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### Authors

Battaglia, V.
Newman, J.



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V.S. Battaglia and J. Newman

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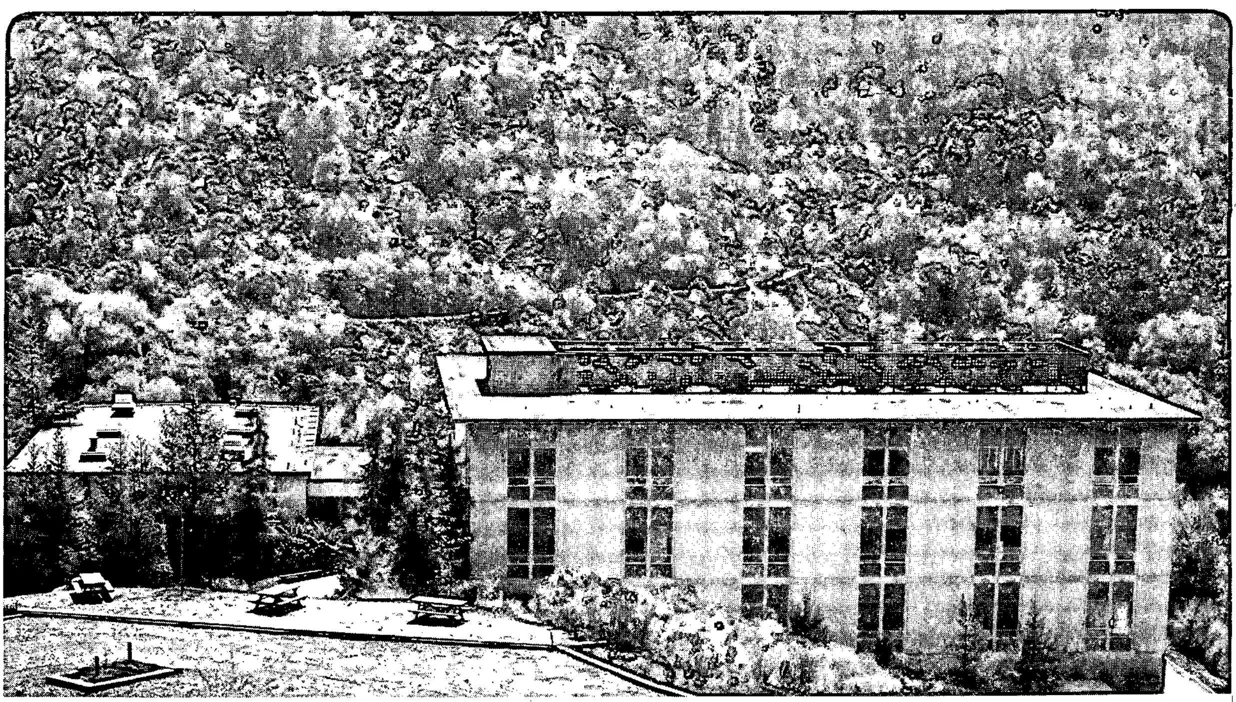



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# Magnetic Field Effects in High-Power Batteries

## I. The Penetration of an Electric Field into a Cylindrical Conductor

Vincent S. Battaglia and John Newman

Department of Chemical Engineering
University of California

and

Materials Sciences Division
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

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# Magnetic Field Effects in High-Power Batteries

## *I: The Penetration of an Electric Field into a Cylindrical Conductor*


### Abstract

The penetration of the axial component of an electric field into a cylindrical conductor is described by an asymptotic solution method for long and short times. The development of the respective current distributions allows for a mathematical comparison of the solution schemes and indicates that the current initially increases at a rate proportional to time until a time of order $\epsilon/\sigma$, subsequently at a rate proportional to the square root of time, and finally levels off exponentially to the steady-state value. Criteria for the proper omission of the displacement current are also given.


### Introduction

In the 70 years since Kapitza[1] used an electrochemical system (a lead-acid bipolar battery) for delivering short pulses of high-power, the design of batteries for this purpose has drawn little attention. However, the ever increasing reliance of the business sector on computer and telecommunication systems has enticed a renewed effort in research of rapid-discharge, high, back-up power batteries. E. Willihnganz[2] studied the effect of high rates of discharge on the negative plate of a lead-acid battery. Gibbard[3] has designed and tested a number of high-power lithium batteries and has presented an overall design criterion.[4] And LaFollette and Bennion discuss the design fundamentals of high-power, pulse-discharged lead acid batteries in terms of experiments[5] and modeling.[6] Researchers are now also asking what is the minimum time required to bring a battery to maximum power.

Inductance, used to denote the effect of a rapidly varying electric field which produces a varying magnetic field which counters with an opposing electric field, governs the rise time of the current. This phenomenon falls under the heading of electrodynamics. LaFollette and Bennion[5] mention that the rigorous way to incorporate magnetic field effects in a battery model is to solve Maxwell's equations of electromagnetism simultaneously with the equations that govern the battery performance. Due to the complicated nature of the equations of electrodynamics and the associated boundary conditions, this is something most modelers would like to avoid. LaFollette *et al.* did not include the inductance in their model after demonstrating that their system was small enough that the inductance effect occurred within the first microsecond and that they were more interested in the 10 to 1000 $\mu$s range. Methods of simplifying the equations for handling the inductance while not neglecting it entirely do exist. McKinney *et al.*[7] include a magnetic field effect of the circuit when analyzing the current from a power source by including a lumped inductance term in their circuit-theory model. Cahan *et al.*[8] performed a parametric study of the impedance of a generic power source as a function of the frequency of the signal. The power source was approximated as a modified, semi-infinite, stripline. Inductance was included by means of a skin-effect. The skin-depth is a measure of the depth of penetration of a sinusoidal signal and is a function of the frequency and conductivity. (For a sinusoidal signal, 95 percent of the signal is limited to 3 skin-depths from the surface.) The skin-depth analysis is applicable for a sinusoidal signal; however, an analogous treatment for a stepped current has not yet been presented. Moreover, these techniques only provide for an adjustment of the total current as a function of time; they do not provide a description of the instantaneous current distribution. The current distribution can play an important role in battery design if a large battery is required for high currents and only a fraction is utilized.

