2\tilde{d}_2)$	$\tilde{\alpha}_2 = 0$
	$\sin k(\pi-2d_1)$	$\alpha_1 = 0$
	$\sin k(\pi-2\tilde{d}_1)$	$\tilde{\alpha}_1 = 0$

In the Cases 23-30 we have used the first two results to obtain the second two. When $d_1 = d_2$ we have the possibility of an eigenvalue problem with one discontinuity. Thus in Cases 1 - 7 we get $\tilde{\alpha}_1 = \alpha_1$ and $\alpha_2 = \tilde{\alpha}_2 = 0$. Similarly in Cases 8 - 10 we have $\tilde{\alpha}_2 = \alpha_2$ and $\alpha_1 = \tilde{\alpha}_1 = 0$. As $d_2 = \tilde{d}_1$ in Cases 11 - 16 we get $\alpha_2 = \tilde{\alpha}_1$ and $\alpha_1 = \tilde{\alpha}_2 = 0$. In Cases 17 - 30 we have $\alpha_1 = \alpha_2 = \tilde{\alpha}_1 = \tilde{\alpha}_2 = 0$. We see that in all of the above cases $\alpha_1 = \tilde{\alpha}_1$ and $\alpha_2 = \tilde{\alpha}_2$. Thus $\alpha_1 = \tilde{\alpha}_1$, $\alpha_2 = \tilde{\alpha}_2$, and we conclude that $\omega_0 = \tilde{\omega}_0$ and $C = 1$.

To continue we must study the Wronskian in more detail. We have

$$\omega(k^2) = -(u_3'(\pi) + Hu_3(\pi)) \ .$$

By using the integral equation for u_3 and the product identities for the trigonometric functions we get

$$\omega(k^2) = \frac{kA_1A_2}{4}(\sin k\pi + \alpha_1 \sin k(\pi-2d_1) \tag{3.4}$$

$$+ \alpha_2 \sin k(\pi - 2d_2) + \alpha_1\alpha_2 \sin k(\pi - 2d_2 - 2d_1) \;\;)$$

$$+ \frac{1}{4}(A_1A_2 [\, -H - h + \frac{1}{2}\int_0^{\pi} q(s)ds \,]$$

$$- A_2b_1 - A_1b_2)\cos k\pi$$

$$+ \frac{1}{4}(A_1A_2\alpha_1 [\, -H + h - \frac{1}{2}\int_0^{d_1} q(s)ds + \frac{1}{2}\int_{d_1}^{\pi} q(s)ds]$$

$$- A_2b_1 + A_1\alpha_1b_2)\cos k(\pi - 2d_1)$$

$$+ \frac{1}{4}(A_1A_2\alpha_2[\, -H + h + \frac{1}{2}\int_0^{d_2} q(s)ds + \frac{1}{2}\int_{d_2}^{\pi} q(s)ds \,]$$

$$- A_1b_2 + A_2\alpha_2b_1)\cos k(\pi - 2d_2)$$

$$+ \frac{1}{4}(A_1A_2\alpha_1\alpha_2 [\, -H - h - \frac{1}{2}\int_0^{d_2} q(s)ds + \frac{1}{2}\int_{d_2}^{\pi} q(s)ds \,]$$

$$+ A_2\alpha_2b_1 - A_1\alpha_1b_2 \;) \cos k(\pi - 2d_2 + 2d_1)$$

$$+ \frac{A_1A_2}{8}(\int_0^{\pi} \cos k(\pi - 2t)q(t)dt - \alpha_1 \int_0^{d_1} \cos k(\pi - 2d_1 + 2t)q(t)dt$$

$$+ \alpha_1 \int_{d_1}^{\pi} \cos k(\pi + 2d_1 - 2t)q(t)dt - \alpha_2 \int_0^{d_2} \cos k(\pi - 2d_2 + 2t)q(t)dt$$

$$+ \alpha_2 \int_{d_2}^{\pi} \cos k(\pi + 2d_2 - 2t)q(t)dt$$

$$+ \alpha_1\alpha_2 \int_0^{d_1} \cos k(\pi - 2d_2 - + 2d_1 - 2t)q(t)dt$$

$$+ \alpha_1\alpha_2 \int_{d_1}^{d_2} \cos k(\pi - 2d_2 - 2d_1 + 2t)q(t)dt$$

$$+ \alpha_1\alpha_2 \int_{d_2}^{\pi} \cos k(\pi + 2d_2 - 2d_1 - 2t)q(t)dt + E$$

where E is $O(\frac{1}{k})$. Note that the arguments of the cosine function in the eight integrands lie in the interval $[\, -k\pi, k\pi]$ and that the arguments of the potential q lie in the interval $[0, \pi]$. Thus each integral can be written as sums of integrals of the form $\int_{t_1}^{t_2} \cos kt q(s(t))\, dt$ where $-\pi \le t_1 < t_2 \le \pi$, see proof of

Lemma 5. We can therefore combine the integrals to obtain a single term of the form $E = \int_0^{\pi} V_1(t)\cos kt\,dt$ where $V_1(t)$ depends on the a_i and the potential q.

The term E consists of all terms of ω which are $O(\frac{1}{k^n})$ for $n \geq 1$. Thus

$$
\begin{aligned}
E = & -H\,(\,u_3(\pi) - \varphi_3(\pi)\,) \\
& + \frac{hb_1}{k}[\,-a_2 \sin kd_1 \sin k(d_2-d_1)\sin k(\pi-d_2) \\
& + \quad a_2^{-1}\sin kd_1 \cos k(d_2-d_1)\cos k(\pi-d_2)\,] \\
& + \frac{b_2}{k}[\,\cos kd_1 \sin k(d_2-d_1) + h\,\{a_1^{-1}\cos kd_1 \sin k(d_2-d_1) \\
& + \quad a_1 \sin kd_1 \cos k(d_2-d_1) + \frac{b_1}{k}\sin kd_1 \sin k(d_2-d_1)\}]\cos k(\pi-d_2) \\
& + \frac{1}{k}\int_0^{d_1}[\,-a_2 b_1 \sin k(d_2-d_1)\sin k(d_1-t)\sin k(\pi-d_2) \\
& \qquad + a_2^{-1} b_1 \cos k(d_2-d_1)\sin k(d_1-t)\cos k(\pi-d_2) \\
& \qquad + b_2\{a_1 \cos k(d_2-d_1)\sin k(d_1-t) + a_1^{-1}\sin k(d_2-d_1)\cos k(d_1-t) \\
& \qquad + \frac{b_1}{k}\sin k(d_2-d_1)\sin k(d_1-t)\}\cos k(\pi-d_2)]\varphi_1(t)q(t)dt \\
& + \frac{1}{k}\int_{d_1}^{d_2} b_2 \sin k(d_2-t)\cos k(\pi-d_2)\varphi_2(t)q(t)dt \\
& + \int_0^{d_1}\frac{\partial G_{31}(\pi,t)}{\partial x}(u_1(t) - \varphi_1(t))q(t)dt \\
& + \int_{d_1}^{d_2}\frac{\partial G_{32}(\pi,t)}{\partial x}(u_2(t) - \varphi_2(t))q(t)dt \\
& + \int_0^{\pi}\frac{\partial G_{33}(\pi,t)}{\partial x}(u_3(t) - \varphi_3(t))q(t)dt
\end{aligned}
$$

Let $k = \sigma + i\tau$. We can use the inequalities

$$\frac{|\sin kx|}{|kx|},\ |\cos kx|,\ |\sin kx| \le e^{|\tau|x}$$

and Lemmas 2 and 3 to conclude $|E(k^2)| \le |k|^{-1}Ce^{|\tau|\pi}$. As E is real for real k and even in k, it follows from the Paley-Wiener theorem that

$$E(k^2) = \int_0^\pi V_2(t)\cos kt\,dt$$

where V_2 is a square integrable function. Thus we can write the Wronskian as

$$\begin{aligned}\omega(k^2) &= \omega_0 + C_0\cos k\pi + C_1\cos k(\pi-2d_1) \\ &+ C_2\cos k(\pi-2d_2) + C_3\cos k(\pi-2d_2+2d_1) + \int_0^\pi V(t)\cos kt\,dt\ .\end{aligned} \tag{3.5}$$

