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Generalized Demazure Modules for the Twisted Current Algebra $^2\tilde{A}_{2l-1}$

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Generalized Demazure Modules for the Twisted Current Algebra ${}^2\tilde{A}_{2l-1}$

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Joseph Page Wagner

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ABSTRACT OF THE DISSERTATION

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by

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, September 2024
Dr. Vyjayanthi Chari, Chairperson

In this thesis, I study certain generalized Demazure modules for a twisted current algebra of type ${}^2\tilde{A}_{2\ell-1}$; that is, the fixed point subalgebra under an order 2 graph automorphism defined on an untwisted affine Lie algebra of type $\tilde{A}_{2\ell-1}$. In particular, I give a presentation of a family of generalized Demazure modules which can be realized as a submodule of the tensor product of two level one Demazure modules. I also show that, in certain cases, this type of generalized Demazure module is in fact isomorphic to a level two Demazure module.

Contents

Chapter 1

Introduction

In [3], a family of indecomposable finite-dimensional graded modules were introduced for current algebras associated to simple Lie algebras. These modules were indexed by an $|R^+|$-tuple of partitions $\xi = (\xi^\alpha)$ where α varies over a set R^+ of positive roots of a simple lie algebra $\mathfrak{g}$. It was shown that, in the case when (ξ^α) was a rectangular partition, these modules were in fact isomorphic to Demazure modules of various levels. This led to a simplification of the defining relations of said Demazure modules.

Later, in [10], a similar family of indecomposable finite-dimensional graded modules were introduced for twisted current algebras. Like in [3], it was shown that, when (ξ^α) was a rectangular partition, these modules were isomorphic to twisted Demazure modules of various levels, leading to a similar simplification of defining relations.

Then, in [2], it was shown that the graded limit of a family of irreducible prime representations of the quantum affine algebra associated to a simple Lie algebra $\mathfrak{g}$ of type D_n is, in certain cases, isomorphic to a generalized Demazure module. That is, a submodule of the tensor product of level one Demazure modules. A presentation of this family of generalized Demazure modules is also proved in this paper.

For this thesis, I will be using the simplified presentation of level one Demazure modules for twisted current algebras from [10], along with the methods outlined in [2], to give a presentation of a family of generalized Demazure modules for a twisted current algebra of type ${}^2\tilde{A}_{2\ell-1}$.

1.1 Simple Lie Algebras

In this thesis, I will denote $\mathbb{C}$ as the field of complex numbers, $\mathbb{Z}$ as the set of integers, and $\mathbb{Z}_+$ as the set of non-negative integers. Given an indeterminate t, let $\mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials, and $\mathbb{C}[t] \subset \mathbb{C}[t, t^{-1}]$ as the set of polynomials with complex coefficients. For two complex vector spaces V and W, I denote their tensor product over $\mathbb{C}$ by $V \otimes W$. Given a complex Lie algebra $\mathfrak{g}$, I denote $\mathbb{U}(\mathfrak{g})$ as the universal enveloping algebra of $\mathfrak{g}$. I also say that a vector space V is $\mathbb{Z}$-graded if V can be expressed as the direct sum $V = \bigoplus_{k\in\mathbb{Z}} V[k]$.

For a simple finite lie algebra $\mathfrak{g}$ of rank n, with $x, y \in \mathfrak{g}$, the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by $\mathrm{ad}(x) = \mathrm{ad}_x$, with $\mathrm{ad}_x(y) = [x, y]$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote $R \subset \mathfrak{h}^*$ as the corresponding set of roots of $\mathfrak{g}$ with simple roots given by $\{\alpha_i : 1 \le i \le n\}$. Let $\kappa : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ denote the killing form, defined by $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$. Restricting κ to $\mathfrak{h}$ induces an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$, as well as a symmetric, non-degenerate form $(\cdot, \cdot)$ on $\mathfrak{h}^*$. For this thesis, I will assume that this form is normalized so that the square of a long root is 4. For $\alpha \in R$, let $d_\alpha = \frac{4}{(\alpha,\alpha)}$ and let $b_\alpha = \frac{(\alpha,\alpha)}{2}$. Note that $b_\alpha = 2$ when α is long, and $b_\alpha = 1$ when α is short.

Along with a set of simple roots $\{\alpha_i : 1 \le i \le n\}$, I also fix a set of fundamental weights $\{\omega_i : 1 \le i \le n\} \subset \mathfrak{h}^*$ such that $(\omega_i, \alpha_j) = \delta_{i,j}$. Let Q denote the $\mathbb{Z}$-span of the simple roots, and denote the $\mathbb{Z}_+$-span by Q^+. Similarly, denote the $\mathbb{Z}$-span of the fundamental weights, called the weight lattice, as P, and denote the $\mathbb{Z}_+$-span by P^+. Then denote the positive roots by $R^+ = R \cap Q^+$. I denote the negative roots by R^-, defined in a similar way. I also denote R_l as the long roots of R, and R_s as the short roots.

Next, I define a partial order on P by $\lambda \le \mu$ iff $\mu - \lambda \in Q^+$. Finally, let $\{x_\alpha^\pm, h_i : \alpha \in R^+, 1 \le i \le n\}$ be a Chevalley Basis of $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the corresponding triangular decomposition. For convenience, I set $x_i^\pm = x_{\alpha_i}^\pm$.

For $\lambda \in P^+$, I denote the finite dimensional irreducible $\mathfrak{g}$-module as $V(\lambda)$. I denote the generator of $V(\lambda)$ as v_λ, subject to the following defining relations:

$$x_i^+ v_\lambda = 0, \quad h_i v_\lambda = \lambda(h_i) v_\lambda, \quad (x_i^-)^{\lambda(h_i)+1} v_\lambda = 0$$

for $i \in I$. These modules allow for a characterization of finite dimensional $\mathfrak{g}$-modules; in particular, any finite dimensional $\mathfrak{g}$-module V can be written as a direct sum of modules $V(\lambda),\ \lambda \in P^+$.

Throughout this thesis, I will make reference to simple, finite-dimensional Lie algebras of two types: a special Lie algebra $\overline{\mathfrak{g}}$ of type A_n, and a symplectic Lie algebra $\mathfrak{g}$ of type C_n.

1.2 (Untwisted) Affine Lie Algebras

To realize an untwisted affine Lie algebra $\tilde{\mathfrak{g}}$ of type $\tilde{A}_n$, I start with a simple Lie algebra $\overline{\mathfrak{g}}$ of type A_n, with root system $R_{\overline{\mathfrak{g}}}$, and denote the Loop algebra as

$$\mathcal{L}(\overline{\mathfrak{g}}) = \overline{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}].$$

This can be made into a Lie algebra by defining the bracket operation: for $x \otimes f(t),\ y \otimes g(t) \in \mathcal{L}(\overline{\mathfrak{g}})$, the bracket operation of $\mathcal{L}(\overline{\mathfrak{g}})$ is defined to be

$$[x \otimes f(t), y \otimes g(t)] = [x, y]_{\overline{\mathfrak{g}}} \otimes f(t)g(t)$$

where $[\cdot, \cdot]_{\overline{\mathfrak{g}}}$ is the bracket operation of $\overline{\mathfrak{g}}$. Then the untwisted affine Lie algebra $\tilde{\mathfrak{g}}$ is given by

$$\tilde{\mathfrak{g}} = \mathcal{L}(\overline{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where c is the canonical central element and d acts as the derivation $t\frac{d}{dt}$, with a bracket operation given by

$$[x \otimes t^r, y \otimes t^s] = [x, y]_{\overline{\mathfrak{g}}} \otimes t^{r+s} + \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) r \delta_{r+s,0} c, \ \ [d, x \otimes t^r] = r(x \otimes t^r), \ \ x, y \in \overline{\mathfrak{g}},\ r, s \in \mathbb{Z}.$$

If $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}$ is a Cartan subalgebra, then the elements of the dual of $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}$, $\alpha_0, \alpha_1, \cdots, \alpha_n \in \tilde{\mathfrak{h}}^*$ can be defined by extending $\alpha_1, \cdots, \alpha_n \in \overline{\mathfrak{h}}^*$ to $\tilde{\mathfrak{h}}^*$ by stating $\alpha_i(c) = 0 = \alpha_i(d)$ for $1 \leq i \leq n$ and defining $\delta \in \tilde{\mathfrak{h}}^*$ by:

$$\delta(h) = 0 \text{ for } h \in \overline{\mathfrak{h}}, \quad \delta(c) = 0, \quad \delta(d) = 1.$$

Remark that $\alpha_0 \in \tilde{\mathfrak{h}}^*$ is defined as $\alpha_0 = -\theta + \delta$, where θ is the longest root in $R_{\bar{\mathfrak{g}}}$.

1.3 Twisted Affine Lie Algebras

Assume that $\overline{\mathfrak{g}}$ has rank $n = 2\ell - 1$ for $\ell \geq 3$. Given an indexing I on a set of simple roots $\{\alpha_i\}_{i \in I}$ of $\overline{\mathfrak{g}}$, let σ be a permutation of I defined by

$$\sigma(i) = 2\ell - i.$$

I can then extend σ to a graph automorphism of $\overline{\mathfrak{g}}$ by setting $\sigma(x_{\alpha_i}) = x_{\alpha_{\sigma(i)}}$ and then extending this action linearly to the rest of $\overline{\mathfrak{g}}$ such that it respects the bracket operation.

I now denote the Twisted Lie Algebra, defined as the fixed point subalgebra under this automorphism, as $\overline{\mathfrak{g}}^\sigma := \{x \in \overline{\mathfrak{g}} | \sigma(x) = x\}$. In this case, when $\overline{\mathfrak{g}}$ is a simple Lie algebra of type $A_{2\ell-1}$ for $\ell \geq 3$, $\overline{\mathfrak{g}}^\sigma$ is isomorphic to a simple Lie algebra of type C_ℓ.

I can now introduce the twisted graph automorphism τ on $\tilde{\mathfrak{g}}$, defined by the following:

$$\tau(x \otimes t^k) = \sigma(x) \otimes (-1)^k t^k \text{ for } x \in \overline{\mathfrak{g}},$$

$$\tau(c) = c, \quad \tau(d) = d.$$

The fixed point subalgebra of $\tilde{\mathfrak{g}}$ under the automorphism τ, denoted as $\hat{\mathfrak{g}}$, is a twisted affine Lie algebra of type ${}^2\tilde{A}_{2\ell-1}$. From here on, unless otherwise specified, assume that $\mathfrak{g}$ is a simple Lie algebra of type C_ℓ. I will use both $\hat{\mathfrak{g}}$ and $\mathfrak{g}$ in the definition of a special twisted current algebra $\mathfrak{Cg}$ of type ${}^2\tilde{A}_{2\ell-1}$.

Letting δ denote the unique non-divisible positive imaginary root in the root system of $\hat{\mathfrak{g}}$, I can then denote the root system of $\hat{\mathfrak{g}}$ as $\hat{R}$ and I have $\hat{R} = \hat{R}^+ \cup \hat{R}^-$, where $\hat{R}^- = -\hat{R}^+$, $\hat{R}^+ = \hat{R}^+_{re} \cup \hat{R}^+_{im}$, $\hat{R}^+_{im} = \mathbb{N}\delta$, $\hat{R}^+_{re} = R^+ \cup (R_s + \mathbb{N}\delta) \cup (R_l + 2\mathbb{N}\delta)$, and $\hat{R}_{re}(\pm) = R^\pm \cup (R^\pm_s + \mathbb{N}\delta) \cup (R^\pm_l + 2\mathbb{N}\delta)$.

Given $\alpha \in \hat{R}^+$, let $\hat{\mathfrak{g}}_\alpha \subset \hat{\mathfrak{g}}$ be the corresponding root space; note that $\hat{\mathfrak{g}}_\alpha \subset \mathfrak{g}$ if $\alpha \in R$. For a non-imaginary root α, I denote x_α as the generator of $\hat{\mathfrak{g}}_\alpha$. I also denote $\hat{\mathfrak{b}}$ as the Borel subalgebra corresponding to $\hat{R}^+$, and $\hat{\mathfrak{n}}^+$ as its nilpotent radical;

$$\hat{\mathfrak{b}} = \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+, \quad \hat{\mathfrak{n}}^\pm = \bigoplus_{\alpha \in \hat{R}^+} \hat{\mathfrak{g}}_{\pm\alpha}.$$

The subalgebras $\mathfrak{b}$ and $\mathfrak{n}^\pm$ of $\mathfrak{g}$ are defined analogously.