In part I of our study, we shall demonstrate the manner in which an electric field penetrates a cylindrical conductor when the conductor is instantaneously subjected to a constant electric field at its surface. This work provides a first approximation to the instantaneous current distribution in a bipolar battery, the typical battery design chosen for delivering high power. In part II, the cylindrical conductor is included in a radial circuit. A transmission-line analysis is used in conjunction with the above treatment to determine the time constant of the complete system. The importance of addressing the complete circuit lies in the fact that the electromagnetic field distribution is not a localized entity but is a function of the entire circuit geometry.

## The Electromagnetic System

The transition of an electrical signal from one amplitude and frequency to another begins at the source of the disturbance and propagates to the rest of the electrical configuration. The new form of the signal travels at the speed of light between the electrical conductors used to direct the electrical power from the source to the intended load. During the initial stages of the transition, the charge on the conductors distributes itself such that the potential around the circuit is consistent with Ohm's law. (Further comments on the initial stages of current start-up are provided by Rosser[9] and Heald.[10])

For conductors of finite conductivity, the electric field penetrates the conductors to within a few skin depths, for an alternating signal, and throughout the conductors, for a direct signal. The rate at which the new signal can penetrate the conductors determines the rate to a periodic or steady state. Using Maxwell's equations, it is the purpose of this paper to describe the penetration of the axial electric field into an electrically conducting medium.

It is well-known that Maxwell's four coupled equations of electromagnetism reduce to two, noncoupled modified wave equations in a linear, homogeneous, conducting medium:

$$\nabla^2 E = \mu\epsilon\frac{\partial^2 E}{\partial t^2} + \mu\sigma\frac{\partial E}{\partial t}, \tag{1}$$

$$\nabla^2 B = \mu\epsilon\frac{\partial^2 B}{\partial t^2} + \mu\sigma\frac{\partial B}{\partial t}. \qquad (2)$$

The solution to these equations for the transient electric field distribution in the conductors and surrounding medium for a finite geometry with a load and source is difficult to obtain. Since we are interested here in the time it takes for the penetration of the electrical signal into the power source, we shall address the specific problem of a step in an electrical signal propagating down an infinitely long coaxial cable. The radial component of the electrical signal propagates between the two conductors at the speed of light while the axial component propagates radially into the inner and outer conductors. With this configuration, the problem of determining the penetration of the electric field into the transmission line is virtually reduced to the mathematically one-dimensional problem of the penetration of an axial electric field component into an infinitely long cylindrical conductor.

From this analysis we shall assume that the bipolar battery configuration can be approximated by the properties of the central wire of the coaxial cable, the return current of the circuit being located in an outer cylindrical can. In a bipolar battery, a circular cathode and anode are separated by a circular ionically conducting separator to form a cell. A number of these cells are then stacked together and separated by cylindrical electronic conductors, such as steel plates, to form the battery, which resembles the Volta pile. Although a cylindrical wire is a reasonable approximation of the bipolar battery geometry, it does not account for the details of the battery. The solid wire obviously lacks the multiple regions of various conductivities, surfaces of capacitance, electromotive force, and chemical reactions, and possible concentration variations found in a battery. However, as a first approximation, the infinite-wire analysis will allow us to predict the rate and the manner in which an electric field penetrates a conductor. And although this falls well short of a complete analysis, a considerable amount of information can still be inferred about the magnetic field effects.

Solutions have been derived for the electric field distribution inside and outside of an infinite wire maintained at a fixed current. A. Marcus[11] provided the field distribution of the infinite wire surrounded by an axially symmetric, perfectly conducting, outer cylinder. Sommerfeld[12] described the field distribution of a steady current in an infinite wire in free space. And D. Marcuse[13] gave the general solution to the electric field distribution of an infinite wire carrying a steady current; the particular solution being a function of the rest of the circuit configuration.

Solutions to the steady-periodic electric field distributions in media of high but finite conductivity also exist. Maxwell[14] solved for the self inductance of a periodic electric field in a cylindrical conductor. Jackson[15] showed the electric field distribution of a periodic electric field penetrating a slab.

The problem that has been solved that closest resembles the problem we address here is that of a linearly increasing current in an infinite conductor. This problem was addressed by Shakur[16] and furthered by Chervenak.[17] However, these analyses assume that the current is uniform through the conductor; Heald[2] raised some further questions of the results.

Typically, when solving the Maxwell equations in electrically conducting media, the first term on the right of equations 1 and 2, referred to as the "displacement current," is omitted.[6, 7, 18, 19, 20] In this analysis, this term is retained in what we shall call the "full solution" and in the "inner solution" and neglected in the "diffusion solution." The development of these solutions will elucidate the effect the displacement current plays in the initial stages of the field penetration and provide criteria for cases when its omission is justified, greatly simplifying the equations.

The full solution maintains all the physics and is applicable for all times; however its open form requires a large number of terms to investigate the short time regime. To probe the short time effects the problem is divided into the three regimes depicted in figure 1. Region I is the inner regime. The solution obtained for this regime highlights the effects of the initial penetration of the electric field traveling at the speed of light into the conductor. Region II is the diffusion regime. After a time of $(O\ \epsilon/\sigma)$ the field penetrates parabolically into the conductor. Region III is the outer regime. This is a long time regime in which the field penetrates to the center of the conductor at the speed of light, but with an essentially negligible magnitude. We shall not refer to this regime again.