Since $\omega_0 = \tilde{\omega}_0$ and $C = 1$, we have

$$\begin{aligned}\omega - \tilde{\omega} &= (C_0-\tilde{C}_0)\cos k\pi + C_1\cos k(\pi-2d_1) - \tilde{C}_1\cos k(\pi-2\tilde{d}_1) \\ &+ C_2\cos k(\pi-2d_2) - C_2\cos k\beta(\pi-2\tilde{d}_2) \\ &+ C_3\cos k(\pi-2d_2+2d_1) - \tilde{C}_3\cos k(\pi-2\tilde{d}_2+2\tilde{d}_1) + \int_0^\pi (V-\tilde{V})\cos kt\,dt\end{aligned} \tag{3.6}$$

To present the idea of the remainder of the proof we consider Case 23. Thus $d_1 \ne \tilde{d}_1$, $d_1 \ne \tilde{d}_2$, $d_1 \ne d_2-d_1$ and $d_1 \ne \tilde{d}_2-\tilde{d}_1$ and we have $\alpha_1 = \alpha_2 = \tilde{\alpha}_1 = \tilde{\alpha}_2 = 0$. We multiply the above equation by $T^{-1}\cos k(\pi-2d_1)$, we integrate with respect to k from $3\bar{c}$ to T and arrive at

$$C_1\left(\frac{1}{2} + O\left(\frac{1}{T}\right)\right) + T^{-1}\int_0^\pi (V - \tilde{C}\tilde{V})\int_{3\bar{c}}^T \cos kt\cos k\beta_1\,dk\,dt = 0$$

where we have used Fubini's theorem to interchange the order of integration. We let T tend to infinity and get $C_1 = 0$. However $\alpha_1 = 0$. Thus $C_1 = \dfrac{-b_1A_2}{4}$ and we conclude that $b_1 = 0$, which contradicts the hypothesis of the lemma.

Now consider Cases 1 - 7. We see that $\alpha_1 = \tilde{\alpha}_1$ for each of these cases and $\alpha_2 = \tilde{\alpha}_2 = 0$. If $\tilde{d}_2 \ne d_2-d_1$ we multiply equation (3.6) by $T^{-1}\sin k(\pi-2\tilde{d}_2)$, integrate and take the limit. We then get $C\tilde{C}_2 = 0$. But $\tilde{\alpha}_2 = 0$ and thus $\tilde{b}_2 = 0$. If $\tilde{d}_2 = d_2-d_1$, then $d_2 \ne \tilde{d}_2-\tilde{d}_1$ and we multiply by $T^{-1}\cos k(\pi-2d_2)$ and proceed

in a similar manner to conclude that $b_2 = 0$. In Cases 8 - 10, $\alpha_1 = \tilde{\alpha}_1 = 0$. As $d_2 \neq \tilde{d}_2$ and $d_1 \neq \tilde{d}_1$, we cannot have both $2d_1 = d_2$ and $2d_1 = \tilde{d}_2$. Thus we can follow the previous technique to conclude either $\tilde{C} = 0$ and $\tilde{b}_1 = 0$ or $\tilde{C}_1 = 0$ and $b_1 = 0$. The Cases 11 - 26 have $\tilde{d}_2 > \tilde{d}_1 = d_2$, and $\tilde{\alpha}_2 = 0$. Thus $\tilde{d}_2 \neq d_2 - d_1$ and we multiply by $T^{-1}\cos k(\pi - \tilde{d}_2)$ to conclude $\tilde{b}_2 = 0$. Finally in cases 27 - 30, we have $\alpha_1 = \alpha_2 = \tilde{\alpha}_1 = \tilde{\alpha}_2 = 0$. Also $\tilde{d}_1 \neq d_2$ and $\tilde{d}_1 \neq d_1$. Thus multiplication by $T^{-1}\cos k(\pi - 2\tilde{d}_1)$ leads to $\tilde{b}_1 = 0$.

We argue by contradiction that the eigenvalues of a problem with two discontinuities cannot have the same eigenvalues as a problem with one or no discontinuities, provided that all discontinuities occur in the interval $(0, \frac{\pi}{2})$. Suppose $\tilde{a}_2^2 = 1$ and $\tilde{b}_2 = 0$ and thus the second problem has at most one discontinuity. It follows as before that $A_1 A_2 = C\tilde{A}_1\tilde{A}_2$. We multiply equation (3.3) by $T^{-2}\sin k(\pi - 2d_2)$ and use our previous method to conclude that

$$\alpha_2 - \tilde{\alpha}_1 = 0 \quad \text{if} \quad d_2 = \tilde{d}_1$$

$$\alpha_2 = 0 \quad \text{if} \quad d_2 \neq \tilde{d}_1$$

In the first case we multiply by $T^{-2}\sin k(\pi - 2d_1)$ to conclude $\alpha_1 = 0$. We then follow the arguments for Cases 11 - 17 to conclude that $b_1 = 0$. Similarly, if $\alpha_2 = 0$ then we can show $b_2 = 0$ as before. Thus both cases lead to a contradiction.

4: An integral equation for the difference of the potentials

In this section we assume that two different eigenvalue problems have the same eigenvaues and derive an integral equation for the difference of the two potentials. In Section 6 we use this integral equation to prove that the potentials are equal.

LEMMA 7: *Let $u = u(x,\lambda)$ be the solution of equation (1.1) that satisfies the initial condition $u = 1$, $u' = h$ at $x = 0$ and the jump conditions (1.3) and (1.4). Let $\tilde{u}$ be defined similarly with a_1, a_2, b_1, b_2, d_1, d_2, h, H and q replaced by $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1$, $\tilde{b}_2$, $\tilde{d}_1$, $\tilde{d}_2$, $\tilde{h}$, $\tilde{H}$ and $\tilde{q}$. Let $\alpha = |\alpha_1| + |\alpha_2| + |\alpha_1\alpha_2| < 1$ and $\tilde{\alpha} = |\tilde{\alpha}_1| + |\tilde{\alpha}_2| + |\alpha_1\alpha_2| < 1$. Set $u_{i-} = u(d_{i-})$ and $\tilde{u}_{i-} = \tilde{u}(\tilde{d}_{i-})$. If $\lambda_j = \tilde{\lambda}_j$ for $j \geq 0$, $H = \tilde{H}$ and $q = \tilde{q}$ almost everywhere in $(\pi/2,\pi)$, then $a_i = \tilde{a}_i$, $d_i = \tilde{d}_i$ for $i=1,2$ and*

$$b_2 - \tilde{b}_2 = -\frac{a_2^2 - a_2^{-2}}{2a_2}\int_{d_2}^{\frac{\pi}{2}}(q - \tilde{q})(t)dt \tag{4.1}$$

$$b_1 - \tilde{b}_1 = -\frac{a_1^2 - a_1^{-2}}{2a_1}\int_{d_1}^{d_2}(q - \tilde{q})(t)dt - \frac{a_1^2 - a_1^{-2}}{2a_2^2 a_1}\int_{d_2}^{\frac{\pi}{2}}(q - \tilde{q})(t)dt \tag{4.2}$$

$$h - \tilde{h} = -\frac{1}{2}\int_0^{d_1}(q - \tilde{q})(t)dt - \frac{1}{2a_1^2}\int_{d_1}^{d_2}(q - \tilde{q})(t)dt \tag{4.3}$$

$$- \frac{1}{2a_1^2 a_2^2}\int_{d_2}^{\frac{\pi}{2}}(q - \tilde{q})(t)dt$$

$$\int_0^{d_1}(q-\tilde{q})(t)\,[u\tilde{u}(t)-\frac{1}{2}]dt \tag{4.4}$$

$$+\int_{d_1}^{d_2}(q-\tilde{q})(t)\,[u\tilde{u}(t)-\frac{1}{2a_1^2}-\frac{a_1^2-a_1^{-2}}{2}u_{1-}\tilde{u}_{1-}]dt$$

$$+\int_{d_2}^{\frac{\pi}{2}}(q-\tilde{q})(t)[u\tilde{u}(t)-\frac{1}{2a_1^2a_2^2}-\frac{a_1^2-a_1^{-2}}{2a_2^2}u_{1-}\tilde{u}_{1-}$$

$$-\frac{a_2^2-a_2^{-2}}{2}u_{2-}\tilde{u}_{2-}]dt$$

$$=0\ .$$

Remark: If the two potentials are equal in the first half of the interval, then it follows from (4.1), (4.2), and (4.3) that $b_1=\tilde{b}_1$, $b_2=\tilde{b}_2$, and $h=\tilde{h}$. If $a_2=1$ and $b_2=\tilde{b}_2=0$, then equation (4.4) reduces to the integral equation considered by Hald and equations (4.2) and (4.3) reduce to the formula for the parameters of an eigenproblem with one discontinuity. If, in addition, $a_1=1$ and $b_1=\tilde{b}_1=0$, then equation (4.4) reduces to the integral equation for the difference of two potentials of a regular Sturm-Liouville problem considered by Hochstadt and Lieberman [30] and Hald [26].

