Lipschitz Embeddings of Random Objects and Related Topics

by

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Abstract

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More than twenty years ago Peter Winkler introduced a fascinating class of dependent or co-ordinate percolation models with his compatible sequences and clairvoyant demon scheduling problems. These, and other problems in this class, can be interpreted either as problems of embedding one sequence into another according to certain rules or as oriented percolation problems in $\mathbb{Z}^2$ where the sites are open or closed according to random variables on the co-ordinate axes. In most situations, these problems are not tractable by the usual tools of independent Bernoulli percolation, and new methods are required. We study several problems in this class and their natural extensions.

We develop a new multi-scale framework flexible enough to solve a number of problems involving embedding random sequences into random sequences. A natural question in this class was considered by Grimmett, Liggett and Richthammer in [24] where they asked whether there exists an increasing M-Lipschitz embedding from one i.i.d. Bernoulli sequence into an independent copy with positive probability. We give a positive answer for large enough M. A closely related problem is to show that two independent Poisson processes on $\mathbb{R}$ are almost surely roughly isometric (or quasi-isometric). Our approach also applies in this case answering a conjecture of Szegedy and of Peled [35]. We also obtain a new proof for Winkler's compatible sequence problem. All these results are obtained as corollaries to an abstract embedding result that can potentially be applied to a number of one-dimensional embedding questions.

We build upon the central idea of the multi-scale construction in the above work to apply it to a different problem. On the complete graph $\mathcal{K}_M$ with $M \geq 3$ vertices consider two independent discrete time random walks $\mathbb{X}$ and $\mathbb{Y}$, choosing their steps uniformly at random. We say that it is possible to **schedule** a pair of trajectories $\mathbb{X} = \{X_1, X_2, \ldots\}$ and $\mathbb{Y} = \{Y_1, Y_2, \ldots\}$, if by delaying their jump times one can keep both walks at distinct vertices forever. It was conjectured by Winkler that for large enough M the set of pairs of trajectories $\{\mathbb{X}, \mathbb{Y}\}$ that can be scheduled has positive measure. Noga Alon translated this problem to the language of coordinate percolation. In this representation Winkler's

conjecture is equivalent to the existence of an infinite open cluster for large enough M. With a multi-scale construction we provide a positive answer for M sufficiently large.

The questions of Lipschitz embedding and rough isometry of random sequences have natural higher dimensional analogues. We consider the higher dimensional analogue of the Lipschitz embedding problem. We show that for M sufficiently large and two independent collections of i.i.d. Bernoulli random variables $\mathbb{X} = \{X_v\}_{v\in\mathbb{Z}^2}$ and $\mathbb{Y} = \{Y_v\}_{v\in\mathbb{Z}^2}$, almost surely there exists an M-Lipschitz embedding of $\mathbb{X}$ into $\mathbb{Y}$. The argument is again multi-scale using similar ideas, but this is technically much more challenging because of the more complicated geometry in two dimensions.

This presents an added difficulty in extending the argument in one dimension to show that copies of two dimensional Poisson processes are almost surely rough isometric. A key ingredient is to show that one can map measurable sets to smaller measurable sets in a bi-Lipschitz manner. This motivates the final problem we consider. We show that for $0 < \gamma, \gamma' < 1$ and for measurable subsets of the unit square with Lebesgue measure γ there exist bi-Lipschitz maps with bounded Lipschitz constant(uniformly over all such sets) which are identity on the boundary and increases the Lebesgue measure of the set to at least $1-\gamma'$.

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In the Spring of 2012, I, along with Allan, started working with Vladas Sidoravicius; a collaboration that continues till date. I am very grateful to Vladas for his kindness and encouragement and sharing his wealth of experience which taught me a lot. Working with Allan and Vladas has been a wonderful learning experience, and I cannot thank them enough for that. Everything in this dissertation is based on joint works with Allan and with Allan and Vladas, and I am grateful to them for permitting the use of the co-authored material in my dissertation.

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Chapter 1

Introduction

Percolation has been studied as a paradigm model for spatial randomness for more than half a century. The deep and rich understanding that emerged in the study of independent Bernoulli percolation is a celebrated success story of contemporary probability. In the mean time several natural questions arising from mathematical physics and theoretical computer science has necessitated the study of models containing more complicated dependent structures, which are not amenable to the tools of Bernoulli percolation. In this dissertation we focus on questions in and around one particular subclass of dependent percolation models, called "co-ordinate percolation", which have received significant attention in the past couple of decades.

In the simplest setting, a co-ordinate percolation model is an oriented percolation model on the positive quadrant of $\mathbb{Z}^2$ (or some modifications thereof), where the vertex $(i, j) \in \mathbb{Z}^2$ is declared to be **open** or **closed** depending on two variables on the co-ordinate axes X_i and Y_j where typically $\{X_i\}_{i\geq 0}$ and $\{Y_j\}_{j\geq 0}$ are independent sequences of i.i.d. random variables. The questions asked about these models are the same ones that are studied in the independent Bernoulli percolation, e.g. whether there exists an infinite open oriented path starting at origin and whether the probability of percolation exhibits a phase transition as certain model parameters are varied. Long range dependent structure makes these models difficult to study and it turns out that many of these models are fundamentally different from the independent Bernoulli percolation models in that these exhibit power law decays for certain tail probabilities as opposed to exponential decays. This is one of the reasons the techniques for independent percolation do not yield useful results in these models.

Models of co-ordinate percolation were first introduced, motivated by problems of statistical physics, in late eighties by B. Tóth under the name "corner percolation" which was later studied by Gábor Pete in [36]. It is curious to observe that a co-ordinate percolation environment was introduced by Diaconis and Freedman earlier in 1981 in the context of studying Julesz conjecture on visually distinguishable patterns in [12]. In early nineties a number of problems in this class were popularised by Peter Winkler, who introduced and later studied several models of this type arising out of considerations in theoretical computer science [11, 38, 34, 10].

Another set of questions in random geometry involving embedding one random sequence into another can also be cast into the framework of co-ordinate percolation. In several of its variants this has been investigated in [24, 20, 18, 16, 31]. The simplest problem in this vein is the problem of embedding one random sequence into another in a Lipschitz manner. This in turn is intimately connected to quasi-isometries of one dimensional random objects [35].

In this dissertation we focus on Lipschitz embeddings of random objects and some related problems. We study Lipschitz embeddings of one dimensional random binary sequences, and the related problem of quasi-isometry of one dimensional Poisson processes and the problem of compatible sequences. Using similar techniques we also investigate a specific model of co-ordinate percolation that arises from Winkler's scheduling problem. We further extend the Lipschitz embedding result to two dimensions, and obtain certain results towards establishing on quasi-isometries to random objects in higher dimensions. We start with formal descriptions of the models we consider.

1.1 Embedding Problems in One Dimension

In the simplest and most natural setting, we first consider the problem of embedding a random one dimensional object into another. We restrict our attention to i.i.d. binary sequences.

1.1.1 Lipschitz Embedding of Binary Sequences

Let $\mathbb{X} = \{X_i\}_{i\in\mathbb{Z}}$ and $\mathbb{Y} = \{Y_i\}_{i\in\mathbb{Z}}$ be two binary sequences. For $M > 0$, we call $\mathbb{X}$ **M-embeddable** in $\mathbb{Y}$ if $\mathbb{X}$ can be embedded into $\mathbb{Y}$ in an M-Lipschitz manner, i.e. there exists a strictly increasing function $\phi : \mathbb{Z} \to \mathbb{Z}$ such that $X_i = Y_{\phi(i)}$ and $1 \le \phi(i) - \phi(i-1) \le M$ for all i. Grimmett, Liggett and Richthammer had the following question as the main open problem in [24] (re-iterated in [20]).

Question 1.1.1. *Let $\mathbb{X} = \{X_i\}_{i\in\mathbb{Z}}$ and $\mathbb{Y} = \{Y_i\}_{i\in\mathbb{Z}}$ be independent sequences of independent identically distributed Ber($\frac{1}{2}$) random variables. Does there exist $M > 0$ such that $\mathbb{X}$ is almost surely M-embeddable into $\mathbb{Y}$?*

The original question of [24] was slightly different asking for a positive probability on the natural numbers with the condition $\phi(0) = 0$, which is implied by a positive answer to the above question (and is equivalent by ergodic theory considerations). In this variant, the question can be cast into the co-ordinate percolation framework as follows.

Consider the following graph with vertex set $\mathbb{Z}^2_{\ge 0}$, i.e., $\{(i,j) \in \mathbb{Z}^2 : i, j \ge 0\}$, with the following edge set. There is an edge between (i,j) and (i',j') if and only if $|i-i'| = 1$ and $1 \le |j-j'| \le M$. For any two binary sequence $\mathbb{X} = \{X_i\}_{i\ge 1}$ and $\mathbb{Y} = \{Y_j\}_{j\ge 1}$ as above, consider the following site percolation on this graph. Call (i,j) **closed** if $X_i \ne Y_j$ and **open** otherwise. It is easy to see that there is an M-embedding ϕ of $\mathbb{X}$ into $\mathbb{Y}$ (with obvious modification of the definition above) with $\phi(0) = 0$ if and only if there is an infinite open oriented path starting at the origin. See Figure 1.1 for the case $M = 3$ where we write the sequences $\mathbb{X}$ and $\mathbb{Y}$ along the x-axis and y-axis respectively. The open and closed vertices are denoted by blue and red respectively.

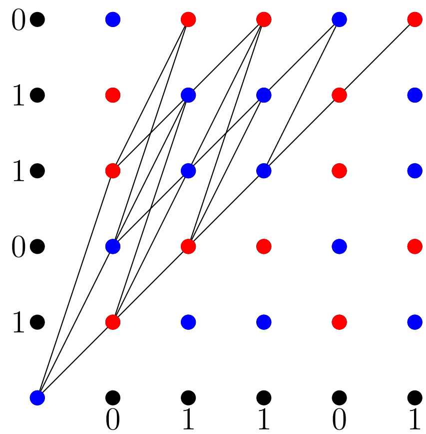


Figure 1.1: Lipschitz embedding as co-ordinate percolation

Of course one can ask the same question with independent Ber(p) sequences for any $p \in (0, 1)$. It was shown in [24] that an affirmative answer to Question 1.1.1 for some $p \in (0, 1)$ is equivalent to an affirmative answer for all $p \in (0, 1)$. Among other results they also showed that the answer to the question is negative for $M = 2$.

In a series of subsequent works Grimmett, Holroyd and their collaborators [13, 20, 21, 23, 22, 26] investigated a range of related problems including when one can embed $\mathbb{Z}^d$ into site percolation in $\mathbb{Z}^D$ and showed that this was possible almost surely for $M = 2$ when $D > d$ and the the site percolation parameter was sufficiently large but almost surely impossible for any M when $D \leq d$. Recently Holroyd and Martin showed that a comb can be embedded in $\mathbb{Z}^2$. Another important series of work in this area involves embedding words into higher dimensional percolation clusters [9, 32, 29, 30]. Despite this impressive progress the question of embedding one random sequence into another had remained open. The difficulty lies in the presence of long strings of ones and zeros on all scales in both sequences which must be paired together.

In Chapter 2 we provide an affirmative answer to Question 1.1.1 for M sufficiently large. See Theorem 2.1.

1.1.2 Rough Isometry of One Dimensional Random Objects

Intimately connected with the embedding problem is the question of a rough, (or quasi-), isometry of two independent Poisson processes. Informally, two metric spaces are roughly isometric if their metrics are equivalent up to multiplicative and additive constants. The formal definition, introduced by Gromov [25] in the case of groups and more generally by Kanai [27], is as follows.

Definition 1.1.2. *We say two metric spaces X and Y are roughly isometric with parameters (M, D, C) if there exists a mapping $T : X \to Y$ such that for any $x_1, x_2 \in X$,*

$$\frac{1}{M} d_X(x_1, x_2) - D \leq d_Y(T(x_1), T(x_2)) \leq M d_X(x_1, x_2) + D,$$

and for all $y \in Y$ there exists $x \in X$ such that $d_Y(T(x), y) \leq C$.

Whether two random metric spaces (or two i.i.d. copies of the same random metric space) are almost surely rough isometric have been asked in several instances. Originally Abért [1] asked whether two independent infinite components of bond percolation on a Cayley graph are roughly isometric. Szegedy asked the problem when these sets are independent Poisson process in $\mathbb{R}$ (see [35] for a fuller description of the history of the problem). An important progress on this question is by Peled [35] who showed that Poisson processes on $[0, n]$ are roughly isometric with parameter $M = \sqrt{\log n}$. The main open question of [35] was the following.

Question 1.1.3. *Let X and Y be independent homogeneous Poisson processes on $\mathbb{R}$ viewed as metric spaces, i.e., X and Y are metric subspaces of $\mathbb{R}$ induced by independent Poisson processes on $\mathbb{R}$. Does there exist (M, D, C) such that almost surely X and Y are (M, D, C)-roughly isometric?*

The same question can also be asked about two independent copies of Bernoulli percolation on $\mathbb{Z}$. That is, considering the random metric subspaces of $\mathbb{Z}$ given by the set of open sites in two copies of site percolation on $\mathbb{Z}$, where each vertex is declared open with probability $p \in (0, 1)$ with different vertices receiving independent assignments. Results of [35] show that an affirmative answer to Question 1.1.3 is equivalent to an affirmative answer for percolation on $\mathbb{Z}$ and the answer is independent of the parameter p.

Again the challenge is to find a good matching on all scales, in this case to the long gaps in the each point processes with ones of proportional length in the other. In Chapter 2 we provide an affirmative answer to Question 1.1.3 for M, D, C sufficiently large. See Theorem 2.2. The isometries we find are also weakly increasing answering a further question of Peled [35].

1.1.3 Compatible Sequence Problem

In a similar vein, we also consider the compatible sequence problem introduced by Peter Winkler. Given two binary sequences $\{X_i\}_{i \in \mathbb{N}}$ and $\{Y_i\}_{i \in \mathbb{N}}$, we say they are **compatible** after removing some zeros from both sequences, there is no index with a 1 in both sequence. Equivalently there exist increasing subsequences $k_1, k_2, \ldots,$ (respectively $k'_1 \ldots$) such that if

$X_j = 1$ then $j = k_i$ for some i (resp. if $Y_j = 1$ then $j = k'_i$) so that for all i, we have $X_{k_i} Y_{k'_i} = 0$.

The question of compatibility can be cast into the co-ordinate percolation frame work as well. Consider the directed graph with vertex set $\mathbb{Z}^2_{>0}$ where we have edges leading from the vertex (i, j) to vertices $(i + 1, j)$, $(i, j + 1)$ and $(i + 1, j + 1)$. Consider the following bond percolation on this graph. If $X_i = 1$ then we declare the edges from (i, j) to $(i + 1, j)$ closed for all j. Similarly if $Y_j = 1$ we declare the edges from (i, j) to $(i, j + 1)$ closed for all i. If $X_i = Y_j$, we declare the edge from (i, j) to $(i + 1, j + 1)$ closed. All other edges are declared open. It is easy to see that the sequences $\{X_i\}$ and $\{Y_j\}$ are compatible if and only if there is an infinite open (oriented) path from $(1, 1)$.

Now suppose $\{X_i\}_{i\in\mathbb{N}}$ and $\{Y_i\}_{i\in\mathbb{N}}$ are independent i.i.d. sequences of Ber(q) random variables where $q \in (0, 1)$. It is not hard to show that if q is sufficiently large then the sequences are compatible with positive probability. Winkler and Kesten independently obtained upper bounds on q that are strictly smaller than $1/2$. The following question was asked by Winkler.

Question 1.1.4. *Does there exist $q > 0$ two independent Ber(q) sequences $\{X_i\}_{i\in\mathbb{N}}$ and $\{Y_i\}_{i\in\mathbb{N}}$ are compatible with positive probability?*

Numerical simulations suggest that the above question has affirmative answer for $q < 0.3$. An affirmative answer for q sufficiently small was given by Gács in [18]. Other recent progress was made on this problem by Kesten et. al. [31] constructing sequences with a positive density of zeros which are compatible with a random sequence with positive probability. We give a new proof of Gács' result in Chapter 2. See Theorem 2.3. Our proof is different and we believe more transperant. We also provide deterministic construction of a sequence with positive density of zeros that is compatible with an i.i.d. Ber(q) sequence with positive probability for q sufficiently small.

In each of Theorems 2.1, 2.2 and 2.3 the main difficulty is the presence of difficult to embed strings of arbitrarily large length. The challenge is to be able to simultaneously match these strings with their suitable partners (in the other sequence) at all length scales. Like essentially all results in this area, our approach is multi-scale using renormalization. The novelty of our approach is that, as far as possible, we ignore the anatomy of what makes different configurations difficult to embed and instead consider simply the probability that they can be embedded into a random block proving recursive power-law estimates for these quantities. It is thus well suited to addressing embedding questions in a range of different models even in the ones where to give a description of bad configurations becomes exceedingly difficult. Indeed, we prove an abstract embedding result for general alphabets satisfying certain conditions. See Theorem 2.4 in Chapter 2. Theorems 2.1, 2.2 and 2.3 follow in turn from this abstract theorem. The general idea of our multi-scale construction yields useful results in other co-ordinate percolation type problems as well.

1.2 Winkler's Scheduling Problem

We consider the following problem introduced by Winkler, which in its original formulation relates to clairvoyant scheduling of two independent random walks on a complete graph. More precisely, on the complete graph $\mathcal{K}_M$ with $M \geq 3$ vertices consider the trajectories of two independent discrete time random walks $\mathbb{X}$ and $\mathbb{Y}$ which move by choosing steps uniformly at random, for convenience we assume the graphs have self-loops at each vertex. We say that it is possible to **schedule** $\mathbb{X}$ and $\mathbb{Y}$ if it is possible to introduce *delays* (i.e., with the knowledge of both $\mathbb{X}$ and $\mathbb{Y}$, at each time only one of the random walks is chosen and allowed to move) in the random walk trajectories such that the random walks never collide. The following question, asked by Winkler, became prominent as the **clairvoyant demon problem**.

Question 1.2.1. *Does there exist $M > 0$ such that it is possible to schedule two independent random walks on $\mathcal{K}_M$ with positive probability?*

For $M = 3$ it was shown in [38, Corollary 3.4] that the answer to the above question is negative, i.e., even with the knowledge of infinite future of both the random walks, it is not possible to schedule the walks on $\mathcal{K}_3$. However an affirmative answer to Question 1.2.1 was conjectured in [11] for large enough M. In particular it is believed based on simulations, that for $M \geq 4$ the set of trajectories $\{\mathbb{X}, \mathbb{Y}\}$ that can be scheduled has positive measure.

Noga Alon translated this problem into the language of coordinate percolation. Namely, let $\mathbb{X} = (X_1, X_2, \ldots)$ and $\mathbb{Y} = (Y_1, Y_2, \ldots)$ be two i.i.d. sequences with

$$\mathbb{P}(X_i = k) = \mathbb{P}(Y_j = k) = \frac{1}{M} \text{ for } \ k = 1, 2, \ldots, M \text{ and for } i, j = 1, 2, \ldots.$$

Consider the following oriented percolation on the subgraph of the two-dimensional Euclidean lattice induced on the vertex set $\mathbb{Z}^2_{>0}$. Call a vertex $(i_1, i_2) \in \mathbb{Z}^2_{>0}$ **closed** if $X_{i_1} = Y_{i_2}$ an call it **open** otherwise. It is curious to notice that this percolation process (for M=2) was introduced much earlier by Diaconis and Freedman [12] in the completely different context of studying visually distinguishable random patterns in connection with Julesez's conjecture. It is easy to observe that a pair of trajectories $\{\mathbb{X}, \mathbb{Y}\}$ can be scheduled if and only if there is an open oriented infinite path starting at the vertex $(1, 1)$. See Figure 1.2 where blue and red vertices respectively denote open and closed ones and open paths from $(1, 1)$ are shown.

This scheduling problem first appeared in the context of distributed computing [11] where it is shown that two independent random walks on a finite connected non-bipartite graph will collide in a polynomial time even if a scheduler tries to keep them apart, unless the scheduler is clairvoyant. In a recent work [3], instead of independent random walks, by allowing coupled random walks, it was shown that a large number of random walks can be made to avoid one another forever. In the context of clairvoyant scheduling of two independent walks, the non-oriented version of the oriented percolation process described above was studied independently in [38] and [4] where it is established that in the non-oriented model there is percolation with positive probability if and only if $M \geq 4$. In [19] it was established that,

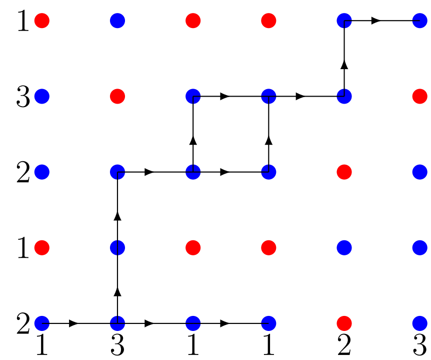


Figure 1.2: Scheduling problem as co-ordinate percolation

if there is percolation, the chance that the cluster dies out after reaching distance n must decay polynomially in n, which showed that, unlike the non-oriented models, this model was fundamentally different from Bernoulli percolation, where such decay is exponential.

In Chapter 3 we prove that for M sufficiently large it is possible to schedule two independent random walks on $\mathcal{K}_M$ with positive probability, thus providing an affirmative answer to Winker's question. See Theorem 3.1. Our proof uses a similar multi-scale structure developed in Chapter 2 but with crucial differences. An exceedingly complex proof of the same result was previously given in [17]. Our proof is different and we believe more transparent. In addition, we believe that our proof is robust and can be applied for more general graphs and also for the situation when we try to schedule multiple random walks.

1.3 Higher Dimensional Embedding Problems

The first natural question one can ask by way of generalising the results in Chapter 1 are the following higher dimensional analogues of Questions 1.1.1 and 1.1.3.

Question 1.3.1. *Let $\mathbb{X} = \{X_i\}_{i\in\mathbb{Z}^d}$ and $\mathbb{Y} = \{Y_i\}_{i\in\mathbb{Z}^d}$ be independent collections of i.i.d. Ber($\frac{1}{2}$) random variables. Does there exist $M > 0$ such that $\mathbb{X}$ is almost surely M-embeddable into $\mathbb{Y}$?*

Question 1.3.2. *Let X and Y be independent homogeneous Poisson processes on $\mathbb{R}^d$ viewed as metric spaces, i.e., X and Y are metric subspaces of $\mathbb{R}^d$ induced by independent Poisson processes on $\mathbb{R}^d$. Does there exist (M, D, C) such that almost surely X and Y are (M, D, C)-roughly isometric?*

Here the definition of M-embedding is the obvious modification of the definition in one dimension where we consider M-Lipschitz maps from $\mathbb{Z}^d$ to $\mathbb{Z}^d$. In Chapter 4 we provide an affirmative answer to Question 1.3.1 for $d = 2$ and M sufficiently large; see Theorem

4.1. The same argument works for $d > 2$ with minor modifications. The argument here is also multi-scale and in spirit similar to the arguments presented in Chapter 2, but this is much more technically challenging than the proof of Theorem 2.1 as in higher dimensions the difficult to embed regions can have very complicated shapes.

1.3.1 Bi-Lipschitz Expansion

The complicated geometry of higher dimensional spaces presents an added challenge while investigating Question 1.3.2. One key feature of the proof of Theorem 2.2 is that we have to map large gaps between points in our Poisson process to relatively smaller gaps. In one dimension it is straightforward to do as the geometry is linear and the gaps are all intervals. But for $d \geq 2$, we need to deal with arbitrary measurable sets. This motivates the following question, which is also of independent interest.

Question 1.3.3. *Fix $0 < \gamma < 1 - \gamma' < 1$. Does there exist $C = C(\gamma, \gamma') > 0$ such that for each measurable subset A of $[0,1]^2$ with Lebesgue measure γ, there exists a bijection $\Phi_A : [0,1]^2 \to [0,1]^2$ which is identity on the boundary and bi-Lipschitz with Lipschitz constant C and such that $\Phi_A(A)$ has Lebesgue measure at least $1 - \gamma'$?*

In Chapter 5 we provide an affirmative answer to this question. See Theorem 5.1. Although our construction of such maps is deterministic, the analysis is probabilistic using martingale techniques. This result lets us map big holes in the Poisson process into smaller holes in a bi-Lipschitz manner while maintaining a uniform control on the Lipschitz constant. See Theorem 5.2 for an illustration of why this result is useful in proving rough isometry of Poisson processes.

1.4 Note on Prior Publication and Collaboration

The results presented in this dissertation were obtained in collaboration with other researchers and some have already been published elsewhere. Chapter 2 is based on a joint work with Allan Sly [8] that has been published by *Probability Theory and Related Fields*. We acknowledge the journal as the first published source of this material. The remaining chapters are based on joint works with Vladas Sidoravicius and Allan Sly. Chapter 3 is based on [7], which is available on Arxiv. Chapter 4 is based on [6], a work in preparation. Chapter 5 is based on [5], which is also available on Arxiv. I express my sincere thanks towards my co-authors for allowing the inclusion of joints works with them in this dissertation.

Chapter 2

Embeddings of One Dimensional Random Objects

In this chapter we study embeddings of one dimensional random objects, i.e., binary sequences or Poisson processes on $\mathbb{R}$. We provide affirmative answers to Questions 1.1.1, 1.1.3 and 1.1.4 using a multi-scale argument. Theorems 2.1, 2.2 and 2.3 are our main results in this chapter.

2.1 Main Results

We start with Lipschitz embedding of binary sequences. We prove the following result.

Theorem 2.1. *Let $\{X_i\}_{i\in\mathbb{Z}}$ and $\{Y_i\}_{i\in\mathbb{Z}}$ be independent sequences of independent identically distributed Ber($\frac{1}{2}$) random variables. For sufficiently large M almost surely there exists a strictly increasing function $\phi : \mathbb{Z} \to \mathbb{Z}$ such that $X_i = Y_{\phi(i)}$ and $1 \le \phi(i) - \phi(i-1) \le M$ for all i.*

Recall Definition 1.1.2 for rough isometry of two metric spaces. We have the following theorem for random metric spaces given by independent Poisson processes on $\mathbb{R}$.

Theorem 2.2. *Let X and Y be independent Poisson processes on $\mathbb{R}$ viewed as metric spaces. There exists (M, D, C) such that almost surely X and Y are (M, D, C)-roughly isometric.*

Our final result is the compatible sequence problem of Winkler. Recall the definition of compatible sequences from § 1.1.3. We give a new proof of the following result of Gács [18].

Theorem 2.3. *For sufficiently small $q > 0$ two independent Ber(q) sequences $\{X_i\}_{i\in\mathbb{N}}$ and $\{Y_i\}_{i\in\mathbb{N}}$ are compatible with positive probability.*

Independent Results: Two other researchers have also solved some of these problems independently. Vladas Sidoravicius [37] solved the same set of problems and described his approach to us. His work is based on a different multi-scale approach, proving that for certain choices of parameters p_1 and p_2 one can see random binary sequence sampled with parameter p_1 in the scenery determined by another binary sequence sampled with parameter p_2, with positive probability. This generalizes the main theorem of [28] and a slight modification of it then implies Theorems 2.1, 2.2 and 2.3.

Shortly before uploading this work to arXiv Peter Gács sent us a draft of his paper [16] solving Theorem 2.1. His approach extends his work on the scheduling problem [17]. The proof is geometric taking a percolation type view and involves a complex multi-scale system of structures. Our work was done completely independently of both.

As already mentioned in Chapter 1, each of Theorems 2.1, 2.2 and 2.3 follows from an abstract embedding result for general alphabets which we now turn to.

2.1.1 General Theorem

To apply to a range of problems we need to consider larger alphabets of symbols. Let $\mathcal{C}^{\mathbb{X}} = \{C_1, C_2, \ldots\}$ and $\mathcal{C}^{\mathbb{Y}} = \{C'_1, C'_2, \ldots\}$ be a pair of countable alphabets and let $\mu^{\mathbb{X}}$ and $\mu^{\mathbb{Y}}$ be probability measures on $\mathcal{C}^{\mathbb{X}}$ and $\mathcal{C}^{\mathbb{Y}}$ respectively.

We will suppose also that we have a relation $\mathcal{R} \subseteq \mathcal{C}^{\mathbb{X}} \times \mathcal{C}^{\mathbb{Y}}$. If $(C_i, C'_k) \in \mathcal{R}$, we denote this by $C_i \hookrightarrow C'_k$. Let $G_0^{\mathbb{X}} \subseteq \mathcal{C}^{\mathbb{X}}$ and $G_0^{\mathbb{Y}} \subseteq \mathcal{C}^{\mathbb{Y}}$ be two given subsets such that $C_i \in G_0^{\mathbb{X}}$ and $C'_k \in G_0^{\mathbb{Y}}$ implies $C_i \hookrightarrow C'_k$. Symbols in $G_0^{\mathbb{X}}$ and $G_0^{\mathbb{Y}}$ will be referred to as "good".

Definitions

Now let $\mathbb{X} = (X_1, X_2, \ldots)$ and $\mathbb{Y} = (Y_1, Y_2, \ldots)$ be two sequences of symbols coming from the alphabets $\mathcal{C}^{\mathbb{X}}$ and $\mathcal{C}^{\mathbb{Y}}$ respectively. We will refer to such sequences as an $\mathbb{X}$-sequence and a $\mathbb{Y}$-sequence respectively. For $1 \leq i_1 < i_2$, we call the subsequence $(X_{i_1}, X_{i_1+1}, \ldots, X_{i_2})$ the "$[i_1, i_2]$-segment" of $\mathbb{X}$ and denote it by $\mathbb{X}^{[i_1,i_2]}$. We call $\mathbb{X}^{[i_1,i_2]}$ a "good" segment if $X_i \in G_0^{\mathbb{X}}$ for $i_1 \leq i \leq i_2$ and similarly for $\mathbb{Y}$.

Let R be a fixed constant. Let $R_0 = 2R$, $R_0^- = 1$, $R_0^+ = 3R^2$.

Definition 2.1.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be sequences as above. Let $X = (X_1, \ldots X_n)$ and $Y = (Y_1, \ldots Y_{n'})$ be segments of $\mathbb{X}$ and $\mathbb{Y}$ respectively. We say X R-embeds or R-maps into Y, denoted $X \hookrightarrow_R Y$ if there exists $0 = i_0 < i_1 < i_2 < \ldots < i_k = n$ and $0 = i'_0 < i'_1 < i'_2 < \ldots < i'_k = n'$ satisfying the following conditions.*

1. *For each r, $r \geq 0$, either $i_{r+1} - i_r = i'_{r+1} - i'_r = 1$ or $i_{r+1} - i_r = R_0$ or $i'_{r+1} - i'_r = R_0$.*
2. *If $i_{r+1} - i_r = i'_{r+1} - i'_r = 1$, then $X_{i_r+1} \hookrightarrow Y_{i'_r+1}$.*
3. *If $i_{r+1} - i_r = R_0$, then $R_0^- \leq i'_{r+1} - i'_r \leq R_0^+$, and both $\mathbb{X}^{[i_r+1,i_{r+1}]}$ and $\mathbb{Y}^{[i'_r+1,i'_{r+1}]}$ are good segments.*
4. *If $i'_{r+1} - i'_r = R_0$, then $R_0^- \leq i_{r+1} - i_r \leq R_0^+$, and both $\mathbb{X}^{[i_r+1,i_{r+1}]}$ and $\mathbb{Y}^{[i'_r+1,i'_{r+1}]}$ are good segments.*

We say that $\mathbb{X}$ *R-embeds or R-maps into* $\mathbb{Y}$*, denoted* $\mathbb{X} \hookrightarrow_R \mathbb{Y}$ *if there exists* $0 = i_0 < i_1 < i_2 < \ldots$ *and* $0 = i'_0 < i'_1 < i'_2 < \ldots$ *satisfying the above conditions.*

Throughout we will use a fixed R defined in Theorem 2.4 and will simply refer to mappings and write that $\mathbb{X} \hookrightarrow \mathbb{Y}$ (or $X \hookrightarrow Y$) except where it is ambiguous. The following elementary observation is useful. Suppose we have $n_0 < n_1 < \ldots n_k$ and $n'_0 < n'_1 < \ldots n'_k$ such that $\mathbb{X}^{[n_r+1,n_{r+1}]} \hookrightarrow \mathbb{Y}^{[n'_r+1,n'_{r+1}]}$ for $0 \leq r < k$, then $\mathbb{X}^{[n_0+1,n_k]} \hookrightarrow \mathbb{Y}^{[n'_0+1,n'_k]}$.

A key element in our proof is tail estimates on the probability that we can map a block X into a random block Y and so we make the following definition.

Definition 2.1.2. *For* $X \in \mathcal{C}^{\mathbb{X}}$*, we define the* embedding probability *of* X *as* $S_0^{\mathbb{X}}(X) = \mathbb{P}(X \hookrightarrow Y|X)$ *where* $Y \sim \mu^{\mathbb{Y}}$*. We define* $S_0^{\mathbb{Y}}(Y)$ *similarly and suppress the notation* $\mathbb{X}, \mathbb{Y}$ *when the context is clear.*

General Theorem

We can now state our general theorem which will imply the main results of this chapter as shown in § 2.2.

Theorem 2.4 (General Theorem)**.** *There exist positive constants* β, δ, m, R *such that for all large enough* L_0 *the following hold. Let* $X \sim \mu^{\mathbb{X}}$ *and* $Y \sim \mu^{\mathbb{Y}}$ *where* $\mu^{\mathbb{X}}$ *and* $\mu^{\mathbb{Y}}$ *are probability distributions on alphabets such that for all* $k \geq L_0$*,*

$$\mu^{\mathbb{X}}(\{C_{k+1}, C_{k+2}, \ldots\}) \leq \frac{1}{k}, \quad \mu^{\mathbb{Y}}(\{C'_{k+1}, C'_{k+2}, \ldots\}) \leq \frac{1}{k}. \tag{2.1.1}$$

Suppose the following conditions are satisfied

1. *For all* $0 < p \leq 1 - L_0^{-1}$*,*

$$\mathbb{P}(S_0^{\mathbb{X}}(X) \leq p) \leq p^{m+1} L_0^{-\beta}, \quad \mathbb{P}(S_0^{\mathbb{Y}}(Y) \leq p) \leq p^{m+1} L_0^{-\beta}. \tag{2.1.2}$$

2. *Most symbols are "good",*

$$\mathbb{P}(X \in G_0^{\mathbb{X}}) \geq 1 - L_0^{-\delta}, \quad \mathbb{P}(Y \in G_0^{\mathbb{Y}}) \geq 1 - L_0^{-\delta}. \tag{2.1.3}$$

Then for $\mathbb{X} = (X_1, X_2, \ldots)$ *and* $\mathbb{Y} = (Y_1, Y_2, \ldots)$*, two sequences of i.i.d. symbols with laws* $\mu^{\mathbb{X}}$ *and* $\mu^{\mathbb{Y}}$ *respectively, we have*

$$\mathbb{P}(\mathbb{X} \hookrightarrow_R \mathbb{Y}) > 0.$$

2.1.2 Proof Outline

The proof makes use of a number of parameters, $\alpha, \beta, \delta, m, k_0, R$ and L_0 which must satisfy a number of relations described in the next subsection. Our proof is multi-scale and divides the sequences into blocks on a series of doubly exponentially growing length scales $L_j = L_0^{\alpha^j}$ for $j \geq 0$ and at each of these levels we define a notion of a "good" block. Single characters in the base sequences $\mathbb{X}$ and $\mathbb{Y}$ constitute the level 0 blocks.

Suppose that we have constructed the blocks up to level j denoting the sequence as $(X_1^{(j)}, X_2^{(j)} \ldots)$. In § 2.3, we give a construction of $(j+1)$-level blocks out of j-level sub-blocks in such way that the blocks are independent and apart from the first block, identically distributed and that the first and last L_j^3 sub-blocks of each block are good. For more details, see § 2.3.

At each level we distinguish a set of blocks to be good. In particular this will be done in such a way that at each level *any* good block maps into *any* other good block. Moreover, any segment of $R_j = 4^j(2R)$ good $\mathbb{X}$-blocks will map into any segment of $\mathbb{Y}$-blocks of length between $R_j^- = 4^j(2 - 2^{-j})$ and $R_j^+ = 4^j R^2(2 + 2^{-j})$ and vice-versa. This property of certain mappings will allow us to avoid complicated conditioning issues. Moreover, being able to map good segments into shorter or longer segments will give us the flexibility to find suitable partners for difficult to embed blocks and to achieve improving estimates of the probability of mapping random j-level blocks $X \hookrightarrow Y$. We describe how to define good blocks in § 2.3 .

The proof then involves a series of recursive estimate at each level given in § 2.4. We ask that at level j the probability that a block is good is at least $1 - L_j^{-\delta}$ so that the vast majority of blocks are good. Furthermore, we show tail bounds on the embedding probabilities showing that for $0 < p \leq 1 - L_j^{-1}$,

$$\mathbb{P}(S_j^{\mathbb{X}}(X) \leq p) \leq p^{m+2^{-j}} L_j^{-\beta}$$

where $S_j^{\mathbb{X}}(X)$ denotes the j-level embedding probability $\mathbb{P}[X \hookrightarrow Y | X]$ for X, Y random independent j-level blocks. We show the analogous bound for $\mathbb{Y}$-blocks as well. This is essentially the best we can hope for – we cannot expect a better than power-law bound here because of the probability of occurrences of sequences of repeating symbols in the base 0-level sequence of length $C \log(L_j^{\alpha})$ for large C. We also ask that good blocks have the properties described above and that the length of blocks satisfy an exponential tail estimate. The full inductive step is given in § 2.4.1. Proving this constitutes the main work of this chapter.

The key quantitative estimate in the chapter is Lemma 2.7.3 which follows directly from the recursive estimates and bounds the chance of a block having an excessive length, many bad sub-blocks or a particularly difficult collection of sub-blocks measured by the product of their embedding probabilities. In order to achieve the improving embedding probabilities at each level we need to take advantage of the flexibility in mapping a small collection of very bad blocks to a large number of possible partners by mapping the good blocks around them into longer or shorter segments using the inductive assumptions. To this effect we define families of mappings between partitions to describe such potential mappings. Because

m is large and we take many independent trials the estimate at the next level improves significantly. Our analysis is split into 5 different cases.

To show that good blocks have the required properties we construct them so that they have at most k_0 bad sub-blocks all of which are "semi-bad" (defined in § 2.3) in particular with embedding probability at least $(1 - \frac{1}{20k_0 R_{j+1}^+})$. We also require that each subsequence of $L_j^{3/2}$ sub-blocks is "strong" in that every semi-bad block maps into a large proportion of the sub-blocks. Under these condition we show that for any good blocks X and Y at least one of our families of mappings gives an embedding. This holds similarly for embeddings of segments of good blocks.

To complete the proof we note that with positive probability $X_1^{(j)}$ and $Y_1^{(j)}$ are good for all j with positive probability. This gives a sequence of embeddings of increasing segments of $\mathbb{X}$ and $\mathbb{Y}$ and by taking a converging subsequential limit we can construct an R-embedding of the infinite sequences completing the proof.

We can also give deterministic constructions using our results. In Section 2.10 we construct a deterministic sequence which has an M-Lipschitz embedding into a random binary sequence in the sense of Theorem 2.1 with positive probability. Similarly, this approach gives a binary sequence with a positive density of ones which is compatible sequence with a random $\mathrm{Ber}(q)$ sequence in the sense of Theorem 2.3 for small enough $q > 0$ with positive probability.

Parameters

Our proof involves a collection of parameters $\alpha, \beta, \delta, k_0, m$ and R which must satisfy a system of constraints. The required constraints are

$$\alpha > 6, \delta > 2\alpha \vee 48, \beta > \alpha(\delta + 1), m > 9\alpha\beta, k_0 > 36\alpha\beta, R > 6(m + 1).$$

To fix on a choice we will set

$$\alpha = 10, \delta = 50, \beta = 600, m = 60000, k_0 = 300000, R = 400000. \tag{2.1.4}$$

Given these choices we then take L_0 to be a sufficiently large integer. We did not make a serious attempt to optimize the parameters or constraints and indeed at times did not in order to simplify the exposition.

2.1.3 Organization of the Chapter

In Section 2.2 we show how to derive Theorems 2.1, 2.2 and 2.3 from our general Theorem 2.4. In Section 2.3 we describe our block constructions and formally define good blocks. In Section 2.4 we state the main recursive theorem and show that it implies Theorem 2.4. In Sections 2.5 and 2.6 we construct a collection of generalized mappings of partitions which we will use to describe our mappings between blocks. In Section 2.7 we prove the main recursive tail estimates on the embedding probabilities. In Section 2.8 we prove the recursive length estimates on the blocks. In Section 2.9 we show that good blocks have the required inductive properties. Finally in Section 2.10 we describe how these results yield deterministic sequences with positive probabilities of M-Lipschitz embedding or being a compatible sequence.

2.2 Applications to Lipschitz Embeddings, Rough Isometries and Compatible Sequences

In this section we show how Theorem 2.4 can be used to derive our three main results. Notice that since Theorem 2.4 does not require $\mathbb{X}$ and $\mathbb{Y}$ to be independent. Hence all the three results shall remain valid even if we drop the assumption of independence between the two sequences.

2.2.1 Lipschitz Embeddings

Defining the sequences $\mathbb{X}$ and $\mathbb{Y}$ and the alphabets $\mathcal{C}^{\mathbb{X}}$ and $\mathcal{C}^{\mathbb{Y}}$

Let $X^* = \{X_i^*\}_{i\geq 1}$ and $Y^* = \{Y_i^*\}_{i\geq 1}$ be two independent sequences of i.i.d. Ber($\frac{1}{2}$) variables. Let M_0 be a large constant which will be chosen later. Let $\widetilde{Y^*} = \{\widetilde{Y_i^*}\}$ be the sequence given by $\widetilde{Y_i^*} = Y^{*[(i-1)M_0+1, iM_0]}$. Now let us divide the $\{0,1\}$ sequences of length M_0 in the following 3 classes.

1. Class $\star$. Let $Z = (Z_1, Z_2, \cdots, Z_{M_0})$ be a sequence of 0's and 1's. A length 2-subsequence (Z_i, Z_{i+1}) is called a "flip" if $Z_i \neq Z_{i+1}$. We say $Z \in \star$ if the number of flips in Z is at least $2R_0^+$.
2. Class $\mathbf{0}$. If $Z = (Z_1, Z_2, \cdots, Z_{M_0}) \notin \star$ and Z contains more 0's than 1's, then $Z \in \mathbf{0}$.
3. Class $\mathbf{1}$. If $Z = (Z_1, Z_2, \cdots, Z_{M_0}) \notin \star$ and Z contains more 1's than 0's, then $Z \in \mathbf{1}$. For definiteness, let us also say $Z \in \mathbf{1}$, if Z contains equal number of 0's and 1's and $Z \notin \star$.

Now set $\mathbb{X} = (X_1, X_2, \ldots) = X^*$ and construct $\mathbb{Y} = (Y_1, Y_2, \ldots)$ from $\widetilde{Y^*}$ as follows. Set $Y_i = \mathbf{0}, \mathbf{1}$ or $\star$ according as whether $\widetilde{Y_i^*} \in \mathbf{0}, \mathbf{1}$ or $\star$.

It is clear from this definition that $\mathbb{X} = (X_1, X_2, \ldots)$ and $\mathbb{Y} = (Y_1, Y_2, \ldots)$ are two independent sequences of i.i.d. symbols coming from the alphabets $\mathcal{C}^{\mathbb{X}}$ and $\mathcal{C}^{\mathbb{Y}}$ having distributions $\mu^{\mathbb{X}}$ and $\mu^{\mathbb{Y}}$ respectively where

$$\mathcal{C}^{\mathbb{X}} = \{0,1\}, \mathcal{C}^{\mathbb{Y}} = \{\mathbf{0}, \mathbf{1}, \star\}.$$

We take $\mu^{\mathbb{X}}$ to be the uniform measure on $\{0,1\}$ and $\mu^{\mathbb{Y}}$ to be the natural measure on $\{\mathbf{0}, \mathbf{1}, \star\}$ induced by the independent Ber($\frac{1}{2}$) variables.

We take the relation $\mathcal{R} \subseteq \mathcal{C}^{\mathbb{X}} \times \mathcal{C}^{\mathbb{Y}}$ to be: $\{0 \hookrightarrow \mathbf{0}, 0 \hookrightarrow \star, 1 \hookrightarrow \mathbf{1}, 1 \hookrightarrow \star\}$ and the good sets $G_0^{\mathbb{X}} = \{0,1\}$ and $G_0^{\mathbb{Y}} = \{\star\}$.

It is now very easy to verify that $\mathcal{C}^{\mathbb{X}}, \mathcal{C}^{\mathbb{Y}}, \mu^{\mathbb{X}}, \mu^{\mathbb{Y}}, \mathcal{R}, G_0^{\mathbb{X}}, G_0^{\mathbb{Y}}$, as defined above satisfies all the conditions described in our abstract framework.

Constructing the Lipschitz Embedding

Now we verify that the the sequences $\mathbb{X}$ and $\mathbb{Y}$ constructed from the binary sequences X^* and Y^* can be used to construct an embedding with positive probability. Note that though we constructed the sequences $\mathbb{X}$ and $\mathbb{Y}$ from i.i.d. Ber($\frac{1}{2}$) sequences X^* and Y^* in the previous subsection, the construction is deterministic and hence can be carried out for any binary sequence. We have the following lemma.

Lemma 2.2.1. *Let $X^* = \{X_i^*\}_{i\geq 1}$ and $Y^* = \{Y_i^*\}_{i\geq 1}$ be two binary sequences. Let $\mathbb{X}$ and $\mathbb{Y}$ be the sequences constructed from X^* and Y^* as above. There is a constant M, such that whenever $\mathbb{X} \hookrightarrow \mathbb{Y}$, there exists a strictly increasing map $\phi : \mathbb{N} \to \mathbb{N}$ such that for all $i, j \in \mathbb{N}$, $X_i = Y_{\phi(i)}$ and $|\phi(i) - \phi(j)| \leq M|i - j|$, $\phi(1) < M/2$.*

Before proceeding with the proof, let us make the following notation. We say $X^* \hookrightarrow_{*M} Y^*$ if a map ϕ satisfying the conditions of the lemma exists. Let us also make the following definition for finite subsequences.

Definition 2.2.2. *Let $X^{*[i_1,i_2]}$ and $Y^{*[i'_1,i'_2]}$ be two segments of X^* and Y^* respectively. We say that $X^{*[i_1,i_2]} \hookrightarrow_{*M} Y^{*[i'_1,i'_2]}$ if there exists a strictly increasing $\widetilde{\phi} : \{i_1, i_1 + 1, \ldots, i_2\} \to \{i'_1, i'_1 + 1, \ldots, i'_2\}$ such that*

(i) $X_k = Y_{\widetilde{\phi}(k)}$ and $k, l \in \{i_1, i_1 + 1, \ldots, i_2\}$ implies $|\phi(k) - \phi(l)| \leq M|k - l|$.

(ii) $\widetilde{\phi}(i_1) - i'_1 \leq M/3$ and $i'_2 - \widetilde{\phi}(i_2) \leq M/3$.

In what follows, we shall always be taking $M \geq 6$. The following observation is trivial.

Observation 2.2.3. *Let $0 = i_0 < i_2 < \ldots$ and $0 = i'_0 < i'_2 < \ldots$ be two increasing sequences of integers. If $X^{*[i_k+1,i_{k+1}]} \hookrightarrow_{*M} Y^{*[i'_k+1,i'_{k+1}]}$ for each $k \geq 0$, then $X^* \hookrightarrow_{*M} Y^*$.*

Proof of Lemma 2.2.1. Let $X^*, Y^*, \mathbb{X}, \mathbb{Y}$ be as in the statement of the Lemma. Let $\mathbb{X} \hookrightarrow \mathbb{Y}$. Let $0 = i_0 < i_1 < i_2 < \ldots$ and $0 = i'_0 < i'_1 < i'_2 < \ldots$ be the two sequences obtained from Definition 2.1.1. The previous observation then implies that it suffices to prove that there exists M (not depending on X^* and Y^*) such that for all $h \geq 0$,

$$X^{*[i_h+1,i_{h+1}]} \hookrightarrow_{*M} Y^{*[i'_h M_0+1,i'_{h+1}M_0]}.$$

Notice that since $\{i_{h+1} - i_h\}$ and $\{i'_{h+1} - i'_h\}$ are bounded sequences, if we can find maps $\phi_h : \{i_h + 1, \ldots, i_{h+1}\} \to \{i'_h M_0 + 1, \ldots, i'_{h+1} M_0\}$ such that $X_i^* = Y_{\phi_h(i)}^*$, then for sufficiently large M and for all h we shall have $X^{*[i_h+1,i_{h+1}]} \hookrightarrow_{*M} Y^{*[i'_h M_0+1,i'_{h+1}M_0]}$. We shall call such a ϕ_h an embedding.

There are three cases to consider.

Case 1: $i_{h+1} - i_h = i'_{h+1} - i'_h = 1$. By hypothesis, this implies $X_{i_h+1} \hookrightarrow Y_{i'_h+1}$. If $X^*_{i_h+1} = 0$ and $Y^{*[i'_h M_0+1, i'_h M_0+M_0]} \in \{\mathbf{0}, \star\}$, then $Y^{*[i'_h M_0+1, i'_h M_0+M_0]}$ must contain at least one 0 and hence an embedding exists. Similarly if $X^*_{i_h+1} = 1$ and $Y^{*[i'_h M_0+1, i'_h M_0+M_0]} \in \{\mathbf{1}, \star\}$ then also an embedding exists.

Case 2: $i'_{h+1} - i'_h = R_0, R_0^- \leq i_{h+1} - i_h \leq R_0^+$. In this case, $Y^{[i'_h+1, i'_{h+1}]}$ is a "good" segment, i.e., $Y^{*[(i'_h+k)M_0+1, (i'_h+k+1)M_0]} \in \star$, for $0 \leq k \leq i'_{h+1} - i'_h - 1$. By what we have already observed it now suffices to only consider the case $i_{h+1} - i_h = R_0^+$. Now by definition of $\star$, there exist an alternating sub-sequence of $2R_0^+$ 0's and $2R_0^+$ 1's in $Y^{*[i'_h M_0+1, (i'_h+1)M_0]}$. It follows that there is an embedding in this case also.

Case 3: $i_{h+1} - i_h = R_0, R_0^- \leq i'_{h+1} - i'_h \leq R_0^+$. Similarly as in Case 2, there exists an embedding in this case as well, we omit the details. □

Proof of Theorem 2.1

We now complete the proof of Theorem 2.1 by using Theorem 2.4.

Proof of Theorem 2.1. Let $\mathcal{C}^{\mathbb{X}}, \mu^{\mathbb{X}}, \mathcal{C}^{\mathbb{Y}}, \mu^{\mathbb{Y}}$ be as described above. Let $X \sim \mu^{\mathbb{X}}$, $Y \sim \mu^{\mathbb{Y}}$. (Notice that $\mu^{\mathbb{Y}}$ implicitly depends on the choice of M_0). Notice that (2.1.1) holds trivially if $L_0 \geq 3$. Let β, δ, m, R, L_0 be given by Theorem 2.4. First we show that there exists M_0 such that (2.1.2) and (2.1.3) hold.

Let $Z = (Z_1, Z_2, \ldots, Z_{M_0})$ be a sequence of i.i.d. Ber($\frac{1}{2}$) variables. Observe that

$$\mathbb{P}(Z \in \star) \geq (1 - 2^{1 - \lfloor \frac{M_0}{2R_0^+} \rfloor})^{2R_0^+} \to 1 \text{ as } M_0 \to \infty.$$

Hence we can choose M_0 large enough such that

$$\mu^{\mathbb{Y}}(\star) \geq \max\{1 - L_0^{-\delta}, 1 - 2^{-(m+1)} L_0^{-\beta}\}. \tag{2.2.1}$$

Since all $\mathbb{X}$ blocks are good and $\star$ is a good $\mathbb{Y}$ block, $\mathbb{P}(X \in G_0^{\mathbb{X}}) = 1$ and $\mathbb{P}(Y \in G_0^{\mathbb{Y}}) \geq \mu^{\mathbb{Y}}(\star) \geq 1 - L_0^{-\delta}$ and hence 2.1.2 holds. For (2.1.3), notice that $S_0^{\mathbb{X}}(X) > 1 - L_0^{-1}$ for all X and $S_0^{\mathbb{Y}}(Y) \geq \frac{1}{2}$ for all Y. Hence $\mathbb{P}(S_0^{\mathbb{Y}}(Y) \leq p) = 0$ if $p \leq \frac{1}{2}$. For $1 - L_0^{-1} \geq p \geq \frac{1}{2}$,

$$\mathbb{P}(S_0^Y(Y) \leq p) \leq \mathbb{P}(Y \neq \star) \leq (\frac{1}{2})^{m+1} L_0^{-\beta} \leq p^{m+1} L_0^{-\beta},$$

and hence (2.1.3) holds.

Now let $X^* = \{X_i^*\}_{i \geq 1}$ and $Y^* = \{Y_i^*\}_{i \geq 1}$ be two independent sequences of i.i.d. $Ber(\frac{1}{2})$ variables. Choosing M_0 as above, construct $\mathbb{X}$, $\mathbb{Y}$ as described in the previous subsection. Then by Theorem 2.4, we have that $\mathbb{P}(\mathbb{X} \hookrightarrow_R \mathbb{Y}) > 0$. Using Lemma 2.2.1 it now follows that for M sufficiently large, we have $\mathbb{P}(X^* \hookrightarrow_{*M} Y^*) > 0$. This gives an embedding for sequences indexed by the natural numbers which can easily be extended to embedding of sequences indexed by the full integers with positive probability. To see that this has probability 1 we note that the event that there exists an embedding is shift invariant and i.i.d. sequences are ergodic with respect to shifts and hence it has probability 0 or 1 completing the proof. □

2.2.2 Rough Isometry

Proposition 2.1 and 2.2 of [35] showed that to show that there exists (M, D, C) such that two Poisson processes on $\mathbb{R}$ are (M, D, C)-roughly isometric almost surely it is sufficient to show that two independent copies of Bernoulli percolation on $\mathbb{Z}$ with parameter $\frac{1}{2}$, viewed as subsets of $\mathbb{R}$, are (M', D', C')-roughly isometric with positive probability for some (M', D', C'). We will solve the percolation problem and thus infer Theorem 2.2.

Defining the sequences $\mathbb{X}$ and $\mathbb{Y}$ and the alphabets $\mathcal{C}^{\mathbb{X}}$ and $\mathcal{C}^{\mathbb{Y}}$

Let $X^* = \{X_i^*\}_{i\geq 0}$ and $Y^* = \{Y_i^*\}_{i\geq 0}$ be two independent sequences of i.i.d. Ber$(\frac{1}{2})$ variables conditioned so that $X_0^* = Y_0^* = 1$. Now let us define two sequences $k_0 < k_1 < k_2 < \ldots$ and $k'_0 < k'_1 < k'_2 < \ldots$ as follows. Let $k_0 = 0$ and $k_{i+1} = \min_{r>k_i} X_r^* = 1$. Similarly let $k'_0 = 0$ and $k'_{i+1} = \min_{r>k'_i} Y_r^* = 1$. Let $\widetilde{X_i^*} = X^{*[k_{i-1},k_i-1]}$ and $\widetilde{Y_i^*} = Y^{*[k'_{i-1},k'_i-1]}$. The elements of the sequences $\{\widetilde{X_i^*}\}_{i\geq 1}$ and $\{\widetilde{Y_i^*}\}_{i\geq 1}$ are sequences consisting of a single 1 followed by a number (possibly none) of 0's. We now divide such sequences into the following classes.

Let $Z = (Z_0, Z_1, \ldots Z_L), (L \geq 0)$ be a sequence of 0's and 1's with $Z_0 = 1$ and $Z_i = 0$ for $0 < i \leq L$. We say that $Z \in C_0$ if $L = 0$ and for $j \geq 1$, we say $Z \in C_j$ if $2^{j-1} \leq L < 2^j$.

Now construct $\mathbb{X} = (X_1, X_2, \ldots)$ from $\widetilde{X^*}$ and $\mathbb{Y} = (Y_1, Y_2, \ldots)$ from $\widetilde{Y^*}$ as follows. Set $X_i = C_j$ if $\widetilde{X_i^*} \in C_j$. Similarly set $Y_i = C_j$ if $\widetilde{Y_i^*} \in C_j$.

It is clear from this definition that $\mathbb{X} = (X_1, X_2, \ldots)$ and $\mathbb{Y} = (Y_1, Y_2, \ldots)$ are two sequences of i.i.d. symbols coming from the alphabets $\mathcal{C}^{\mathbb{X}}$ and $\mathcal{C}^{\mathbb{Y}}$ having distributions $\mu^{\mathbb{X}}$ and $\mu^{\mathbb{Y}}$ respectively where

$$\mathcal{C}^{\mathbb{X}} = \mathcal{C}^{\mathbb{Y}} = \{C_0, C_1, C_2, \ldots\}$$

and $\mu^{\mathbb{X}} = \mu^{\mathbb{Y}}$ is given by

$$\mu^{\mathbb{X}}(\{C_j\}) = \mu^{\mathbb{Y}}(\{C_j\}) = \mathbb{P}(Z \in C_j)$$

where $Z = Z^{*[0,i-1]}$, $Z_0^* = 0$, Z_t^* are of i.i.d. Ber$(\frac{1}{2})$ variables for $t \geq 1$ and $i = \min\{k > 0 : Z_k^* = 1\}$.

We take the relation $\mathcal{R} \subseteq \mathcal{C}^{\mathbb{X}} \times \mathcal{C}^{\mathbb{Y}}$ to be: $C_k \hookrightarrow C_{k'}$ if $|k - k'| \leq M_0$. The "good" sets are defined to be $G_0^{\mathbb{X}} = G_0^{\mathbb{Y}} = \{C_j : j \leq M_0\}$. It is now very easy to verify that $\mathcal{C}^{\mathbb{X}}, \mathcal{C}^{\mathbb{Y}}, \mu^{\mathbb{X}}, \mu^{\mathbb{Y}}, \mathcal{R}, G_0^{\mathbb{X}}, G_0^{\mathbb{Y}}$, as defined above satisfy all the conditions described in our abstract framework.

Existence of the Rough Isometry

Lemma 2.2.4. *Let $X^* = \{X_i^*\}_{i\geq 0}$ and $Y^* = \{Y_i^*\}_{i\geq 0}$ be two binary sequences with $X_0^* = Y_0^* = 1$. Let $N_{X^*} = \{i : X_i^* = 1\}$ and $N_{Y^*} = \{i : Y_i^* = 1\}$. Let $\mathbb{X}$ and $\mathbb{Y}$ be the sequences* *constructed from X^* and Y^* as above. Then there exist constants (M', D', C'), such that whenever $\mathbb{X} \hookrightarrow_R \mathbb{Y}$, there exists $\phi : N_{X^*} \to N_{Y^*}$ such that $\phi(0) = 0$ and*

(i) For all $t, s \in N_{X^*}$,

$$\frac{1}{M'}|t-s| - D' \leq |\phi(t) - \phi(s)| \leq M'|t-s| + D'.$$

(ii) For all $t \in N_{Y^*}$, $\exists s \in N_{X^*}$ such that $|t - \phi(s)| \leq C'$.

That is, $\mathbb{X} \hookrightarrow_R \mathbb{Y}$ implies N_{X^*} and N_{Y^*} (viewed as subsets of $\mathbb{R}$) are (M', D', C')-roughly isometric.

Proof. Suppose that $\mathbb{X} \hookrightarrow \mathbb{Y}$ and let $0 = i_0 < i_1 < i_2 < \dots$ and $0 = i'_0 < i'_1 < i'_2 < \dots$ be the two sequences satisfying the conditions of Definition 2.1.1. Let $0 = k_0 < k_1 < k_2 < \dots$ and $0 = k'_0 < k'_1 < k'_2 < \dots$ be the sequences described in the previous subsection while defining $\mathbb{X}$ and $\mathbb{Y}$. For $r \geq 1$, define $X_r^{**} = X^{*[k_{i_{r-1}}, k_{i_r}-1]}$ and $Y_r^{**} = Y^{*\left[k'_{i'_{r-1}}, k'_{i'_r}-1\right]}$, i.e., X_r^{**} is the segment of X^* corresponding to $\mathbb{X}^{[i_{r-1}+1, i_r]}$ and Y_r^{**} is the segment of Y^* corresponding to $\mathbb{Y}^{[i'_{r-1}+1, i'_r]}$ Define $N_{X,r} = N_{X^*} \cap [k_{i_{r-1}}, k_{i_r} - 1]$ and $N_{Y,r} = N_{Y^*} \cap [k'_{i'_{r-1}}, k'_{i'_r} - 1]$. Notice that by construction, for each r, $X^*_{k_{i_{r-1}}} = 1$ and $Y^*_{k'_{i'_{r-1}}} = 1$, i.e., $k_{i_{r-1}} \in N_{X,r} \subseteq N_{X^*}$ and $k'_{i'_{r-1}} \in N_{Y,r} \subseteq N_{Y^*}$.

Now let us define $\phi : N_{X^*} \to N_{Y^*}$ as follows. If $s \in N_{X,r}$, define $\phi(s) = k'_{i'_{r-1}}$. Clearly $\phi(0) = 0$. We show now that for $M' = 2^{M_0+2}R_0^+$, $D' = 2^{2M_0+3}(R_0^+)^2$ and $C' = 2^{M_0+1}R_0^+$, the map defined as above satisfies the conditions in the statement of the lemma.