Consider the algebra

$$\mathfrak{k} = (\mathfrak{h} \oplus \mathbb{C}d) \oplus \hat{\mathfrak{n}}^+ \oplus \mathfrak{n}^-.$$

The twisted current algebra $\mathfrak{Cg}$ can then be defined as the following ideal of $\mathfrak{k}$:

$$\mathfrak{Cg} = \mathfrak{h} \oplus \hat{\mathfrak{n}}^+ \oplus \mathfrak{n}^-$$

with triangular decomposition

$$\mathfrak{Cg} = \mathfrak{Cn}^+ \oplus \mathfrak{Ch} \oplus \mathfrak{Cn}^-,$$

where

$$\mathfrak{Ch} = \mathfrak{Ch}_+ \oplus \mathfrak{h}, \quad \mathfrak{Ch}_+ = \bigoplus_{k>0} \hat{\mathfrak{g}}_{k\delta}, \quad \mathfrak{Cn}^\pm = \bigoplus_{\alpha \in \hat{R}_{re}(\pm)} \hat{\mathfrak{g}}_{\pm\alpha}.$$

Note that, for any $\alpha \in R^+$, there is $\overline{\alpha} \in R^+_{\overline{\mathfrak{g}}}$ such that $\overline{\alpha}|_{\mathfrak{h}} = \alpha$. Thus, fixing a Chevalley basis $\{X^\pm_\alpha,\ H_i\ :\ i \in I,\ \alpha \in R^+_{\overline{\mathfrak{g}}}\}$ for $\overline{\mathfrak{g}}$ enables us to realize $\mathfrak{Cg}$ as a subalgebra of $\mathcal{L}(\overline{\mathfrak{g}})$ via the following [10]:

For $r \in \mathbb{Z}_+$ and $\alpha \in R^+$,

$$x_{\pm\alpha + b_\alpha r\delta} = \left(X^\pm_{\overline{\alpha}} + (-1)^{b_\alpha r} X^\pm_{\sigma(\overline{\alpha})}\right) \otimes t^{b_\alpha r}$$

$$h_{\alpha, r\delta} = \left(H_\alpha + (-1)^r H_{\sigma(\overline{\alpha})}\right) \otimes t^r$$

$$h_{i, r\delta} = H_i \otimes t^r + H_{2n-i} \otimes (-t)^r.$$

Remark that $\alpha_i^\vee = h_{i,0}$ for $i \in I$. Note that the element d defines a $\mathbb{Z}_+$-graded structure on $\mathfrak{Cg}$: for $\alpha \in \hat{R}$, $\hat{\mathfrak{g}}_\alpha$ has grade k if

$$[d, x_\alpha] = k$$

or, equivalently, if $\alpha(d) = k$. Note that the eigenvalues of d are all integers, and if $\hat{\mathfrak{g}}_\alpha \subset \mathfrak{Cg}$, then the eigenvalues are non-negative. This also defines a grading on $\mathbb{U}(\mathfrak{Cg})$; In particular, for $\gamma_1, \cdots, \gamma_k \in R$, the element $(x_{\gamma_1 + r_1\delta})(x_{\gamma_2 + r_2\delta}) \cdots (x_{\gamma_k + r_k\delta})$ has grade $r_1 + r_2 + \cdots r_k$.

Since $\mathfrak{Cg}$ is graded, I can also introduce the notion of a graded $\mathfrak{Cg}$ module. V is considered to be a graded $\mathfrak{Cg}$ module if it is $\mathbb{Z}$-graded and the action of $\mathfrak{Cg}$ respects this grading; that is, for $\beta \in R$,

$$(x_{\beta+s\delta})V[r] \subset V[r+s].$$

I now denote the grade shift operator as τ_s^*, which maps $V[r] \to V[r+s]$ for $r, s \in \mathbb{Z}$. That is, for a $\mathfrak{Cg}$-graded module V, I have that $\tau_s^* V$ is the graded $\mathfrak{Cg}$ module V where the graded pieces are shifted uniformly by s, but the action of $\mathfrak{Cg}$ remains unchanged.

1.4 Local Weyl Module and Demazure Module

For $\lambda \in P^+$, the local Weyl module, $W_{\text{loc}}(\lambda)$, is defined as the cyclic $\mathfrak{Cg}$-module generated by w_λ subject to the following relations:

$$\mathfrak{Cn}^+ w_\lambda = 0, \quad \mathfrak{Ch}_+ w_\lambda = 0, \quad h_{\alpha,0} w_\lambda = \lambda(\alpha^\vee) w_\lambda, \quad (x_{-\alpha})^{\lambda(\alpha^\vee)+1} w_\lambda = 0, \tag{1.4.1}$$

for all $\alpha \in R^+$ [10]. By declaring the grade of w_λ to be 0, $W_{\text{loc}}(\lambda)$ becomes a graded $\mathfrak{Cg}$-module. Remark that the 0th graded piece, $W_{\text{loc}}(\lambda)[0]$, is $V(\lambda)$.

Now, let $(l, \lambda) \in \mathbb{Z}_+ \times P^+$. For any $\alpha \in R^+$, I write $\lambda(\alpha^\vee) = (s_\alpha - 1)l + m_\alpha$, $0 < m_\alpha \leq l$. Then by Theorem 5 in [10], the level l Demazure module is defined as the quotient of $W_{\text{loc}}(\lambda)$ by the submodule generated by the elements:

$$\{(x_{-\alpha+b_\alpha s_\alpha \delta})w_\lambda : \alpha \in R^+\} \bigcup \{(x_{-\alpha+b_\alpha(s_\alpha-1)\delta})^{m_\alpha+1} w_\lambda : \alpha \in R^+,\ m_\alpha < l\}. \tag{1.4.2}$$

Consequently, for special twisted current algebras, I have that level one Demazure modules are isomorphic to local Weyl modules (initially proven in [5]).

I will also use an equivalent presentation of $D(l, \lambda)$ given in [5]. Let Φ_0 be the root system of C_ℓ and $\Phi_1 = (\Phi_0)_s$, i.e., the short roots of Φ_0. The following was proved in [5]

Proposition 1. *As a module for* $\mathfrak{Cg}$ *the Demazure module* $D(l,\lambda)$ *is isomorphic to the cyclic* $\mathbb{U}(\mathfrak{Cg})$*-module generated by a vector* $v \neq 0$ *subject to the following relations:*

For $\beta \in \Phi_j^+,\ 0 \leq j \leq 1$ *I have:*

$$(\mathfrak{Cn}_j^+ \otimes t^j\mathbb{C}[t^2])v = 0 \tag{1.4.3}$$

$$(x_\beta^- \otimes t^{2s+j})^{k_\beta+1}v = 0 \text{ where } s \geq 0,\ k_\beta = max\left\{0, \langle\lambda,\beta^\vee\rangle - \frac{2(2s+j)}{\langle\beta,\beta\rangle}l\right\} \tag{1.4.4}$$

$$(h \otimes t^{2s+j})v = \delta_{j,0}\delta_{s,0}\lambda(h)v\ \forall\ \ h \in \mathfrak{h}_j,\ s \geq 0. \tag{1.4.5}$$

Finally, I can introduce the generalized Demazure modules for $\mathfrak{Cg}$. First, consider the tensor product $\tau_s^* D(l,\lambda) \otimes \tau_{s'}^* D(l',\lambda')$, and then take the $\mathfrak{Cg}$ module through $w_\lambda \otimes w_{\lambda'}$. In this thesis, I will give a presentation of the family of generalized Demazure modules of the form

$$D(\lambda,\mu) := \mathbb{U}(\mathfrak{Cg})(w_\lambda \otimes w_\mu) \subset D(1,\lambda) \otimes D(1,\mu),$$

with certain restrictions on the pair $(\lambda,\mu) \in P^+ \times P^+$.

The following result is proven as in [2] by replacing affine with twisted affine:

Lemma 2. *There exists a (unique up to scalars) map* $\eta_{\lambda,\mu} : D(\lambda,\mu) \to D(2,\lambda+\mu) \to 0$*, of* $\mathfrak{Cg}$*-modules extending the assignment* $w_\lambda \otimes w_\mu \to w_{2,\lambda+\mu}$. □

Chapter 2

Main Results

Keeping the notation introduced in the previous chapter, with $\mathfrak{g}$ a simple Lie algebra of type C_ℓ and $\mathfrak{C}\mathfrak{g}$ a twisted current algebra of type ${}^2\tilde{A}_{2\ell-1}$, I denote the following roots of R^+:

$$\alpha_{i,j} = \alpha_i + \cdots + \alpha_j, \quad 1 \le i \le j \le \ell - 1$$

$$\beta_{i,j} = \alpha_i + \cdots \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{\ell-1}) + \alpha_\ell, \quad 1 \le i < j \le \ell$$

$$\beta_j = 2(\alpha_j + \cdots + \alpha_{\ell-1}) + \alpha_\ell, \quad 1 \le j \le \ell.$$

Note that

$$R^+ = \{\alpha_{i,j} : 1 \le i \le j \le \ell - 1\} \sqcup \{\beta_{i,j} : 1 \le i < j \le \ell\} \sqcup \{\beta_j : 1 \le j \le \ell\}.$$

Furthermore, for $\lambda \in P^+$, I have

$$\lambda(\alpha_{i,j}^\vee) = \lambda(\alpha_i^\vee) + \cdots + \lambda(\alpha_j^\vee)$$

$$\lambda(\beta_{i,j}^\vee) = \lambda(\alpha_i^\vee) + \cdots + \lambda(\alpha_{j-1}^\vee) + 2(\lambda(\alpha_j^\vee) + \cdots + \lambda(\alpha_\ell^\vee))$$

$$\lambda(\beta_j^\vee) = \lambda(\alpha_j^\vee) + \cdots + \lambda(\alpha_\ell^\vee).$$

2.1 Interlacing Pairs

Let

$$P^+(1) = \{\lambda \in P^+ : \lambda(\alpha_i^\vee) \leq 1, \quad 1 \leq i \leq \ell\}.$$

Note that any $\lambda \in P^+(1)$ can be written uniquely (up to order) as a sum $\lambda = \lambda_1 + \lambda_2$ where $\lambda_k \in P^+(1)$ for $k = 1, 2$ such that the following is satisfied for $1 \leq i \leq j \leq \ell$:

$$\lambda_r(\alpha_i^\vee) = 1 = \lambda_r(\alpha_j^\vee) \implies \lambda_p(\alpha_s^\vee) = 1 \text{ for some } i < s < j,\ \{r, p\} = \{1, 2\}.$$

I call $(\lambda_1, \lambda_2) \in P^+ \times P^+$ an *interlacing pair* if $\lambda_1 + \lambda_2 \in P^+(1)$, and the preceding condition holds.

Examples. The pairs $(\omega_i, 0)$ for $0 \leq i \leq \ell$ and the elements of the set $\{(\omega_i, \omega_j) : 0 \leq i \neq j \leq \ell\}$ are interlacing. The pair $(\omega_1 + \omega_4, \omega_5 + \omega_6)$ is not interlacing, but the pair $(\omega_1 + \omega_5, \omega_4 + \omega_6)$ is.

For an interlacing pair (λ_1, λ_2) with $\lambda = \lambda_1 + \lambda_2$, if $\lambda = 0$, set $p = p' = p'' = 0$. If $\lambda = \omega_j$, set $p = j$ and $p' = p'' = 0$. If $\lambda = \omega_i + \omega_j$ with $i > j$, set $p = i$, $p' = j$, and $p'' = 0$. If $\lambda(\alpha_{1,\ell-1}^\vee + \alpha_\ell^\vee) \geq 3$, let $p > p' > p''$ be maximal such that $\lambda(\alpha_{p''}^\vee + \alpha_{p'}^\vee + \alpha_p^\vee) = 3$. I now define $\nu \in P^+$ as (λ_1, λ_2)-*compatible* if $\nu(\alpha_{p-1}^\vee) > 0$ whenever $p' \neq p - 1$.

Throughout the rest of this chapter, I will assume that (λ_1, λ_2) *is an interlacing pair, that* $\lambda = \lambda_1 + \lambda_2$*, and that* ν *is* (λ_1, λ_2)*-compatible. Furthermore, the property of interlacing pairs allows me to assume without loss of generality that whenever* $1 \leq p \leq \ell$ *is maximal such that* $\lambda(\alpha_p^\vee) > 0$*, then* $\lambda_1(\alpha_p^\vee) = \lambda(\alpha_p^\vee)$.

The following lemma was proved in [4], and will be useful for later.