Proof: Since the two Sturm-Liouville problems have the same eigenvalues, it follows from Lemma 6 that $a_i=\tilde{a}_i$ and $d_i=\tilde{d}_i$ for $n=1,2$. Moreover, since the eigenvalues are equal, we have that

$$(u\tilde{u}'-\tilde{u}u')'+(q-\tilde{q})u\tilde{u}=0$$

where u and $\tilde{u}$ are eigenfunctions with the same eigenvalue. By integrating the above equation from 0 to π, using integration by parts, the boundary conditions and the jump conditions we obtain

$$(h-\tilde{h})+a_1(b_1-\tilde{b}_1)u_{1-}\tilde{u}_{1-}+a_2(b_2-\tilde{b}_2)u_{2-}\tilde{u}_{2-} \tag{4.5}$$

$$+ \int_0^{\frac{\pi}{2}} (q - \tilde{q})(t) u \tilde{u}(t) dt = 0 \ .$$

Let $k = \sqrt{\lambda} = \sigma + i\tau$. We define the function $\Phi(\lambda)$ to be the left-hand side of equation (4.5). Then $\Phi(\lambda) = 0$ for any eigenvalue λ of the two problems. We will show that $\Phi(\lambda) \equiv 0$. Since u and $\tilde{u}$ are entire functions of λ, Φ is an entire function of λ. Let $\psi = \frac{\Phi}{\omega}$ where ω is the Wronskian of the two eigenproblems. From Lemma 4 we have $|\omega| > \text{constant}\sqrt{|\lambda|} e^{|\tau|\pi}$ on the contour Γ_n for all n sufficiently large. Let c be defined as in Lemma 3 and let $\bar{c} = \max(c, \tilde{c})$. From Lemma 3 follows

$$\begin{aligned} |\Phi| &\leq 2\bar{c}(1 + a_1 4 e^{2|\tau| d_1} + a_2 9 A_1^2 e^{2|\tau| d_2} + 11^2 A_1^2 A_2^2 e^{2|\tau| \frac{\pi}{2}}) \\ &\leq 250 \bar{c} A_1^2 A_2^2 e^{|\tau|\pi} \end{aligned}$$

since $A_i \geq 2$. Both Φ and ω are entire functions of λ and $\Phi(\lambda) = 0$ for every zero of ω. Moreover, the zeros of ω are simple. Thus $\psi = \frac{\Phi}{\omega}$ is an entire function and

$$|\psi| \leq \frac{C\bar{c} A_1 A_2}{\sqrt{|\lambda|}}$$

on the curve Γ_n for n sufficiently large and C constant. By the maximum modulus principle ψ is bounded. Since ψ becomes arbitrarily small we have $\psi \equiv 0$ and it follows that $\Phi \equiv 0$.

We write Φ as

$$\begin{aligned} \Phi(\lambda) = h - \tilde{h} &+ a_1(b_1 - \tilde{b}_1)\varphi_1^2(d_1) + a_2(b_2 - \tilde{b}_2)\varphi_2^2(d_2) \\ &+ \int_0^{\frac{\pi}{2}} (q - \tilde{q})(t) \varphi_1^2(t) dt + E \end{aligned}$$

where $\varphi = \varphi_1$ for $0 \leq t \leq d_1$; φ_2 for $d_1 < t < d_2$ and φ_3 for $d_2 < t < \frac{\pi}{2}$. The term E is given by

$$E = a_1(b_1 - \tilde{b}_1)[(u_{1^-} - \varphi_1(d_1))\tilde{u}_1 + \varphi_1(d_1)(\tilde{u}_{1^-} - \varphi_1(d_1))]$$
$$+ a_2(b_2 - \tilde{b}_2)[(u_{2^-} - \varphi_2(d_2))\tilde{u}_{2^-} + \varphi_2(d_2)(\tilde{u}_{2^-} - \varphi_2(d_2))]$$
$$+ \int_0^{\frac{\pi}{2}} (q - \tilde{q})(t)[(u(t) - \varphi(t))\tilde{u}(t) + \varphi(t)(\tilde{u}(t) - \varphi(t)))]dt$$

Let $\lambda > 0$ and $k \geq 3\bar{c}$. It follows from Lemma 3 that

$$|E| \leq \frac{a_1 2\bar{c}}{|k|}(12\bar{c}) + \frac{2a_2\bar{c}}{|k|}(30A_1^2\bar{c}) + \frac{2\bar{c}}{|k|}(616A_1^2A_2^2\bar{c})$$
$$\leq \frac{1400\bar{c}^2}{|k|}A_1^2A_2^2$$

and thus E is $O(\frac{1}{k})$ for k real.

By using the definition of φ and trigonometric identities we get

$$\Phi(\lambda) = A + B\cos 2kd_1 + C\cos 2kd_2 + D\cos 2k(d_2 - d_1) \tag{4.6}$$
$$+ F\cos 2k(d_2 - 2d_1) + I + E = 0$$

where

$$A = h - \tilde{h} + \frac{a_1(b_1 - \tilde{b}_1)}{2} + \frac{a_2(a_1^2 + a_1^{-2})}{4}(b_2 - \tilde{b}_2) + \frac{1}{2}\int_0^{d_1}(q - \tilde{q})(t)dt$$
$$+ \frac{a_1^2 + a_1^{-2}}{4}\int_{d_1}^{d_2}(q - \tilde{q})(t)dt + \frac{(a_1^2 + a_1^{-2})(a_2^2 + a_2^{-2})}{8}\int_{d_2}^{\frac{\pi}{2}}(q - \tilde{q})(t)dt$$

$$B = \frac{a_1(b_1 - \tilde{b}_1)}{2} + \frac{a_2(a_1^2 - a_1^{-2})(b_2 - \tilde{b}_2)}{4} + \frac{(a_1^2 + a_1^{-2})}{4}\int_{d_1}^{d_2}(q - \tilde{q})(t)dt$$
$$\frac{(a_1^2 - a_1^{-2})(a_2^2 + a_2^{-2})}{8}\int_{d_2}^{\frac{\pi}{2}}(q - \tilde{q})(t)dt$$

$$C = \frac{A_1^2 a_2(b_2 - \tilde{b}_2)}{8} + \frac{A_1^2(a_2^2 - a_2^{-2})}{16}\int_{d_2}^{\frac{\pi}{2}}(q - \tilde{q})(t)dt$$

$$D = \frac{(a_1^2 - a_1^{-2})a_2(b_2 - \tilde{b}_2)}{4} + \frac{(a_1^2 - a_1^{-2})(a_2^2 - a_2^{-2})}{8}\int_{d_2}^{\frac{\pi}{2}}(q - \tilde{q})(t)dt$$

$$F = \frac{(a_1 - a_1^{-1})^2 a_2(b_2 - \tilde{b}_2)}{8} + \frac{(a_1 - a_1^{-1})^2(a_2^2 - a_2^{-2})}{16}\int_{d_2}^{\frac{\pi}{2}}(q - \tilde{q})dt$$