Proof of (i). First consider the case where $s, t \in N_{X,r}$ for some r. If $s \neq t$ then clearly $\mathbb{X}^{[i_{r-1}+1, i_r]}$ is a good segment and hence $|s-t| \leq 2^{M_0}R_0^+$. Clearly $|\phi(s) - \phi(t)| = 0$. It follows that for the specified choice of M' and D',

$$\frac{1}{M'}|t-s| - D' \leq |\phi(t) - \phi(s)| \leq M'|t-s| + D'.$$

Let us now consider the case $s \in N_{X,r_1}$, $t \in N_{X,r_2}$ where $r_1 < r_2$. Clearly then $k_{i_{r_1-1}} \leq s < k_{i_{r_1}}$ and $k_{i_{r_2-1}} \leq t < k_{i_{r_2}}$. Also notice that by choice of D', for any good segment $\mathbb{X}^{[i_h+1, i_{h+1}]}$ we must have $|k_{i_{h+1}} - k_{i_h}| \leq 2^{M_0}R_0^+ \leq \frac{D'}{2M'}$. Further if for some h, $i_{h+1} = i_h + 1$, we must have that $|N_{X,h+1}| = 1$. It follows that $s \leq k_{i_{r_1-1}} + \frac{D'}{M'}$ and $t \leq k_{i_{r_2-1}} + \frac{D'}{M'}$. It is clear from the definitions that $\phi(s) = k'_{i'_{r_1-1}}$ and $\phi(t) = k'_{i'_{r_2-1}}$.

Then we have,

$$|\phi(t) - \phi(s)| = \sum_{h=r_1}^{r_2-1} |k'_{i'_h} - k'_{i'_{h-1}}|$$

and

$$\sum_{h=r_1}^{r_2-1} |k_{i_h} - k_{i_{h-1}}| - \frac{D'}{M'} \leq |t-s| \leq \sum_{h=r_1}^{r_2-1} |k_{i_h} - k_{i_{h-1}}| + \frac{D'}{M'}.$$

It now follows from the definitions that for each h,

$$\frac{1}{M'}|k_{i_h} - k_{i_{h-1}}| \leq |k'_{i'_h} - k'_{i'_{h-1}}| \leq M'|k_{i_h} - k_{i_{h-1}}|.$$

Adding this over $h = r_1, \ldots, r_2 - 1$, we get that

$$\frac{1}{M'}|t - s| - D' \leq |\phi(t) - \phi(s)| \leq M'|t - s| + D'$$

in this case as well, which completes the proof of (i).

Proof of (ii). Let $t \in N_{Y^*}$ and let r be such that $k'_{i'_r} \leq t < k'_{i'_{r+1}}$. Now if $i'_{r+1} - i'_r = 1$ we must have $t = k'_{i'_r}$ and hence $t = \phi(s)$ where $s = k_{i_r} \in N_{X^*}$ and hence (ii) holds for t. If $i'_{r+1} - i'_r \neq 1$ we must have that $\mathbb{Y}^{[i_r+1, i_{r+1}]}$ is a good segment and hence $k'_{i'_{r+1}} - k'_{i'_r} \leq 2^{M_0} R_0^+$. Setting $s = k_{i_r} \in N_{X^*}$ we see that $\phi(s) = k'_{i'_r}$ and hence $|t - \phi(s)| \leq 2^{M_0} R_0^+ \leq C'$, completing the proof of (ii). □

Proof of Theorem 2.2

Now we prove Theorem 2.2 using Theorem 2.4.

Proof of Theorem 2.2. Let $\mathcal{C}^{\mathbb{X}}, \mu^{\mathbb{X}}, \mathcal{C}^{\mathbb{Y}}, \mu^{\mathbb{Y}}$ be as described above. Let $X \sim \mu^{\mathbb{X}}$, $Y \sim \mu^{\mathbb{Y}}$. Notice first that

$$\mu^{\mathbb{X}}(C_0) = \mu^{\mathbb{Y}}(C_0) = \frac{1}{2} \text{ and } \mu^{\mathbb{X}}(C_j) = \mu^{\mathbb{Y}}(C_j) = (\frac{1}{2})^{2^{j-1}} - (\frac{1}{2})^{2^j}$$

for $j \geq 1$, hence (2.1.1) is satisfied for for all k. We first show that there exists M_0 such that (2.1.2) and (2.1.3) hold if β, δ, m, R and L_0 are constants such that the conclusion of Theorem 2.4 holds.

First observe that everything is symmetric in $\mathbb{X}$ and $\mathbb{Y}$. Clearly, we can take M_0 sufficiently large so that $S_0^{\mathbb{X}}(X) \geq 1 - L_0^{-1}$ for $X = C_k$ for all $k \leq M_0$.

Now suppose $X = C_k$, where $k > M_0$.

$$S_0^{\mathbb{X}}(X) = \sum_{k':|k'-k| \leq M_0} \mu^{\mathbb{Y}}(C'_k) = (\frac{1}{2})^{2^{k-M_0-1}} - (\frac{1}{2})^{2^{k+M_0}}.$$

Let us fix $p \leq 1 - L_0^{-1}$. Then we have

$$\begin{aligned}
\mathbb{P}(S_0^{\mathbb{X}}(X) \leq p) &\leq \sum_{k>M_0} \mu^{\mathbb{X}}(C_k) I((\frac{1}{2})^{2^{k-M_0-1}} - (\frac{1}{2})^{2^{k+M_0}} \leq p) \\
&\leq \sum_{k>M_0} \mu^{\mathbb{X}}(C_k) I((\frac{1}{2})^{2^{k-M_0}} \leq p) \\
&\leq \sum_{k\geq M_0+\log_2(-\log_2 p)} (\frac{1}{2})^{2^{k-1}} - (\frac{1}{2})^{2^k} \\
&\leq (\frac{1}{2})^{2^{M_0+\log_2(-\log_2 p)-1}} (\frac{1}{2})^{2^{M_0-1}(-\log_2 p)} \\
&= p^{2^{M_0-1}} \leq p^{m+1}(1-L_0^{-1})^{2^{M_0-1}-m-1} \leq p^{m+1} L_0^{-\beta}
\end{aligned}$$

for M_0 sufficiently large. Also, since $\sum_k \mu^{\mathbb{X}}(C_k) = 1$, by choosing M_0 sufficiently large we can make $\sum_{k\leq M_0} \mu^{\mathbb{X}}(C_k) \geq 1 - L_0^{-\delta}$.

Hence there exist some constant M_0 for which both (2.1.2) and (2.1.3) hold. This together with Lemma 2.2.4 and Theorem 2.4 implies a rough isometry with positive probability for two independent copies of site percolation on $\mathbb{N}\cup\{0\}$ and hence on $\mathbb{Z}$. The comments at the beginning of this subsection then show that the conditional results of [35] extend this result to Poisson processes on $\mathbb{R}$ proving Theorem 2.2. □

2.2.3 Compatible Sequences

Defining the alphabets $\mathcal{C}^{\mathbb{X}}$ and $\mathcal{C}^{\mathbb{Y}}$

Let $\mathbb{X} = \{X_i\}_{i\geq 1}$ and $\mathbb{Y} = \{Y_i\}_{i\geq 1}$ be two independent sequences of i.i.d. Ber(q) variables. Let us take $\mathcal{C}^{\mathbb{X}} = \mathcal{C}^{\mathbb{Y}} = \{0, 1\}$. The measures $\mu^{\mathbb{X}}$ and $\mu^{\mathbb{Y}}$ are induced by the distribution of X_i's and Y_i's, i.e., $\mu^{\mathbb{X}}(\{1\}) = \mu^{\mathbb{Y}}(\{1\}) = q$ and $\mu^{\mathbb{X}}(\{0\}) = \mu^{\mathbb{Y}}(\{0\}) = 1 - q$. It is then clear that $\mathbb{X}$ and $\mathbb{Y}$ are two independent sequences of i.i.d. symbols coming from the alphabets $\mathcal{C}^{\mathbb{X}}$ and $\mathcal{C}^{\mathbb{Y}}$ having distributions $\mu^{\mathbb{X}}$ and $\mu^{\mathbb{Y}}$ respectively.

We define the relation $\mathcal{R} \subseteq \mathcal{C}^{\mathbb{X}} \times \mathcal{C}^{\mathbb{Y}}$ by

$$\{0 \hookrightarrow 0, 0 \hookrightarrow 1, 1 \hookrightarrow 0\}.$$

Finally the "good" symbols are defined by $G_0^{\mathbb{X}} = G_0^{\mathbb{Y}} = \{0\}$. It is clear that all the conditions in the definition of our set-up is satisfied by this structure.

Existence of the compatible map

Lemma 2.2.5. *Let $\mathbb{X} = \{X_i\}_{i\geq 1}$ and $\mathbb{Y} = \{Y_i\}_{i\geq 1}$ be two binary sequences. Suppose $\mathbb{X} \hookrightarrow \mathbb{Y}$. Then there exist $D, D' \subseteq \mathbb{N}$ such that,*

(i) For all $i \in D$, $X_i = 0$, for all $i' \in D'$, $Y_{i'} = 0$.

(ii) *Let* $\mathbb{N} - D = k_1 < k_2 < \dots$ *and* $\mathbb{N} - D' = k'_1 < k'_2 < \dots$. *Then for each* i, $X_{k_i} \neq Y_{k'_i}$ *and hence* $X_{k_i} Y_{k'_i} = 0$.

Proof. The sets D and D' denote the set of sites we will delete. Let $0 = i_0 < i_1 < i_2 < \dots$ and $0 = i'_0 < i'_1 < i'_2 < \dots$ be the sequences satisfying the properties listed in Definition 2.1.1. Let $H_1^* = \{h : i_{h+1} - i_h = R_0\}$, $H_2^* = \{h : i'_{h+1} - i'_h = R_0\}$, $H_3^* = \{h : i_{h+1} - i_h = i'_{h+1} - i'_h = 1, X_{i_h+1} = Y_{i'_h+1} = 0\}$. Let $H^* = \cup_{i=1}^{3} H_i^*$. Now define

$$D = \bigcup_{h \in H^*} [i_h + 1, i_{h+1}] \cap \mathbb{N}, \;\; D' = \bigcup_{h \in H^*} [i'_h + 1, i'_{h+1}] \cap \mathbb{N}. \tag{2.2.2}$$

It is clear from Definition 2.1.1 that D, D' defined as above satisfies the conditions in the statement of the lemma. □

Proof of Theorem 2.3

Now we complete proof of Theorem 2.3 using Theorem 2.4.

Proof of Theorem 2.3. Let $\mathcal{C}^{\mathbb{X}}, \mu^{\mathbb{X}}, \mathcal{C}^{\mathbb{Y}}, \mu^{\mathbb{Y}}$ be as described above. Let $X \sim \mu^{\mathbb{X}}$, $Y \sim \mu^{\mathbb{Y}}$. (Notice that $\mu^{\mathbb{X}}$, $\mu^{\mathbb{Y}}$ implicitly depend on q) Let β, δ, m, R, L_0 be constants such that the conclusion of Theorem 2.4 holds. Take $q_0 = L_0^{-\delta}$. Let $q \leq q_0$. Clearly, then, for any $X \in \mathcal{C}^{\mathbb{X}}$ (resp. for any $Y \in \mathcal{C}^{\mathbb{Y}}$) we have $S_0^{\mathbb{X}}(X) \geq 1 - q \geq 1 - L_0^{-1}$ (resp. $S_0^{\mathbb{Y}}(Y) \geq 1 - L_0^{-1}$). Hence, (2.1.2) is vacuously satisfied. That (2.1.3) holds follows directly from the definitions. Notice that since the alphabet sets are finite (2.1.1) trivially holds. Theorem 2.3 now follows from Lemma 2.2.5 and Theorem 2.4. □

2.3 The Multi-scale Structure

Let $\mathbb{X}$, $\mathbb{Y}$, $\mathcal{C}^{\mathbb{X}}, \mathcal{C}^{\mathbb{Y}}, G_0^{\mathbb{X}}, G_0^{\mathbb{Y}}$ be as described in the previous section. As we have described in § 2.1.2 before, our strategy of proof of Theorem 2.4 is to partition the sequences $\mathbb{X}$ and $\mathbb{Y}$ into blocks at each level $j \geq 1$. Because of the symmetry between $\mathbb{X}$ and $\mathbb{Y}$ we only describe the procedure to form the blocks for the sequence $\mathbb{X}$. For each $j \geq 1$, we write $\mathbb{X} = (X_1^{(j)}, X_2^{(j)}, \dots)$ where we call each $X_i^{(j)}$ a level j $\mathbb{X}$-block. Most of the time we would clearly state that something is a level j block and drop the superscript j. Each of the $\mathbb{X}$-block at level j is a concatenation of a number of level $(j-1)$ $\mathbb{X}$-blocks, where level 0 blocks are just the characters of the sequence $\mathbb{X}$. At each level, we also have a recursive definition of "*good*" blocks. Let $G_j^{\mathbb{X}}$ and $G_j^{\mathbb{Y}}$ denote the set of good $\mathbb{X}$-blocks and good $\mathbb{Y}$-blocks at j-th level respectively. Now we are ready to describe the recursive construction of the blocks $X_i^{(j)}$. for $j \geq 1$.

2.3.1 Recursive Construction of Blocks

We only describe the construction for $\mathbb{X}$. Let us suppose we have already constructed the blocks of partition upto level j for some $j \geq 0$ and we have $X = (X_1^{(j)}, X_2^{(j)}, \ldots)$. Also assume we have defined the "good" blocks at level j, i.e., we know $G_j^{\mathbb{X}}$. We can start off the recursion since both these assumptions hold for $j = 0$. We describe how to partition $\mathbb{X}$ into level $(j+1)$ blocks: $\mathbb{X} = (X_1^{(j+1)}, X_2^{(j+2)}, \ldots)$.

Recall that $L_{j+1} = L_j^{\alpha} = L_0^{\alpha^{j+1}}$. Suppose the first k ($k \geq 0$) blocks $X_1^{(j+1)}, \ldots, X_k^{(j+1)}$ at level $(j+1)$ has already been constructed and suppose that the rightmost level j-subblock of $X_k^{(j+1)}$ is $X_m^{(j)}$. Then $X_{k+1}^{(j+1)}$ consists of the sub-blocks $X_{m+1}^{(j)}, X_{m+2}^{(j)}, \ldots, X_{m+l+L_j^3}^{(j)}$ where $l > L_j^3 + L_j^{\alpha-1}$ is selected in the following manner. Let $W_{k+1,j+1}$ be a geometric random variable having $\text{Geom}(L_j^{-4})$ distribution and independent of everything else. Then

$$l = \min\{s \geq L_j^3 + L_j^{\alpha-1} + W_{k+1,j+1} : X_{m+s+i} \in G_j^{\mathbb{X}} \text{ for } 1 \leq i \leq 2L_j^3\}.$$

That such an l is finite with probability 1 will follow from our recursive estimates.

Put simply, our block construction mechanism at level $(j+1)$ is as follows:
Starting from the right boundary of the previous block, we include L_j^3 many sub-blocks, then further $L_j^{\alpha-1}$ many sub-blocks, then a Geom(L_j^{-4}) many sub-blocks. Then we wait for the first occurrence of a run of $2L_j^3$ many consecutive good sub-blocks, and end our block at the midpoint of this run.

This somewhat complex choice of block structure is made for several reasons. It guarantees stretches of good sub-blocks at both ends of the block thus ensuring these are not problematic when trying to embed one block into another. The fact that good blocks can be mapped into shorter or longer stretches of good blocks then allows us to line up sub-blocks in a potential embedding in many possible ways which is crucial for the induction. Our blocks are not of fixed length. It is potentially problematic to our approach if conditional on a block being long that it contains many bad blocks. Thus we added the geometric term to the length. This has the effect that given that the block is long, it is most likely because the geometric random variable is large, not because of the presence of many bad blocks. Finally, the construction means that block will be independent.

We now record two simple but useful properties of the blocks thus constructed in the following observation. Once again a similar statement holds for $\mathbb{Y}$-blocks.

Observation 2.3.1. *Let $\mathbb{X} = (X_1^{(j+1)}, X_2^{(j+1)}, \ldots) = (X_1^{(j)}, X_2^{(j)}, \ldots)$ denote the partition of $\mathbb{X}$ into blocks at levels $(j+1)$ and j respectively. Then the following hold.*

1. *Let $X_i^{(j+1)} = (X_{i_1}^{(j)}, X_{i_1+1}^{(j)}, \ldots X_{i_1+l}^{(j)})$. For $i \geq 1$, $X_{i_1+l+1-k}^{(j)} \in G_j^{\mathbb{X}}$ for each k, $1 \leq k \leq L_j^3$. Further, if $i > 1$, then $X_{i_1+k-1}^{(j)} \in G_j^{\mathbb{X}}$ for each k, $1 \leq k \leq L_j^3$. That is, all blocks at level $(j+1)$, except possibly the leftmost one ($X_1^{(j+1)}$), are guaranteed to have at least L_j^3 "good" level j sub-blocks at either end. Even $X_1^{(j+1)}$ ends in L_j^3 many good sub-blocks.*

2. *The blocks* $X_1^{(j+1)}, X_2^{(j+1)}, \ldots$ *are independently distributed. In fact,* $X_2^{(j+1)}, X_3^{(j+1)}, \ldots$ *are independently and identically distributed according to some law, say* $\mu_{j+1}^{\mathbb{X}}$. *Furthermore, conditional on the event* $\{X_i^{(k)} \in G_k^{\mathbb{X}}$ *for* $i = 1, 2, \ldots, L_k^3,$ *for all* $k \leq j\}$, *the* $(j+1)$*-th level blocks* $X_1^{(j+1)}, X_2^{(j+1)}, \ldots$ *are independently and identically distributed according to the law* $\mu_{j+1}^{\mathbb{X}}$.

From now on whenever we say "a (random) $\mathbb{X}$-block at level j", we would imply that it has law $\mu_j^{\mathbb{X}}$, unless explicitly stated otherwise. Similarly let us denote the corresponding law of "a (random) $\mathbb{Y}$-block at level j" by $\mu_j^{\mathbb{Y}}$. Also for convenience, we assume $\mu_0^{\mathbb{X}} = \mu^{\mathbb{X}}$ and $\mu_0^{\mathbb{Y}} = \mu^{\mathbb{Y}}$.

Also, for $j \geq 0$, let $\mu_{j,G}^{\mathbb{X}}$ denote the conditional law of an $\mathbb{X}$ block at level j, given that it is in $G_j^{\mathbb{X}}$. We define $\mu_{j,G}^{\mathbb{Y}}$ similarly.

We observe that we can construct a block with law $\mu_{j+1}^{\mathbb{X}}$ (resp. $\mu_{j+1}^{\mathbb{Y}}$) in the following alternative manner without referring to the the sequence $\mathbb{X}$ (resp. $\mathbb{Y}$).

Observation 2.3.2. *Let* $X_1, X_2, X_3, \ldots$ *be a sequence of independent level* j $\mathbb{X}$*-blocks such that* $X_i \sim \mu_{j,G}^{\mathbb{X}}$ *for* $1 \leq i \leq L_j^3$ *and* $X_i \sim \mu_j^{\mathbb{X}}$ *for* $i > L_j^3$. *Now let* W *be a Geom*(L_j^{-4}) *variable independent of everything else. Define as before*

$$l = \min\{i \geq L_j^3 + L_j^{\alpha-1} + W : X_{i+k} \in G_j^{\mathbb{X}} \text{ for } 1 \leq k \leq 2L_j^3\}.$$

Then $X = (X_1, X_2, \ldots, X_{l+L_j^3})$ *has law* $\mu_{j+1}^{\mathbb{X}}$.

Whenever we have a sequence $X_1, X_2, \ldots$ satisfying the condition in the observation above, we shall call X the (random) level $(j+1)$ block constructed from $X_1, X_2, \ldots$ and we shall denote the corresponding geometric variable to be W_X and $T_X = l - L_j^3 - L_j^{\alpha-1}$.

2.3.2 Embedding Probabilities and Semi-bad Blocks

Now we make some definitions that we are going to use throughout our proof.

Definition 2.3.3. *For* $j \geq 0$, *let* X *be a block of* $\mathbb{X}$ *at level* j *and let* Y *be a block of* $\mathbb{Y}$ *at level* j. *We define the embedding probability of* X *to be* $S_j^{\mathbb{X}}(X) = \mathbb{P}(X \hookrightarrow Y|X)$. *Similarly we define* $S_j^{\mathbb{Y}}(Y) = \mathbb{P}(X \hookrightarrow Y|Y)$. *As noted above the law of* Y *is* $\mu_j^{\mathbb{Y}}$ *in the definition of* $S_j^{\mathbb{X}}$ *and the law of* X *is* $\mu_j^{\mathbb{X}}$ *in the definition of* $S_j^{\mathbb{Y}}$.

Notice that $j = 0$ in the above definitions corresponds to the definition we had in Section 2.1.1.

Definition 2.3.4. *Let* X *be an* $\mathbb{X}$*-block at level* j. *It is called "semi-bad" if* $X \notin G_j^{\mathbb{X}}$, $S_j^{\mathbb{X}}(X) \geq 1 - \frac{1}{20k_0 R_{j+1}^+}$, $|X| \leq 10L_j$ *and* $C_k \notin X$ *for any* $k > L_j^m$. *Here* $|X|$ *denotes the number of* $\mathcal{C}^{\mathbb{X}}$ *characters in* X. *A "semi-bad"* $\mathbb{Y}$ *block at level* j *is defined similarly.*

We denote the set of all semi-bad $\mathbb{X}$-blocks (resp. $\mathbb{Y}$-blocks) at level j by $SB_j^{\mathbb{X}}$ (resp. $SB_j^{\mathbb{Y}}$).

Definition 2.3.5. *Let $\widetilde{Y} = (Y_1, \ldots, Y_n)$ be a sequence of consecutive $\mathbb{Y}$ blocks at level j. $\widetilde{Y}$ is said to be a "strong sequence" if for every $X \in SB_j^{\mathbb{X}}$*

$$\#\{1 \leq i \leq n : X \hookrightarrow Y_i\} \geq n(1 - \frac{1}{10k_0R_{j+1}^+}).$$

Similarly a "strong" $\mathbb{X}$-sequence can also be defined.

2.3.3 Good Blocks

To complete the description, we need now give the definition of "good" blocks at level $(j+1)$ which we have alluded to above. With the definitions from the preceding section, we are now ready to give the recursive definition of a "good" block as follows. Suppose we already have definitions of "good" blocks upto level j (i.e., characterized $G_k^{\mathbb{X}}$ for $k \leq j$). Good blocks at level $(j+1)$ are then defined in the following manner. As usual we only give the definition for $\mathbb{X}$-blocks, the definition for $\mathbb{Y}$ is exactly similar.

Let $X^{(j+1)} = (X_1^{(j)}, X_2^{(j)}, \ldots, X_n^{(j)})$ be a $\mathbb{X}$ block at level $(j+1)$. Notice that we can form blocks at level $(j+1)$ since we have assumed that we already know $G_j^{\mathbb{X}}$. Then we say $X^{(j+1)} \in G_{j+1}^{\mathbb{X}}$ if the following conditions hold.

(i) It starts with L_j^3 good sub-blocks, i.e., $X_i^{(j)} \in G_j^{\mathbb{X}}$ for $1 \leq i \leq L_j^3$.

(ii) It contains at most k_0 bad sub-blocks. $\#\{1 \leq i \leq n : X_i \notin G_j^{\mathbb{X}}\} \leq k_0$.

(iii) For each $1 \leq i \leq n$ such that $X_i \notin G_j^{\mathbb{X}}$, $X_i \in SB_j^{\mathbb{X}}$, i.e., the bad sub-blocks are only semi-bad.

(iv) Every sequence of $\lfloor L_j^{3/2} \rfloor$ consecutive level j sub-blocks is "strong".

(v) The length of the block satisfies $n \leq L_j^{\alpha-1} + L_j^5$.

Finally we define "segments" of a sequence of consecutive $\mathbb{X}$ or $\mathbb{Y}$ blocks at level j. Notice that for $j = 0$ the following definition reduces to the definition given in § 2.1.1.

Definition 2.3.6. *Let $\widetilde{X} = (X_1, X_2, \ldots)$ be a sequence of consecutive $\mathbb{X}$-blocks. For $i_2 > i_1 \geq 1$, we call the subsequence $(X_{i_1}, X_{i_1+1}, \ldots, X_{i_2})$ the "$[i_1, i_2]$-segment" of $\widetilde{X}$ denoted by $\widetilde{X}^{[i_1,i_2]}$. The "$[i_1, i_2]$-segment" of a sequence of $\mathbb{Y}$ blocks is also defined similarly. Also a segment is called a "good" segment if it consists of all good blocks.*

2.4 Recursive Estimates

Our proof of the general theorem depends on a collection of recursive estimates, all of which are proved together by induction. In this section we list these estimates for easy reference. The proof of these estimates are provided in the next section. We recall that for all $j > 0$ $L_j = L_{j-1}^{\alpha} = L_0^{\alpha^j}$ and for all $j \geq 0$, $R_j = 4^j(2R)$, $R_j^- = 4^j(2-2^{-j})$ and $R_j^+ = 4^j R^2(2+2^{-j})$. For $j = 0$, this definition of R_j, R_j^+ and R_j^- agrees with the definition given in § 2.1.1.

2.4.1 Tail Estimate

I. Let $j \geq 0$. Let X be a $\mathbb{X}$-block at level j and let $m_j = m + 2^{-j}$. Then

$$\mathbb{P}(S_j^{\mathbb{X}}(X) \leq p) \leq p^{m_j} L_j^{-\beta} \quad \text{for} \quad p \leq 1 - L_j^{-1}. \tag{2.4.1}$$

Let Y be a $\mathbb{Y}$-block at level j. Then

$$\mathbb{P}(S_j^{\mathbb{Y}}(Y) \leq p) \leq p^{m_j} L_j^{-\beta} \quad \text{for} \quad p \leq 1 - L_j^{-1}. \tag{2.4.2}$$

2.4.2 Length Estimate

II. For X be an $\mathbb{X}$-block at at level $j \geq 0$,

$$\mathbb{E}[\exp(L_{j-1}^{-6}(|X| - (2-2^{-j})L_j))] \leq 1. \tag{2.4.3}$$

Similarly for Y, a $\mathbb{Y}$-block at level j, we have

$$\mathbb{E}[\exp(L_{j-1}^{-6}(|Y| - (2-2^{-j})L_j))] \leq 1. \tag{2.4.4}$$

For the case $j = 0$ we interpret equations (2.4.3) and (2.4.4) by setting $L_{-1} = L_0^{\alpha^{-1}}$.

2.4.3 Properties of Good Blocks

III. "Good" blocks map to good blocks, i.e.,

$$X \in G_j^{\mathbb{X}}, Y \in G_j^{\mathbb{Y}} \Rightarrow X \hookrightarrow Y. \tag{2.4.5}$$

IV. Most blocks are "good".

$$\mathbb{P}(X \in G_j^{\mathbb{X}}) \geq 1 - L_j^{-\delta}. \tag{2.4.6}$$

$$\mathbb{P}(Y \in G_j^{\mathbb{Y}}) \geq 1 - L_j^{-\delta}. \tag{2.4.7}$$

V. Good blocks can be compressed or expanded.
   Let $\widetilde{X} = (X_1, X_2, \ldots)$ be a sequence of $\mathbb{X}$-blocks at level j and $\widetilde{Y} = (Y_1, Y_2, \ldots)$ be a sequence of $\mathbb{Y}$-blocks at level j. Further we suppose that $\widetilde{X}^{[1,R_j^+]}$ and $\widetilde{Y}^{[1,R_j^+]}$ are "good segments". Then for every t with $R_j^- \le t \le R_j^+$,

$$\widetilde{X}^{[1,R_j]} \hookrightarrow \widetilde{Y}^{[1,t]} \text{ and } \widetilde{X}^{[1,t]} \hookrightarrow \widetilde{Y}^{[1,R_j]}. \tag{2.4.8}$$

Theorem 2.4.1 (Recursive Theorem)**.** *For α, β, δ, m, k_0 and R as in equation (2.1.4), the following holds for all large enough L_0. If the recursive estimates (2.4.1), (2.4.2), (2.4.3), (2.4.4), (2.4.5), (2.4.6), (2.4.7) and (2.4.8) hold at level j for some $j \ge 0$ then all the estimates hold at level $(j+1)$ as well.*

Before giving a proof of Theorem 2.4.1 we show how using this theorem we can prove the general theorem.

Proof of Theorem 2.4. Let $\mathbb{X} = (X_1, X_2, \ldots)$, $\mathbb{Y} = (Y_1, Y_2, \ldots)$ be as in the statement of the theorem. Let for $j \ge 0$, $\mathbb{X} = (X_1^{(j)}, X_2^{(j)}, \ldots)$ denote the partition of $\mathbb{X}$ into level j blocks as described above. Similarly let $\mathbb{Y} = (Y_1^{(j)}, Y_2^{(j)}, \ldots)$ denote the partition of $\mathbb{Y}$ into level j blocks. Let β, δ, m, R, L_0 be as in Theorem 2.4.1. Recall that the characters are the blocks at level 0, i.e., $X_i^{(0)} = X_i$ and $Y_i^{(0)} = Y_i$ for all $i \ge 1$. Hence the hypotheses of Theorem 2.4 implies that (2.4.1), (2.4.2), (2.4.6), (2.4.7) hold for $j = 0$. It follows from definition that (2.4.5) and (2.4.8) also hold at level 0. That (2.4.3) and (2.4.4) hold for $j = 0$ is trivial. Hence the estimates $I - V$ hold at level j for $j = 0$. Using Theorem 2.4.1, it now follows that (2.4.1), (2.4.2), (2.4.3), (2.4.4), (2.4.5), (2.4.6), (2.4.7) and (2.4.8) hold for each $j \ge 0$.

Let $\mathcal{T}_j^{\mathbb{X}} = \{X_k^{(j)} \in G_j^{\mathbb{X}}, 1 \le k \le L_j^3\}$ be the event that the first L_j^3 blocks at level j are good. Notice that on the event $\cap_{k=0}^{j-1} \mathcal{T}_k^{\mathbb{X}} = \mathcal{T}_{j-1}^{\mathbb{X}}$, $X_1^{(j)}$ has distribution $\mu_j^{\mathbb{X}}$ by Observation 2.3.1 and so $\{X_i^{(j)}\}_{i \ge 1}$ is i.i.d. with distribution $\mu_j^{\mathbb{X}}$. Hence it follows from equation (2.4.6) that $\mathbb{P}(\mathcal{T}_j^{\mathbb{X}} | \cap_{k=0}^{j-1} \mathcal{T}_k^{\mathbb{X}}) \ge (1 - L_j^{-\delta})^{L_j^3}$. Similarly defining $\mathcal{T}_j^{\mathbb{Y}} = \{Y_k^{(j)} \in G_j^{\mathbb{Y}}, 1 \le k \le L_j^3\}$ we get using (2.4.7) that $\mathbb{P}(\mathcal{T}_j^{\mathbb{Y}} | \cap_{k=0}^{j-1} \mathcal{T}_k^{\mathbb{Y}}) \ge (1 - L_j^{-\delta})^{L_j^3}$.

Let $\mathcal{A} = \cap_{j \ge 0} (\mathcal{T}_j^{\mathbb{X}} \cap \mathcal{T}_j^{\mathbb{Y}})$. It follows from above that $\mathbb{P}(\mathcal{A}) > 0$ since $\delta > 3$ and L_0 sufficiently large. Also, notice that, on $\mathcal{A}$, $X_1^{(j)} \hookrightarrow Y_1^{(j)}$ for each $j \ge 0$. Since $|X_1^{(j)}|, |Y_1^{(j)}| \to \infty$ as $j \to \infty$, it follows that there exists a subsequence $j_n \to \infty$ such that there exist R-embeddings of $X_1^{(j_n)}$ into $Y_1^{(j_n)}$ with associated partitions $(i_0^n, i_1^n, \ldots, i_{\ell_n}^n)$ and $(i'^n_0, i'^n_1, \ldots, i'^n_{\ell_n})$ with $\ell_n \to \infty$ satisfying the conditions of Definition 2.1.1 and such that for all $r \ge 0$ we have that $i_r^n \to i_r^\star$ and $i'^n_r \to i'^\star_r$ as $n \to \infty$. These limiting partitions of $\mathbb{N}$, $(i_0^\star, i_1^\star, \ldots)$ and $(i'^\star_0, i'^\star_1, \ldots)$, satisfy the conditions of Definition 2.1.1 implying that $\mathbb{X} \hookrightarrow_R \mathbb{Y}$. It follows that $\mathbb{P}(\mathbb{X} \hookrightarrow \mathbb{Y}) > 0$. $\square$

The remainder of the chapter is devoted to the proof of the estimates in the induction. Throughout these sections we assume that the estimates $I - V$ hold for some level $j \ge 0$ and then prove the estimates at level $j+1$. Combined they complete the proof of Theorem 2.4.1.

From now on, in every Theorem, Proposition and Lemma we state, we would implicitly assume the hypothesis that all the recursive estimates hold upto level j, the parameters satisfy the constraints described in § 2.1.2 and L_0 is sufficiently large.

2.5 Notation for maps: Generalised Mappings

Since in our estimates we will need to map segments of sub-blocks to segments of sub-blocks we need a notation for constructing such mappings. Let $A, A' \subseteq \mathbb{N}$, be two sets of consecutive integers. Let $A = \{n_1+1, \ldots, n_1+n\}$, $A' = \{n_1'+1, \ldots, n_1'+n'\}$. Let

$$\mathcal{P}_A = \{P : P = \{n_1 = i_0 < i_1 < \ldots < i_z = n_1 + n\}\}$$

denote the set of partitions of A. For $P = \{n_1 = i_0 < i_1 < \ldots < i_z = n_1 + n\} \in \mathcal{P}_A$, let us denote the "length" of P by $l(P) = z$. Also let the set of all blocks of P, be denoted by $\mathcal{B}(P) = \{[i_r + 1, i_{r+1}] \cap \mathbb{Z} : 0 \leq r \leq z - 1\}$.

2.5.1 Generalised Mappings

Now let Υ denote a "generalised mapping" which assigns to the tuple (A, A'), a triplet (P, P', τ), where $P \in \mathcal{P}_A$, $P' \in \mathcal{P}_{A'}$, with $l(P) = l(P')$, and $\tau : \mathcal{B}(P) \mapsto \mathcal{B}(P')$ be the unique increasing bijection from the blocks of P to the blocks of P'. Let $P = \{n_1 = i_0 < i_1 < \ldots < i_{l(P)} = n_1 + n\}$ and $P' = \{n_1' = i_0 < i_1' < \ldots < i'_{l(P')} = n_1' + n'\}$. Then by "$\tau$ is an increasing bijection" we mean that $l(P) = l(P') = z$ (say), and $\tau([i_r + 1, i_{r+1}] \cap \mathbb{Z}) = [i_r' + 1, i'_{r+1}] \cap \mathbb{Z}$. A generalised mapping Υ of (A, A') (say, $\Upsilon(A, A') = (P, P', \tau)$) is called "admissible" if the following holds.

Let $\{x\} \in \mathcal{B}(P)$ is a singleton. Then $\tau(\{x\}) = \{y\}$ (say) is also a singleton. Similarly, if $\{y\} \in \mathcal{B}(P')$ is a singleton, then $\tau^{-1}(\{y\})$ is also a singleton. Note that since we already require τ to be a bijection, it makes sense to talk about τ^{-1} as a function here.

If $\tau(\{x\}) = \{y\}$ or $\tau^{-1}(\{y\}) = x$, we simply denote this by $\tau(x) = y$ and $\tau^{-1}(y) = x$ respectively.

Let $B \subseteq A$ and $B' \subseteq A'$ be two subsets of A, A' respectively. An admissible generalized mapping Υ of (A, A') is called of class G^j with respect to (B, B') (we denote this by saying $\Upsilon(A, A', B, B')$ is admissible of class G^j) if it satisfies the following conditions:

(i) If $x \in B$, then the singleton $\{x\} \in \mathcal{B}(P)$. Similarly if $y \in B'$, then $\{y\} \in \mathcal{B}(P')$.

(ii) If $i_{r+1} > i_r + 1$ (equivalently, $i'_{r+1} > i'_r + 1$), then $(i_{r+1} - i_r) \wedge (i'_{r+1} - i'_r) > L_j$ and $\frac{1-2^{-(j+5/4)}}{R} \leq \frac{i'_{r+1}-i'_r}{i_{r+1}-i_r} \leq R(1 + 2^{-(j+5/4)})$.

(iii) For all $x \in B$, $\tau(x) \notin B'$.

Similarly, an admissible generalised mapping $\Upsilon(A, A') = (P, P', \tau)$ is called of *Class* H_1^j with respect to B if it satisfies the following conditions:

(i) If $x \in B$, then $\{x\} \in \mathcal{B}(P)$.

(ii) If $i_{r+1} > i_r + 1$ (equivalently, $i'_{r+1} > i'_r + 1$), then $(i_{r+1} - i_r) \wedge (i'_{r+1} - i'_r) > L_j$ and $\frac{1-2^{-(j+5/4)}}{R} \leq \frac{i'_{r+1}-i'_r}{i_{r+1}-i_r} \leq R(1 + 2^{-(j+5/4)})$.

(iii) For all $x \in B$, $n' - L_j^3 \geq \tau(x) - n_1 > L_j^3$.

Finally, an admissible generalised mapping $\Upsilon^j(A, A') = (P, P', \tau)$ is called of *Class* H_2^j with respect to B if it satisfies the following conditions:

(i) If $x \in B$, then $\{x\} \in \mathcal{B}(P)$.

(ii) $L_j^3 < \tau(x) - n_1 \leq n' - L_j^3$ for all $x \in B$.

(iii) If $[i_h+1, i_{h+1}] \cap \mathbb{Z} \in \mathcal{B}(P)$ and $i_h+1 \neq i_{h+1}$ then $i_{h+1} - i_h = R_j$ and $R_j^- \leq i'_{h+1} - i'_h \leq R_j^+$.

2.5.2 Generalised Mapping Induced by a Pair of Partitions

Let A, A', B, B' be as above. By a "marked partition pair" of (A, A') we mean a triplet (P_*, P'_*, Z) where $P_* = \{n_1 = i_0 < i_1 < \ldots < i_{l(P_*)} = n_1 + n\} \in \mathcal{P}_A$ and $P'_* = \{n'_1 = i_0 < i'_1 < \ldots < i'_{l(P'_*)} = n'_1 + n'\} \in \mathcal{P}_{A'}$, $l(P_*) = l(P'_*)$ and $Z \subseteq [l(P_*) - 1]$ is such that $r \in Z \Rightarrow i_{r+1} - i_r = i'_{r+1} - i'_r$.

It is easy to see that a "marked partition pair" induces a Generalised mapping Υ of (A, A', B, B') in the following natural way.

Let P be the partition of A whose blocks are given by

$$\mathcal{B}(P) = \cup_{r \in Z}\{\{i\} : i \in [i_r + 1, i_{r+1}] \cap \mathbb{Z}\} \cup_{r \notin Z} \{[i_r + 1, i_{r+1}] \cap \mathbb{Z}\}.$$

Similarly let P' be the partition of A' whose blocks are given by

$$\mathcal{B}(P') = \cup_{r \in Z}\{\{i'\} : i' \in [i'_r + 1, i'_{r+1}] \cap \mathbb{Z}\} \cup_{r \notin Z} \{[i'_r + 1, i'_{r+1}] \cap \mathbb{Z}\}.$$

Clearly, $l(P_*) = l(P'_*)$ and the condition in the definition of Z implies that $l(P) = l(P')$. Let τ denote the increasing bijection from $\mathcal{B}(P)$ to $\mathcal{B}(P')$. Clearly in this case $\Upsilon(A, A') = (P, P', \tau)$ is a generalised mapping and is called the generalised mapping induced by the marked partition pair (P_*, P'_*, Z).

The following lemma gives condition under which an induced generalised mapping is admissible. The proof is straightforward and hence omitted.

Lemma 2.5.1. *Let A, A', B, B' be as above. Let $P_* = \{n_1 = i_0 < i_1 < \ldots < i_{l(P_*)} = n_1 + n\} \in \mathcal{P}(A)$ and $P'_* = \{n'_1 = i_0 < i'_1 < \ldots < i'_{l(P'_*)} = n'_1 + n'\} \in \mathcal{P}'_*$ be partitions of A and A' respectively of equal length. Let $B_{P_*} = \{r : B \cap [i_r + 1, i_{r+1}] \neq \emptyset\}$ and $B_{P'_*} = \{r : B' \cap [i'_r + 1, i_{r+1}] \neq \emptyset\}$. Let us suppose the following conditions hold.*

(i) $(P_*, P'_*, B_{P_*} \cup B_{P'_*})$ is a marked partition pair.

(ii) For $r \notin (B_{P_*} \cup B_{P'_*})$, $(i_{r+1}-i_r) \wedge (i'_{r+1}-i'_r) > L_j$ *and* $\frac{1-2^{-(j+5/4)}}{R} \leq \frac{i'_{r+1}-i'_r}{i_{r+1}-i_r} \leq R(1+2^{-(j+5/4)})$.

(iii) $B_{P_*} \cap B_{P'_*} = \emptyset$,

then the induced generalised mapping $\Upsilon(A, A', B, B')$ *is admissible of Class* G^j.

The usefulness of making these abstract definitions follow from the following lemma and next couple of propositions.

Lemma 2.5.2. *Let* $X = (X_1, X_2, \ldots)$ *be a sequence of* $\mathbb{X}$ *blocks at level* j *and* $Y = (Y_1, Y_2, \ldots)$ *be a sequence of* $\mathbb{Y}$ *blocks at level* j. *Further suppose that* n, n' *are such that* $X^{[1,n]}$ *and* $Y^{[1,n']}$ *are both "good" segments,* $n > L_j$ *and* $\frac{1-2^{-(j+5/4)}}{R} \leq \frac{n'}{n} \leq R(1+2^{-(j+5/4)})$. *Then* $X^{[1,n]} \hookrightarrow Y^{[1,n']}$.

Proof. Let us write $n = kR_j + r$ where $0 \leq r < R_j$ and $k \in \mathbb{N} \cup \{0\}$. Now let $s = [\frac{n'-r}{k}]$. Define $0 = t_0 < t_1 < t_2 < \ldots t_k = n' - r \leq t_{k+1} = n'$ such that for all $i \leq k$, $t_i - t_{i-1} = s$ or $s+1$.

Claim: $R_j^- \leq s \leq R_j^+ - 1$.

Proof of Claim. From $\frac{n'}{n} \leq R(1+2^{-(j+5/4)})$ it follows that,

$$ks \leq n' \leq nR(1+2^{-(j+5/4)}) \leq (k+1)R_jR(1+2^{-(j+5/4)}).$$

Since $n > L_j$ and L_0 is sufficiently large we have $\frac{1}{k} \leq \frac{2R_j}{L_j} \leq 2^{-(j+13/4)}(2^{1/4}-1)$, it follows from the above that

$$\begin{aligned}
s &\leq \left(1+\frac{1}{k}\right) R_j^+ \frac{(1+2^{-(j+5/4)})}{(1+2^{-(j+1)})} \\
&\leq R_j^+ \left(1+\frac{1}{k}\right)\left(1 - \frac{2^{-(j+5/4)}(2^{1/4}-1)}{1+2^{-(j+1)}}\right) \\
&\leq R_j^+ \left(1+\frac{1}{k}\right)\left(1-2^{-(j+9/4)}(2^{1/4}-1)\right) \\
&\leq R_j^+ \left(1-2^{-(j+9/4)}(2^{1/4}-1)+\frac{1}{k}\right) \\
&\leq R_j^+ \left(1-2^{-(j+13/4)}(2^{1/4}-1)\right) \\
&\leq R_j^+ \left(1-\frac{1}{R_j^+}\right),
\end{aligned}$$

the last inequality follows as $2^jR^2(2^{1/4}-1) \geq 2^{9/4}$ for all j since $R > 10$. Hence $s \leq R_j^+ - 1$.

To prove the other inequality in the claim, we note that it follows from $\frac{1-2^{-(j+5/4)}}{R} \leq \frac{n'}{n}$ that

$$(k+1)s + R_j \geq n' \geq n\frac{(1-2^{-(j+5/4)})}{R} \geq kR_j\frac{(1-2^{-(j+5/4)})}{R}.$$

This in turn implies that

$$\begin{aligned} s &\geq \frac{kR_j(1-2^{-(j+5/4)})}{(k+1)R} - \frac{R_j}{k+1} \\ &\geq R_j^- \left(\frac{k}{k+1}\frac{(1-2^{-(j+5/4)})}{(1-2^{-(j+1)})} - \frac{R}{(k+1)(1-2^{-(j+1)})}\right) \\ &\geq R_j^- \left(\frac{k}{k+1}(1+2^{-(j+5/4)})(2^{1/4}-1) - \frac{2R}{k+1}\right) \\ &\geq R_j^- \end{aligned}$$

where the last inequality follows from the fact that for L_0 sufficiently large we have for all $j \geq 0$, $k \geq \frac{L_j}{2R_j} \geq \frac{(2R+1)2^{j+5/4}}{2^{1/4}-1}$. This completes the proof of the claim.

Now, from (2.4.5) and (2.4.8) it follows that, $X^{[iR_j+1,(i+1)R_j]} \hookrightarrow Y^{[t_i+1,t_{i+1}]}$ for $0 \leq i < k$ and $X^{[kR_j+1,n]} \hookrightarrow Y^{[t_k+1,t_{k+1}]}$. The lemma follows. □

Let $X = (X_1, X_2, X_3, \ldots, X_n)$ be an $\mathbb{X}$-block (or a segment of $\mathbb{X}$-blocks) at level $(j+1)$ where the X_i's denote the j-level sub-blocks constituting it. Similarly, let a $\mathbb{Y}$-block (or a segment of $\mathbb{Y}$-blocks) at level $(j+1)$ be denoted by $Y = (Y_1, Y_2, Y_3, \ldots, Y_{n'})$. Let $B_X = \{i : X_i \notin G_j^{\mathbb{X}}\} = \{l_1 < l_2 < \ldots < l_{K_X}\}$ denote the positions of "bad" level j $\mathbb{X}$-subblocks. Similarly let $B_Y = \{i : X_i \notin G_j^{\mathbb{Y}}\} = \{l'_1 < l'_2 < \ldots < l'_{K_Y}\}$ be the positions of "bad" Y-blocks.

We next state an easy proposition.

Proposition 2.5.3. *Let X, Y, B_X, B_Y be as above. Suppose there exists a generalised mapping Υ given by $\Upsilon([n],[n'],B_X,B_Y) = (P, P', \tau)$ which is admissible and is of Class G^j. Further, suppose, for $1 \leq i \leq K_X$, $X_{l_i} \hookrightarrow Y_{\tau(l_i)}$ and for each $1 \leq i \leq K_Y$, $X_{\tau^{-1}(l'_i)} \hookrightarrow Y_{l'_i}$. Then $X \hookrightarrow Y$.*

Proof. Let P, P' be as in the statement of the proposition with $l(P) = l(P') = z$. Let us fix r, $0 \leq r \leq z-1$. From the definition of an admissible mapping, it follows that there are 3 cases to consider.

- $i_{r+1} - i_r = i'_{r+1} - i'_r = 1$ and either $i_r + 1 \in B_X$ or $i'_r + 1 \in B_Y$. In either case it follows from the hypothesis that $X_{i_r+1} \hookrightarrow Y_{i'_r+1}$.
- $i_{r+1} - i_r = i'_{r+1} - i'_r = 1$, $i_r + 1 \notin B_X$, $i'_r + 1 \notin B_Y$. In this case $X_{i_r+1} \hookrightarrow Y_{i'_r+1}$ follows from the inductive hypothesis (2.4.5).
- $i_{r+1} - i_r \neq 1$. In this case both $X^{[i_r+1,i_{r+1}]}$ and $Y^{[i'_r+1,i'_{r+1}]}$ are good segments, and it follows from Lemma 2.5.2 that $X^{[i_r+1,i_{r+1}]} \hookrightarrow Y^{[i'_r+1,i'_{r+1}]}$.

Hence for all r, $0 \le r \le z-1$, $X^{[i_r+1,i_{r+1}]} \hookrightarrow Y^{[i'_r+1,i'_{r+1}]}$. It follows that $X \hookrightarrow Y$, as claimed. □

In the same vein, we state the following Proposition whose proof is essentially similar and hence omitted.

Proposition 2.5.4. *Let X, Y, B_X be as before. Suppose there exists a generalised mapping Υ given by $\Upsilon([n],[n']) = (P, P', \tau)$ which is admissible and is of Class H_1^j or H_2^j with respect to B_X. Further, suppose, for $1 \le i \le K_X$, $X_{l_i} \hookrightarrow Y_{\tau(l_i)}$ and for each $i' \in [n'] \setminus \{\tau(l_i) : 1 \le i \le K_X\}$, $Y_{i'} \in G_j^{\mathbb{Y}}$. Then $X \hookrightarrow Y$.*

2.6 Constructions

In this section we provide the necessary constructions of generalised mappings which we would use in later sections to prove different estimates on probabilities that certain $\mathbb{X}$-blocks can be mapped to certain $\mathbb{Y}$-blocks.

Proposition 2.6.1. *Let $j \ge 0$ and $n, n' > L_j^{\alpha-1}$ such that*

$$\frac{1-2^{-(j+7/4)}}{R} \le \frac{n'}{n} \le R(1+2^{-(j+7/4)}). \tag{2.6.1}$$

Let $B = \{l_1 < l_2 < \ldots < l_{k_x}\} \subseteq [n]$ and $B' = \{l'_1 < l'_2 < \ldots l'_{k_y}\} \subseteq [n']$ be such that $l_1, l'_1, (n-l_{k_x}), (n'-l'_{k_y}) > L_j^3$, $k_x, k_y \le k_0 R_{j+1}^{+}$. Then there exist a family of admissible generalised mappings Υ_h for $1 \le h \le L_j^2$, such $\Upsilon_h([n],[n'],B,B') = (P_h, P'_h, \tau_h)$ is of class G^j and such that for $1 \le h \le L_j^2$, $1 \le i \le k_x$, $1 \le r \le k_y$, $\tau_h(l_i) = \tau_1(l_i) + h - 1$ and $\tau_h^{-1}(l'_r) = \tau_1^{-1}(l'_r) - h + 1$.

To prove Proposition 2.6.1 we need the following two lemmas.

Lemma 2.6.2. *Assume the hypotheses of Proposition 2.6.1. Then there exists a piecewise linear increasing bijection $\psi : [0,n] \mapsto [0,n']$ and two partitions Q and Q' of $[0,n]$ and $[0,n']$ respectively of equal length (= q, say), given by $Q = \{0 = t_0 < t_1 < \ldots t_{q-1} < t_q = n\}$ and $Q' = \{0 = \psi(t_0) < \psi(t_1) < \ldots \psi(t_{q-1}) < \psi(t_q) = n'\}$ satisfying the following properties:*

1. *None of the intervals $[t_{r-1}, t_r]$ intersect both B and $\psi^{-1}(B')$ but each intersects $B \cup \psi^{-1}(B')$. Hence, none of the intervals $[\psi(t_{r-1}), \psi(t_r)]$ intersect both B' and $\psi(B)$ but each intersects $B' \cup \psi(B)$.*
2. *For all a, b; $0 \le a < b \le n$, $\frac{1-2^{-(j+3/2)}}{R} \le \frac{\psi(b)-\psi(a)}{b-a} \le R(1+2^{-(j+3/2)})$.*
3. *Suppose $i \in (B \cup \psi^{-1}(B')) \cap [t_{r-1}, t_r]$. Then $|i-t_{r-1}| \wedge |t_r - i| \ge L_j^{9/4}$. Similarly if $i' \in (B' \cup \psi(B)) \cap [\psi(t_{r-1}), \psi(t_r)]$, then $|i' - \psi(t_{r-1})| \wedge |\psi(t_r) - i| \ge L_j^{9/4}$.*

Proof. Let us define a family of maps $\psi_s : [0,n] \to [0,n']$, $0 \le s \le L_j^{5/2}$ as follows:

$$\psi_s(x) = \begin{cases} x\frac{L_j^3/2+s}{L_j^3/2} & \text{if} \quad x \le L_j^3/2 \\ L_j^3/2 + s + \frac{n'-L_j^3}{n-L_j^3}(x - L_j^3/2) & \text{if} \quad L_j^3/2 \le x \le n - L_j^3/2 \\ n' - (n-x)(\frac{L_j^3/2-s}{L_j^3/2}) & \text{if} \quad n - L_j^3/2 \le x \le n. \end{cases} \tag{2.6.2}$$

It is easy to see that ψ_s is a piecewise linear bijection for each s with the piecewise linear inverse being given by

$$\psi_s^{-1}(y) = \begin{cases} y\frac{L_j^3/2}{L_j^3/2+s} & \text{if} \quad y \le L_j^3/2 + s \\ L_j^3/2 + \frac{n-L_j^3}{n'-L_j^3}(y - L_j^3/2 - s) & \text{if} \quad L_j^3/2 + s \le y \le n' - L_j^3/2 + s \\ n - (n'-x)(\frac{L_j^3/2}{L_j^3/2-s}) & \text{if} \quad n' - L_j^3/2 + s \le y \le n'. \end{cases} \tag{2.6.3}$$

Notice that since $\alpha > 4$, for L_0 sufficiently large, we get from (2.6.1) that $\frac{1-2^{-(j+13/8)}}{R} \le \frac{n'-L_j^3}{n-L_j^3} \le R(1+2^{-(j+13/8)})$. Since each ψ_s is piecewise linear, it follows that each ψ_s satisfies condition (2) in the statement of the lemma.

Let S be distributed uniformly on $[0, L_j^{5/2}]$, and consider the random map ψ_S. Let

$$E = \{|\psi_S(i) - i'| \ge 2L_j^{9/4}, |i - \psi_S^{-1}(i')| \ge 2L_j^{9/4} \forall i \in B, \forall i' \in B'\}.$$

It follows that for $i \in B, i' \in B'$, $\mathbb{P}(|\psi_S(i) - i'| < 2L_j^{9/4}) \le \frac{8RL_j^{9/4}}{L_j^{5/2}} = \frac{8R}{L_j^{1/4}}$. Similarly $\mathbb{P}(|i - \psi_S^{-1}(i')| < 2L_j^{9/4}) \le \frac{8R}{L_j^{1/4}}$. Using $k_x, k_y \le k_0 R_{j+1}^+$, a union bound now yields

$$\mathbb{P}(E) \ge 1 - \frac{16Rk_0^2(R_{j+1}^+)^2}{L_j^{1/4}} > 0$$

for L_0 large enough. It follows that there exists $s_0 \in [0, L_j^{5/2}]$ such that $|\psi_{s_0}(i) - i'| \ge 2L_j^{9/4}, |i - \psi_{s_0}^{-1}(i')| \ge 2L_j^{9/4} \forall i \in B, i' \in B'$.

Setting $\psi = \psi_{s_0}$ it is now easy to see that for sufficiently large L_0 there exists $0 = t_0 < t_1 < \ldots < t_q = n \in [0,n]$ satisfying the properties in the statement of the lemma. One way to do this is to choose t_k's at the points $\frac{l_i + \psi^{-1}(l'_{i'})}{2}$ where i, i' are such that there does not exist any point in the set $B \cup \psi^{-1}(B')$ in between l_i and $\psi^{-1}(l'_{i'})$. That such a choice satisfies the properties $(1) - (3)$ listed in the lemma is easy to verify. □

Lemma 2.6.3. *Assume the hypotheses of Proposition 2.6.1. Then there exist partitions P_* and P'_* of $[n]$ and $[n']$ of equal length ($= z$, say) given by $P_* = \{0 = i_0 < i_1 < \ldots < i_z = n\}$ and $P'_* = \{0 = i'_0 < i'_1 < \ldots < i'_z = n'\}$ such that if we denote $B_{P_*} = \{r : B \cap [i_r + 1, i_{r+1}] \neq \emptyset\}$ and $B'_{P'_*} = \{r : B' \cap [i'_r + 1, i'_{r+1}] \neq \emptyset\}$ then all the following properties hold.*

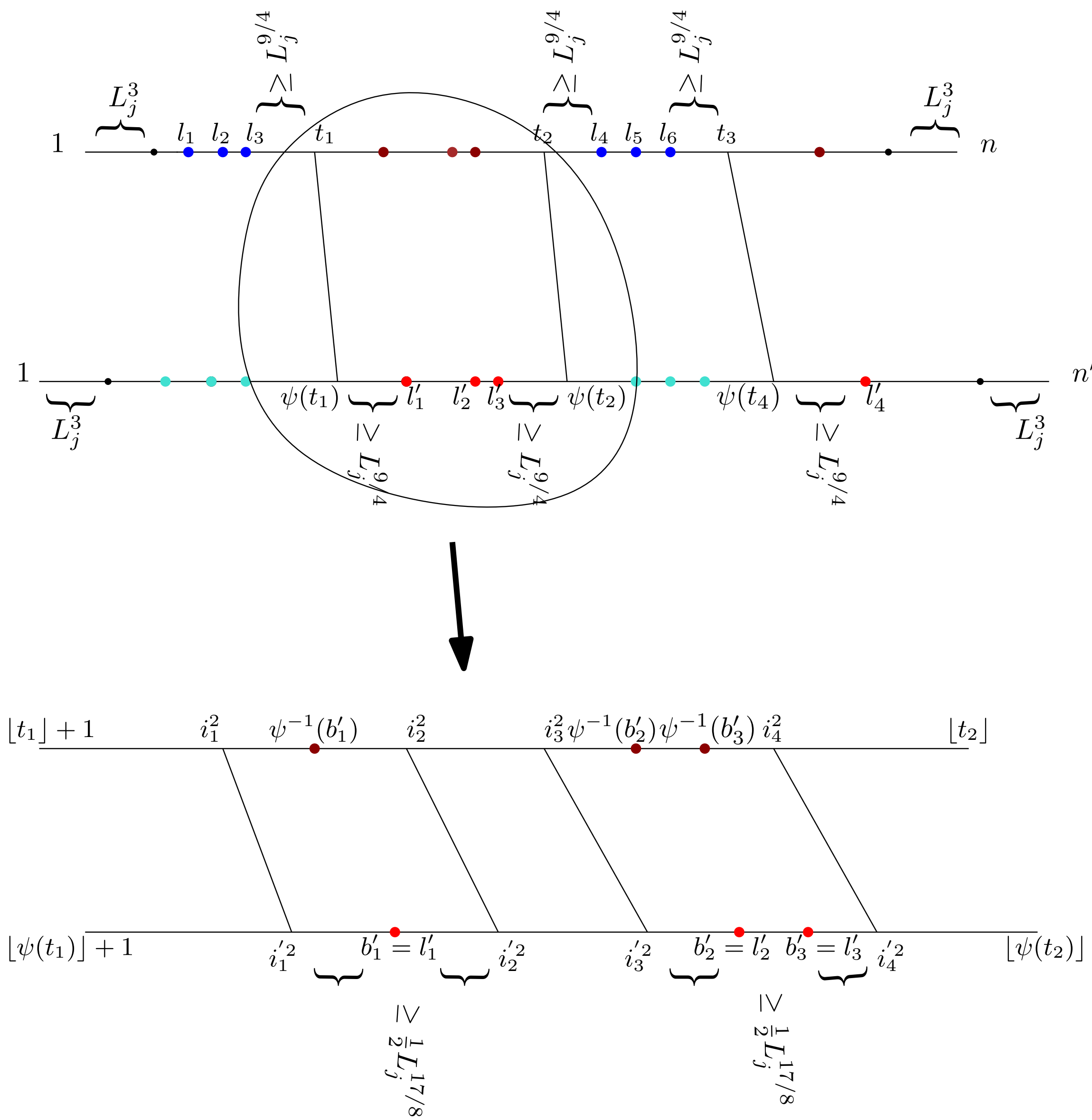


Figure 2.1: Partitions described in Lemma 2.6.2 and Lemma 2.6.3

1. *$(P_*, P'_*, B_{P_*} \cup B'_{P'_*})$ is a marked partition pair.*

2. *For $r \notin B_{P_*} \cup B'_{P'_*}$, $(i_{r+1} - i_r) \wedge (i'_{r+1} - i'_r) \geq \frac{L_j^{17/8}}{4R}$ and $\frac{1-2^{-(j+7/5)}}{R} \leq \frac{i'_r - i'_{r-1}}{i_r - i_{r-1}} \leq R(1 + 2^{-(j+7/5)})$.*

3. *$B_{P_*} \cap B'_{P'_*} = \emptyset$, $0, z-1 \notin B_{P_*} \cup B_{P'_*}$, $B_{P_*} \cup B_{P'_*}$ does not contain consecutive integers.*

4. *If $l_i \in [i_r+1, i_{r+1}]$, then $|l_i - i_r| \wedge |l_i - i_{r+1}| > \frac{1}{2}L_j^{17/8}$. Similarly if $l'_i \in [i'_r+1, i'_{r+1}]$, then $|l'_i - i'_r| \wedge |l'_i - i'_{r+1}| > \frac{1}{2}L_j^{17/8}$.*

Proof. Choose a map ψ and partitions Q, Q' as given by Lemma 2.6.2. Let us fix an interval $[t_{r-1}, t_r]$, $1 \le r \le q$. We need to consider two cases.