## Full Solution

For the geometry described above, the characteristic length is the radius of the inner conductor, $r_o$. Defining the electric diffusivity of the propagating electric field as the reciprocal of the electric conductivity times the permeability, $1/\mu\sigma$, a characteristic time is developed as the radius of the wire squared divided by the electric diffusivity. Thus, substitution of the dimensionless parameters

$$\xi = r/r_o \text{ and } \tau = t/r_o^2\mu\sigma,$$

into equation 1 gives it the form

$$\frac{\partial^2 E}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial E}{\partial \xi} = \delta^2 \frac{\partial^2 E}{\partial \tau^2} + \frac{\partial E}{\partial \tau}, \tag{3}$$

where $\delta = 1/\mu\sigma r_o\sqrt{\mu\epsilon}$. $\delta$ can be interpreted as the ratio of the displacement current to the conduction current. To simplify matters, we assume that initially the electric field is zero in the wire; then, at time 0, a step change in the electric field with axial component of magnitude $E_o$ at a certain position $z$ and $r_o$ is imposed. This signal, as it proceeds along the coaxial line, will be felt initially only on the inner wall of the outer conductor and outer wall of the inner conductor. If the outer conductor is a perfect conductor, as in this example, the rate of propagation of the electric field throughout the inner conductor will be the limiting effect to steady state. Thus, in this investigation we shall treat only the propagation of the field into the inner conductor. The conditions are described in mathematical terms as:

boundary conditions

$E$ is well-behaved at $\xi = 0$;

$E = E_o$ at $\xi = 1$;

initial conditions

$E = 0$ at $\tau = 0$;

$\frac{\partial E}{\partial \tau} = 0$ at $\tau = 0$.

The Laplace transformed solution of equation 3 is

$$\frac{\bar{E}(s)}{E_o} = \frac{I_o[(\delta^2 s^2 + s)^{\frac{1}{2}}\xi]}{s\, I_o[(\delta^2 s^2 + s)^{\frac{1}{2}}]}, \qquad (4)$$

where $I_p(x)$ is the modified Bessel function of the first kind, of order $p$. This solution can then be inverted by the method of residues.

Regardless of the solution scheme used, the radial penetration of the axial electric field component is given by*

$$\frac{E}{E_o} = \sum_{k=1}^{\infty} \frac{2\, J_o(\lambda_k \xi)}{\lambda_k\, J_1(\lambda_k)} \left\{ 1 - \left[ \cosh(\omega_k \tau) + \frac{\sinh(\omega_k \tau)}{2\delta^2 \omega_k} \right] \exp(-\tau/2\delta^2) \right\}, \qquad (5)$$

where $\omega_k = [1-(2\lambda_k \delta)^2]^{\frac{1}{2}}/2\delta^2$, $J_p(x)$ is the Bessel function of the

*When $2\lambda_k \delta > 1$, $\omega_k$ becomes a pure imaginary number. Equation 5 continues to apply.

first kind of order $p$, and $\lambda_k$ is the $k^{th}$ zero of $J_o(x)$. Although this solution is mathematically correct, it is not of a beneficial form for investigating short-time results. We say this because more and more terms of the summation are required for an accurate numerical solution at shorter and shorter times. With this in mind, we shall reapproach the problem using an asymptotic solution scheme to develop a long-time and a short-time solution. As will be shown, the long-time or diffusion solution is that which is felt throughout the conductor for times proportional to $r_o^2\mu\sigma$ (for a 10-cm-thick lead wire this is $\approx$ 60 ms), whereas, the short-time or inner solution is that which is felt just at the outer edge of the conductor at a time scale proportional to $\epsilon/\sigma$ (for the same lead wire this is $\approx 1\times10^{-18}$ s). This approach will allow us to describe the development of the electric field for all time regimes. Since the first term on the right of equation 3 is usually neglected, $\delta$ was chosen as the stretching parameter.

## Diffusion Solution

The variables in equation 3 have been made dimensionless such that they are of O(1) in the diffusion region. It follows that, as a first approximation, the first term on the right of equation 3 can be neglected for $\delta^2 \ll 1$. The equation reduces to the form of the transient diffusion equation in cylindrical coordinates found in the mass-transfer literature,

$$\frac{\partial^2 E}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial E}{\partial \xi} = \frac{\partial E}{\partial \tau}. \tag{6}$$

This is a major simplification; the highest derivative with respect to time is omitted. However, the equation takes a more manageable form, and one that we are familiar solving. We shall address the implications of this mathematical simplification, shortly.

Equation 6 is solved by separation of variables with the boundary and first of the initial conditions listed above equation 4 to yield

$$\frac{E}{E_o} = \sum_{k=1}^{\infty} \frac{2\, J_o(\lambda_k \xi)}{\lambda_k\, J_1(\lambda_k)} \left\{ 1 - \exp(-\lambda_k^2 \tau) \right\}. \tag{7}$$

Equation 7 is plotted as a function of $\xi$ for different fractions of the time constant in figure 2. The current distribution is directly related to the electric field distribution through Ohm's law, $i = \sigma E$. Other than the absence of the hyperbolic cosines and sines and $\omega$'s of the full solution, this solution does not appear to be much of an improvement over the "full solution", equation 5, since it too is of open form. Nevertheless, from this solution it is apparent that the time constant for the exponential decay of the long-time solution is

$$\tau_o = \frac{r_o^2 \mu \sigma}{\lambda_1^2}. \tag{8}$$

This is the same time regime that is approximated by a the lumped-inductance analysis where the current approaches the steady-state current exponentially as the time over the inductive time constant

$L/R$. Thus, $L/R$ is equal to $\tau_o$.