The term I is of the form

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} C_1(q - \tilde{q})(t)\cos 2kt dt \\
&\quad + \int_{d_1}^{\frac{\pi}{2}}(q - \tilde{q})(t)[C_2\cos 2k(t - 2d_1) + C_3\cos 2k(t - d_1)]dt \\
&\quad + \int_{d_2}^{\frac{\pi}{2}}(q - \tilde{q})(t)[C_4\cos 2k(t - 2d_2) + C_5\cos 2k(t + 2d_1 - 2d_2) \\
&\qquad + C_6\cos 2k(t - d_2) + C_7\cos 2k(t + d_1 - d_2) \\
&\qquad + C_8\cos 2k(t - d_1 - d_2) + C_9\cos 2k(t - 2d_2 + d_1)]dt
\end{aligned}$$

The coefficients C_i depend on the parameters a_1 and a_2. Note that C_1, C_2, and C_3 may be defined differently on the intervals $(0,d_1)$, (d_1,d_2) and $(d_2,\frac{\pi}{2})$. The calculation of the C_i is straightforward but will not be presented here. Except for the factor k, the arguments of the cosine functions in the integrand of I lie in the interval $(-\pi,\pi)$. Therefore we may follow our previous procedure from the proof of Lemma 5 to obtain

$$I = \int_0^{\frac{\pi}{2}} V(t)\cos 2kt dt$$

where V depends on the coefficients C_i and the values of the potentials q and $\tilde{q}$ in the interval $(0,\pi)$. We will now show that $A = B = C = D = F = 0$. Recall that $0 < d_1 < d_2$. We multiply equation (4.6) by $T^{-1}\cos 2kd_2$ and integrate the resulting equation from $3\bar{c}$ to T with respect to K. Then our estimate of E and Fubini's theorem give

$$(A + B + D + F)\, O(\frac{1}{T}) + C(\frac{1}{2} + O(\frac{1}{T}))$$

$$+ \int_0^{\frac{\pi}{2}} V(t)\, \frac{1}{T}\int_{3\pi}^{T} \cos 2kt \cos 2kd_2 dkdt + O(\frac{\log T}{T}) = 0 \ .$$

The integrand $T^{-1}\int_{3\pi}^{T} \cos 2kd_2 \cos 2ktdk$ is bounded by 1 and tends to zero as T tends to infinity if $t \neq d_2$. We let T tend to infinity and apply Lebesque's theorem of dominated convergence to get

$$\int_0^{\frac{\pi}{2}} V(t)\, T^{-1} \int_{3\pi}^{T} \cos 2kd \cos 2ktdkdt \ \to 0 \ .$$

Thus $C = 0$ and we obtain equation (4.1). Now $\alpha_1 C = 2D$ and $\alpha_1^2 C = F$. Hence $C = D = F = 0$ and equation (4.6) can be simplified to

$$A + B\cos 2kd_1 + I + E = 0$$

We multiply by $T^{-1}\cos 2kd_1$ and follow the same procedure as before to conclude $B = 0$. This and equation (4.1) lead to (4.2).

We now show $A = 0$. Recall that $I = \int_0^{\frac{\pi}{2}} V(t)\cos 2ktdt$. By the Riemann-Lebesgue lemma we have I tends to zero as k tends to infinity. Since $E = O(\frac{1}{k})$, we can let k become arbitrarily large and conclude $A = 0$. We combine this result with equations (4.1) and (4.2) to arrive at (4.3). We then use the definition of Φ to obtain the integral equation (4.4).

This completes the proof of Lemma 7.

5: The product of eigenfunctions

We derive a formula for the product of the eigenfunctions of two different eigenproblems. This formula will allow us to obtain an integral equation for the difference of the two potentials which is independent of λ.

LEMMA 8: *Let u, $\tilde{u}$, u_- and $\tilde{u}_-$ be defined as in Lemma 7 and assume that $d_1=\tilde{d}_1$, $d_2=\tilde{d}_2$ and $a_1=\tilde{a}_1$, $a_2=\tilde{a}_2$. Let $k=\sqrt{\lambda}$. Then there exists a bounded function $\overline{K}(x,t)$ such that*

$$u\tilde{u}-\frac{1}{2}=\frac{1}{2}\cos 2kx+\frac{1}{2}\int_0^x \overline{K}(x,t)\cos 2kt\,dt \tag{5.1}$$

for $0\le x<d_1$.

$$u\tilde{u}-\frac{1}{2a_1^2}-\frac{a_1^2-a_1^{-2}}{2}u_-\tilde{u}_-=\frac{1}{2}\Big[\frac{A_1^2}{4}\cos 2kx+\frac{a_1^2-a_1^{-2}}{2}\cos 2k(x-d_1) \tag{5.2}$$
$$+\frac{(a_1-a_1^{-1})^2}{4}\cos 2k(x-2d_1)\Big]+\frac{1}{2}\int_0^x \overline{K}(x,t)\cos 2kt\,dt$$

for $d_1<x<d_2$.

$$u\tilde{u}-\frac{1}{2a_1^2a_2^2}-\frac{(a_1^2-a_1^{-2})}{2a_2^2}u_{1-}\tilde{u}_{1-}-\frac{(a_2^2-a_2^{-2})}{2}u_{2-}\tilde{u}_{2-} \tag{5.3}$$
$$=\frac{A_1^2A_2^2}{32}\cos 2kx+\frac{(a_1^2-a_1^{-2})A_2^2}{16}\cos 2k(x-d_1)$$
$$+\frac{(a_1^2+a_1^{-1})(a_2^2-a_2^{-2})}{8}\cos 2k(x-d_2)+\frac{(a_1-a_1^{-1})^2A_2^2}{32}\cos 2k(x-2d_1)$$
$$+\frac{A_1^2(a_2-a_2^{-1})^2}{32}\cos 2k(x-2d_2)+\frac{(a_1^2-a_1^{-2})(a_2^2-a_2^{-2})}{16}\cos 2k(x-d_2-d_1)$$

$$+ \frac{(a_1^2-a_1^{-2})(a_2^2-a_2^{-2})}{16}\cos 2k(x-d_2+d_1)$$

$$+ \frac{(a_1^2-a_1^{-2})(a_2^2-a_2^{-2})^2}{16}\cos 2k(x-2d_2+d_1)$$

$$+ \frac{(a_1-a_1^{-1})^2(a_2-a_2^{-1})^2}{32}\cos 2k(x-2d_2+2d_1)$$

for $d_2 < x \leq \pi$.

Remark: The representation for $0 \leq x < d_1$ is well known, see Levitan [...]. The representation on the second interval was derived by Hald in his study of the eigenvalue problems with one discontinuity. We will prove only equation (5.3). We use the integral representation of u from Lemma 5.

Proof: We write the left-hand side of equation (5.3) as

$$u_3(x)\,\tilde{u}_3(x) - A - Bu_1(d_1)\,\tilde{u}_1(d_1) - Cu_2(d_2)\,\tilde{u}_2(d_2)$$

From Lemma 5 we see that this is equivalent to

$$\varphi_3^2(x) - A - B\varphi_1^2(d_1) - C\varphi_2^2(d_2) + E$$

where

$$E = \int_0^x (K+\tilde{K})(x,t)\varphi_3(x)\cos kt\,dt + \int_0^x\int_0^x K(x,t)\tilde{K}(x,s)\cos kt\cos ks\,ds\,dt$$
$$-B\left[\int_0^{d_2}(K+\tilde{K})(d_2,t)\varphi_2(d_2)\cos kt\,dt + \int_0^{d_2}\int_0^{d_2}K(d_2,t)\tilde{K}(d_2,s)\cos kt\cos ks\,ds\,dt\right]$$
$$-C\left[\int_0^{d_1}(K+\tilde{K})(d_1,t)\varphi_1(d_1)\cos kt\,dt + \int_0^{d_1}\int_0^{d_1}K(d_1,t)\tilde{K}(d_1,s)\cos kt\cos ks\,ds\,dt\right]$$

and both K and $\tilde{K}$ are bounded.