Case 1: $B_r := B \cap [t_{r-1}, t_r] = \{b_1 < b_2 < \ldots < b_{k_r}\} \neq \emptyset$.
Clearly $k_r \le k_0 R^+_{j+1}$. We now define a partition $P^r = \{\lfloor t_{r-1} \rfloor = i_0^r < i_1^r < \ldots < i_{z_r}^r = \lfloor t_r \rfloor\}$ of $[\lfloor t_{r-1} \rfloor + 1, \lfloor t_r \rfloor]$ as follows.

- $i_1^r = b_1 - \lfloor L_j^{17/8} \rfloor$.
- For $h \ge 1$, if $[i_{h-1}^r, i_h^r] \cap B_r = \emptyset$, then define, $i_{h+1}^r = \min\{i \ge i_h^r + \lfloor L_j^{17/8} \rfloor : B_r \cap [i - \lfloor L_j^{17/8} \rfloor, i + 3\lfloor L_j^{17/8} \rfloor] = \emptyset\}$.
- For $h \ge 1$, if $[i_{h-1}^r, i_h^r] \cap B_r \neq \emptyset$, define $i_{h+1}^r = \min\{i \ge i_h^r + 2\lfloor L_j^{17/8} \rfloor : i + \lfloor L_j^{17/8} \rfloor + 1 \in B_r\}$ or $\lfloor t_r \rfloor$ if no such i exists.

Notice that the construction implies that $i_{z_r-1}^r = b_{k_r} + \lfloor L_j^{17/8} \rfloor + 1$. Also $i_{h+1}^r - i_h^r \ge 2\lfloor L_j^{17/8} \rfloor$ for all h. Also notice that this implies that alternate blocks of this partition intersect B_r and hence $z_r \le 2k_0 R^+_{j+1} + 2$. It also follows that the total length of the blocks intersecting B_r is at most $8k_0 R^+_{j+1} L_j^{17/8}$.

Now we construct a corresponding partition $P'^r = \{\lfloor \psi(t_{r-1}) \rfloor = i_0'^r < i_1'^r < \ldots < i_{z_r}'^r = \lfloor \psi(t_r) \rfloor\}$ of $[\lfloor \psi(t_{r-1}) \rfloor + 1, \lfloor \psi(t_r) \rfloor]$ as follows.

- $i_1'^r = \lfloor \psi(i_1^r) \rfloor$.
- For $1 \le h \le z_r - 2$, $i_{h+1}'^r = i_h'^r + (i_{h+1}^r - i_h^r)$, when $B_r \cap [i_h^r, i_{h+1}^r] \neq \emptyset$, and $i_{h+1}'^r = i_h'^r + \lfloor \psi(i_{h+1}^r) - \psi(i_h^r) \rfloor$ otherwise.

Notice that condition (2) of Lemma 2.6.2 and the preceding observation implies that

$$|(i_{z_r}'^r - i_{z_r-1}'^r) - (\psi(i_{z_r}^r) - \psi(i_{z_r-1}^r))| \le 4R(8k_0 R^+_{j+1} L_j^{17/8} + 2k_0 R^+_{j+1} + 2).$$

This together with conditions (2) and (3) of Lemma 2.6.2 implies that for L_0 sufficiently large P'^r is a valid partition of $[\lfloor \psi(t_{r-1}) \rfloor + 1, \lfloor \psi(t_r) \rfloor]$ such that for all h

$$\frac{1 - 2^{-(j+7/5)}}{R} \le \frac{i_{h+1}'^r - i_h'^r}{i_{h+1}^r - i_h^r} \le R(1 + 2^{-(j+7/5)}).$$

Case 2: $B'_r := B' \cap [\lfloor \psi(t_{r-1}) \rfloor, \lfloor \psi(t_r) \rfloor] = \{b'_1 < b'_2 < \ldots < b'_{k'_r}\} \neq \emptyset$.
Clearly $k'_r \le k_0 R^+_{j+1}$. In this case, we start with defining a partition $P'^r = \{\lfloor \psi(t_{r-1}) \rfloor = i_0'^r < i_1'^r < \ldots < i_{z_r}'^r = \lfloor \psi(t_r) \rfloor\}$ of $[\lfloor \psi(t_{r-1}) \rfloor + 1, \lfloor \psi(t_r) \rfloor]$ as follows.

- $i_1^{\prime r} = b_1' - \lfloor L_j^{17/8} \rfloor$.
- For $h \geq 1$, if $[i_{h-1}^{\prime r}, i_h^{\prime r}] \cap B_r' = \emptyset$, then define, $i_{h+1}^{\prime r} = \min\{i \geq i_h^{\prime r} + \lfloor L_j^{17/8} \rfloor : B_r' \cap [i - \lfloor L_j^{17/8} \rfloor, i + 3\lfloor L_j^{17/8} \rfloor] = \emptyset\}$.
- For $h \geq 1$, if $[i_{h-1}^{\prime r}, i_h^{\prime r}] \cap B_r' \neq \emptyset$, define $i_{h+1}^{\prime r} = \min\{i \geq i_h^{\prime r} + 2\lfloor L_j^{17/8} \rfloor : i + \lfloor L_j^{17/8} \rfloor + 1 \in B_r'\}$ or $\lfloor \psi(t_r) \rfloor$ if no such i exists.

As before, next we construct a corresponding partition $P^r = \{\lfloor t_{r-1} \rfloor = i_0^r < i_1^r < \ldots < i_{z_r}^r = \lfloor t_r \rfloor\}$ of $[\lfloor t_{r-1} \rfloor + 1, \lfloor t_r \rfloor]$ as follows.

- $i_1^r = \lfloor \psi^{-1}(i_1^{\prime r}) \rfloor$.
- For $z_r - 2 \geq h \geq 1$, $i_{h+1}^r = i_h^r + (i_{h+1}^{\prime r} - i_h^{\prime r})$, provided $B_r' \cap [i_h^{\prime r}, i_{h+1}^{\prime r}] \neq \emptyset$, and $i_{h+1}^r = i_h^r + \lfloor \psi^{-1}(i_{h+1}^{\prime r}) - \psi^{-1}(i_h^{\prime r}) \rfloor$, otherwise.

As before it can be verified that the procedure described above gives a valid partition of $[\lfloor t_{r-1} \rfloor + 1, \lfloor (t_r) \rfloor]$ such that for L_0 large enough we have for every h

$$\frac{1 - 2^{-(j+7/5)}}{R} \leq \frac{i_{h+1}^{\prime r} - i_h^{\prime r}}{i_{h+1}^r - i_h^r} \leq R(1 + 2^{-(j+7/5)}).$$

Let us define, $P_* = \cup_r P^r$ and $P_*' = \cup_r P'^r$ where $\cup P^r$ denotes the partition containing the points of all P^r's (or alternatively, $\mathcal{B}(\cup P^r) = \cup_r \mathcal{B}(P^r)$). It is easy to check that (P_*, P_*') satisfies the properties (1)-(4) listed in the statement of the lemma. □

The procedure for constructing (P_*, P_*') as described in Lemma 2.6.2 and Lemma 2.6.3 is illustrated in Figure 2.1. The upper figure illustrates a function ψ and partitions $0 = t_0 < t_1 < \ldots < t_q = n'$ and $0 = \psi(t_0) < \psi(t_1) < \ldots < \psi(t_q) = n'$ as described in Lemma 2.6.2. The lower figure illustrates the further sub-division of an interval $[t_1, t_2]$ as described in Lemma 2.6.3. The neighbourhoods of b_1', b_2', b_3' are mapped rigidly so above we have $i_2^2 - i_1^2 = i_2^{\prime 2} - i_1^{\prime 2}$ and $i_4^2 - i_3^2 = i_4^{\prime 2} - i_3^{\prime 2}$.

Proof of Proposition 2.6.1. Construct the partitions P_* and P_*' of $[n]$ and $[n']$ respectively as in Lemma 2.6.3. Let $P_* = \{0 = i_0 < i_1 < \ldots < i_{z-1} < i_z = n\}$ and $P_*' = \{0 = i_0' < i_1' < \ldots < i_{z-1}' < i_z' = n'\}$. For $1 \leq h \leq L_j^2$ we let $i_r^h = i_r$ and so $P_*^h = P_*$ while we define $i_r^{\prime h} = i_r' + h - 1$ for $1 \leq r \leq z - 1$ so that $P_*^{\prime h} = \{0 = i_0^{\prime h} < i_1^{\prime h} < \ldots < i_{z-1}^{\prime h} < i_z^{\prime h} = n'\}$.

First we observe that conditions (2) and (3) Lemma 2.6.3 implies that in the above definition is consistent and gives rise to a valid partition pair $(P_*^h, P_*^{\prime h})$ for each h, $1 \leq h \leq L_j^2$. From item (4), in the statement of Lemma 2.6.3 it follows that for each h, $(P_*^h, P_*^{\prime h}, B_{P_*^h} \cup B_{P_*^{\prime h}})$ forms a marked partition pair. Furthermore, for each h, if L_0 is sufficiently large this marked partition pair satisfies

1. For $r \notin B_{P_*^h} \cup B_{P_*'^h}$, $(i_{r+1}-i_r) \wedge (i'_{r+1}-i'_r) > L_j^2$ and $\frac{1+2^{-(j+5/4)}}{R} \leq \frac{i'_{r+1}-i'_r}{i_{r+1}-i_r} \leq R(1+2^{-(j+5/4)})$.
2. $B_{P_*^h} \cap B_{P_*'^h} = \emptyset$.

Using Lemma 2.5.1, for each h, the generalized mapping $\Upsilon_h([n],[n'],B,B') = (P_h, P'_h, \tau_h)$ induced by the marked partition pair $(P_*^h, P_*'^h, B_{P_*} \cup B_{P'_*})$ is an admissible mapping of class G^j. It follows easily from definitions that for $1 \leq h \leq L_j^2$, $1 \leq i \leq k_x$, $1 \leq r \leq k_y$, $\tau_h(l_i) = \tau_1(l_i) + h - 1$ and $\tau_h^{-1}(l'_r) = \tau_1^{-1}(l'_r) - h + 1$. This procedure is illustrated in Figure 2.2. □

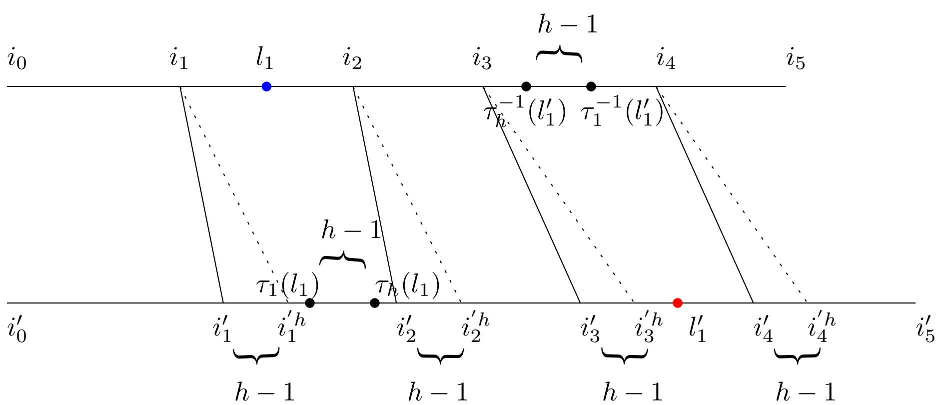


Figure 2.2: Construction of generalized mappings (P_h, P'_h, τ_h) from (P_*, P_*) as described in the proof of Proposition 2.6.1

Proposition 2.6.4. *For $j \geq 0$, let $n, n' > L_j^{\alpha-1}$ be such that $\frac{1}{R} \leq \frac{n'}{n} \leq R$. Let $B = \{l_1 < l_2 < \ldots < l_{k_x}\} \subseteq [n]$ be such that $l_1, (n - l_{k_x}) > L_j^3$, $k_x \leq k_0$. Then there exist a family of admissible mappings Υ_h for $1 \leq h \leq L_j^2$, $\Upsilon_h([n],[n']) = (P_h, P'_h, \tau_h)$ which are of Class H_1^j with respect to B such that for $1 \leq h \leq L_j^2$, $1 \leq i \leq k_x$ we have that $\tau_h(l_i) = \tau_1(l_i) + h - 1$.*

Proof. This proof is a minor modification of the proof of Proposition 2.6.1. Clearly, as in the proof of Proposition 2.6.1, we can construct L_j^2 admissible mappings $\Upsilon_h^*(A, A') = (P_h^*, P_h'^*)$ which are of Class G^j with respect to $(B, \emptyset)$ where $A = \{\lfloor L_j^3/2 \rfloor + 1, \lfloor L_j^3/2 \rfloor + 2, ..., n - \lfloor L_j^3/2 \rfloor\}$ and $A' = \{L_j^3 + 1, L_j^3 + 2, ..., n' - L_j^3\}$. Denote $P_h^* = \{\lfloor L_j^3/2 \rfloor + 1 = i_0^h < i_1^h < \ldots < i_{z-1}^h < i_z^h = n - \lfloor L_j^3/2 \rfloor\}$. Define the partition P_h of $[n]$ as $P_h = \{0 < i_0^h < i_1^h < \ldots < i_{z-1}^h < i_z^h < n\}$, that is with segments of length $\lfloor L_j^3/2 \rfloor$ added to each end of P_h^*. Define P'_h similarly by adding segments of length L_j^3 to each end of $P_h'^*$. It can be easily checked that since L_0 is sufficiently large, for each h, $1 \leq h \leq L_j^2$, (P_h, P'_h, τ_h) is an admissible mapping which is of *Class* H_1^j with respect to B such that for $1 \leq h \leq L_j^2$, $1 \leq i \leq k_x$ we have that $\tau_h(l_i) = \tau_1(l_i) + h - 1$. □

Proposition 2.6.5. *Let For $j \geq 0$, $n, n' > L_j^{\alpha-1}$ be such that $\frac{5}{3R} \leq \frac{n'}{n} \leq \frac{3R}{5}$. Let $B = \{l_1 < l_2 < \ldots < l_{k_x}\} \subseteq [n]$ be such that $l_1, (n - l_{k_x}) > L_j^3$, $k_x \leq \frac{n-2L_j^3}{10R_j^+}$. Then there exist an admissible mapping $\Upsilon([n],[n']) = (P, P', \tau)$ which is of Class H_2^j with respect to B.*

To prove this proposition we need the following lemma.

Lemma 2.6.6. *Assume the hypotheses of Proposition 2.6.5. Then there exists partitions P and P' of $[n]$ and $[n']$ of equal length ($= z$, say) given by $P_* = \{0 = i_0 < i_1 = L_j^3 < \ldots i_{z-1} = n - L_j^3 < i_z = n\}$ and $P'_* = \{0 = i'_0 < i'_1 = L_j^3 < \ldots < i'_{z-1} = n' - L_j^3 < i'_z = n'\}$ satisfying the following properties:*

1. *(P_*, P'_*, B^*) is a marked partition pair for some $B^* \supseteq B_P \cup \{0, z-1\}$ where $B_P = \{h : [i_h + 1, i_{h+1}] \cap B \neq \emptyset\}$.*
2. *For $h \notin B^*$, $(i_{h+1} - i_h) = R_j$ and $R_j^- \leq i'_{h+1} - i'_h \leq R_j^+$.*

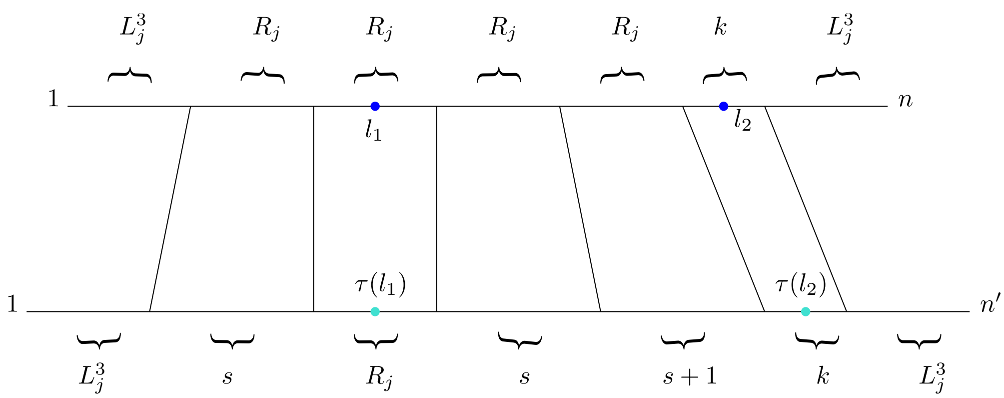


Figure 2.3: Marked Partition pair of $[n]$ and $[n']$ as described in Lemma 2.6.6 and the induced generalised mapping

Proof. Let us write $n = 2L_j^3 + kR_j + r$ where $0 \leq r < R_j$ and $k \in \mathbb{N}$. Construct the partition $P_* = \{0 = i_0 < i_1 = L_j^3 < \ldots i_{z-1} = n - L_j^3 < i_z = n\}$ where we set $i_h = L_j^3 + (h-1)R_j$ for $h = 2, 3, \ldots, (k+1)$ and $z = (k+2)$ or $(k+3)$ depending on whether $r = 0$ or not. For the remainder of this proof we assume that $r > 0$. In the case $r = 0$, the same proof works with the obvious modifications.

Now define $B^* = B_P \cup \{0, z-1\} \cup \{k+1\}$.

Clearly

$$\sum_{h \in B_P \cup \{k+1\}} (i_{h+1} - i_h) \leq R_j \left(\frac{n - 2L_j^3}{10R_j^+} + 1 \right) \leq \frac{n - 2L_j^3}{9R} \tag{2.6.4}$$

for L_0 sufficiently large.

Also notice that since $\alpha > 4$, for L_0 sufficiently large $\frac{3}{2R} \leq \frac{n'-2L_j^3}{n-2L_j^3} \leq \frac{2R}{3}$.

Now let

$$s = \left\lfloor \frac{(n' - 2L_j^3) - \sum_{h \in B_P \cup \{k+1\}} (i_{h+1} - i_h)}{k + 1 - |B_P \cup \{k+1\}|} \right\rfloor.$$

Claim: $R_j^- \leq s \leq R_j^+ - 1$.

Proof of Claim. Clearly, $|B_P \cup \{k+1\}| \leq \frac{(n-2L_j^3)}{10R_j^+} + 1 \leq \frac{(n-2L_j^3)}{9R_j^+} \leq \frac{(k+1)R_j}{9R_j^+}$. Hence $k+1-|B_P \cup \{k+1\}| \geq (k+1)(1 - \frac{R_j}{9R_j^+}) \geq \frac{8}{9}(k+1)$. It follows that

$$\begin{aligned} s &\leq \frac{n' - 2L_j^3}{\frac{8}{9}(k+1)} = \frac{(n-2L_j^3)\frac{n'-2L_j^3}{n-2L_j^3}}{\frac{8}{9}(k+1)} \\ &\leq \frac{18(k+1)RR_j}{24(k+1)} = \frac{3}{4}RR_j \leq R_j^+ - 1. \end{aligned}$$

To prove the other inequality let us observe using (2.6.4),

$$\begin{aligned} s &\geq \frac{(n' - 2L_j^3) - \frac{(n-2L_j^3)}{9R}}{(k+1)} - 1 \\ &\geq \frac{(n-2L_j^3)\frac{3}{2R} - \frac{(n-2L_j^3)}{9R}}{(k+1)} - 1 \\ &\geq \frac{25kR_j}{18(k+1)R} - 1 \geq \frac{4R_j}{3R} - 1 \geq \frac{2^{2j+3}}{3} - 1 \geq 2^{2j+1} - 2^j = R_j^- \end{aligned}$$

for all $j \geq 0$, since for L_0 sufficiently large and $n > L_j^{\alpha-1}$, we have $k \geq \frac{L_j}{2R_j}$ and $\frac{25k}{18(k+1)} \geq \frac{4}{3}$. This completes the proof of the claim.

Coming back to the proof of the lemma let us denote the set $\{1, 2, \ldots, k+1\} \setminus (B_P \cup \{k+1\}) = \{w_1 < w_2 < \ldots < w_d\}$ where $d = k+1-|B_P \cup \{k+1\}|$. Also let us write

$$(n' - 2L_j^3) - \sum_{h \in B_P \cup \{k+1\}} (i_{h+1} - i_h) = s(k+1-|B_P \cup \{k+1\}|) + r'; \quad 0 \leq r' < k+1-|B_P \cup \{k+1\}|.$$

Now we define $P'_* = \{0 = i'_0 < i'_1 = L_j^3 < \ldots < i'_{z-1} = n' - L_j^3 < i'_z = n'\}$. We define i'_h inductively as follows.

- Set $i'_1 = L_j^3$.
- For $h \in B_P \cup \{k+1\}$, define $i'_{h+1} = i'_h + (i_{h+1} - i_h)$.

- If $h = w_t$ for some t, then define $i'_{h+1} = i'_h + (s+1)$ if $t > d - r'$, and $i'_{h+1} = i'_h + s$, otherwise.

Now from the definition of s, it is clear that $i'_{k+2} = n' - L_j^3$, as asserted. It now clearly follows that (P_*, P'_*) is a pair of partitions of $([n], [n'])$ as asserted in the statement of the Lemma. That (P_*, P'_*, B^*) is a marked partition pair is clear. It follows from the claim just proved that (P_*, P'_*) satisfies condition (2) in the statement of the Lemma. This procedure for forming the marked partition pair (P_*, P'_*) is illustrated in Figure 2.3. □

Proof of Proposition 2.6.5. Construct the partitions (P_*, P'_*) as given by Lemma 2.6.6. Consider the generalized mapping $\Upsilon([n], [n']) = (P, P', \tau)$ induced by the marked partition pair (P_*, P'_*, B^*). It follows that $B^* \supseteq \{0, z-1\}$ that Υ is an admissible mapping which of class H_2^j with respect to B. □

2.7 Tail Estimate

The most important of our inductive hypotheses is the following recursive estimate.

Theorem 2.7.1. *Assume that the inductive hypothesis holds up to level j. Let X and Y be random $(j+1)$-level blocks according to $\mu_{j+1}^{\mathbb{X}}$ and $\mu_{j+1}^{\mathbb{Y}}$. Then*

$$\mathbb{P}(S_{j+1}^{\mathbb{X}}(X) \le p) \le p^{m_{j+1}} L_{j+1}^{-\beta}, \quad \mathbb{P}(S_{j+1}^{\mathbb{Y}}(Y) \le p) \le p^{m_{j+1}} L_{j+1}^{-\beta}$$

for $p \le 1 - L_{j+1}^{-1}$ and $m_{j+1} = m + 2^{-(j+1)}$.

There is of course a symmetry between our X and Y bounds and for conciseness all our bounds will be stated in terms of X and $S_{j+1}^{\mathbb{X}}$ but will similarly hold for Y and $S_{j+1}^{\mathbb{Y}}$. For the rest of this section we shall drop the superscript $\mathbb{X}$ and denote $S_{j+1}^{\mathbb{X}}$ (resp. $S_j^{\mathbb{X}}$) simply by S_{j+1} (resp. S_j).

The block X is constructed from an i.i.d. sequence of j-level blocks $X_1, X_2, \ldots$ conditioned on the event $X_i \in G_j^{\mathbb{X}}$ for $1 \le i \le L_j^3$ as described in Section 2.3. The construction also involves a random variable $W_X \sim \mathrm{Geom}(L_j^{-4})$ and let T_X denote the number of extra sub-blocks of X, that is the length of X is $L_j^{\alpha-1} + 2L_j^3 + T_X$. Let K_X denote the number of bad sub-blocks of X and denote their positions by $\ell_1, \ldots, \ell_{K_X}$. We define $Y_1, \ldots, W_Y, T_Y$ and K_Y similarly and denote the positions of the bad blocks by $\ell'_1, \ldots, \ell'_{K_Y}$. The proof of Theorem 2.7.1 is divided into 5 cases depending on the number of bad sub-blocks, the total number of sub-blocks of X and how "bad" the sub-blocks are.

Let $K_X(t) = \sum_{i=L_j^3+1}^{L_j^3+L_j^{\alpha-1}} I(X_i \notin G_j^{\mathbb{X}})$ and $\mathcal{G}_X(t) = -\sum_{i=L_j^3+1}^{L_j^{\alpha-1}+t} I(X_i \notin G_j^{\mathbb{X}}) S_j(X_i)$. Our inductive bounds allow the following stochastic domination description of $K_X(t)$ and $\mathcal{G}_X(t)$.

Lemma 2.7.2. *Let $\widetilde{K}_X = \widetilde{K}_X(t)$ be distributed as a $\mathrm{Bin}(L_j^{\alpha-1} + t, L_j^{-\delta})$ and let $\mathfrak{S} = \mathfrak{S}(t) = \sum_{i=1}^{\widetilde{K}_X(t)} (1 + U_i)$ where U_i are i.i.d. rate m_j exponentials. Then,*

$$(K_X(t), \mathcal{G}_X(t)) \preceq (\widetilde{K}_X, \mathfrak{S})$$

where $\preceq$ denotes stochastic domination w.r.t. the partial order in $\mathbb{R}^2$ given by (via a slight abuse of notation) $(x,y) \preceq (x',y')$ iff $x \le x', y \le y'$.

Proof. If V_i are i.i.d. Bernoulli with probability $L_j^{-\delta}$, by the inductive assumption and the fact that $\beta > \delta$ we have that for all i, $I(X_i \notin G_j^{\mathbb{X}}) \preceq V_i$ and hence

$$\big(I(X_i \notin G_j^{\mathbb{X}}), -I(X_i \notin G_j^{\mathbb{X}}) \log S_j(X_i)\big) \preceq (V_i, V_i(1+U_i))$$

since for $x > 1$

$$\mathbb{P}[-\log S_j(X_i) \ge x] \le L_j^{-\beta} e^{-x m_j} < L_j^{-\delta} e^{-(x-1)m_j} = \mathbb{P}[V_i(1+U_i) \ge x].$$

Summing over $L_j^3 + 1 \le i \le L_j^3 + L_j^{\alpha-1} + t$ completes the result. □

Using Lemma 2.7.2 we can bound the probability of blocks having large length, number of bad sub-blocks or small $\prod_{i=1}^{K_X} S_j(X_{\ell_i})$. This is the key estimate of the chapter.

Lemma 2.7.3. *For all $t', k', x \ge 0$ we have that*

$$\mathbb{P}\left[T_X \ge t', K_X \ge k', -\log \prod_{i=1}^{K_X} S_j(X_{\ell_i}) \ge x\right] \le 2L_j^{-\delta k'/4} \exp\left(-x m_{j+1} - \frac{1}{2} t' L_j^{-4}\right).$$

Proof. If $T_X = t$ and $K_X = k$ then $W_X \ge (t - 2kL_j^3) \vee 0$. Hence when $K_X = 0$

$$\mathbb{P}\left[T_X \ge t', K_X = 0\right] \le \mathbb{P}[W_X \ge t'] = (1 - L_j^{-4})^{t'} \le \exp[-\frac{2}{3} t' L_j^{-4}] \tag{2.7.1}$$

and of course $-\log \prod_{i=1}^{K_X} S_j(X_{\ell_i}) = 0$.

By Lemma 2.7.2 and the fact that $\mathbb{P}[W_X \ge (t - 2kL_j^3)]$ increases with k we have that

$$\begin{aligned}
&\mathbb{P}\left[T_X \ge t', K_X \ge k', -\log \prod_{i=1}^{K_X} S_j(X_{\ell_i}) \ge x\right] \\
&= \sum_{k=k'}^{\infty} \sum_{t=t'}^{\infty} \mathbb{P}\left[T_X = t, K_X(t) = k, \mathcal{G}_X(t) \ge x\right] \\
&\le \sum_{k=k'}^{\infty} \sum_{t=t'}^{\infty} \mathbb{P}[W_X \ge (t - 2kL_j^3), K_X(t) = k, \mathcal{G}_X(t) \ge x] \\
&= \sum_{k=k'}^{\infty} \sum_{t=t'}^{\infty} \mathbb{P}[W_X \ge (t - 2kL_j^3)]\mathbb{P}[K_X(t) = k, \mathcal{G}_X(t) \ge x] \\
&\le \sum_{k=k'}^{\infty} \sum_{t=t'}^{\infty} \mathbb{P}[W_X \ge (t - 2kL_j^3)]\mathbb{P}[\widetilde{K}_X(t) = k, \mathfrak{S}(t) \ge x] \\
&\le \sum_{k=k'}^{\infty} \sum_{t=t'}^{\infty} \exp[-\frac{2}{3}(t - 2kL_j^3)L_j^{-4}]\mathbb{P}[\widetilde{K}_X(t) = k, \mathfrak{S}(t) \ge x].
\end{aligned} \tag{2.7.2}$$

Since $\widetilde{K}_X$ is binomially distributed,

$$\mathbb{P}[\widetilde{K}_X(t) = k] = \binom{L_j^{\alpha-1} + t}{k} L_j^{-\delta k} \left(1 - L_j^{-\delta}\right)^{L_j^{\alpha-1}+t-k}. \tag{2.7.3}$$

If $k \geq 1$, conditional on $\widetilde{K}_X = k$, we have that $\mathfrak{S} - \widetilde{K}_X$ has distribution $\Gamma(k, 1/m_j)$ and so

$$\mathbb{P}[\mathfrak{S} \geq x \mid \widetilde{K}_X(t) = k] = \int_{(x-k)\vee 0}^{\infty} \frac{m_j^k}{(k-1)!} y^{k-1} \exp(-y m_j) dy. \tag{2.7.4}$$

Observe that $\frac{m_j^k}{m_{j+1}(k-1)!} y^{k-1} \exp(-y 2^{-(j+1)})$ is proportional to the density of a $\Gamma(k, 2^{j+1})$ which is maximized at $2^{j+1}(k-1)$. Hence

$$\begin{aligned} \max_{y \geq 0} \frac{m_j^k}{m_{j+1}(k-1)!} y^{k-1} \exp(-y 2^{-(j+1)}) &\leq \frac{m_j^k}{m_{j+1}(k-1)!} (2^{j+1}(k-1))^{k-1} \exp(-(k-1)) \\ &\leq (2^{j+1} m_j)^k, \end{aligned} \tag{2.7.5}$$

since by Stirling's approximation $\frac{(k-1)^{k-1}}{(k-1)!\exp(k-1)} \leq 1$. Since $m_{j+1} = m_j - 2^{-(j+1)}$, substituting (2.7.5) into (2.7.4) we get that

$$\begin{aligned} \mathbb{P}[\mathfrak{S} \geq x \mid \widetilde{K}_X(t) = k] &\leq (2^{j+1} m_j)^k \int_{(x-k)\vee 0}^{\infty} m_{j+1} \exp(-y m_{j+1}) dy \\ &\leq (m_j 2^{j+1} e^{m_{j+1}})^k \exp(-x m_{j+1}). \end{aligned} \tag{2.7.6}$$

Combining (2.7.3) and (2.7.6) we get that,

$$\begin{aligned} &\mathbb{P}[\widetilde{K}_X(t) = k, \mathfrak{S}(t) \geq x] \\ &\quad \leq \binom{L_j^{\alpha-1} + t}{k} L_j^{-\delta k} \left(1 - L_j^{-\delta}\right)^{L_j^{\alpha-1}+t-k} (m_j 2^{j+1} e^{m_{j+1}})^k \exp(-x m_{j+1}) \\ &\quad \leq \frac{\left(1 - L_j^{-\delta}\right)^{L_j^{\alpha-1}+t-k}}{\left(1 - L_j^{-\delta/2}\right)^{L_j^{\alpha-1}+t-k}} (L_j^{-\delta/2} m_j 2^{j+1} e^{m_{j+1}})^k \exp(-x m_{j+1}) \end{aligned} \tag{2.7.7}$$

since

$$\binom{L_j^{\alpha-1} + t}{k} L_j^{-\delta/2k} \left(1 - L_j^{-\delta/2}\right)^{L_j^{\alpha-1}+t-k} = \mathbb{P}[\mathrm{Bin}(L_j^{\alpha-1} + t, L_j^{-\delta/2}) = k] < 1.$$

Now for large enough L_0,

$$\frac{\left(1 - L_j^{-\delta}\right)^{L_j^{\alpha-1}+t-k}}{\left(1 - L_j^{-\delta/2}\right)^{L_j^{\alpha-1}+t-k}} \leq \exp(2(L_j^{\alpha-1} + t) L_j^{-\delta/2}) \leq 2 \exp(2t L_j^{-\delta/2}), \tag{2.7.8}$$

since $\delta/2 > \alpha$. As $L_j = L_0^{\alpha^j}$, for large enough L_0 we have that $L_j^{-\delta/2} m_j 2^{j+1} e^{m_{j+1}} \leq \frac{1}{10} L_j^{-\delta/3}$ and so combining (2.7.7) and (2.7.8) we have that

$$\mathbb{P}[\widetilde{K}_X(t) = k, \mathfrak{S}(t) \geq x] \leq \frac{2}{10^k} \exp(2tL_j^{-\delta/2}) L_j^{-\delta k/3} \exp(-x m_{j+1}). \tag{2.7.9}$$

Finally substituting this into (2.7.2) we get that if $k' \geq 1$,

$$\begin{aligned} &\mathbb{P}\left[T_X \geq t', K_X \geq k', -\log \prod_{i=1}^{K_X} S_j(X_{\ell_i}) \geq x\right] \\ &\quad \leq \sum_{k=k'}^{\infty} \sum_{t=t'}^{\infty} \frac{2}{10^k} \exp[-\frac{2}{3}(t - 2kL_j^3) L_j^{-4} + 2tL_j^{-\delta/2}] L_j^{-\delta k/3} \exp(-x m_{j+1}) \\ &\quad \leq L_j^4 \exp[-\frac{1}{2} t' L_j^{-4}] L_j^{-\delta k'/3} \exp(-x m_{j+1}). \end{aligned} \tag{2.7.10}$$

for large enough L_0 since $\delta/2 > 4$. Since $\delta/3 - \delta/4 > 4$, we get that for $k' \geq 1$,

$$\mathbb{P}\left[T_X \geq t', K_X \geq k', -\log \prod_{i=1}^{K_X} S_j(X_{\ell_i}) \geq x\right] \leq L_j^{-\delta k'/4} \exp(-x m_{j+1} - \frac{1}{2} t' L_j^{-4})$$

which together with (2.7.1) completes the result. □

We now move to our five cases. In each one we will use a different mapping (or mappings) to get good lower bounds on the probability that $X \hookrightarrow Y$ given X.

2.7.1 Case 1

The first case is the generic situation where the blocks are of typical length, have few bad sub-blocks whose embedding probabilities are not too small. This case holds with high probability. We define the event $\mathcal{A}_{X,j+1}^{(1)}$ to be the set of $(j+1)$ level blocks such that

$$\mathcal{A}_{X,j+1}^{(1)} := \left\{ X : T_X \leq \frac{RL_j^{\alpha-1}}{2}, K_X \leq k_0, \prod_{i=1}^{K_X} S_j(X_{\ell_i}) > L_j^{-1/3} \right\}.$$

Lemma 2.7.4. *The probability that $X \in \mathcal{A}_{X,j+1}^{(1)}$ is bounded below by*

$$\mathbb{P}[X \notin \mathcal{A}_{X,j+1}^{(1)}] \leq L_{j+1}^{-3\beta}.$$

Proof. By Lemma 2.7.3

$$\mathbb{P}\left[T_X > \frac{RL_j^{\alpha-1}}{2}\right] \leq 2 \exp\left(-\frac{RL_j^{\alpha-5}}{4}\right) \leq \frac{1}{3} L_{j+1}^{-3\beta}. \tag{2.7.11}$$

since $\alpha > 5$ and L_0 is large. Again by Lemma 2.7.3

$$\mathbb{P}[K_X > k_0] \leq 2L_j^{-\delta k_0/4} = 2L_{j+1}^{-\delta k_0/(4\alpha)} \leq \frac{1}{3}L_{j+1}^{-3\beta}, \tag{2.7.12}$$

since $k_0 > 36\alpha\beta$. Finally again by Lemma 2.7.3,

$$\mathbb{P}[\prod_{i=1}^{K_X} S_j(X_{\ell_i}) \leq L_j^{-1/3}] \leq 2L_j^{-m_{j+1}/3} \leq \frac{1}{3}L_{j+1}^{-3\beta}, \tag{2.7.13}$$

since $\frac{1}{3}m_{j+1} > \frac{1}{3}m > 3\alpha\beta$. Combining (2.7.11), (2.7.12) and (2.7.13) completes the result. □

Lemma 2.7.5. *We have that*

$$\mathbb{P}[X \hookrightarrow Y \mid Y \in \mathcal{A}^{(1)}_{Y,j+1}, X \in \mathcal{A}^{(1)}_{X,j+1}, X] \geq \frac{3}{5}, \tag{2.7.14}$$

and that

$$\mathbb{P}[X \hookrightarrow Y \mid X \in \mathcal{A}^{(1)}_{X,j+1}, Y \in \mathcal{A}^{(1)}_{Y,j+1}] \geq 1 - L_{j+1}^{-3\beta}. \tag{2.7.15}$$

Proof. We first prove equation (2.7.15) where we do not condition on X. Suppose that $X \in \mathcal{A}^{(1)}_{X,j+1}, Y \in \mathcal{A}^{(1)}_{Y,j+1}$. Let us condition on the block lengths T_X, T_Y, the number of bad sub-blocks, K_X, K_Y, their locations, $\ell_1, \ldots, \ell_{K_X}$ and $\ell'_1, \ldots, \ell'_{K_Y}$ and the bad-sub-blocks themselves. Denote this conditioning by

$$\mathcal{F} = \{X \in \mathcal{A}^{(1)}_{X,j+1}, Y \in \mathcal{A}^{(1)}_{Y,j+1}, T_X, T_Y, K_X, K_Y, \ell_1, \ldots, \ell_{K_X}, \ell'_1, \ldots, \ell'_{K_Y}, \\ X_{\ell_1}, \ldots, X_{\ell_{K_X}}, Y_{\ell'_1}, \ldots, Y_{\ell'_{K_Y}}\}.$$

By Proposition 2.6.1 we can find L_j^2 admissible generalized mappings $\Upsilon_h([L_j^{\alpha-1} + 2L_j^3 + T_X], [L_j^{\alpha-1} + 2L_j^3 + T_Y])$ with associated τ_h for $1 \leq h \leq L_j^2$ which are of class G^j with respect to $B = \{\ell_1 < \ldots < \ell_{K_X}\}, B' = \{\ell'_1 < \ldots < \ell'_{K_Y}\}$. By construction we have that $\tau_h(\ell_i) = \tau_1(\ell_i) + h - 1$ and in particular each position ℓ_i is mapped to L_j^2 distinct sub-blocks by the map, none of which is equal to one of the $\ell'_{i'}$. Similarly for the τ_h^{-1}. Hence we can construct a subset $\mathcal{H} \subset [L_j^2]$ with $|\mathcal{H}| = L_j < \lfloor L_j^2/2k_0^2 \rfloor$ so that for all $i_1 \neq i_2$ and $h_1, h_2 \in \mathcal{H}$ we have that $\tau_{h_1}(\ell_{i_1}) \neq \tau_{h_2}(\ell_{i_2})$ and $\tau_{h_1}^{-1}(\ell'_{i_1}) \neq \tau_{h_2}^{-1}(\ell'_{i_2})$, that is that all the positions bad blocks are mapped to are distinct.

By construction all the $Y_{\tau_h(\ell_i)}$ are uniformly chosen good j-blocks conditional on $\mathcal{F}$ and since $S_j(X_{\ell_i}) \geq L_j^{-1/3}$ we have that

$$\mathbb{P}[X_{\ell_i} \hookrightarrow Y_{\tau_h(\ell_i)} \mid \mathcal{F}] \geq S_j(X_{\ell_i}) - \mathbb{P}[Y_{\tau_h(\ell_i)} \notin G_j^{\mathbb{Y}}] \geq \frac{1}{2}S_j(X_{\ell_i}). \tag{2.7.16}$$

Similarly we have

$$\mathbb{P}[X_{\tau_h^{-1}(\ell_i')} \hookrightarrow Y_{\ell_i'} \mid \mathcal{F}] \geq S_j(Y_{\ell_i'}) - \mathbb{P}[X_{\tau_h^{-1}(\ell_i')} \notin G_j^{\mathbb{X}}] \geq \frac{1}{2} S_j(Y_{\ell_i'}). \tag{2.7.17}$$

Let $\mathcal{D}_h$ denote the event

$$\mathcal{D}_h = \left\{ X_{\ell_i} \hookrightarrow Y_{\tau_h(\ell_i)} \text{ for } 1 \leq i \leq K_X, X_{\tau_h^{-1}(\ell_i')} \hookrightarrow Y_{\ell_i'} \text{ for } 1 \leq i \leq K_Y \right\}.$$

By Proposition 2.5.3 if one of the $\mathcal{D}_h$ hold then $X \hookrightarrow Y$. Conditional on $\mathcal{F}$, for $h \in \mathcal{H}$, the $\mathcal{D}_h$ are independent and by (2.7.16) and (2.7.17),

$$\mathbb{P}[\mathcal{D}_h \mid \mathcal{F}] \geq \prod_{i=1}^{K_X} \frac{1}{2} S_j(X_{\ell_i}) \prod_{i=1}^{K_Y} \frac{1}{2} S_j(Y_{\ell_i'}) \geq 2^{-2k_0} L_j^{-2/3}. \tag{2.7.18}$$

Hence

$$\mathbb{P}[X \hookrightarrow Y \mid \mathcal{F}] \geq \mathbb{P}[\cup_{h \in \mathcal{H}} \mathcal{D}_h \mid \mathcal{F}] \geq 1 - \left(1 - 2^{-2k_0} L_j^{-2/3}\right)^{L_j} \geq 1 - L_{j+1}^{-3\beta}. \tag{2.7.19}$$

Now removing the conditioning we get equation (2.7.15). To prove equation (2.7.14) we proceed in the same way but note that since it involves conditioning on the good sub-blocks of X, equation (2.7.17) no longer holds and further the events $X_{\tau_h^{-1}(\ell_i')} \hookrightarrow Y_{\ell_i'}$ are no longer conditionally independent. So we will condition on Y having no bad blocks so

$$\mathcal{F} = \{X \in \mathcal{A}_{X,j+1}^{(1)}, Y \in \mathcal{A}_{Y,j+1}^{(1)}, X, T_X, T_Y, K_X, K_Y = 0, \ell_1, \ldots, \ell_{K_X}, X_{\ell_1}, \ldots, X_{\ell_{K_X}}\}.$$

By the above argument then

$$\mathbb{P}[X \hookrightarrow Y \mid \mathcal{F}] \geq \mathbb{P}[\cup_{h \in \mathcal{H}} \mathcal{D}_h \mid \mathcal{F}(X)] \geq 1 - \left(1 - 2^{-k_0} L_j^{-1/3}\right)^{L_j} \geq 1 - L_{j+1}^{-3\beta}. \tag{2.7.20}$$

Hence

$$\begin{aligned}\mathbb{P}[X \hookrightarrow Y \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, X, T_Y] &\geq \mathbb{P}[X \hookrightarrow Y \mid \mathcal{F}] \cdot \mathbb{P}[K_Y = 0 \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, T_Y] \\ &\geq \left(1 - L_{j+1}^{-3\beta}\right) \cdot \mathbb{P}[K_Y = 0 \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, T_Y].\end{aligned}$$

Removing the conditioning on T_Y we get

$$\begin{aligned}\mathbb{P}[X \hookrightarrow Y \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, X] &\geq \left(1 - L_{j+1}^{-3\beta}\right) \cdot \mathbb{P}[K_Y = 0 \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}] \\ &\geq \left(1 - L_{j+1}^{-3\beta}\right) \cdot (1 - L_{j+1}^{-3\beta} - 2L_j^{-\delta/4}) \geq \frac{3}{5}\end{aligned}$$

for large enough L_0, where the penultimate inequality follows from Lemma 2.7.3 and Lemma 2.7.4. This completes the proof of the lemma. □

Lemma 2.7.6. *When* $\frac{1}{2} \leq p \leq 1 - L_{j+1}^{-1}$

$$\mathbb{P}(S_{j+1}(X) \leq p) \leq p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. By Lemma 2.7.4 and 2.7.5 we have that

$$\begin{aligned}\mathbb{P}(\mathbb{P}[X \not\hookrightarrow Y \mid X] \geq L_{j+1}^{-1}) &\leq \mathbb{P}[X \not\hookrightarrow Y] L_{j+1} \\ &\leq \Big(\mathbb{P}[X \not\hookrightarrow Y \mid X \in \mathcal{A}_{X,j+1}^{(1)}, Y \in \mathcal{A}_{Y,j+1}^{(1)}] \\ &\qquad + \mathbb{P}[X \notin \mathcal{A}_{X,j+1}^{(1)}] + \mathbb{P}[Y \notin \mathcal{A}_{Y,j+1}^{(1)}]\Big) L_{j+1} \\ &\leq 3L_{j+1}^{1-3\beta} \leq 2^{-m_{j+1}} L_{j+1}^{-\beta}\end{aligned}$$

where the first inequality is by Markov's inequality. This implies the lemma. □

2.7.2 Case 2

The next case involves blocks which are not too long and do not contain too many bad sub-blocks but whose bad sub-blocks may have very small embedding probabilities. We define the class of blocks $\mathcal{A}_{X,j+1}^{(2)}$ as

$$\mathcal{A}_{X,j+1}^{(2)} := \left\{ X : T_X \leq \frac{RL_j^{\alpha-1}}{2}, K_X \leq k_0, \prod_{i=1}^{K_X} S_j(X_{\ell_i}) \leq L_j^{-1/3} \right\}.$$

Lemma 2.7.7. *For* $X \in \mathcal{A}_{X,j+1}^{(2)}$,

$$S_{j+1}(X) \geq \min\left\{ \frac{1}{2}, \frac{1}{10} L_j \prod_{i=1}^{K_X} S_j(X_{\ell_i}) \right\}$$

Proof. Suppose that $X \in \mathcal{A}_{X,j+1}^{(2)}$. Let $\mathcal{E}$ denote the event

$$\mathcal{E} = \{W_Y \leq L_j^{\alpha-1}, T_Y = W_Y\}.$$

Then by definition of W_Y, $\mathbb{P}[W_Y \leq L_j^{\alpha-1}] \geq 1-(1-L_j^{-4})^{L_j^{\alpha-1}} \geq 9/10$ while by the definition of the block boundaries the event $T_Y = W_Y$ is equivalent to their being no bad sub-blocks amongst $Y_{L_j^3+L_j^{\alpha-1}+W_Y+1}, \ldots, Y_{L_j^3+L_j^{\alpha-1}+W_Y+2L_j^3}$, that is that we don't need to extend the block because of bad sub-blocks. Hence $\mathbb{P}[T_Y = W_Y] \geq (1-L_j^{-\delta})^{2L_j^3} \geq 9/10$. Combining these we have that

$$\mathbb{P}[\mathcal{E}] \geq 8/10. \tag{2.7.21}$$

On the event $T_Y = W_Y$ we have that the blocks $Y_{L_j^3+1}, \ldots, Y_{L_j^3+L_j^{\alpha-1}+T_Y}$ are uniform j-blocks since the block division did not evaluate whether they are good or bad.

Similarly to Lemma 2.7.5, by Proposition 2.6.4 we can find L_j^2 admissible generalized mappings $\Upsilon_h([L_j^{\alpha-1}+2L_j^3+T_X],[L_j^{\alpha-1}+2L_j^3+T_Y])$ for $1\le h\le L_j^2$ with associated τ_h which are of class H_1^j with respect to $B=\{\ell_1<\ldots<\ell_{K_X}\}$. For all h and i, $L_j^3+1\le \tau_h(\ell_i)\le L_j^3+L_j^{\alpha-1}+T_Y$. As in Lemma 2.7.5 we can construct a subset $\mathcal{H}\subset[L_j^2]$ with $|\mathcal{H}|=L_j<\lfloor L_j^2/k_0^2\rfloor$ so that for all $i_1\neq i_2$ and $h_1,h_2\in\mathcal{H}$ we have that $\tau_{h_1}(\ell_{i_1})\neq\tau_{h_2}(\ell_{i_2})$, that is that all the positions bad blocks are mapped to are distinct. We will estimate the probability that one of these maps work.

In trying out these h different mappings there is a subtle conditioning issue since a map failing may imply that Y_{τ_h} is not good. As such we condition on an event $\mathcal{D}_h\cup\mathcal{G}_h$ which holds with high probability. Let $\mathcal{D}_h$ denote the event

$$\mathcal{D}_h=\left\{X_{\ell_i}\hookrightarrow Y_{\tau_h(\ell_i)} \text{ for } 1\le i\le K_X\right\}.$$

and let

$$\mathcal{G}_h=\left\{Y_{\tau_h(\ell_i)}\in G_j^{\mathbb{Y}} \text{ for } 1\le i\le K_X\right\}.$$

Then

$$\mathbb{P}[\mathcal{D}_h\cup\mathcal{G}_h\mid X,\mathcal{E}]\ge\mathbb{P}[\mathcal{G}_h\mid X,\mathcal{E}]\ge(1-L_j^{-\delta})^{k_0}\ge 1-2k_0L_j^{-\delta}.$$

and since they are conditionally independent given X and $\mathcal{E}$,

$$\mathbb{P}[\cap_{h\in\mathcal{H}}(\mathcal{D}_h\cup\mathcal{G}_h)\mid X,\mathcal{E}]\ge(1-L_j^{-\delta})^{k_0L_j}\ge 9/10. \tag{2.7.22}$$

Now

$$\mathbb{P}[\mathcal{D}_h\mid X,\mathcal{E},(\mathcal{D}_h\cup\mathcal{G}_h)]\ge\mathbb{P}[\mathcal{D}_h\mid X,\mathcal{E}]=\prod_{i=1}^{K_X}S_j(X_{\ell_i})$$

and hence

$$\begin{aligned}\mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h\mid X,\mathcal{E},\cap_{h\in\mathcal{H}}(\mathcal{D}_h\cup\mathcal{G}_h)]&\ge 1-\left(1-\prod_{i=1}^{K_X}S_j(X_{\ell_i})\right)^{L_j}\\&\ge\frac{9}{10}\wedge\frac{1}{4}L_j\prod_{i=1}^{K_X}S_j(X_{\ell_i})\end{aligned} \tag{2.7.23}$$

since $1-e^{-x}\ge x/4\wedge 9/10$ for $x\ge 0$. Furthermore, if

$$\mathcal{M}=\{\exists h_1\neq h_2\in\mathcal{H}:\mathcal{D}_{h_1}\setminus\mathcal{G}_{h_1},\mathcal{D}_{h_2}\setminus\mathcal{G}_{h_2}\},$$

then

$$\begin{aligned}\mathbb{P}[\mathcal{M}\mid X,\mathcal{E},\cap_{h\in\mathcal{H}}(\mathcal{D}_h\cup\mathcal{G}_h)]&\le\binom{L_j}{2}\mathbb{P}[\mathcal{D}_h\setminus\mathcal{G}_h\mid X,\mathcal{E},\cap_{h\in\mathcal{H}}(\mathcal{D}_h\cup\mathcal{G}_h)]^2\\&\le\binom{L_j}{2}2\left(\prod_{i=1}^{K_X}S_j(X_{\ell_i})\wedge 2k_0L_j^{-\delta}\right)^2\\&\le 2k_0L_j^{-(\delta-2)}\prod_{i=1}^{K_X}S_j(X_{\ell_i}).\end{aligned} \tag{2.7.24}$$

Finally let $\mathcal{J}$ denote the event

$$\mathcal{J} = \left\{Y_k \in G_j^{\mathbb{Y}} \text{ for all } k \in \{L_j^3+1,\ldots,L_j^3+L_j^{\alpha-1}+T_Y\} \setminus \cup_{h\in\mathcal{H},1\le i\le K_X}\{\tau_h(\ell_i)\}\right\}.$$

Then

$$\mathbb{P}[\mathcal{J} \mid X,\mathcal{E}] \ge \left(1-L_j^{-\delta}\right)^{2L_j^{\alpha-1}} \ge 9/10. \tag{2.7.25}$$

If $\mathcal{J}, \cup_{h\in\mathcal{H}}\mathcal{D}_h$ and $\cap_{h\in\mathcal{H}}(\mathcal{D}_h\cup\mathcal{G}_h)$ all hold and $\mathcal{M}$ does not hold then we can find at least one $h\in\mathcal{H}$ such that $\mathcal{D}_h$ holds and $\mathcal{G}_{h'}$ holds for all $h'\in\mathcal{H}\setminus\{h\}$. Then by Proposition 2.5.4 we have that $X\hookrightarrow Y$. Hence by (2.7.22), (2.7.23), (2.7.24), and (2.7.25) and the fact that $\mathcal{J}$ is conditionally independent of the other events that

$$\begin{aligned}
\mathbb{P}[X\hookrightarrow Y \mid X,\mathcal{E}] &\ge \mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h, \cap_{h\in\mathcal{H}}(\mathcal{D}_h\cup\mathcal{G}_h), \mathcal{J},\ \neg\mathcal{M} \mid X,\mathcal{E}] \\
&= \mathbb{P}[\mathcal{J} \mid X,\mathcal{E}]\mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h,\ \neg\mathcal{M} \mid X,\mathcal{E},\cap_{h\in\mathcal{H}}(\mathcal{D}_h\cup\mathcal{G}_h)] \\
&\quad\times \mathbb{P}[\cap_{h\in\mathcal{H}}(\mathcal{D}_h\cup\mathcal{G}_h) \mid X,\mathcal{E}] \\
&\ge \frac{81}{100}\left[\left(\frac{9}{10}\wedge\frac{1}{4}L_j\prod_{i=1}^{K_X}S_j(X_{\ell_i})\right) - 2k_0L_j^{-(\delta-2)}\prod_{i=1}^{K_X}S_j(X_{\ell_i})\right] \\
&\ge \frac{7}{10}\wedge\frac{1}{5}L_j\prod_{i=1}^{K_X}S_j(X_{\ell_i}).
\end{aligned}$$

Combining with (2.7.21) we have that

$$\mathbb{P}[X\hookrightarrow Y \mid X] \ge \frac{1}{2}\wedge\frac{1}{10}L_j\prod_{i=1}^{K_X}S_j(X_{\ell_i}),$$

which completes the proof. □

Lemma 2.7.8. *When* $0<p<\frac{1}{2}$,

$$\mathbb{P}(X\in\mathcal{A}_{X,j+1}^{(2)}, S_{j+1}(X)\le p) \le \frac{1}{5}p^{m_{j+1}}L_{j+1}^{-\beta}$$

Proof. We have that

$$\begin{aligned}
\mathbb{P}(X\in\mathcal{A}_{X,j+1}^{(2)}, S_{j+1}(X)\le p) &\le \mathbb{P}\left[\frac{1}{10}L_j\prod_{i=1}^{K_X}S_j(X_{\ell_i})\le p\right] \\
&\le 2\left(\frac{10p}{L_j}\right)^{m_{j+1}} \le \frac{1}{5}p^{m_{j+1}}L_{j+1}^{-\beta}
\end{aligned} \tag{2.7.26}$$

where the first inequality holds by Lemma 2.7.7, the second by Lemma 2.7.3 and the third holds for large enough L_0 since $m_{j+1}>m>\alpha\beta$. □

2.7.3 Case 3

The third case allows for a greater number of bad sub-blocks. The class of blocks $\mathcal{A}^{(3)}_{X,j+1}$ is defined as

$$\mathcal{A}^{(3)}_{X,j+1} := \left\{ X : T_X \leq \frac{RL_j^{\alpha-1}}{2}, k_0 \leq K_X \leq \frac{L_j^{\alpha-1} + T_X}{10R_j^+} \right\}.$$

Lemma 2.7.9. *For* $X \in \mathcal{A}^{(3)}_{X,j+1}$,

$$S_{j+1}(X) \geq \frac{1}{2} \prod_{i=1}^{K_X} S_j(X_{\ell_i})$$

Proof. The proof is a simpler version of Lemma 2.7.7 where this time we only need consider a single map Υ. Suppose that $X \in \mathcal{A}^{(3)}_{X,j+1}$. Again let $\mathcal{E}$ denote the event

$$\mathcal{E} = \{W_Y \leq L_j^{\alpha-1}, T_Y = W_Y\}.$$

Similarly to (2.7.21) we have that,

$$\mathbb{P}[\mathcal{E}] \geq 8/10. \tag{2.7.27}$$

On the event $T_Y = W_Y$ we have that the blocks $Y_{L_j^3+1}, \ldots, Y_{L_j^3+L_j^{\alpha-1}+T_Y}$ are uniform j-blocks since the block division did not evaluate whether they are good or bad.

By Proposition 2.6.5 we can find an admissible generalized mapping $\Upsilon([L_j^{\alpha-1} + 2L_j^3 + T_X], [L_j^{\alpha-1} + 2L_j^3 + T_Y])$ with associated τ which are of class H_2^j with $B = \{\ell_1 < \ldots < \ell_{K_X}\}$ so that for all i, $L_j^3 + 1 \leq \tau(\ell_i) \leq L_j^3 + L_j^{\alpha-1} + T_Y$. We estimate the probability that this gives an embedding.

Let $\mathcal{D}$ denote the event

$$\mathcal{D} = \left\{ X_{\ell_i} \hookrightarrow Y_{\tau(\ell_i)} \text{ for } 1 \leq i \leq K_X \right\}.$$

By definition,

$$\mathbb{P}[\mathcal{D} \mid X, \mathcal{E}] = \prod_{i=1}^{K_X} S_j(X_{\ell_i}) \tag{2.7.28}$$

Let $\mathcal{J}$ denote the event

$$\mathcal{J} = \left\{ Y_k \in G_j^{\mathbb{Y}} \text{ for all } k \in \{L_j^3 + 1, \ldots, L_j^3 + L_j^{\alpha-1} + T_X\} \setminus \cup_{1 \leq i \leq K_X} \{\tau(\ell_i)\} \right\}.$$

Then for large enough L_j,

$$\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \geq \left(1 - L_j^{-\delta}\right)^{2L_j^{\alpha-1}} \geq 9/10. \tag{2.7.29}$$

If $\mathcal{D}$ and $\mathcal{J}$ hold then by Proposition 2.5.4 we have that $X \hookrightarrow Y$. Hence by (2.7.28) and (2.7.29) and the fact that $\mathcal{D}$ and $\mathcal{J}$ are conditionally independent we have that,

$$\begin{aligned}\mathbb{P}[X \hookrightarrow Y \mid X, \mathcal{E}] &\geq \mathbb{P}[\mathcal{D}, \mathcal{J} \mid X, \mathcal{E}] \\ &= \mathbb{P}[\mathcal{D} \mid X, \mathcal{E}]\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \\ &\geq \frac{9}{10} \prod_{i=1}^{K_X} S_j(X_{\ell_i}).\end{aligned}$$

Combining with (2.7.27) we have that

$$\mathbb{P}[X \hookrightarrow Y \mid X] \geq \frac{1}{2} \prod_{i=1}^{K_X} S_j(X_{\ell_i}),$$

which completes the proof. □

Lemma 2.7.10. *When* $0 < p \leq \frac{1}{2}$,

$$\mathbb{P}(X \in \mathcal{A}^{(3)}_{X,j+1}, S_{j+1}(X) \leq p) \leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. We have that

$$\begin{aligned}\mathbb{P}(X \in \mathcal{A}^{(3)}_{X,j+1}, S_{j+1}(X) \leq p) &\leq \mathbb{P}\left[K_X \geq k_0, \frac{1}{2}\prod_{i=1}^{K_X} S_j(X_{\ell_i}) \leq p\right] \\ &\leq 2\,(2p)^{m_{j+1}}\,L_j^{-\delta k_0/4} \leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta} \qquad (2.7.30)\end{aligned}$$

where the first inequality holds by Lemma 2.7.9, the second by Lemma 2.7.3 and the third holds for large enough L_0 since $\delta k_0 > 4\alpha\beta$. □

2.7.4 Case 4

Case 4 is the case of blocks of long length but not too many bad sub-blocks (at least with a density of them smaller than $(10R_j^+)^{-1}$). The class of blocks $\mathcal{A}^{(4)}_{X,j+1}$ is defined as

$$\mathcal{A}^{(4)}_{X,j+1} := \left\{ X : T_X > \frac{R L_j^{\alpha-1}}{2}, K_X \leq \frac{L_j^{\alpha-1} + T_X}{10 R_j^+} \right\}.$$

Lemma 2.7.11. *For* $X \in \mathcal{A}^{(4)}_{X,j+1}$,

$$S_{j+1}(X) \geq \prod_{i=1}^{K_X} S_j(X_{\ell_i}) \exp(-3 T_X L_j^{-4}/R)$$

Proof. The proof is a modification of Lemma 2.7.9 allowing the length of Y to grow at a slower rate than X. Suppose that $X \in \mathcal{A}^{(4)}_{X,j+1}$ and let $\mathcal{E}(X)$ denote the event

$$\mathcal{E}(X) = \{W_Y = \lfloor 2T_X/R \rfloor, T_Y = W_Y\}.$$

Then by definition $\mathbb{P}[W_Y = \lfloor 2T_X/R \rfloor] = L_j^{-4}(1-L_j^{-4})^{\lfloor 2T_X/R \rfloor}$. Similarly to Lemma 2.7.7, $\mathbb{P}[T_Y = W_Y \mid W_Y] \geq (1-L_j^{-\delta})^{2L_j^3} \geq 9/10$. Combining these we have that

$$\mathbb{P}[\mathcal{E}(X)] \geq \frac{9}{10} L_j^{-4}(1-L_j^{-4})^{\lfloor 2T_X/R \rfloor}. \tag{2.7.31}$$

By Proposition 2.6.5 we can find an admissible generalized mapping $\Upsilon([L_j^{\alpha-1} + 2L_j^3 + T_X], [L_j^{\alpha-1} + 2L_j^3 + T_Y])$ with associated τ which is of class H_2^j with respect to $B = \{\ell_1 < \ldots < \ell_{K_X}\}$ so that for all i, $L_j^3 + 1 \leq \tau(\ell_i) \leq L_j^3 + L_j^{\alpha-1} + T_Y$. We again estimate the probability that this gives an embedding.