Lemma 3. *For all* $1 \le i \le j \le \ell - 1$ *and* (λ_1, λ_2) *interlacing, I have*

$$|(\lambda_1 - \lambda_2)(\alpha_{i,j}^\vee)| \le 1,$$

and

$$|(\lambda_1 - \lambda_2)(\alpha^\vee)| \le d_\alpha \textit{ for all other } \alpha \in R^+.$$

2.2 Presentation of $V(\lambda_1 + \nu, \lambda_2 + \nu)$

For an interlacing pair (λ_1, λ_2) with $\lambda = \lambda_1 + \lambda_2$ and a (λ_1, λ_2)-compatible $\nu \in P^+$, I set

$$R(\lambda_1, \lambda_2) = \{\beta_{i,j} \in R^+ : (\lambda_1 - \lambda_2)(\beta_{i,j}^\vee) = \pm 2\}$$

and define $V(\lambda_1 + \nu, \lambda_2 + \nu)$ to be the $\mathfrak{Cg}$-module generated by $w_{\lambda_1+\nu,\lambda_1+\nu}$ satisfying the following defining relations. For $\alpha \in R^+$ and α_i with $1 \le i \le \ell$,

$$\mathfrak{Cn}^+ w_{\lambda_1+\nu,\lambda_2+\nu} = 0,\ \mathfrak{Ch}_+ w_{\lambda_1+\nu,\lambda_2+\nu} = 0,\ h_{\alpha,0} w_{\lambda_1+\nu,\lambda_2+\nu} = (\lambda+2\nu)(\alpha^\vee) w_{\lambda_1+\nu,\lambda_2+\nu}, \quad (2.2.1)$$

$$(x_{-\alpha_i})^{(\lambda+2\nu)(\alpha_i^\vee)+1} w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \quad (2.2.2)$$

$$(x_{-\alpha+\max\{r_{1,\alpha}, r_{2,\alpha}\}\delta}) w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \quad (2.2.3)$$

$$(x_{-\alpha+b_\alpha(s_\alpha-1)\delta})^{m_\alpha+1} w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \quad (2.2.4)$$

$$(x_{-\beta+s_\beta\delta})^2 w_{\lambda_1+\nu,\lambda_2+\nu} = 0 \quad \beta \in R(\lambda_1, \lambda_2), \quad (2.2.5)$$

where s_α and m_α are the unique positive integers such that $(\lambda + 2\nu)(\alpha^\vee) = 2(s_\alpha - 1) + m_\alpha$, $0 < m_\alpha \le 2$, and $r_{j,\alpha} = b_\alpha((\lambda_j + \nu)(\alpha^\vee))$ for $j \in \{1, 2\}$.

I can define a grading on $V(\lambda_1 + \nu, \lambda_2 + \nu)$ by declaring the grade of $w_{\lambda_1+\nu,\lambda_2+\nu}$ to be 0. Relations (2.2.1) and (2.2.2) show that $V(\lambda_1 + \nu, \lambda_2 + \nu)$ is a quotient of the local Weyl module $W_{\text{loc}}(\lambda + 2\nu)$.

Lemma 4. *The assignments* $w_{\lambda_1+\nu,\lambda_2+\nu} \to w_{2,\lambda+2\nu}$ *and* $w_{\lambda_1+\nu,\lambda_2+\nu} \to w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}$ *define surjective maps of* $\mathfrak{C}\mathfrak{g}-$*modules.*

$$\psi_{\lambda_1+\nu,\lambda_2+\nu} : V(\lambda_1+\nu,\lambda_2+\nu) \to D(2,\lambda+2\nu), \quad \phi_{\lambda_1+\nu,\lambda_2+\nu} : V(\lambda_1+\nu,\lambda_2+\nu) \to D(\lambda_1+\nu,\lambda_2+\nu)$$

$$\textit{and } \psi_{\lambda_1+\nu,\lambda_2+\nu} = \eta_{\lambda_1+\nu,\lambda_2+\nu} \circ \phi_{\lambda_1+\nu,\lambda_2+\nu}.$$

Proof. I'll begin with $\phi_{\lambda_1+\nu,\lambda_2+\nu}$. First, note that $(x_{-\alpha+k\delta})w_{\lambda_j+\nu} = 0$ for $k \geq r_{j,\alpha}, j \in \{1,2\},\ \alpha \in R^+$. Thus, relation (2.2.3) holds in $D(\lambda_1+\nu,\lambda_2+\nu)$; that is,

$$(x_{-\alpha+\max\{r_{1,\alpha},r_{2,\alpha}\}\delta})(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}) = 0.$$

As for relation (2.2.2), note that

$$(x_{-\alpha_i})^{(\lambda+2\nu)(\alpha_i^\vee)+1}(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu})$$
$$= \sum_{k=0}^{(\lambda+2\nu)(\alpha_i^\vee)+1} \binom{(\lambda+2\nu)(\alpha_i^\vee)+1}{k} (x_{-\alpha_i})^{(\lambda+2\nu)(\alpha_i^\vee)+1-k} w_{\lambda_1+\nu} \otimes (x_{-\alpha_i})^k w_{\lambda_2+\nu}.$$

Since $(x_{-\alpha_i})^{(\lambda_j+\nu)(\alpha_i^\vee)+1}w_{\lambda_j+\nu} = 0$ for $j = 1,2$, for values of $k \leq (\lambda_2+\nu)(\alpha_i^\vee)$, the first part of the tensor product is 0, and for $k > (\lambda_2+\nu)(\alpha_i^\vee)$, the second part of the tensor product is 0. Hence, each term in this sum is 0; therefore,

$$(x_{-\alpha_i})^{(\lambda+2\nu)(\alpha_i^\vee)+1}(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}) = 0.$$

Next, I will prove relation (2.2.4) holds. This will be done in several cases. First, recall that s_α and m_α are the unique non-negative integers such that $(\lambda+2\nu)(\alpha^\vee) = 2(s_\alpha - 1) + m_\alpha$ with $0 < m_\alpha \leq 2$, and that

$$(x_{-\alpha+b_\alpha(s_\alpha-1)\delta})^{m_\alpha+1}(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu})$$
$$= \sum_{k=0}^{m_\alpha+1} (x_{-\alpha+b_\alpha(s_\alpha-1)\delta})^k w_{\lambda_1+\nu} \otimes (x_{-\alpha+b_\alpha(s_\alpha-1)\delta})^{m_\alpha+1-k} w_{\lambda_2+\nu}.$$

In each case, I will show that every term in this sum is equal to 0.

For the first case, suppose α is short and that $(\lambda + 2\nu)(\alpha^\vee) \equiv 0 \bmod_2$. Then $m_\alpha = 2$ and $b_\alpha = 1$. For $l = 1$ and $2s + j = b_\alpha(s_\alpha - 1)$, I have

$$\langle \lambda_1 + \nu, \alpha^\vee \rangle - \frac{2(2s+j)}{\langle \alpha, \alpha \rangle} l = (\lambda_1 + \nu)(\alpha^\vee) - (s_\alpha - 1)$$

$$= (\lambda_1 + \nu)(\alpha^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee)}{2} + 1 = \frac{(\lambda_1 - \lambda_2)(\alpha^\vee)}{2} + 1.$$

Similarly,

$$\langle \lambda_2 + \nu, \alpha^\vee \rangle - \frac{2(2s+j)}{\langle \alpha, \alpha \rangle} l = \frac{(\lambda_2 - \lambda_1)(\alpha^\vee)}{2} + 1.$$

By Lemma 3, $|(\lambda_1 - \lambda_2)(\alpha^\vee)| \leq d_\alpha$. By assumption, $(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee) \equiv 0 \bmod_2$, and hence, $(\lambda_1 - \lambda_2)(\alpha^\vee) \equiv 0 \bmod_2$. Thus, $(\lambda_1 - \lambda_2)(\alpha^\vee) \in \{0, \pm 2\}$. Suppose $(\lambda_1 - \lambda_2)(\alpha^\vee) = 2$. Then $(\lambda_2 - \lambda_1)(\alpha^\vee) = -2$ and by relation (1.4.4), I have

$$(x_{-\alpha + b_\alpha(s_\alpha - 1)\delta})^3 w_{\lambda_1 + \nu} = 0 \text{ and } (x_{-\alpha + b_\alpha(s_\alpha - 1)\delta}) w_{\lambda_2 + \nu} = 0.$$

Hence,

$$(x_{-\alpha + b_\alpha(s_\alpha - 1)\delta})^{m_\alpha + 1}(w_{\lambda_1 + \nu} \otimes w_{\lambda_2 + \nu}) = 0.$$

The argument is symmetric when $(\lambda_1 - \lambda_2)(\alpha^\vee) = -2$. Alternatively, if $(\lambda_1 - \lambda_2)(\alpha^\vee) = 0$, then $(\lambda_2 - \lambda_1)(\alpha^\vee) = 0$, and by relation (1.4.4), I have $(x_{-\alpha + b_\alpha(s_\alpha - 1)}\delta)^2 w_{\lambda_i + \nu} = 0$ for $i = 1, 2$ and again the relation holds.

Now suppose α is short and that $(\lambda + 2\nu)(\alpha^\vee) \equiv 1 \bmod_2$. Then $m_\alpha = 1$ and $b_\alpha = 1$. For $l = 1$ and $2s + j = b_\alpha(s_\alpha - 1)$, I have

$$\langle \lambda_1 + \nu, \alpha^\vee \rangle - \frac{2(2s+j)}{\langle \alpha, \alpha \rangle} l = (\lambda_1 + \nu)(\alpha^\vee) - (s_\alpha - 1) = (\lambda_1 + \nu)(\alpha^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee) - 1}{2}$$
$$= \frac{(\lambda_1 - \lambda_2)(\alpha^\vee) + 1}{2}.$$

Similarly,

$$\langle \lambda_2 + \nu, \alpha^\vee \rangle - \frac{2(2s+j)}{\langle \alpha, \alpha \rangle} l = \frac{(\lambda_2 - \lambda_1)(\alpha^\vee) + 1}{2}.$$

Again, I use the fact that $|(\lambda_1 - \lambda_2)(\alpha^\vee)| \leq d_\alpha$ along with my assumption that $(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee) \equiv 1 \bmod_2$ to conclude that $(\lambda_1 - \lambda_2)(\alpha^\vee) \in \{\pm 1\}$. If $(\lambda_1 - \lambda_2)(\alpha^\vee) = 1$, then $(\lambda_2 - \lambda_1)(\alpha^\vee) = -1$ and by relation (1.4.4), I have

$$(x_{-\alpha + (s_\alpha - 1)\delta})^2 w_{\lambda_1 + \nu} = 0 = (x_{-\alpha + (s_\alpha - 1)\delta}) w_{\lambda_2 + \nu}$$

and hence, $(x_{-\alpha + b_\alpha(s_\alpha - 1)\delta})^{m_\alpha + 1}(w_{\lambda_1 + \nu} \otimes w_{\lambda_2 + \nu}) = 0$. Again, the argument is symmetric when $(\lambda_1 - \lambda_2)(\alpha^\vee) = -1$.

Now suppose α is a long root and that $(\lambda + 2\nu)(\alpha^\vee) \equiv 0 \bmod_2$. Then $m_\alpha = 2$ and $b_\alpha = 2$. For $l = 1$ and $2s + j = b_\alpha(s_\alpha - 1)$, I have

$$\langle \lambda_1 + \nu, \alpha^\vee \rangle - \frac{2(2s+j)}{\langle \alpha, \alpha \rangle} l = (\lambda_1 + \nu)(\alpha^\vee) - (s_\alpha - 1)$$
$$= (\lambda_1 + \nu)(\alpha^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee)}{2} + 1.$$

Similarly,

$$\langle \lambda_2 + \nu, \alpha^\vee \rangle - \frac{2(2s+j)}{\langle \alpha, \alpha \rangle} l = (\lambda_2 + \nu)(\alpha^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee)}{2} + 1.$$

Lemma 3, along with my assumption that $(\lambda + 2\nu)(\alpha^\vee) \equiv 0 \bmod_2$, implies that $(\lambda_1 - \lambda_2)(\alpha^\vee) = 0$, and hence $\lambda_1(\alpha^\vee) = \lambda_2(\alpha^\vee)$. Thus,

$$(\lambda_1 + \nu)(\alpha^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee)}{2} + 1 = 1 = (\lambda_2 + \nu)(\alpha^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee)}{2} + 1,$$

so by relation (1.4.4), $(x_{-\alpha + b_\alpha(s_\alpha - 1)\delta})^2 w_{\lambda_i + \nu} = 0$ for $i = 1, 2$. Therefore,

$$(x_{-\alpha + b_\alpha(s_\alpha - 1)\delta})^{m_\alpha + 1}(w_{\lambda_1 + \nu} \otimes w_{\lambda_2 + \nu}) = 0.$$

Finally, suppose α is long and $(\lambda + 2\nu)(\alpha^\vee) \equiv 1 \bmod_2$. For $l = 1$ and $2s + j = b_\alpha(s_\alpha - 1)$, I have

$$\langle \lambda_1 + \nu, \alpha^\vee \rangle - \frac{2(2s + j)}{\langle \alpha, \alpha \rangle} l = (\lambda_1 + \nu)(\alpha^\vee) - (s_\alpha - 1) = (\lambda_1 + \nu)(\alpha^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee) + 1}{2} + 1.$$

Similarly,

$$\langle \lambda_2 + \nu, \alpha^\vee \rangle - \frac{2(2s + j)}{\langle \alpha, \alpha \rangle} l = (\lambda_2 + \nu)(\alpha^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee) + 1}{2} + 1.$$

In this case, I have $|(\lambda_1 - \lambda_2)(\alpha^\vee)| \leq 1$ and $m_\alpha = 1$, and thus $\lambda_1(\alpha^\vee) = \lambda_2(\alpha^\vee) \pm 1$. Suppose $\lambda_1(\alpha^\vee) = \lambda_2(\alpha^\vee) - 1$. Then

$$(\lambda_1 + \nu)(\alpha^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee) + 1}{2} + 1 = (\lambda_1 + \nu)(\alpha^\vee) - (\lambda_1 + \nu)(\alpha^\vee) = 0,$$

and

$$(\lambda_2 + \nu)(\alpha^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\alpha^\vee) + 1}{2} + 1 = (\lambda_2 + \nu)(\alpha^\vee) - (\lambda_2 + \nu)(\alpha^\vee) + 1 = 1.$$

Thus, by relation (1.4.4),

$$(x_{-\alpha + b_\alpha(s_\alpha - 1)\delta}) w_{\lambda_1 + \nu} = 0 = (x_{-\alpha + b_\alpha(s_\alpha - 1)\delta})^2 w_{\lambda_2 + \nu},$$

so the relation holds. The case when $\lambda_1(\alpha^\vee) = \lambda_2 + 1$ is symmetric.