Consider the first, third and fifth terms of E. We use the product rule for cosine functions to write

$$\varphi_3(x)\cos kt = \frac{A_1A_2}{8}[\cos k(x+t) + \cos k(x-t) + \alpha_1\{\cos k(x-2d_1+t)$$

$$+ \cos k(x-2d_1-t)\} + \alpha_2\{\cos k(x-2d_2+t) + \cos k(x-2d_2-t)\}$$

$$+ \alpha_1\alpha_2\{\cos k(x-2d_2+2d_1+t) + \cos k(x-2d_2+2d_1-t)\}]$$

for $0 \le t \le x$.

$$\varphi_2(d_2)\cos kt = \frac{A_1}{4}[\cos k(d_2+t) + \cos k(d_2-t)$$

$$+ \alpha_1\{\cos k(d_2 - 2d_1+t) + \cos k(d_2 - 2d_1-t)\}]$$

for $0 \le t \le d_2$.

$$\varphi_1(d_1)\cos kt = \frac{1}{2}\cos k(d_1+t) + \cos k(d_1-t)$$

for $0 \le t \le d_1$.

Since $x > d_2$, the arguments of each cosine function appearing above lies in the interval $(-x,x)$, except for the factor $2k$. We can therefore use the argument in the proof of Lemma 5 and conclude that the sum of the first, third and fifth terms in E can be written as

$$\int_0^x \bar{K}_1(x,t)\cos 2kt\,dt$$

where $\bar{K}_1$ will depend on the sum of $K(x,s)$ and $\tilde{K}(x,s)$ for $0 \le s \le x$. As K and $\tilde{K}$ are bounded, $\bar{K}_1$ is bounded.

Next consider the second term of E. Let $K_2(s,t) = K(x,s)\tilde{K}(x,t)$. Then the term can be written as

$$\frac{1}{2}\int_{d_2}^x\int_{d_2}^x K_2(s,t)(\cos k(t+s) + \cos k(t-s))\,ds\,dt$$

We have $-2x \le t+s \le 2x$ and $-2x \le t-s \le 2x$. We can therefore use the arguments of Lemma 5 to change the order of integration, change variables and reflect about the t-axis to write this term of E as $\int_0^x \overline{K}_2(x,t)\cos 2kt\,dt$. By a similar argument we show that the fourth and sixth terms of E can be written in the same form. Thus $E = \int_0^x \overline{K}(x,t)\cos 2kt\,dt$ where $\overline{K}$ is a bounded kernel.

We complete the proof of Lemma 8 by using the definitions of φ and the trigonometric identities to write $\varphi_3^2(x)$, $\varphi_2^2(d_2)$, and $\varphi_1^2(d_1)$ as a linear combination of cosine functions. The calculations are straightforward, but tedious and will be omitted here.

6: Completion of Proof

We will show that the difference of the potentials satisfies a homogeneous Volterra integral equation on the interval $[0, \frac{\pi}{2}]$. We can then conclude that the two potentials are equal almost everywhere. The method for obtaining the explicit integral equation depends on the position of the discontinuities. There are 16 cases, each having several subcases, see Figure II.6.1. We present the technique of the proof and show that it works for all $d_1 < d_2 < \frac{\pi}{2}$.

Let $Q = q - \tilde{q}$. We assume $H = \tilde{H}$, $\lambda_j = \tilde{\lambda}_j$ for $j \geq 0$ and $q = \tilde{q}$ almost everywhere in $[\frac{\pi}{2}, \pi]$. Then from Lemma 6 we have $a_1 = \tilde{a}_1$, $a_2 = \tilde{a}_2$, $d_1 = \tilde{d}_1$, and $d_2 = \tilde{d}_2$. From Lemmas 7 and 8 follows

$$
\begin{aligned}
&\int_0^{d_1} Q(t)[\cos 2kt + \int_0^t \bar{K}(t,s)\cos 2ks\,ds]dt \\
&+ \int_{d_1}^{d_2} Q(t)[C_1\cos 2kt + C_2\cos 2k(t - d_1) \\
&\qquad + C_3\cos 2k(t - 2d_1) + \int_0^t \bar{K}(t,s)\cos 2ks\,ds]dt \\
&+ \int_{d_2}^{\frac{\pi}{2}} Q(t)[C_4\cos 2kt + C_5\cos 2k(t - d_1) \\
&+ C_6\cos 2k(t - d_2) + C_7\cos 2k(t - 2d_1) \\
&+ C_8\cos 2k(t - 2d_2) + C_9\cos 2k(t - d_2 + d_1) \\
&+ C_{10}\cos 2k(t - d_1 - d_2) + C_{11}\cos 2k(t - 2d_2 + d_1) \\
&+ C_{12}\cos 2k(t + 2d_1 - 2d_2) + \int_0^t \bar{K}(t,s)\cos 2ks\,ds]dt \\
&= 0
\end{aligned}
$$

The coefficients C_i depend on a_1, a_1^{-1}, a_2, and a_2^{-1} and are given in Lemma 8. In the remainder of the proof, zero will mean zero almost everywhere.

We will now change variables to obtain an equation of the form

$$\int_0^{\frac{\pi}{2}} [F(t) + \int_t^{\frac{\pi}{2}} \bar{K}(s,t)Q(s)ds]\cos 2kt dt = 0$$

which holds for all k. Since $\cos 2kt$ is complete in $[0, \frac{\pi}{2}]$, we will then have

$$F(t) + \int_0^t \bar{K}(s,t)Q(s)ds = 0$$

in $[0, \frac{\pi}{2}]$. The form of F will allow us to conclude that $Q = 0$.

We first consider the terms with $\bar{K}$. Since $\bar{K}$ is bounded and Q is integrable, Fubini's theorem allows us to change the order of integration to write

$$\int_0^{d_1} Q(t)\int_0^t \bar{K}(t,s)\cos 2ks\ ds\ dt + \int_{d_1}^{d_2} Q(t)\int_0^t \bar{K}(t,s)\cos 2ks\ ds\ dt$$
$$+ \int_{d_2}^{\frac{\pi}{2}} Q(t)\int_0^t \bar{K}(t,s)\cos 2ks\ ds\ dt$$
$$= \int_0^{\frac{\pi}{2}} \cos 2kt [\int_t^{\frac{\pi}{2}} \bar{K}(s,t)Q(s)ds]dt\ .$$

Next we consider the terms

$$\int_{s_1}^{s_2} Q(t)\cos 2ks(t)dt$$

and show that each can be written as a sum of integrals of the form

$$\int_{t_1}^{t_2} Q(\beta(t))\cos 2kt dt$$

where $0 \le t_1 < t_2 \le \frac{\pi}{2}$. Specifically we have

$$\int_0^{d_1} Q(t)\cos 2kt dt = \int_0^{d_1} Q(t)\cos 2kt dt \tag{6.1}$$
$$\text{for } 0 < d_1 < d_2 < \frac{\pi}{2}.$$

$$\int_{d_1}^{d_2} Q(t)\cos 2kt dt = \int_{d_1}^{d_2} Q(t)\cos 2kt dt \tag{6.2}$$

$$\text{for } 0 \le d_1, d_2 < \frac{\pi}{2}.$$

$$\int_{d_1}^{d_2} Q(t)\cos 2k(t - d_1)dt = \int_0^{d_2 - d_1} Q(t + d_1)\cos 2kt dt \tag{6.3}$$

$$\text{for } 0 < d_1 < d_2 < \frac{\pi}{2}.$$

$$\int_{d_1}^{d_2} Q(t)\cos 2k(t - 2d_1)dt \tag{6.4}$$

$$= \int_{2d_1 - d_2}^{d_1} Q(-t + 2d_1)\cos 2kt dt$$

$$\text{for } 2d_1 > d_2$$

$$= \int_0^{d_1} Q(-t + 2d_1)\cos 2kt dt + \int_0^{d_2 - 2d_1} Q(t + 2d_1)\cos 2kt dt$$

$$\text{for } 2d_1 \le d_2.$$

$$\int_{d_2}^{\frac{\pi}{2}} Q(t)\cos 2kt dt = \int_{d_2}^{\frac{\pi}{2}} Q(t)\cos 2kt dt \tag{6.5}$$

$$\text{for } 0 < d_1 < d_2 < \frac{\pi}{2}.$$

$$\int_{d_2}^{\frac{\pi}{2}} Q(t)\cos 2k(t - d_1)dt = \int_{d_2 - d_1}^{\frac{\pi}{2} - d_1} Q(t + d_1)\cos 2kt dt \tag{6.6}$$