Defining $\mathcal{D}$ and $\mathcal{J}$ as in Lemma 2.7.9 and we again have that (2.7.28) holds. Then for large enough L_0,

$$\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}(X)] \geq \left(1-L_j^{-\delta}\right)^{L_j^{\alpha-1} + \lfloor 2T_X/R \rfloor + 2L_j^3} \geq \exp\left(-2L_j^{-\delta}(L_j^{\alpha-1} + \lfloor 2T_X/R \rfloor + 2L_j^3)\right). \tag{2.7.32}$$

If $\mathcal{D}$ and $\mathcal{J}$ hold then by Proposition 2.5.4 we have that $X \hookrightarrow Y$. Hence by (2.7.28) and (2.7.32) and the fact that $\mathcal{D}$ and $\mathcal{J}$ are conditionally independent we have that,

$$\begin{aligned}\mathbb{P}[X \hookrightarrow Y \mid X, \mathcal{E}] &\geq \mathbb{P}[\mathcal{D} \mid X, \mathcal{E}]\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \\ &\geq \exp\left(-2L_j^{-\delta}(L_j^{\alpha-1} + \lfloor 2T_X/R \rfloor + 2L_j^3)\right) \prod_{i=1}^{K_X} S_j(X_{\ell_i}).\end{aligned}$$

Combining with (2.7.31) we have that for large enough L_0

$$\mathbb{P}[X \hookrightarrow Y \mid X] \geq \exp(-3T_X L_j^{-4}/R) \prod_{i=1}^{K_X} S_j(X_{\ell_i}),$$

since $T_X L_j^{-4} = \Omega(L_j^{\alpha-6})$ and $\delta > 5$ which completes the proof. ∎

Lemma 2.7.12. *When* $0 < p \leq \frac{1}{2}$,

$$\mathbb{P}(X \in \mathcal{A}^{(4)}_{X,j+1}, S_{j+1}(X) \leq p) \leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. We have that

$$\begin{aligned}
\mathbb{P}(X \in \mathcal{A}^{(4)}_{X,j+1}, S_{j+1}(X) \leq p) &\leq \sum_{t=\frac{RL_j^{\alpha-1}}{2}+1}^{\infty} \mathbb{P}\left[T_X = t, \prod_{i=1}^{K_X} S_j(X_{\ell_i}) \exp(-3tL_j^{-4}/R) \leq p\right] \\
&\leq \sum_{t=\frac{RL_j^{\alpha-1}}{2}+1}^{\infty} 2\left(p\exp(3tL_j^{-4}/R)\right)^{m_{j+1}} \exp\left(-\frac{1}{2}tL_j^{-4}\right) \\
&\leq \frac{1}{5}p^{m_{j+1}}L_{j+1}^{-\beta}
\end{aligned} \tag{2.7.33}$$

where the first inequality holds by Lemma 2.7.11, the second by Lemma 2.7.3 and the third holds for large enough L_0 since $3m_{j+1}/R < \frac{1}{2}$ and so for large enough L_0,

$$\sum_{t=RL_j^{\alpha-1}/2+1}^{\infty} \exp\left(-tL_j^{-4}\left(\frac{1}{2} - \frac{3m_{j+1}}{R}\right)\right) < \frac{1}{10}L_{j+1}^{-\beta}.$$

□

2.7.5 Case 5

The final case involves blocks with a large density of bad sub-blocks. The class of blocks $\mathcal{A}^{(5)}_{X,j+1}$ is defined as

$$\mathcal{A}^{(5)}_{X,j+1} := \left\{X : K_X > \frac{L_j^{\alpha-1} + T_X}{10R_j^+}\right\}.$$

Lemma 2.7.13. *For $X \in \mathcal{A}^{(5)}_{X,j+1}$,*

$$S_{j+1}(X) \geq \exp(-2T_X L_j^{-4}) \prod_{i=1}^{K_X} S_j(X_{\ell_i})$$

Proof. The proof follows by minor modifications of Lemma 2.7.11. We take $\mathcal{E}(X)$ to denote the event

$$\mathcal{E}(X) = \{W_Y = T_X, T_Y = W_Y\}.$$

and get a bound of

$$\mathbb{P}[\mathcal{E}(X)] \geq \frac{9}{10}L_j^{-4}(1 - L_j^{-4})^{T_X}. \tag{2.7.34}$$

We take as our mapping the complete partitions $\{0 \leq 1 \leq 2 \leq \ldots \leq 2L_j^3 + L_j^{\alpha-1} + T_X\}$ and $\{0 \leq 1 \leq 2 \leq \ldots \leq 2L_j^3 + L_j^{\alpha-1} + T_Y\}$ and so are simply mapping sub-blocks to sub-blocks. The new bound for $\mathcal{J}$ becomes

$$\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}(X)] \geq \left(1 - L_j^{-\delta}\right)^{L_j^{\alpha-1}+T_X+2L_j^3} \geq \exp\left(-2L_j^{-\delta}(L_j^{\alpha-1} + T_X + 2L_j^3)\right). \tag{2.7.35}$$

Proceeding as in Lemma 2.7.11 then yields the result. □

Lemma 2.7.14. *When* $0 < p \le \frac{1}{2}$,

$$\mathbb{P}(X \in \mathcal{A}^{(5)}_{X,j+1}, S_{j+1}(X) \le p) \le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. First note that since $\alpha > 4$,

$$L_j^{-\frac{\delta}{40R_j^+}} = L_0^{-\frac{\delta\alpha^j}{40R_j^+}} \to 0$$

as $j \to \infty$. Hence for large enough L_0,

$$\sum_{t=0}^{\infty} \left(\exp(2m_{j+1}L_j^{-4}) L_j^{-\frac{\delta}{40R_j^+}} \right)^t < 2. \tag{2.7.36}$$

We have that

$$\begin{aligned}
&\mathbb{P}(X \in \mathcal{A}^{(5)}_{X,j+1}, S_{j+1}(X) \le p) \\
&\qquad \le \sum_{t=0}^{\infty} \mathbb{P}\left[T_X = t, K_X > \frac{L_j^{\alpha-1}+t}{10R_j^+}, \prod_{i=1}^{K_X} S_j(X_{\ell_i}) \exp(-2tL_j^{-4}) \le p \right] \\
&\le p^{m_{j+1}} \sum_{t=0}^{\infty} 2\left(\exp(2m_{j+1}tL_j^{-4})\right) L_j^{-\frac{\delta(L_j^{\alpha-1}+t)}{40R_j^+}} \\
&\le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}
\end{aligned} \tag{2.7.37}$$

where the first inequality holds be by Lemma 2.7.13, the second by Lemma 2.7.3 and the third follows by (2.7.36) and the fact that

$$L_j^{-\frac{\delta L_j^{\alpha-1}}{40R_j^+}} \le \frac{1}{20} L_{j+1}^{-\beta},$$

for large enough L_0. □

2.7.6 Proof of Theorem 2.7.1

We now put together the five cases to establish the tail bounds.

Proof of Theorem 2.7.1. The case of $\frac{1}{2} \le p \le 1 - L_{j+1}^{-1}$ is established in Lemma 2.7.6. By Lemma 2.7.5 and Lemma 2.7.4 we have that $S_{j+1}(X) \ge \frac{1}{2}$ for all $X \in \mathcal{A}^{(1)}_{X,j+1}$ since L_0 is sufficiently large. Hence we need only consider $0 < p < \frac{1}{2}$ and cases 2 to 5. By Lemmas 2.7.8, 2.7.10, 2.7.12 and 2.7.14 then

$$\mathbb{P}(S_{j+1}(X) \le p) \le \sum_{l=2}^{5} \mathbb{P}(X \in \mathcal{A}^{(l)}_{X,j+1}, S_{j+1}(X) \le p) \le p^{m_{j+1}} L_{j+1}^{-\beta}.$$

The bound for $S^{\mathbb{Y}}_{j+1}$ follows similarly. □

2.8 Length Estimate

Theorem 2.8.1. *Let X be an $\mathbb{X}$ block at level $(j+1)$ we have that*

$$\mathbb{E}[\exp(L_j^{-6}(|X|-(2-2^{-(j+1)})L_{j+1}))] \leq 1. \tag{2.8.1}$$

and hence for $x \geq 0$,

$$\mathbb{P}(|X| > ((2-2^{-(j+1)})L_{j+1} + xL_j^6) \leq e^{-x}. \tag{2.8.2}$$

Proof. By the inductive hypothesis we have for X, a random $\mathbb{X}$-block at level j,

$$\mathbb{E}[\exp(L_{j-1}^{-6}(|X|-(2-2^{-j})L_j))] \leq 1. \tag{2.8.3}$$

It follows that

$$\mathbb{E}[\exp(L_{j-1}^{-6}(|X|-(2-2^{-j})L_j))|X \in G_j^{\mathbb{X}}] \leq \mathbb{P}[X \in G_j^{\mathbb{X}}]^{-1} \leq \frac{1}{1-L_j^{-\delta}} \leq 1+2L_j^{-\delta}, \tag{2.8.4}$$

since L_0 is large enough. Since $0 \leq L_j^{-6} \leq 2L_j^{-6} \leq L_{j-1}^{-6}$ Jensen's inequality and equation (2.8.3) imply that

$$\mathbb{E}[\exp(2L_j^{-6}(|X|-(2-2^{-j})L_j))] \leq 1 \tag{2.8.5}$$

and similarly

$$\mathbb{E}[\exp(L_j^{-6}(|X|-(2-2^{-j})L_j))|X \in G_j^{\mathbb{X}}] \leq 1+2L_j^{-\delta}. \tag{2.8.6}$$

Let $\widetilde{X} = (X_1, X_2, \ldots)$ be a sequence of independent $\mathbb{X}$-blocks at level j with the distribution specified by $X_i \sim \mu_{j,G}^{\mathbb{X}}$ for $i = 1, \ldots, L_j^3$ and $X_i \sim \mu_j^{\mathbb{X}}$ for $i > L_j^3$. Let $X = (X_1, X_2, \ldots, X_{L_j^{\alpha-1}+2L_j^3+T_X})$ be the $(j+1)$ level $\mathbb{X}$-block obtained from $\widetilde{X}$. Then since T_X is independent of the first L_j^3 sub-blocks we have

$$\begin{aligned}
\mathbb{E}[\exp(L_j^{-6}|X|)] &= \mathbb{E}[\sum_{t=0}^{\infty}\exp(L_j^{-6}\sum_{i=1}^{2L_j^3+L_j^{\alpha-1}+t}|X_i|)I[T_X=t]] \\
&= \mathbb{E}\left[\exp\left(L_j^{-6}\sum_{i=1}^{L_j^3}|X_i|\right)\right] \\
&\quad \cdot \sum_{t=0}^{\infty}\mathbb{P}[T_X=t]^{\frac{1}{2}}\mathbb{E}\left[\exp\left(2L_j^{-6}\sum_{i=L_j^3+1}^{2L_j^3+L_j^{\alpha-1}+t}|X_i|\right)\right]^{\frac{1}{2}},
\end{aligned}$$

using Hölder's Inequality. Now using (2.8.5), (2.8.6) and Lemma 2.7.3 it follows from the above equation that

$$\begin{aligned}\mathbb{E}[\exp(L_j^{-6}|X|)] &\leq 2\left(1+2L_j^{-\delta}\right)^{L_j^3}\sum_{t=0}^{\infty}\exp\left(L_j^{-5}(2-2^{-j})(L_j^{\alpha-1}+2L_j^3+t)-\frac{1}{4}tL_j^{-4}\right)\\ &\leq 4\exp\left((2-2^{-j})(L_j^{\alpha-6}+2L_j^{-2})\right)\sum_{t=0}^{\infty}\left(\exp(L_j^{-5}(2-2^{-j})-\frac{1}{4}L_j^{-4})\right)^t\\ &\leq \exp\left((2-2^{-(j+1)})L_j^{\alpha-6}\right),\end{aligned}$$

since $\alpha > 6$. It follows that

$$\mathbb{E}[\exp(L_j^{-6}(|X|-(2-2^{-(j+1)})L_{j+1}))] \leq 1 \tag{2.8.7}$$

while equation (2.8.2) follows by Markov's inequality which completes the proof of the theorem.

□

2.9 Estimates for Good Blocks

2.9.1 Most Blocks are Good

Theorem 2.9.1. *Let X be a $\mathbb{X}$-block at level $(j+1)$. Then $\mathbb{P}(X \in G_{j+1}^{\mathbb{X}}) \geq 1 - L_{j+1}^{-\delta}$. Similarly for $\mathbb{Y}$-block Y at level $(j+1)$, $\mathbb{P}(Y \in G_{j+1}^{\mathbb{Y}}) \geq 1 - L_{j+1}^{-\delta}$.*

Before proving the theorem we need the following lemma to show that a sequence of $\lfloor L_j^{3/2} \rfloor$ independent level j subblocks is with high probability strong.

Lemma 2.9.2. *$X = (X_1, \ldots X_{\lfloor L_j^{3/2} \rfloor})$ be a sequence of $\lfloor L_j^{3/2} \rfloor$ independent subblocks at level j. Then*

(a)

$$\mathbb{P}(X \text{ is “strong”}) \geq 1 - e^{-\frac{L_j^{5/4}}{2}}.$$

(b) Let, for $i = 1, 2, \ldots L_j^{3/2}$, $\mathcal{E}_i = \{X^{[1,i]}$ is “good” $\}$. Then for each i,

$$\mathbb{P}(X \text{ is “strong”}|\mathcal{E}_i) \geq 1 - e^{-\frac{L_j^{5/4}}{2}}.$$

Proof. We only prove part (b). Part (a) is similar. Let Y be a fixed semi-bad block at level j. Each of the events $\{X_k \hookrightarrow Y\}$ are independent, they are independent conditional on $\mathcal{E}_i$ as well. Now, for $k > i$

$$\mathbb{P}(X_k \hookrightarrow Y|\mathcal{E}_i) \geq 1 - 1/20k_0R_{j+1}^{+} \tag{2.9.1}$$

and for $k \leq i$

$$\mathbb{P}(X_k \hookrightarrow Y|\mathcal{E}_i) \geq (1 - 1/20k_0 R^+_{j+1} - L_j^{-\delta}). \tag{2.9.2}$$

Since L_0 is sufficiently large, we have $L_j^\delta > 60k_0 R^+_{j+1}$. It then follows that, conditional on $\mathcal{E}_i$,

$$\#\{k : X_k \hookrightarrow Y\} \succeq V$$

where V has a $\mathrm{Bin}(\lfloor L_j^{3/2} \rfloor, (1 - 1/15k_0 R^+_{j+1}))$ distribution. Using Hoeffding's inequality, we get

$$\begin{aligned}
&\mathbb{P}(\#\{k : X_k \hookrightarrow Y\} \geq \lfloor L_j^{3/2} \rfloor (1 - 1/10k_0 R^+_{j+1})|\mathcal{E}_i) \\
&\geq \mathbb{P}(V \geq \lfloor L_j^{3/2} \rfloor (1 - 1/10k_0 R^+_{j+1})) \geq 1 - 2e^{-\frac{\lfloor L_j^{3/2} \rfloor}{450k_0^2 (R^+_{j+1})^2}} \geq 1 - e^{-L_j^{5/4}}
\end{aligned}$$

for L_0 sufficiently large. Since the length of a semi-bad block at level j can be at most $10L_j$, and semi-bad blocks can contain only the first L_j^m many characters, there can be at most $L_j^{10mL_j}$ many semi-bad blocks at level j.

Hence, using a union bound we get, for each i,

$$\mathbb{P}(X \text{ is "strong"}|\mathcal{E}_i) \geq 1 - e^{10mL_j \log L_j} e^{-L_j^{5/4}} \geq 1 - e^{-\frac{L_j^{5/4}}{2}}$$

for large enough L_0, completing the proof of the lemma. □

Proof of Theorem 2.9.1. To avoid repetition, we only prove the theorem for $\mathbb{X}$-blocks.

Recall the notation of Observation 2.3.2 with $(X_1, X_2, X_3, ...)$ a sequence of independent $\mathbb{X}$-blocks at level j with the first L_j^3 conditioned to be good and $X \sim \mu^{\mathbb{X}}_{j+1}$ be the $(j+1)$-th level block constructed from them. Let W_X be the $\mathrm{Geom}(L_j^{-4})$ variable associated with X and T_X be the number of excess blocks. Let us define the following events.

$$A_1 = \{(X_i, X_{i+1}, \ldots X_{i+\lfloor L_j^{3/2} \rfloor}) \text{is a strong sequence for } 1 \leq i \leq 2L_j^{\alpha-1}\}.$$

$$A_2 = \{\#\{1 \leq i \leq L_j^{\alpha-1} + 2L_j^3 + T_X : X_i \notin G_j^{\mathbb{X}}\} \leq k_0\}.$$

$$A_3 = \{\#\{1 \leq i \leq 2L_j^{\alpha-1} : X_i \notin G_j^{\mathbb{X}} \cup SB_j^{\mathbb{X}}\} = 0\}.$$

$$A_4 = \{T_X \leq L_j^5 - 2L_j^3\}.$$

From the definition of good blocks it follows that

$$\mathbb{P}(X \in G^{\mathbb{X}}_{j+1}) \geq \mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4). \tag{2.9.3}$$

Now, to complete the proof we need suitable estimates for the quantities $\mathbb{P}(A_i)$, $i = 1, 2, 3, 4$, each of which we now compute.

• Let $\widetilde{X_i} = (X_{i+1}, X_{i+2}, \ldots, X_{i+\lfloor L_j^{3/2} \rfloor})$. From Lemma 2.9.2, it follows that for each i,

$$\mathbb{P}(\widetilde{X_i} \text{ “is strong” }) \geq 1 - e^{-\frac{L_j^{5/4}}{2}}.$$

It follows that

$$\mathbb{P}[A_1^c] \leq 2L_j^{\alpha-1} e^{-\frac{L_j^{5/4}}{2}} \leq \frac{1}{10} L_{j+1}^{-\delta} \tag{2.9.4}$$

since L_0 is sufficieciently large.

• By Lemma 2.7.3 we have that

$$\mathbb{P}[A_2] \geq 1 - L_j^{-\delta k_0/4} \geq 1 - \frac{1}{10} L_{j+1}^{-\delta} \tag{2.9.5}$$

since $k_0 > 4\alpha$.

• From the definition of semi-bad blocks, we know that for $i > L_j^3$,

$$\mathbb{P}(X_i \notin G_j^{\mathbb{X}} \cup SB_j^{\mathbb{X}}) \leq \mathbb{P}(S_j^{\mathbb{X}}(X_i) \leq 1 - \frac{1}{20k_0 R_{j+1}^+}) + \mathbb{P}(|X_i| > 10L_j)$$
$$+ \mathbb{P}(C_k \in X_i \text{ for some } k > L_j^m).$$

Claim: We have

$$\mathbb{P}(C_k \in X_i \text{ for some } k > L_j^m) \leq \mu^{\mathbb{X}}(\{C_{L_j^m+1}, C_{L_j^m+2}, \ldots\})\mathbb{E}(|X_i|). \tag{2.9.6}$$

Proof of Claim. Let A_r denote the event that $\{C_k \in X_r^{(j)}$ for some $k > L_j^m\}$ where $X_r^{(j)}$ denotes the r-th block at level j. Observation 2.3.1 and strong law of large numbers then imply

$$\lim \frac{1}{n} \sum_{r=1}^{n} I(A_r) = \mathbb{P}(C_k \in X_i \text{ for some } k > L_j^m) \text{ a.s. .} \tag{2.9.7}$$

Let B_s denote the event that $\{X_s^{(0)} = C_k$ for some $k > L_j^m\}$ where $X_s^{(0)}$ denotes the s-th element of the sequence $\mathbb{X}$, i.e., the s-th block at level 0. Observe that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} I(A_r) \leq \limsup_{N\to\infty} \frac{\sum_{s=1}^{N} I(B_s)}{\max\{t : \sum_{h=1}^{t} |X_h^{(j)}| \leq N\}} \text{ a.s. .} \tag{2.9.8}$$

Dividing the numerator and denominator of the right hand side of (2.9.8) by N and using strong law of large numbers again we get that the a.s. limit of the right hand side of (2.9.8) is $\mu^{\mathbb{X}}(\{C_{L_j^m+1}, C_{L_j^m+2}, \ldots\})\mathbb{E}(|X_i|)$. Comparing (2.9.7) and (2.9.8) completes the proof of the claim.

Using (2.1.1) and (2.4.3), it follows that $\mathbb{P}(C_k \in X_i$ for some $k > L_j^m) \leq 3L_j^{1-m} \leq L_j^{-\beta}$ for L_0 large enough and since $m > 2 + \beta$.

Since for L_0 sufficiently large, $1-\frac{1}{20k_0R^+_{j+1}} \leq 1-L_{j+1}^{-1}$, using (2.4.1) and (2.4.3) we see that

$$\mathbb{P}(X_i \notin G_j^{\mathbb{X}} \cup SB_j^{\mathbb{X}}) \leq (1-\frac{1}{20k_0R^+_{j+1}})^m L_j^{-\beta} + \mathbb{P}(|X_i| > 10L_j) + L_j^{-\beta} \leq 3L_j^{-\beta}$$

since $\alpha > 6$.

Hence it follows that

$$\mathbb{P}[A_3^c] \leq 6L_j^{\alpha-\beta-1} \leq \frac{1}{10}L_{j+1}^{-\delta} \tag{2.9.9}$$

for sufficiently large L_0 since $\beta > \alpha\delta + \alpha - 1$.

• By Lemma 2.7.3 we have that

$$\mathbb{P}[A_4] \geq 1 - 2\exp(-\frac{1}{4}L_j) \geq 1 - \frac{1}{10}L_{j+1}^{-\delta}. \tag{2.9.10}$$

Now from (2.9.3), (2.9.4), (2.9.5), (2.9.9), (2.9.10) it follows that,

$$\mathbb{P}(X \in G_{j+1}^{\mathbb{X}}) \geq 1 - \sum_{i=1}^{4}\mathbb{P}[A_i^c] \geq 1 - L_{j+1}^{-\delta},$$

completing the proof of the theorem. □

2.9.2 Mappings of Good Segments

Theorem 2.9.3. *Let $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \ldots)$ be a sequence of $\mathbb{X}$-blocks at level $(j+1)$ and $\widetilde{Y} = (Y_1, Y_2, \ldots)$ be a sequence of $\mathbb{Y}$-blocks at level $(j+1)$. Further we suppose that $\widetilde{X}^{[1,R^+_{j+1}]}$ and $\widetilde{Y}^{[1,R^+_{j+1}]}$ are "good segments". Then for every t with $R^-_{j+1} \leq t \leq R^+_{j+1}$,*

$$\widetilde{X}^{[1,R_{j+1}]} \hookrightarrow \widetilde{Y}^{[1,t]} \text{ and } \widetilde{X}^{[1,t]} \hookrightarrow \widetilde{Y}^{[1,R^+_{j+1}]}. \tag{2.9.11}$$

Proof. Let us fix t with $R^-_{j+1} \leq t \leq R^+_{j+1}$. We only prove that $\widetilde{X}^{[1,R_{j+1}]} \hookrightarrow \widetilde{Y}^{[1,t]}$, the other part follows similarly.

Let $\widetilde{X}^{[1,R_{j+1}]} = (X_1, X_2, \ldots, X_n)$ be the decomposition of $\widetilde{X}^{[1,R_{j+1}]}$ into level j blocks. Similarly let $\widetilde{Y}^{[1,t]} = (Y_1, Y_2, \ldots, Y_{n'})$ denote the decomposition of $\widetilde{Y}^{[1,t]}$, into level j blocks.

Before proceeding with the proof, we make the following useful observations. Since both $\widetilde{X}^{[1,R_{j+1}]}$ and $\widetilde{Y}^{[1,t]}$ are good segments, it follows that $R_{j+1}L_j^{\alpha-1} \leq n \leq R_{j+1}(L_j^{\alpha-1} + L_j^5)$, and $tLj^{\alpha-1} \leq n' \leq t(L_j^{\alpha-1} + L_j^5)$. Since L_0 large enough and $\alpha > 6$, we have

$$\frac{1-2^{-(j+7/4)}}{R} \leq \frac{n'}{n} \leq R(1+2^{-(j+7/4)}). \tag{2.9.12}$$

Let $B_X = \{1 \leq i \leq n : X_i \notin G_j^{\mathbb{X}}\} = \{l_1 < l_2 < \ldots < l_{K_X}\}$ denote the positions of "bad" $\mathbb{X}$-blocks. Similarly, let $B_Y = \{1 \leq i \leq n' : Y_i \notin G_j^{\mathbb{Y}}\} = \{l'_1 < l'_2 < \ldots < l'_{K_Y}\}$ denote the positions of "bad" $\mathbb{Y}$-blocks. Notice that $K_X, K_Y \leq k_0 R_{j+1}^+$. Using Proposition 2.6.1 we can find a family of admissible generalised mappings Υ_h, $1 \leq h \leq L_j^2$ which are of *Class* G^j with respect to (B_X, B_Y), given by $\Upsilon_h([n],[n'],B_X,B_Y) = (P_h, P'_h, \tau_h)$ such that for all h, $1 \leq h \leq L_j^2$, $1 \leq i \leq K_X$, $1 \leq r \leq K_Y$, $\tau_h(l_i) = \tau_1(l_i) + h - 1$ and $\tau_h^{-1}(l'_r) = \tau_1^{-1}(l'_r) - h + 1$. At this point we need the following Lemma.

Lemma 2.9.4. *Let $\Upsilon_h = (P_h, P'_h, \tau_h), 1 \leq h \leq L_j^2$ be the set of generalised mappings as described above. Then there exists $1 \leq h_0 \leq L_j^2$, such that $X_{l_i} \hookrightarrow Y_{\tau_{h_0}(l_i)}$ for all $1 \leq i \leq K_X$ and $X_{\tau_{h_0}^{-1}(l'_i)} \hookrightarrow Y_{l'_i}$ for all $1 \leq i \leq K_Y$.*

Proof. Once again we appeal to the probabilistic method. First observe that for any fixed i, $\{\tau_h(l_i) : h = 1, 2, \ldots L_j^2\}$ is a set of L_j^2 consecutive integers. Notice that the j-th level sub-blocks corresponding to these indices need not belong to the same $(j+1)$-th level block. However, they can belong to at most 2 consecutive $(j+1)$-level blocks (both of which are good). Suppose the number of sub-blocks belonging to the two different blocks are a and b, where $a + b = L_j^2$. Now, by the strong sequence assumption, these L_j^2 blocks the must contain at least $\lfloor \frac{a}{L_j^{3/2}} \rfloor + \lfloor \frac{b}{L_j^{3/2}} \rfloor \geq \lfloor L_j^{1/2} \rfloor - 2$ many disjoint strong sequences of length $\lfloor L_j^{3/2} \rfloor$. By definition of strong sequences then, there exist, among these L_j^2 sub-blocks, at least $(L_j^{1/2} - 3) L_j^{3/2} (1 - \frac{1}{10 k_0 R_{j+1}^+})$ many to which X_{l_i} can be successfully mapped, i.e.,

$$\#\{h : X_{l_i} \hookrightarrow Y_{\tau_h(l_i)}\} \geq (L_j^{1/2} - 3) L_j^{3/2} (1 - \frac{1}{10 k_0 R_{j+1}^+}). \tag{2.9.13}$$

Now, choosing H uniformly at random from $\{1, 2, \ldots, L_j^2\}$, it follows from (2.9.13) that for each i, $1 \leq i \leq K_X$

$$\mathbb{P}(X_{l_i} \hookrightarrow Y_{\tau_H(l_i)}) \geq (1 - 3/L_j^{1/2})(1 - \frac{1}{10 k_0 R_{j+1}^+}) \geq 1 - \frac{1}{10 k_0 R_{j+1}^+} - \frac{3}{L_j^{1/2}}. \tag{2.9.14}$$

Similar arguments show that for all $i \in \{1, 2, \ldots, K_Y\}$,

$$\mathbb{P}(X_{\tau_H^{-1}(l'_i)} \hookrightarrow Y_{l'_i}) \geq 1 - \frac{1}{10 k_0 R_{j+1}^+} - \frac{3}{L_j^{1/2}}. \tag{2.9.15}$$

A union bound then gives,

$$\mathbb{P}\left(X_{l_i} \hookrightarrow Y_{\tau_H(l_i)} : 1 \leq i \leq K_X,\ X_{\tau_H^{-1}(l'_i)} \hookrightarrow Y_{(l'_i)} : 1 \leq i \leq K_Y\right) \geq 1 - 2 k_0 R_{j+1}^+ (\frac{1}{10 k_0 R_{j+1}^+} + \frac{3}{L_j^{1/2}}),$$

and the right hand side is always positive for L_0 sufficiently large. The lemma immediately follows from this. □

The proof of Theorem 2.9.3 can now be completed using Proposition 2.5.3. □

2.9.3 Good Blocks Map to Good Blocks

Theorem 2.9.5. *Let $X \in G^{\mathbb{X}}_{j+1}, Y \in G^{\mathbb{Y}}_{j+1}$, then $X \hookrightarrow Y$.*

The theorem follows from a simplified version of the proof of Theorem 2.9.3 so we omit the proof.
We have now completed all the parts of the inductive step. Together these establish Theorem 2.4.1.

2.10 Explicit Constructions

Our proof provides an implicit construction of a deterministic sequence $(X_1, \ldots)$ which embeds into $\mathbb{Y}$ with positive probability. We will describe a deterministic algorithm, based on our proof, which for any n will return the first n co-ordinates of the sequence in finite time. Though it is not strictly necessary, we will restrict discussion to the case of finite alphabets. It can easily be seen from our construction and proof that any good j-level block can be extended into a $(j+1)$-level good block and so the algorithm proceeds by extending one good block to one of the next level. As such one only needs to show that we can identify all the good blocks at level j in a finite amount of time.

We will also recover all semi-bad blocks. By our construction all good and semi-bad blocks are of bounded length so there is only a finite space to examine. To determine if a block is good at level $j+1$ one needs to count how many of its sub-blocks at level j are good and verify that the others are semi-bad. It also requires that it has "strong subsequences". This can be computed if we have a complete list of semi-bad j-level blocks.

Determining if X, a $(j+1)$-level block, is semi-bad requires calculating its length and its embedding probability. For this we need to run over all possible $(j+1)$-level blocks, calculate their probability and then test if X maps into them. By the definition of an R-embedding we need only consider those of length at most $O(R|X|)$ so this can be done in finite time.

With this listing of all good blocks one can then construct in an arbitrary manner a sequence in which the first block is good at all levels which will have a positive probability of an R-embedding into a random sequence. From this construction, and the reduction in § 2.2, we can construct a deterministic sequences which has an M-Lipschitz embedding into a random binary sequence in the sense of Theorem 2.1 with positive probability. Similarly, this approach gives a binary sequence with a positive density of ones which is compatible sequence with a random Ber(q) sequence in the sense of Theorem 2.3 for small enough $q > 0$ with positive probability.

Chapter 3

Scheduling of Random Walks on a Complete Graph

In this chapter we study Winkler's scheduling problem and provide an affirmative answer to Question 1.2.1 for M sufficiently large. Recall the problem in the language of co-ordinate percolation. Let $\mathbb{X} = (X_1, X_2, \ldots)$ and $\mathbb{Y} = (Y_1, Y_2, \ldots)$ be two i.i.d. sequences with

$$\mathbb{P}(X_i = k) = \mathbb{P}(Y_j = k) = \frac{1}{M} \text{ for } \ k = 1, 2, \ldots, M \text{ and for } i, j = 1, 2, \ldots.$$

Define an oriented percolation process on $\mathbb{Z}_+ \times \mathbb{Z}_+$: the vertex $(i_1, i_2) \in \mathbb{Z}^2_{>0}$ is **closed** if $X_{i_1} = Y_{i_2}$ and is **open** otherwise. The issue of settling Winkler's conjecture then translates to proving that for M sufficiently large, there is percolation with positive probability, which is our main result in this chapter. For $\mathbb{X}$ and $\mathbb{Y}$ as above, we say $\mathbb{X} \longleftrightarrow \mathbb{Y}$ if there exists an infinite open oriented path starting from $(1, 1)$.

Theorem 3.1. *For all M sufficiently large, $\mathbb{P}(\mathbb{X} \longleftrightarrow \mathbb{Y}) > 0$, thus clairvoyant scheduling is possible.*

To prove Theorem 3.1, we build upon the methods of Chapter 2, using a similar multi-scale structure, but with crucial adaptations. The most crucial difference comes in the definition of good blocks. Unlike in Chapter 2 we work here directly with the percolation picture, which necessitates considerations of different types of connections across a square in the plane. Also notice that, it is impossible to define good blocks in this model in such a manner that good block are typical and a good block in one sequence can always be matched to any good block in the other sequence. Our multi-scale structure needs to be adapted to circumvent these difficulties.

3.1 Outline of the Proof

As already mentioned, the proof of Theorem 3.1 is based on a multi-scale argument similar to the one appearing in Chapter 2. As there we divide the original sequences into blocks of doubly exponentially growing length scales $L_j = L_0^{\alpha^j}$, for $j \geq 1$, and at each of these levels j we have a definition of a "good" block. The multi-scale structure that we construct has a number of parameters, $\alpha, \beta, \delta, m, k_0, R$ and L_0 which must satisfy a number of relations. For our purposes we shall take these parameters to be identical to ones defined in (2.1.4). Single characters in the original sequences $\mathbb{X}$ and $\mathbb{Y}$ constitute the level 0 blocks.

Suppose that we have constructed the blocks up to level j denoting the sequence of blocks of level j as $(X_1^{(j)}, X_2^{(j)} \ldots)$. We construction of $(j+1)$-level blocks out of j-level sub-blocks in such way that the blocks are independent and, apart from the first block, identically distributed, this construction is identical to the one appearing in Chapter 2. Construction of blocks at level 1 has slight difference from the general construction.

At each level we have a definition which distinguishes some of the blocks as good. This is designed in such a manner that at each level, if we look at the rectangle in the lattice determined by a good block X and a random block Y, then, with high probability, it will have many open paths with varying slopes through it. For a precise definition see Definitions 3.2.4 and 3.2.5. Having these paths with different slopes will help achieve improving estimates of the probability of the event of having a path from the bottom left corner to the top right corner of the lattice rectangle determined by random blocks X and Y, denoted by $[X \overset{c,c}{\longleftrightarrow} Y]$, at higher levels.

The proof then involves a series of recursive estimates at each level, given in § 3.3. We require that at level j the probability of a block being good is at least $1 - L_j^{-\delta}$, so that the vast majority of blocks are good. Furthermore, we obtain tail bounds on $\mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y \mid X)$ by showing that for $0 < p \leq \frac{3}{4} + 2^{-(j+3)}$,

$$\mathbb{P}(\mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y \mid X) \leq p) \leq p^{m+2^{-j}} L_j^{-\beta},$$

where β and m are parameters mentioned at the beginning of this section. We show the similar bound for $\mathbb{Y}$-blocks as well. We also ask that the length of blocks satisfy an exponential tail estimate. The full inductive step is given in § 3.3. Proving this constitutes the main work in this chapter.

We use the key quantitative estimate provided by Lemma 3.6.2, which bounds the probability of a block having: $a)$ an excessive length, $b)$ too many bad sub-blocks, $c)$ a particularly difficult collection of sub-blocks, where we quantify the difficulty of a collection of bad sub-blocks $\{X_i\}_{i=1}^k$ by the value of $\prod_{i=1}^k \mathbb{P}[X_i \overset{c,c}{\longleftrightarrow} Y|X]$, where Y is a random block at the same level. This lemma is essentially identical to Lemma 2.7.3.

In order to achieve the improvement on the tail bounds of $\mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y \mid X)$ at each level, we take advantage of the flexibility in trying a large number of potential positions to cross the rectangular strips determined by each member of a small collection of bad sub-blocks, obtained by using the recursive estimates on probabilities of existence of paths of varying slopes through rectangles determined by collections of good sub-blocks.

To this effect we also build upon the notion of generalised mappings developed in Chapter 2 to describe such potential mappings. Our analysis is split into 5 different cases. To push through the estimate of the probability of having many open paths of varying slopes at a higher level, we make some finer geometric constructions. To complete the proof we note that $X_1^{(j)}$ and $Y_1^{(j)}$ are good for all j with positive probability. Using the definition of good blocks and a compactness argument we conclude the existence of an infinite open path with positive probability.

Parameters

Our proof involves a collection of parameters $\alpha, \beta, \delta, k_0, m$ and R which must satisfy a system of constraints. The required constraints are identical to the ones listed in § 2.1.2.

$$\alpha > 6, \delta > 2\alpha \vee 48, \beta > \alpha(\delta+1), m > 9\alpha\beta, k_0 > 36\alpha\beta, R > 6(m+1).$$

To fix on a choice we will the parameters to be the same ones given in (2.1.4). Recalling from § 2.1.2 we take

$$\alpha = 10, \delta = 50, \beta = 600, m = 60000, k_0 = 300000, R = 400000. \tag{3.1.1}$$

Given these choices we then take L_0 to be a sufficiently large integer. We did not make a serious attempt to optimize the parameters or constraints, sometimes for the sake of clarity of exposition.

3.2 The Multi-scale Structure

Our strategy for the proof of Theorem 3.1 is to partition the sequences $\mathbb{X}$ and $\mathbb{Y}$ into blocks at each level $j \geq 1$. For each $j \geq 1$, we write $\mathbb{X} = (X_1^{(j)}, X_2^{(j)}, \ldots)$ where we call each $X_i^{(j)}$ a level j $\mathbb{X}$-block, similarly we write $\mathbb{Y} = (Y_1^{(j)}, Y_2^{(j)}, \ldots)$. Most of the time we would clearly state that something is a level j block and drop the superscript j. Each of the $\mathbb{X}$-block (resp. $\mathbb{Y}$-block) at level $(j+1)$ is a concatenation of a number of level j $\mathbb{X}$-blocks, where the level 0 blocks are just the elements of the original sequence.

3.2.1 Recursive Construction of Blocks

Level 1 blocks are constructed inductively as follows:

Suppose the first k blocks $X_1^{(1)}, \ldots, X_k^{(1)}$ at level 1 have already been constructed and suppose that the rightmost element of $X_k^{(1)}$ is $X_{n_k}^{(0)}$. Then $X_{n_k+1}^{(1)}$ consists of the elements $X_{n_k+1}^{(0)}, X_{n_k+2}^{(0)}, \ldots, X_{n_k+l}^{(0)}$ where

$$l = \min\{t \geq L_1 : X_{n_k+t}^{(0)} = 1 \bmod 4 \text{ and } X_{n_k+t+1}^{(0)} = 0 \bmod 4\}. \tag{3.2.1}$$

The same definition holds for $k = 0$, assuming $n_0 = -1$. Recall that $L_1 = L_0^{\alpha}$.

Similarly, suppose the first k $\mathbb{Y}$-blocks at level 1 are $Y_1^{(1)}, \ldots, Y_k^{(1)}$ and also suppose that the rightmost element of $Y_k^{(1)}$ is $Y_{n_k}^{(0)}$. Then $Y_{k+1}^{(1)}$ consists of the elements $Y_{n_k+1}^{(0)}, Y_{n_k+2}^{0)}, \ldots,$ $Y_{n_k+l}^{(0)}$ where

$$l = \min\{t \geq L_1 : Y^{(0)}_{n_k+t} = 3 \bmod 4 \text{ and } Y^{(0)}_{n_k+t+1} = 2 \bmod 4\}. \tag{3.2.2}$$

We shall denote the length of an $\mathbb{X}$-block X (resp. a $\mathbb{Y}$-block Y) at level 1 by $L_X = L_1 + T^{(1)}_X$ (resp. $L_Y = L_1 + T^{(1)}_Y$). Notice that this construction, along with Assumption 1, ensures that the blocks at level one are independent and identically distributed.

At each level $j \geq 1$, we also have a recursive definition of "*good*" blocks (see Definition 3.2.7). Let $G^{\mathbb{X}}_j$ and $G^{\mathbb{Y}}_j$ denote the set of good $\mathbb{X}$-blocks and good $\mathbb{Y}$-blocks at j-th level respectively. Now we are ready to describe the recursive construction of the blocks $X^{(j)}_i$ and $Y^{(j)}_i$ for $j \geq 2$.

The construction of blocks at level $j \geq 2$ is identical to the one described in § 2.3. We recall it here for ready reference. Let us suppose we have already constructed the blocks of partition up to level j for some $j \geq 1$ and we have $X = (X^{(j)}_1, X^{(j)}_2, \ldots)$. Also assume we have defined the "good" blocks at level j, i.e., we know $G^{\mathbb{X}}_j$. We describe how to partition $\mathbb{X}$ into level $(j+1)$ blocks: $\mathbb{X} = (X^{(j+1)}_1, X^{(j+1)}_2, \ldots)$.

Suppose the first k blocks $X^{(j+1)}_1, \ldots, X^{(j+1)}_k$ at level $(j+1)$ has already been constructed and suppose that the rightmost level j-subblock of $X^{(j+1)}_k$ is $X^{(j)}_m$. Then $X^{(j+1)}_{k+1}$ consists of the sub-blocks $X^{(j)}_{m+1}, X^{(j)}_{m+2}, \ldots, X^{(j)}_{m+l+L^3_j}$ where $l > L^3_j + L^{\alpha-1}_j$ is selected in the following manner. Let $W_{k+1,j+1}$ be a geometric random variable having $\text{Geom}(L^{-4}_j)$ distribution and independent of everything else. Then

$$l = \min\{s \geq L^3_j + L^{\alpha-1}_j + W_{k+1,j+1} : X_{m+s+i} \in G^{\mathbb{X}}_j \text{ for } 1 \leq i \leq 2L^3_j\}.$$

That such an l is finite with probability 1 will follow from our recursive estimates. The case $k = 0$ is dealt with as before. Observe that as the recursive construction is identical to the one in § 2.3. Observation 2.3.1 holds for this model as well.

We use the same notations used there. From now on whenever we say "a (random) $\mathbb{X}$-block at level j", we would imply that it has law $\mu^{\mathbb{X}}_j$, unless explicitly stated otherwise. Similarly let us denote the corresponding law of "a (random) $\mathbb{Y}$-block at level j" by $\mu^{\mathbb{Y}}_j$.

Also, for $j > 0$, let $\mu^{\mathbb{X}}_{j,G}$ denote the conditional law of an $\mathbb{X}$ block at level j, given that it is in $G^{\mathbb{X}}_j$. We define $\mu^{\mathbb{Y}}_{j,G}$ similarly. Note also that Observation 2.3.2 holds for this model too.

As before, whenever we have a sequence $X_1, X_2, \ldots$ satisfying the condition in the observation above, we shall call X the (random) level $(j+1)$ block constructed from $X_1, X_2, \ldots$. and we shall denote the corresponding geometric variable by W_X and set $T_X = l - L^3_j - L^{\alpha-1}_j$.

We still need to define good blocks to complete the structure, we now move towards this direction.

3.2.2 Corner to Corner, Corner to Side and Side to Side Mapping Probabilities

Now we make some definitions that we are going to use throughout our proof. Let $X = (X^{(j)}_{s+1}, X^{(j)}_{s+2}, \ldots, X^{(j)}_{s+l_X}) = (X^{(0)}_{a_1}, \ldots, X^{(0)}_{a_2})$ be a level $(j+1)$ $\mathbb{X}$-block $(j \geq 1)$ where $X^{(j)}_i$'s and $X^{(0)}_i$ are the level j sub-blocks and the level 0 sub-blocks constituting it respectively. Similarly let $Y = (Y^{(j)}_{s'+1}, Y^{(j)}_{s'+2}, ..., Y^{(j)}_{s+l_Y}) = (Y^{(0)}_{b_1}, \ldots, Y^{(0)}_{b_2})$ is a level $(j+1)$ $\mathbb{Y}$-block. Let us consider the lattice rectangle $[a_1, a_2] \times [b_1, b_2] \cap \mathbb{Z}^2$, and denote it by $X \times Y$. It follows from (3.2.1) and (3.2.2) that sites at all the four corners of this rectangle are open.

Definition 3.2.1 (Corner to Corner Path)**.** *We say that there is a corner to corner path in $X \times Y$, denoted by*

$$X \overset{c,c}{\longleftrightarrow} Y,$$

if there is an open oriented path in $X \times Y$ from (a_1, b_1) to (a_2, b_2).

A site (x, b_2) and respectively a site (a_2, y), on the top, respectively on the right side of $X \times Y$, is called "reachable from bottom left site" if there is an open oriented path in $X \times Y$ from (a_1, b_1) to that site.

Further, the intervals $[a_1, a_2]$ and $[b_1, b_2]$ will be partitioned into "chunks" $\{C^X_r\}_{r\geq 1}$ and $\{C^Y_r\}_{r\geq 1}$ respectively in the following manner. Let for any $\mathbb{X}$-block $\widetilde{X}$ at any level $j \geq 1$,

$$\mathcal{I}(\widetilde{X}) = \{a \in \mathbb{N} : \widetilde{X} \text{ contains the level 0 block } X^{(0)}_a\}.$$

Let $X = (X^{(j)}_{s+1}, X^{(j)}_{s+2}, \ldots, X^{(j)}_{s+l_X})$, and $n_X := \lfloor l_X / L_j^4 \rfloor$. Similarly we define $n'_Y := \lfloor l_Y / L_j^4 \rfloor$.

Definition 3.2.2 (Chunks)**.** *The discrete segment $C^X_k \subset \mathcal{I}(X)$ defined as*

$$C^X_k := \begin{cases} \cup_{t=(k-1)L_j^4+1}^{kL_j^4} \mathcal{I}(X^{(j)}_{s+t}), & k = 1, \ldots, n_X - 1; \\ \cup_{t=k_X L_j^4+1}^{l_X} \mathcal{I}(X^{(j)}_{s+t}), & k = n_X; \end{cases} \tag{3.2.3}$$

is called the k^{th} chunk of X.

By C^X and C^Y we denote the set of all chunks $\{C^X_k\}_{k=1}^{n_X}$ and $\{C^Y_k\}_{k=1}^{n'_Y}$ of X and Y respectively. In what follows the letters $\mathcal{T}, \mathcal{B}, \mathcal{L}, \mathcal{R}$ will stand for "top", "bottom", "left", and "right", respectively. Define:

$$C^X_{\mathcal{B}} = C^X \times \{1\}, \quad C^X_{\mathcal{T}} = C^X \times \{n'_Y\},$$
$$C^Y_{\mathcal{L}} = \{1\} \times C^Y, \quad C^Y_{\mathcal{R}} = \{n_X\} \times C^Y.$$

Definition 3.2.3 (Entry/Exit Chunk, Slope Conditions)**.** *A pair $(C^X_k, 1) \in C^X_{\mathcal{B}}$, $k \in [L_j, n_X - L_j]$ is called an entry chunk (from the bottom) if it satisfies the slope condition*

$$\frac{1 - 2^{-(j+4)}}{R} \leq \frac{n'_Y - 1}{n_X - k} \leq R(1 + 2^{-(j+4)}). \tag{3.2.4}$$

Similarly, $(1, C^Y_k) \in C^Y_{\mathcal{L}}$, $k \in [L_j, n'_Y - L_j]$, is called an entry chunk (from the left) if it satisfies the slope condition

$$\frac{1-2^{-(j+4)}}{R} \leq \frac{n'_Y - k}{n_X - 1} \leq R(1+2^{-(j+4)}). \tag{3.2.5}$$

The set of all entry chunks is denoted by $\mathcal{E}_{in}(X,Y) \subseteq (C^X_{\mathcal{B}} \cup C^Y_{\mathcal{L}})$. The set of all exit chunks $\mathcal{E}_{out}(X,Y)$ is defined in a similar fashion.

We call $(e_1, e_2) \in (C^X_B \cup C^Y_L) \times (C^X_T \cup C^Y_R)$ is an "entry-exit pair of chunks" if the following conditions are satisfied. Without loss of generality assume $e_1 = (C^X_k, 1) \in C^X_B$ and $e_2 = (n'_X, C^Y_{k'}) \in C^Y_R$. Then (e_1, e_2) is called an "entry-exit pair" if $k \in [L_j, n_X - L_j]$, $k' \in [L_j, n'_Y - L_j]$ and they satisfy the slope condition

$$\frac{1-2^{-(j+4)}}{R} \leq \frac{k'-1}{n_X - k} \leq R(1+2^{-(j+4)}). \tag{3.2.6}$$

Let us denote the set of all "entry-exit pair of chunks" by $\mathcal{E}(X,Y)$.

Definition 3.2.4 (Corner to Side and Side to Corner Path)**.** *We say that there is a corner to side path in $X \times Y$, denoted by*

$$X \overset{c,s}{\longleftrightarrow} Y$$

if for each $(C^X_k, n_X), (n'_Y, C^Y_{k'}) \in \mathcal{E}_2(X,Y)$

$$\#\{a \in C^X_k : (a, b_2) \text{ is reachable from } (a_1, b_1) \text{ in } X \times Y\ \} \geq \left(\frac{3}{4} + 2^{-(j+5)}\right)|C^X_k|,$$

$$\#\{b \in C^Y_k : (a_2, b) \text{ is reachable from } (a_1, b_1) \text{ in } X \times Y\ \} \geq \left(\frac{3}{4} + 2^{-(j+5)}\right)|C^Y_k|.$$

Side to corner paths in $X \times Y$, denoted $X \overset{s,c}{\longleftrightarrow} Y$ is defined in the same way except that in this case we want paths from the bottom or left side of the rectangle $X \times Y$ to its top right corner and use $\mathcal{E}_1(X,Y)$ instead of $\mathcal{E}_2(X,Y)$.

Condition S: Let $(e_1, e_2) \in \mathcal{E}(X,Y)$. Without loss of generality we assume $e_1 = (C^X_{k_1}, 1) \in C^X_B$ and $e_2 = (n_X, C^Y_{k_2}) \in C^Y_R$. (e_1, e_2) is said to satisfy condition S if there exists $A \subseteq C^X_{k_1}$ with $|A| \geq \left(\frac{3}{4} + 2^{-(j+5)}\right)|C^X_{k_1}|$ and $B \subseteq C^Y_{k_2}$ with $|B| \geq \left(\frac{3}{4} + 2^{-(j+5)}\right)|C^Y_{k_2}|$ such that for all $a \in A$ and for all $b \in B$ there exist an open path in $X \times Y$ from (a, b_1) to (a_2, b). Condition S is defined similarly for the other cases.

Definition 3.2.5 (Side to Side Path)**.** *We say that there is a side to side path in $X \times Y$, denoted by*

$$X \overset{s,s}{\longleftrightarrow} Y$$

if each $(e_1, e_2) \in \mathcal{E}(X,Y)$ satisfies condition S.

It will be convenient for us to define corner to corner, corner to side, and side to side paths not only in rectangles determined by one $\mathbb{X}$-block and one $\mathbb{Y}$-block. Consider a $j+1$-level $\mathbb{X}$-block $X = (X_1, X_2, \ldots, X_n)$ and a $j+1$-level $\mathbb{Y}$ block $Y = (Y_1, \ldots, Y_{n'})$ where X_i, Y_i are j level subblocks constituting it. Let $\widetilde{X}$ (resp. $\widetilde{Y}$) denote a sequence of consecutive sub-blocks of X (resp. Y), e.g., $\widetilde{X} = (X_{t_1}, X_{t_1+1}, \ldots, X_{t_2})$ for $1 \leq t_1 \leq t_2 \leq n$. Call $\widetilde{X}$ to be a *segment* of X. Let $\widetilde{X} = (X_{t_1}, X_{t_1+1}, \ldots, X_{t_2})$ be a segment of X and let $\widetilde{Y} = (Y_{t'_1}, Y_{t'_1+1}, \ldots, Y_{t'_2})$ be a segment of Y. Let $\widetilde{X} \times \widetilde{Y}$ denote the rectangle in $\mathbb{Z}^2$ determined by $\widetilde{X}$ and $\widetilde{Y}$. Also let $X_{t_1} = (X^{(0)}_{a_1}, \ldots, X^{(0)}_{a_2})$, $X_{t_2} = (X^{(0)}_{a_3}, \ldots, X^{(0)}_{a_4})$, $Y_{t'_1} = (Y^{(0)}_{b_1}, \ldots, Y^{(0)}_{a_2})$, $Y_{t'_2} = (Y^{(0)}_{b_3}, \ldots, Y^{(0)}_{b_4})$.

- We denote by $\widetilde{X} \overset{c,c}{\longleftrightarrow} \widetilde{Y}$, the event that there exists an open oriented path from the bottom left corner to the top right corner of $\widetilde{X} \times \widetilde{Y}$.
- Let $\widetilde{X} \overset{c,s,*}{\longleftrightarrow} \widetilde{Y}$ denote the event that

  $$\left\{\#\{b \in [b_3, b_4] : (a_2, b) \text{ is reachable from } (a_1, b_1)\} \geq (\frac{3}{4} + 2^{-(j+7/2)})(b_4 - b_3)\right\} \text{ and}$$

  $$\left\{\#\{a \in [a_3, a_4] : (a, b_4) \text{ is reachable from } (a_1, b_1)\} \geq (\frac{3}{4} + 2^{-(j+7/2)})(a_4 - a_3)\right\}.$$

  $\widetilde{X} \overset{s,c,*}{\longleftrightarrow} \widetilde{Y}$ is defined in a similar manner.
- We set $\widetilde{X} \overset{s,s,*}{\longleftrightarrow} \widetilde{Y}$ to be the following event. There exists $A \subseteq [a_1, a_2]$ with $|A| \geq (\frac{3}{4} + 2^{-(j+7/2)})(a_2 - a_1)$, $A' \subseteq [a_3, a_4]$ with $|A'| \geq (\frac{3}{4} + 2^{-(j+7/2)})(a_4 - a_3)$, $B \subseteq [b_1, b_2]$ with $|B| \geq (\frac{3}{4} + 2^{-(j+7/2)})(b_2 - b_1)$ and $B' \subseteq [b_3, b_4]$ with $|B| \geq (\frac{3}{4} + 2^{-(j+7/2)})(b_4 - b_3)$ such that for all $a \in A, a' \in A'$, $b \in B$, $b' \in B'$ we have that (a_4, b') and $(a', b_4))$ are reachable from (a, b_1) and (a_1, b).

Definition 3.2.6 (Corner to Corner Connection probability)**.** *For $j \geq 1$, let X be an $\mathbb{X}$-block at level j and let Y be a $\mathbb{Y}$-block at level j. We define the corner to corner connecting probability of X to be $S^{\mathbb{X}}_j(X) = \mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y|X)$. Similarly we define $S^{\mathbb{Y}}_j(Y) = \mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y|Y)$.*

As noted above the law of Y is $\mu^{\mathbb{Y}}_j$ in the definition of $S^{\mathbb{X}}_j$ and the law of X is $\mu^{\mathbb{X}}_j$ in the definition of $S^{\mathbb{Y}}_j$.

3.2.3 Good Blocks

To complete the description, we need to give the definition of "good" blocks at level j for each $j \geq 1$ which we have alluded to above. With the definitions from the preceding section, we are now ready to give the recursive definition of a "good" block as follows. As usual we only give the definition for $\mathbb{X}$-blocks, the definition for $\mathbb{Y}$ is similar.

Let $X^{(j+1)} = (X^{(j)}_1, X^{(j)}_2, \ldots, X^{(j)}_n)$ be an $\mathbb{X}$ block at level $(j+1)$. Notice that we can form blocks at level $(j+1)$ since we have assumed that we already know $G^{\mathbb{X}}_j$.

Definition 3.2.7 (Good Blocks). *We say $X^{(j+1)}$ is a good block at level $(j+1)$ (denoted $X^{(j+1)} \in G^{\mathbb{X}}_{j+1}$) if the following conditions hold.*

(i) It starts with L_j^3 good sub-blocks, i.e., $X_i^{(j)} \in G_j^{\mathbb{X}}$ for $1 \le i \le L_j^3$. (This is required only for $j > 0$, as there are no good blocks at level 0 this does not apply for the case $j = 0$).

(ii) $\mathbb{P}(X \overset{s,s}{\longleftrightarrow} Y|X) \ge 1 - L_{j+1}^{-2\beta}$

(iii) $\mathbb{P}(X \overset{c,s}{\longleftrightarrow} Y|X) \ge 9/10 + 2^{-(j+4)}$ and $\mathbb{P}(X \overset{s,c}{\longleftrightarrow} Y|X) \ge 9/10 + 2^{-(j+4)}$.

(iv) $S_j^{\mathbb{X}}(X) \ge 3/4 + 2^{-(j+4)}$.

(v) The length of the block satisfies $n \le L_j^{\alpha-1} + L_j^5$.

3.3 Recursive Estimates

Our proof of the theorem depends on a collection of recursive estimates, all of which are proved together by induction. In this section we list these estimates for easy reference. The proof of these estimates are provided in the next few sections. We recall that for all $j > 0$ $L_j = L_{j-1}^{\alpha} = L_0^{\alpha^j}$.

3.3.1 Tail Estimate

I. Let $j \ge 1$. Let X be a $\mathbb{X}$-block at level j and let $m_j = m + 2^{-j}$. Then

$$\mathbb{P}(S_j^{\mathbb{X}}(X) \le p) \le p^{m_j} L_j^{-\beta} \quad \text{for} \quad p \le \frac{3}{4} + 2^{-(j+3)}. \tag{3.3.1}$$

Let Y be a $\mathbb{Y}$-block at level j. Then

$$\mathbb{P}(S_j^{\mathbb{Y}}(Y) \le p) \le p^{m_j} L_j^{-\beta} \quad \text{for} \quad p \le \frac{3}{4} + 2^{-(j+3)}. \tag{3.3.2}$$

3.3.2 Length Estimate

II. For X an $\mathbb{X}$-block at at level $j \ge 0$,

$$\mathbb{E}[\exp(L_{j-1}^{-6}(|X| - (2 - 2^{-j})L_j))] \le 1. \tag{3.3.3}$$

Similarly for Y, a $\mathbb{Y}$-block at level j, we have

$$\mathbb{E}[\exp(L_{j-1}^{-6}(|Y| - (2 - 2^{-j})L_j))] \le 1. \tag{3.3.4}$$

3.3.3 Probability of Good Blocks

III. Most blocks are "good".

$$\mathbb{P}(X \in G_j^{\mathbb{X}}) \geq 1 - L_j^{-\delta}. \tag{3.3.5}$$

$$\mathbb{P}(Y \in G_j^{\mathbb{Y}}) \geq 1 - L_j^{-\delta}. \tag{3.3.6}$$

3.3.4 Consequences of the Estimates

For now let us assume that the estimates $I - III$ hold at some level j. Then we have the following consequences (we only state the results for $\mathbb{X}$, but similar results hold for $\mathbb{Y}$ as well).

Lemma 3.3.1. *Let us suppose (3.3.1) and (3.3.5) hold at some level j. Then for all $X \in G_j^{\mathbb{X}}$ we have the following.*

(i)

$$\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid Y \in G_j^{\mathbb{Y}}, X] \geq \frac{3}{4} + 2^{-(j+7/2)}. \tag{3.3.7}$$

(ii)

$$\begin{aligned} \mathbb{P}[X \overset{c,s}{\longleftrightarrow} Y \mid Y \in G_j^{\mathbb{Y}}, X] &\geq \frac{9}{10} + 2^{-(j+7/2)}, \\ \mathbb{P}[X \overset{s,c}{\longleftrightarrow} Y \mid Y \in G_j^{\mathbb{Y}}, X] &\geq \frac{9}{10} + 2^{-(j+7/2)}. \end{aligned} \tag{3.3.8}$$

(iii)

$$\mathbb{P}[X \overset{s,s}{\longleftrightarrow} Y \mid Y \in G_j^{\mathbb{Y}}, X] \geq 1 - L_j^{-\beta}. \tag{3.3.9}$$

Proof. We only prove (3.3.9), other two are similar. We have

$$\mathbb{P}[X \overset{s,s}{\not\longleftrightarrow} Y \mid Y \in G_j^{\mathbb{Y}}, X] \leq \frac{\mathbb{P}[X \overset{s,s}{\not\longleftrightarrow} Y \mid X]}{\mathbb{P}[Y \in G_j^{\mathbb{Y}}]} \leq L_j^{-2\beta}(1 - L_j^{-\delta})^{-1} \leq L_j^{-\beta}$$

which implies (3.3.9). □

Theorem 3.3.2 (Recursive Theorem)**.** *There exist positive constants α, β, δ, m, k_0 and R such that for all large enough L_0 the following holds. If the recursive estimates (3.3.1), (3.3.2), (3.3.3), (3.3.4), (3.3.5), (3.3.6) and hold at level j for some $j \geq 1$ then all the estimates hold at level $(j+1)$ as well.*

We will choose the parameters as in equation (3.1.1). Before giving a proof of Theorem 3.3.2 we show how using this theorem we can prove the main theorem. To use the recursive theorem we first need to show that the estimates I and II hold at the base level $j = 1$. Because of the obvious symmetry between $\mathbb{X}$ and $\mathbb{Y}$ we need only show that (3.3.1), (3.3.3) and (3.3.5) hold for $j = 1$ if M is sufficiently large.