We now solve for the short-time solution of the diffusion region to determine how the field increases with time for relatively short times (and for future comparison with the long-time solution of the inner region.) Solving equation 6 by Laplace transforms,

$$\frac{\bar{E}(s)}{E_o} = \frac{I_o(s^{\frac{1}{2}}\xi)}{s\, I_o(s^{\frac{1}{2}})}, \tag{9}$$

expanding this solution for large $s$, and then inverting it, we obtain the short-time solution of the diffusion region,

$$\frac{E}{E_o} = \frac{1}{\xi^{\frac{1}{2}}}\left[1 - \frac{9\tau}{32\xi^2} + \frac{3675\tau^2}{2048\xi^4} - O(\frac{\tau^3}{\xi^6})\right] \mathrm{erfc}\frac{1-\xi}{\sqrt{4\tau}} \tag{10}$$

where $\tau \ll 1$. One should note that fewer terms of this solution are needed as $\xi \longrightarrow 1$ and $\tau \longrightarrow 0$. Thus, at short times the electric field penetrates as the erfc of distance from the edge of the conductor divided by the square root of time. We thus define a penetration depth, which is analogous to the skin-depth of a sinusoidal signal, as

$$d_p = \sqrt{t/\mu\sigma}. \tag{11}$$

The solutions developed in this, the diffusion regime, contain all the physics of equation 1 that occur on the time frame of $\tau_o$. We should now like to investigate the consequences of dropping the second order time derivative that provided these results and to elucidate the physics that occur *immediately* following the application of the electric field.

## Inner Solution

To obtain the short-time or inner solution, we shall first define the variable $\xi' = 1 - \xi$, a dimensionless distance from the edge of the conductor, and then define new independent variables by stretching $\xi'$ and $\tau$ by powers of $\delta$. That is, let

$$\bar{\xi} = \frac{1 - \xi}{\delta^n} \quad \text{and} \quad \bar{\tau} = \frac{\tau}{\delta^m}.$$

Setting m = 2 and n = 1, substituting them into equation 3, and keeping terms multiplied by the smallest powers of $\delta$, we obtain the equation

$$\frac{\partial^2 E}{\partial \bar{\xi}^2} = \frac{\partial^2 E}{\partial \bar{\tau}^2} + \frac{\partial E}{\partial \bar{\tau}}, \tag{12}$$

with boundary conditions,

$E = E_o$ , at $\bar{\xi} = 0$ ;

$E$ is well behaved, as $\bar{\xi} \longrightarrow \infty$ ;

and initial conditions,

$E = 0$ , at $\bar{\tau} = 0$ ;

$\dfrac{\partial E}{\partial \bar{\tau}} = 0$ , at $\bar{\tau} = 0$.

(Note the absence of the $1/\xi$ term on the left of equation 12, implying that the curvature of the wire has little effect in this time régime.) These equations were solved by the Laplace-transform technique to give

$$\bar{E}(s) = E_o \frac{\exp[-(s^2 + s)^{1/2}\,\bar{\xi}]}{s}. \tag{13}$$

Applying the complex inversion formula to this solution while integrating around the branch cut between 0 and -1, we get the solution

$$E = E_o \left[ 1 - \frac{1}{\pi} \int_0^1 \frac{e^{-x\bar{\tau}} \sin[\sqrt{|x|\,|1-x|}\;\bar{\xi}]}{x}\, dx \right] u(\bar{\tau}-\bar{\xi}), \tag{14}$$

where $u(\bar{\tau}-\bar{\xi})$ is the Heaviside function.[21] We extract more information from this time regime by expanding equation 13 for large and small values of $s$ to obtain the short and long time solutions, respectively. Expanding 13 for large values of $s$ and taking the inverse Laplace transform to get the short-time solution of the inner region gives

$$E = E_o e^{-\bar{\xi}/2} \left[ 1 + \frac{\bar{\xi}}{8}(\bar{\tau}-\bar{\xi}) - \frac{\bar{\xi}(8-\bar{\xi})}{256}\,(\bar{\tau}-\bar{\xi})^2 + \frac{\bar{\xi}(120-24\bar{\xi}+\bar{\xi}^2)}{18432}\,(\bar{\tau}-\bar{\xi})^3 + O(\bar{\xi}\,\bar{\tau}^{-4}) \right] u(\bar{\tau}-\bar{\xi}). \tag{15}$$

This solution indicates that the electric field initially penetrates the conductor essentially as a front, traveling at the speed of light, with a magnitude that is exponentially damped with distance from the surface. The characteristic length of penetration is $2r_o\delta$. Beyond the distance of $\bar{\tau} = \bar{\xi}$ the field is zero.

Allowing $s$ to approach 0 in equation 13 and inverting to obtain the long-time solution of the inner region gives

$$\frac{E}{E_o} = \text{erfc}\left[\frac{\bar{\xi}}{(4\bar{\tau})^{1/2}}\right]. \tag{16}$$

Thus after an initial period of $\epsilon/\sigma$ the field penetrates as the erfc of the distance divided by the square root of time. This is of the same form as the solution obtained as the short-time solution in the diffusion region, equation 11.