$$\text{for } 0 < d_1 < d_2 < \frac{\pi}{2}.$$

$$\int_{d_2}^{\frac{\pi}{2}} Q(t)\cos 2k(t - d_2)dt = \int_0^{\frac{\pi}{2} - d_2} Q(t + d_2)\cos 2kt dt \tag{6.7}$$

$$\text{for } 0 < d_1 < d_2 < \frac{\pi}{2}.$$

$$\int_{d_2}^{\frac{\pi}{2}} Q(t)\cos 2k(t-2d_1)dt \tag{6.8}$$

$$= \int_0^{d_2-2d_1} Q(-t+2d_1)\cos 2ktdt + \int_0^{\frac{\pi}{2}-2d_1} Q(t+2d_1)\cos 2ktdt$$

$$\text{for } 0<d_2 \le 2d_1 < \frac{\pi}{2}$$

$$= \int_{2d_1-\frac{\pi}{2}}^{2d_1-d_2} Q(t+2d_1)\cos 2ktdt$$

$$\text{for } 0<d_2<\frac{\pi}{2}\le 2d_1.$$

$$\int_{d_2}^{\frac{\pi}{2}} Q(t)\cos 2kt(t-2d_2)dt \tag{6.9}$$

$$= \int_0^{d_2} Q(-t+2d_2)\cos 2ktdt + \int_0^{\frac{\pi}{2}-2d_2} Q(t+2d_2)\cos 2ktdt$$

$$\text{for } 0<2d_2\le\frac{\pi}{2}$$

$$= \int_{2d_2-\frac{\pi}{2}}^{d_2} Q(-t+2d_2)\cos 2ktdt$$

$$\text{for } \frac{\pi}{2}<2d_2.$$

$$\int_{d_2}^{\frac{\pi}{2}} Q(t)\cos 2k(t+d_1-d_2) = \int_{d_1}^{\frac{\pi}{2}-d_2+d_1} \cos 2ktdt \tag{6.10}$$

$$\text{for } 0<d_1<d_2<\frac{\pi}{2}.$$

$$\int_{d_2}^{\frac{\pi}{2}} Q(t)\cos 2k(t-d_1-d_2)dt \tag{6.11}$$

$$= \int_0^{d_1} Q(-t+d_1+d_2)\cos 2ktdt + \int_0^{\frac{\pi}{2}-d_1-d_2} Q(t+d_1+d_2)\cos 2ktdt$$

$$\text{for } 0<d_1 + d_2 \leq \frac{\pi}{2}$$

$$= \int_{d_1+d_2-\frac{\pi}{2}}^{d_1} Q(-t + d_1 + d_2)\cos 2kt dt$$

$$\text{for } \frac{\pi}{2} < d_1 + d_2.$$

$$\int_{d_2}^{\frac{\pi}{2}} Q(t)\cos 2k(t - 2d_2 + d_1)dt \tag{6.12}$$

$$= \int_0^{d_2-d_1} Q(-t + 2d_2 - d_1)\cos 2kt dt + \int_0^{\frac{\pi}{2}-2d_2+d_1} Q(t + 2d_2 - d_1)\cos kt dt$$

$$\text{for } 0 < 2d_2 - d_1 \leq \frac{\pi}{2}$$

$$= \int_0^{d_2-d_1} Q(t + 2d_2 - d_1)\cos 2kt dt + \int_0^{2d_2-d_1-\frac{\pi}{2}} Q(-t + 2d_2 - d_1)\cos 2kt dt$$

$$\text{for } \frac{\pi}{2} < 2d_2 - d_1.$$

$$\int_{d_2}^{\frac{\pi}{2}} Q(t)\cos 2k(t + 2d_1 - 2d_2)dt \tag{6.13}$$

$$= \int_{2d_1-d_2}^{\frac{\pi}{2}+2d_1-2d_2} Q(t - 2d_1 + 2d_2)\cos 2kt dt$$

$$\text{for } 0<d_2<2d_1$$

$$= \int_0^{d_2-2d_1} Q(-t - 2d_1 + 2d_2)\cos 2kt dt + \int_0^{\frac{\pi}{2}+2d_1-2d_2} Q(t - 2d_1 + 2d_2)\cos 2kt dt$$

$$\text{for } 0<2d_1 \leq d_2 \quad 2d_2 - 2d_1 < \frac{\pi}{2}$$

$$= \int_{2d_2-2d_1-\frac{\pi}{2}}^{d_2-2d_1} Q(-t - 2d_1 + 2d_2)\cos 2kt dt$$

$$\text{for } 0<2d_1 \leq d_2 \quad \frac{\pi}{2} \leq 2d_2 - 2d_1).$$

Let d_1, d_2 be given. Then we can choose the appropriate form of each integral. There are 16 cases which are shown in Figure 6.1. We can combine integrals to obtain a homogeneous equation of the form

$$\int_{t_0=0}^{t_1}[F_1(t)+\int_t^{\frac{\pi}{2}}\overline{K}(s,t)Q(s)ds]\cos 2kt dt + \int_{t_1}^{t_2}[F_2(t)+\int_t^{\frac{\pi}{2}}\overline{K}(s,t)Q(s)ds]\cos 2kt dt$$

$$+ \ldots\ldots\ldots + \int_{t_{m-1}}^{t_m=\frac{\pi}{2}}[F_m(t) + \int_t^{\frac{\pi}{2}}\overline{K}(s,t)Q(s)ds]\cos 2kt dt = 0 \ .$$

Note that the limits of integration must be ordered to combine the integrals. This creates subcases in each of the 16 cases mentioned above and the t_i depend upon the subcase. The functions

$$F_i(t) = C_{io}\,Q(t)+\sum_j C_{ij}\,Q\,(\beta_{ij}(t)) \tag{6.14}$$

where j is summed from 1 to m_i. The C_{ij} depend on α_1, α_1^{-1}, α_2 and α_2^{-1} and are piecewise constant with a finite number of discontinuities. In particular we refer to Lemma 8 and see that C_{i0} is 1 for $0 \le t < d_1$, $A_1^2/8$ for $d_1 < t < d_2$ and $A_1^2 A_2^2/32$ for $d_2 < t \le \pi/2$. We will need $C_{i0} > 0$.

Let $F = F_i$ for $t \in [t_{i-1}, t_i]$. Since the $\{\cos 2kx\}$ are complete on $(0, \frac{\pi}{2})$ we conclude

$$F(t) + \int_0^t \overline{K}(s,t)Q(s)ds = 0 \tag{6.15}$$

$$0 \le t \le \frac{\pi}{2}$$

We now show that $F_m(t) = C_4 Q(t)$. Consider the upper limits of the integrals given in equations (6.1)-(6.13), excluding (6.5). They are contained in the set

$$\{d_1,\ d_2,\ d_2 - d_1,\ d_2 - 2d_1,\ 2d_1 - d_2,\ \frac{\pi}{2} - d_1,\ \frac{\pi}{2} - 2d_1,$$

$$\frac{\pi}{2} - d_2,\ \frac{\pi}{2} - 2d_2 + d_1,\ 2d_2 - d_1 - \frac{\pi}{2},\ \frac{\pi}{2} - d_1 - d_2,$$

$$\frac{\pi}{2} + 2d_1 - 2d_2,\ \pi - 2d_2,\ \frac{\pi}{2} + d_1 - d_2\}$$

Recall that the cases are chosen so that the upper limits are non-negative. For $0 < d_1 < d_2 < \frac{\pi}{2}$ we have each element in the above set strictly less than $\pi/2$. Thus the only contribution to the integral equation (6.14) in the interval $[t_{m-1}, \frac{\pi}{2}]$ comes from equation (6.5). and $F_m = C_4\ Q(t)$. We have

$$C_4\ Q(t) + \int_t^{\frac{\pi}{2}} \overline{K}(s,t)\ Q(s)ds = 0$$

almost everywhere in $[t_{m-1}, \frac{\pi}{2}]$. This is a homogeneous Volterra integral equation and we can conclude $Q(t) = 0$ for $t_{m-1} \le t \le \frac{\pi}{2}$.

To complete the theorem we show that there exists an $\varepsilon > 0$ such that given $Q = 0$ in $[T, \frac{\pi}{2}]$, we can conclude $Q = 0$ in $[\delta, \frac{\pi}{2}]$ where $\delta = \max(0, T-\varepsilon)$. Since $N\varepsilon > \frac{\pi}{2}$ for some N, we will then have $Q = 0$ in $[0, \frac{\pi}{2}]$.