3.3.5 Proving the Recursive Estimates at Level 1

Let $X = (X_1^{(0)}, X_{(2)}^{(0)}, \ldots, X_{(L_1+T_X^{(1)})}^{(0)}) \sim \mu_1^{\mathbb{X}}$ be an $\mathbb{X}$-block at level 1. Similarly denote a $\mathbb{Y}$-block at level 1 by $Y = (Y_1^{(0)}, Y_{(2)}^{(0)}, \ldots, Y_{(L_1+T_Y^{(1)})}^{(0)}) \sim \mu_1^{\mathbb{Y}}$.

Theorem 3.3.3. *For all sufficiently large L_0, if M (depending on L_0) is sufficiently large, then*

$$\mathbb{P}(S_j^{\mathbb{X}}(X) \leq p) \leq p^{m+2^{-1}} L_1^{-\beta} \quad \text{for} \quad p \leq \frac{3}{4} + 2^{-4}, \tag{3.3.10}$$

and

$$\mathbb{P}(X \in G_j^{\mathbb{X}}) \geq 1 - L_1^{-\delta}. \tag{3.3.11}$$

Theorem 3.3.3 is proved using the following Lemmas. Without loss of generality we shall assume that M is a multiple of 4.

Lemma 3.3.4. *Let X be an $\mathbb{X}$ block at level 1 as above. Then we have for all $l \geq 1$,*

$$\mathbb{P}(T_X^{(1)} \geq l) \leq \left(\frac{15}{16}\right)^{\frac{l-1}{2}}. \tag{3.3.12}$$

Further we have,

$$\mathbb{E}[\exp(L_0^{-6}(|X| - \frac{3}{2}L_1))] \leq 1. \tag{3.3.13}$$

Proof. It follows from the construction of blocks at level 1 that $T_X^{(1)} \preceq 2V$ where V has a Geom(1/16) distribution, (3.3.12) follows immediately from this. To prove (3.3.13) we notice the following two facts.

$$\begin{aligned}\mathbb{P}[\exp(L_0^{-6}(|X| - 3L_1/2)) \geq \frac{1}{2}] &\leq \mathbb{P}[|X| \geq \frac{3}{2}L_1 - L_0^6 \log 2] \leq \mathbb{P}[|X| \geq 5/4 L_1] \\ &\leq (15/16)^{\frac{L_1}{10}} \leq 1/4\end{aligned}$$

for L_0 large enough using (3.3.12). Also, for all $x \geq 0$ using (3.3.12),

$$\mathbb{P}[\frac{|X| - 3/2L_1}{L_0^6} \geq x] \leq \left(\frac{15}{16}\right)^{xL_0^6/2 + L_1/4} \leq \frac{1}{10}\exp(-3x).$$

Now it follows from above that

$$\begin{aligned}
\mathbb{E}[\exp(L_0^{-6}(|X|-3/2L_1))] &= \int_0^\infty \mathbb{P}[\exp(L_0^{-6}(|X|-3/2L_1)) \geq y]\ dy \\
&= \int_0^{\frac{1}{2}} \mathbb{P}[\exp(L_0^{-6}(|X|-3/2L_1)) \geq y]\ dy \\
&+ \int_{\frac{1}{2}}^1 \mathbb{P}[\exp(L_0^{-6}(|X|-3/2L_1)) \geq y]\ dy \\
&+ \int_1^\infty \mathbb{P}[\exp(L_0^{-6}(|X|-3/2L_1)) \geq y]\ dy \\
&\leq \frac{1}{2}+\frac{1}{8}+\frac{1}{10}\int_0^\infty \mathbb{P}[(L_0^{-6}(|X|-3/2L_1)) \geq z]e^z\ dz \\
&\leq \frac{1}{2}+\frac{1}{8}+\frac{1}{10} \leq 1.
\end{aligned}$$

This completes the proof. □

We define $\mathcal{A}_{X,1}^{(1)}$ to be the set of level 1 $\mathbb{X}$-blocks defined by

$$\mathcal{A}_{X,1}^{(1)} := \left\{X : T_X^{(1)} \leq 100mL_1\right\}.$$

It follows from Lemma 3.3.4 that for L_0 sufficiently large

$$\mathbb{P}(X \in \mathcal{A}_{X,1}^{(1)}) \geq 1 - L_1^{-3\beta}. \tag{3.3.14}$$

Lemma 3.3.5. *For M sufficiently large, the following inequalities hold for each $X \in \mathcal{A}_{X,1}^{(1)}$.*

(i)

$$\mathbb{P}[X \xleftrightarrow{c,c} Y \mid X] \geq \frac{3}{4} + 2^{-4}. \tag{3.3.15}$$

(ii)

$$\mathbb{P}[X \xleftrightarrow{c,s} Y \mid X] \geq \frac{9}{10} + 2^{-4} \text{ and } \mathbb{P}[X \xleftrightarrow{s,c} Y \mid X] \geq \frac{9}{10} + 2^{-4}. \tag{3.3.16}$$

(iii)

$$\mathbb{P}[X \xleftrightarrow{s,s} Y \mid X] \geq 1 - L_1^{-2\beta}. \tag{3.3.17}$$

Proof. Let Y be a level 1 block constructed out of the sequence $Y_1^{(0)}, \ldots$. Let $\mathcal{C}(X)$ be the event

$$\left\{Y_i^{(0)} \neq X_{i'}^{(0)} \forall\ i, i', i \in [(10m+1)L_1], i' \in [L_1 + T_X^{(1)}]\right\}.$$

Let $\mathcal{E}$ denote the event

$$\left\{Y \in \mathcal{A}_{Y,1}^{(1)}\right\}.$$

Using the definition of the sequence $Y_1^{(0)}, \ldots$ and the $\mathbb{Y}$-version of (3.3.14) we get that

$$\mathbb{P}[\mathcal{C}(X) \cap \mathcal{E} \mid X] \geq \left(1 - \frac{4(100m+1)L_1}{M}\right)^{(100m+1)L_1} - L_1^{-3\beta} \geq \max\left\{1 - L_1^{-2\beta}, \frac{9}{10} + 2^{-4}\right\}$$

for M large enough.

Since $X \overset{s,s}{\longleftrightarrow} Y$, $X \overset{s,c}{\longleftrightarrow} Y$, $X \overset{c,s}{\longleftrightarrow} Y$, $X \overset{c,c}{\longleftrightarrow} Y$ each hold if $\mathcal{C}(X)$ and $\mathcal{E}$ both hold, the lemma follows immediately. □

Lemma 3.3.6. *If M is sufficiently large then*

$$\mathbb{P}(\mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y \mid X) \leq p) \leq p^{m+\frac{1}{2}} L_1^{-\beta} \quad \textit{for} \quad p \leq \frac{3}{4} + 2^{-4}. \tag{3.3.18}$$

Proof. Since L_1 is sufficiently large, (3.3.15) implies that it suffices to consider the case $p < \frac{1}{500}$ and $X \notin \mathcal{A}_{X,1}^{(1)}$. We prove that for $p < \frac{1}{500}$

$$\mathbb{P}[\mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y \mid X) \leq p, X \notin \mathcal{A}_{X,1}^{(1)}] \leq p^{m+2^{-1}} L_1^{-\beta}. \tag{3.3.19}$$

Let $\mathcal{E}(X)$ denote the event

$$\{T_Y^{(1)} = \lfloor \frac{1}{50m} T_X^{(1)} \rfloor, Y_i^{(0)} \neq 2 \bmod 4. \forall i \in [L_1 + 1, L_1 + T_Y^{(1)}]\}$$

It follows from definition that

$$\mathbb{P}[\mathcal{E}(X) \mid X] \geq \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{\frac{T_X^{(1)}}{50m}}. \tag{3.3.20}$$

Now let D_k denote the event that

$$D_k = \{Y_k^{(0)} \neq X_{i'}^{(0)} \forall\ i' \in [50km, 50(k+2)m \wedge T_Y^{(1)}\}.$$

Let

$$D = \bigcap_{k=1}^{L_1 + T_Y^{(1)}} D_k$$

It follows that

$$\mathbb{P}[D_k \mid X, \mathcal{E}(X)] \geq (1 - \frac{400m}{M}).$$

Since D_k are independent conditional on X and $\mathcal{E}(X)$

$$\mathbb{P}[D \mid X, \mathcal{E}(X)] \geq (1-400m/M)^{L_1+T_X^{(1)}/50m}.$$

It follows that

$$\begin{aligned}\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid X] &\geq \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{\frac{T_X^{(1)}}{50m}} \left(1-\frac{200m}{M}\right)^{L_1+T_X^{(1)}/50m} \\ &\geq \frac{1}{20}\left(\frac{7}{10}\right)^{\frac{T_X^{(1)}}{50m}}\end{aligned}$$

for M sufficiently large.

It follows that

$$\begin{aligned}\mathbb{P}[\mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y \mid X) \leq p, X \notin \mathcal{A}_{X,1}^{(1)}] &\leq \mathbb{P}[T_X^{(1)} \geq (50m\frac{\log 20p}{\log\frac{7}{10}}) \vee 100mL_1] \\ &\leq \left(\frac{15}{16}\right)^{20m\frac{\log 20p}{\log\frac{7}{10}}} \wedge \left(\frac{15}{16}\right)^{40mL_1} \\ &\leq (20p)^{2m} \wedge \left(\frac{15}{16}\right)^{40mL_1} \leq p^{m+2^{-1}} L_1^{-\beta}\end{aligned}$$

since $(15/16)^{10} < 7/10$ and L_0 is sufficiently large and $m > 100$. □

Proof of Theorem 3.3.3. We have established (3.3.10) in Lemma 3.3.6. That (3.3.11) holds follows from Lemma 3.3.5 and (3.3.14) noting $\beta > \delta$. □

Now we prove Theorem 3.1 using Theorem 3.3.2.

Proof of Theorem 3.1. Let $\mathbb{X} = (X_1, X_2, \ldots)$, $\mathbb{Y} = (Y_1, Y_2, \ldots)$ be as in the statement of the theorem. Let for $j \geq 1$, $\mathbb{X} = (X_1^{(j)}, X_2^{(j)}, \ldots)$ denote the partition of $\mathbb{X}$ into level j blocks as described above. Similarly let $\mathbb{Y} = (Y_1^{(j)}, Y_2^{(j)}, \ldots)$ denote the partition of $\mathbb{Y}$ into level j blocks. Let β, δ, m, R be as in Theorem 3.3.2. It follows form Theorem 3.3.3 that for all sufficiently large L_0, estimates I and II hold for $j = 1$ for all sufficiently large M. Hence the Theorem 3.3.2 implies that if L_0 is sufficiently large then I and II hold for all $j \geq 1$ for M sufficiently large.

Let $\mathcal{T}_j^{\mathbb{X}} = \{X_k^{(j)} \in G_j^{\mathbb{X}}, 1 \leq k \leq L_j^3\}$ be the event that the first L_j^3 blocks at level j are good. Notice that on the event $\cap_{k=1}^{j-1} \mathcal{T}_k^{\mathbb{X}}$, $X_1^{(j)}$ has distribution $\mu_j^{\mathbb{X}}$ by Observation 2.3.1 and so $\{X_i^{(j)}\}_{i\geq 1}$ is i.i.d. with distribution $\mu_j^{\mathbb{X}}$. Hence it follows from equation (3.3.5) that $\mathbb{P}(\mathcal{T}_j^{\mathbb{X}} | \cap_{k=1}^{j-1} \mathcal{T}_k^{\mathbb{X}}) \geq (1-L_j^{-\delta})^{L_j^3}$. Similarly defining $\mathcal{T}_j^{\mathbb{Y}} = \{Y_k^{(j)} \in G_j^{\mathbb{Y}}, 1 \leq k \leq L_j^3\}$ we get using (3.3.6) that $\mathbb{P}(\mathcal{T}_j^{\mathbb{Y}} | \cap_{k=0}^{j-1} \mathcal{T}_k^{\mathbb{Y}}) \geq (1-L_j^{-\delta})^{L_j^3}$.

Let $\mathcal{A} = \cap_{j\geq 0}(\mathcal{T}_j^{\mathbb{X}} \cap \mathcal{T}_j^{\mathbb{Y}})$. It follows from above that $\mathbb{P}(\mathcal{A}) > 0$ since $\delta > 3$. Let $\mathcal{A}_{j+1} = \cap_{k\leq j}(\mathcal{T}_k^{\mathbb{X}} \cap \mathcal{T}_k^{\mathbb{Y}})$. It follows from (3.3.7) and (3.3.5) that

$$\mathbb{P}[X_1^{(j+1)} \overset{c,c}{\longleftrightarrow} Y_1^{(j+1)} \mid \mathcal{A}_{j+1}] \geq \frac{3}{4} + 2^{-(j+9/2)} - 2L_{j+1}^{-\delta} \geq \frac{3}{4}.$$

Let $\mathcal{B}_{j+1}$ denote the event

$$\mathcal{B}_{j+1} = \{\exists \text{ an open path from } (0,0) \to (m,n) \text{ for some } m,n \geq L_{j+1}\}.$$

Then $\mathcal{B}_{j+1}\downarrow$ and $\mathcal{B}_{j+1} \supseteq \{X_1^{(j+1)} \overset{c,c}{\longleftrightarrow} Y_1^{(j+1)}\}$. It follows that

$$\mathbb{P}[\cap \mathcal{B}_{j+1}] \geq \liminf \mathbb{P}[X_1^{(j+1)} \overset{c,c}{\longleftrightarrow} Y_1^{(j+1)}] \geq \frac{3}{4}\mathbb{P}[\mathcal{A}] > 0.$$

A standard compactness argument shows that $\cap \mathcal{B}_{j+1} \subseteq \{\mathbb{X} \leftrightarrow \mathbb{Y}\}$ and hence $\mathbb{P}[\mathbb{X} \leftrightarrow \mathbb{Y}] > 0$, which completes the proof of the theorem. □

The remainder of the chapter is devoted to the proof of the estimates in the induction. Throughout these sections we assume that the estimates $I - III$ hold for some level $j \geq 1$ and then prove the estimates at level $j+1$. Combined they complete the proof of Theorem 3.3.2.

From now on, in every Theorem, Proposition and Lemma we state, we would implicitly assume the hypothesis that all the recursive estimates hold upto level j, the parameters satisfy the constraints described in § 3.1 and L_0 is sufficiently large.

3.4 Geometric Constructions

We shall join paths across blocks at a lower level two form paths across blocks at a higher level. The general strategy will be as follows. Suppose we want to construct a path across $X \times Y$ where X, Y are level $j+1$ blocks. Using the recursive estimates at level j we know we are likely to find many paths across $X_i \times Y$ where X_i is a good sub-block of X. So we need to take special care to ensure that we can find open paths crossing bad-subblocks of X (or Y). To show the existence of such paths, we need some geometric constructions, which we shall describe in this section. We start with the following definition.

Definition 3.4.1 (Admissible Assignments)**.** *Let $I_1 = [a+1, a+t] \cap \mathbb{Z}$ and $I_2 = [b+1, b+t'] \cap \mathbb{Z}$ be two intervals of consecutive positive integers. Let $I_1^* = [a + L_j^3 + 1, a + t - L_j^3] \cap \mathbb{Z}$ and $I_2^* = [b + L_j^3 + 1, b + t' - L_j^3] \cap \mathbb{Z}$. Also let $B \subseteq I_1^*$ and $B' \subseteq I_2^*$ be given. We call $\Upsilon(I_1, I_2, B, B') = (H, H', \tau)$ to be an admissible assignment at level j of (I_1, I_2) w.r.t. (B, B') if the following conditions hold.*

- *(i) $B \subseteq H = \{a_1 < a_2 < \cdots < a_\ell\} \subseteq I_1$ and $B' \subseteq H' = \{b_1 < b_2 < \cdots < b_\ell\} \subseteq I_2^*$ with $\ell = |B| + |B'|$.*
- *(ii) $\tau(a_i) = b_i$ and $\tau(B) \cap B' = \emptyset$.*

(iii) Set $a_0 = a, a_{\ell+1} = a + t + 1$*;* $b_0 = b, b_{\ell+1} = b + t' + 1$*. Then we have for all* $i \geq 0$

$$\frac{1 - 2^{-(j+7/2)}}{R} \leq \frac{b_{i+1} - b_i - 1}{a_{i+1} - a_i - 1} \leq R(1 + 2^{-(j+7/2)}).$$

The following proposition concerning the existence of admissible assignment follows from the results in § 2.6. We omit the proof.

Proposition 3.4.2. *Assume the set-up in Definition 3.4.1. We have the following.*

(i) Suppose we have

$$\frac{1 - 2^{-(j+4)}}{R} \leq \frac{t'}{t} \leq R(1 + 2^{-(j+4)}).$$

Also suppose $|B|, |B'| \leq 3k_0$*. Then there exist* L_j^2 *level* j *admissible assignments* (H_i, H_i', τ_i) *of* (I_1, I_2) *w.r.t.* (B, B') *such that for all* $x \in B$*,* $\tau_i(x) = \tau_1(x) + i - 1$ *and for all* $y \in B'$*,* $\tau_i^{-1}(y) = \tau_1^{-1}(y) - i + 1$*.*

(ii) Suppose

$$\frac{3}{2R} \leq \frac{t'}{t} \leq \frac{2R}{3}$$

and $|B| \leq \frac{t - 2L_j^3}{10R_j^+}$*. Then there exists an admissible assignment* (H, H', τ) *at level* j *of* (I_1, I_2) *w.r.t.* $(B, \emptyset)$*.*

Constructing suitable admissible assignments will let us construct different types of open paths in different rectangles. To demonstrate this we first define the following somewhat abstract set-up.

3.4.1 Admissible Connections

Assume the set-up in Definition 3.4.1. Consider the lattice $A = I_1 \times I_2$. Let $\mathcal{B} = (B_{i_1,i_2})_{(i_1,i_2) \in A}$ be a collection of finite rectangles where $B_{i_1,i_2} = [n_{i_1}] \times [n'_{i_2}]$. Let $A \otimes \mathcal{B}$ denote the bi-indexed collection

$$\{((a_1, b_1), (a_2, b_2)) : (a_1, a_2) \in A, (b_1, b_2) \in B_{a_1,a_2}\}.$$

We think of $A \otimes \mathcal{B}$ as a $\sum_{i_1} n_{i_1} \times \sum_{i_2} n'_{i_2}$ rectangle which is further divided into rectangles indexed by $(i_1, i_2) \in A$ in the obvious manner.

Definition 3.4.3 (Route)**.** *A* ***route*** P *at level* j *in* $A \otimes \mathcal{B}$ *is a sequence of points*

$$\left\{((v_i, b^{1,v_i}), (v_i, b^{2,v_i}))\right\}_{i \in [\ell]}$$

in $A \otimes \mathcal{B}$ *satisfying the following conditions.*

(i) $V(P) = \{v_1, v_2, \ldots, v_\ell\}$ *is an oriented path from* $(a+1, b+1)$ *to* $(a+t, b+t')$ *in* A.

(ii) *Let* $v_i = (v_i^1, v_i^2)$. *For each* i, $b^{1,v_i} \in [L_{j-1}, n_{v_i^1} - L_{j-1}] \times \{1\} \cup \{1\} \times [L_{j-1}, n'_{v_i^2} - L_{j-1}]$ *and* $b^{2,v_i} \in [L_{j-1}, n_{v_i^1} - L_{j-1}] \times \{n'_{v_i^2}\} \cup \{n_{v_i^1}\} \times [L_{j-1}, n'_{v_i^2} - L_{j-1}]$ *except that* $b^{1,v_1} = (1,1)$ *and* $b^{2,v_\ell} = (n_{v_\ell^1}, n'_{v_{ell}^2})$ *are also allowed.*

(iii) *For each* i *(we drop the superscript* v_i*), let* $b^1 = (b_1^1, b_2^1)$ *and* $b^2 = (b_1^2, b_2^2)$. *Then for each* i, *we have* $\frac{1-2^{-(j+3)}}{R} \leq \frac{b_2^2 - b_2^1}{b_1^2 - b_1^1} \leq R(1 + 2^{-(j+3)})$.

(iv) b^{2,v_i} *and* $b^{1,v_{i+1}}$ *agree in one co-ordinate.*

A route P *defined as above is called a route in* $A \otimes B$ *from* (v_1, b^{1,v_1}) *to* (v_ℓ, b^{2,v_ℓ}). *We call* P *a* ***corner to corner route*** *if* $b^{1,v_1} = (1,1)$ *and* $b^{2,v_\ell} = (n_{v_\ell^1}, n'_{v_\ell^2})$. *For* $k \in I_2$, *the* k*-section of the route* P *is defined to be the set of* $k' \in I_1$ *such that* $(k', k) \in V(P)$.

Now gluing together these routes one can construct corner to corner (resp. corner to side or side to side) paths under certain circumstances. We make the following definition to that end.

Definition 3.4.4 (Admissible Connections)**.** *Consider the above set-up. Let*

$$S_{in} = [L_{j-1}, n_{a+1} - L_{j-1}] \times \{1\} \cup \{1\} \times [L_{j-1}, n'_{b+1} - L_{j-1}]$$

and

$$S_{out} = [L_{j-1}, n_{a+t} - L_{j-1}] \times \{n'_{b+t'}\} \cup \{n_{a+t}\} \times [L_{j-1}, n'_{b+t} - L_{j-1}].$$

Suppose for each $b \in S_{out}$ *there exists a level* j *route* P^b *in* $A \otimes \mathcal{B}$ *from* $(1,1)$ *to* b. *The collection* $\mathcal{P} = \{P^b\}$ *is called a corner to side admissible connection in* $A \otimes \mathcal{B}$. *A side to corner admissible connection is defined in a similar manner. Now suppose for each* $b \in S_{in}$, $b' \in S_{out}$ *there exists a level* j *route* $P^{b,b'}$ *in* $A \otimes \mathcal{B}$ *from* b *to* b'. *The collection* $\mathcal{P} = \{P^{b,b'}\}$ *in this case is called a side to side admissible connection in* $A \otimes \mathcal{B}$. *We also define* $V(\mathcal{P}) = \cup_{P \in \mathcal{P}} V(P)$.

The usefulness of having these abstract definitions is demonstrated by the next few lemmata. These follow directly from definition and hence we shall omit the proofs.

Now let $X = (X_1, X_2, ..., X_t)$ be an $\mathbb{X}$-blocks at level $j+1$ with X_i being the j-level subblocks constituting it. Let X_i consisting of n_i many chunks of $(j-1)$-level subblocks. Similarly let $Y = (Y_1, Y_2, ..., Y_{t'})$ be a $\mathbb{Y}$-block at level $j+1$ with j-level subblocks Y_i consisting of n'_i many chunks of $(j-1)$ level subblocks. Then we have the following lemmata. Set $A = [t] \times [t']$. Define $\mathcal{B} = \{B_{i,j}\}$ where $B_{i_1,i_2} = [n'_{i_1}] \times [n'_{i_2}]$.

Lemma 3.4.5. *Consider the set-up described above. Let* $H = \{a_1 < a_2 < \cdots < a_\ell\} \subseteq [t]$ *and* $H' = \{b_1 < b_2 < \cdots < b_\ell\}$. *Set* $\widetilde{X}_{(s)} = (X_{a_s+1}, \ldots, X_{a_{s+1}-1})$ *and* $\widetilde{Y}_{(s)} = (Y_{b_s+1}, \ldots, Y_{b_{s+1}-1})$. *Suppose further that for each* s, $\widetilde{X}_{(s)} \overset{c,c}{\longleftrightarrow} \widetilde{Y}_{(s)}$ *and* $X_{a_s} \overset{c,c}{\longleftrightarrow} Y_{b_s}$. *Then we have* $X \overset{c,c}{\longleftrightarrow} Y$.

The next lemma gives sufficient conditions under which we have $\widetilde{X}_{(s)} \overset{c,c}{\longleftrightarrow} \widetilde{Y}_{(s)}$.

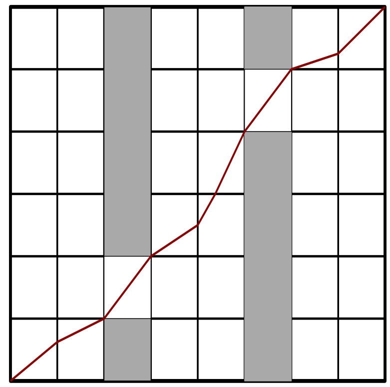


Figure 3.1: Corner to Corner Paths

Lemma 3.4.6. *In the above set-up, let $I_1^s = [a_s+1, a_{s+1}-1]$, $I_2^s = [a_s+1, a_{s+1}-1]$. Set $A^s = I_1^s \times I_2^s$ and let $\mathcal{B}^s$ be the restriction of $\mathcal{B}$ to A^s. Suppose there exists a corner to corner route P in $A^s \otimes B^s$ such that $X_{a_s+1} \overset{c,s}{\longleftrightarrow} Y_{b_s+1}$, $X_{a_{s+1}-1} \overset{s,c}{\longleftrightarrow} Y_{b_{s+1}-1}$ and for all other $(v_1, v_2) \in V(P)$ $X_{v_1} \overset{s,s}{\longleftrightarrow} Y_{v_2}$. Then $\widetilde{X}_{(s)} \overset{c,c}{\longleftrightarrow} \widetilde{Y}_{(s)}$.*

The above lemmata are immediate from definition. Now we turn to corner to side, side to corner and side to side connections. We have the following lemma.

Lemma 3.4.7. *Consider the set-up as above. Suppose X and Y contain n_X and n_Y many chunks respectively. Further suppose that none of the subblocks X_i or Y_i contain more than $3L_j$ level 0 subblocks.*

(i) Suppose for every exit chunk in $\mathcal{E}_{out}(X,Y)$ the following holds. For concreteness consider the chunk (k, n_Y). Let T_k denote the set of all i such that X_i is contained in C_k^X. There exists $T_k^* \subseteq T_k$ with $|T_k^*| \geq (1 - 10k_0L_j^{-1})|T_k|$ such that for all $r \in T_k^*$ and $\widetilde{X} = (X_1, \ldots, X_r)$ we have $\widetilde{X} \overset{c,s,*}{\longleftrightarrow} Y$.

Then we have $X \overset{c,s}{\longleftrightarrow} Y$.

(ii) A similar statement holds for $X \overset{s,c}{\longleftrightarrow} Y$.

(iii) Suppose for every pair of entry-exit chunks in $\mathcal{E}(X,Y)$ the following holds. For concreteness consider the pair of entry-exit chunks $((k_1,1),(n_X,k_2))$. Let T_{k_1} (resp. T'_{k_2}) denote the set of all i such that X_i (resp. Y_i) is contained in $C_{k_1}^X$ (resp. $C_{k_2}^Y$). There exists $T_{k_1,*} \subseteq T_{k_1}$, $T'_{k_2,*} \subseteq T'_{k_2}$ with $|T_{k_1,*}| \geq (1-10k_0L_j^{-1})|T_{k_1}|$, $|T'_{k_2,*}| \geq (1-10k_0L_j^{-1})|T'_{k_2}|$ such that for all $r \in T_{k_1,*}$, $r' \in T'_{k_2,*}$ and $\widetilde{X} = (X_r, \ldots, X_t)$, $\widetilde{Y} = (Y_1, \ldots, Y_{r'})$ we have $\widetilde{X} \overset{s,s,*}{\longleftrightarrow} \widetilde{Y}$.

Then we have $X \overset{s,s}{\longleftrightarrow} Y$.

Proof. Parts (*i*) and (*ii*) are straightforward from definitions. Part (*iii*) follows from definitions by noting the following consequence of planarity. Suppose there are open oriented paths in $\mathbb{Z}^2$ from $v_1 = (x_1, y_1)$ to $v_2 = (x_2, y_2)$ and also from $v_3 = (x_3, y_1)$ to $v_4(x_2, y_3)$ such that $x_1 < x_3 < x_2$ and $y_1 < y_2 < y_3$. Then these paths must intersect and hence there are open paths from v_1 to v_4 and also from v_2 to v_3. The condition on the length of sub-blocks is used to ensure that none of the subblocks in $T_{k_1} \setminus T_{k_1,*}$ are extremely long. $\square$

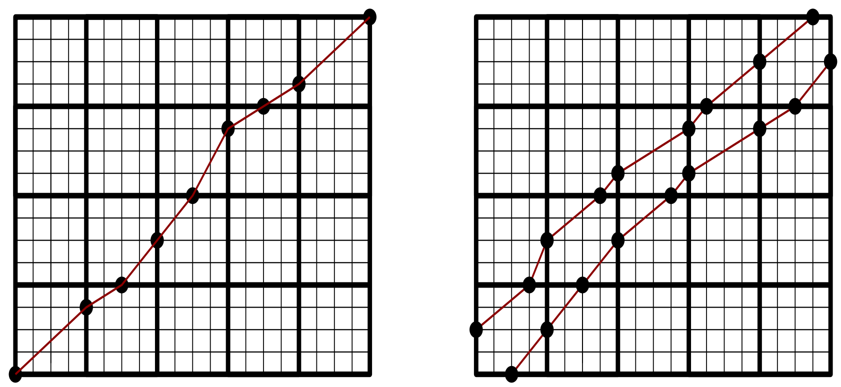


Figure 3.2: Corner to Corner and Side to Side routes

The next lemma gives sufficient conditions for $\widetilde{X} \overset{c,s,*}{\longleftrightarrow} \widetilde{Y}$ and $\widetilde{X} \overset{s,s,*}{\longleftrightarrow} \widetilde{Y}$ in the set-up of the above lemma. This lemma also easily follows from definitions.

Lemma 3.4.8. *Assume the set-up of Lemma 3.4.7. Let* $\widetilde{X} = (X_{t_1}, X_{t_1+1}, \ldots, X_{t_2})$ *and* $\widetilde{Y} = (Y_{t'_1}, \ldots, Y_{t'_2})$. *Let* $H = \{a_1 < a_2 < \cdots < a_\ell\} \subseteq [t_1, t_2]$ *and* $H' = \{b_1 < b_2 < \cdots < b_\ell\} \subseteq [t'_1, t'_2]$. *Set* $\widetilde{X}_{(s)} = (X_{a_s+1}, \ldots, X_{a_{s+1}-1})$ *and* $\widetilde{Y}_{(s)} = (Y_{b_s+1}, \ldots, Y_{b_{s+1}-1})$ *(*a_0, b_0 *etc. are defined in the natural way).*

(i) Suppose that for each $s < \ell$, $\widetilde{X}_{(s)} \overset{c,c}{\longleftrightarrow} \widetilde{Y}_{(s)}$ *and* $\widetilde{X}_{(\ell)} \overset{c,s,*}{\longleftrightarrow} \widetilde{Y}_{(\ell)}$. *Also suppose for each* s, $X_{a_s} \overset{c,c}{\longleftrightarrow} Y_{b_s}$. *Then we have* $\widetilde{X} \overset{c,s,*}{\longleftrightarrow} \widetilde{Y}$.

(ii) A similar statement holds for $\widetilde{X} \overset{s,c,*}{\longleftrightarrow} \widetilde{Y}$.

(iii) Suppose that for each $s \in [\ell - 1]$, $\widetilde{X}_{(s)} \overset{c,c}{\longleftrightarrow} \widetilde{Y}_{(s)}$, $\widetilde{X}_{(0)} \overset{s,c,*}{\longleftrightarrow} \widetilde{Y}_{(0)}$ $\widetilde{X}_{(\ell)} \overset{c,s,*}{\longleftrightarrow} \widetilde{Y}_{(\ell)}$. *Also suppose for each* s, $X_{a_s} \overset{c,c}{\longleftrightarrow} Y_{b_s}$. *Then we have* $\widetilde{X} \overset{s,s,*}{\longleftrightarrow} \widetilde{Y}$.

Now we give sufficient conditions for $\widetilde{X} \overset{c,s,*}{\longleftrightarrow} \widetilde{Y}$ and $\widetilde{X} \overset{s,c,*}{\longleftrightarrow} \widetilde{Y}$ in terms of routes.

Lemma 3.4.9. *In the above set-up, further suppose that none of the level* $(j-1)$ *sub-blocks of* X_{t_1}, X_{t_2}, $Y_{t'_1}$, $Y_{t'_2}$ *contain more than* $3L_{j-1}$ *level* 0 *sub-blocks. Set* $I_1^s = [a_s + 1, a_{s+1} - 1]$, $I_2^s = [a_s + 1, a_{s+1} - 1]$. *Set* $A^s = I_1^s \times I_2^s$ *and let* $\mathcal{B}^s$ *be the restriction of* $\mathcal{B}$ *to* A^s. *Suppose* *there exists a corner to side admissible connection $\mathcal{P}$ in $A^s \otimes B^s$ such that $X_{a_s+1} \overset{c,s}{\longleftrightarrow} Y_{b_s+1}$ and for all other $(v_1, v_2) \in V(\mathcal{P})$ $X_{v_1} \overset{s,s}{\longleftrightarrow} Y_{v_2}$. Then $\widetilde{X}_{(s)} \overset{c,s,*}{\longleftrightarrow} \widetilde{Y}_{(s)}$. Similar statements hold for $\widetilde{X}_{(s)} \overset{s,c,*}{\longleftrightarrow} \widetilde{Y}_{(s)}$ and $\widetilde{X}_{(s)} \overset{s,s,*}{\longleftrightarrow} \widetilde{Y}_{(s)}$.*

Proof. Proof is immediate from definition of admissible connections and the inductive hypotheses (this is where we need the assumption on the lengths of $j-1$ level subblocks). For $\widetilde{X}_{(s)} \overset{s,s,*}{\longleftrightarrow} \widetilde{Y}_{(s)}$, we again need to use planarity as before. □

Now we connect it up with the notion of admissible assignments defined earlier in this section. Consider the set-up in Lemma 3.4.5. Let $B_1 \subseteq I_1 = [t]$, $B_2 \subseteq I_2 = [t']$, let $B_1^* \supseteq B_1$ (resp. $B_2^* \supseteq B_2$) be the set containing elements of B_1 (resp. B_2) and its neighbours. Let Υ be a level j admissible assignment of (I_1, I_2) w.r.t. (B_1^*, B_2^*) with associated τ. Suppose $H = \tau^{-1}(B_2) \cup B_1$ and $H' = B_2^* \cup \tau(B_2)$. We have the following lemmata.

Lemma 3.4.10. *Consider $(\widetilde{X}_{(s)}, \widetilde{Y}_{(s)})$ in the above set-up. There exists a corner to corner route P in $A^s \otimes \mathcal{B}^s$. Further for each $k \in I_2^s$, there exist sets $H_k^\tau \subseteq I_1^s$ with $|H_k^\tau| \leq L_j$ such that the k-section of the route P is contained in H_k^τ for all k. In the special case where $t = t'$ and $\tau(i) = i$ for all i, one can take $H_k^\tau = \{k-1, k, k+1\}$. Further Let $A' \subseteq A^s$ with $|A'| \leq k_0$. Suppose further that for all $v = (v_1, v_2) \in A'$ and for $i \in \{s, s+1\}$ we have $||v - (a_i, b_i)||_\infty \geq k_0 R^3 10^{j+8}$. Then we can take $V(P) \cap A' = \emptyset$.*

Proof. This lemma is a consequence of Lemma 3.4.12 below. □

Lemma 3.4.11. *In the above set-up, consider $(\widetilde{X}_{(s)}, \widetilde{Y}_{(s)})$. Assume for each $i \in [a_s+1, a_{s+1}-1]$, $i' \in [b_s+1, b_{s+1}-1]$ we have $L_{j-1}^{\alpha-5} \leq n_i, n'_{i'} \leq L_{j-1}^{\alpha-5} + L_{j-1}$. Let $A' \subseteq A^s$ with $|A'| \leq k_0$. Suppose further that for all $v = (v_1, v_2) \in A'$ and for $i \in s, s+1$ we have $||v - (a_i, b_i)||_\infty \geq k_0 R^3 10^{j+8}$. Assume also $a_{s+1} - a_s, b_{s+1} - b_s \geq 5^{j+6} R$. Then there exists a corner to side (resp. side to corner, side to side) admissible connection $\mathcal{P}$ in $A^s \otimes \mathcal{B}^s$ such that $V(P) \cap A' = \emptyset$.*

Proof. This lemma also follows from Lemma 3.4.12 below. □

Lemma 3.4.12. *Let $A \otimes \mathcal{B}$ be as in Definition 3.4.3. Assume that $\frac{1-2^{-(j+7/2)}}{R} \leq \frac{t'}{t} \leq R(1+2^{-(j+7/2)})$, and $L_{j-1}^{\alpha-5} + L_{j-1} \geq n_i, n'_{i'} \geq L_{j-1}^{\alpha-5}$. Then the following holds.*

(i) There exists a corner to corner route P in $A \otimes \mathcal{B}$ where $V(P) \subseteq R(A)$ where

$$R(A) = \{v = (v_1, v_2) \in A : |v - (a + xt, b + xt')|_1 \leq 50 \text{ for some } x \in [0, 1]\}.$$

(ii) Further, if $t, t' \geq 5^{j+6} R$, then there exists a corner to side (resp. side to corner, side to side) admissible connection $\mathcal{P}$ with $V(\mathcal{P}) \subseteq R(A)$.

(iii) Let A' be a given subset of A with $|A'| \leq k_0$ such that $A' \bigcap ([k_0 R^3 10^{j+8}] \times [k_0 R^3 10^{j+8}] \cup ([n - k_0 R^3 10^{j+8}, n] \times [n' - k_0 R^3 10^{j+8}, n']) = \emptyset$. Then there is a corner to corner route P in $A \otimes \mathcal{B}$ such that $V(P) \cap A' = \emptyset$. Further, if $t, t' \geq 5^{j+6} R$, then there exists a corner to side (resp. side to corner, side to side) admissible connection $\mathcal{P}$ with $V(\mathcal{P}) \cap A' = \emptyset$.

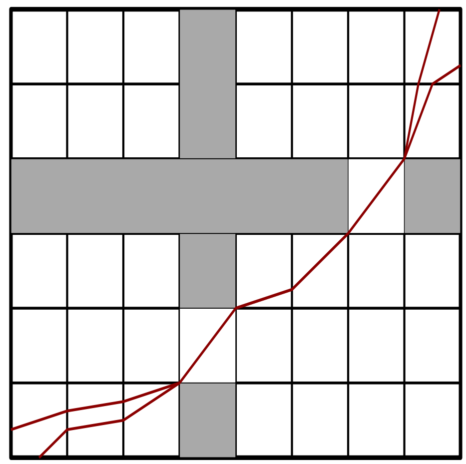


Figure 3.3: Side to Side admissible connections

Proof. Without loss of generality, for this proof we shall assume $a = b = 0$. We prove (i) first. Let $y_i = \lfloor it'/t \rfloor + 1$ for $i \in [t]$ and let $x_i = \lceil it/t' \rceil$ for $i \in [t']$. Define $\widetilde{y_i} = (it'/t - y_i + 1)$ and $\widetilde{x_i} = (it/t' - x_i + 1)$.

Define $y_i^* = \lfloor \widetilde{y_i} n'_{y_i} \rfloor + 1$ and $x_i^* = \lceil \widetilde{x_i} n_{x_i} \rceil$. Observe that it follows from the definitions that $y_i^* \in [n'_{y_i}]$ and $x_i^* \in [n_{x_i}]$. Now define $y_i^{**} = y_i^*$ if $y_i^* \in [L_{j-1}, n'_{y_i} - L_{j-1}]$. If $y_i^* \in [L_{j-1}]$ define $y_i^{**} = L_{j-1}$, if $y_i^* \in [n'_{y_i} - L_{j-1}, n'_{y_i}]$ define $y_i^{**} = n'_{y_i} - L_{j-1}$. Similarly define $x_i^{**} = x_i^*$ if $x_i^* \in [L_{j-1}, n_{x_i} - L_{j-1}]$. If $x_i^* \in [L_{j-1}]$ define $x_i^{**} = L_{j-1}$, if $x_i^* \in [n_{x_i} - L_{j-1}, n_{x_i}]$ define $x_i^{**} = n_{x_i} - L_{j-1}$. Now for $i \in [t-1], i' \in [t'-1]$ consider points $((i, n_i), (y_i, y_i^{**}))$, $((i+1, 1), (y_i, y_i^{**}))$, $((x_{i'}, x_{i'}^{**}), (i', n'_{i'}))$, $((x_{i'}, x_{i'}^{**}), (i'+1, 1))$ along with the two corner points. We construct a corner to corner route using these points.

Let us define $V(P) = \{(i, y_i), (x_{i'}, i') : i \in [t-1], i' \in [t'-1]\} \cup \{(t, t')\}$. We notice that either $y_1 = 1$ or $x_1 = 1$. It is easy to see that the vertices in $V(P)$ defines an oriented path from $(1,1)$ to (t, t') in A. Denote the path by $(v^1, v^2, \ldots v^{t+t'-1})$. For $v = v^r, r \in [2, t+t'-2]$, we define points b^{1,v^r} and b^{2,v^r} as follows. Without loss of generality assume $v = v^r = (i, y_i)$. Then either $v^{r-1} = (i-1, y_i) = (i-1, y_{i-1})$ or $v^{r-1} = (i, y_i - 1) = (x_{y_i - 1}, y_i - 1)$. If $v^{r-1} = (i-1, y_{i-1})$, then define $\{(b_1^{1,v}, b_2^{1,v}), (b_1^{2,v}, b_2^{2,v})\}$ by $b_1^{1,v} = 1$, $b_2^{1,v} = y_{i-1}^{**}$, $b_1^{2,v} = n_i$, $b_2^{2,v} = y_i^{**}$. If $v^{t-1} = (x_{y_i-1}, y_i - 1)$ then define $K^v = \{(b_1^{1,v}, b_2^{1,v}), (b_1^{2,v}, b_2^{2,v})\}$ by $b_1^{1,v} = x_{y_i-1}^{**}$, $b_2^{1,v} = 1$, $b_1^{2,v} = n_i$, $b_2^{2,v} = y_i^{**}$. To prove that this is indeed a route we only need to check the slope condition in Definition 3.4.3 in both the cases. We do that only for the latter case and the former one can be treated similarly.

Notice that from the definition it follows that the slope between the points (in $\mathbb{R}^2$) $(\widetilde{x}_{y_i-1}, 0)$ and $(1, \widetilde{y_i})$ is $\frac{t'}{t}$. We need to show that

$$\frac{1 - 2^{-(j+3)}}{R} \le \frac{b_2^2 - b_2^1}{b_1^2 - b_1^1} = \frac{y_i^{**} - 1}{n_i - x_{y_i-1}^{**}} \le R(1 + 2^{-(j+3)})$$

where once more we have dropped the superscript v for convenience. Now if $x^*_{y_i-1} \in [n_i - L_{j-1}, n_i]$ and $y^*_i \in [L_{j-1}]$ then from definition it follows that $\frac{b^2_2-b^1_2}{b^2_1-b^1_1} = 1$ and hence the slope condition holds. Next let us suppose $y^*_i \in [L_{j-1}]$ but $x^*_{y_i-1} \notin [n_i - L_{j-1}, n_i]$. Then clearly, $\frac{y^{**}_i-1}{n_i-x^{**}_{y_i-1}} \leq 1$. Also notice that in this case $x^*_{y_i-1} > L_{j-1}$ and $y^{**}_i - 1 \geq n'_{y_i}\widetilde{y}_i(1 - L^{-1}_{j-1})$. It follows that

$$1 - \frac{x^{**}_{y_i-1}}{n_i} = 1 - \frac{x^*_{y_i-1}}{n_i} \leq 1 - \widetilde{x}_{y_i-1} + \frac{1}{n_i} \leq (1 - \widetilde{x}_{y_i-1})(1 + L^{-1}_{j-1}).$$

Hence

$$\frac{y^{**}_i - 1}{n_i - x^{**}_{y_i-1}} \geq \frac{n'_{y_i}}{n_i}\frac{\widetilde{y}_i}{1 - \widetilde{x}_{y_i-1}}\frac{1 - L^{-1}_{j-1}}{1 + L^{-1}_{j-1}} \geq \frac{t'}{t}\frac{L^{\alpha-5}_{j-1}(1 - L^{-1}_{j-1})}{(L^{\alpha-5}_{j-1} + L_{j-1})(1 + L^{-1}_{j-1})} \geq \frac{1 - 2^{-(j+3)}}{R}$$

for L_0 sufficiently large. The case where $y^*_i \notin [L_{j-1}]$ but $x^*_{y_i-1} \in [n_i - L_{j-1}, n_i]$ can be treated similarly.

Next we treat the case where $x^*_{y_i-1} \in [L_{j-1} + 1, n_i - L_{j-1} - 1]$ and $y^*_i \in [L_{j-1} + 1, n'_{y_i} - L_{j-1} - 1]$. Here we have similarly as before

$$(1 - L^{-1}_{j-1})(1 - \widetilde{x}_{y_i-1}) \leq 1 - \frac{x^{**}_{y_i-1}}{n_i} \leq (1 - \widetilde{x}_{y_i-1})(1 + L^{-1}_{j-1})$$

and

$$\widetilde{y}_i(1 + 2L^{-1}_{j-1}) \geq \frac{y^{**}_i - 1}{n'_{y_i}} \geq \widetilde{y}_i(1 - 2L^{-1}_{j-1}).$$

It follows as before that

$$\frac{(1 + 2Lj - 1^{-1})}{1 - L^{-1}_{j-1}}\frac{n'_{y_i}}{n_i}\frac{n'}{n} \geq \frac{y^{**}_i - 1}{n_i - x^{**}_{y_i-1}} \geq \frac{n'_{y_i}}{n_i}\frac{t'}{t}\frac{(1 + 2Lj - 1^{-1})}{1 - L^{-1}_{j-1}}$$

and hence

$$R(1 + 2^{-(j+3)})\frac{y^{**}_i - 1}{n_i - x^{**}_{y_i-1}} \geq \frac{1 - 2^{-(j+3)}}{R}$$

for L_0 sufficiently large.

Other cases can be treated in similar vein and we only provide details in the case where $y^*_i \in [n'_{y_i} - L_{j-1}, n'_{y_i}]$ and $x^*_{y_i-1} \in [L_{j-1}]$. In this case we have that

$$\widetilde{y}_i(1 - \frac{2L_{j-1}}{n'_{y_i}}) \leq \frac{y^{**}_i - 1}{n'_{y_i}} \leq \widetilde{y}_i.$$

We also have that

$$(1 - \widetilde{x}_{y_i-1})(1 - \frac{L_{j-1}}{n_i})1 - \frac{x^{**}_{y_i-1}}{n_i} \leq 1 - \widetilde{x}_{y_i-1}.$$

Combining these two relations we get as before that

$$R(1+2^{-(j+3)})\frac{y_i^{**}-1}{n_i-x_{y_i-1}^{**}} \geq \frac{1-2^{-(j+3)}}{R}$$

for L_0 sufficiently large.

Thus we have constructed a corner to corner route in $A \otimes \mathcal{B}$. From the definitions it follows easily that for P as above $V(P) \subseteq R(A)$ and hence proof of (i) is complete.

Proof of (ii) is similar. Say, for the side to corner admissible connection, for a given $b \in S_{in}$, in stead of starting with the line $y=(t'/t)x$, we start with the line passing through $(b_1/n_1, 0)$ and (t,t'), and define $\widetilde{x}_i$, $\widetilde{y}_i$ to be the intersection of this line with the lines $y=i$ and $x=i$ respectively. Rest of the proof is almost identical, we use the fact $t,t' > 5^{j+6}R$ to prove that the slope of this new line is still sufficiently close to t'/t.

For part (iii), instead of a straight line we start with a number of piecewise linear functions which approximate $V(P)$. By taking a large number of such choices, it follows that for one of the cases $V(P)$ must be disjoint with the given set A', we omit the details. □

Finally we show that if we try a large number of admissible assignments, at least one of them must obey the hypothesis in Lemma 3.4.10 and Lemma 3.4.11 regarding A'

Lemma 3.4.13. *Assume the set-up in Proposition 3.4.2. Let $\Upsilon_h, h \in [L_j^2]$ be the family of admissible assignments of (I_1, I_2) w.r.t. (B, B') described in Proposition 3.4.2(i). Fix any arbitrary $\mathcal{T} \subset [L_j^2]$ with $|\mathcal{T}| = R^6 k_0^5 10^{2j+20}$. Then for every $S \subset I_1 \times I_2$ with $|S| = k_0$, there exist $h_0 \in \mathcal{T}$ such that*

$$\min_{x\in B_X, y\in B_Y, s\in S}\{|(x,\tau_{h_0}(x)) - s|, |(\tau_{h_0}^{-1}(y), y)| - s|\} \geq 2k_0 R^3 10^{j+8}.$$

Proof. Call $(x,y) \in I_1 \times I_2$ *forbidden* if there exist $s \in S$ such that $|(x,y)-s| \leq 2k_0R^3 10^{j+8}$. For each $s \in S$, let $B_s \subset I_1 \times I_2$ denote the set of vertices which are *forbidden* because of s, i.e., $B_s = \{(x,y) : |(x,y)-s| \leq 2k_0R^3 10^{j+8}\}$. Clearly $|B_s| \leq 10^{2j+18}k_0^2R^6$. So the total number of forbidden vertices is $\leq 10^{2j+18}k_0^3R^6$. Since $|B|, |B'| \leq k_0$, there exists $\mathcal{H} \subset \mathcal{T}$ with $|\mathcal{H}| = 10^{2j+19}R^6k_0^4$ such that for all $x, x' \in B$, $x \neq x'$, $y, y' \in B'$, $y \neq y'$, $h_1, h_2 \in \mathcal{H}$, we have $\tau_{h_1}(x) \neq \tau_{h_2}(x')$ and $\tau_{h_1}^{-1}(y) \neq \tau_{h_2}^{-1}(y')$. Now for each $x \in B$ (resp. $y \in B'$), $(x, \tau_h(x))$ (resp. $(\tau_h^{-1}(y), y)$) can be *forbidden* for at most $10^{2j+18}k_0^3R^6$ many different $h \in \mathcal{H}$. Hence,

$$\# \bigcup_{x\in B, y\in B'} \{h \in \mathcal{H} : (x,\tau_h(x)) \text{ or } (\tau_h^{-1}(y), y) \text{ is forbidden}\} \leq 2 \times 10^{2j+18}R^6k_0^4 < |\mathcal{H}|.$$

It follows that there exist $h_0 \in \mathcal{H}$ which satisfies the condition in the statement of the lemma. □

3.5 Length Estimate

Theorem 3.5.1. *Let X be an $\mathbb{X}$ block at level $(j+1)$ we have that*

$$\mathbb{E}[\exp(L_j^{-6}(|X| - (2 - 2^{-(j+1)})L_{j+1}))] \leq 1. \tag{3.5.1}$$

and hence for $x \geq 0$,

$$\mathbb{P}(|X| > ((2 - 2^{-(j+1)})L_{j+1} + xL_j^6)) \leq e^{-x}. \tag{3.5.2}$$

The proof of this theorem is identical to the proof of Theorem 2.8.1. We omit the proof.

3.6 Corner to Corner Estimate

In this section we prove the recursive tail estimate for the corner to corner connection probabilities.

Theorem 3.6.1. *Assume that the inductive hypothesis holds up to level j. Let X and Y be random $(j+1)$-level blocks according to $\mu_{j+1}^{\mathbb{X}}$ and $\mu_{j+1}^{\mathbb{Y}}$. Then*

$$\mathbb{P}\left(\mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y|X) \leq p\right) \leq p^{m_{j+1}} L_{j+1}^{-\beta}, \quad \mathbb{P}\left(\mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y|Y) \leq p\right) \leq p^{m_{j+1}} L_{j+1}^{-\beta}$$

for $p \leq \frac{3}{4} + 2^{-(j+4)}$ and $m_{j+1} = m + 2^{-(j+1)}$.

Due to the obvious symmetry between our X and Y bounds and for brevity all our bounds will be stated in terms of X and $S_{j+1}^{\mathbb{X}}$ but will similarly hold for Y and $S_{j+1}^{\mathbb{Y}}$. For the rest of this section we drop the superscript $\mathbb{X}$ and denote $S_{j+1}^{\mathbb{X}}$ (resp. $S_j^{\mathbb{X}}$) simply by S_{j+1} (resp. S_j).

The block X is constructed from an i.i.d. sequence of j-level blocks $X_1, X_2, \ldots$ conditioned on the event $X_i \in G_j^{\mathbb{X}}$ for $1 \leq i \leq L_j^3$ as described in Section 3.2. The construction also involves a random variable $W_X \sim \mathrm{Geom}(L_j^{-4})$ and let T_X denote the number of extra sub-blocks of X, that is the length of X is $L_j^{\alpha-1} + 2L_j^3 + T_X$. Let K_X denote the number of bad sub-blocks of X, and let

$$B_X = \{i \in [L_j^{\alpha-1} + 2L_j^3 + T_X] : X_i \notin G_j^{\mathbb{X}}\}$$

denote the position.s of the bad sub-blocks. Let us also denote the positions of bad subblock of X and their neighbours by $\{\ell_1 < \ell_2 < \cdots < \ell_{K'_X}\}$, where K'_X denotes the number of such blocks. Trivially, $K'_X \leq 3K_X$. We define Y, W_Y, T_Y and K_Y similarly.The proof of Theorem 3.6.1 is divided into 5 cases depending on the number of bad sub-blocks, the total number of sub-blocks of X and how "bad" the sub-blocks are.

We note here that the proof of Theorem 3.6.1 follows along the same general line of argument as the proof of Theorem 2.7.1, with significant adaptations resulting from the specifics of the model and especially the difference in the definition of good blocks. As such this section is similar to Section § 2.7.

The following key lemma provides a bound for the probability of blocks having large length, number of bad sub-blocks or small $\prod_{i \in B_X} S_j(X_i)$.

Lemma 3.6.2. *For all $t', k', x \geq 0$ we have that*

$$\mathbb{P}\left[T_X \geq t', K_X \geq k', -\log \prod_{i \in B_X} S_j(X_i) > x\right] \leq 2L_j^{-\delta k'/4} \exp\left(-x m_{j+1} - \frac{1}{2} t' L_j^{-4}\right).$$

The proof of this Lemma is same as the proof of Lemma 2.7.3 and we omit the details. We now proceed with the 5 cases we need to consider.

3.6.1 Case 1

The first case is the scenario where the blocks are of typical length, have few bad sub-blocks whose corner to corner corner to corner connection probabilities are not too small. This case holds with high probability.

We define the event $\mathcal{A}^{(1)}_{X,j+1}$ to be the set of $(j+1)$ level blocks such that

$$\mathcal{A}^{(1)}_{X,j+1} := \left\{X : T_X \leq \frac{RL_j^{\alpha-1}}{2}, K_X \leq k_0, \prod_{i \in B_X} S_j(X_i) > L_j^{-1/3}\right\}.$$

The following Lemma is an easy corollary of Lemma 3.6.2 and the choices of parameters, we omit the proof.

Lemma 3.6.3. *The probability that $X \in \mathcal{A}^{(1)}_{X,j+1}$ is bounded below by*

$$\mathbb{P}[X \notin \mathcal{A}^{(1)}_{X,j+1}] \leq L_{j+1}^{-3\beta}.$$

Lemma 3.6.4. *We have that for all $X \in \mathcal{A}^{(1)}_{X,j+1}$,*

$$\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid Y \in \mathcal{A}^{(1)}_{Y,j+1}, X] \geq \frac{3}{4} + 2^{-(j+3)}, \tag{3.6.1}$$

Proof. Suppose that $X \in \mathcal{A}^{(1)}_{X,j+1}$ with length $L_j^{\alpha-1} + 2L_j^3 + T_X$. Let B_X denote the location of bad subblocks of X. let K'_X be the number of bad sub-blocks and their neighbours and let set of their locations be $B^* = \{\ell_1 < \cdots < \ell_{K'_X}\}$. Notice that $K'_X \leq 3k_0$. We condition on $Y \in \mathcal{A}^{(1)}_{Y,j+1}$ having no bad subblocks. Denote this conditioning by

$$\mathcal{F} = \{Y \in \mathcal{A}^{(1)}_{Y,j+1}, T_Y, K_Y = 0\}.$$

Let $I_1 = [L_j^{\alpha-1} + 2L_j^3 + T_X]$ and $I_2 = [L_j^{\alpha-1} + 2L_j^3 + T_Y]$. By Proposition 3.4.2(i), we can find L_j^2 admissible assignments Υ_h at level j w.r.t. $(B^*, \emptyset)$, with associated τ_h for $1 \leq h \leq L_j^2$, such that $\tau_h(\ell_i) = \tau_1(\ell_i) + h - 1$ and in particular each block ℓ_i is mapped to L_j^2 distinct sub-blocks. Hence we get $\mathcal{H} \subset [L_j^2]$ of size $L_j < \lfloor L_j^2 / 9k_0^2 \rfloor$ so that for all $i_1 \neq i_2$ and $h_1, h_2 \in \mathcal{H}$ we have that $\tau_{h_1}(\ell_{i_1}) \neq \tau_{h_2}(\ell_{i_2})$, that is that all the positions bad blocks and their neighbours are mapped to are distinct.

Our construction ensures that all $Y_{\tau_h(\ell_i)}$ are uniformly chosen good j-blocks conditional on $\mathcal{F}$ and since $S_j(X_{\ell_i}) \geq L_j^{-1/3}$ we have that if $X_{\ell_i} \notin G_j^{\mathbb{X}}$,

$$\mathbb{P}[X_{\ell_i} \xleftrightarrow{c,c} Y_{\tau_h(\ell_i)} \mid \mathcal{F}] \geq S_j(X_{\ell_i}) - \mathbb{P}[Y_{\tau_h(\ell_i)} \notin G_j^{\mathbb{X}}] \geq \frac{1}{2} S_j(X_{\ell_i}). \tag{3.6.2}$$

Also if $X_{\ell_i} \in G_j^{\mathbb{X}}$ then from the recursive estimates it follows that

$$\mathbb{P}[X_{\ell_i} \xleftrightarrow{c,c} Y_{\tau_h(\ell_i)} \mid \mathcal{F}] \geq \frac{3}{4};$$

$$\mathbb{P}[X_{\ell_i} \xleftrightarrow{c,s} Y_{\tau_h(\ell_i)} \mid \mathcal{F}] \geq \frac{9}{10};$$

$$\mathbb{P}[X_{\ell_i} \xleftrightarrow{s,c} Y_{\tau_h(\ell_i)} \mid \mathcal{F}] \geq \frac{9}{10}.$$

If $X_{\ell_i} \notin G_j^{\mathbb{X}}$, or, if neither $X_{\ell_i - 1}$ nor $X_{\ell_i + 1}$ is $\in G_j^{\mathbb{X}}$, let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{ X_{\ell_i} \xleftrightarrow{c,c} Y_{\tau_h(\ell_i)} \right\}.$$

If $X_{\ell_i}, X_{\ell_i+1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{ X_{\ell_i} \xleftrightarrow{c,s} Y_{\tau_h(\ell_i)} \right\}.$$

If $X_{\ell_i}, X_{\ell_i-1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{ X_{\ell_i} \xleftrightarrow{s,c} Y_{\tau_h(\ell_i)} \right\}.$$

Let $\mathcal{D}_h$ denote the event

$$\mathcal{D}_h = \bigcap_{i=1}^{K'_X} \mathcal{D}_{h,i}.$$

Further, $\mathcal{S}$ denote the event

$$\mathcal{S} = \left\{ X_k \xleftrightarrow{s,s} Y_{k'} \forall k \in [L_j^{\alpha-1} + 2L_j^3 + T_X] \setminus B_X, \forall k' \in [L_j^{\alpha-1} + 2L_j^3 + T_Y] \right\}.$$

Also let

$$\mathcal{C}_1 = \left\{ X_1 \xleftrightarrow{c,s} Y_1 \right\} \text{ and } \mathcal{C}_2 = \left\{ X_{L_j^{\alpha-1} + 2L_j^3 + T_X} \xleftrightarrow{s,c} Y_{L_j^{\alpha-1} + 2L_j^3 + T_Y} \right\}.$$

By Lemma 3.4.5, Lemma 3.4.6 and Lemma 3.4.10 if $\cup_{h\in\mathcal{H}}\mathcal{D}_h, \mathcal{S}, \mathcal{C}_1, \mathcal{C}_2$ all hold then $X \overset{c,c}{\longleftrightarrow} Y$. Conditional on $\mathcal{F}$, for $h \in \mathcal{H}$, the $\mathcal{D}_h$, $\mathcal{C}_1$, $\mathcal{C}_2$ are independent and and by (3.6.2) and the recursive estimates ,

$$\mathbb{P}[\mathcal{D}_h \mid \mathcal{F}] \geq 2^{-5k_0}3^{2k_0}L_j^{-1/3}. \tag{3.6.3}$$

Hence

$$\mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h \mid \mathcal{F}] \geq 1 - \left(1 - 2^{-5k_0}3^{2k_0}L_j^{-1/3}\right)^{L_j} \geq 1 - L_{j+1}^{-3\beta}. \tag{3.6.4}$$

It follows from the recursive estimates that

$$\mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h, \mathcal{C}_1, \mathcal{C}_2 \mid \mathcal{F}] \geq \left(\frac{9}{10}\right)^2 \left(1 - L_{j+1}^{-3\beta}\right) \tag{3.6.5}$$

Also a union bound using the recursive estimates at level j gives

$$\mathbb{P}[\neg\mathcal{S} \mid \mathcal{F}] \leq (1 + \frac{R}{2})^2 L_j^{2\alpha-2}L_j^{-2\beta} \leq L_j^{-\beta}. \tag{3.6.6}$$

It follows that

$$\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid \mathcal{F}] \geq \mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h, \mathcal{C}_1, \mathcal{C}_2, \mathcal{S}] \geq \left(\frac{9}{10}\right)^2 \left(1 - L_{j+1}^{-3\beta}\right) - L_j^{-\beta}. \tag{3.6.7}$$

Hence

$$\begin{aligned}\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, X, T_Y] &\geq \mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid \mathcal{F}] \cdot \mathbb{P}[K_Y = 0 \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, T_Y] \\ &\geq \left(0.81(1 - L_{j+1}^{-3\beta}) - L_j^{-\beta}\right)\mathbb{P}[K_Y = 0 \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, T_Y].\end{aligned}$$

Removing the conditioning on T_Y we get

$$\begin{aligned}\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, X] &\geq \left(\left(\frac{9}{10}\right)^2 \left(1 - L_{j+1}^{-3\beta}\right) - L_j^{-\beta}\right) \cdot \mathbb{P}[K_Y = 0 \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}] \\ &\geq \left(\left(\frac{9}{10}\right)^2 \left(1 - L_{j+1}^{-3\beta}\right) - L_j^{-\beta}\right) \cdot \left(1 - L_{j+1}^{-3\beta} - 2L_j^{-\delta/4}\right) \\ &\geq \frac{3}{4} + 2^{-(j+1)}\end{aligned}$$

for large enough L_0, where the penultimate inequality follows from Lemma 3.6.2 and Lemma 3.6.3. This completes the lemma. □

Lemma 3.6.5. *When* $\frac{1}{2} \leq p \leq \frac{3}{4} + 2^{-(j+4)}$

$$\mathbb{P}(S_{j+1}(X) \leq p) \leq p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. By Lemma 3.6.3 and 3.6.4 we have that for all $X \in \mathcal{A}^{(1)}_{X,j+1}$

$$\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid X] \geq \mathbb{P}[Y \in \mathcal{A}^{(1)}_{Y,j+1}]\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid X, Y \in \mathcal{A}^{(1)}_{Y,j+1}] \geq \frac{3}{4} + 2^{-(j+4)}. \tag{3.6.8}$$

Hence if $\frac{1}{2} \leq p \leq \frac{3}{4} + 2^{-(j+4)}$

$$\begin{aligned}\mathbb{P}(\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid X] \leq p) &\leq \mathbb{P}[X \notin \mathcal{A}^{(1)}_{X,j+1}] \\ &\leq L_{j+1}^{-3\beta} \leq 2^{-m_{j+1}} L_{j+1}^{-\beta} \leq p^{m_{j+1}} L_{j+1}^{-\beta}.\end{aligned}$$

□

3.6.2 Case 2

The next case involves blocks which are not too long and do not contain too many bad sub-blocks but whose bad sub-blocks may be very bad in the since that corner to corner connection probabilities of those might be really small. We define the class of blocks $\mathcal{A}^{(2)}_{X,j+1}$ as

$$\mathcal{A}^{(2)}_{X,j+1} := \left\{ X : T_X \leq \frac{RL_j^{\alpha-1}}{2}, K_X \leq k_0, \prod_{i \in B_X} S_j(X_i) \leq L_j^{-1/3} \right\}.$$

Lemma 3.6.6. *For* $X \in \mathcal{A}^{(2)}_{X,j+1}$,

$$S_{j+1}(X) \geq \min \left\{ \frac{1}{2}, \frac{1}{10} \left(\frac{3}{4}\right)^{2k_0} L_j \prod_{i \in B_X} S_j(X_i) \right\}$$

Proof. Suppose that $X \in \mathcal{A}^{(2)}_{X,j+1}$. Let $\mathcal{E}$ denote the event

$$\mathcal{E} = \{W_Y \leq L_j^{\alpha-1}, T_Y = W_Y\}.$$

Then by definition of W_Y, $\mathbb{P}[W_Y \leq L_j^{\alpha-1}] \geq 1 - (1 - L_j^{-4})^{L_j^{\alpha-1}} \geq 9/10$ while by the definition of the block boundaries the event $T_Y = W_Y$ is equivalent to their being no bad sub-blocks amongst $Y_{L_j^3+L_j^{\alpha-1}+W_Y+1}, \ldots, Y_{L_j^3+L_j^{\alpha-1}+W_Y+2L_j^3}$, that is that we don't need to extend the block because of bad sub-blocks. Hence $\mathbb{P}[T_Y = W_Y] \geq (1 - L_j^{-\delta})^{2L_j^3} \geq 9/10$. Combining these we have that

$$\mathbb{P}[\mathcal{E}] \geq 8/10. \tag{3.6.9}$$

By our block construction procedure, on the event $T_Y = W_Y$ we have that the blocks $Y_{L_j^3+1}, \ldots, Y_{L_j^3+L_j^{\alpha-1}+T_Y}$ are uniform j-level blocks.