To summarize, when an electric field with an axial component of magnitude $E_o$ is applied to the surface of a conductor at time 0, the field penetrates the conductor at the speed of light with a magnitude that is dramatically damped within in a distance of $\delta r_o$. This occurs within the time 0+ and $\delta^2 r_o^2 \mu\sigma = \epsilon/\sigma$. This is a very short time frame. For comparison, the time it takes light to reach the center of the conductor is $\delta r_o^2 \mu\sigma$. After this initial period, the field penetrates the conductor as the erfc of the distance from the surface divided by the square root of four times the time divided by $r_o^2 \mu\sigma$. Finally, as $t$ approaches $r_o^2 \mu\sigma$, the field approaches the uniform, steady-state field distribution.

From this analysis, we shall develop the solutions of the total current as a function of time.

## Current

From the expressions given for the electric field, one can determine the instantaneous total current. This value is derived from Ohm's law and integration of $i$ over the cross-sectional area,

$$I = 2\pi r_o^2 \sigma \int_0^1 E \, \xi d\xi . \tag{17}$$

Solving for the instantaneous current allows us to compare the results of the above cases more readily by eliminating the independent variable $\xi$. The instantaneous current for the full solution and diffusion solution are, accordingly,

Full $$\frac{I}{\pi r_o^2 \sigma E_o} = 1 - \sum_{k=1}^{\infty} \frac{4}{\lambda_k^2} \left[ \cosh(\omega_k \tau) + \frac{\sinh(\omega_k \tau)}{2\delta^2 \omega_k} \right] \exp\left[ -\frac{\tau}{2\delta^2} \right], \tag{18}$$

Diffusion $$\frac{I}{\pi r_o^2 \sigma E_o} = 1 - \sum_{k=1}^{\infty} \frac{4}{\lambda_k^2} \exp(-\lambda_k^2 \tau). \tag{19}$$

Another approach to the full solution is first to integrate the Laplace transform of the electric field, equation 4, with respect to $\xi$ from 0 to 1. Using the method of residues, we obtain the above solution; but expanding this equation for large $s$ and inverting gives a solution useful at short times,

$$\begin{aligned}
\frac{I}{\pi r_o^2 \sigma E_o} = {} & \frac{2\tau e^{-\tau/2\delta^2}}{\delta} [I_o(\tau/2\delta^2) + I_1(\tau/2\delta^2)] \\
& - [\tau - \delta^2(1 - e^{-\tau/\delta^2})] - \frac{\tau^2 e^{-\tau/2\delta^2}}{6\delta} [I_1(\tau/2\delta^2) + I_2(\tau/2\delta^2)] \\
& - \frac{1}{8} [\tau^2 - 2\delta^2 \tau (2 + e^{-\tau/\delta^2}) + 6\delta^4 (1 - e^{-\tau/\delta^2})] \\
& - \frac{5\tau^3 e^{-\tau/2\delta^2}}{48\delta} [I_2(\tau/2\delta^2) + I_3(\tau/2\delta^2)] + O(\tau^3)
\end{aligned} \tag{20}$$

for $\tau \ll 1$. Allowing $s$ to get *very* large provides the very-short-time solution,

$$\frac{I}{\pi r_o^2 \sigma E_o} = 2\tau/\delta . \tag{21}$$

In the diffusion region, if we start with the transformed solution of the electric field, equation 10, integrate over the cross-sectional area of the conductor, take the limit as $s$ gets very large, and invert it, we get the short-time solution to the diffusion region,

$$\frac{I}{\pi r_o^2 \sigma E_o} = \left(\frac{2\tau}{\pi}\right)^{1/2} . \tag{22}$$

For the inner region, we shall first integrate equation 13 with respect to $\bar{\xi}$ from 0 to $\infty$ and then take the inverse Laplace transform. Here, the current is given as

Inner $$\frac{I}{\pi r_o^2 \sigma E_o} = 2\delta\bar{\tau} e^{-\bar{\tau}/2}[I_o(\bar{\tau}/2) + I_1(\bar{\tau}/2)] . \tag{23}$$

Since $\bar{\tau} = \tau/\delta^2$, we see that the first term of the expanded full solution, equation 20, is equivalent to the short-time solution given here. The rest of the terms of equation 20 are corrective terms for large $\tau$ which tend to bend the solution down to the long-time solution. If we take the limit of the Laplace transformed solution of the inner region, equation 13, for very small s, and again integrate with respect to $\bar{\xi}$ from 0 to $\infty$, we get the long-time solution of the inner region,

$$\frac{I}{\pi r_o^2 \sigma E_o} = \delta \left(\frac{2\bar{\tau}}{\pi}\right)^{\frac{1}{2}}, \tag{24}$$

and for *very* large s, we get the short-time solution of the inner region,

$$\frac{I}{\pi r_o^2 \sigma E_o} = 2\delta\bar{\tau}. \tag{25}$$

From this analysis of the current distribution, we see that at short times ($t<\epsilon/\sigma$) the current is proportional to time to the first power and at long times ($t>\epsilon/\sigma$) is proportional to time to the one half power until $t$ is of order $r_o^2\mu\sigma$ where it approaches the steady-state value (see figure 3). In the former case, the short-time solution of the inner region overlaps the very-short-time solution of the full solution. In the latter case, the long-time solution of the inner region overlaps the short-time solution of the diffusion region. We can conclude that the "displacement current," which is responsible for this difference in solutions, is important only for $\bar{\tau} = \tau/\delta^2 \leq O(1)$ where $\delta^2 \ll 1$. One further note, figure 2 shows that the current approaches the final steady-state asymptotically with an exponential time constant of $r_o^2\mu\sigma$. This is as predicted by the lumped-inductance analysis — the only thing in this work predicted by the lumped-inductance analysis.