Consider the integrals in equations (6.3), (6.4) and (6.6)-(6.13). The general form is $\int_{t_1}^{t_2} Q(\beta(t)) cos\, 2kt dt$ where $\beta(t) = t + \gamma$ or $\beta(t) = -t + t_2 + \gamma$. By inspection, we see that $\gamma > 0$. Let ε be less than the smallest γ for the various integrals. Assume $Q = 0$ in $[T, \frac{\pi}{2}]$. Equation (6.15) holds for $0 \le t \le pi/2$. In particular we have

$$F(t) + \int_t^{\frac{\pi}{2}} \overline{K}(s,t)\ Q(s)ds = 0$$

$$\text{in} \quad [T-\varepsilon, \frac{\pi}{2}]$$

where F is given by (6.14). But $\beta_{ij}(t) \geq T$ for $T-\varepsilon \leq t \leq T$. Thus

$$F(t) = C_{i0}\, Q(t)$$

in this interval.

This completes the proof of the theorem.

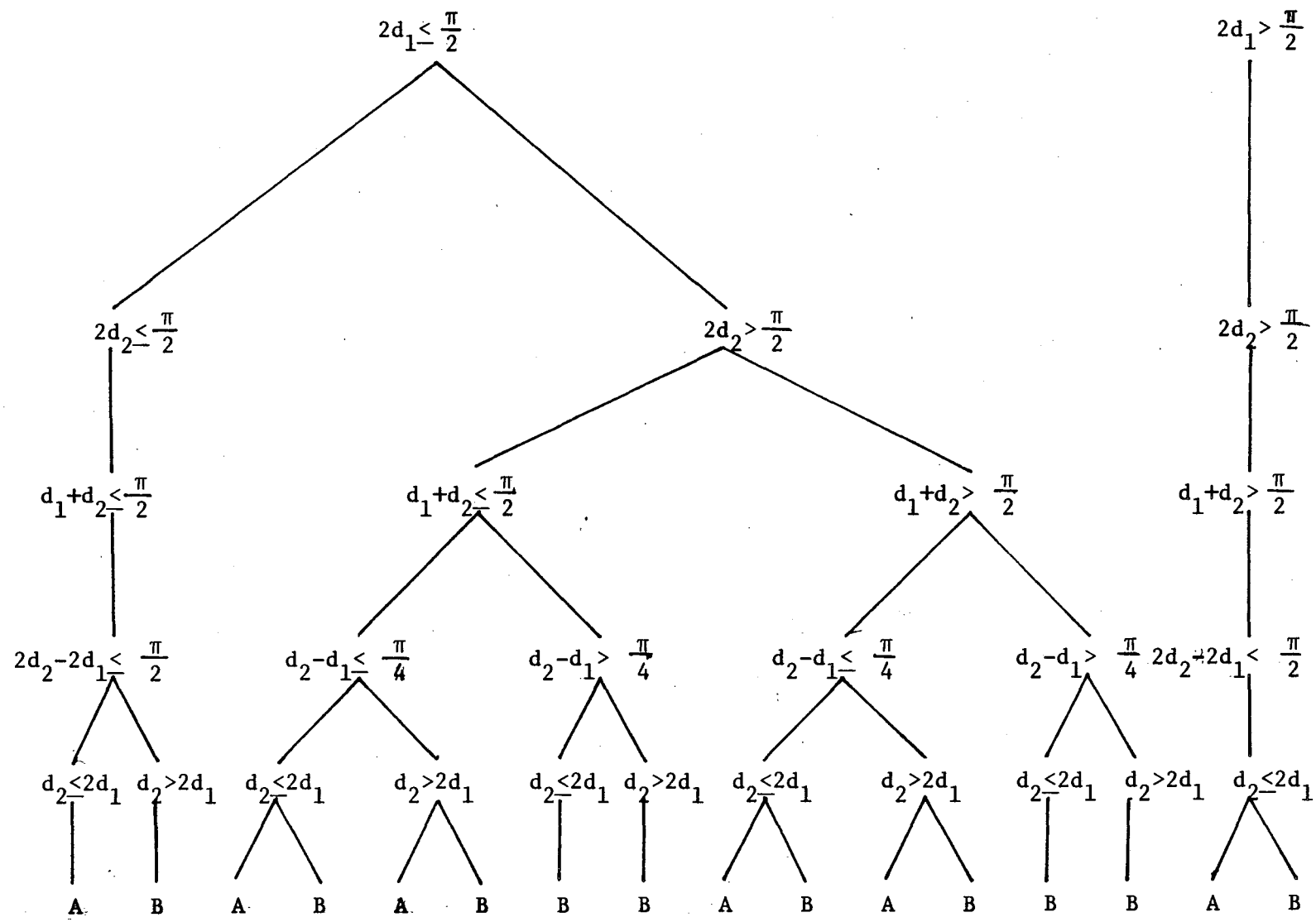


Figure II.6.1. The 16 Main Cases "A" → $2d_2-d_1 \leq \frac{\pi}{2}$ "B" → $2d_2-d_1 > \frac{\pi}{2}$

7: Application to the inverse problem for the mantle

In this section we apply Theorem 1 to the inverse problem for the mantle. We assume that the density and S-wave velocity of the Earth are discontinuous at the boundary between the core and the mantle. We show that if the velocity of the S-waves is known in the mantle and in the crust and if the density is given in the lower mantle then the periods of the torsional oscillations of one angular order determine the density in the upper mantle uniquely. If the S-wave velocity is known only in the lower mantle then two torsional spectra determine the density and the S-wave velocity uniquely in the upper mantle. These results are stated more precisely in next two theorems.

THEOREM 2: *Let $R_c < R_1 < R_2 < R$ be given. Assume ρ and β are positive and twice continuously differentiable for $R_c \le r \le R_1$, $R_1 \le r \le R_2$ and $R_2 \le r \le R$ and that $1/2 \le \frac{\rho_+\beta_+}{\rho_-\beta_-} \le 2$ at $r = R_1$ and R_2. Consider the eigenvalue problem*

$$-(r^4\rho\beta^2\dot{u})^{\cdot} + (l+2)(l-1)r^2\rho\beta^2 u = \omega^2 r^4 \rho u \tag{7.1}$$
$$\dot{u}(R_c) = \dot{u}(R) = 0$$

with the continuity requirements

$$u_+ = u_- \qquad r^4\rho_+\beta_+^2\dot{u}_+ = r^4\rho_-\beta_-^2\dot{u}_- \tag{7.2}$$

at $r = R_1$ and R_2. Here $\dot{u} = \frac{du}{dr}$ and $u_+(R_i) = \lim_{r\to R_i^+} u(r)$. Let $\beta(r)$ be given in the three intervals. Choose r_0 such that

$$\int_{R_c}^{r_0} \beta^{-1}dr = \int_{r_0}^{R} \beta^{-1}dr$$

and let $r_0 < R_1 < R_2 < R$. Assume that ρ is given for $R_c \le r \le r_0$. Then one

spectrum $\{\omega_n^2\}$ *uniquely determines* ρ *for* $r_0 \le r \le R$.

Remark: Equation (7.1) is derived from applying separation of variables to the equation for the torsional oscillations of a spherically symmetric non-rotating Earth where the Earth consists of an isotropic, perfect elastic material. See [2] and Part I, Section 1. The conditions (7.2) come from the requirement of continuity of displacement and stress at a spherical interface. In practice, $R \approx 6371$ km, $R_c \approx 3473$ km and r_0 lies at a depth of about 1300 km. The discontinuity at R_2 is known as the Mohorovicic discontinuity and occurs at a depth of approximately 33 km. The position of the discontinuity at R_1 is not as well established. We follow Model A and let $R_1 \approx 5958$ km. We refer to the region above the core and below r_0 as the lower mantle. The upper mantle lies between r_0 and R_1 and the crust lies between the Mohorovicic discontinuity and the surface. Our lower mantle is contained in the region in which Bullen assumed that the density satisfied the Adams-Williamson equation. The restriction at the discontinuities corresponds to our conditions for α_1 and α_2 in Theorem 1. The factor 2 will insure the hypotheses of Theorem 1 but can be relaxed. The proof of Theorem 2 is based on Theorem 1. It follows the proof for the continuous case, see Hald [26].