Lastly, I'll show that relation (2.2.5) holds in $D(\lambda_1+\nu, \lambda_2+\nu)$ as well; that is,

$$(x_{-\beta+s_\beta\delta})^2(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}) = 0 \text{ for } \beta \in R(\lambda_1, \lambda_2).$$

First, assume $\beta \in R(\lambda_1, \lambda_2)$. Then for $l = 1$ and $2s + j = s_\beta$, I have

$$\langle \lambda_1 + \nu, \beta^\vee \rangle - (2s + j) = (\lambda_1 + \nu)(\beta^\vee) - \frac{(\lambda_1 + \lambda_2 + 2\nu)(\beta^\vee)}{2} = \frac{(\lambda_1 - \lambda_2)(\beta^\vee)}{2}.$$

Similarly,

$$\langle \lambda_2 + \nu, \beta^\vee \rangle - (2s + j) = \frac{(\lambda_2 - \lambda_1)(\beta^\vee)}{2}.$$

Now since $\beta \in R(\lambda_1, \lambda_2)$, I have $(\lambda_1 - \lambda_2)(\beta^\vee) = \pm 2$. By my convention, $\lambda_1(\alpha_p^\vee) = \lambda(\alpha_p^\vee)$, so I can conclude that $(\lambda_1 - \lambda_2)(\beta^\vee) = 2$, in which case $(\lambda_2 - \lambda_1)(\beta^\vee) = -2$. Then by relation (1.4.4), $(x_{-\beta+s_\beta\delta})^2 w_{\lambda_1+\nu} = 0$ and $(x_{-\beta+s_\beta\delta}) w_{\lambda_2+\nu} = 0$; hence,

$$(x_{-\beta+s_\beta\delta})^2(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}) = 0 \text{ for } \beta \in R(\lambda_1, \lambda_2).$$

Now that the existence of $\phi_{\lambda_1+\nu,\lambda_2+\nu}$ has been established, the map $\psi_{\lambda_1+\nu,\lambda_2+\nu}$ is obvious.

□

2.3 Main Theorem

The following is the main result of this thesis.

Theorem 5. *Let $(\lambda_1, \lambda_2) \in P^+ \times P^+$ be an interlacing pair with $\lambda = \lambda_1 + \lambda_2$, and let $\nu \in P^+$ be*

(λ_1, λ_2)-compatible. The map

$$\phi_{\lambda_1+\nu,\lambda_2+\nu} : V(\lambda_1 + \nu, \lambda_2 + \nu) \to D(\lambda_1 + \nu, \lambda_2 + \nu)$$

is an isomorphism.

2.4 First Reduction

This theorem will be proved in several steps. The first reduction is the following proposition which provides a condition for the generalized Demazure module to be isomorphic to a Demazure module.

Proposition 6. *If* $\lambda = \omega_{i-1} + \omega_i$ *for* $0 \le i \le \ell$*, then for all* (λ_1, λ_2)*-compatible* $\nu \in P^+$*, I have*

$$V(\lambda_1 + \nu, \lambda_2 + \nu) \cong D(2, 2\nu + \lambda) \cong D(\lambda_1 + \nu, \lambda_2 + \nu).$$

2.5 β_λ and ν_0

Suppose that $\lambda \neq \omega_{i-1} + \omega_i$ for $0 \le i \le \ell$, and set

$$\beta_\lambda = \beta_{s,p}, \text{ with } s = \begin{cases} p-1 & p' \neq p-1 \\ p'' & p' = p-1. \end{cases}$$

Observe that $\lambda_1(\beta_\lambda^\vee) = 3 - \delta_{s,p-1}$ and $\lambda_2(\beta_\lambda^\vee) = 1 - \delta_{s,p-1}$.

Lemma 7. *Suppose that* $\lambda \neq \omega_{i-1} + \omega_i$ *for* $0 \le i \le \ell$*. Then* $\lambda_1 - \beta_\lambda \in P^+$ *and there exists* $\nu_0 \in P^+$ *such that* $(\lambda_1 - \beta_\lambda - \nu_0, \lambda_2 - \nu_0)$ *is an interlacing pair.*

Proof. With my assumptions, it is clear to see that

$$\lambda_1 - \beta_\lambda = \lambda_1 - \omega_p + (1 - \delta_{s,p-1})\omega_{p-1} - (1 - \delta_{s,p-1})\omega_s + \omega_{s-1} \in P^+.$$

Taking $\nu_0 = \lambda_2(\alpha_{s-1}^\vee)\omega_{s-1} + (1 - \delta_{s,p-1})\lambda_2(\alpha_{p-1}^\vee)\omega_{p-1}$, it is easy to verify that $(\lambda_1 - \beta_\lambda - \nu_0, \lambda_2 - \nu_0)$ is interlacing, and that $\nu_0 + \nu$ is $(\lambda_1 - \beta_\lambda - \nu_0, \lambda_2 - \nu_0)$-compatible. □

2.6 Second Reduction

The next reduction is the following proposition which establishes an upper bound on the dimension of $V(\lambda_1 + \nu, \lambda_2 + \nu)$.

Proposition 8. *Suppose that* $\lambda \neq \omega_{i-1} + \omega_i$ *for* $0 \leq i \leq \ell$. *Then there exists a right exact sequence of* $\mathfrak{Cg}$*-modules*

$$\tau^*_{r_{1,\beta_\lambda}-1} V(\lambda_1 + \nu - \beta_\lambda, \lambda_2 + \nu) \to V(\lambda_1 + \nu, \lambda_2 + \nu) \to D(2, \lambda + 2\nu) \to 0$$

with $w_{\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu} \to (x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta}) w_{\lambda_1+\nu,\lambda_2+\nu}$.

2.7 Inclusion of Level One Demazure Modules

Assuming Proposition 6 and Proposition 8, I complete the proof of Theorem 5 via an induction with respect to the partial order on P^+. The minimal elements with respect to this order are 0 and ω_1, and Proposition 6 shows that induction begins. It also establishes the theorem when $\lambda = \omega_{i-1} + \omega_i$ for $0 \leq i \leq \ell$. Hence, it suffices to prove the inductive step when $\lambda \neq \omega_{i-1} + \omega_i$. The following result is necessary to complete the proof of the inductive step.

Lemma 9. *There exists an inclusion of* $\mathfrak{Cg}$ *modules*

$$\tau^*_{r_{1,\beta_\lambda}-1} D(1, \lambda_1 + \nu - \beta_\lambda) \hookrightarrow D(1, \lambda_1 + \nu),$$

which sends $w_{\lambda_1+\nu-\beta_\lambda} \to (x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta)}) w_{\lambda_1+\nu}$.

Proof. Since it was proven in [5] that level one Demazure modules of special twisted current algebras are isomorphic to local Weyl modules, it suffices to show that $w := (x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta)}) w_{\lambda_1+\nu}$ satisfies the relations in (1.4.1).

Now, suppose $\alpha_i \in R^+$ is simple. Then

$$(x_{\alpha_i+k\delta})w = (x_{\alpha_i+k\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu}$$

$$= (x_{-(\beta_\lambda-\alpha_i)+(k+r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu}.$$

If $\beta_\lambda - \alpha_i \notin R^+$ or if $k > 0$, the above equations equals 0, so assume $\beta_\lambda - \alpha_i \in R^+$ and $k = 0$. Then I must have either $i = p$ or $i = s$, in which case

$$(\lambda_1 + \nu)((\beta_\lambda - \alpha_i)^\vee) \leq (\lambda_1 + \nu)(\beta_\lambda^\vee) - 1 = r_{1,\beta_\lambda} - 1.$$

Hence,

$$(x_{-(\beta_\lambda-\alpha_i)+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu} = 0.$$

Now for $\alpha \in R^+$ and $k \geq 0$, consider

$$(h_{\alpha,k\delta})w = (h_{\alpha,k\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu}.$$

If $k > 0$, then I have

$$(x_{\beta_\lambda+(r_{1,\beta_\lambda}-1+k)\delta})w_{\lambda_1+\nu} = 0$$

and if $k = 0$, the relation is trivial. Thus, the first three relations of (1.4.1) hold. Finally, the last relation holds because the modules are all finite-dimensional. □

2.8 Main Induction Argument

Lemma 4, Proposition 8, and the inductive hypothesis establish the following inequalities:

$$\dim D(\lambda_1 + \nu, \lambda_2 + \nu) \leq \dim V(\lambda_1 + \nu, \lambda_2 + \nu)$$

$$\dim V(\lambda_1 + \nu, \lambda_2 + \nu) \leq \dim D(2, 2\nu + \lambda) + \dim D(\lambda_1 + \nu - \beta_\lambda, \lambda_2 + \nu).$$

The inductive step follows if I prove that

$$\dim D(\lambda_1+\nu,\lambda_2+\nu) = \dim D(2,2\nu+\lambda) + \dim D(\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu).$$

Observe that Lemma 9 gives an inclusion

$$0 \to D(1,\lambda_1+\nu-\beta_\lambda)\otimes D(1,\lambda_2+\nu) \to D(1,\lambda_1+\nu)\otimes D(1,\lambda_2+\nu),$$

which sends

$$w_{\lambda_1+\nu-\beta_\lambda}\otimes w_{\lambda_2+\nu} \to ((x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu})\otimes w_{\lambda_2+\nu}.$$

Since $r_{1,\beta_\lambda}-1 = (\lambda_1+\nu)(\beta_\lambda^\vee)-1 \geq (\lambda_2+\nu)(\beta_\lambda^\vee)$, the relations in (1.4.2) show that

$$(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})(w_{\lambda_1+\nu}\otimes w_{\lambda_2+\nu}) = ((x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu})\otimes w_{\lambda_2+\nu}.$$

Hence, I have an inclusion

$$\iota : D(\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu) \hookrightarrow D(\lambda_1+\nu,\lambda_2+\nu)$$

and it suffices to prove that the corresponding quotient is isomorphic to $D(2,2\nu+\lambda)$. By Lemma 4 I have the following surjective maps:

$$V(\lambda_1+\nu,\lambda_2+\nu) \twoheadrightarrow D(\lambda_1+\nu,\lambda_2+\nu) \twoheadrightarrow D(2,2\nu+\lambda).$$

These maps are all unique up to scalars and Proposition 8 shows that the kernel of the composite map is generated by the element $(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$. Hence, the kernel of

$$D(\lambda_1+\nu,\lambda_2+\nu) \twoheadrightarrow D(2,2\nu+\lambda)$$

is generated by $(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})(w_{\lambda_1+\nu}\otimes w_{\lambda_2+\nu})$. But this means that the latter kernel is precisely the image of ι and hence the corresponding quotient is isomorphic to $D(2,2\nu+\lambda)$ as needed.

Chapter 3

Proof of Proposition 6

I shall assume throughout this chapter that (λ_1, λ_2) *is interlacing, and that* $\lambda = \lambda_1 + \lambda_2$. *I shall also assume that, when there exists* p *maximal such that* $\lambda(\alpha_p^\vee) = 1$, *I have* $\lambda_1(\alpha_p^\vee) = 1$.

3.1 Minimal Element of $R(\lambda_1, \lambda_2)$

Note that Lemma 3 shows that $R(\lambda_1, \lambda_2) = \emptyset$ if $\lambda = \omega_{i-1} + \omega_i$ for $0 \leq i \leq \ell$. The following result establishes the converse.

Lemma 10. *Suppose that* $\lambda \neq \omega_{i-1} + \omega_i$ *for* $0 \leq i \leq \ell$. *Then* $\beta_\lambda \in R(\lambda_1, \lambda_2)$ *and more generally,*

$$\beta_{i,j} \in R(\lambda_1, \lambda_2) \iff \beta_{i,j} = \alpha_{i,s-1} + \alpha_{j,p-1} + \beta_\lambda,$$

and

$$(\lambda_1 - \lambda_2)(\alpha_{i,s-1}^\vee) = 0 = (\lambda_1 - \lambda_2)(\alpha_{j,p-1}^\vee).$$

Proof. Recall that $\beta_\lambda = \beta_{s,p}$, with $s = \begin{cases} p-1 & p' \neq p-1 \\ p'' & p' = p-1 \end{cases}$ where $p'' < p' < p$ are maximal such that $\lambda(\alpha^\vee_{p''} + \alpha^\vee_{p'} + \alpha^\vee_p) = 3$.