## Summary

This analysis describes the penetration of a steadily applied electric field into a cylindrical conductor from the initial time to steady state. This is recognized as a crude first approximation to the current distribution in a bipolar battery. The solution scheme used introduced two distinct time regimes referred to as the inner and diffusion solutions. A long-time and short-time solution of the electric field distribution were determined in each regime and then confirmed as consistent by direct comparison and comparison of the instantaneous current distributions. The current initially penetrates the wire at a rate proportional to time up to a time of the order of $\epsilon/\sigma$ and then at a rate proportional to the square root of time. It is this same order of time after which the omission of the displacement current is justified. The lumped inductance analysis is appropriate for times on the order of $r_o^2\mu\sigma$.

It part II we shall include the cylindrical wire in a radial circuit. That analysis will describe the interactions of the fields in the conductor with the rest of the circuit in terms of the time constant of the complete system.

(It should be noted that all calculations in this paper have been performed assuming that $\sigma$, $\epsilon$, and $\mu$ are independent of time; *i.e.*, the conductor is nondispersive.)

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# List of Symbols

*Roman*

| | |
|---|---|
| $d_p$ | penetration depth, m |
| $E$ | axial electric field component, V/m |
| $E_o$ | magnitude of axial electric field component at the outer edge of conductor, V/m |
| $I$ | total current, A |
| $i$ | current density, $A/m^2$ |
| $L$ | inductance, V-s/A |
| $r$ | distance from center of wire, m |
| $r_o$ | radius of wire, m |
| $R$ | resistance, $\Omega$ |
| $s$ | Laplace transform variable, $s^{-1}$ |
| $t$ | time, s |
| $x$ | variable of integration |

*Greek*

| | |
|---|---|
| $\delta$ | dimensionless stretching parameter |
| $\epsilon$ | permittivity, $C^2/N\text{-}m^2$ |
| $\lambda_k$ | $k^{th}$ zero of Bessel function $J_o(x)$ |
| $\mu$ | permeability, $N/A^2$ |
| $\pi$ | 3.14159265358979 |
| $\sigma$ | electric conductivity, S/m |
| $\tau$ | dimensionless time |
| $\tau_o$ | dimensionless time constant |

$\bar{\tau}$ stretched dimensionless time

$\xi$ dimensionless distance from center

$\bar{\xi}$ stretched dimensionless distance from edge

## References

1. P. L. Kapitza, "A Method of Producing Strong Magnetic Fields," *Proc. Roy. Soc. Lond.*, **105A**, 691-710 (1924).

2. E. Willihnganz, "The Voltage of the Sponge Lead Plate During Discharge of the Storage Battery," *Trans. Electrochem. Soc.*, **79**, 243-251 (1941).

3. H. F. Gibbard, "High Temperature, High Pulse Power Lithium Batteries," *Journal of Power Sources*, **26**, 81-91 (1989).

4. H. F. Gibbard, "Ultra-High-Power Batteries," *Electrochemical Society Proc.*, **86-12**, 193-205 (1986).

5. R. M. LaFollette and D. N. Bennion, "Design Fundamentals of High Power Density, Pulsed Discharge, Lead Acid Batteries: I Experimental," *J. Electrochem. Soc.*, **137**, 3693-3701 (1990).

6. R. M. LaFollette and D. N. Bennion, "Design Fundamentals of High Power Density, Pulsed Discharge, Lead Acid Batteries: II Modeling," *J. Electrochem. Soc.*, **137**, 3701-3707 (1990).

7. B. McKinney, W. Tiedemann, and J. Newman, "Short Circuit Currents in Uninterruptible Power Source Batteries," *Proceedings of the Symposium on Advances in Lead-Acid Batteries*, Pennington N. J., 3336-3347 (1984).

8. B. D. Cahan, M. L. Daroux, and E. B. Yeager, "Effect of Physical and Geometrical Factors on the Impedance of Electrochemical Power Sources," *Proc. - Electrochemical Soc.*, **87**-12, 137-151 (1987).

9. W. G. V. Rosser, "What Makes an Electric Current Flow," *Am. J. Phys.*, 31, 884 (1963).

10. M. A. Heald, "Electric Fields and Charges in Elementary Circuits," *Am. J. Phys.*, **52**, 522-526 (1984).

11. A. Marcus, "The Electric Field Associated with a Steady Current in Long Cylindrical Conductor," *Am. J. Phys.*, **9**, 225-226 (1941).

12. A. Sommerfeld, *Electrodynamics*, Academic Press, New York, 125-133 (1952).

13. D. Marcuse, "Comment on 'The Electric Field Outside a Long Straight Wire Carrying a Steady Current,'" *Am. J. Phys.*, 38, 935-936 (1970).

14. J. C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd Edition, Dover Publications, Inc., New York (1954).

15. J. D. Jackson, *Classical Electrodynamics*, 2nd Edition, John Wiley & Sons, New York (1975).

16. M. A. Shakur, "Solution to the Problem on Page 785," *Am. J. Phys.*, 52, 849 (1984).

17. J. G. Chervenak, "The Back Electric Field from a Long Wire," *Am. J. Phys.*, 54, 946-947 (1986).

18. P. Silvester, "AC Resistance and Reactance of Isolated Rectangular Conductors," IEEE Transactions on Power Apparatus and Systems, **86**, 770-774 (1967).

19. J. R. Carson, "Wave Propagation over Parallel Wires: The Proximity Effect." *Phil. Mag.*, **41**, 607-633 (1921).

20. J. R. Gosselin, P. Rochon, and N. Gauthier, "Study of Eddy Currents in a Cylindrical Wire: An Undergraduate Laboratory Experiment," Am. J. Phys., **50**, 440-443 (1982).