Define I_1, I_2, B_X and B^* as in the proof of Lemma 3.6.4. Also set $[L_j^{\alpha-1}+2L_j^3+T_X] \setminus B_X = G_X$. Using Proposition 3.4.2 again we can find L_j^2 level j admissible assignments Υ_h of (I_1, I_2) w.r.t. $(B^*, \emptyset)$ for $1 \leq h \leq L_j^2$ with associated τ_h. As in Lemma 3.6.4 we can construct a subset $\mathcal{H} \subset [L_j^2]$ with $|\mathcal{H}| = L_j < \lfloor L_j^2/9k_0^2 \rfloor$ so that for all $i_1 \neq i_2$ and $h_1, h_2 \in \mathcal{H}$ we have that $\tau_{h_1}(\ell_{i_1}) \neq \tau_{h_2}(\ell_{i_2})$, that is that all the positions bad blocks are assigned to are distinct. We will estimate the probability that one of these assignments work.

In trying out these L_j different assignments there is a subtle conditioning issue since conditioned on an assignment not working (e.g., the event $X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)}$ failing) the distribution of $Y_{\tau_h(l_i)}$ might change. As such we condition on an event $\mathcal{D}_h \cup \mathcal{G}_h$ which holds with high probability.

If $X_{\ell_i} \notin G_j^{\mathbb{X}}$, or, if neither $X_{\ell_i - 1}$ nor X_{ℓ_i+1} is $\in G_j^{\mathbb{X}}$, let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{ X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \right\}.$$

If $X_{\ell_i}, X_{\ell_i+1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{ Y_{\tau_h(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{c,s}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \forall k \in G_X \right\}.$$

If $X_{\ell_i}, X_{\ell_i-1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{ Y_{\tau_h(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{s,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \forall k \in G_X \right\}.$$

Let $\mathcal{D}_h$ denote the event

$$\mathcal{D}_h = \bigcap_{i=1}^{K'_X} \mathcal{D}_{h,i}.$$

Further, let

$$\mathcal{G}_h = \left\{ Y_{\tau_h(\ell_i)} \in G_j^{\mathbb{Y}} \text{ and } Y_{\tau_h(\ell_i)} \overset{s,s}{\longleftrightarrow} X_k \text{ for } 1 \leq i \leq K'_X, k \in G_X \right\}.$$

Then it follows from the recursive estimates and since $\beta > \alpha + \delta + 1$ that

$$\mathbb{P}[\mathcal{D}_h \cup \mathcal{G}_h \mid X, \mathcal{E}] \geq \mathbb{P}[\mathcal{G}_h \mid X, \mathcal{E}] \geq 1 - 10 k_0 L_j^{-\delta}.$$

and since they are conditionally independent given X and $\mathcal{E}$,

$$\mathbb{P}[\cap_{h \in \mathcal{H}} (\mathcal{D}_h \cup \mathcal{G}_h) \mid X, \mathcal{E}] \geq (1 - 10 k_0 L_j^{-\delta})^{L_j} \geq 9/10. \tag{3.6.10}$$

Now

$$\mathbb{P}[\mathcal{D}_h \mid X, \mathcal{E}, (\mathcal{D}_h \cup \mathcal{G}_h)] \geq \mathbb{P}[\mathcal{D}_h \mid X, \mathcal{E}] \geq \left(\frac{3}{4}\right)^{2k_0} \prod_{i \in B_X} S_j(X_i)$$

and hence

$$\begin{aligned} \mathbb{P}[\cup_{h \in \mathcal{H}} \mathcal{D}_h \mid X, \mathcal{E}, \cap_{h \in \mathcal{H}} (\mathcal{D}_h \cup \mathcal{G}_h)] &\geq 1 - \left(1 - \left(\frac{3}{4}\right)^{2k_0} \prod_{i \in B_X} S_j(X_i)\right)^{L_j} \\ &\geq \frac{9}{10} \wedge \frac{1}{4} \left(\frac{3}{4}\right)^{2k_0} L_j \prod_{i \in B_X} S_j(X_i) \end{aligned} \tag{3.6.11}$$

since $1 - e^{-x} \geq x/4 \wedge 9/10$ for $x \geq 0$. Furthermore, if

$$\mathcal{M} = \{\exists h_1 \neq h_2 \in \mathcal{H} : \mathcal{D}_{h_1} \setminus \mathcal{G}_{h_1}, \mathcal{D}_{h_2} \setminus \mathcal{G}_{h_2}\},$$

then

$$\begin{aligned}\mathbb{P}[\mathcal{M} \mid X, \mathcal{E}, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)] &\leq \binom{L_j}{2}\mathbb{P}[\mathcal{D}_h \setminus \mathcal{G}_h \mid X, \mathcal{E}, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)]^2 \\ &\leq \binom{L_j}{2}\left(2\left(\frac{3}{4}\right)^{2k_0}\prod_{i\in B_X} S_j(X_i) \wedge 2L_j^{-\delta}\right)^2 \\ &\leq L_j^{-(\delta-2)}\left(\frac{3}{4}\right)^{2k_0}\prod_{i\in B_X} S_j(X_i). \end{aligned} \tag{3.6.12}$$

Let $\mathcal{J}_I = \mathcal{J}_1$ and $\mathcal{J}_F = \mathcal{J}_{L_j^{\alpha-1}+2L_j^3+T_Y}$ denote the events

$$\mathcal{J}_I = \left\{X_1 \overset{c,s}{\longleftrightarrow} Y_1 \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_1 \text{ for all } k \in G_X\right\};$$

$$\mathcal{J}_F = \left\{X_{L_j^{\alpha-1}+2L_j^3+T_Y} \overset{s,c}{\longleftrightarrow} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \forall k \in G_X\right\}.$$

For $k \in \{2, \ldots L_j^{\alpha-1} + 2L_j^3 + T_Y - 1,\} \setminus \cup_{h\in\mathcal{H},1\leq i\leq K_X'}\{\tau_h(\ell_i)\}$, let $\mathcal{J}_k$ denote the event

$$\mathcal{J}_k = \left\{Y_k \in G_j^{\mathbb{Y}}, X_{k'} \overset{s,s}{\longleftrightarrow} Y_k \text{ for all } k' \in G_X\right\}.$$

Finally let

$$\mathcal{J} = \bigcap_{k\in[L_j^{\alpha-1}+2L_j^3+T_Y]\setminus\cup_{h\in\mathcal{H},1\leq i\leq K_X'}\{\tau_h(\ell_i)\}} \mathcal{J}_k.$$

Then it follows from the recursive estimates and the fact that $\mathcal{J}_k$ are conditionally independent that

$$\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \geq \left(\frac{9}{10}\right)^2\left(1 - RL_j^{\alpha-1-\beta}\right)^{2L_j^{\alpha-1}} \geq 3/4. \tag{3.6.13}$$

If $\mathcal{J}, \cup_{h\in\mathcal{H}}\mathcal{D}_h$ and $\cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)$ all hold and $\mathcal{M}$ does not hold then we can find at least one $h \in \mathcal{H}$ such that $\mathcal{D}_h$ holds and $\mathcal{G}_{h'}$ holds for all $h' \in \mathcal{H} \setminus \{h\}$. Then by Lemma 3.4.10 as before we have that $X \overset{c,c}{\longleftrightarrow} Y$. Hence by (3.6.10), (3.6.11), (3.6.12), and (3.6.13) and the fact that $\mathcal{J}$ is conditionally independent of the other events that

$$\begin{aligned}\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid X, \mathcal{E}] &\geq \mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h), \mathcal{J}, \neg\mathcal{M} \mid X, \mathcal{E}] \\ &= \mathbb{P}[\mathcal{J} \mid X, \mathcal{E}]\mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h, \neg\mathcal{M} \mid X, \mathcal{E}, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)] \\ &\quad \times \mathbb{P}[\cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h) \mid X, \mathcal{E}] \\ &\geq \frac{27}{40}\left[\frac{9}{10} \wedge \frac{1}{4}\left(\frac{3}{4}\right)^{2k_0} L_j \prod_{i\in B_X} S_j(X_i) - L_j^{-(\delta-2)}\prod_{i\in B_X} S_j(X_i)\right] \\ &\geq \frac{3}{5} \wedge \frac{1}{5}L_j\left(\frac{3}{4}\right)^{2k_0}\prod_{i\in B_X} S_j(X_i).\end{aligned}$$

Combining with (3.6.9) we have that

$$\mathbb{P}[X \hookrightarrow Y \mid X] \geq \frac{1}{2} \wedge \frac{1}{10}\left(\frac{3}{4}\right)^{2k_0} L_j \prod_{i \in B_X} S_j(X_i),$$

which completes the proof. ☐

Lemma 3.6.7. *When* $0 < p < \frac{1}{2}$,

$$\mathbb{P}(X \in \mathcal{A}^{(2)}_{X,j+1}, S_{j+1}(X) \leq p) \leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. We have that

$$\begin{aligned}\mathbb{P}(X \in \mathcal{A}^{(2)}_{X,j+1}, S_{j+1}(X) \leq p) &\leq \mathbb{P}\left[\frac{1}{10}\left(\frac{3}{4}\right)^{2k_0} L_j \prod_{i \in B_X} S_j(X_i) \leq p\right] \\ &\leq 2\left(\frac{10p}{L_j}\left(\frac{4}{3}\right)^{2k_0}\right)^{m_{j+1}} \leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta} \end{aligned} \tag{3.6.14}$$

where the first inequality holds by Lemma 3.6.6, the second by Lemma 3.6.2 and the third holds for large enough L_0 since $m_{j+1} > m > \alpha\beta$. ☐

3.6.3 Case 3

The third case allows for a greater number of bad sub-blocks. The class of blocks $\mathcal{A}^{(3)}_{X,j+1}$ is defined as

$$\mathcal{A}^{(3)}_{X,j+1} := \left\{ X : T_X \leq \frac{RL_j^{\alpha-1}}{2}, k_0 \leq K_X \leq \frac{L_j^{\alpha-1} + T_X}{10R_j^+} \right\}.$$

Lemma 3.6.8. *For* $X \in \mathcal{A}^{(3)}_{X,j+1}$,

$$S_{j+1}(X) \geq \frac{1}{2}\left(\frac{3}{4}\right)^{2K_X} \prod_{i \in B_X} S_j(X_i)$$

Proof. For this proof we only need to consider a single admissible assignment Υ. Suppose that $X \in \mathcal{A}^{(3)}_{X,j+1}$. Again let $\mathcal{E}$ denote the event

$$\mathcal{E} = \{W_Y \leq L_j^{\alpha-1}, T_Y = W_Y\}.$$

Similarly to (3.6.9) we have that,

$$\mathbb{P}[\mathcal{E}] \geq 8/10. \tag{3.6.15}$$

As before we have, on the event $T_Y = W_Y$, the blocks $Y_{L_j^3+1}, \ldots, Y_{L_j^3+L_j^{\alpha-1}+T_Y}$ are uniform j-blocks since the block division did not evaluate whether they are good or bad.

Set I_1, I_2, B_X, G_X and B^* as in the proof of Lemma 3.6.6. By Proposition 3.4.2 we can find a level j admissible assignment Υ of (I_1, I_2) w.r.t. (B^*, ϕ) with associated τ so that for all i, $L_j^3 + 1 \le \tau_h(\ell_i) \le L_j^3 + L_j^{\alpha-1} + T_Y$. We estimate the probability that this assignment works.

If $X_{\ell_i} \notin G_j^{\mathbb{X}}$, or, if neither X_{ℓ_i-1} nor X_{ℓ_i+1} is $\in G_j^{\mathbb{X}}$, let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{ X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau(\ell_i)} \right\}.$$

If $X_{\ell_i}, X_{\ell_i+1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{ Y_{\tau(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{c,s}{\longleftrightarrow} Y_{\tau(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{\tau(\ell_i)} \forall k \in G_X \right\}.$$

If $X_{\ell_i}, X_{\ell_i-1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{ Y_{\tau(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{s,c}{\longleftrightarrow} Y_{\tau(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} \} Y_{\tau(\ell_i)} \forall k \in G_X \right\}.$$

Let $\mathcal{D}$ denote the event

$$\mathcal{D} = \bigcap_{i=1}^{K'_X} \mathcal{D}_i.$$

By definition and the recursive estimates,

$$\mathbb{P}[\mathcal{D} \mid X, \mathcal{E}] \ge \left(\frac{3}{4}\right)^{2K_X} \prod_{i \in B_X} S_j(X_i) \tag{3.6.16}$$

Let $\mathcal{J}_I = \mathcal{J}_1$ and $\mathcal{J}_F = \mathcal{J}_{L_j^{\alpha-1}+2L_j^3+T_Y}$ denote the events

$$\mathcal{J}_I = \left\{ X_1 \overset{c,s}{\longleftrightarrow} Y_1 \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_1 \text{ for all } k \in G_X \right\};$$

$$\mathcal{J}_F = \left\{ X_{L_j^{\alpha-1}+2L_j^3+T_Y} \overset{s,c}{\longleftrightarrow} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \forall k \in G_X \right\}.$$

For $k \in \{2, \ldots L_j^{\alpha-1} + 2L_j^3 + T_Y - 1,\} \setminus \cup_{1 \le i \le K'_X} \{\tau(\ell_i)\}$, let $\mathcal{J}_k$ denote the event

$$\mathcal{J}_k = \left\{ Y_k \in G_j^{\mathbb{Y}}, X_{k'} \overset{s,s}{\longleftrightarrow} Y_k \text{ for all } k' \in G_X \right\}.$$

Finally let

$$\mathcal{J} = \bigcap_{k \in [L_j^{\alpha-1}+2L_j^3+T_Y] \setminus \cup_{1 \le i \le K'_X} \{\tau(\ell_i)\}} \mathcal{J}_k.$$

From the recursive estimates

$$\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \ge \frac{3}{4}. \tag{3.6.17}$$

If $\mathcal{D}$ and $\mathcal{J}$ hold then by Lemma 3.4.10 we have that $X \overset{c,c}{\longleftrightarrow} Y$. Hence by (3.6.16) and (3.6.17) and the fact that $\mathcal{D}$ and $\mathcal{J}$ are conditionally independent we have that,

$$\begin{aligned}\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid X, \mathcal{E}] &\geq \mathbb{P}[\mathcal{D}, \mathcal{J} \mid X, \mathcal{E}] \\ &= \mathbb{P}[\mathcal{D} \mid X, \mathcal{E}]\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \\ &\geq \frac{3}{4}\left(\frac{3}{4}\right)^{2K_X} \prod_{i \in B_X} S_j(X_i).\end{aligned}$$

Combining with (3.6.15) we have that

$$\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid X] \geq \frac{1}{2}\left(\frac{3}{4}\right)^{2K_X} \prod_{i \in B_X} S_j(X_i),$$

which completes the proof. □

Lemma 3.6.9. *When* $0 < p \leq \frac{1}{2}$,

$$\mathbb{P}(X \in \mathcal{A}^{(3)}_{X,j+1}, S_{j+1}(X) \leq p) \leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. We have that

$$\begin{aligned}\mathbb{P}(X \in \mathcal{A}^{(3)}_{X,j+1}, S_{j+1}(X) \leq p) &\leq \mathbb{P}\left[K_X > k_0, \frac{1}{2}\left(\frac{3}{4}\right)^{2K_X} \prod_{i \in B_X} S_j(X_i) \leq p\right] \\ &\leq \sum_{k=k_0}^{\infty} \mathbb{P}\left[K_X = k, \prod_{i \in B_X} S_j(X_i) \leq 2p\left(\frac{4}{3}\right)^{2k}\right] \\ &\leq 2\sum_{k=k_0}^{\infty} \left(2p\left(\frac{4}{3}\right)^{2k}\right)^{m_{j+1}} L_j^{-\delta k/4} \leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta} \qquad (3.6.18)\end{aligned}$$

where the first inequality holds by Lemma 3.6.8, the third follows from Lemma 3.6.2 and the last one holds for large enough L_0 since $\delta k_0 > 4\alpha\beta$. □

3.6.4 Case 4

In Case 4 we allow blocks of long length but not too many bad sub-blocks. The class of blocks $\mathcal{A}^{(4)}_{X,j+1}$ is defined as

$$\mathcal{A}^{(4)}_{X,j+1} := \left\{X : T_X > \frac{R L_j^{\alpha-1}}{2}, K_X \leq \frac{L_j^{\alpha-1} + T_X}{10 R_j^{+}}\right\}.$$

Lemma 3.6.10. *For* $X \in \mathcal{A}^{(4)}_{X,j+1}$,

$$S_{j+1}(X) \geq \left(\frac{3}{4}\right)^{2K_X} \prod_{i \in B_X} S_j(X_i) \exp(-3T_X L_j^{-4}/R)$$

Proof. In this proof we allow the length of Y to grow at a slower rate than that of X. Suppose that $X \in \mathcal{A}^{(4)}_{X,j+1}$ and let $\mathcal{E}(X)$ denote the event

$$\mathcal{E}(X) = \{W_Y = \lfloor 2T_X/R \rfloor, T_Y = W_Y\}.$$

Then by definition $\mathbb{P}[W_Y = \lfloor 2T_X/R \rfloor] = L_j^{-4}(1 - L_j^{-4})^{\lfloor 2T_X/R \rfloor}$. Similarly to Lemma 3.6.6, $\mathbb{P}[T_Y = W_Y \mid W_Y] \geq (1 - L_j^{-\delta})^{2L_j^3} \geq 9/10$. Combining these we have that

$$\mathbb{P}[\mathcal{E}(X)] \geq \frac{9}{10} L_j^{-4}(1 - L_j^{-4})^{\lfloor 2T_X/R \rfloor}. \tag{3.6.19}$$

Set I_1, I_2, B_X, B^* as before. By Proposition 3.4.2 we can find an admissible assignment at level j, Υ of (I_1, I_2) w.r.t. $(B^*, \emptyset)$ with associated τ so that for all i, $L_j^3 + 1 \leq \tau(\ell_i) \leq L_j^3 + L_j^{\alpha-1} + T_Y$. We again estimate the probability that this assignment works.

We need to modify the definition of $\mathcal{D}$ and $\mathcal{J}$ in this case since the length of X could be arbitrarily large. For $k \in [L_j^{\alpha-1} + 2L_j^3 + T_Y] \setminus \tau(B_X)$, let $H_k^\tau \subseteq [L_j^{\alpha-1} + 2L_j^3 + T_Y] \setminus B_X$ be the sets given by Lemma 3.4.10 such that $|H_k^\tau| \leq L_j$ and there exists a τ-compatible admissible route with k-sections contained in H_k^τ for all k. We define $\mathcal{D}$ and $\mathcal{J}$ in this case as follows.

If $X_{\ell_i} \notin G_j^{\mathbb{X}}$, or, if neither $X_{\ell_i - 1}$ nor $X_{\ell_i + 1}$ is $\in G_j^{\mathbb{X}}$, let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{ X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau(\ell_i)} \right\}.$$

If $X_{\ell_i}, X_{\ell_i+1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{ Y_{\tau(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{c,s}{\longleftrightarrow} Y_{\tau(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{\tau(\ell_i)} \forall k \in H^\tau_{\tau(l_i)} \right\}.$$

If $X_{\ell_i}, X_{\ell_i-1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{ Y_{\tau(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{s,c}{\longleftrightarrow} Y_{\tau(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{\tau(\ell_i)} \forall k \in H^\tau_{\tau(l_i)} \right\}.$$

Let $\mathcal{D}$ denote the event

$$\mathcal{D} = \bigcap_{i=1}^{K'_X} \mathcal{D}_i.$$

Let $\mathcal{J}_I = \mathcal{J}_1$ and $\mathcal{J}_F = \mathcal{J}_{L_j^{\alpha-1} + 2L_j^3 + T_Y}$ denote the events

$$\mathcal{J}_I = \left\{ X_1 \overset{c,s}{\longleftrightarrow} Y_1 \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_1 \text{ for all } k \in H_1^\tau \right\};$$

$$\mathcal{J}_F = \left\{ X_{L_j^{\alpha-1}+2L_j^3+T_Y} \stackrel{s,c}{\longleftrightarrow} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \text{ and } X_k \stackrel{s,s}{\longleftrightarrow} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \forall k \in H^{\tau}_{L_j^{\alpha-1}+2L_j^3+T_Y} \right\}.$$

For $k \in \{2, \dots L_j^{\alpha-1} + 2L_j^3 + T_Y - 1, \} \setminus \cup_{1\le i\le K'_X}\{\tau(\ell_i)\}$, let $\mathcal{J}_k$ denote the event

$$\mathcal{J}_k = \left\{ Y_k \in G_j^{\mathbb{Y}}, X_{k'} \stackrel{s,s}{\longleftrightarrow} Y_k \text{ for all } k' \in H_k^{\tau} \right\}.$$

Finally let

$$\mathcal{J} = \bigcap_{k\in[L_j^{\alpha-1}+2L_j^3+T_Y]\setminus \cup_{1\le i\le K'_X}\{\tau(\ell_i)\}} \mathcal{J}_k.$$

If $\mathcal{D}$ and $\mathcal{J}$ hold then by Lemma 3.4.10 we have that $X \stackrel{c,c}{\longleftrightarrow} Y$. It is easy to see that, in this case (3.6.16) holds. Also we have for large enough L_0,

$$\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}(X)] \ge \frac{3}{4}\left(1-2L_j^{-\delta}\right)^{L_j^{\alpha-1}+\lfloor 2T_X/R\rfloor+2L_j^3} \ge \frac{1}{4}\exp\left(-2L_j^{-\delta}(L_j^{\alpha-1}+\lfloor 2T_X/R\rfloor+2L_j^3)\right). \tag{3.6.20}$$

Hence by (3.6.16) and (3.6.20) and the fact that $\mathcal{D}$ and $\mathcal{J}$ are conditionally independent we have that,

$$\begin{aligned}\mathbb{P}[X \stackrel{c,c}{\longleftrightarrow} Y \mid X, \mathcal{E}] &\ge \mathbb{P}[\mathcal{D} \mid X, \mathcal{E}]\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \\ &\ge \frac{1}{4}\exp\left(-L_j^{-\delta}(L_j^{\alpha-1}+\lfloor 2T_X/R\rfloor+2L_j^3)\right)\left(\frac{3}{4}\right)^{2K_X}\prod_{i\in B_X} S(X_i).\end{aligned}$$

Combining with (3.6.19) we have that

$$\mathbb{P}[X \stackrel{c,c}{\longleftrightarrow} Y \mid X] \ge \exp(-3T_X L_j^{-4}/R)\left(\frac{3}{4}\right)^{2K_X}\prod_{i\in B_X} S_j(X_i),$$

since $T_X L_j^{-4} = \Omega(L_j^{\alpha-6})$ and $\delta > 5$ which completes the proof. □

Lemma 3.6.11. *When* $0 < p \le \frac{1}{2}$,

$$\mathbb{P}(X \in \mathcal{A}^{(4)}_{X,j+1}, S_{j+1}(X) \le p) \le \frac{1}{5}p^{m_{j+1}}L_{j+1}^{-\beta}$$

Proof. Set $t_0 = \frac{RL_j^{\alpha-1}}{2} + 1$ and for $k \ge k_0$, set

$$S(k) = \left(\frac{3}{4}\right)^{2k}\prod_{i\in B_X} S_j(X_i).$$

We have that

$$\begin{aligned}\mathbb{P}(X \in \mathcal{A}^{(4)}_{X,j+1}, S_{j+1}(X) \le p) &\le \sum_{t=t_0}^{\infty}\sum_{k=k_0}^{\infty} \mathbb{P}\left[T_X = t, K_X = k, S(k)\exp(-3tL_j^{-4}/R) \le p\right] \\ &\le \sum_{t=t_0}^{\infty}\sum_{k=k_0}^{\infty} 2\left(\frac{4^{2k}p}{3^{2k}}\right)^{m_{j+1}} \exp(3m_{j+1}tL_j^{-4}/R - tL_j^{-4}/2)L_j^{-\delta k/4} \\ &\le \frac{1}{5}p^{m_{j+1}}L_{j+1}^{-\beta} \end{aligned} \tag{3.6.21}$$

where the first inequality holds by Lemma 3.6.10, the second by Lemma 3.6.2 and the third holds for large enough L_0 since $3m_{j+1}/R < \frac{1}{2}$ and so for large enough L_0, $(4/3)^{2(m+1)}L_j^{-\delta/4} \le 1/2$ and

$$\sum_{t=RL_j^{\alpha-1}/2+1}^{\infty} \exp\left(-tL_j^{-4}\left(\frac{1}{2} - \frac{3m_{j+1}}{R}\right)\right) < \frac{1}{10}L_{j+1}^{-\beta}.$$

□

3.6.5 Case 5

It remains to deal with the case involving blocks with a large density of bad sub-blocks. Define the class of blocks $\mathcal{A}^{(5)}_{X,j+1}$ is as

$$\mathcal{A}^{(5)}_{X,j+1} := \left\{X : K_X > \frac{L_j^{\alpha-1} + T_X}{10R_j^+}\right\}.$$

Lemma 3.6.12. *For $X \in \mathcal{A}^{(5)}_{X,j+1}$,*

$$S_{j+1}(X) \ge \exp(-2T_XL_j^{-4})\left(\frac{3}{4}\right)^{2K_X}\prod_{i\in B_X} S_j(X_i)$$

Proof. The proof is a minor modification of the proof of Lemma 3.6.10. We take $\mathcal{E}(X)$ to denote the event

$$\mathcal{E}(X) = \{W_Y = T_X, T_Y = W_Y\}.$$

and get a bound of

$$\mathbb{P}[\mathcal{E}(X)] \ge \frac{9}{10}L_j^{-4}(1 - L_j^{-4})^{T_X}. \tag{3.6.22}$$

We consider the admissible assignment Υ given by $\tau(i) = i$ for $i \in B^*$. It follows from Lemma 3.4.10 that in this case we can define $H_k^\tau = k-1, k, k+1$. We define $\mathcal{D}$ and $\mathcal{J}$ as before. The new bound for $\mathcal{J}$ becomes

$$\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}(X)] \ge \frac{3}{4}\left(1 - 2L_j^{-\delta}\right)^{L_j^{\alpha-1}+T_X+2L_j^3} \ge \frac{1}{4}\exp\left(-2L_j^{-\delta}(L_j^{\alpha-1} + T_X + 2L_j^3)\right). \tag{3.6.23}$$

We get the result proceeding as in the proof of Lemma 3.6.10. □

Lemma 3.6.13. *When* $0 < p \le \frac{1}{2}$,

$$\mathbb{P}(X \in \mathcal{A}^{(5)}_{X,j+1}, S_{j+1}(X) \le p) \le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. First note that since $\alpha > 4$,

$$L_j^{-\frac{\delta}{50R_j^+}} = L_0^{-\frac{\delta\alpha^j}{50R_j^+}} \to 0$$

as $j \to \infty$. Hence for large enough L_0,

$$\sum_{t=0}^{\infty} \left(\exp(2m_{j+1}L_j^{-4}) L_j^{-\frac{\delta}{50R_j^+}} \right)^t < 2. \tag{3.6.24}$$

Set $k_* = \frac{L_j^{\alpha-1}+t}{10R_j^+}$ and for $k \ge k_*$ set $S(k) = \left(\frac{3}{4}\right)^{2k} \prod_{i \in B_X} S_j(X_i)$. We have that

$$\begin{aligned}
\mathbb{P}(X \in \mathcal{A}^{(5)}_{X,j+1}, S_{j+1}(X) \le p) &\le \sum_{t=0}^{\infty} \sum_{k=k_*}^{\infty} \mathbb{P}\left[T_X = t, K_X = k, S(k)\exp(-2tL_j^{-4}) \le p\right] \\
&\le p^{m_{j+1}} \sum_{t=0}^{\infty} \sum_{k=k_*}^{\infty} 2\left(\exp(2m_{j+1}tL_j^{-4})\right) \left(\left(\frac{16}{9}\right)^{m_j+1} L_j^{-\frac{\delta}{4}} \right)^k \\
&\le p^{m_{j+1}} \sum_{t=0}^{\infty} 4\left(\exp(2m_{j+1}tL_j^{-4})\right) L_j^{-\frac{L_j^{\alpha-1}+t}{50R_j^+}} \\
&\le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta} \qquad (3.6.25)
\end{aligned}$$

where the first inequality holds be by Lemma 3.6.12, the second by Lemma 3.6.2 and the third follows since L_0 is sufficiently large and the last one by (3.6.24) and the fact that

$$L_j^{-\frac{\delta L_j^{\alpha-1}}{50R_j^+}} \le \frac{1}{40} L_{j+1}^{-\beta},$$

for large enough L_0. □

3.6.6 Proof of Theorem 3.6.1

Putting together all the five cases we now prove Theorem 3.6.1.

Proof of Theorem 3.6.1. The case of $\frac{1}{2} \le p \le 1 - L_{j+1}^{-1}$ is established in Lemma 3.6.5. By Lemma 3.6.4 we have that $S_{j+1}(X) \ge \frac{1}{2}$ for all $X \in \mathcal{A}^{(1)}_{X,j+1}$. Hence we need only consider

$0 < p < \frac{1}{2}$ and cases 2 to 5. By Lemmas 3.6.7, 3.6.9, 3.6.11 and 3.6.13 then

$$\mathbb{P}(S_{j+1}(X) \le p) \le \sum_{l=2}^{5} \mathbb{P}(X \in \mathcal{A}^{(l)}_{X,j+1}, S_{j+1}(X) \le p) \le p^{m_{j+1}} L_{j+1}^{-\beta}.$$

The bound for $S^{\mathbb{Y}}_{j+1}$ follows similarly. □

3.7 Side to Corner and Corner to Side Estimates

The aim of this section is to show that for a large class of $\mathbb{X}$- blocks (resp. $\mathbb{Y}$-blocks), $\mathbb{P}(X \overset{c,s}{\longleftrightarrow} Y \mid X)$ and $\mathbb{P}(X \overset{s,c}{\longleftrightarrow} Y \mid X)$ (resp. $\mathbb{P}(X \overset{c,s}{\longleftrightarrow} Y \mid Y)$ and $\mathbb{P}(X \overset{s,c}{\longleftrightarrow} Y \mid Y)$) is large. We shall state and prove the result only for $\mathbb{X}$-blocks.

Here we need to consider a different class of blocks where the blocks have few bad subblocks whose corner to corner connection probabilities are not too small, where the excess number of subblocks is of smaller order than the typical length and none of the subblocks, and their chunks contain too many level 0 blocks. This case holds with high probability. Let X be a level $(j+1)$ X-block constructed out of the independent sequence of j level blocks $X_1, X_2, \ldots$ where the first L_j^3 ones are conditioned to be good.

For $i = 1, 2, \ldots, L_j^{\alpha-1} + 2L_j^3 + T_X$, let $\mathcal{G}_i$ denote the event that all level $j-1$ subblocks contained in X_i contains at most $3L_{j-1}$ level 0 blocks, and X_i contains at most $3L_j$ level 0 blocks. Let $\mathcal{G}_X$ denote the event that for all good blocks X_i contained in X, $\mathcal{G}_i$ holds. We define $\mathcal{A}^{(*)}_{X,j+1}$ to be the set of $(j+1)$ level blocks such that

$$\mathcal{A}^{(*)}_{X,j+1} := \left\{ X : T_X \le L_j^5 - 2L_j^3, K_X \le k_0, \prod_{i \in B_X} S_j(X_i) > L_j^{-1/3}, \mathcal{G}_X \right\}.$$

It follows from Theorem 3.5.1 that $\mathbb{P}[\mathcal{G}_X^c]$ is exponentially small in L_{j-1} and hence we shall be able to safely ignore this conditioning while calculating probability estimates since L_0 is sufficiently large.

Similarly to Lemma 3.6.3 it can be proved that

$$\mathbb{P}[X \in \mathcal{A}^{(*)}_{X,j+1}] \ge 1 - L_{j+1}^{-3\beta}. \tag{3.7.1}$$

We have the following proposition.

Proposition 3.7.1. *We have that for all $X \in \mathcal{A}^{(*)}_{X,j+1}$,*

$$\mathbb{P}[X \overset{c,s}{\longleftrightarrow} Y \mid Y \in \mathcal{A}^{(*)}_{Y,j+1}, X] \ge \frac{9}{10} + 2^{-(j+15/4)},$$
$$\mathbb{P}[X \overset{s,c}{\longleftrightarrow} Y \mid Y \in \mathcal{A}^{(1)}_{Y,j+1}, X] \ge \frac{9}{10} + 2^{-(j+15/4)}. \tag{3.7.2}$$

We shall only prove the corner to side estimate, the other one follows by symmetry. Suppose that $X \in \mathcal{A}^{(*)}_{X,j+1}$ with length $L_j^{\alpha-1} + 2L_j^3 + T_X$, define B_X, B^*,K'_X, T_Y and K_Y as in the proof of Lemma 3.6.4. We condition on $Y \in \mathcal{A}^{(*)}_{Y,j+1}$ having no bad subblocks. Denote this conditioning by

$$\mathcal{F} = \{Y \in \mathcal{A}^{(*)}_{Y,j+1}, T_Y, K_Y = 0\}.$$

Let n_X and n_Y denote the number of chunks in X and Y respectively. We first prove the following lemma.

Lemma 3.7.2. *Consider an exit chunk (k, n_Y) (resp. (n_X, k)) in $\mathcal{E}_{out}(X, Y)$. Fix $t \in [L_j^{\alpha-1} + 2L_j^3 + T_X]$ contained in C_k^X such that $[t, t - L_j^3] \cap B_X = \emptyset$ (resp. fix $t' \in [L_j^{\alpha-1} + 2L_j^3 + T_Y]$ contained in C_k^Y). Consider $\widetilde{X} = (X_1, \ldots, X_t)$ (or $\widetilde{Y} = (Y_1, \ldots, Y_{t'})$). Then there exists an event $\mathcal{S}_t$ with $\mathbb{P}[S_t \mid \mathcal{F}] \geq 1 - L_j^{-\alpha}$ and on S_t, $\mathcal{F}$ and $\{X_1 \overset{c,s}{\longleftrightarrow} Y_1\}$ we have $\widetilde{X} \overset{c,s,*}{\longleftrightarrow} Y$ (resp. $\mathcal{S}_{t'}$ with $\mathbb{P}[S_{t'} \mid \mathcal{F}] \geq 1 - L_j^{-\alpha}$ and on $S_{t'}$, $\mathcal{F}$ and $\{X_1 \overset{c,s}{\longleftrightarrow} Y_1\}$ we have $X \overset{c,s,*}{\longleftrightarrow} \widetilde{Y}$).*

Proof. We shall only prove the first case, the other case follows by symmetry. Set $I_1 = [t]$, $I_2 = [L_j^{\alpha-1} + 2L_j^3 + T_Y]$. Also define $B_{\widetilde{X}}$ and B^* as in the proof of Lemma 3.6.4. The slope condition in the definition of $\mathcal{E}_{out}(X, Y)$, and the fact that B_X is disjoint with $[t - L_j^3, t]$ implies that by Proposition 3.4.2 we can find L_j^2 admissible generalized mappings Υ_h of (I_1, I_2) with respect to $(B^*, \emptyset)$ with associated τ_h for $1 \leq h \leq L_j^2$ as in the proof of Lemma 3.6.4. As in there, we construct a subset $\mathcal{H} \subset [L_j^2]$ with $|\mathcal{H}| = L_j < \lfloor L_j^2/3k_0 \rfloor$ so that for all $i_1 \neq i_2$ and $h_1, h_2 \in \mathcal{H}$ we have that $\tau_{h_1}(\ell_{i_1}) \neq \tau_{h_2}(\ell_{i_2})$.

For $h \in \mathcal{H}, i \in B^*$, define the events $\mathcal{D}_{h,i}$ similarly as in the proof of Lemma 3.6.4. Set

$$\mathcal{D}_h = \bigcap_{i=1}^{K'_X} \mathcal{D}_{h,i}^k \text{ and } \mathcal{D} = \bigcup_{h \in \mathcal{H}} \mathcal{D}_h^k.$$

Further, $\mathcal{S}$ denote the event

$$\mathcal{S} = \left\{ X_k \overset{s,s}{\longleftrightarrow} Y_{k'} \forall k \in [t] \setminus \{\ell_1, \ldots, \ell_{K'_X}\}, \forall k' \in [L_j^{\alpha-1} + 2L_j^3 + T_Y] \right\}.$$

Same arguments as in the proof of yields

$$\mathbb{P}[\mathcal{D} \mid \mathcal{F}] \geq 1 - L_{j+1}^{-3\beta} \tag{3.7.3}$$

and

$$\mathbb{P}[\neg \mathcal{S} \mid \mathcal{F}] \leq 4L_j^{2\alpha-2} L_j^{-2\beta} \leq L_j^{-\beta}. \tag{3.7.4}$$

Now it follows from Lemma 3.4.8 and Lemma 3.4.11, that on $\{X_1 \overset{c,s}{\longleftrightarrow} Y_1\}$, $\mathcal{S}, \mathcal{D}$ and $\mathcal{F}$, we have $\widetilde{X} \overset{c,s,*}{\longleftrightarrow} Y$. The proof of the Lemma is completed by setting $\mathcal{S}_t = \mathcal{S} \cap \mathcal{D}$. □

Now we are ready to prove Proposition 3.7.1.

Proof of Proposition 3.7.1. Fix an exit chunk (k, n_Y) or (n_X, k') in $\mathcal{E}_{out}(X, Y)$. In the former case set T_k to be the set of all blocks X_t contained in C_k^X such that $[t, t - L_j^3] \cap B_X = \emptyset$, in the later case set $T'_{k'}$ to be the set of all blocks $Y_{t'}$ conttained in $C_{k'}^Y$. Notice that the number of blocks contained in T_k is at least $(1 - 2k_0 L_j^{-1})$ fraction of the total number of blocks contained in C_k^X. For $t \in T_k$ (resp. $t' \in T'_{k'}$), let S_t (resp. $S_{t'}$) be the event given by Lemma 3.7.2 Hence it follows from Lemma 3.4.7(i), that on $\{X_1 \overset{c,s}{\longleftrightarrow} Y_1\} \bigcap \cap_{k,T_k} \mathcal{S}_t \bigcap \cap_{k',T_{k'}} \mathcal{S}_{t'}$, we have $X \overset{c,s}{\longleftrightarrow} Y$. Taking a union bound and using Lemma 3.7.2 and also using the recursive lower bound on $\mathbb{P}[X_1 \overset{c,s}{\longleftrightarrow} Y_1]$ yields,

$$\mathbb{P}[X \overset{c,s}{\longleftrightarrow} Y \mid \mathcal{F}, X] \geq \frac{9}{10} + 2^{-(j+31/8)}.$$

The proof can now be completed by removing the conditioning on T_Y and proceeding as in Lemma 3.6.4. □

3.8 Side to Side Estimate

In this section we estimate the probability of having a side to side path in $X \times Y$. We work in the set up of previous section. We have the following theorem.

Proposition 3.8.1. *We have that*

$$\mathbb{P}[X \overset{s,s}{\longleftrightarrow} Y \mid X \in \mathcal{A}_{X,j+1}^{(*)}, Y \in \mathcal{A}_{Y,j+1}^{(*)}] \geq 1 - L_{j+1}^{-3\beta}. \tag{3.8.1}$$

Suppose that $X \in \mathcal{A}_{X,j+1}^{(*)}, Y \in \mathcal{A}_{Y,j+1}^{(*)}$. Let T_X, T_Y, B_X, B_Y, G_X, G_Y be as before. Let $B_1^* = \{\ell_1 < \cdots < \ell_{K'_X}\}$ and $B_2^* = \{\ell'_1 < \cdots < \ell'_{K'_Y}\}$ denote the locations of bad blocks and their neighbours in X and Y respectively. Let us condition on the block lengths T_X, T_Y, B_1^*, B_2^* and the bad-sub-blocks and their neighbours themselves. Denote this conditioning by

$$\begin{aligned}\mathcal{F} = \{X \in \mathcal{A}_{X,j+1}^{(1)}, Y \in \mathcal{A}_{Y,j+1}^{(1)}, T_X, T_Y, K'_X, K'_Y, \ell_1, \ldots, \ell_{K'_X}, \ell'_1, \ldots, \ell'_{K'_Y},\\ X_{\ell_1}, \ldots, X_{\ell_{K'_X}}, Y_{\ell'_1}, \ldots, Y_{\ell'_{K'_Y}}\}.\end{aligned}$$

Let

$$B_{X,Y} = \{(k, k') \in G_X \times G_Y : X_k \overset{s,s}{\not\longleftrightarrow} Y_{k'}\}$$

and $N_{X,Y} = |B_{X,Y}|$. Let $\mathcal{S}$ denote the event $\{N_{X,Y} \leq k_0\}$. We first prove the following lemma.

Lemma 3.8.2. *Let n_X and n_Y denote the number of chunks in X and Y respectively. Fix an entry exit pair of chunks. For concreteness, take $((k, 1), (n_X, k')) \in \mathcal{E}(X, Y)$. Fix $t \in [L_j^{\alpha-1} + 2L_j^3 + T_X]$ and $t' \in [L_j^{\alpha-1} + 2L_j^3 + T_Y]$ such that X_t is contained in C_k^X, $Y_{t'}$ contained in $C_{k'}^Y$ also such that $[t, t + L_j^3] \cap B_X = \emptyset$. Also let $A_{t,t'}$ denote the event that*

$[t,t+L_j^3]\times[1,L_j^3]\cup[L_j^{\alpha-1}+T_X+L_j^3,L_j^{\alpha-1}+T_X+2L_j^3]\times[t'-L_j^3,t']$ is disjoint with $B_{X,Y}$. Set $\widetilde{X}=(X_t,X_{t+1},\ldots,X_{L_j^{\alpha-1}+T_X+2L_j^3})$ and $\widetilde{Y}=(Y_1,Y_2,\ldots,Y_{t'})$, call such a pair $(\widetilde{X},\widetilde{Y})$ to be a proper section *of (X,Y). Then there exists an event $S_{t,t'}$ with $\mathbb{P}[S_{t,t'}\mid\mathcal{F}]\geq 1-L_{j+1}^{-4\beta}$ and such that on $S\cap S_{t,t'}\cap A_{t,t'}$, we have $\widetilde{X}\overset{s,s,*}{\longleftrightarrow}\widetilde{Y}$.*

Proof. Set $I_1=[t,L_j^{\alpha-1}+T_X+2L_j^3]\cap\mathbb{Z}$, $I_2=[1,t']\cap\mathbb{Z}$. By Proposition 3.4.2 we can find L_j^2 admissible assignments mappings Υ_h with associated τ_h of (I_1,I_2) w.r.t. $(B_1^*\cap I_1,B_2^*\cap I_2)$ such that we have $\tau_h(\ell_i)=\tau_1(\ell_i)+h-1$ and $\tau_h^{-1}(\ell_i')=\tau_1^{-1}(\ell_i')-h+1$. As before we can construct a subset $\mathcal{H}\subset[L_j^2]$ with $|\mathcal{H}|=10k_0L_j<\lfloor L_j^2/36k_0^2\rfloor$ so that for all $i_1\neq i_2$ and $h_1,h_2\in\mathcal{H}$ we have that $\tau_{h_1}(\ell_{i_1})\neq\tau_{h_2}(\ell_{i_2})$ and $\tau_{h_1}^{-1}(\ell_{i_1}')\neq\tau_{h_2}^{-1}(\ell_{i_2}')$, that is that all the positions bad blocks and their neighbours are assigned to are distinct.

Hence we have for all $h\in\mathcal{H}$

$$\mathbb{P}[X_{\ell_i}\overset{c,c}{\longleftrightarrow}Y_{\tau_h(\ell_i)}\mid\mathcal{F}]\geq\frac{1}{2}S_j(X_{\ell_i});\tag{3.8.2}$$

$$\mathbb{P}[X_{\tau_h^{-1}(\ell_i')}\overset{c,c}{\longleftrightarrow}Y_{\ell_i'}\mid\mathcal{F}]\geq\frac{1}{2}S_j(Y_{\ell_i'}).\tag{3.8.3}$$

If $X_{\ell_i}\notin G_j^{\mathbb{X}}$, or, if neither X_{ℓ_i-1} nor X_{ℓ_i+1} is $\in G_j^{\mathbb{X}}$, let $\mathcal{D}_{h,i,X}$ denote the event

$$\mathcal{D}_{h,i,X}=\left\{X_{\ell_i}\overset{c,c}{\longleftrightarrow}Y_{\tau_h^{k,k'}(\ell_i)}\right\}.$$

If $X_{\ell_i},X_{\ell_i+1}\in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i,X}$ denote the event

$$\mathcal{D}_{h,i,X}=\left\{X_{\ell_i}\overset{c,s}{\longleftrightarrow}Y_{\tau_h^{k,k'}(\ell_i)}\right\}.$$

If $X_{\ell_i},X_{\ell_i-1}\in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i,X}$ denote the event

$$\mathcal{D}_{h,i,X}=\left\{X_{\ell_i}\overset{s,c}{\longleftrightarrow}Y_{\tau_h^{k,k'}(\ell_i)}\right\}.$$

Let $\mathcal{D}_{h,X}$ denote the event

$$\mathcal{D}_{h,X}=\bigcap_{i=1}^{K_X'}\mathcal{D}_{h,i,X}$$

Let us define the event $\mathcal{D}_{h,Y}$ similarly and let

$$\mathcal{D}_h=\mathcal{D}_{h,X}\cap\mathcal{D}_{h,Y}$$

Finally, let

$$\mathcal{D}=\left\{\sum_{h\in\mathcal{H}}\mathbf{1}_{\mathcal{D}_h}\geq R^6k_0^5 10^{2j+20}\right\}.$$

Conditional on $\mathcal{F}$, for $h \in \mathcal{H}$, the $\mathcal{D}_h$ are independent and by (3.8.2), (3.8.3) and the recursive estimates ,

$$\mathbb{P}[\mathcal{D}_h \mid \mathcal{F}] \geq 2^{-10k_0}3^{4k_0}L_j^{-2/3}. \tag{3.8.4}$$

Hence using a large deviation estimate for binomial tail probabilities we get,

$$\mathbb{P}[\mathcal{D} \mid \mathcal{F}] \geq \mathbb{P}[\mathrm{Bin}(10k_0L_j, 2^{-10k_0}3^{4k_0}L_j^{-2/3}) \geq R^6k_0^5 10^{2j+20}\}] \geq 1 - L_{j+1}^{-4\beta} \tag{3.8.5}$$

for L_0 sufficiently large. Now it follows from Lemma 3.4.13 and Lemma 3.4.11 that if $\mathcal{D}$, $\mathcal{S}$, and $A_{t,t'}$ all holds than $\widetilde{X} \overset{s,s,*}{\longleftrightarrow} \widetilde{Y}$. This completes the proof of the lemma. □

Before proving Proposition 3.8.1, we need the following lemma bounding the probability of $\mathcal{S}$.

Lemma 3.8.3. *We have*

$$\mathbb{P}[\neg S \mid \mathcal{F}] \leq \frac{1}{3}L_{j+1}^{-3\beta}. \tag{3.8.6}$$

Proof. Let for $k' \in G_Y$,

$$V_{k'}^Y = I\left[\left\{\#\left\{k \in G_X : X_k \overset{s,s}{\not\longrightarrow} Y_{k'}\right\} \geq 1\right\}\right].$$

It follows from taking a union bound and using the recursive estimates that

$$\mathbb{P}[V_{k'}^Y = 1 \mid \mathcal{F}, X] \leq 2L_j^{\alpha-1-\beta}.$$

Since $V_{k'}^Y$ are conditionally independent given X and $\mathcal{F}$, a stochastic domination argument yields

$$\mathbb{P}[\sum_{k'} V_{k'}^Y \geq k_0^{1/2} \mid X, \mathcal{F}] \leq \mathbb{P}[\mathrm{Bin}(2L_j^{\alpha-1}, 2L_j^{\alpha-1-\beta}) \geq k_0^{1/2}].$$

Using a Chernoff bound and setting $\lambda = \frac{1}{4}k_0^{1/2}L_j^{-2\alpha+2+\beta}$ (note $\lambda > 1$ as $\beta > 2\alpha$ and L_0 is large enough) we get

$$\begin{aligned}
\mathbb{P}[\sum_{k'} V_{k'}^Y \geq k_0^{1/2} \mid \mathcal{F}, X] &\leq \exp\left(4L_j^{2\alpha-2-\beta}(\lambda - 1 - \lambda\log\lambda)\right) \\
&\leq \exp\left(-2L_j^{2\alpha-2-\beta}\lambda\log\lambda\right) \\
&\leq \left(\frac{1}{4}k_0^{1/2}L_j^{-2\alpha+2+\beta}\right)^{k_0^{1/2}/2} \leq \frac{1}{6}L_{j+1}^{-3\beta}
\end{aligned}$$

for L_0 large enough since $k_0^{1/2}(\beta + 2 - 2\alpha) > 6\alpha\beta$.

Removing the conditioning on X we get,

$$\mathbb{P}[\sum_{k'} V_k^Y \geq k_0^{1/2} \mid \mathcal{F}] \leq \frac{1}{6} L_{j+1}^{-3\beta}.$$

Defining V_k^X's similarly we get

$$\mathbb{P}[\sum_{k} V_k^X \geq k_0^{1/2} \mid \mathcal{F}] \leq \frac{1}{6} L_{j+1}^{-3\beta}.$$

Since on $\mathcal{F}$,

$$\neg\mathcal{S} \subseteq \{\sum_k V_k^X \geq k_0^{1/2}\} \cup \{\sum_k V_k^X \geq k_0^{1/2}\},$$

the lemma follows. □

Now we are ready to prove Proposition 3.8.1.

Proof of Proposition 3.8.1. Consider the set-up of Lemma 3.8.2. Let T_k (resp. $T'_{k'}$) denote the set of indices t (resp. t') such that X_t is contained in C_k^X (resp. $Y_{t'}$ is contained in $C_{k'}^Y$). It is easy to see that there exists $T_{k,*} \subset T_k$ (resp. $T'_{k',*} \subset T'_{k'}$) with $|T_{k,*}| \geq (1 - 10k_0 L_j^{-1})|T_k|$ (resp. $|T'_{k',*}| \geq (1 - 10k_0 L_j^{-1})|T'_{k'}|$) such that for all $t \in T_{k,*}$ and for all $t' \in T'_{k',*}$, $\widetilde{X}$ and $\widetilde{Y}$ defined as in Lemma 3.8.2 satisfies that $(\widetilde{X}, \widetilde{Y})$ is a *proper section* of (X, Y) and $A_{t,t'}$ holds.

It follows now by taking a union bound over all $t \in T_k$, $t' \in T'_{k'}$, and all pairs of entry exit chunks in $\mathcal{E}(X, Y)$ and using Lemma 3.4.7 that

$$\mathbb{P}[X \stackrel{s,s}{\longleftrightarrow} Y \mid \mathcal{F}] \geq 1 - \frac{1}{3} L_{j+1}^{-3\beta} - 4L_j^{2\alpha} L_{j+1}^{-3\beta} \geq 1 - L_{j+1}^{-3\beta} \tag{3.8.7}$$

for L_0 sufficiently large since $\beta > 2\alpha$. Now removing the conditioning we get (3.8.1). □

3.9 Good Blocks

Now we are ready to prove that a block is good with high probability.

Theorem 3.9.1. *Let X be a $\mathbb{X}$-block at level $(j+1)$. Then $\mathbb{P}(X \in G_{j+1}^{\mathbb{X}}) \geq 1 - L_{j+1}^{-\delta}$. Similarly for $\mathbb{Y}$-block Y at level $(j+1)$, $\mathbb{P}(Y \in G_{j+1}^{\mathbb{Y}}) \geq 1 - L_{j+1}^{-\delta}$.*

Proof. To avoid repetition, we only prove the theorem for $\mathbb{X}$-blocks. Let X be a $\mathbb{X}$-block at level $(j+1)$ with length $L_j^{\alpha-1}$

Let the events $A_i, i = 1, \ldots 5$ be defined as follows.

$$A_1 = \left\{T_X \leq L_j^5 - 2L_j^3\right\}.$$

$$A_2 = \left\{\mathbb{P}[X \stackrel{c,c}{\longleftrightarrow} Y \mid X] \geq \frac{3}{4} + 2^{-(j+4)}\right\}.$$

$$A_3 = \left\{ \mathbb{P}[X \overset{c,s}{\longleftrightarrow} Y \mid X] \geq \frac{9}{10} + 2^{-(j+4)} \right\}.$$

$$A_4 = \left\{ \mathbb{P}[X \overset{s,c}{\longleftrightarrow} Y \mid X] \geq \frac{9}{10} + 2^{-(j+4)} \right\}.$$

$$A_5 = \left\{ \mathbb{P}[X \overset{s,s}{\longleftrightarrow} Y \mid X] \geq 1 - L_j^{2\beta} \right\}.$$

From Lemma 3.6.2 it follows that

$$\mathbb{P}[A_1^c] \leq L_{j+1}^{-3\beta}.$$

From Lemma 3.6.3 and 3.6.4 it follows that

$$\mathbb{P}[A_2^c] \leq L_{j+1}^{-3\beta}.$$

From (3.7.1) and Proposition 3.7.1 it follows that

$$\mathbb{P}[A_3^c] \leq L_{j+1}^{-3\beta}, \ \mathbb{P}[A_4^c] \leq L_{j+1}^{-3\beta}.$$

Using Markov's inequality, it follows from Proposition 3.8.1

$$\begin{aligned}
\mathbb{P}[A_5^c] &= \mathbb{P}[\mathbb{P}[X \overset{s,s}{\not\longleftrightarrow} Y \mid X] \geq L_{j+1}^{-2\beta}] \\
&\leq \mathbb{P}[X \overset{s,s}{\not\longleftrightarrow} Y] L_{j+1}^{2\beta} \\
&\leq \left(\mathbb{P}[X \overset{s,s}{\not\longleftrightarrow} Y, X \in \mathcal{A}_{X,j+1}^{(*)}, Y \in \mathcal{A}_{Y,j+1}^{(*)}] + \mathbb{P}[X \notin \mathcal{A}_{X,j+1}^{(*)}] + \mathbb{P}[Y \notin \mathcal{A}_{Y,j+1}^{(*)}] \right) L_{j+1}^{2\beta} \\
&\leq 3L_{j+1}^{-\beta}.
\end{aligned}$$

Putting all these together we get

$$\mathbb{P}[X \in G_{j+1}^{\mathbb{X}}] \geq \mathbb{P}[\cap_{i=1}^{5} A_i] \geq 1 - L_{j+1}^{-\delta}$$

for L_0 large enough since $\beta > \delta$. $\square$

Chapter 4

Lipschitz Embedding in Higher Dimensions

In this chapter we study the problem of Lipschitz embedding of a collection of i.i.d. Bernoulli variables indexed by higher dimensional Euclidean lattices. This is a natural generalization of corresponding one dimensional question studies in Chapter 2. Let $\mathbb{X} = \{X_v\}_{v\in\mathbb{Z}^d}$ and $\mathbb{Y} = \{Y_v\}_{v\in\mathbb{Z}^d}$ be collections of binary entries indexed by $\mathbb{Z}^d$. We say $\mathbb{X}$ can be M-embedded in $\mathbb{Y}$ if there exists an injective map $\phi : \mathbb{Z}^d \to \mathbb{Z}^d$ such that $X_v = Y_{\phi(v)}$ $\forall v \in \mathbb{Z}^d$ and $||\phi(v_1) - \phi(v_2)|| \leq M||v_1 - v_2||$ $\forall v_1, v_2 \in \mathbb{Z}^d$ where $||\cdot||$ denotes the Euclidean norm in $\mathbb{Z}^d$. The primary question we investigate is the following. Suppose $\mathbb{X}$ and $\mathbb{Y}$ are independent collection of i.i.d. Bernoulli variables. Does there exist M sufficiently large such that $\mathbb{X}$ can be M-embedded in $\mathbb{Y}$ almost surely? This question was answered affirmatively for $d = 1$ in [8]. Our main theorem provides an affirmative answer to Question 1.3.1 for $d = 2$.

Theorem 4.1. *Let $\mathbb{X} = \{X_v\}_{v\in\mathbb{Z}^d}$ and $\mathbb{Y} = \{Y_v\}_{v\in\mathbb{Z}^d}$ be collections of i.i.d. Ber($\frac{1}{2}$) random variables. For $d = 2$, there exists $M > 0$ such that $\mathbb{X}$ can be M-embedded in $\mathbb{Y}$, denoted $\mathbb{X} \hookrightarrow_M \mathbb{Y}$, almost surely.*

By ergodicity, the event $\mathbb{X} \hookrightarrow_M \mathbb{Y}$ is a $0-1$ event, and hence to prove Theorem 4.1 it suffices to prove that $\mathbb{P}[\mathbb{X} \hookrightarrow_M \mathbb{Y}] > 0$ for M sufficiently large. This is what we shall prove. It will be clear from our proof that the same argument works for any dimensions $d \geq 2$ with minor modifications. We stick to $d = 2$ for the purpose of notational convenience.

4.1 Outline of the Proof

Our proof again is based on multi-scale analysis and in spirit is similar to the argument used in Chapter 2. The main challenge, as in one dimension, is to match the difficult to embed regions in $\mathbb{X}$ to their suitable partners in $\mathbb{Y}$ simultaneously at all scales. This is technically much more challenging because the difficult to embed regions can have many different shapes and complicated geometries in higher dimensions.