The subtlety of Theorems 2 and 3 lies in the fact that the transformed equation given by (7.1) and (7.2) may have fewer discontinuities than the earth model. This is the case if $\rho\beta$ and $\frac{d(\sqrt{\rho\beta})}{dx}$ are continuous where ρ and β are discontinuous. I know of no earth models having this property, but we investigate this case for the sake of completeness.

Proof: Let ρ, β and $\tilde{\rho}$, $\tilde{\beta}$ be two earth models such that $\rho = \tilde{\rho}$ for $R_c \le r \le r_0$, $\beta = \tilde{\beta}$ in the mantle and in the crust and $\omega_n^2 = \tilde{\omega}_n^2$ for a fixed angular order l. We will show that equations (7.1) and (7.2) for each earth model can be transformed to a Sturm-Liouville problem with at most two discontinuities and that the two transformed equations must have the same number of discontinuities. We will then show that the potentials, the boundary conditions and the jump conditions of the two transformed equations are equal. Note that if β and $\tilde{\beta}$ are discontinuous at R_1 and R_2, then the positions of the discontinuities are known and we will not use the full strength of Theorem 1.

Let $K = \frac{1}{\pi}\int_{R_c}^{R} \beta^{-1}(r)dr$ and $x = \psi(z) = \frac{1}{K}\int_0^z \frac{1}{\beta(R-\xi)}d\xi$. Note that x is continuous in the interval $(0,\pi)$ but not differentiable at $R-R_1$ and $R-R_2$. Let $f(x) = r^2\sqrt{\rho\beta}$ and $y(x) = f(x)u(r)$. This is the same transformation as we used in Section 1, Part I. We make the substitution in equation (7.1) and obtain

$$-y'' + qy = \lambda y \tag{7.2}$$

$$y' - hy = y' + Hy = 0 \ .$$

Here $q = v + (l+2)(l-1)w$ where $v = \frac{f''}{f}$ and $w = \frac{K^2\beta^2}{r^2}$, $\lambda = K^2\omega^2$, $h = \frac{f'}{f}$ at $x = 0$ and $H = \frac{f_{,}'}{f}$ at $x = \pi$. Note that $x(R-r_0) = \frac{\pi}{2}$ and that the interval (r_0,R) is mapped to $(0,\frac{\pi}{2})$. Let $d_1 = x(R-R_2)$ and $d_2 = x(R-R_1)$. Since $r_0 < R_1 < R_2 < R$ we see that $0 < d_1 < d_2 < \frac{\pi}{2}$. The continuity requirements (7.2) become

$$\frac{y_+}{f_+} = \frac{y_-}{f_-} \tag{7.3}$$

$$f_+y_+ - f_+'y_+ = f_-y_-' - f_-'y_-$$

at $x = d_1$ and d_2. Thus in the notation of Theorem 1, $a_i = \frac{f_+}{f_-}$ and

$b_i = \frac{f_+'}{f_-} - \frac{f_-'}{f_+}$ evaluated at $x = d_i$. We have now transformed the differential equation for the torsional modes of the Earth with two interior discontinuities to the eigenvalue problem considered in Theorem 1.

It has been proved by Hald that the eigenvalues of an eigenproblem with one discontinuity cannot be the same as those for a continuous problem and we have shown that the eigenvalues of an eigenproblem with two discontinuities cannot equal those for an eigenproblem with one or no discontinuities, see Lemma 6. Thus the transformed equations for the two earth models must have the same number of discontinuities.

We now show that all the hypotheses, except those concerning the discontinuities, of Theorem 1 are satisfied. Since both ρ and β are known for $R_c \le r \le r_0$, we can determine the potentials q for $\frac{\pi}{2} \le x \le \pi$ and the constant H. Since $\rho = \tilde{\rho}$ and $\beta = \tilde{\beta}$ in this interval we have $q = \tilde{q}$ for $\pi/2 \le \pi$ and $H = \tilde{H}$. The conditions on the jumps in ρ and β give $1/\sqrt{2} < a_i \le \sqrt{2}$ which implies that $\alpha \le \frac{7}{9} < 1$. Since $K = \tilde{K}$ the two Sturm-Liouville problems have the same eigenvalues: $\lambda_j = \tilde{\lambda}_j$. If the transformed equations has one or no discontinuities we apply the theory by Hald to conclude that $q = \tilde{q}$, see [26], [28]. If the transformed equation has two discontinuities, we apply Theorem 1 to obtain the same result.

Since $q = \tilde{q}$ and $\beta = \tilde{\beta}$ we have $v = \tilde{v}$. Thus both f and $\tilde{f}$ satisfy satisfy the differential equation

$$f'' = vf \tag{7.4}$$

with the same initial conditions at $x = \frac{\pi}{2}$. We can therefore solve equation (7.4) to conclude $f = \tilde{f}$ for $d_2 \le x \le \frac{\pi}{2}$. In particular $f_+ = \tilde{f}_+$ and $f_+' = \tilde{f}_+'$ at

$x = d_2$.

Since $a_2 = \tilde{a}_2$ and $b_2 = \tilde{b}_2$ we get $f_- = \tilde{f}_-$ and $f_-' = \tilde{f}_-'$. Note that $a_2 = \tilde{a}_2 = 1$ and $b_2 = \tilde{b}_2 = 0$ in the degenerate case. We now solve (7.4) for f and $\tilde{f}$ with the initial conditions determined from f_+, f_+', a_2 and b_2 and to conclude $f = \tilde{f}$ for $d_1 \le x \le d_2$. Similarly, we get $f = \tilde{f}$ for $0 \le x \le d_1$. Finally since $\psi = \tilde{\psi}$ we conclude

$$\rho(r) = \frac{f^2(\psi(R-r))}{r^4\beta(r)} = \frac{\tilde{f}^2(\psi(R-r))}{r^4\tilde{\beta}(r)} = \tilde{\rho}(r) \ .$$

This completes the proof of Theorem 2.

We now show that if the density and velocity are given in the lower mantle and there are at most two discontinuities in the upper mantle, then the position of the discontinuities and the density and S-wave velocity are uniquely determined in the upper mantle and crust by two torsional spectra.

THEOREM 3. *Let $R_c < R_1 < R_2 < R$ and assume that ρ and β are positive and twice continuously differentiable for $R_c \le r \le R_1$, $R_1 \le r \le R_2$ and $R_2 \le r \le R$. Consider the eigenvalue problem (7.1) with the continuity conditions (7.2). Let $K = \lim(n/\omega_n)$ where $\{\omega_n^2(l)\}$ is the spectrum for a fixed value of l. Assume that ρ and β are given for $R_c \le r \le r_0$, where r_0 is determined by*

$$\int_{R_c}^{r_0} \beta^{-1}(r)\, dr = \frac{\pi}{2} K \ .$$

Let $r_0 < R_1 < R_2$. Then ρ and β are uniquely determined by two torsional spectra $\{\omega_n^2(l_1)\}$ and $\{\omega_n^2(l_2)\}$.

Proof: We show that the inverse transformation is uniquely determined. This will determine β for $R_c < r < R$. Then we can use the proof of Theorem 2 to conclude that the density is uniquely determined in the mantle. The proof

follows the continuous case, see Hald [26] and the details are omitted.

From the inverse Sturm-Liouville theory we get $q_{l_1} = \tilde{q}_{l_1}$ and $q_{l_2} = \tilde{q}_{l_2}$. We solve

$$w = \frac{q_{l_2} - q_{l_1}}{(l_2+l_1+1)(l_2-l_1)} .$$

Then $w = \tilde{w}$. The inverse transformation is given by $r = R-\Phi(x)$ where

$$\Phi(x)' = K\beta = (R-\Phi)\sqrt{w} \tag{7.5}$$

with the initial conditions $\Phi(0)=R$. Since $w = K^2\beta^2/r^2$, w will be piecewise continuous. Thus we can solve equation (7.5) for Φ and β. This implies that ρ and β are uniquely determined.

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