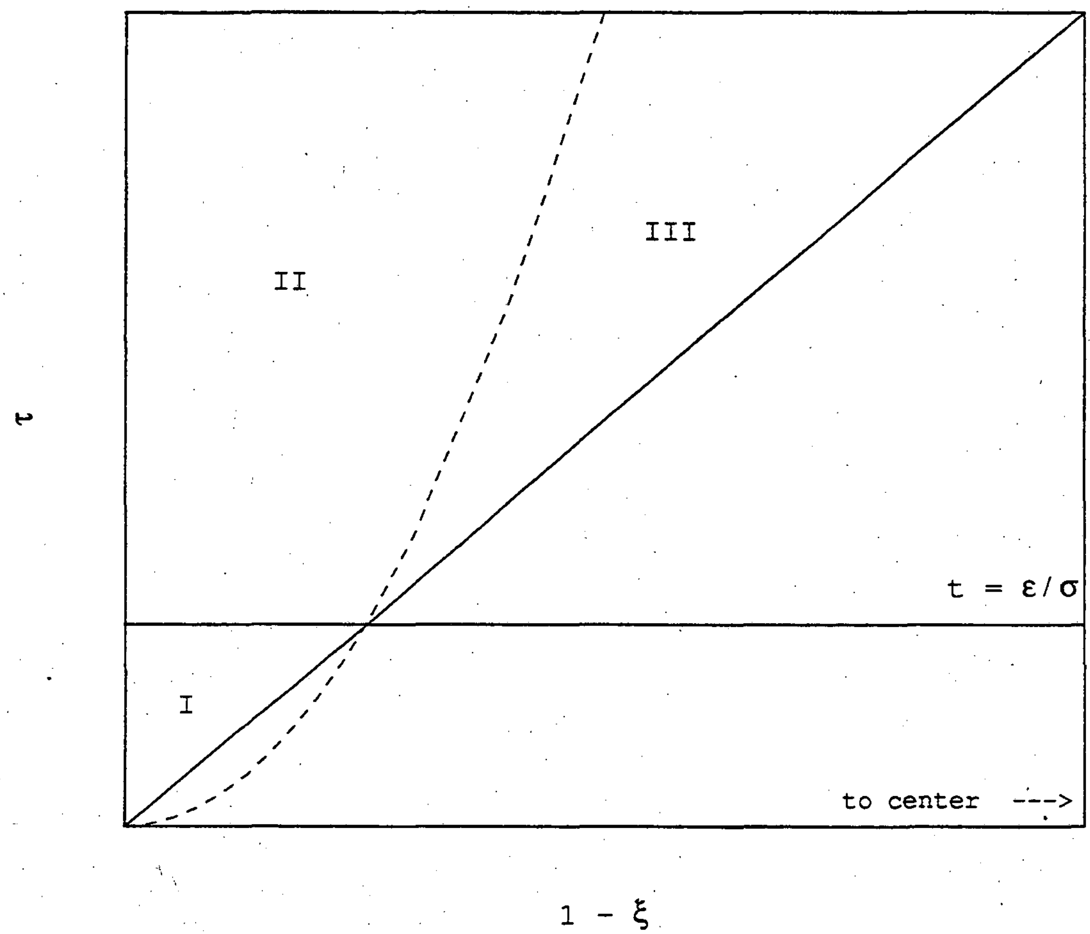
21. M. Abromowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, Inc., New York (1965).

Figure Captions:

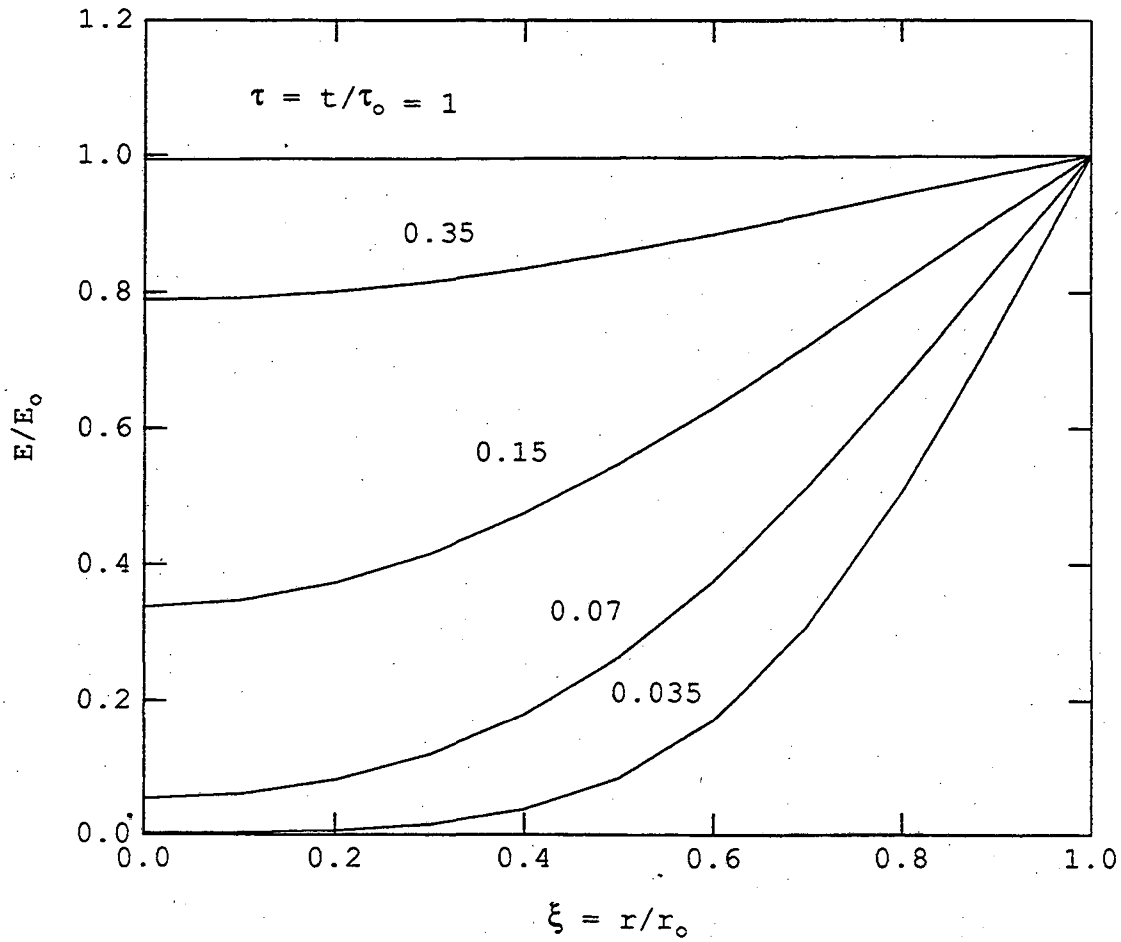
Figure 1. Solution regimes in the space-time coordinate.