Our proof is multi-scale and divides the collections $\mathbb{X}$ and $\mathbb{Y}$ into blocks on a series of doubly exponentially growing length scales $L_j = L_0^{\alpha^j}$ for $j \geq 0$. A block of level j is typically (approximately) a square of side length L_j, though we also allow blocks of more complicated shapes and larger sizes. At each of these levels we define a notion of a "good" block. Single characters in $\mathbb{X}$ constitute the level 0 blocks and in $\mathbb{Y}$ squares of a fixed (large) size make level 0 blocks.

Suppose that we have constructed the blocks up to level j. In § 4.2, we give a construction of $(j+1)$-level blocks as a union of j-level sub-blocks in such way that the blocks are identically distributed, non neighbouring blocks are independent and there are no bad j-level subblocks very close to the boundary of a $(j+1)$-level block. To ensure the last condition we need to allow blocks to be of larger size, and in certain cases blocks will approximate a connected union of squares of size L_j. For more details, see § 4.2.

At each level we distinguish a set of blocks to be good. In particular this will be done in such a way that at each level $(j+1)$ for *any* good block X in $\mathbb{X}$ and *any* good block Y in $\mathbb{Y}$, their j-level bad sub-blocks can be matched with suitable partners via a bi-Lipschitz map of Lipschitz constant $(1+10^{-(j+4)})$ (this is termed as embedding at level $(j+1)$). Flexibility in choosing this map gives us an improved chance to find suitable partners for difficult to embed blocks at higher levels. We describe how to define good blocks in § 4.2.9. We also define components which are unions of blocks such that different components containing bad sub-blocks are separated by good components which are just single good blocks.

The proof then involves a series of recursive estimates at each level given in § 4.3. We ask that at level j the probability that a block is good is at least $1 - L_j^{-\gamma}$, conditioned on a subset (possibly empty) of other level j blocks and hence a vast majority of the blocks are good. Furthermore, we show tail bounds on the embedding probabilities showing that for $0 < p \leq 1 - L_j^{-1}$,

$$\mathbb{P}(S_j^{\mathbb{X}}(X) \leq p, V_X \geq v) \leq p^{m+2^{-j}} L_j^{-\beta} L_j^{-\gamma(v-1)}$$

where $S_j^{\mathbb{X}}(X)$ denotes the j-level embedding probability of a j level component X, and V_X denote the number of squares of size L_j that X approximates, see § 4.2.8 for a formal definition. We show the analogous bounds for $\mathbb{Y}$-blocks as well. The full inductive step is given in § **??**. Proving this constitutes the main work of the chapter.

The key quantitative estimate in the chapter is Lemma 4.5.2 which follows directly from the recursive estimates, and bounds the chance of a block having a large size, many bad sub-blocks or a particularly difficult collection of sub-blocks measured by the product of their embedding probabilities. In order to achieve the improving embedding probabilities at each level we need to take advantage of the flexibility in mapping a small collection of bad blocks to a large number of possible partners in a Lipschitz manner with appropriate Lipschitz constants. To this effect we define families of maps between blocks to describe such potential maps. Because m is large and we take many independent trials the estimate at the next level improves significantly. Our analysis is split into 4 different cases.

To show that good blocks at level $(j+1)$ have the required properties, we construct them so that the total size of bad subcomponents contained in them is at most k_0 and all of which are "semi-bad" (defined in § 4.2.8) in particular with embedding probability close to 1. We also require that every semi-bad block maps into a large proportion of the sub-blocks in every $L_j^{3/2} \times L_j^{3/2}$ square of j level blocks contained in a $(j+1)$-level block. Under these conditions we show that good blocks can always be mapped to any other good block.

To complete the proof we note that with positive probability the blocks surrounding the origin are good at each level. The proof is then completed using a standard compactness argument.

4.1.1 Parameters

Our proof involves a collection of parameters $\alpha, \beta, \gamma, k_0, m$ and v_0 which must satisfy a system of constraints. The required constraints are

$$\alpha > 6, \gamma > 40\alpha, \beta > 1500\alpha\gamma, k_0 > 6000\alpha\gamma, v_0 > 3000\alpha,$$
$$8\gamma(v_0 - 1) > 3\alpha\beta, m \geq 9\alpha\beta + 3\alpha\gamma v_0, \gamma k_0 > 300\alpha\beta, k_0 > 10\gamma, (1 - 10^{-10})^{4v_0} > \frac{9}{10}.$$

To fix on a choice we will set

$$\alpha = 8, \gamma = 350, \beta = 4500000, v_0 = 45000, m = 15 \times 10^7, k_0 = 13 \times 10^6. \tag{4.1.1}$$

Given these choices we then take L_0 to be a sufficiently large integer. We did not make a serious attempt to optimize the parameters or constraints, often aiming to keep the exposition more transparent.

4.1.2 Organization of the Chapter

Rest of this chapter is organised as follows. In Section 4.2 we describe our block constructions and formally define good blocks. In Section 4.3 we state the main recursive theorem and show that it implies Theorem 4.1. In Section 4.4 we construct a collection of bi-Lipschitz functions which we will use to describe our mappings between blocks. In Section 4.5 we prove the main recursive tail estimates on the embedding probabilities. In Section 4.7 we show that good blocks have the required inductive properties thus completing the induction.

4.2 The Multi-scale Structure

For reasons of notational convenience that will momentarily be clear, without loss of generality, we shall take our sequence to be indexed by a translate of $\mathbb{Z}^2$ rather than $\mathbb{Z}^2$ itself. Let $\iota = (1/2, 1/2)$. Let $\mathbb{X} = \{X_v\}_{v \in \iota + \mathbb{Z}^2}$ and $\mathbb{Y} = \{Y_v\}_{v \in \iota + \mathbb{Z}^2}$ be collections of i.i.d. Ber($\frac{1}{2}$) random variables.

As mentioned above, our argument for proof of Theorem 4.1 is multi-scale and depends of partitioning $\mathbb{X}$ and $\mathbb{Y}$ into blocks at level j-for each $j \geq 0$. The blocks are constructed recursively. For the purpose of our construction we shall work with $\mathbb{R}^2$ rather than $\mathbb{Z}^2$. At each level j we shall partition $\mathbb{R}^2$ into disjoint (except at the boundary) random regions $\{\mathscr{B}_{\alpha}^{j,\mathbb{X}}\}_{\alpha\in I_1}$ and $\{\mathscr{B}_{\alpha}^{j,\mathbb{Y}}\}_{\alpha\in I_2}$ respectively for $\mathbb{X}$ and $\mathbb{Y}$.

We shall interchangeably use the term blocks at level j (for $\mathbb{X}$, say) to refer to the regions $\mathscr{B}_{\alpha}^{j,\mathbb{X}}$ or the collection of random variables contained in these regions: $\{X_u : u \in \mathscr{B}_{\alpha}^{j,\mathbb{X}}\}$.

Our blocks will be indexed by elements in a random partition of $\mathbb{Z}^2$.

4.2.1 Blocks at Level 0

We start with describing the construction of blocks at level 0. Construction of blocks at level 0 are different for $\mathbb{X}$ and $\mathbb{Y}$. Also level 0 blocks are deterministic (i.e. the regions corresponding to them are deterministic) and indexed by vertices in $\mathbb{Z}^2$.

For each $u = (u_1, u_2) \in \mathbb{Z}^2$, the $\mathbb{X}$-block at level 0 indexed by u, denoted by $X^0(u)$ corresponds to the region $[u_1, u_1+1] \times [u_2, u_2+1]$.

Let $M_0 \in \mathbb{N}$ denote some large constant to be determined later. For each $u = (u_1, u_2) \in \mathbb{Z}^2$, the $\mathbb{Y}$-block at level 0 indexed by u, denoted by $Y^0(u)$ corresponds to the region $[u_1M_0, (u_1+1)M_0] \times [u_2M_0, (u_2+1)M_0]$.

For $U \subseteq \mathbb{Z}^2$, the collection of blocks $\{X^0(u) : u \in U\}$ will be denoted by X_U^0 (and similarly for Y_U^0).

Observe that level 0 blocks are independent for both $\mathbb{X}$ and $\mathbb{Y}$. Level 0 blocks are fundamental units of our multi-scale structure. All the blocks at higher scales will be unions of blocks at level 0. For the rest of this construction, we rescale space for $\mathbb{Y}$ such that blocks at level 0 become unit squares. Under this rescaling construction of higher level blocks are performed identically for $\mathbb{X}$ and $\mathbb{Y}$.

Good Blocks at Level 0

As we have mentioned before, at each scale of the multi-scale construction, we shall designate a set of blocks in both $\mathbb{X}$ and $\mathbb{Y}$ to as **good**. At level 0, each $\mathbb{X}$-block will be good. For $u \in \mathbb{Z}^2$, $Y^0(u)$ is called good if we have the fraction of both 0's and 1's contained in $Y^0(u)$ is at least 1/3 i.e.,

$$\#\{v \in Y^0(u) : Y_v = 1\} \wedge \#\{v \in Y^0(u) : Y_v = 0\} \geq \frac{M_0^2}{3}.$$

4.2.2 An Overview of the Recursive Construction

After rescaling blocks at level 0 the recursive construction of blocks at higher levels is identical for both $\mathbb{X}$ and $\mathbb{Y}$. Without loss of generality, we shall restrict ourself to construction of the blocks for $\mathbb{X}$ for levels $j \geq 1$. Our recursive block construction algorithm is fairly complex and has many elements to it. To facilitate the reader, before giving the formal definition, in this subsection we give a rough description of how the construction goes and make a list of different terms associated with the construction for easy reference.

- **Cells:** Cells at level j are basic units of construction at level j. These are squares, indexed by $\mathbb{Z}^2$, of size L_j, that $\mathbb{R}^2$ is divided into. Denote the cell corresponding to $u \in \mathbb{Z}^2$ by $B^j(u)$. Recall that $L_j = L_0^{\alpha^j}$ is the doubly exponentially increasing length scale.

- **Buffer Zones:** Buffer zones are regions around the boundary of a cell, which should be thought of fattened versions of the boundaries of cells.

- **Lattice Blocks:** At each level j we partition $\mathbb{Z}^2$ as a (random) union of lattice animals (connected finite subsets). The elements of this are called lattice blocks. Let the set of lattice blocks at level j be $\mathcal{H}_j = \mathcal{H}$. The blocks at level j are indexed by elements of $\mathcal{H}$, typically denoted $X = X_H^j, H \in \mathcal{H}$. For H that is a union of elements in $\mathcal{H}$, X_H^j will denote the union of the corresponding blocks.

- **Ideal Multi-blocks:** For a lattice block H at level j, we call $\cup_{u \in H} B^j(u)$ an ideal multi-block.

- **Domains and Boundary Curves of Blocks:** Domains of blocks at level j are small bi-Lipschitz perturbation of ideal multi-blocks. These are formed in such a way that the boundaries of the domains are nice (in some sense to be specified later). For a block X, we typically denote its domain by $\hat{U}_X$ and the curve corresponding to the boundary of $\hat{U}_X$ by C_X.

- **Blocks:** We shall define regions $\widetilde{U}_X$, that are unions of smaller level blocks and these will define blocks. The regions $\widetilde{U}_X$ will be defined as approximations of the regions $\hat{U}_X$ defined above. We shall denote the term block interchangeably for the region defining it as well as the collection of random variables in the region. For a block X, we shall denote by V_X the size of a block, i.e., the size of the lattice block corresponding to it.

- **Good and Bad Blocks:** At each level, we designate some of the blocks to be good (depending on the configuration), other blocks are called bad. Good blocks will always correspond to lattice blocks of size 1, but the converse need not be true.

- **Components of blocks:** We also form components of blocks at each level, where a component is a connected union of a number of blocks such that two components containing bad blocks are not neighbouring. Components are deterministically determined given the blocks and the identity of good blocks. For a component X, the size of it, i.e., the total size of all lattice blocks contained in that component will be denote by V_X.

- **Semi-bad Components:** Components are called bad if they contain one or more bad blocks. Some of the bad components are designated as semi-bad component, depending on the configuration.

Observe that, at level 0, lattice blocks are all singletons. Cells, domains and blocks are all the same and boundary curves are just the boundaries of cells. Now we give a detailed description of how we construct each of the steps above for $j \geq 1$.

4.2.3 Cells and Buffer Zones

Definition 4.2.1 (Cells at level j). *For $j \geq 1$, set $L_j = L_{j-1}^{\alpha} = L_0^{\alpha^j}$. For $j \geq 1$ and $u = (u_1, u_2) \in \mathbb{Z}^2$, we define $B^j(u) = [u_1 L_j, (u_1+1)L_j] \times [u_2 L_j, (u_2+1)L_j]$. These squares which partition $\mathbb{R}^2$, will be called **cells at level** j.*

Observe that cells at level j are squares of doubly exponentially growing length L_j. Also observe that cells are nested across j, i.e., a cell at level $j \geq 1$ is a union of $L_j^{2\alpha-2}$ many cells at level $(j-1)$. The above definition is illustrated in Figure 4.1.

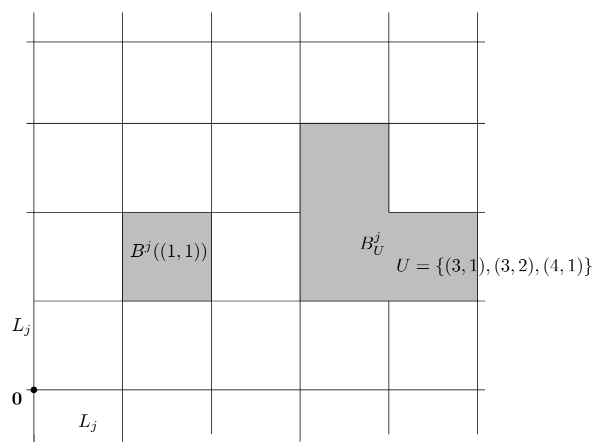


Figure 4.1: Cells and multi-cells at level j

The basic philosophy of constructing the blocks here is similar to that in [8]: we want the region around the boundary of the blocks at level j to consist of 'good' subblocks at level $(j-1)$. Because of the more complicated geometry of $\mathbb{R}^2$ (as compared to the real line considered in [8]) we shall need to consider cells of different shapes and sizes at a given level. This motivates the following sequence of definitions.

Definition 4.2.2 (Lattice animals and Shapes). *A connected finite subset of vertices in $\mathbb{Z}^2$ is called a **lattice animal**. Two lattice animals U and U' are said to have the same **shape** if there is a translation from $\mathbb{Z}^2$ to itself that takes U to U'.*

We shall use the term shape also to identify equivalence classes of lattice animals having the same shape.

Two cells $B^j(u)$ and $B^j(u')$ are called **neighbouring** if they share a common side, i.e., if u and u' are neighbours in $\mathbb{Z}^2$.

Definition 4.2.3 (Multi-cells at level j)**.** *For a lattice animal $U \subset \mathbb{Z}^2$, we call $B_U^j := \cup_{u\in U} B^j(u)$ a* ***multi-cell*** *at level j corresponding to the lattice animal U.*

The size of a multi-cell at level j is defined to be $|U|$, i.e., the number of cells contained in it. The boundary of B_U^j shall be denoted by ∂B_U^j.

Our blocks at levels $j \geq 1$ will be suitable perturbations of certain j level multi-cells (ideal multi-blocks) that ensure that there are no bad $(j-1)$ level subblocks near the boundary. To define the appropriate notion of perturbation we need to consider slightly thinned and fattened versions of cells at levels $j \geq 1$.

Definition 4.2.4 (Buffer zones of cells)**.** *Consider the squares*

$$B^{j,int}(\mathbf{0}) := [L_{j-1}^5, L_j - L_{j-1}^5]^2;$$

$$B^{j,ext}(\mathbf{0}) := [-L_{j-1}^5, L_j + L_{j-1}^5]^2.$$

For $j \geq 1$, call $B^{j,int}(\mathbf{0})$ the ***interior*** *and $B^{j,ext}(\mathbf{0})$ the* ***blow up*** *of the j-level cell $B^j(\mathbf{0})$.*

For $u = (u_1, u_2) \in \mathbb{Z}^2$, define the interior and blow up of the cell $B^j(u)$ by setting

$$B^{j,int}(u) := (u_1 L_j, u_2 L_j) + B^{j,int}(\mathbf{0});$$
$$B^{j,ext}(u) := (u_1 L_j, u_2 L_j) + B^{j,ext}(\mathbf{0}).$$

We call $\Delta B^j(u) := B^{j,ext}(u) \setminus B^{j,int}(u)$ the ***buffer*** *for the cell $B^j(u)$. We write $\Delta B^j(u)$ as the (non-disjoint) union of 4 rectangles called the* ***top, left, bottom*** *and* ***right*** *buffer zone denoted $\Delta B^{j,T}(u)$, $\Delta B^{j,L}(u)$, $\Delta B^{j,B}(u)$ and $\Delta B^{j,R}(u)$ respectively. Define $\Delta B^{j,T}(u) = \Delta B^j(u) \cap \Delta B^j(u')$ where $u' = u + (0,1)$, rest are defined similarly.*

Observe that if u and u' are neighbours in $\mathbb{Z}^2$, then $B^j(u)$ and $B^j(u')$ has one rectangular buffer zone (e.g. $\Delta B^{j,T}(u)$) in common, and conversely every rectangular buffer zone is shared between two neighbouring cells. If u and u' are neighbours in the closed packed lattice of $\mathbb{Z}^2$ then also their buffer zones intersect. See Figure 4.2 for illustration of this definition.

We next extend the definition of buffer zone to multi-cells at level j.

Definition 4.2.5 (Buffer zones of Multi-cells)**.** *Fix $j \geq 1$, and a lattice animal $U \subset \mathbb{Z}^2$. Consider B_U^j, the multi-cell corresponding to U at level j. For $u \in U$, and $\star \in \{T, L, B, R\}$, we call $\Delta B^{j,\star}(u)$ an* ***outer buffer zone*** *of B_U^j if this buffer zone is shared with a cell outside B_U^j. The buffer $\Delta B^j(U)$, of the multi-cell B_U^j is defined as the union of all outer buffer zones of $B^j(u)$ for $u \in U$. The interior and blow up of B_U^j is defined similarly as above.*

This is illustrated in Figure 4.3.

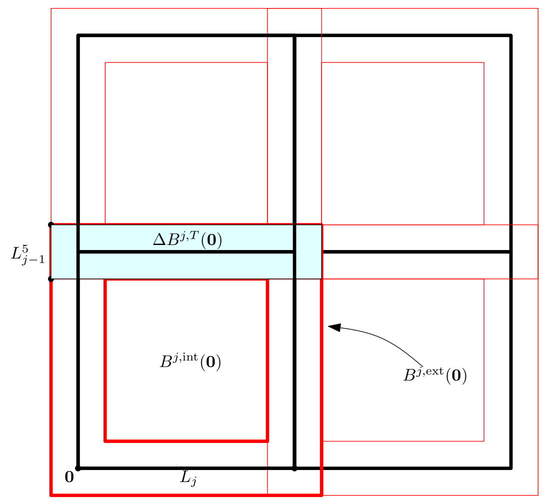


Figure 4.2: Buffer Zones of Cells

4.2.4 Recursive construction of blocks I: Forming Ideal Multi-blocks

In this subsection we describe how to recursively construct the blocks at levels $j \geq 1$. Suppose that blocks have already been constructed for some $j \geq 0$. Also suppose that the good blocks at level j have been specified. Further assume that other elements of the structure at level j have also been constructed. In particular this means components have been identified with bad and semi-bad components also being specified at level j. We now describe how to construct the structure at level $(j+1)$. Notice that the blocks and good blocks at level 0 has already been defined. We postpone the precise definitions of components and semi-bad components for the moment.

Conjoined Buffer Zones

Our first step is to construct the lattice blocks and ideal multi-blocks at level $(j+1)$. We start with the following observation. For each $u \in \mathbb{Z}^2$, by recursive construction, there exists a set $H(u) = H^j(u) \subseteq \mathbb{Z}^2$ containing such that X^j_H is a component at level j.

To construct the ideal multiblocks at level $(j+1)$ we start with the following definition.

Definition 4.2.6 (Conjoined buffer zone and Conjoined cells)**.** *Fix neighbouring vertices $u, u' \in \mathbb{Z}^2$, consider the shared buffer zone denoted by $\Delta B^{j+1}(u,u')$ between cells $B^{j+1}(u)$ and $B^{j+1}(u')$. We call the buffer zone $\Delta B^{j+1}(u,u')$* ***conjoined*** *if one of the following conditions fail.*

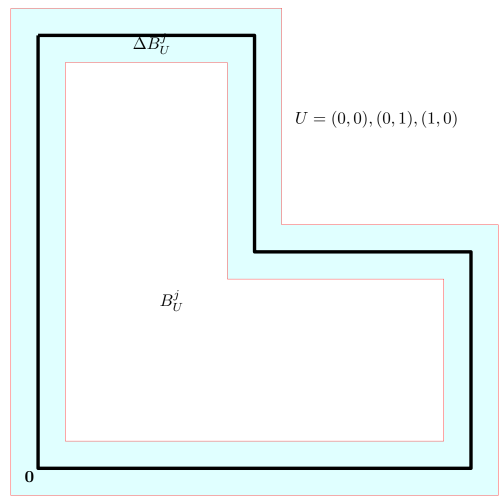


Figure 4.3: Buffer Zones of a Multi-cell

i. Let $T \subseteq \mathbb{Z}^2$ be such that $B_T^j = \Delta B^{j+1}(u,u')$. Then we have

$$\#\{t \in T : X_{H(t)}^j \text{ is a bad component}\} \leq k_0.$$

That is, the total size of bad level j-components contained in the buffer zone is at most k_0.

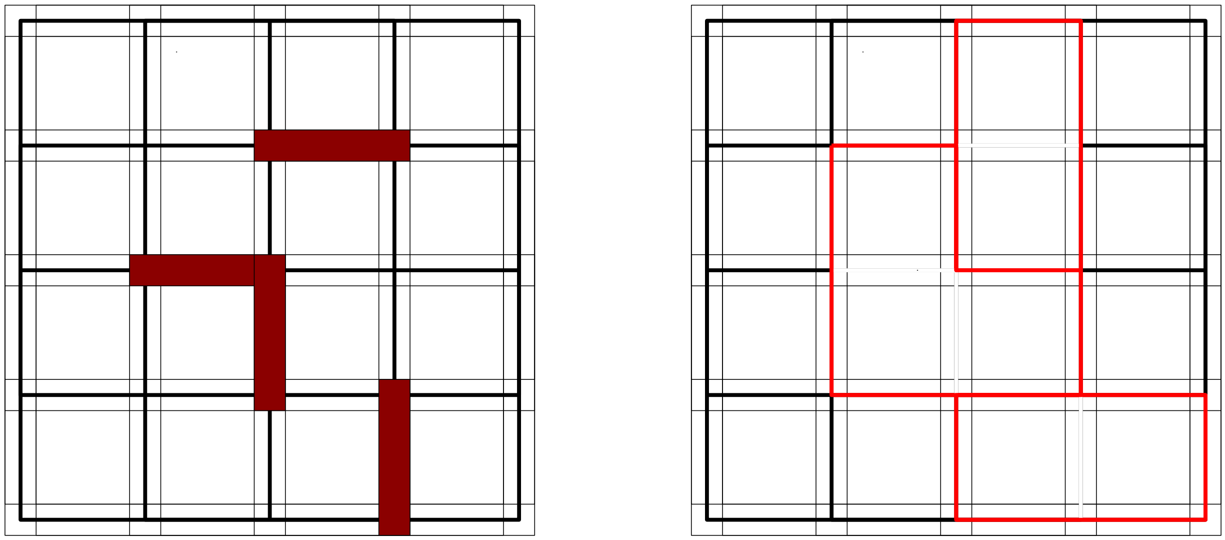
ii. All the bad components contained in the buffer zone are semi-bad.

Call the $(j+1)$-level cells $B^{j+1}(u)$ and $B^{j+1}(u')$ conjoined if $\Delta B^{j+1}(u,u')$ is conjoined.

Using the notion of conjoined cells above we now define the ideal multi-blocks at level $j+1$ with the property that if two cells sharing a conjoined buffer zone are necessarily contained in the same ideal multi-block. More formally we define the following.

Definition 4.2.7 (Lattice Blocks and Ideal multi-blocks at level $j+1$)**.** *Consider the following bond percolation on $\mathbb{Z}^2$. For u, u' neighbours in $\mathbb{Z}^2$, we keep the edge between u and u' if $B^{j+1}(u)$ and $B^{j+1}(u')$ are conjoined. The connected components of this percolation are called the lattice blocks at level $(j+1)$. For a lattice block U at level $(j+1)$, we call B_U^{j+1} an ideal multi-block at level $(j+1)$.*

It will follow from our probabilistic estimates that almost surely all lattice blocks are finite. The definition of Ideal multi-blocks is illustrated in Figure 4.4. The conjoined buffer zones and the ideal multi-blocks are marked in the figure.



(a) Conjoined buffer zones (b) Ideal multi-blocks

Figure 4.4: Formation of Ideal Multi-blocks. Ideal multi-blocks of size bigger than 1 are marked

4.2.5 Recursive construction of blocks II: Constructing Domains

Let $\mathcal{H} = \mathcal{H}^{j+1,\mathbb{X}}$ denote the lattice blocks of $\mathbb{X}$ at level $(j+1)$ constructed as above. Clearly $\mathcal{H}$ is a partition of $\mathbb{Z}^2$ and $\{B_H^{j+1}\}_{H\in\mathcal{H}}$ is a partition of $\mathbb{R}^2$. As alluded to above, the blocks at level $(j+1)$ will be indexed by $\mathcal{H}$ and will be "approximations" to the ideal multi-blocks B_H^{j+1}. To construct the blocks at level $(j+1)$, we first start with constructing domains of blocks which will be some smooth perturbations of the ideal multiblocks B_H^{j+1}.

Potential Boundary Curves

Ideally we would have liked to use the ideal multi-blocks as our blocks at level $(j+1)$, but in that case it is not possible to guarantee that the j-level subblocks near the boundary will be good. Hence depending on the distribution of j-level subblocks in the buffer zone we would choose boundaries for our blocks. We want the number of possible curves that could serve as boundaries to be limited and hence we first construct a family of curves through buffer zones.

Let $(\mathbb{Z}^2, E^2)$ denote the usual nearest neighbour lattice on $\mathbb{Z}^2$. The family of curves we construct would be indexed by $\{(\ell_v, s_v) : v \in \mathbb{Z}^2, s_e : e \in E^2\}$ where each $\ell_v, s_e \in [2k_0]$ and each $s_v \in \{1,2\}$. Here is the rough meaning of the above indexing. Observe that the buffer zone is union of mutually parallel horizontal and vertical strips, which can be thought of as a fattened version of the graph $(\mathbb{Z}^2, E^2)$. That is, consider the horizontal strips $S^1_{v_1} = \mathbb{R} \times [v_1 L_{j+1} - L_j^5, v_1 L_{j+1} + L_j^5]$ for $v_1 \in \mathbb{Z}$ and the vertical strips $S^2_{v_2} = [v_2 L_{j+1} -$

$L_j^5, v_2L_{j+1} + L_j^5] \times \mathbb{R}$. So the vertex $v = (v_1, v_2)$ corresponds to the square $S_v = S^1_{v_1} \cap S^2_{v_2}$ and an edge would correspond to the rectangle connecting two such squares. Roughly the parameters ℓ_v and s_v determine the curve in the square S_v whereas s_e determines the curve in the region of the buffer zones corresponding to the edge $e \in E^2$.

Curves we construct through $S^1_{v_1}$ (say) will be images of the horizontal line $y = v_1L_{j+1}$ under some mild perturbation, and a similar statement is true for vertical strips of buffer zones. Without loss of generality we describe the construction of these maps of S^1_0, rest are obtained by translation. Curves through vertical strips are defined similarly.

Let $v = (v_1, 0)$. Define points $p^-_{v,\ell} = (v_1L_{j+1} - \ell 100^{-(j+5)}L^5_j, 0)$ and $p^+_{v,\ell} = (v_1L_{j+1} + \ell 100^{-(j+5)}L^5_j, 0)$ for $\ell \in [2k_0]$. Let $T_{\ell,v}$ denote the square whose centre is $(v_1L_{j+1}, 0)$ and has a side length $2\ell 100^{-(j+5)}L^5_j$. Also let e denote the edge between v and $v' = v + (1, 0)$. Denote by $T_{\ell_1,\ell_2,e}$ the rectangle $[v_1L_{j+1} + \ell_1 100^{-(j+5)}L^5_j, 0), (v_1 + 1)L_{j+1} - \ell_2 100^{-(j+5)}L^5_j, 0)] \times [-L^5_j/2, L^5_j/2]$. Also let $R^1_{\ell,v} = R_{\ell,v}$ denote the straightline segment in the intersection of $T_{\ell,v}$ and the x-axis. Further let $R^1_{\ell_1,\ell_2,e}$ denote the straightline segment in the intersection of $T_{\ell_1,\ell_2,e}$ and the x-axis.

Now suppose we choose ℓ_v and $\ell_{v'}$ to be corresponding parameters to our curves. Then the curve passes through points $p_1 = p^-_{v,\ell_v}$ and $p_2 = p^+_{v,\ell_v}$ (and also through points $p_3 = p^-_{v',\ell_{v'}}$ and $p_4 = p^+_{v',\ell_{v'}}$). The curve between the points p_1 and p_2 is determined by the choice of s_v and the curve between the points p_2 and p_3 is determined by the choice of s_e. Fix ℓ_1, $\ell_2 \in [2k_0]$. Fix functions $F^s_{\ell_1,v}$ for $s \in \{1, 2\}$ and $F^s_{\ell_1,\ell_2,e}$ for $s \in [2k_0]$ satisfying the following properties (we shall suppress the subscript v and e in the following):

i. F^s_ℓ (resp. $F^s_{\ell_1,\ell_2}$) is a bijection from T_ℓ (resp. T_{ℓ_1,ℓ_2}) to itself.

ii. F^s_ℓ (resp. $F^s_{\ell_1,\ell_2}$) is identity on the boundary of T_ℓ (resp. T_{ℓ_1,ℓ_2}) and is bi-Lipschitz with Lipschitz constant $1 + 10^{-(j+10)}$.

iii. For all ℓ_1, ℓ_2 we have $F^1_{\ell_1}$ (resp. $F^1_{\ell_1,\ell_2}$) is the identity map.

iv. Let R^1_ℓ (resp. $R^1_{\ell_1,\ell_2}$) denote the straight line segment formed by the intersection of the x-axis with T_ℓ (resp. T_{ℓ_1,ℓ_2}). We have that $R^2_\ell = F^2_\ell(R^1_\ell)$ (resp. $R^s_{\ell_1,\ell_2} = F^s_{\ell_1,\ell_2}(R^1_{\ell_1,\ell_2})$ for each $s \in [2k_0] \setminus \{1\}$) is contained in the strip $\mathbb{R} \times [-100^{-(j+6)}L^5_j, 100^{-(j+6)}L^5_j]$.

v. The ℓ_∞ distance between R_ℓ and $R_{\ell'}$ for $\ell \neq \ell'$ (resp. between $R^s_{\ell_1,\ell_2}$ and $R^{s'}_{\ell_1,\ell_2}$ for $s \neq s'$) is at least $10L^4_j$ on the interval $[p^+_{v,\ell_1} + L^4_j, p^-_{v+1,\ell_2} - L^4_j]$.

We shall omit the proof of the following basic lemma which easily follows from the fact L_0 is sufficiently large and L_j grows doubly exponentially.

Lemma 4.2.8. *For all $\ell, \ell_1, \ell_2 \in [2k_0]$, there exist functions F^s_ℓ and $F^s_{\ell_1,\ell_2}$ satisfying the properties listed above.*

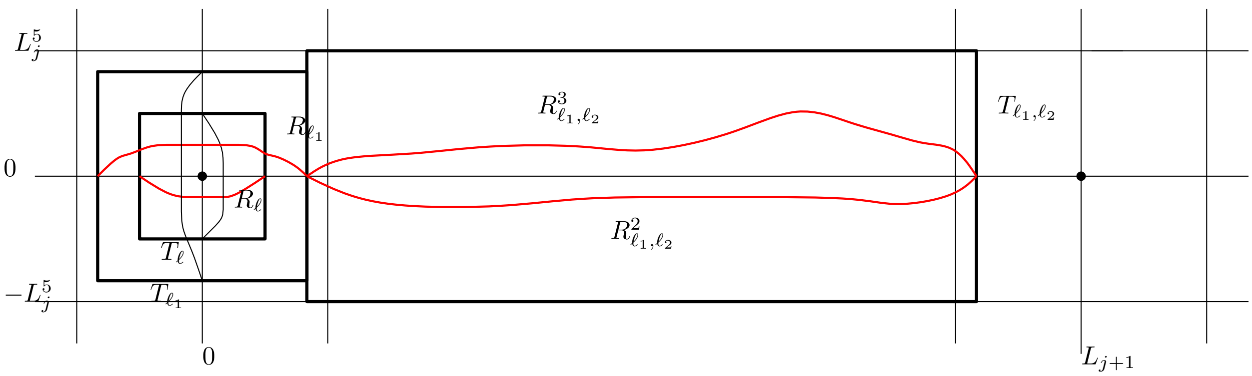


Figure 4.5: Potential boundary curves through a buffer zone

See Figure 4.5 for an illustration of the above construction.

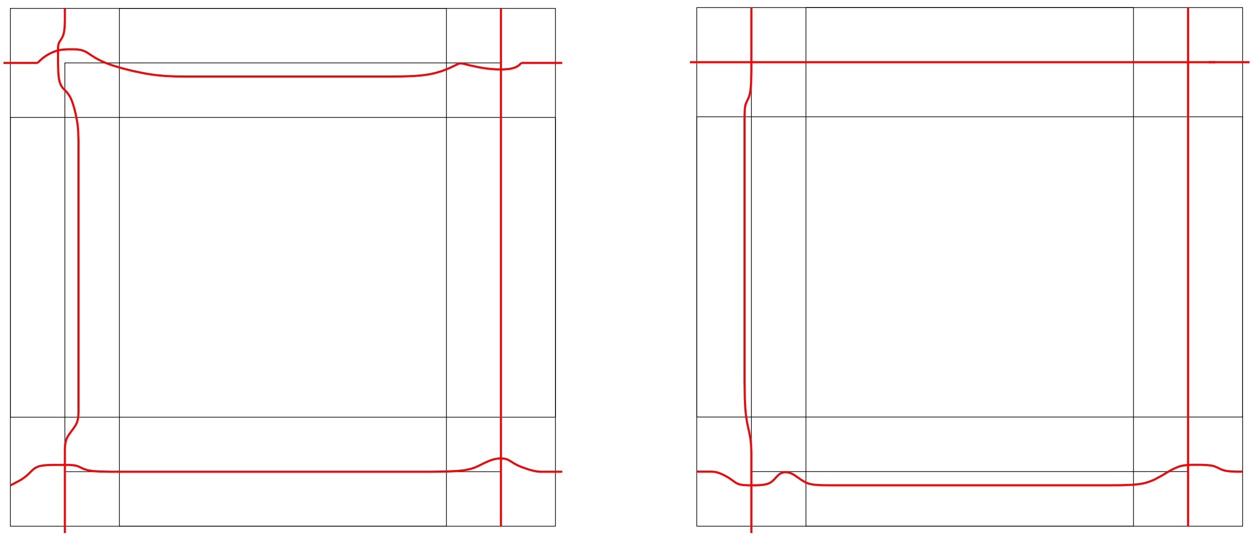
We do similar constructions for vertical strips of buffer zones as well using the same maps $F^s_{\ell,v}$ for the squares $T_{\ell,v}$. Observe the following. For each choice of $\{\ell_v, s_v\}_{v\in\mathbb{Z}^2}$ and $\{s_e\}_{e\in E^2}$ we get one curve contained in each horizontal and vertical buffer zone strip. The family of such curves are called **potential boundary curves**. When we restrict to one buffer zone, the family is called **potential boundary curves** through that buffer zone.

Fix $u \in \mathbb{Z}^2$. Now observe that if we restrict to the buffer zone $\Delta B^{j+1,B}(u)$, then a potential boundary curve through $\Delta B^{j,B}(u)$ is determined by $\ell_u, \ell_{u'}, s_u, s_{u'}$ and s_e where $u' = u + (1, 0)$ and e is the edge joining u and u', (except at the extremities). In particular, a potential boundary curve through the buffer zone of a cell is determined by choices of ℓ and s along the corners and edges of the cell. See Figure 4.6.

Definition 4.2.9 (Potential Boundary Curves of a multi-cell and Potential Domains)**.** *Fix a multi-cell B_U^{j+1} at level $j + 1$. Each choice of potential boundary curves through each of the outer buffer zones of B_U^{j+1} determines a simple closed curve C through the buffer zone of B_U^{j+1}. These curves are called the potential boundary curves of the multi-cell B_U^{j+1}. The region surrounded by C is called a potential domain of the multi-cell B_U^{j+1}.*

It follows from the construction, that the number of potential boundary curves of the multi-cell B_U^{j+1} is at most $(8k_0)^{16|U|k_0^2}$.

It is clear from our construction that associated with each potential boundary curve there is a unique bijection from $\mathbb{R}^2$ to itself which is bi-Lipschitz with Lipschitz constant $(1 + 10^{-(j+5)})$. Let F denote such a map. Then for all multi-cell B_U^{j+1}, $F(\partial B_U^{j+1})$ is the potential boundary curve through the buffer zone of B_U^{j+1} induced by the potential boundary



(a) (b)

Figure 4.6: Two choices of potential boundary curves of a (multi) cell of size 1

curve corresponding to F That is, potential boundary curves are small perturbations of the boundaries of multi-cells. We make a formal definition for this.

Definition 4.2.10 (Canonical Maps)**.** *For a multi-cell B_U^{j+1} and for any potential boundary curve C through ΔB_U^{j+1}, there exists a unique bi-Lipschitz map $F = F_C$ on B_U^{j+1} with Lipschitz constant $(1 + 10^{-(j+5)})$ such that $F(\partial B_U^{j+1}) = C$. These maps and their inverses are called canonical maps. That is, a canonical map is a map that transforms a multi-cell B_U^{j+1} to a potential domain U_C and vice-versa. Observe also that the family of canonical maps only depend on the shape of U upto translation. For two potential boundary curves C_1, C_2 of the multi-cell B_U^{j+1}, the maps $F_{C_2} \circ F_{C_1}^{-1}$ from U_{C_1} to U_{C_2} are also called canonical maps.*

Valid Boundary Curves and Domains

Recall that we have already constructed the ideal multi-blocks at level $(j+1)$. Our next order of business is to stochastically choose one boundary curve through the outer buffer zones of each ideal multi-block satisfying certain conditions. This curve will be called the boundary curve at level $(j+1)$ and the potential domain corresponding to this choice of boundary will be called domain. Since the outer buffer zones of ideal multi-blocks are not conjoined, the choice of a boundary curve through these boils down to choosing $\{(\ell_v, s_v)\}_{v \in V^*}$ and $\{s_e\}_{e \in E^*}$. Here $V^* \subseteq \mathbb{Z}^2$ is the set of all vertices corresponding to the squares (intersection of a horizontal and a vertical buffer zone) such that not all of the four buffer zones intersecting at that square are conjoined and E^* denotes the edges in E^2 that correspond to non-conjoined buffer zones.

Recall that we want to choose our boundaries so that they are away from the j level bad components. To this end we restrict our choices to **valid** boundary curves defined below.

For $v \in V^*$, we call (ℓ_v, s_v) (where $\ell_v \in [2k_0]$ and $s_v \in [2]$) **valid** if there does not exist any bad j level component within distance $10L_j^4$ of the boundary of $T_{v,\ell}$ and $F^s(R^*)$, where R^* is the intersection of the boundaries of $(j+1)$-level cells with $T_{v,\ell}$.

Let e be the edge connecting neighbouring vertices $v, v' \in V^*$. For a valid choice of (ℓ_v, s_v) and $(\ell_{v'}, s_{v'})$ we call $(\ell_v, s_v, \ell_{v'}, s_{v'}, s_e)$ **valid** if $R^s_{\ell_1,\ell_2,e}$ does not have any j level bad-component within distance L^4 of it.

The following observation is immediate from the definition of conjoined block.

Observation 4.2.11. *For all $v \in V^*$, there exist valid choices of (ℓ_v, s_v). Also for all $e = (v, v') \in E^*$, and for all valid choices of (ℓ_v, s_v) and $(\ell_{v'}, s_{v'})$ there exist s_e such that $(\ell_v, s_v, \ell_{v'}, s_{v'}, s_e)$ is valid.*

Given V^* and E^*, we choose a valid boundary curve randomly independently of everything else as follows.

- For each $v \in V^*$, choose a valid (ℓ_v, s_v).
- If there exist valid (ℓ_v, s_v) with $s_v = 1$ choose one such with probability at least $(1 - 10^{-(j+10)})$.
- For $e = (v, v') \in E^*$, choose s_e such that $(\ell_v, s_v, \ell_{v'}, s_{v'}, s_e)$ is valid.
- If $s_e = 1$ leads to a valid choice, then choose it with probability at least $(1 - 10^{-(j+10)})$.
- The probability of each valid choice must be at least $(8k_0)^{-4vk_0^2} 100^{-(j+10)}$.

This choice leads to a boundary curve, which we shall call the boundary curve at level $(j+1)$. The following important properties of the boundary curve as chosen above is easy to see and recorded as an observation for easy reference.

Observation 4.2.12 (Domains)**.** *Let $\mathcal{H} = \mathcal{H}^{j+1}$ denote the set of lattice blocks of $\mathbb{X}$ at level $(j+1)$. The boundary curve partitions $\mathbb{R}^2$ (in a weak sense) into closed connected regions $\{\hat{U}_X\}_{U \in \mathcal{H}}$, called domains, which have the following properties.*

i. For each $U \in \mathcal{H}$, $\hat{U}_X$ contains the interior of the ideal multi-block B_U^{j+1} and is contained in the blow-up of the B_U^{j+1}.

ii. Given $\mathcal{H}$, for $U_1, U_2 \in \mathcal{H}$ such that U_1 and U_2 are non-neighbouring, the choice of $\hat{U}_{1X}$ and $\hat{U}_{2X}$ are independent.

iii. There is a canonical map F, which is a bi-Lipschitz bijection from $\mathbb{R}^2 \to \mathbb{R}^2$ with Lipschitz constant $(1 + 10^{-(j+5)})$ such that $F(B_U^{j+1}) = U_X$ for all $U \in \mathcal{H}$, and such that F is identity everywhere except near the boundaries of ideal multi-blocks.

iv. There are no j-level bad components near the boundaries of the domains.

v. If there are no bad j level component in the buffer zone of the ideal multi-block B_U^{j+1}, then with probability at least $(1-10^{-(j+10)})^{4|U|}$, the canonical map F is identity on B_U^{j+1}.

See Figure 4.7 for an illustration of domain constructions. Bad level j components are marked in red.

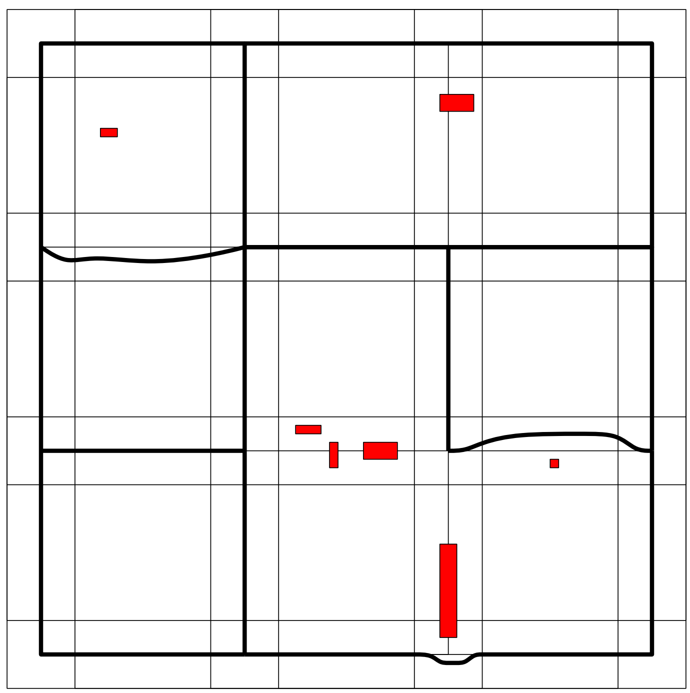


Figure 4.7: Domains at level $j+1$

4.2.6 Recursive Construction of Blocks III: Forming Blocks out of Domains

Notice that we have constructed the domains in such a way that boundaries of domains at level $(j+1)$ avoid the bad components at level j. However observe also that domains at level $j+1$ are not necessarily unions of blocks at level j. This is why we cannot use domains as blocks themselves and have to do one more level of approximation.

Let $\{\hat{U}_X\}_{U\in\mathcal{H}}$ denote the set of domains of $\mathbb{X}$ at level $(j+1)$. Define $\widetilde{U} \subseteq \mathbb{Z}^2$ to be the set of all vertices u of $\mathbb{Z}^2$ such that the j-level cell $B^j(u)$ is contained in $\hat{U}_X$ or the north east corner of $B^j(u)$ is contained in $\hat{U}_X$. Then define the block at level $j+1$ corresponding to the lattice block U, denoted by X_U^{j+1} to be equal to $X_{\widetilde{U}}^j$. Notice that this is well defined because by construction $\widetilde{U}$ is a union of lattice blocks for $\mathbb{X}$ at level j. The set of all blocks at level $(j+1)$ is $\{X_U^{j+1} : U \in \mathcal{H}^{j+1}\}$. Notice that blocks at level $(j+1)$ are union of blocks at level j with none of the j level bad subcomponents close to the boundary of the $(j+1)$ level blocks. Suppose $V \subseteq \mathbb{Z}^2$ is such that X_V^j is a bad component at level j. Then the distance of V from the boundary of $\widetilde{U}$ is at least L_j^3. We record some useful properties of the blocks in the following observation.

Observation 4.2.13 (Properties of Blocks)**.** *The blocks constructed as above satisfy the following conditions.*

i. Each block corresponds to a unique ideal multi-block, contains its interior and is contained in its blow-up.

ii. The distance between any bad j-level subblock contained in a $j+1$-level block is at least L_j^3 level j cells.

iii. Suppose B_H^{j+1} and $B_{H'}^{j+1}$ are two multi-cells that do not share a buffer zone. Condition on the event $\mathcal{E} = \mathcal{E}(H, H')$ that none of the external buffer zones of B_H^{j+1} and $B_{H'}^{j+1}$ are conjoined. Clearly, on $\mathcal{E}$ we have that H and H' are both unions of lattice blocks for $\mathbb{X}$ at level $(j+1)$. Then conditioned on $\mathcal{E}$, we have that $\{X_H^{j+1}\}$ and $\{X_{H'}^{j+1}\}$ are independent.

4.2.7 Geometry of a Block: Components

To complete the description of block construction, it remains to define good blocks at level $j \geq 1$. Before we give the recursive definition of the good blocks, it is necessary to introduce certain definitions and notations regarding the geometry of the multi-blocks.

Bad Components of blocks

Fix $j \geq 0$. Suppose that blocks and good blocks are already defined up to level j. Recall that good blocks at level j always correspond to lattice blocks of size 1. Let $\mathcal{H} = \{H(u)\}_{u \in \mathbb{Z}^2}$ denote the family of lattice blocks at level j, i.e., $H(u)$ denotes the lattice block containing u. Our objective is to group the neighbouring bad blocks together. To this end we make the following definition.

Definition 4.2.14 (Lattice Components)**.** *Let $\mathcal{Q} = \{Q(u)\}_{u \in \mathbb{Z}^2}$ be the family of subsets having the following properties.*

i. $Q(u) = \cup_{v \in Q(u)} H(v)$, i.e., elements of $\mathcal{Q}$ form a partition of $\mathbb{Z}^2$, where each element is a union of lattice blocks.

ii. *If $|Q(u)| > 1$, then $Q(u)$ must contain H such that X_H^j is a bad block at level j.*

iii. *If $|Q(u)| > 1$ or if $X_{Q(u)}^j$ is a bad block at level j, then for all neighbours v of $Q(u)$ in the closed packed lattice of $\mathbb{Z}^2$, $X_v^j = X_{\{v\}}^j$ is a good block at level j.*

iv. *If there are are vertices $v, v' \in Q(u)$ which are not neighbours in the usual Euclidean lattice but neighbours in the close packed lattice of $\mathbb{Z}^2$, then the 2×2 square containing v and v' is also contained in $Q(u)$.*

v. *The family $\{Q(u)\}_{u \in \mathbb{Z}^2}$ is the maximal family having properties i.-iv. above, i.e., any other family having the same properties must consist of unions of elements of $\mathcal{Q}$.*

*Elements of $\mathcal{Q}$ are called **lattice components** at level j.*

It is easy to see that $\mathcal{Q}$ is well defined. For $Q \in \mathcal{Q}$, we call X_Q^j a **component** of $\mathbb{X}$ at level j. Notice that a component is always a union of blocks at level j. We call X_Q^j a **bad component** at level j if it contains a bad block at level j. Often we shall denote the component $X_{Q(u)}^j$ by $X^{*,j}(u)$. The following observation is easy but useful.

Observation 4.2.15. *If $|Q(u)| \geq k > 1$, there exists $Q^* \subseteq Q(u)$ with $|Q^*| \geq \lceil \frac{k}{25} \rceil$ such that elements of Q^* are non-neighbouring and for all $v \in Q^*$, $X_{H(v)}^j$ is a bad block at level j.*

Notice that once we know the blocks at level j, and also know which blocks at level j are good, we can work out what the components at level j are, as the components only depend on the geometry of locations of the bad blocks at level j and not on the anatomy of the blocks themselves. See Figure 4.8 for an illustration. The bad blocks are marked as well as the boundary of the components.

Sub-blocks and Subcomponents

Let X_U^j be a j-level block or component. Then $|U|$ shall denote the size of the block or component. Now suppose $j \geq 1$. Let $U' \subset \mathbb{Z}^2$ be such that $X_{U'}^{j-1} = X_U^j$. For $V \subset U'$ such that V is a lattice block (resp. lattice component at level $(j-1)$) we call X_V^{j-1} a sub-block (resp. sub-component) at level $(j-1)$ of the the j-level block/component X_U^j. Notice that by construction we have that all bad subcomponents are away from the boundary of the component X_U^j.

4.2.8 Embedding, Embedding Probabilities and Semi-bad components

Embedding at level 0

For $v, v' \in \mathbb{Z}^2$, suppose X_v^0 and $Y_{v'}^0$ are blocks at level 0. We call $Y_{v'}^0 \in \mathbf{0}$ if $Y_{v'}^0$ is not good and $Y_{v'}^0$ contains more 0's than 1's. Similarly $Y_{v'}^0 \in \mathbf{1}$ if $Y_{v'}^0$ is not good and $Y_{v'}^0$ contains at least as many 1's as 0's.

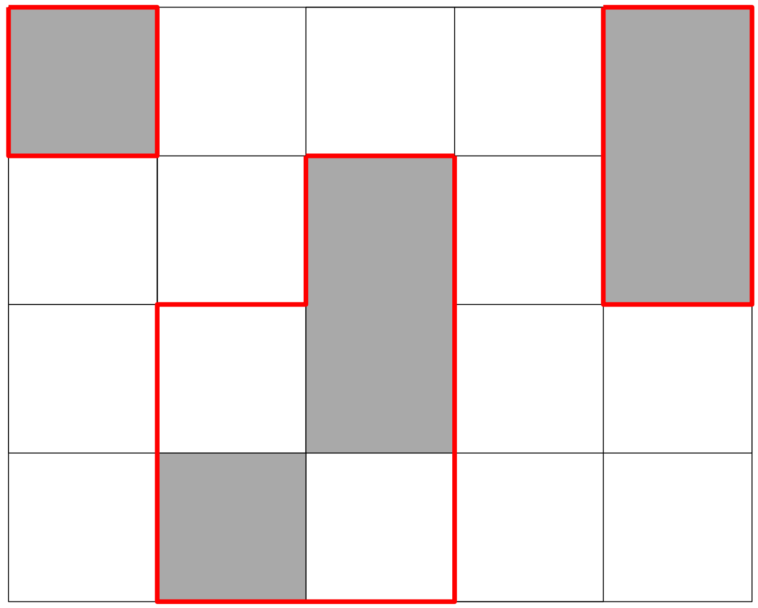


Figure 4.8: Blocks and Components: Bad blocks are marked in gray, Boundaries of components are also marked

For $v, v' \in \mathbb{Z}^2$, we call X_v^0 embeds into $Y_{v'}^0$, denoted $X_v^0 \hookrightarrow Y_{v'}^0$, if one of the following three conditions hold.

i. $Y_{v'}^0$ is a good block at level 0.

ii. $X_{\iota+v} = 0$ and $Y_{v'}^0 \in \mathbf{0}$.

iii. $X_{\iota+v} = 1$ and $Y_{v'}^0 \in \mathbf{1}$.

Let U and U' lattice animals. Let $h : \mathbb{Z}^2 \to \mathbb{Z}^2$ denote the translation that sends U to U'. Then we say $X_U^0 \hookrightarrow Y_{U'}^0$ if $X_u \hookrightarrow Y_{h(u)}$ for all $u \in U$.

Notice that at level 0, the component X_U^0 always corresponds to the ideal multi-block B_U^0. This is no longer true for $j \geq 1$ as the boundaries can have different shapes. So we need to make a more complicated recursive definition at level $j \geq 1$.

Embedding at higher levels

Fix $j \geq 1$. Suppose U is a union of lattice blocks $\mathbb{X}$ at level j. Suppose also that $V \subseteq \mathbb{Z}^2$ is a union of lattice blocks for Y at level j. Suppose further that U and V have the same shape. We want to define an event X_U^j embeds into Y_V^j, denoted by $X_U^j \hookrightarrow Y_V^j$.

Modulo a translation from $\mathbb{R}^2 \to \mathbb{R}^2$ that takes B_U^j to B_V^j, we can assume that $U = V$. Define the domain of X_U^j to be the union of the domains of the j level blocks contained in X_U^j, denote it by $\hat{U}_X$. Define $\hat{U}_Y$, the domain of Y_U^j, in a similar manner. To define the embedding we need to define bi-Lipschitz maps that take $\hat{U}_X$ to $\hat{U}_Y$. Notice that we already have one such candidate map, namely the canonical map that takes $\hat{U}_X$ to $\hat{U}_Y$. We shall consider small perturbations of that map.

Definition 4.2.16 (α-canonical maps)**.** *Let X_U^j, Y_U^j, $\hat{U}_X$, $\hat{U}_Y$ be as above. Let $T_1, T_2, \ldots, T_k \subseteq \mathbb{Z}^2$ be such that $X_{T_1}^{(j-1)}, \ldots, X_{T_k}^{(j-1)}$ are unions of blocks of $\mathbb{X}$ at level $(j-1)$ with domains $\hat{T}_{i,X}$ for $i \in [k]$. Similarly let $T'_1, T'_2, \ldots, T'_{k'}$ be such that $Y_{T'_1}^{(j-1)}, \ldots, Y_{T'_k}^{(j-1)}$ are unions of blocks of $\mathbb{Y}$ at level $(j-1)$ with domains $\hat{T'}_{i,Y}$ for $i \in [k']$. Let F be the canonical map from $\hat{U}_X$ to $\hat{U}_Y$. Then we call $G_\theta = \theta \circ F$ to be an α-canonical map from $\hat{U}_X$ to $\hat{U}_Y$ (with respect to $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ and $\mathcal{T}' = \{T'_1, \ldots T'_{k'}\}$) if the following conditions are satisfied.*

i. θ is a bijection from $\hat{U}_Y$ to itself that is identity on the boundary of $\hat{U}_Y$ and is bi-Lipschitz with Lipschitz constant $(1 + 10^{-(j+10)})$.

ii. There exists $\{S_i : i \in [k]\}$ (resp. $\{S'_i : i \in [k']\}$) such that S_i has the same shape as T_i (resp. S'_i has the same shape as T'_i) such that $Y_{S_i}^{j-1}$ is a union of $j-1$ level blocks of $\mathbb{Y}$ with domain $\hat{S}_{i,Y}$ (resp. $X_{S'_i}^{j-1}$ is a union of $j-1$ level blocks of $\mathbb{X}$ with domain $\hat{S'}_{i,X}$) such that $G_\theta(\hat{T}_{i,X}) = \hat{S}_{i,Y}$ for all $i \in [k]$ and $G_\theta(\hat{S'}_{i,X}) = T'_{i,Y}$ for all $i \in [k']$.

iii. G_θ restricted to T_i (resp. S'_i) coincides with the canonical map from $\hat{T}_{i,X}$ to $\hat{S}_{i,Y}$ (resp. from $\hat{S}'_{i,X}$ to $\hat{T}'_{i,Y}$).

Notice that an α-canonical map by definition is a bi-Lipschitz map with Lipschitz constant $(1 + 10^{-(j+5)})$. In the above setting denote $S_i = G_\theta(T_i)$ and $S'_i = G_\theta^{-1}(T'_i)$.

Observe that an α-canonical map maps a domain to a domain of same shape while matching up certain sub-blocks in $\mathbb{X}$ (resp. in $\mathbb{Y}$) to sub-blocks of same shapes in $\mathbb{Y}$ (resp. in $\mathbb{X}$). For embedding, we want to match up all bad sub-blocks by an α-canonical map as above. We define embedding at level j formally as follows. Assume that we have defined embedding at levels upto $(j-1)$.

Definition 4.2.17 (Embedding at level j)**.** *Let $X = X_U^j$, $Y = Y_U^j$, $\hat{U}_X$, $\hat{U}_Y$ be as above. Let $T_1, T_2, \ldots, T_k \subseteq \mathbb{Z}^2$ be such that $X_{T_1}^{(j-1)}, \ldots, X_{T_k}^{(j-1)}$ are unions of blocks of $\mathbb{X}$ at level $(j-1)$ containing all $(j-1)$ level bad sub-blocks. Similarly let $T'_1, T'_2, \ldots, T'_{k'}$ be such that $Y_{T'_1}^{(j-1)}, \ldots, Y_{T'_{k'}}^{(j-1)}$ are unions of blocks of $\mathbb{Y}$ at level $(j-1)$ containing all bad sub-blocks. We say X embeds into Y, denoted $X \hookrightarrow Y$ if there exist $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$, and $\mathcal{T}' = \{T'_1, \ldots T'_{k'}\}$ as above and there exists an α-canonical map G_θ from $\hat{U}_X$ to $\hat{U}_Y$ with respect to $\mathcal{T}$ and $\mathcal{T}'$ such that for all $i \in [k]$, $X_{T_i}^{(j-1)} \hookrightarrow Y_{G_\theta(T_i)}^{(j-1)}$ and for all $i \in [k']$ we have $X_{G_\theta^{-1}(T'_i)}^{(j-1)} \hookrightarrow Y_{T'_i}^{(j-1)}$.*

In the situation of the above definition, we say G_θ **gives an embedding of** X **into** Y. The following sufficient condition for embedding given in terms of components will be useful for us.

Lemma 4.2.18. *Let* $X = X_U^j$, $Y = Y_U^j$, $\hat{U}_X$, $\hat{U}_Y$ *be as above. Let* $T_1, T_2, \ldots, T_k \subseteq \mathbb{Z}^2$ *be such that* $X_{T_1}^{(j-1)}, \ldots, X_{T_k}^{(j-1)}$ *are all bad level* $(j-1)$ *components contained in* X. *Let* $W \subseteq \mathbb{Z}^2$ *be such that* $Y = Y_W^{j-1}$. *Suppose there exists an* α*-canonical map* G_θ *from* $\hat{U}_X$ *to* $\hat{U}_Y$ *with respect to* $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ *and* $\emptyset$ *such that for all* $i \in [k]$, $X_{T_i}^{(j-1)} \hookrightarrow Y_{G_\theta(T_i)}^{(j-1)}$ *and for all* $u \in W \setminus \cup_i G_\theta(T_i)$, Y_u^j *is a good block at level* $j-1$. *Then* $X \hookrightarrow Y$.

Proof. Follows immediately from Definition 4.2.17. □

Random Blocks and Embedding Probabilities

Observe that at a fixed level j the distribution of the blocks and components is translation invariant. That is, there exist laws $\mu_j^{\mathbb{X}}$ (resp. $\mu_j^{\mathbb{Y}}$) such that for all $u \in \mathbb{Z}^2$, the j-level component $X^{*,j}(u)$ (resp. $Y^{*,j}(u)$) has the law $\mu_j^{\mathbb{X}}$ (resp. $\mu_j^{\mathbb{Y}}$).

Fix a component $X^* = X_U^j$ at level j. Let $A_{\text{valid}}^{\mathbb{Y}}$ denote the event that the external buffer zones of B_U^j are not conjoined in $\mathbb{Y}$. On $A_{\text{valid}}^{\mathbb{Y}}$, clearly $Y^* = Y_U^j$ is a union of j level blocks in $\mathbb{Y}$. Denote the embedding probability of the component X^*

$$S_j^{\mathbb{X}}(X^*) = \mathbb{P}[X^* \hookrightarrow Y^*, A_{\text{valid}}^{\mathbb{Y}} \mid X^*]. \tag{4.2.1}$$

In a similar vein we define the embedding probability of a j-level $\mathbb{Y}$-component Y^* by

$$S_j^{\mathbb{Y}}(Y^*) = \mathbb{P}[X^* \hookrightarrow Y^*, A_{\text{valid}}^{\mathbb{X}} \mid Y^*]. \tag{4.2.2}$$

We shall drop the superscripts $\mathbb{X}$ or $\mathbb{Y}$ when it will be clear from the context which block we are talking about. The embedding probabilities are very important quantities for us. The key of our multi-scale proof rests on proving recursive power law tail estimates for $S_j(X^*)$ when X^* is distributed according to $\mu_j^{\mathbb{X}}$ and similarly for $S_j(Y^*)$.