By my convention, I have $\lambda_1(\alpha^\vee_p) = 1$, so by the interlacing property of (λ_1, λ_2), I have $\lambda_2(\alpha^\vee_{p'}) = 1 = \lambda_1(\alpha^\vee_{p''})$. It is easy to see that $(\lambda_1 - \lambda_2)(\alpha^\vee_{\beta_\lambda}) = 2$, and a calculation shows that

$$\beta_{i,j} \in R(\lambda_1, \lambda_2) \implies i \leq s \text{ or } s < j \leq p,$$

which shows that $\beta_{i,j} = \alpha_{i,s-1} + \alpha_{j,p-1} + \beta_\lambda$. Since $\beta_{i,p} \in R^+$ if $i < s$, I have

$$(\lambda_1 - \lambda_2)(\beta^\vee_{i,p}) = (\lambda_1 - \lambda_2)(\alpha^\vee_{i,s-1}) + 2.$$

Note that Lemma 3 forces $(\lambda_1 - \lambda_2)(\alpha^\vee_{i,s-1}) \in \{-1, 0\}$. Similarly, if $j < p$, I have $\beta_{s,j} = \beta_\lambda + \alpha_{j,p-1}$ and $(\lambda_1 - \lambda_2)(\alpha^\vee_{j,p-1}) \in \{-1, 0\}$. If $\beta_{i,j} \in R(\lambda_1, \lambda_2)$, then $(\lambda_1 - \lambda_2)(\beta^\vee_{i,j}) = \pm 2$, hence $(\lambda_1 - \lambda_2)(\alpha^\vee_{i,s-1}) = 0 = (\lambda_1 - \lambda_2)(\alpha^\vee_{j,p-1})$, as needed. □

3.2 Kernel of $\psi_{\lambda_1+\nu,\lambda_2+\nu}$

By observing the relations of $D(2, \lambda + 2\nu)$, it is easy to see that the kernel K of the map $\psi_{\lambda_1+\nu,\lambda_2+\nu}$ is generated by the elements

$$(x_{-\alpha+b_\alpha s_\alpha \delta}) w_{\lambda_1+\nu,\lambda_2+\nu}$$

where $b_\alpha s_\alpha < \max\{r_{1,\alpha}, r_{2,\alpha}\}$.

Lemma 11. *For $\alpha \in R^+$, $b_\alpha s_\alpha < \max\{r_{1,\alpha}, r_{2,\alpha}\} \iff \alpha \in R(\lambda_1, \lambda_2)$.*

Proof. Note that $(\lambda_i + \nu)(\alpha^\vee) = \frac{r_{i,\alpha}}{b_\alpha}$. Thus, I have

$$(\lambda + 2\nu)(\alpha^\vee) = 2(s_\alpha - 1) + m_\alpha = \frac{r_{1,\alpha} + r_{2,\alpha}}{b_\alpha},$$

$$s_\alpha = \frac{r_{1,\alpha} + r_{2,\alpha}}{2b_\alpha} + 1 - \frac{m_\alpha}{2},$$

and

$$(\lambda_1 - \lambda_2)(\alpha^\vee) = \frac{r_{1,\alpha} - r_{2,\alpha}}{b_\alpha}.$$

First, suppose $\alpha \in R^+$ is a long root. Then by Lemma 3, I have

$$\left| \frac{r_{1,\alpha} - r_{2,\alpha}}{b_\alpha} \right| = \left| \frac{r_{1,\alpha} - r_{2,\alpha}}{2} \right| \leq 1.$$

Since $r_{1,\alpha} - r_{2,\alpha}$ is necessarily even, $r_{1,\alpha} - r_{2,\alpha} \in \{0, \pm 2\}$. If $r_{1,\alpha} - r_{2,\alpha} = 0$, then

$$s_\alpha = \frac{r_{1,\alpha}}{2} + 1 - \frac{m_\alpha}{2}.$$

Since $s_\alpha \in \mathbb{Z}_+$ and $r_{1,\alpha}$ is even, I must have $m_\alpha = 2$, so this simplifies to

$$s_\alpha = \frac{r_{1,\alpha}}{2}.$$

Thus, $b_\alpha s_\alpha = \max\{r_{1,\alpha}, r_{2,\alpha}\}$.

Next, consider the case when $r_{1,\alpha} - r_{2,\alpha} = 2$. Then I have

$$s_\alpha = \frac{2r_{2,\alpha} + 2}{4} + 1 - \frac{m_\alpha}{2} = \frac{r_{2,\alpha} + 1}{2} + 1 - \frac{m_\alpha}{2}.$$

Since $r_{2,\alpha}$ is even, I must have $m_\alpha = 1$, and hence

$$s_\alpha = \frac{r_{2,\alpha}}{2} + 1.$$

Therefore

$$b_\alpha s_\alpha = r_{2,\alpha} + 2 = r_{1,\alpha} = \max\{r_{1,\alpha}, r_{2,\alpha}\}.$$

The argument is symmetric when $r_{1,\alpha} - r_{2,\alpha} = -2$.

Now suppose α is short. Lemma 3 shows that

$$|r_{1,\alpha} - r_{2,\alpha}| \leq 2.$$

Since $r_{i,\alpha}$ is not necessarily even when α is short, without loss of generality, I have three cases to consider.

Case 1: $r_{1,\alpha} = r_{2,\alpha}$. Then

$$s_\alpha = r_{1,\alpha} + 1 - \frac{m_\alpha}{2}.$$

Again I must have $m_\alpha = 2$, and thus $b_\alpha s_\alpha = r_{1,\alpha} = \max\{r_{1,\alpha}, r_{2,\alpha}\}$.

Case 2: $r_{1,\alpha} = r_{2,\alpha} + 1$. Then

$$s_\alpha = r_{2,\alpha} + \frac{3 - m_\alpha}{2}.$$

In this case, I must have $m_\alpha = 1$, and hence

$$b_\alpha s_\alpha = r_{2,\alpha} + 1 = r_{1,\alpha} = \max\{r_{1,\alpha}, r_{2,\alpha}\}.$$

Case 3: $r_{1,\alpha} = r_{2,\alpha} + 2$. Note, this is only possible for $\alpha = \beta_{i,j}$ for some $1 \leq i < j \leq \ell$ since by Lemma 3, $|(\lambda_1 - \lambda_2)(\alpha_{i,j}^\vee)| \leq 1$. In this case I have

$$s_\alpha = r_{2,\alpha} + 2 - \frac{m_\alpha}{2}.$$

Then $m_\alpha = 2$, and I have

$$b_\alpha s_\alpha = r_{2,\alpha} + 1 < r_{1,\alpha} = \max\{r_{1,\alpha}, r_{2,\alpha}\}.$$

This occurs precisely when $\alpha \in R(\lambda_1, \lambda_2)$. Moreover, in this case, $b_\alpha s_\alpha = r_{1,\alpha} - 1$. $\square$

Assume that $\lambda = \omega_{i-1} + \omega_i$ for $0 \leq i \leq \ell$. Then by Lemma 3, $R(\lambda_1, \lambda_2) = \emptyset$, and hence by Lemma 11, I have

$$V(\lambda_1 + \nu, \lambda_2 + \nu) \cong D(2, 2\nu + \lambda).$$

Since the maps in Lemma 4 are unique up to scalars, it follows that the map

$$V(\lambda_1 + \nu, \lambda_2 + \nu) \twoheadrightarrow D(\lambda_1 + \nu, \lambda_2 + \nu) \twoheadrightarrow D(2, 2\nu + \lambda)$$

is an isomorphism, and hence all maps are isomorphisms. Thus, Proposition 6 is proved.

Chapter 4

Proof of Proposition 8

I shall again assume throughout this chapter that (λ_1, λ_2) *is interlacing, and that* $\lambda = \lambda_1 + \lambda_2$. *I shall also assume that, when there exists* p *maximal such that* $\lambda(\alpha_p^\vee) = 1$, *I have* $\lambda_1(\alpha_p^\vee) = 1$.

4.1 β_λ and the Kernel of $\psi_{\lambda_1+\nu,\lambda_2+\nu}$

I begin by considering the map $\psi_{\lambda_1+\nu,\lambda_2+\nu} : V(\lambda_1 + \nu, \lambda_2 + \nu) \to D(2, \lambda + 2\nu)$. As Lemma 11 shows that the kernel K of $\psi_{\lambda_1+\nu,\lambda_2+\nu}$ is generated by

$$(x_{-\beta+b_\beta s_\beta \delta})w_{\lambda_1+\nu,\lambda_2+\nu} \text{ for } \beta \in R(\lambda_1, \lambda_2),$$

I can now proceed with the proof of Proposition 8 by first proving that, in fact,

$$K = \mathbb{U}(\mathfrak{Cg})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}.$$

To this end, let $\beta_{i,j} \in R(\lambda_1, \lambda_2)$, and assume that $i \le s-1$ or $j \le p-1$ (else, $\beta_{i,j} = \beta_\lambda$ and there's nothing to prove). By Lemma 10, I can write $\beta_{i,j} = \beta_\lambda + \alpha_{i,s-1} + \alpha_{j,p-1}$.

Because of the defining relations

$$(x_{-\alpha_{i,s-1}+(\lambda_1+\nu)(\alpha_{i,s-1}^\vee)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0 = (x_{-\alpha_{j,p-1}+(\lambda_1+\nu)(\alpha_{j,p-1}^\vee)\delta})w_{\lambda_1+\nu,\lambda_2+\nu},$$

I have the following equivalences:

$$(x_{-\alpha_{i,s-1}+(\lambda_1+\nu)(\alpha_{i,s-1}^\vee)\delta})(x_{-\alpha_{j,p-1}+(\lambda_1+\nu)(\alpha_{j,p-1}^\vee)\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= (x_{-(\beta_\lambda+\alpha_{i,s-1}+\alpha_{j,p-1})+((\lambda_1+\nu)(\beta_\lambda^\vee+\alpha_{i,s-1}^\vee+\alpha_{j,p-1}^\vee)-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= (x_{-\beta_{i,j}+((\lambda_1+\nu)(\beta_{i,j}^\vee)-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}.$$

4.2 Map from $V(\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu)\to K$

The next step in the proof of Proposition 8 is to establish the existence of the map

$$V(\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu)\to K\to 0$$

by showing that the element $(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$ satisfies all of the defining relations of the element $w_{\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu}\in V(\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu)$. This will be done over several different cases. I will begin by showing that $(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$ satisfies the local Weyl module relations; that is, relations (2.2.1) and (2.2.2).

4.2.1 Relations (2.2.1) and (2.2.2)

The first of the local Weyl module relations I will show is that

$$(x_{\alpha_i+r\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}=0 \text{ for } r\geq 0.$$

Since $(x_{\alpha_i+r\delta})w_{\lambda_1+\nu,\lambda_2+\nu}=0$, the relation is immediate if $\beta_\lambda-\alpha_i\notin R^+$. Thus, I'll assume that $\beta_\lambda-\alpha_i\in R^+$ and show that

$$(x_{-(\beta_\lambda-\alpha_i)+(r_{1,\beta_\lambda}-1+r)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}=0.$$

Since $\beta_\lambda = \beta_{s,p}$ and $\beta_\lambda - \alpha_i \in R^+$, I must have either $i = p$ or $i = s$. If $i = s = p''$ or $i = p$, I have that $(\lambda_1 - \lambda_2)(\beta_\lambda^\vee) = 2$, and $\lambda_1(\alpha_i^\vee) = 1$. Thus, $\max\{r_{1,\beta_\lambda - \alpha_i}, r_{2,\beta_\lambda-\alpha_i}\} = r_{1,\beta_\lambda-\alpha_i} \leq r_{1,\beta_\lambda} - 1$.

Now, when $i = s = p-1$, since ν is (λ_1, λ_2) compatible, I must have $\nu(\alpha_{p-1}^\vee) \geq 1$. In this case, I have $\beta_\lambda - \alpha_{p-1} = \beta_p$, and hence,

$$r_{1,\beta_p} = 2(\lambda_1 + \nu)(\beta_p) = 2 + 2\nu(\beta_p^\vee) \leq 2 + 2\nu(\beta_p^\vee) + \nu(\alpha_{p-1}^\vee) - 1$$

$$= r_{1,\beta_\lambda} - 1 = \max\{r_{1,\beta_\lambda-\alpha_i}, r_{2,\beta_\lambda-\alpha_i}\}.$$

Therefore,

$$(x_{-(\beta_\lambda-\alpha_i)+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0,$$

and hence,

$$(x_{-(\beta_\lambda-\alpha_i)+(r_{1,\beta_\lambda}-1+r)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

for $r \geq 0$. Finally, it is clear to see that

$$(h_{i,r\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = \delta_{r,0}(\lambda + 2\nu - \beta_\lambda)(\alpha_i^\vee)(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu},$$

and relation (2.2.2) holds because $V(\lambda_1+\nu, \lambda_2+\nu)$ is finite dimensional.