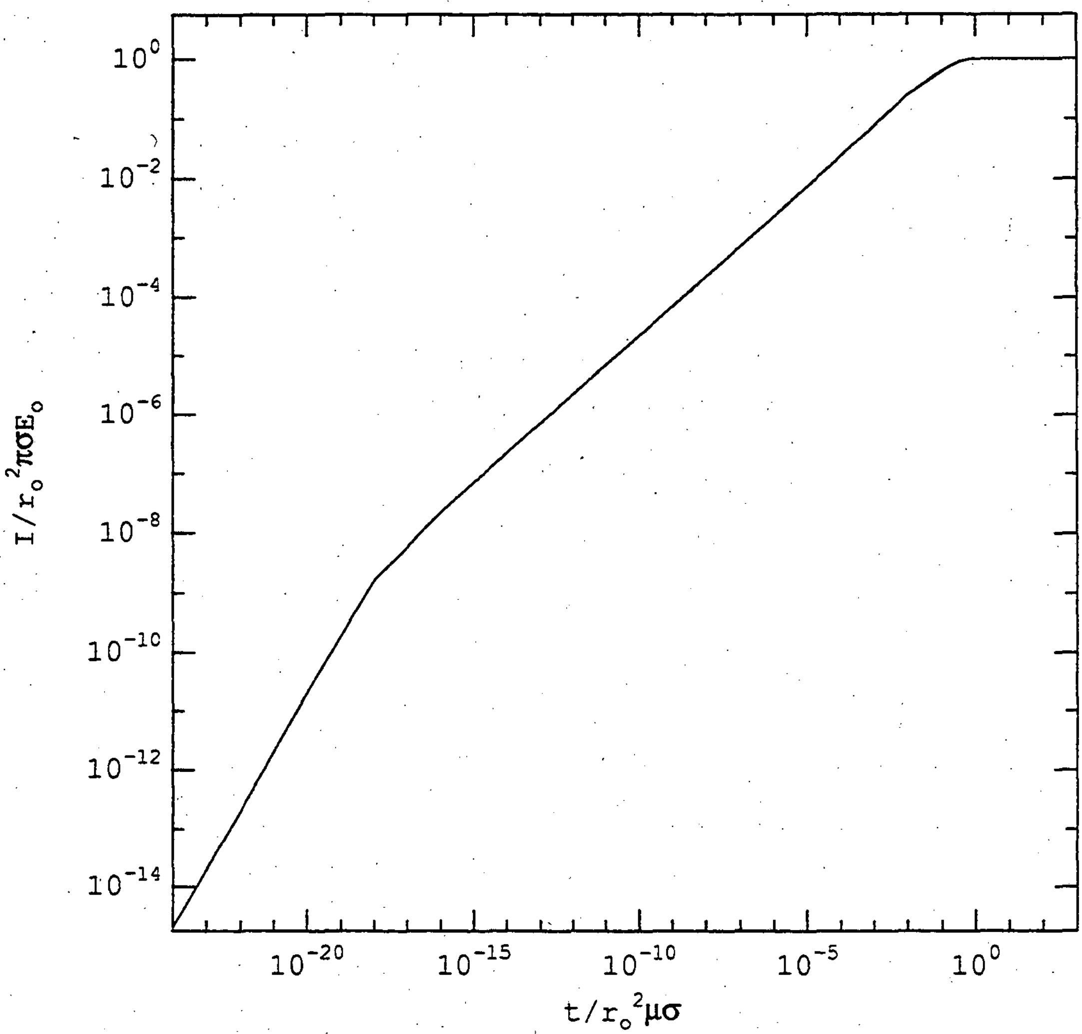
Figure 2. Axial electric field distribution divided by the magnitude of the electric field at the surface *versus* the dimensionless distance from the center of a cylindrical conductor.

Figure 3. Dimensionless current *versus* dimensionless time for $\delta = 1\times10^{-9}$.



III
II
t = ε/σ
I
to center --->
1 - ξ


1.2
1.0
0.8
0.6
0.4
0.2
0.0
$\tau = t/\tau_o = 1$
0.35
0.15
0.07
0.035
E/E$_o$
0.0
0.2
0.4
0.6
0.8
1.0
$\xi = r/r_o$



$10^{0}$
$10^{-2}$
$10^{-4}$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
$I/r_o^2\pi\sigma E_o$
$10^{-20}$
$10^{-15}$
$10^{-10}$
$10^{-5}$
$10^{0}$
$t/r_o^2\mu\sigma$

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
TECHNICAL INFORMATION DEPARTMENT
BERKELEY, CALIFORNIA 94720