4.2.2 Relation (2.2.3)

Next, let $\alpha \in R^+$ and set $r'_{1,\alpha} = b_\alpha(\lambda_1 + \nu - \beta_\lambda)(\alpha^\vee)$. I'll show that relation (2.2.3) holds; that is,

$$(x_{-\alpha+(\max\{r'_{1,\alpha}, r_{2,\alpha}\})\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

This will be done in several cases. First, suppose that $\beta_\lambda(\alpha^\vee) = 0$. Then $r_{1,\alpha} = r'_{1,\alpha}$, so the relation is immediate if $\beta_\lambda + \alpha \notin R^+$. Thus, I assume that $\beta_\lambda + \alpha \in R^+$. This is only possible for $\beta_\lambda = \beta_{p'',p}$, and in this case, I must have $\alpha = \alpha_{p'',p-1}$. However, note that $\lambda_k(\alpha^\vee_{p'',p-1}) = 1$ for $k \in \{1, 2\}$, and thus

$$\max\{r'_{1,\alpha_{p'',p-1}}, r_{2,\alpha_{p'',p-1}}\} + r_{1,\beta_\lambda} - 1 = 3 + 2\nu(\beta^\vee_{p''}) \equiv 1 \mod 2.$$

Because $\beta_\lambda + \alpha_{p'',p-1} = \beta_p$ and β_p is a long root, I conclude that

$$[(x_{-\alpha_{p'',p-1}+(\max\{r'_{1,\alpha_{p'',p-1}}, r_{2,\alpha_{p'',p-1}}\})\delta}), (x_{-\beta_\lambda+((\lambda_1+\nu)(\beta^\vee_\lambda)-1)\delta})] = 0,$$

and thus, the relation holds.

Now suppose $\beta_\lambda(\alpha^\vee) = -1$. By lemma 3, $|(\lambda_1 - \lambda_2)(\alpha^\vee)| \leq 1$. Hence,

$$r'_{1,\alpha} = \max\{r'_{1,\alpha}, r_{2,\alpha}\} \geq \max\{r_{1,\alpha}, r_{2,\alpha}\}, \text{ and } r_{1,\alpha+\beta_\lambda} = \max\{r_{1,\alpha+\beta_\lambda}, r_{2,\alpha+\beta_\lambda}\}.$$

Thus, the relation is again immediate unless $\beta_\lambda + \alpha \in R^+$, so I assume $\beta_\lambda + \alpha \in R^+$. Note that when $\beta_\lambda = \beta_{p'',p}$, I must have either $\alpha = \alpha_{i,p''-1}$ for $1 \leq i \leq p'' - 1$ or $\alpha = \alpha_{i,p-1}$ for $1 \leq i \neq p'' \leq p - 1$, and when $\beta_\lambda = \beta_{p-1,p}$, I must have $\alpha = \alpha_{i,p-2}$ for $1 \leq i \leq p - 2$. In all of these cases,

$$r'_{1,\alpha} = (\lambda_1 + \nu - \beta_\lambda)(\alpha^\vee) = (\lambda_1 + \nu)(\alpha^\vee) + 1$$

and hence,

$$r'_{1,\alpha} + r_{1,\beta_\lambda} - 1 = (\lambda_1 + \nu)(\alpha^\vee) + (\lambda_1 + \nu)(\beta^\vee_\lambda)$$

$$= (\lambda_1 + \nu)(\alpha^\vee + \beta^\vee_\lambda) = b_\alpha(\lambda_1 + \nu)((\alpha + \beta_\lambda)^\vee) = r_{1,\alpha+\beta_\lambda}.$$

Thus, the relation holds.

Now consider the case when $\beta_\lambda(\alpha^\vee) = 1$. Then $\beta_\lambda + \alpha \notin R^+$, and either $\beta_\lambda - \alpha \in R^+$ or $\alpha - \beta_\lambda \in R^+$. Note, if $r_{2,\alpha} \geq r_{1,\alpha}$, then $r_{2,\alpha} = \max\{r'_{1,\alpha}, r_{2,\alpha}\} = \max\{r_{1,\alpha}, r_{2,\alpha}\}$ and the relation is immediate. Suppose first that $\beta_\lambda - \alpha \in R^+$; I will begin with the case that

$r_{1,\alpha} = r_{2,\alpha} + 2$. Then I must have $\alpha = \beta_p$. Thus, $b_\alpha = 2$ and since $\lambda(\alpha^\vee) = 1$, I have that $m_\alpha = 1$; hence, $(\lambda + 2\nu)(\alpha^\vee) = 2(s_\alpha - 1) + m_\alpha$ implies that $2\nu(\alpha^\vee) = b_\alpha(s_\alpha - 1)$. Therefore, $r'_{1,\alpha} = b_\alpha(\lambda_1 + \nu - \beta_\lambda)(\alpha^\vee) = 2(\nu(\alpha^\vee)) = b_\alpha(s_\alpha - 1)$. Now, by relation (2.2.4), it is clear that

$$(x_{-(\beta_\lambda - \alpha) + (r_{1,\beta_\lambda - \alpha} + 1)\delta})(x_{-\alpha + b_\alpha(s_\alpha - 1)\delta})^2 w_{\lambda_1 + \nu, \lambda_2 + \nu} = 0.$$

Thus, I conclude that

$$0 = (x_{-(\beta_\lambda - \alpha) + (r_{1,\beta_\lambda - \alpha} + 1)\delta})(x_{-\alpha + r'_{1,\alpha}\delta})^2 w_{\lambda_1 + \nu, \lambda_2 + \nu}$$

$$= 2(x_{-\alpha + r'_{1,\alpha}\delta})(x_{-\beta_\lambda + (r_{1,\beta_\lambda} - 1)\delta}) w_{\lambda_1 + \nu, \lambda_2 + \nu},$$

showing that the relation holds. Similarly, if $r_{1,\alpha} = r_{2,\alpha} + 1$, then I either have $\alpha = \beta_{i,p}$ for $s < i \leq p'$, $\alpha = \alpha_{p,j}$ for $p \leq j \leq \ell$, or $\alpha = \alpha_{s,k}$ for $s \leq k < p'$. In all three cases, I have that $b_\alpha = 1$ and $m_\alpha = 1$. When $\alpha = \alpha_{p,j}$ or $\alpha = \alpha_{s,k}$, I have that $\lambda(\alpha^\vee) = 1$; hence, $(\lambda + 2\nu)(\alpha^\vee) = 2(s_\alpha - 1) + m_\alpha$ implies that $2\nu(\alpha^\vee) = 2(s_\alpha - 1)$, so $\nu(\alpha^\vee) = s_\alpha - 1$. Additionally, in this case, $r'_{1,\alpha} = (\lambda_1 + \nu - \beta_\lambda)(\alpha^\vee) = \nu(\alpha^\vee)$. Thus,
$r'_{1,\alpha} = s_\alpha - 1 = b_\alpha(s_\alpha - 1)$, so by the same argument, the relation holds in these two cases.

Lastly, when $\alpha = \beta_{i,p}$, I have $\lambda_2(\beta_{i,p}^\vee) = 1$ and $\lambda_1(\beta_{i,p}^\vee) = 2$; hence,
$1 + \nu(\beta_{i,p}^\vee) = s_{\beta_{i,p}} - 1$. In this case, $r'_{1,\alpha} = 1 + \nu(\beta_{i,p}^\vee) = b_\alpha(s_{\beta_{i,p}} - 1)$ as well, so the relation holds in all three cases.

Now suppose $\alpha - \beta_\lambda \in R^+$. If $r_{1,\alpha} = r_{2,\alpha} + 2$, then either $\alpha = \beta_{i,j} > \beta_\lambda$ or $\alpha = \beta_s$, and if $r_{1,\alpha} = r_{2,\alpha} + 1$, then either $\alpha = \beta_{i,p}$ for $1 \leq i \leq s - 1$ or $\alpha = \beta_{i,s}$ for $1 \leq i \leq s - 1$. In any case, since $\lambda_1(\beta_\lambda^\vee) = \lambda_2(\beta_\lambda^\vee) + 2$, I have that $(\lambda_1 + \nu)(\beta_\lambda^\vee) - 1 = s_{\beta_\lambda}$, and hence, $r_{1,\alpha - \beta_\lambda} + s_{\beta_\lambda} = r_{1,\alpha} - 1$.

Thus, by relation (2.2.5),

$$0 = (x_{-(\alpha-\beta_\lambda)+(r_{1,\alpha-\beta_\lambda})\delta})(x_{-\beta_\lambda+s_{\beta_\lambda}\delta})^2 w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= 2(x_{-\alpha+r_{1,\alpha}-1})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= 2(x_{-\alpha+r'_{1,\alpha}})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu},$$

and therefore, the relation holds.

Now when $\beta_\lambda(\alpha^\vee) = 2$, α must be β_λ, and I have

$$(x_{-\beta_\lambda+r'_{1,\beta_\lambda}\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= (x_{-\beta_\lambda+(r_{1,\beta_\lambda}-2)\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}.$$

Note that $(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-2)\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$ in $W_{\text{loc}}(\lambda+2\nu)$, and hence, the relation also holds in $V(\lambda_1+\nu, \lambda_2+\nu)$.

4.2.3 Relation (2.2.4)

Next, I must show that relation in (2.2.4) holds, i.e., for $\alpha \in R^+$

$$(x_{-\alpha+(b_\alpha(s'_\alpha-1))\delta})^{m'_\alpha+1}(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

where s'_α and m'_α are the unique nonnegative integers satisfying

$$(\lambda + 2\nu - \beta_\lambda)(\alpha^\vee) = 2(s'_\alpha - 1) + m'_\alpha.$$

This will also be done in several cases. First, if $\beta_\lambda(\alpha^\vee) = 0$, then $\alpha + \beta_\lambda \notin R^+$, $m'_\alpha = m_\alpha$, and $s'_\alpha = s_\alpha$, so the relation holds.

Now suppose $\beta_\lambda(\alpha^\vee) = -1$. Again, if $\beta_\lambda + \alpha \notin R^+$ the relation is immediate, so assume that $\beta_\lambda + \alpha \in R^+$. As before, if $\beta_\lambda = \beta_{p'',p}$, I must have either $\alpha = \alpha_{i,p''-1}$ for

$1 \leq i \leq p'' - 1$, or $\alpha = \alpha_{i,p-1}$ for $1 \leq i \neq p'' \leq p-1$, and if $\beta_\lambda = \beta_{p-1,p}$, then $\alpha = \alpha_{i,p-2}$ for $1 \leq i \leq p-2$. In any case, I have two subcases to consider. First, if $m'_\alpha = 1$, then $m_\alpha = 2$ and $s_\alpha = s'_\alpha - 1$. Since $m_\alpha = 2$, $(\lambda + 2\nu)(\alpha^\vee)$ is even, so by Lemma 3, I have $\lambda_1(\alpha^\vee) = \lambda_2(\alpha^\vee)$, and so $r_{1,\alpha} = r_{2,\alpha} = s_\alpha$. Thus,

$$(x_{-\alpha+(b_\alpha(s'_\alpha-1))\delta})^{m'_\alpha+1}(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= (x_{-\alpha+s_\alpha\delta})^2(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= (x_{-\alpha+s_\alpha\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda,\nu}-1)\delta})(x_{-\alpha+s_\alpha\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$+(x_{-(\beta_\lambda+\alpha)+(r_{1,\beta_\lambda,\nu}+s_\alpha-1)\delta})(x_{-\alpha+s_\alpha\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

by relation (2.2.3).

Now, if $m'_\alpha = 2$, then $m_\alpha = 1$ and $s_\alpha = s'_\alpha$, and thus,

$$(x_{-\alpha+(s'_\alpha-1)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= (x_{-(\beta_\lambda+\alpha)+(r_{1,\beta_\lambda}+s_\alpha-2)\delta})(x_{-\alpha+(s_\alpha-1)\delta})^2 w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$+(x_{-\alpha+(s_\alpha-1)\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})(x_{-\alpha+(s_\alpha-1)\delta})^2 w_{\lambda_1+\nu,\lambda_2+\nu}$$

with both terms equal to 0 by relation (2.2.4).

Next, suppose $\beta_\lambda(\alpha^\vee) = 1$. Then either $\beta_\lambda - \alpha \in R^+$ or $\alpha - \beta_\lambda \in R^+$. First, consider the case when $\beta_\lambda - \alpha \in R^+$. In this case, either $\alpha = \beta_{i,p}$ for $s < i \leq p'$, $\alpha = \alpha_{p,j}$ for $p \leq j \leq \ell$, $\alpha = \alpha_{s,k}$ for $s \leq k < p'$, or $\alpha = \beta_p$. If $\alpha = \beta_{i,p}$, then $(\lambda + 2\nu)(\beta_{i,p}^\vee) = 3 + 2\nu(\beta_{i,p}^\vee)$, so I have $m_{\beta_{i,p}} = 1$; hence, $m'_{\beta_{i,p}} = 2$ and $s_{\beta_{i,p}} = s'_{\beta_{i,p}} + 1$. Similarly, if $\alpha = \alpha_{s,k}$ or if $\alpha = \alpha_{p,j}$, then $(\lambda + 2\nu)(\alpha^\vee) = 1 + 2\nu(\alpha^\vee)$, in which case $m_\alpha = 1$, and hence $m'_\alpha = 2$ and $s_\alpha = s'_\alpha + 1$.

For the subalgebras $\mathfrak{sl}_2[t] \subset \mathfrak{Cg}$ corresponding to the short roots $\alpha = \beta_{i,p}$, $\alpha = \alpha_{s,k}$, and $\alpha = \alpha_{p,j}$, I have the following.

Lemma 12. *There exists a homomorphism of $\mathfrak{sl}_2[t]$ modules $D(2, \lambda+2\nu) \to V(\lambda_1+\nu, \lambda_2+\nu)$ mapping generator to generator.*

Proof. By Theorem 2 in [3], I need to show

$$(x_{-\alpha+s_\alpha\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0 \tag{4.2.1}$$

$$(x_{\alpha+(s_\alpha-1)\delta})^{m_\alpha+1}w_{\lambda_1+\nu,\lambda_1+\nu} = 0 \tag{4.2.2}$$

for $\alpha = \beta_{i,p}$, $\alpha = \alpha_{p,j}$, and $\alpha = \alpha_{s,k}$.

Note, (4.2.2) is immediate by relation (2.2.4). To show (4.2.1) holds, observe:

$$s_{\beta_{i,p}} = \frac{1}{2}((\lambda + 2\nu)(\beta_{i,p}^\vee) + 1)$$

$$= \frac{1}{2}(r_{1,\beta_{i,p}} + r_{2,\beta_{i,p}} + 1)$$

$$= r_{1,\beta_{i,p}} = \max\{r_{1,\beta_{i,p}}, r_{2,\beta_{i,p}}\},$$

and similarly, when $\alpha = \alpha_{s,k}$ or $\alpha = \alpha_{p,j}$,

$$s_\alpha = 1 + \nu(\alpha^\vee) = r_{1,\alpha} = \max\{r_{1,\alpha}, r_{2,\alpha}\}.$$

Hence, (4.2.1) holds by relation (2.2.3). □

Using this Lemma, along with the presentation of Demazure modules given in 3.5 of [3] (originally defined in [6] and [11]), I have that

$$(x_{-\alpha+(s_\alpha-2)\delta})^4 w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

for $\alpha = \beta_{i,p}$, $\alpha = \alpha_{p,j}$, and $\alpha = \alpha_{s,k}$. Also note that, in all three cases,

$$r_{1,\beta_\lambda} - s_\alpha + 1 \geq r_{1,(\beta_\lambda-\alpha)} = \max\{r_{1,(\beta_\lambda-\alpha)}, r_{2,(\beta_\lambda-\alpha)}\}.$$

Hence, I have

$$0 = (x_{-(\beta_\lambda-\alpha)+(r_{1,\beta_\lambda}-s_\alpha+1)\delta})(x_{-\alpha+(s_\alpha-2)\delta})^4 w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= (x_{-\alpha+(s_\alpha-2)\delta})^3 (x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta}) w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$+(x_{-\alpha+(s_\alpha-2)\delta})^4 (x_{-(\beta_\lambda-\alpha)+(r_{1,\beta_\lambda}-s_\alpha+1)\delta}) w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= (x_{-\alpha+(s_\alpha-2)\delta})^3 (x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta}) w_{\lambda_1+\nu,\lambda_2+\nu}.$$

Now suppose $\alpha = \beta_p$. Then $\lambda + 2\nu(\beta_p^\vee) = 1 + 2\nu(\beta_p^\vee)$, so I have $m_{\beta_p} = 1$, $s_{\beta_p} = \nu(\beta_p^\vee) + 1$, and $b_{\beta_p} = 2$. Hence, in this case I must show that

$$(x_{-\beta_p+2(s_{\beta_p}-2)\delta})^3 (x_{-\beta_\lambda+r_{1,\beta_\lambda}-1)\delta}) w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

To accomplish this, I'll prove another, similar lemma; that is, for the subalgebra $\mathfrak{sl}_2[t^2] \subset \mathfrak{C}\mathfrak{g}$ corresponding to the long root $\alpha = \beta_p$,

Lemma 13. *There exists a homomorphism of $\mathfrak{sl}_2[t^2]$ modules $D(2, \lambda + 2\nu) \to V(\lambda_1 + \nu, \lambda_2 + \nu)$ mapping generator to generator.*

Proof. To prove this lemma, I must show that the following equations hold in $V(\lambda_1+\nu,\lambda_2+\nu)$:

$$(x_{-\beta_p+2s_{\beta_p}\delta})w_{\lambda_1+\nu,\lambda_2+\nu}=0 \tag{4.2.3}$$

$$(x_{\beta_p+2(s_{\beta_p}-1)\delta})^{m_\alpha+1}w_{\lambda_1+\nu,\lambda_1+\nu}=0. \tag{4.2.4}$$

Note, I have $\lambda_1(\beta_p^\vee)=1$ and $\lambda_2(\beta_p^\vee)=0$, so $2s_{\beta_p}=2+2\nu(\beta_p^\vee)$. Hence,

$$\max\{r_{1,\beta_p},r_{2,\beta_p}\}=r_{1,\beta_p}=2((\lambda_1+\nu)(\beta_p^\vee))=2+2\nu(\beta_p^\vee)=2s_{\beta_p}.$$

Thus, (4.2.3) holds by relation (2.2.3), and (4.2.4) holds by relation (2.2.4). □

Using this Lemma, I can conclude that

$$(x_{-\beta_p+2(s_{\beta_p}-2)\delta})^4 w_{\lambda_1+\nu,\lambda_2+\nu}=0.$$

Since $4+\nu(\alpha_{s,p-1}^\vee)>\max\{r_{1,\alpha_{s,p-1}},r_{2,\alpha_{s,p-1}}\}$, I can therefore conclude the following:

$$0=(x_{-\alpha_{s,p-1}+(4+\nu(\alpha_{s,p-1}^\vee))\delta})(x_{-\beta_p+2(s_{\beta_p}-2)\delta})^4 w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$=(x_{-\beta_p+2(s_{\beta_p}-2)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$+(x_{-\beta_p+2(s_{\beta_p}-2)\delta})^4(x_{-\alpha_{s,p-1}+(4+\nu(\alpha_{s,p-1}^\vee))\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$=(x_{-\beta_p+2(s_{\beta_p}-2)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

and hence, $(x_{-\alpha+(b_\alpha(s'_\alpha-1))\delta})^{m'_\alpha+1}(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}=0$ whenever $\beta_\lambda-\alpha\in R^+$.

Now suppose $\alpha-\beta_\lambda\in R^+$. Then either $\alpha=\beta_{i,p}$ for $1\le i\le s-1$, $\alpha=\beta_{i,s}$ for $1\le i\le s-1$, or $\alpha=\beta_s$. Suppose first that either $\alpha=\beta_{i,p}$ or $\alpha=\beta_{i,s}$ for $1\le i\le s-1$, and that $m_\alpha=2$. Then $b_\alpha=1$, $m'_\alpha=1$, and $s_\alpha=s'_\alpha$, so I must show that:

$$(x_{-\alpha+(s'_\alpha-1)\delta})^2(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}=0.$$

Let $\alpha_1' = \beta_\lambda$. If $\alpha = \beta_{i,p}$, let $\alpha_2' = \alpha_{i,s-1},\ \alpha_1' + \alpha_2' = \beta_{i,p}$, and $s_\alpha = s_{\beta_{i,p}}$, and if $\alpha = \beta_{i,s}$, let $\alpha_2' = \alpha_{i,p-1},\ \alpha_1' + \alpha_2' = \beta_{i,s},$ and $s_\alpha = s_{\beta_{i,s}}$. Now, consider the lie algebra $\mathfrak{sl}_3[t]$ with roots $\alpha_1',\ \alpha_2',$ and $\alpha_1' + \alpha_2'$, and define M to be the $\mathfrak{sl}_3[t]$ module generated by a vector m with the following relations:

$$(x_{\alpha_i' + \mathbb{C}[t]\delta})m = 0$$

$$(h_{\alpha_i', k\delta})m = \delta_{k,0}((2r-2)\omega_1 + (2s-2r+2)\omega_2)(\alpha_i'^{\vee})m$$

$$(x_{-\alpha_1' + r_{1,\beta_\lambda}\delta})m = 0$$

$$(x_{-\alpha_2' + (s_\alpha - r_{1,\beta_\lambda}+1)\delta})m = 0$$

$$(x_{-(\alpha_1'+\alpha_2') + (s_\alpha+1)\delta})m = 0.$$

It is known that this module is isomorphic to the generalized demazure module; namely it can be realized as the submodule

$$\mathbb{U}(\mathfrak{sl}_3[t])(w_1 \otimes w_2) \subset (W_{\text{loc}}^{\mathfrak{sl}_3}(r_{1,\beta_\lambda}\omega_1 + (s_\alpha - r_{1,\beta_\lambda}+1)\omega_2) \otimes W_{\text{loc}}^{\mathfrak{sl}_3}((r_{1,\beta_\lambda}-2)\omega_1 + (s_\alpha - r_{1,\beta_\lambda}+1)\omega_2))$$

where w_1 is the highest weight vector of $W_{\text{loc}}^{\mathfrak{sl}_3}(r_{1,\beta_\lambda}\omega_1 + (s_\alpha - r_{1,\beta_\lambda}+1)\omega_2)$ and w_2 the highest weight vector of $W_{\text{loc}}^{\mathfrak{sl}_3}((r_{1,\beta_\lambda}-2)\omega_1 + (s_\alpha - r_{1,\beta_\lambda}+1)\omega_2)$.

Define a map $M \to V(\lambda_1 + \nu, \lambda_2 + \nu)$ by sending generator to generator. Then the $\mathfrak{sl}_3[t]$ map is well defined; First, it is clear that

$$(x_{\alpha_i' + \mathbb{C}[t]\delta})(w_{\lambda_1+\nu,\lambda_2+\nu}) = 0 \quad \text{and}$$

$$(h_{\alpha_i', k\delta})(w_{\lambda_1+\nu,\lambda_2+\nu}) = ((2r_{1,\beta_\lambda}-2)\omega_1 + (2s_\alpha - 2r_{1,\beta_\lambda}+2)\omega_2)(\alpha_i'^{\vee})\delta_{k,0}(w_{\lambda_1+\nu,\lambda_2+\nu}).$$

Then, by relation (2.2.3), I also have the following:

$$(x_{-\alpha_1'+r_{1,\beta_\lambda}\delta})(w_{\lambda_1+\nu,\lambda_2+\nu}) = 0,$$

$$(x_{-\alpha_2'+(s_\alpha-r_{1,\beta_\lambda}+1)\delta})(w_{\lambda_1+\nu,\lambda_2+\nu}) = 0,$$

$$(x_{-(\alpha_1'+\alpha_2')+(s_\alpha+1)\delta})(w_{\lambda_1+\nu,\lambda_2+\nu}) = 0.$$

Now observe:

$$(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})(w_1\otimes w_2)$$

$$= (x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1\otimes w_2$$

$$+(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})^2 w_1\otimes(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_2$$

$$+2(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1\otimes(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})w_2$$

$$+2(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})w_1\otimes(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_2$$

$$+(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1\otimes(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})^2 w_2$$

$$+w_1\otimes(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_2.$$

Note, since $(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})w_2 = 0 = (x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_2$, this can be immediately reduced to:

$$(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})(w_1\otimes w_2) = (x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1\otimes w_2.$$

Now consider the $\mathfrak{sl}_2$ for the simple root $\alpha_1'+\alpha_2'$ and the $\mathfrak{sl}_2[t]$ module $W_{\text{loc}}(s_\alpha)$ with highest weight vector w. I claim there exists a map $W_{\text{loc}}(s_\alpha)\to\mathbb{U}(\mathfrak{sl}_2[t])(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1$ defined by $w\to(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1$.

Observe:

$$(x_{\alpha_1'+\alpha_2'+k\delta})(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1$$

$$= (x_{\alpha_2'+(k+r_{1,\beta_\lambda}-1)\delta})w_1 + (x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})(x_{\alpha_1'+\alpha_2'+k\delta})w_1 = 0,$$

$$(h_{(\alpha_1'+\alpha_2'),k\delta})(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1$$

$$= \delta_{k,0}s_\alpha(x_{-\alpha_1'+(r_{1,\beta_\lambda}+k-1)\delta})w_1,$$

and

$$(x_{-(\alpha_1'+\alpha_2')+s_\alpha\delta})(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1 = 0.$$

Thus, the map is well defined. Finally, since $(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})^2w = 0$, I can conclude that

$$(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1 = 0,$$

hence

$$(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})m = 0,$$

and therefore,

$$(x_{-\alpha+(s_\alpha'-1)\delta})^2(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

for both $\alpha = \beta_{i,p}$ and $\alpha = \beta_{i,s}$ with $m_\alpha = 2$.

Now suppose that either $\alpha = \beta_{i,p}$ or $\alpha = \beta_{i,s}$ for $1 \leq i \leq s$ and that $m_\alpha = 1$. Then $b_\alpha = 1$, $m_\alpha' = 2$, and $s_\alpha - 1 = s_\alpha'$, so I must show that:

$$(x_{-\alpha+(s_\alpha-2)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

Let $\alpha_1' = \beta_\lambda$. If $\alpha = \beta_{i,p}$, let $\alpha_2' = \alpha_{i,s-1}$, $\alpha_1' + \alpha_2' = \beta_{i,p}$, and $s_\alpha = s_{\beta_{i,p}}$, and if $\alpha = \beta_{i,s}$, let $\alpha_2' = \alpha_{i,p-1}$, $\alpha_1' + \alpha_2' = \beta_{i,s}$, and $s_\alpha = s_{\beta_{i,s}}$.

Now, consider the Lie algebra $\mathfrak{sl}_3[t]$ with roots $\alpha_1',\ \alpha_2',$ and $\alpha_1'+\alpha_2'$, and define M to be the $\mathfrak{sl}_3[t]$ module generated by a vector m with the following relations:

$$(x_{\alpha_i'+\mathbb{C}[t]\delta})m = 0,$$

$$(h_{\alpha_i',k\delta})m = \delta_{k,0}((2r_{1,\beta_\lambda}-2)\omega_1 + (2s_\alpha - 2r_{1,\beta_\lambda}+1)\omega_2)(\alpha_i'^{\vee})m,$$

$$(x_{-\alpha_1'+r_{1,\beta_\lambda}\delta})m = 0,$$

$$(x_{-\alpha_2'+(s_\alpha - r_{1,\beta_\lambda}+1)\delta})m = 0,$$

$$(x_{-(\alpha_1'+\alpha_2')+s_\alpha\delta})m = 0.$$

It is known that this module is isomorphic to the generalized demazure module; namely it can be realized as the submodule

$$\mathbb{U}(\mathfrak{sl}_3[t])(w_1\otimes w_2) \subset (W_{\mathrm{loc}}^{\mathfrak{sl}_3}(r_{1,\beta_\lambda}\omega_1+(s_\alpha-r_{1,\beta_\lambda})\omega_2)\otimes W_{\mathrm{loc}}^{\mathfrak{sl}_3}((r_{1,\beta_\lambda}-2)\omega_1+(s_\alpha-r_{1,\beta_\lambda}+1)\omega_2))$$

where w_1 is the highest weight vector of $W_{\mathrm{loc}}^{\mathfrak{sl}_3}(r_{1,\beta_\lambda}\omega_1+(s_\alpha-r_{1,\beta_\lambda})\omega_2)$ and w_2 the highest weight vector of $W_{\mathrm{loc}}^{\mathfrak{sl}_3}((r_{1,\beta_\lambda}-2)\omega_1+(s_\alpha-r_{1,\beta_\lambda}+1)\omega_2)$.

Define a map $M \to V(\lambda_1+\nu, \lambda_2+\nu)$ by sending generator to generator. Then the $\mathfrak{sl}_3[t]$ map is well defined; First, it is clear that

$$(x_{\alpha_i'+\mathbb{C}[t]\delta})(w_{\lambda_1+\nu,\lambda_2+\nu}) = 0$$

and

$$(h_{\alpha_i',k\delta})(w_{\lambda_1+\nu,\lambda_2+\nu}) = \delta_{k,0}((2r_{1,\beta_\lambda}-2)\omega_1 + (2s_\alpha - 2r_{1,\beta_\lambda}+2)\omega_2)(\alpha_i'^{\vee})(w_{\lambda_1+\nu,\lambda_2+\nu}).$$

Then by relation (2.2.3), I also have the following:

$$(x_{-\alpha_1'+r_{1,\beta_\lambda}\delta})(w_{\lambda_1+\nu,\lambda_2+\nu}) = 0,$$

$$(x_{-\alpha_2'+(s_\alpha-r_{1,\beta_\lambda}+1)\delta})(w_{\lambda_1+\nu,\lambda_2+\nu}) = 0,$$

$$(x_{-(\alpha_1'+\alpha_2')+s_\alpha\delta})(w_{\lambda_1+\nu,\lambda_2+\nu}) = 0.$$

Now observe:

$$(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})^3(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})(w_1\otimes w_2)$$

$$= (x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})^3(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1\otimes w_2$$

$$+3(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1\otimes (x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})w_2$$

$$+3(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1\otimes (x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})^2w_2$$

$$+(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1\otimes (x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})^3w_2$$

$$+(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})^3w_1\otimes (x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_2$$

$$+3(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})^2w_1\otimes (x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_2$$

$$+3(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})w_1\otimes (x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_2$$

$$+w_1\otimes (x_{-(\alpha_1'+\alpha_2')+(s-2)\delta})^3(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_2.$$

Note, since $(x_{-(\alpha_1'+\alpha_2')+(s-2)\delta})^3w_1 = (x_{-(\alpha_1'+\alpha_2')+(s-2)\delta})^2w_2 = (x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_2 = 0$, this can immediately be reduced to:

$$(x_{-(\alpha_1'+\alpha_2')+(s-2)\delta})^3(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})(w_1\otimes w_2)$$

$$= 3(x_{-(\alpha_1'+\alpha_2')+(s-2)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1\otimes (x_{-(\alpha_1'+\alpha_2')+(s-2)\delta})w_2.$$

Now consider the $\mathfrak{sl}_2$ with simple root $\alpha_1'+\alpha_2'$ and the $\mathfrak{sl}_2[t]$ module $W_{\text{loc}}(s_\alpha-1)$ with highest weight vector w. I claim there exists a map $W_{\text{loc}}(s_\alpha-1) \to \mathbb{U}(\mathfrak{sl}_2[t])(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1$ defined by $w \to (x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1$.

Observe:

$$(x_{\alpha_1'+\alpha_2'+k\delta})(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1$$

$$= (x_{\alpha_2'+(k+r_{1,\beta_\lambda}-1)\delta})w_1 + (x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})(x_{\alpha_1'+\alpha_2'+k\delta})w_1 = 0,$$

$$(h_{(\alpha_1'+\alpha_2'),k\delta})(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1$$

$$= \delta_{k,0}(s_\alpha-1)(x_{-\alpha_1'+(r_{1,\beta_\lambda}+k-1)\delta})w_1,$$

and

$$(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-1)\delta})(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1 = 0.$$

Thus, the map is well defined. Finally, since $(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})^2 w = 0$, I can conclude that

$$(x_{-(\alpha_1'+\alpha_2')+(s_\alpha-2)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})w_1 = 0,$$

hence

$$(x_{-(\alpha_1'+\alpha_2')+(s-2)\delta})^2(x_{-\alpha_1'+(r_{1,\beta_\lambda}-1)\delta})m = 0,$$

and therefore,

$$(x_{-\alpha+(s_\alpha-2)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

whenever $\alpha = \beta_{i,p}$ and $\alpha = \beta_{i,s}$.

Finally, suppose $\alpha = \beta_s$. Then $m_{\beta_s} = 1$, so $m'_{\beta_s} = 2$ and $s_{\beta_s} = s'_{\beta_s} + 1$. Thus, I want to show that

$$(x_{-\beta_s+2(s_{\beta_s}-2)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

I have that:

$$(h_{\beta_s,k\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 2(s_{\beta_s}-1)\delta_{k,0}(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

and

$$(x_{\beta_s+k\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0 \text{ for } k \geq 0$$

Hence, there exists a map

$$W_{\text{loc}}^{\mathfrak{sl}_2}(2(s_{\beta_s}-1)) \to \mathbb{U}(\mathfrak{sl}_2[t^2])(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$$

sending the generator $w_{2(s_{\beta_s}-1)}$ of $W_{\text{loc}}^{\mathfrak{sl}_2}(2(s_{\beta_s}-1))$ to $(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}$. Then, since $(x_{-\beta_s+2(s_{\beta_s}-2)\delta})^3 w_{2(s_{\beta_s}-1)} = 0$, I conclude that

$$(x_{-\beta_s+2(s_{\beta_s}-2)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

Therefore, I have

$$(x_{-\alpha+b_\alpha(s'_\alpha-1)\delta})^{m'_\alpha+1}(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

whenever $\beta_\lambda(\alpha^\vee) = 1$.

The last case I must consider is when $\beta_\lambda(\alpha^\vee) = 2$. This occurs only when $\alpha = \beta_\lambda$. Since $m_{\beta_\lambda} = 2$, I have $m'_{\beta_\lambda} = m_{\beta_\lambda}$ and $s_{\beta_\lambda} = s'_{\beta_\lambda} + 1 = r_{1,\beta_\lambda} - 1$. Thus, I must show that

$$(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-3)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

Consider the lie algebra $\mathfrak{sl}_2[t]$ with simple root β_λ and define M to be the $\mathfrak{sl}_2[t]$ module generated by m subject to the following relations:

$$(x_{\beta_\lambda+\mathbb{C}[t]\delta})m = 0$$

$$(h_{\beta_\lambda,k\delta})m = \delta_{k,0}((\lambda + 2\nu)(\beta_\lambda{}^\vee))m$$

$$(x_{-\beta_\lambda+(r_{1,\beta_\lambda})\delta})m = 0.$$

It is known that this module is isomorphic to the generalized Demazure module; namely it can be realized as the submodule

$$\mathbb{U}(\mathfrak{sl}_2[t])(w_1 \otimes w_2) \subset (W^{\mathfrak{sl}_2}_{\text{loc}}(r_{1,\beta_\lambda}) \otimes W^{\mathfrak{sl}_2}_{\text{loc}}(r_{1,\beta_\lambda} - 2))$$

where w_1 is the highest weight vector of $W^{\mathfrak{sl}_2}_{\text{loc}}(r_{1,\beta_\lambda})$ and w_2 the highest weight vector of $W^{\mathfrak{sl}_2}_{\text{loc}}(r_{1,\beta_\lambda} - 2)$. Define a map $M \to V(\lambda_1 + \nu, \lambda_2 + \nu)$ by sending generator to generator. Then the $\mathfrak{sl}_2[t]$ map is well defined; First, it is clear that

$$(x_{\beta_\lambda+\mathbb{C}[t]\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

$$(h_{\beta_\lambda,k\delta})m = \delta_{k,0}((\lambda + 2\nu)(\beta_\lambda{}^\vee))w_{\lambda_1+\nu,\lambda_2+\nu}.$$

Then, by relation (2.2.3), I also have

$$(x_{-\beta_\lambda+(r_{1,\beta_\lambda})\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

Now, since $(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_2 = 0$ and $(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-3)\delta})^2 w_2 = 0$, I have that

$$(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-3)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})(w_1 \otimes w_2)$$

$$= (x_{-\beta_\lambda+(r_{1,\beta_\lambda}-3)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_1 \otimes w_2$$

$$+3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-3)\delta})^2(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_1 \otimes (x_{-\beta_\lambda+(r_{1,\beta_\lambda}-3)\delta})w_2.$$

I claim there exists a map $W_{\text{loc}}^{\mathfrak{sl}_2}(r_{1,\beta_\lambda}-2) \to \mathbb{U}(\mathfrak{sl}_2[t])(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_1$ defined by $w \to (x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_1$.

Observe:

$$(x_{\beta_\lambda+\mathbb{C}[t]\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_1 = 0,$$

$$(h_{\beta_\lambda,k\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_1$$

$$= \delta_{k,0}(r_{1,\beta_\lambda}-2)(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_1,$$

and

$$(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-2)\delta})(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_1 = 0.$$

Hence, the map is well defined, and therefore I can conclude that

$$(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-3)\delta})^2(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_1 = 0,$$

and thus,

$$(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-3)\delta})^3(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

4.2.4 Relation (2.2.5)

Finally, I must show that relation (2.2.5) holds; that is, that

$$(x_{-\beta+s'_\beta\delta})^2(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}=0$$

for $\beta \in R(\lambda'_1, \lambda'_2)$, where $\lambda'_1 = \lambda_1 - \beta_\lambda - \nu_0,\ \lambda'_2 = \lambda_2 - \nu_0$, and ν_0 is as defined in Lemma 7. Using Lemma 3 along with the fact that $\lambda_1(\beta^\vee) \geq \lambda_2(\beta^\vee)$, it becomes clear that $\beta \in R(\lambda'_1, \lambda'_2)$ if and only if $\beta \in R(\lambda_1, \lambda_2)$ and $\beta_\lambda(\beta^\vee) = 0$. Thus, $\beta_\lambda + \beta \notin R^+$, and $s'_\beta = s_\beta$. Hence, I have that

$$(x_{-\beta+s'_\beta\delta})^2(x_{-\beta_\lambda+(r_{1,\beta_\lambda}-1)\delta})w_{\lambda_1+\nu,\lambda_2+\nu}=0,$$

and therefore, the relation holds. This concludes the proof of Proposition 8.

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