Semi-bad Components and Airports

It will be useful for us to classify the bad components at level j into two types **semi-bad** and **really bad**. A semi-bad component will be one which is not too large in size and has a sufficiently high embedding probability. We define it only for $\mathbb{X}$-components, semi-bad $\mathbb{Y}$-components are defined in a similar fashion. For a component X we shall denote by V_X its size, that is the size of the multi-cell it corresponds to.

Definition 4.2.19 (Semi-bad Components)**.** *A component* $X = X_U^j$ *at level* j *is said to be semi-bad if it satisfies the following conditions.*

i. $V_X = |U| \leq v_0$.

ii. $S_j^{\mathbb{X}}(X) \geq 1 - \frac{1}{v_0^5 k_0^4 100^j}$.

An **airport** is a region such that most locations in it can be embedded into any semi-bad component. The formal definition is as follows.

Definition 4.2.20 (Airports)**.** *A square S of $L_{j-1}^{3/2} \times L_{j-1}^{3/2}$ many $j-1$ level cells contained in a j level component of $\mathbb{X}$ is called an airport if for all level $j-1$ semi-bad component $Y^* = Y_U^{j-1}$ the following condition holds.*

- *Fix any $H \subseteq S$ having the same shape as U. Let the event that $X^* = X_H^{j-1}$ is a union of blocks at level $j-1$ be denoted by A_{valid}^H. We have*

$$\#\{H : A_{valid}^H, X^* \hookrightarrow Y^*\} \geq (1 - v_0^{-2} k_0^{-4} 100^{-j}) N(S, U)$$

  where $N(S,U)$, denote the number of multi-cells in S having the same shape as U.

Airports are defined for $\mathbb{Y}$ blocks in an analogous manner.

4.2.9 Good Blocks

To complete the construction of the multi-scale structure, we need to define good blocks at level $j \geq 1$. With the definitions from the preceding subsections, we are now ready to give the recursive definition. Fix $j \geq 1$. Suppose we already have constructed the structure up to level $j-1$. As usual, in the following definition, we only consider $\mathbb{X}$ blocks; j-level good blocks for $\mathbb{Y}$ are defined similarly.

Definition 4.2.21 (Good Blocks)**.** *A block $X = X(u)$ at level j is said to be good if the following conditions hold.*

i. X has size 1, i.e., $B^j(u)$ is an ideal multi-block.

ii. The total sizes of $j-1$ level bad components contained in X is at most k_0.

iii. All the bad components contained in X are semi-bad.

iv. All $L_{j-1}^{3/2} \times L_{j-1}^{3/2}$ squares of $(j-1)$ level cells contained in X are airports.

4.3 Recursive Estimates

Our proof of Theorem 4.1 depends on a collection of recursive estimates, all of which are proved together by induction. In this section we list these estimates. The proof of these estimates are provided over the next few sections. We recall that for all $j > 0$, $L_j = L_{j-1}^{\alpha} = L_0^{\alpha^j}$.

- **Tail Estimate:** Let $j \geq 0$. Let $X = X_U^j$ be a $\mathbb{X}$-component at level j (having the distribution $\mu_j^{\mathbb{X}}$) and let $m_j = m + 2^{-j}$. Recall $V_X = |U|$. Then

$$\mathbb{P}(S_j^{\mathbb{X}}(X) \leq x, V_X \geq v) \leq x^{m_j} L_j^{-\beta} L_j^{-\gamma(v-1)} \text{ for } 0 < x \leq 1 - L_j^{-1} \text{ for all } v \geq 1. \quad (4.3.1)$$

Let $Y = Y_U^j$ be a $\mathbb{Y}$-component at level j (having the distribution $\mu_j^{\mathbb{X}}$). Recall $V_X = |U|$. Then

$$\mathbb{P}(S_j^{\mathbb{Y}}(Y) \leq x, V_Y \geq v) \leq x^{m_j} L_j^{-\beta} L_j^{-\gamma(v-1)} \text{ for } 0 < x \leq 1 - L_j^{-1} \text{ for all } v \geq 1. \quad (4.3.2)$$

- **Size Estimate:** Let $j \geq 0$. Let $X = X_U^j$ (resp. $Y = Y_U^j$) be a $\mathbb{X}$-component at level j having the distribution $\mu_j^{\mathbb{X}}$ (resp. $\mathbb{Y}$-component at level j having the distribution $\mu_j^{\mathbb{Y}}$). Then

$$\mathbb{P}(V_X \geq v) \leq L_j^{-\gamma(v-1)} \text{ for } v \geq 1. \quad (4.3.3)$$

$$\mathbb{P}(V_Y \geq v) \leq L_j^{-\gamma(v-1)} \text{ for } v \geq 1. \quad (4.3.4)$$

- **Good Block Estimate:**

  - Good blocks embed into to good blocks, i.e., for all good j-level block X and for all good j-level block Y we have

$$X \hookrightarrow Y. \quad (4.3.5)$$

  - Conditioned on a partial set of outside blocks, blocks are good with high probability. Fix $u \in \mathbb{Z}^2$. Let $V \subseteq \mathbb{Z}^2 \setminus \{u\}$. Let $\mathcal{F}_V^{\mathbb{X}}$ (resp. $\mathcal{F}_V^{\mathbb{Y}}$) denote the conditioning $\mathcal{F}_V^{\mathbb{X}} = \{X_V^j, X^j(u) \notin X_V^j\}$ (resp. $\mathcal{F}_V^{\mathbb{Y}} = \{Y_V^j, Y^j(u) \notin Y_V^j\}$), i.e. we condition on partial set of j level blocks corresponding to ideal blocks excluding $B^j(u)$ such that these blocks are not the block corresponding to $B^j(u)$. Then we have the following for all $u \in \mathbb{Z}^2$ and for all $V \subseteq \mathbb{Z}^2 \setminus \{u\}$.

$$\mathbb{P}[X_u^j \text{ is good} \mid \mathcal{F}_V^{\mathbb{X}}] \geq 1 - L_j^{-\gamma}. \quad (4.3.6)$$

$$\mathbb{P}[Y_u^j \text{ is good} \mid \mathcal{F}_V^{\mathbb{Y}}] \geq 1 - L_j^{-\gamma}. \quad (4.3.7)$$

Theorem 4.3.1 (Recursive Theorem). *For α, β, γ, m, k_0 and v_0 as in equation (4.1.1), the following holds for all large enough L_0. If the recursive estimates (4.3.1), (4.3.2), (4.3.3), (4.3.4), (4.3.5), (4.3.6) and (4.3.7) hold at level j for some $j \geq 0$ then all the estimates hold at level $(j+1)$ as well.*

We shall prove Theorem 4.3.1 over the next few sections. Before that we prove that these estimates indeed hold at level $j = 0$.

Theorem 4.3.2. *Fix α, β, γ, m, k_0, v_0 and L_0 such that the conclusion of Theorem 4.3.1 holds. Then for M_0 sufficiently large depending on all the parameters the estimates (4.3.1), (4.3.2), (4.3.3), (4.3.4), (4.3.5), (4.3.6) and (4.3.7) hold for $j = 0$.*

Proof. Observe that (4.3.5) for $j = 0$ follows from the definition of good blocks at level 0. Recall that blocks at level 0 are independent and hence by taking M_0 sufficiently large we make sure that (4.3.7) holds for $j = 0$, and (4.3.6) holds vacuously. As a matter of fact, by taking M_0 sufficiently large, we can ensure that $\mathbb{P}[Y^0(u) \text{ is good}] \geq 1 - L_0^{-20\beta}$. Notice that components of $\mathbb{X}$ all have size 1 and hence (4.3.3) also holds trivially. For a component $X = X^0(u)$ we have $S_0^{\mathbb{X}}(X) \geq \mathbb{P}[Y^0(u) \text{ is good}] \geq 1 - L_0^{-1}$, and hence (4.3.1) also holds.

Now look at the component $Y = Y^{*,0}(u) = Y_U^0$. If $V_Y = v > 1$, there are at least $\frac{v}{9}$ many bad blocks contained in Y. Since blocks are independent, it follows by summing over all lattice animals containing u of size v that $\mathbb{P}[V_Y \geq v] \leq L_j^{-10\beta(v-1)}$. Also notice that, $S_0^{\mathbb{Y}}(Y) = 1$ if Y is good, $S_0^{\mathbb{Y}}(Y) = \frac{1}{2^v}$ otherwise. Hence it suffices to prove (4.3.2) for $x = \frac{1}{2^v}$ and $v \geq 1$. For $x \leq \frac{1}{2}$, it follows that for $\mathbb{P}[S_0^{\mathbb{Y}}(Y) \leq x, V_Y \geq v] \leq \mathbb{P}[V_Y \geq \max\{v, \log_2 x\}] \leq x^{m_{j+1}} L_0^{-\beta} L_0^{-\gamma(v-1)}$ because L_0 is sufficiently large. This establishes (4.3.2) for $j = 0$. □

4.3.1 Proof of the Main Theorem

Before proceeding with the proof of Theorem 4.3.1, we show how Theorem 4.3.1 and Theorem 4.3.2 can be used to deduce Theorem 4.1.

Proof of Theorem 4.1. Notice that by ergodic theory considerations it suffices to prove that $\mathbb{P}[\mathbb{X} \hookrightarrow_M \mathbb{Y}] > 0$ for some M. Fix α, β, γ, m, k_0, v_0, L_0 and M_0 in such a way that conclusions of both Theorem 4.3.1 and Theorem 4.3.2 holds. This implies that the recursive estimates (4.3.1), (4.3.2), (4.3.5), (4.3.3), (4.3.4), (4.3.6) and (4.3.7) hold for all $j \geq 0$.

Let $u_1 = (0,0)$, $u_2 = (0,-1)$, $u_3 = (-1,1)$ and $u_4 = (-1,0)$. So $\{X_{u_i}^j : i \in \{1,2,3,4\}\}$ denote the blocks surrounding the origin at level j. Let us denote the domains of these blocks by $D_{u_i}^{j,\mathbb{X}}$ respectively. Define $D_{u_i}^{j,\mathbb{Y}}$ similarly. For $j \geq 0$, let $\mathcal{A}_j^{\mathbb{X}}$ (resp. $\mathcal{A}_j^{\mathbb{Y}}$) denote the following event that for all $i \in [4]$ we have $D_{u_i}^{j,\mathbb{X}} = B_{u_i}^j$ and $X_{u_i}^j$ is good (resp. $D_{u_i}^{j,\mathbb{Y}} = B_{u_i}^j$ and $Y_{u_i}^j$ is good). The proof is now completed using the following three propositions. □

Proposition 4.3.3. *Suppose (4.3.1), (4.3.2), (4.3.5), (4.3.3), (4.3.4), (4.3.6) and (4.3.7) hold for all $j \geq 0$. Then*

$$\mathbb{P}\left[\bigcap_{j\geq 0}(\mathcal{A}_j^{\mathbb{X}} \cap \mathcal{A}_j^{\mathbb{Y}})\right] > 0.$$

Proof. Notice that $\mathcal{A}_j^{\mathbb{X}} = \mathcal{A}_j^{\mathbb{X},G} \cap \mathcal{A}_j^{\mathbb{X},\partial}$ where $\mathcal{A}_j^{\mathbb{X},G}$ denotes the event the the four blocks around origin at level j are good and $\mathcal{A}_j^{\mathbb{X},\partial}$ denotes the event that $D_{u_i}^{j,\mathbb{X}} = B^j(u_i)$ for all i. Clearly $\mathbb{P}[\mathcal{A}_0^{\mathbb{X},\partial}] = 1$. Now for $j \geq 1$, let $\mathcal{A}_j^{\mathbb{X},B}$ denote the event that the j-level buffer zones in these blocks (12 in number) only contain good $(j-1)$ level blocks. Clearly from construction of the blocks

$$\mathbb{P}[\mathcal{A}_j^{\mathbb{X},\partial}] \geq (1-10^{-(j+2)})\mathbb{P}[\mathcal{A}_j^{\mathbb{X},B}].$$

It follows from (4.3.6) that for $j \geq 1$

$$\mathbb{P}[\mathcal{A}_j^{\mathbb{X},B}] \geq 1 - 12L_{j-1}^{\alpha+3-\gamma}.$$

Since $\gamma > \alpha + 3$ we get

$$\mathbb{P}[\mathcal{A}_j^{\mathbb{X},\partial}] \geq 1 - 10^{-(j+3/2)}$$

by taking L_0 sufficiently large. Combining these estimates we get for all $j \geq 0$,

$$\mathbb{P}[\mathcal{A}_j^{\mathbb{X}}] \geq 1 - 10^{-(j+1)}.$$

By the obvious symmetry between $\mathbb{X}$ and $\mathbb{Y}$ the same lower bound also holds for $\mathbb{P}[\mathcal{A}_j^{\mathbb{Y}}]$. The proposition follows by taking a union bound over $\mathbb{X}$, $\mathbb{Y}$ and all $j \geq 0$. □

Proposition 4.3.4. *Fix $J \in \mathbb{N}$. On $\bigcap_{J \geq j \geq 0}(\mathcal{A}_j^{\mathbb{X}} \cap \mathcal{A}_j^{\mathbb{Y}})$, there exists a map $\Phi = \Phi_J : [-L_J, L_j]^2 \to [-L_J, L_J]^2$ satisfying the following conditions.*

i. *$\Phi(0) = 0$ and Φ is identity on the boundary.*

ii. *Φ is bi-Lipschitz with Lipschitz constant 10.*

iii. *For each level 0 bad $\mathbb{Y}$-block $Y^0(u')$ contained in $[-\frac{1}{2}L_J, \frac{1}{2}L_J]^2$, there is a $\mathbb{X}$-block $X^0(u)$ at level 0 such that $\Phi(u) = u'$ and $X^0(u) \hookrightarrow Y^0(u')$.*

We postpone the proof of Proposition 4.3.4 for the moment.

Proposition 4.3.5. *Suppose for all $J \in \mathbb{N}$, there exists a Φ_J satisfying the conditions in Proposition 4.3.4. Then there exists a $20M_0$-Lipschitz injection ϕ from $\mathbb{Z}^2 \to \mathbb{Z}^2$ such that $X_{\iota+v} = Y_{\iota+\phi(v)}$ for all $v \in \mathbb{Z}^2$.*

Proof. Fix $J \in \mathbb{N}$. Define $\phi^J : [-\frac{1}{4}L_j, \frac{1}{4}L_J]^2 \cap \mathbb{Z}^2 \to \mathbb{Z}^2$ as follows.

Case 1: For $u \in \mathbb{Z}^2$ suppose $\Phi_J(u) = v = (v_1, v_2)$ be such that Y_v^0 is a bad level 0 block of $\mathbb{Y}$. Clearly, there exists $v' \in \mathbb{Z}^2$ such that $\iota + v' \in [v_1M_0, (v_1+1)M_0] \times [v_2M_0, (v_2+1)M_0]$ and $X_{\iota+u} = Y_{\iota+v'}$. Choose such a v' arbitrarily and set $\phi^J(u) = v'$.

Case 2: If for $u \in \mathbb{Z}^2$ Case 1 does not hold then there exists $v \in \mathbb{Z}^2$ such that $||v - \Phi_J(u)|| \leq 1$ and $Y^0(v)$ is a good block. Clearly there are many (at least $M_0^2/3$ in number) $\iota + v' \in [v_1M_0, (v_1+1)M_0] \times [v_1M_0, (v_1+1)M_0]$ such that $X_{\iota+u} = Y_{\iota+v'}$. Choose one such v' arbitrarily and set $\phi^J(u) = v'$. Since the number of sites u that correspond to v in the above manner is limited it follows that such a ϕ^J can be chosen to be an injection.

Notice that ϕ^J is $20M_0$-Lipschitz and also observe that as Φ^J is identity at the origin it follows that $||\phi^J(0)|| \leq M_0$. It now follows by a compactness argument that there exists an injective map $\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ which is $20M_0$-Lipschitz and such that $X_{\iota+v} = Y_{\iota+\phi(v)}$ for all $v \in \mathbb{Z}^2$. □

It remains to prove Proposition 4.3.4 which will follow from the next lemma.

Lemma 4.3.6. *Assume the set-up of Proposition 4.3.4. Fix $0 < j \leq J$. Suppose there exists a map $\phi_j : [-L_J, L_j]^2 \to [-L_J, L_J]^2$ satisfying the following conditions.*

i. $\phi_j(0) = 0$ and ϕ_j is identity on the boundary.We take ϕ_J to be the identity map.

ii. ϕ_j is bi-Lipschitz with Lipschitz constant C_j.

iii. There exists $\{X^j_U\}_{U\in I_1}$ with respective domains $\{\hat{U}_X\}_{U\in I_1}$ containing all bad level j blocks of $\mathbb{X}$ contained in $[-L_J(1-10^{-(j+1)}), L_J(1-10^{-(j+1)})]^2$. Also there exists $\{Y^j_{U'}\}_{U'\in I_2}$ with respective domains $\{\hat{U'}_Y\}_{U'\in I_2}$ containing all bad level j blocks of $\mathbb{Y}$ contained in $[-L_J(1-10^{-(j+1)}), L_J(1-10^{-(j+1)})]^2$.

Also all $U \in I_1$, there exists $f(U)$ such that ϕ_j restricted to $\hat{U}_X$ is a canonical map to the domain $\widehat{f(U)}_Y$ of $Y^j_{f(U)}$ and such that such that $X^j_U \hookrightarrow Y^j_{f(U)}$. Further for each $U' \in I_2$, there exists $f^{-1}(U)$ such that $\phi_j^{-1}(\hat{U'}_Y)$ is the domain $\widehat{f^{-1}(U')}_X$ of the block $X^j_{f^{-1}(U)}$ and such that ϕ_j restricted to $\widehat{f^{-1}(U')}_X$ is a canonical map and $X^j_{f^{-1}(U)} \hookrightarrow Y^j_{U'}$.

Then there exists a function $\psi_{j-1} : [-L_J, L_J]^2 \to [-L_J, L_J]^2$ with $\psi_{j-1}(0) = 0$, ψ_{j-1} identity on the boundary, bi-Lipschitz with Lipschitz constant $(1+10^{-(j+9)})$ such that $\phi_{j-1} := \psi_{j-1}\circ\phi_j$ satisfies all the above conditions with j replaced by $j-1$ (with setting $C_{j-1} = C_j(1+10^{-(j+9)})$), that is ϕ_{j-1} matches up all the bad $(j-1)$ level components in a Lipschitz manner.

Notice that Proposition 4.3.4 follows from Lemma 4.3.6 using induction and definition of good block and embedding. Now we prove Lemma 4.3.6.

Proof of Lemma 4.3.6. We shall construct ψ_{j-1} satisfying the requirements of the lemma. The strategy we adopt is the following. Denote $\mathcal{B} = \{\hat{U}_X : U \in I_1; \widehat{f^{-1}(U')}_X : U' \in I_2\}$. Set $B = [-L_J, L_J]^2 \setminus \cup_{A\in\mathcal{B}} A$. For each $A \in \mathcal{B} \cup \{B\}$, we shall construct a function $\psi^A : \phi_j(A) \to \phi_j(A)$ that is a Lipschitz bijection with Lipschitz constant $(1 + 10^{-(j+9)})$ and is identity on the boundary of A. We shall take ψ_{j-1} to be the unique function on $[-L_J, L_J]^2$such that its restriction to A equals ψ^A for each $A \in \mathcal{B} \cup \{B\}$. We shall then verify that ψ_{j-1} constructed as such satisfies the conditions of the lemma.

First fix $A \in \mathcal{B}$. We describe how to construct ψ^A. Without loss of generality, take $A = hatU_X$ and hence $\phi_j(A) = \widehat{f(U)}_Y$. By definition of $X^j_U \hookrightarrow Y^j_{f(U)}$, there exists an α-canonical map G_θ from $\hat{U}_X \to \widehat{f(U)}_Y$ that gives the embedding. Take $\psi^A = theta$. Clearly ψ^A is identity on the boundary and is a Lipschitz bijection with Lipschitz constant $(1+10^{-(j+9)})$, and also satisfies the hypothesis of the Lemma for all $(j-1)$ level bad blocks in X^j_U and $Y^j_{f(U)}$ by Definition 4.2.17.

So it suffices to define ψ^B in such a way that all the bad components at level $(j-1)$ that are not contained in any j-level bad sub-block are matched up.

Let $\{X_W^{j-1}\}_{W\in I_1'}$ denote the set of all $(j-1)$-level bad components of $\mathbb{X}$ contained in $[-L_j(1-10^{-(j)}), L_j(1-10^{-(j)})]^2$ and not contained in A for any $A \in \mathcal{B}$, let $\hat{W}_X$ denote their respective domains. Similarly let $\{Y_{W'}^{j-1}\}_{W'\in I_2'}$ denote the set of all $(j-1)$-level bad components of $\mathbb{Y}$ contained in $[L_j(1-10^{-(j)}), L_j(1-10^{-(j)})]^2$ and not contained in $\phi_j(A)$, let $\hat{W'}_Y$ denote their respective domains. We shall only describe how to match up the X_W^{j-1}'s; the $\mathbb{Y}$-components can be taken care of similarly.

Notice that since all these are contained in good j-level blocks, these are all away from the boundaries of B and $\phi_j(B)$ respectively. Also it follows from the definition of good blocks that there cannot be too many bad components close together. It is easy to see that one can find squares $S_k \subseteq \phi_j(B)$ such that $\phi_j(W_X)$ are all contained in the union of S_k, are at distance at least L_{j-1}^3 from the boundary of S_k's and such that S_k does not intersect any of the bad $(j-1)$-level components of $\mathbb{Y}$. Also it can be ensured that for a fixed k, the total size of components X_W^{j-1} such that $\phi_j(W_X)$ is contained in S_k is not more than k_0. Since by definition of good block the components $X_W^{(j-1)}$ are all semi-bad, and the region $\phi_j(S_k)$ contains enough airports it follows that it is possible to define a map $\psi^{S_k}: \phi(S_k) \to \phi(S_k)$ which is identity on the boundary and ψ^{S_k} gives an embedding $X_W^{j-1} \hookrightarrow Y_{g(W)}^{j-1}$ for all the bad $j-1$ level components contained in S_k. Now gluing together all such maps we get the required ψ^B that matches up all the bad $j-1$ level components contained in B. We omit the details, see the proof of Proposition 4.4.2 for a similar construction. This completes the proof.

□

The remainder of the chapter is devoted to the proof of the estimates in the induction statement. Throughout these sections we assume that the estimates (4.3.1), (4.3.2), (4.3.5), (4.3.3), (4.3.4), (4.3.6) and (4.3.7) hold for some level $j \geq 0$ and then prove the estimates at level $j+1$. Combined they will complete the proof of Theorem 4.3.1.

From now on, in every Theorem, Proposition and Lemma we state, we would implicitly assume the hypothesis that all the recursive estimates hold upto level j, the parameters satisfy the constraints described in § 2.1.2 and L_0 is sufficiently large.

4.4 Geometric Constructions

To show the existence of embeddings we need to construct α-canonical maps having different properties. In this section we develop different geometric constructions which shall imply the existence of α-canonical maps in different cases. We start with the following simple case where the blocks are not moved around but only the boundaries of domains are adjusted.

More specifically, our aim is the following. Consider a potential domain $\widetilde{U}$ at level $(j+1)$. We want to construct an α-canonical map from $\widetilde{U}$ to itself that takes a given set of potential domains of j-level multi-cells contained in $\widetilde{U}$ that are away from the boundary to any other given set of potential domains of the same multi-cells. We have the following proposition.

Proposition 4.4.1. *Fix a lattice animal $U \subseteq \mathbb{Z}^2$. Fix a level $(j+1)$ potential domain $\widetilde{U}$ corresponding to the multi-cell B_U^{j+1}, i.e., $C = C(\widetilde{U})$, the boundary of $\widetilde{U}$ is a potential boundary curve through the buffer zone of B_U^{j+1}. Let $U_1 \subseteq \mathbb{Z}^2$ be such that $B_{U_1}^j$ denotes the collection of all level j-cells that intersect $\widetilde{U}$. Let $U_2 \subseteq U_1$ be the set of all vertices in U_1 such that the distance from the boundary of U_1 is at least $\frac{L_j^3}{2}$. Let $\mathcal{T} = \{T_1, T_2, \dots T_\ell\}$ be a set of disjoint subsets of U_2. Let $\{U_{T_i}\}_{i\in[\ell]}$ (resp. $\{\hat{U}_{T_i}\}_{i\in[\ell]}$) denote a set of potential domains corresponding to the j-level multi-cells $B_{T_i}^j$ that are compatible i.e., there exists a canonical map Γ (resp. $\hat{\Gamma}$) at level j such that $\Gamma(B_{T_i}^j) = U_{T_i}$ (resp. $\hat{\Gamma}(B_{T_i}^j) = \hat{U}_{T_i}$). Then there exists an α-canonical map Υ with respect to $\mathcal{T}$ and $\emptyset$ such that $\Upsilon(T_i) = T_i$ for each $i \in [\ell]$.*

Proof. Notice that the canonical map from $\widetilde{U}$ to itself is the identity map. Also observe that without loss of generality we can assume that $U = \{\mathbf{0}\}$ and $\mathcal{T} = \{\{u\} : u \in U_2\}$. Let $U_2 \subseteq U_3 \subseteq \mathbb{Z}^2$ be such that U_3 contains all sites at a distance 2 from U_2. It is clear from the construction of potential domains that Γ and $\hat{\Gamma}$ as in the statement of the proposition can be chosen such that both Γ and $\hat{\Gamma}$ are identity outside $B_{U_3}^j$. Define the map Υ on $\widetilde{U}$ by $\Upsilon := \hat{\Gamma} \circ \Gamma^{-1}$. It follows from definition that

i. Υ-is identity on $\widetilde{U} \setminus B_{U_3}^j$, in particular on the boundary of $\widetilde{U}$.

ii. Υ is bi-Lipschitz with Lipschitz constant $(1 + 10^{-(j+5)})$.

iii. $\Upsilon(U_{T_i}) = \hat{U}_{T_i}$ for all i.

It follows then from the definition of α-canonical maps that $\Upsilon = \Upsilon \circ Id$ is an α-canonical map from $\widetilde{U}$ to itself with respect to $\mathcal{T}$ and $\emptyset$ such that $\Upsilon(T_i) = T_i$ for all i. This completes the proof. □

An α-canonical map as above will be referred to as a $*$-canonical map. Figure 4.9 illustrates this construction.

Now we want to move to a more complicated construction of α-canonical maps, where we want to match up a non-trivial subset of bad blocks in both $\mathbb{X}$ and $\mathbb{Y}$. We have the following proposition.

Proposition 4.4.2. *Fix $U \subseteq \mathbb{Z}^2$. Consider $\widetilde{U}_1$ and $\widetilde{U}_2$ to be any two potential domains corresponding to the $j+1$-level multi-cell B_U^{j+1}. Let $U_{1,1} \subseteq \mathbb{Z}^2$ (resp. $U_{1,2} \subseteq \mathbb{Z}^2$) be such that $B_{U_{1,1}}^j$ (resp. $B_{U_{1,2}}^j$) denotes the collection of all level j-cells that intersect $\widetilde{U}_1$ (resp. $\widetilde{U}_2$). Let $U_{2,1} \subseteq U_{1,1}$ (resp. $U_{2,2} \subseteq U_{1,2}$) be the set of all vertices in $U_{1,1}$ (resp. $U_{1,2}$) such that*

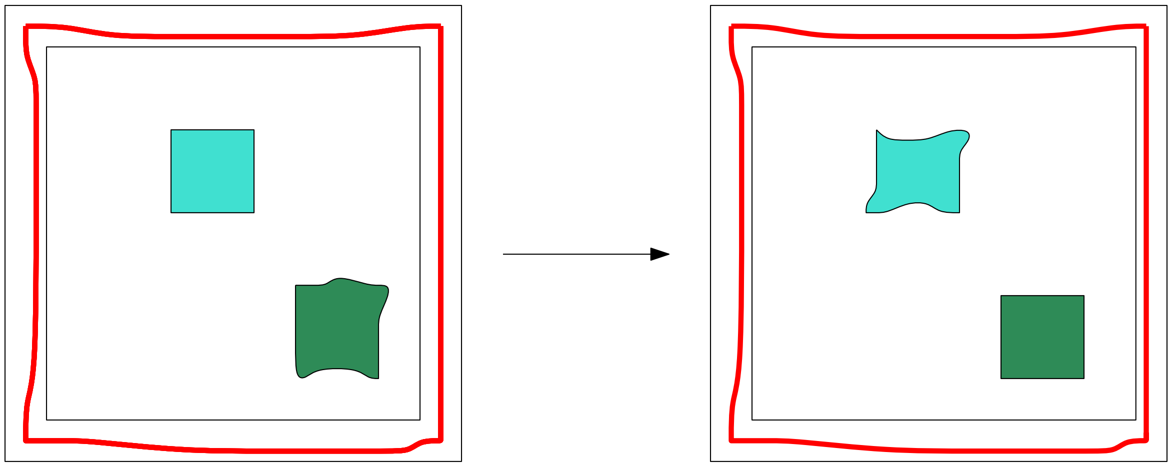


Figure 4.9: A $*$-canonical map

the distance from the boundary of $U_{1,1}$ (resp. $U_{1,2}$) is at least $\frac{L_j^3}{2}$. Let $\mathcal{T} = \{T_1, T_2, \dots T_{\ell_1}\}$ and $\mathcal{T}' = \{T_1', T_2', \dots T_{\ell_2}'\}$ be a set of disjoint and non-neighbouring subsets of $U_{2,1}$ and $U_{2,2}$ respectively such that $\sum |T_i| \leq v_0 k_0$ and $\sum |T_i'| \leq v_0 k_0$. Then there exists a sequence of α-canonical maps $\{\Upsilon_{h_1,h_2}\}_{(h_1,h_2)\in[L_j^2]^2}$ from $\widetilde{U}_1$ to $\widetilde{U}_2$ with respect to $\mathcal{T}$ and $\mathcal{T}'$ satisfying the following conditions.

i. *For each $i \in [\ell_1]$, $\Upsilon_{h_1,h_2}(T_i) = (h_1 - 1, h_2 - 1) + \Upsilon_{1,1}(T_i)$ and for each $i \in [\ell_2]$, $\Upsilon^{-1}_{h_1,h_2}(T_i') = -(h_1 - 1, h_2 - 1) + \Upsilon^{-1}_{1,1}(T_i')$.*

ii. *For all $h = (h_1, h_2)$ for all $i \in [\ell_1], i' \in [\ell_2]$ we have T_i and $\Upsilon_h^{-1}(T_{i'}')$ are disjoint and non-neighbouring.*

Proof. For $i \in [\ell_1]$ (resp. $i' \in [\ell_2]$) Let U_{T_i} (resp. $U_{T_{i'}'}$) be the domain corresponding to $B^j_{T_i}$ (resp. $B^j_{T_{i'}'}$). Also let F denote the canonical map at level $(j+1)$ that takes $\widetilde{U}_1$ to $\widetilde{U}_2$. Since $\widetilde{U}_{2,1}$ and $\widetilde{U}_{2,2}$ are away from the boundaries of $U_{1,1}$ and $U_{1,2}$ respectively (by a distance of order L_j^3) and the total sizes of $\mathcal{T}$ and $\mathcal{T}'$ are bounded (independent of L_j) it follows that for L_j sufficiently large there exists a function $\Omega : \widetilde{U}_2 \to \widetilde{U}_2$ satisfying the following properties.

i. Ω is identity on the boundary of $\widetilde{U}_2$, and bi-Lipschitz with Lipschitz constant $(1 + 10^{-(j+10)})$.

ii. There exists squares $S_1, S_2, \dots S_k \subseteq \widetilde{U}_2$ with the following properties.

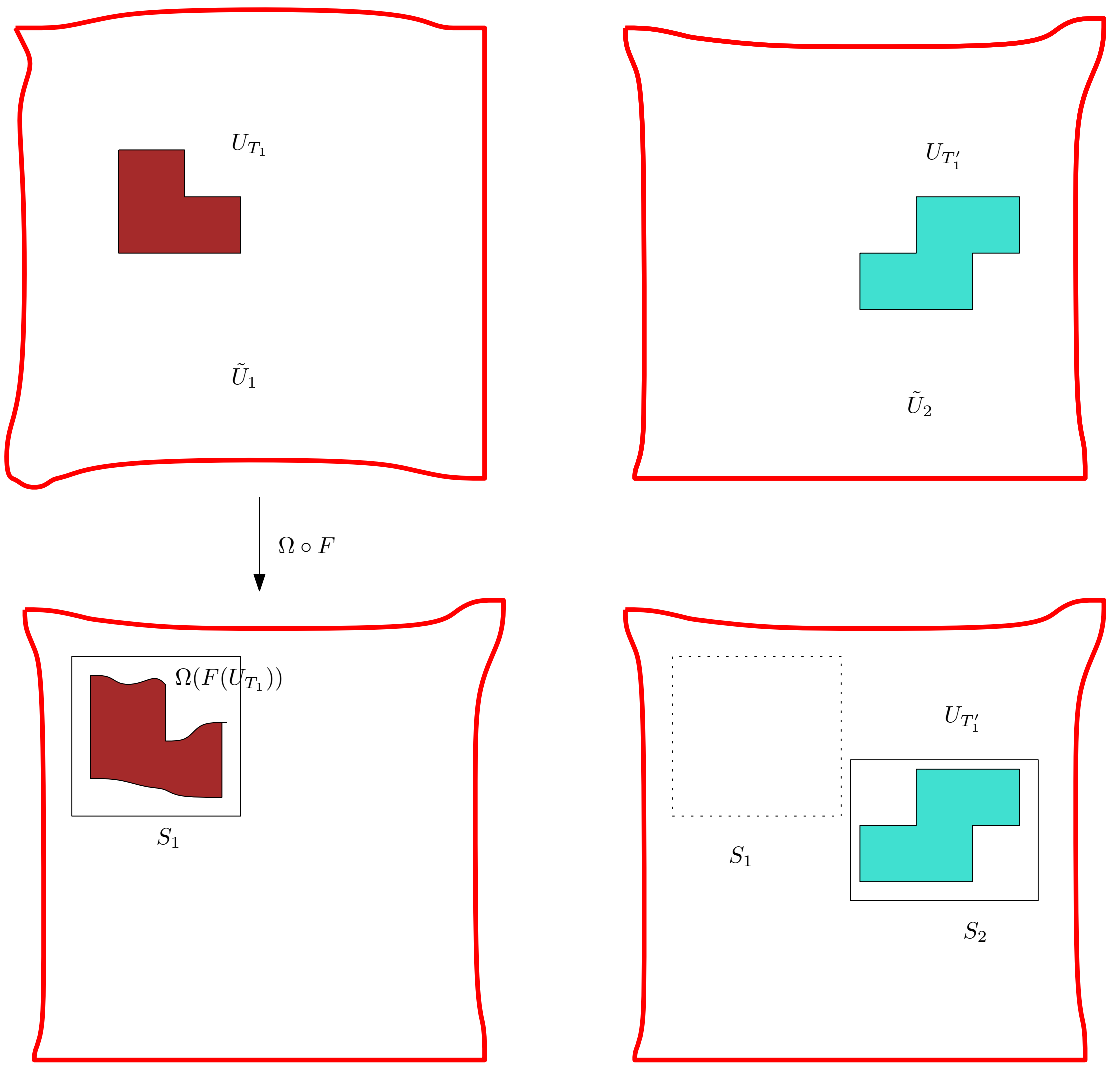


Figure 4.10: Construction of S_i as described in the proof of Proposition 4.4.2

- We have that
$$(\cup_i \Omega \circ F(U_{T_i})) \bigcup (\cup_{i'} U_{T'_i}) \subseteq \cup_{i=1}^k S_i.$$
- For a fixed $i \in [k]$, S_i intersects at most one of $\cup_i \Omega \circ F(U_{T_i})$ and $\cup_{i'} U_{T'_{i'}}$.
- Distance between S_i and $S_{i'}$ for $i \neq i'$ is at least $L_j^{7/2}$.
- The distance between the boundary of S_i and the sets $F(U_{T_\ell})$ or $U_{T'_{\ell'}}$ contained in it is at least $L_j^{7/2}$.

See Figure 4.10 for the above construction. For $(h_1, h_2) \in [L_j^2]^2$ We shall construct functions $\rho_{h_1,h_2} : \widetilde{U}_2 \to \widetilde{U}_2$ such that each ρ_{h_1,h_2} is identity except on the interior of $\cup_i S_i$. Eventually we shall show that $\Upsilon_{h_1,h_2} := \rho_{h_1,h_2} \circ \Omega \circ F$ will be the sequence of α-canonical maps satisfying the conditions in the statement of the proposition. Without loss of generality we describe below how to construct ρ_{h_1,h_2} on S_1, similar constructions work for the other S_i.

Now consider S_1. Without loss of generality assume that S_1 contains only $\Omega \circ F(U_{T_1})$, more general cases can be handled in a similar manner. Fix W, a translate of T_1 such that the distance of $F(U_{T_1})$ from B_W^j is at most $2L_j$. For $h = (h_1, h_2)$ set $W_h = (h_1, h_2) + W$. Let the domain corresponding to the multi-cell $B_{W_h}^j$ be denoted by $\widetilde{W}_h$. Let G_h denote the canonical map from U_{T_1} to $\widetilde{W}_h$. On $\Omega \circ F(U_{T_1})$, set $\rho_h := G_h \circ F^{-1} \circ \Omega^{-1}$. Clearly ρ_h is bi-Lipschitz with Lipschitz constant $1 + 10^{-(j+7)}$, also since $F(U_{T_1})$ and $\widetilde{W}_h$ are sufficiently far from the boundary of S_1, it follows that ρ_h can be extended to S_1 in such a way that ρ_h is bi-Lipschitz on S_1 with Lipschitz constant $1 + 10^{-(j+7)}$ and is identity on the boundary of S_1. See Figure 4.11.

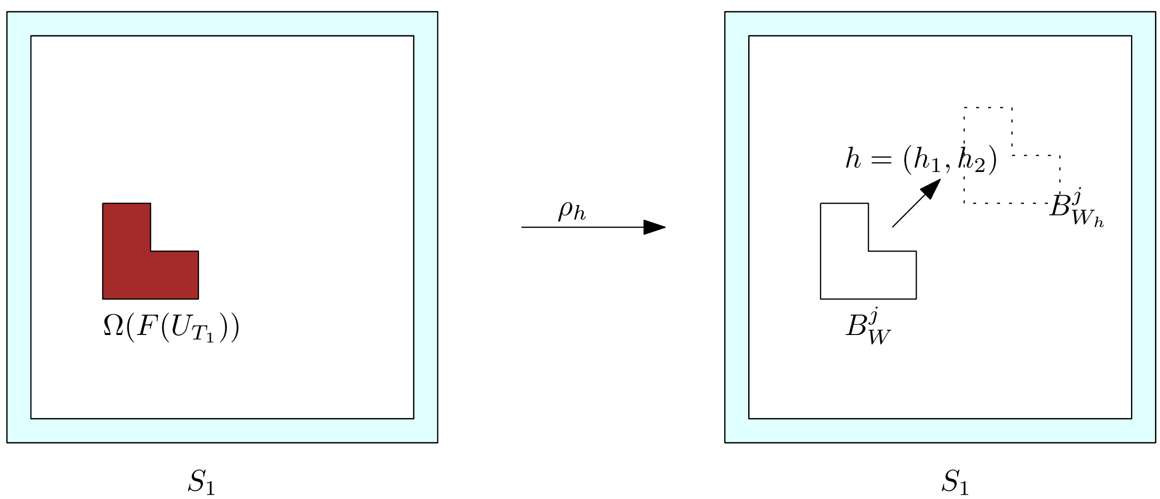


Figure 4.11: Construction of ρ_h on S_1 in the proof of Proposition 4.4.3

It is now easy to check that Υ_h as defined above does indeed produce a sequence of α-canonical maps satisfying the conditions in the proposition. This completes the proof. □

Finally we want to construct α-canonical maps that match up bad sub-components and ensures that interior of the corresponding multi-cell is mapped into the interior of the multi-cell itself. This property is needed to make sure certain boundaries are valid; see Lemma 4.5.8. We have the following proposition.

Proposition 4.4.3. *Fix $U \subseteq \mathbb{Z}^2$. Let $\widetilde{U}_1$ be any potential domain corresponding to the $j+1$-level multi-cell B_U^{j+1}. Let $U_1 \subseteq \mathbb{Z}^2$ be such that $B_{U_1}^j$ denotes the collection of all level j-cells that intersect $\widetilde{U}_1$. Let $U_2 \subseteq U_1$ be the set of all vertices in U_1 such that the distance from the boundary of U_1 is at least $\frac{L_j^3}{2}$. Let $\mathcal{T} = \{T_1, T_2, \ldots T_{\ell_1}\}$ be a set of disjoint and non-neighbouring subsets of U_2 such that $\sum |T_i| \leq v_0 k_0$. Let $U_3 \subseteq U_2$ be such that $B_{U_3}^j = B_U^{j+1,int}$. Then there exists a sequence of α-canonical maps $\{\Upsilon_{h_1,h_2}\}_{(h_1,h_2)\in[L_j^2]^2}$ from $\widetilde{U}_1$ to B_U^{j+1} with respect to $\mathcal{T}$ and $\emptyset$ satisfying the following conditions.*

i. For each $i \in [\ell_1]$, $\Upsilon_{h_1,h_2}(T_i) = (h_1-1, h_2-1) + \Upsilon_{1,1}(T_i)$.

ii. For T_i not contained in $U_2 \setminus U_3$, and for all $h = (h_1, h_2)$ we have $\Upsilon_h(T_i) \subseteq U_3$.

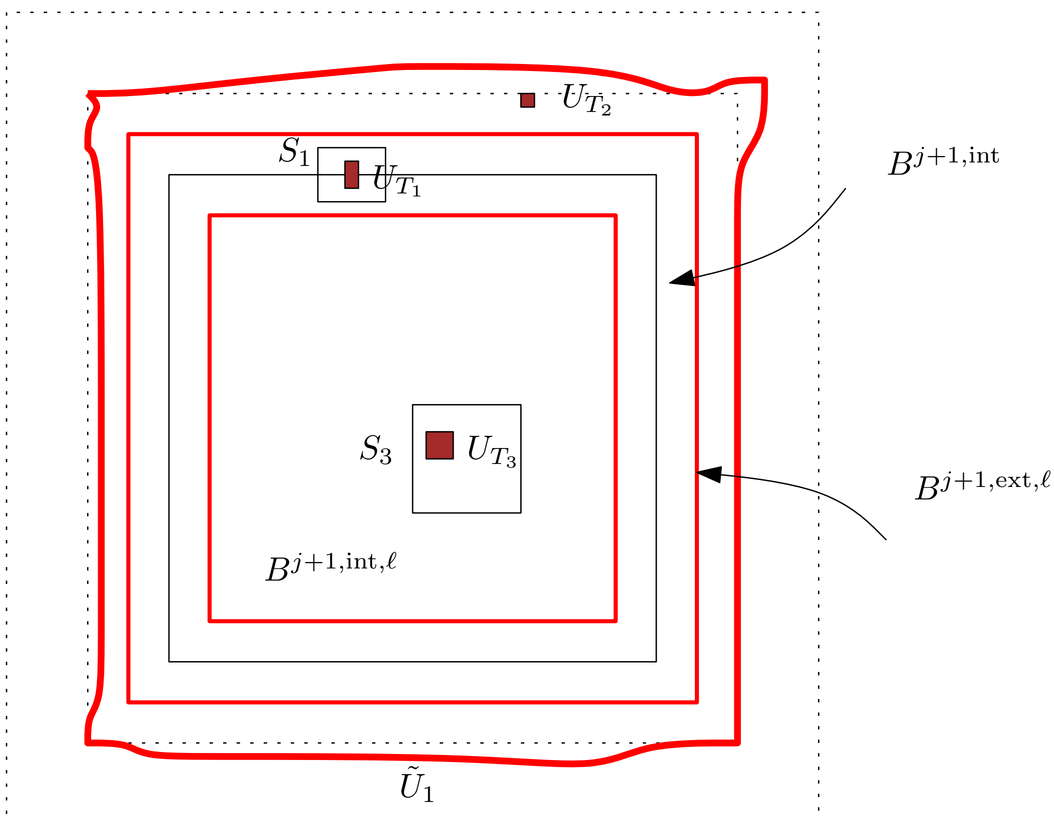


Figure 4.12: Construction of S_i as described in the proof of Proposition 4.4.3

Proof. This proof is similar to the proof of Proposition 4.4.2 except that we have to do some extra work to ensure condition ii. above. We use the same notations for domains as in the proof of Proposition 4.4.2. Let F be the canonical map that takes $\widetilde{U}_1$ to B_U^{j+1}. Without loss of generality, we take U to be the singleton $\{\mathbf{0}\}$. Define the squares $B^{j+1,\text{int},\ell} = [L_j^5 + \ell L_j^4, L_{j+1} - L_j^5 - \ell L_j^4]^2$ and $B^{j+1,\text{ext},\ell} = [L_j^5 - \ell L_j^4, L_{j+1} - L_j^5 + \ell L_j^4]^2$ such that $0 < \ell < 100k_0 v_0$ and such that the distance of $F(U_{T_i})$'s from the boundaries of $B^{j+1,\text{int},\ell}$ and $B^{j+1,\text{ext},\ell}$ is at least L_j^4. See Figure 4.12. Observe that by construction of canonical maps F is identity on $B^{j+1,\text{ext},\ell}$. Now as in the proof of Proposition 4.4.2, it is not hard to see that there exist squares $S_1, S_2, \ldots S_k \subseteq \widetilde{U}_2$ with the following properties.

- We have that
$$\cup_i F(U_{T_i}) \subseteq \cup_{i=1}^k S_i.$$
- The distance between the boundaries of S_i and the sets $F(U_{T_\ell})$ contained in it is at least $L_j^{7/2}$.
- The distance between S_i and the boundaries of $B^{j+1,\text{int},\ell}$ and $B^{j+1,\text{ext},\ell}$ is at least $L_j^{7/2}$.

For $h = (h_1, h_2) \in [L_j^2]^2$, as before our strategy is to construct ρ_h that is identity except on the interiors of S_i and such that $\Upsilon_h = \rho_h \circ F$ is an α-canonical map satisfying the conditions of the proposition. We construct ρ_h separately on squares S_i. If $S_i \subseteq \widetilde{U}_2 \setminus B^{j+1,\text{ext},\ell}$ or $S_i \subseteq B^{j+1,\text{int},\ell}$ then the construction of ρ_h proceeds as in the proof of Proposition 4.4.2. We specify below the changes we need to consider if $S_i \subseteq B^{j+1,\text{ext},\ell} \setminus B^{j+1,\text{int},\ell}$. Without loss of generality take $S_1 \subseteq B^{j+1,\text{ext},\ell} \setminus B^{j+1,\text{int},\ell}$ and also that U_{T_1} is the only one (among U_{T_i}'s) that is contained in S_1.

Recall that we only need to worry about condition ii. in the statement of the proposition being violated if T_1 is not contained in $U_2 \setminus U_3$. Let us assume that to be the case. Notice that the assumptions on S_1 and U_{T_1} implies that there exists W which is a translate of T_1 such that B_W^j has distance at most $10L_j^3$ from $U_{T_1} = F(U_{T_1})$ for all $h \in [L_j^2]^2$ and $W_h := h + W$ we have that $W_h \subseteq U_3$. See Figure 4.13. With this choice of W, construct ρ_h exactly as in the proof of Proposition 4.4.2 and it is easy to verify that $\Upsilon_h = \rho_h \circ F$ satisfies the conclusion of the proposition. This completes the proof. □

4.5 Tail Estimates

The most important of our inductive hypotheses is the following recursive estimate. Let $X = X_U^{j+1}$ and $Y = Y_{U'}^{j+1}$ be random $(j+1)$-level components in $\mathbb{X}$ and $\mathbb{Y}$ having laws $\mu_{j+1}^{\mathbb{X}}$ and $\mu_{j+1}^{\mathbb{Y}}$ respectively. Let V_X, V_Y denote the sizes of X and Y respectively. We have the following theorem establishing (4.3.1) and (4.3.2) at level $j+1$.

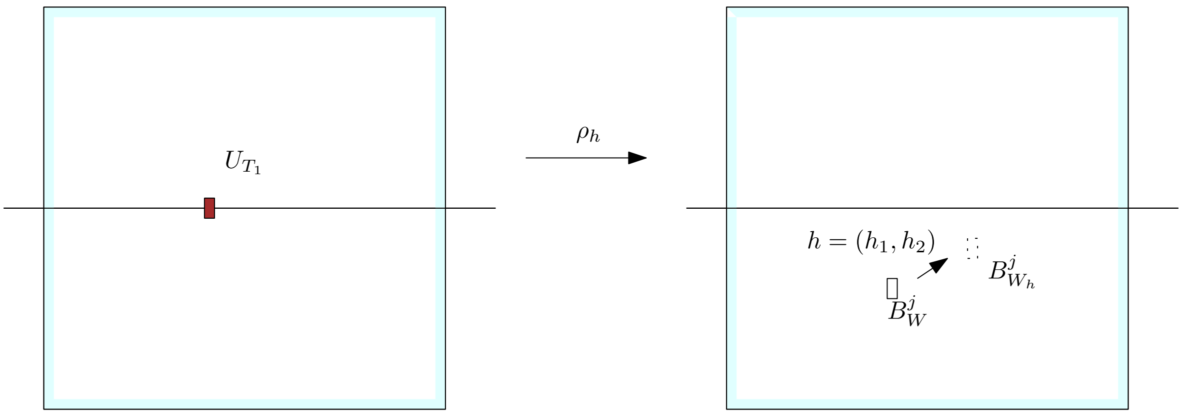


Figure 4.13: Construction of ρ_h on S_1 in the proof of Proposition 4.4.3

Theorem 4.5.1. *In the above set-up, we have for all $v \geq 1$ and all $p \leq 1 - L_{j+1}^{-1}$*

$$\mathbb{P}(S_{j+1}^{\mathbb{X}}(X) \leq p, V_X \geq v) \leq p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)};$$
$$\mathbb{P}(S_{j+1}^{\mathbb{Y}}(Y) \leq p, V_Y \geq v) \leq p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)}$$

where $m_{j+1} = m + 2^{-(j+1)}$.

Due to an obvious symmetry between our X and Y bounds, we shall state all our bounds in terms of X and $S_{j+1}^{\mathbb{X}}$ but will similarly hold for Y and $S_{j+1}^{\mathbb{Y}}$. We shall drop the superscript $\mathbb{X}$ for the rest of this section.

As a consequence of translation invariance we can assume without loss of generality that $X = X_U^{j+1} = X^{*,j+1}(\mathbf{0})$, i.e., U is the lattice component containing the origin. Let U_X denote the domain of X. Also let $\widetilde{U} \subseteq \mathbb{Z}^2$ be such that $X_{\widetilde{U}}^j = X_U^{j+1} = X$. Let $X_{U_1}^j, X_{U_2}^j$, $X_{\widetilde{U}_{N_X}}^j$ denote the j-level bad subcomponents of X. Let K_X denote the total size of the bad subcomponents, i.e., $K_X = \sum_{i=1}^{N_X} |U_i|$. Our first order of business is to obtain a bound on the probability that a component X has either large V_X, or large K_X or small $\prod_{i=1}^{N_X} S_j(X_{U_i}^j)$. The following lemma is the key estimate of the chapter.

Lemma 4.5.2. *Let X be as above. For all $v' \geq 1, k, x \geq 0$ we have that*

$$\mathbb{P}\left[V_X \geq v', K_X \geq k, -\log \prod_{i=1}^{N_X} S_j(X_{U_i}^j) > x\right] \leq 500 L_j^{-\gamma k/10} \exp(-x m_{j+1}) L_{j+1}^{-9\gamma(v'-1)}.$$

For brevity of notation we shall write $S^*(X) = \prod_{i=1}^{N_X} S_j(X^j_{U_i})$. For $v \geq 1$, let $\mathcal{H}_v$ denote the set of all lattice animals of size v containing $\mathbf{0}$. Clearly, we have

$$\mathbb{P}\left[V_X \geq v', K_X \geq k, -\log S^*(X) > x\right] = \sum_{v=v'}^{\infty} \sum_{H \in \mathcal{H}_v} \mathbb{P}\left[U = H, K_X \geq k, -\log S^*(X) > x\right]. \tag{4.5.1}$$

To begin with, let us analyse the event $\{U = H\}$. Let $\hat{W} \subseteq \mathbb{Z}^2$ be such that $B^j_{\hat{W}} = B^{j+1,\mathrm{ext}}_H$, i.e., $\hat{W}$ corresponds to the j-level cells contained in the blow-up of the level $(j+1)$ ideal multi-block B^{j+1}_H. Observe that on $\{U = H\}$, there exists a subset $H^* \subseteq H$ with at least $\lceil \frac{v}{25} \rceil$ vertices that are non neighbouring (in the closed packed lattice of $\mathbb{Z}^2$) and such that for all $h \in H^*$, the ideal multi-block containing B^{j+1}_H must correspond to a bad block. Hence, for each $h \in H^*$, one of the following there events must hold for the cell $B^{j+1}(h)$: (a) it has a conjoined buffer zone or the total size of j level bad components contained in its blow up is at least k_0, (b) it contains a really bad j-level subblock, (c) it fails the airport condition. Hence at least one of these conditions must must hold for at least $\frac{v}{75}$ many $(j+1)$-level cells among the cells corresponding to the vertices of H^*. Hence

$$\{U = H\} \subseteq A_1 \cup A_2 \cup A_3$$

where A_i are defined as follows.

- Let A_1 denote the event that total size of j-level bad components contained in $X^j_{\hat{W}}$ is at least $\frac{k_0 v}{75}$.
- Let A_2 denote the event that the total number of really bad components contained in $X^j_{\hat{W}}$ is at least $\frac{v}{75}$.
- Finally let A_3 denote the event that there exists a subset $H' \subseteq H$ of non-neighbouring vertices with $|H'| = \frac{v}{75}$ such that $\cap_{h \in H'} S_h$ holds where S_h is the following event. For $h \in H'$, let G_h be such that $B^j_{G_h} = B^{j+1,\mathrm{ext}}_h$. Then S_h denotes the event that the following two conditions hold.
    i. The total size of bad components at level j contained in $B^j_{G_h}$ is at most k_0.
    ii. There exists a square $S \subseteq G_h$ of size $L_j^{3/2}$ such that B^j_S is not an airport at level j.

Fix $v \geq v'$ and $H \in \mathcal{H}_v$ for now. The corresponding term in the right hand side of (4.5.1) can be upper bounded by

$$\sum_{i=1}^{3} \mathbb{P}[-\log S^*(X) > x, K_X \geq k, A_i].$$

We shall treat the three cases separately.

Lemma 4.5.3. *In the above set-up, we have*

$$\mathbb{P}[-\log S^*(X) > x, K_X \geq k, A_1] \leq 2\exp(-xm_{j+1})L_j^{-\gamma k/10}L_{j+1}^{-10\gamma(v-1)}.$$

Proof. Fix $k' \geq k$. Fix a collection $\mathcal{T}_{k'} = \{T_1, T_2, \cdots, T_n\}$ of non-neighbouring subsets of $\hat{W}$ with $\sum_i |T_i| = k'$. Let $\mathcal{F}_{\mathcal{T}_{k'}}$ denote the event that $X^j_{T_i}$ is a j-level bad component of $\mathbb{X}$ for each i. It follows that we have

$$\mathbb{P}\left[U = H, K_X \geq k, -\log S^*(X) > x\right] \leq \sum_{k'=k}^{\infty}\sum_{\mathcal{T}_{k'}}\mathbb{P}[-\log\prod_{i=1}^{n} S^j(X^j_{T_i}) > x, \mathcal{F}_{\mathcal{T}_{k'}}, U = H]. \tag{4.5.2}$$

Notice that on the event $\mathcal{F}_{\mathcal{T}_{k'}}$, $X^j_{T_i}$ are independent. Observe that on A_1, we have $K_X \geq \frac{k_0 v}{24}$. Now fix $\mathcal{T}_{k'}$. Set $t_i = |T_i|$. Let $\mathscr{V}_i, i = 1, 2, \ldots, n$ be a sequence of independent random variables with $\mathrm{Ber}(L_j^{-\gamma t_i/2})$ distribution. Let $\mathscr{R}_i, i = 1, 2, \ldots, n$ be a sequence of i.i.d. $\exp(m_j)$ random variables independent of $\{\mathscr{V}_i\}$. It follows from the recursive estimates that

$$-\log S_j(X^j_{T_i})1_{\{X^j_{T_i} \text{ bad component}\}} \preceq \mathscr{V}_i(1 + \mathscr{R}_i)$$

for all i where $\preceq$ denotes stochastic domination. It follows that

$$\begin{aligned}\mathbb{P}[-\log\prod_{i=1}^{n} S_j(X^j_{T_i}) > x, \mathcal{F}_{\mathcal{T}_{k'}}, A_1] &\leq \mathbb{P}[\mathscr{V}_i = 1 \forall i]\mathbb{P}[\sum_{i=1}^{n}(1 + \mathscr{R}_i) > x] \\ &\leq L_j^{-\gamma k'/2}\mathbb{P}[\sum_{i=1}^{n}\mathscr{R}_i > x - n]. \end{aligned} \tag{4.5.3}$$

Now observe that $\sum_{i=1}^{n}\mathscr{R}_i$ has a Gamma(n, m_j) distribution and hence

$$\mathbb{P}[\sum_{i=1}^{n}\mathscr{R}_i > x - n] = \int_{(x-n)\vee 0}^{\infty}\frac{m_j^n}{(n-1)!}y^{n-1}\exp(-ym_j)dy. \tag{4.5.4}$$

Following the proof of Lemma 7.3 in [8] it follows from this that

$$\mathbb{P}[\sum_{i=1}^{n}\mathscr{R}_i > x - n] \leq (m_j 2^{j+1}e^{m_{j+1}})^n\exp(-xm_{j+1}). \tag{4.5.5}$$

Since L_j grows doubly exponentially and $n \leq k'$ and $k' > vk_0/24$ it follows from (4.5.3) that for L_0 sufficiently large we have

$$\mathbb{P}[-\log\prod_{i=1}^{n} S_j(X_{T_i}) > x, \mathcal{F}_{\mathcal{T}_{k'}}, A_1] \leq L_j^{-\gamma k'/4}\exp(-xm_{j+1}) \leq L_j^{-\gamma k'/8}\exp(-xm_{j+1})L_{j+1}^{-10\gamma(v-1)} \tag{4.5.6}$$

as $k_0 > 6000\alpha\gamma$.

Now notice that total number of choices for $\mathcal{T}_{k'}$ is bounded by $16^v 3^{k'} L_{j+1}^{k'}$ hence summing over all such choices and then summing over all k' from k to ∞ we get the desired result as $\gamma > 40\alpha$ and L_j is sufficiently large. $\square$

Lemma 4.5.4. *In the set-up of Lemma 4.5.3, we have*

$$\mathbb{P}[-\log S^*(X) > x, K_X \geq k, A_2] \leq \exp(-xm_{j+1}) L_j^{-\gamma k/10} L_{j+1}^{-10\gamma(v-1)}.$$

Proof. Fix $k' \geq k$ and $\mathcal{T}_{k'}$ as in the proof of Lemma 4.5.4. Fix a subset $\mathcal{N}$ of $[n]$ with $|\mathcal{N}| = \frac{v}{24}$. Now, for $i \in \mathcal{N}$, $X^j_{T_i}$ can be a really bad component in one of two ways: (a) $t_{i_\ell} \geq v_0$ and (b) $S^j(X_{T_{i_\ell}}) \leq 1 - L_j^{-1}$. Observe that it follows from the recursive estimates that for all $i \in \mathcal{N}$ we have

$$-\log S_j(X^j_{T_i}) 1_{\{X^j_{T_i} \text{ really bad component}\}} \preceq \mathscr{W}_i(1 + \mathscr{R}_i)$$

where $\{\mathscr{W}_i\}_{i \in \mathcal{N}}$ is a sequence of i.i.d. $\mathrm{Ber}(L_j^{-\gamma t_i/4 - (\gamma v_0/4 \wedge \beta/2)})$ distribution. It follows that

$$\begin{aligned}
\mathbb{P}[-\log \prod_{i=1}^n S_j(X^j_{T_i}) > x, \mathcal{F}_{\mathcal{T}_{k'}}, A_2] &\leq \sum_{\mathcal{N}} \mathbb{P}[\mathscr{V}_i = 1 \forall i \in [n] \setminus \mathcal{N}, \mathscr{W}_i = 1 \forall i \in \mathcal{N}] \\
&\quad \times \mathbb{P}[\sum_{i=1}^n (1 + \mathscr{R}_i) > x] \\
&\leq \sum_{\mathcal{N}} L_j^{-\gamma k'/4 - \frac{v}{300}(\gamma v_0 \wedge 2\beta)} \mathbb{P}[\sum_{i=1}^n \mathscr{R}_i > x - n] \\
&\leq \binom{k'}{\frac{v}{75}} L_j^{-\gamma k'/4 - \frac{v}{300}(\gamma v_0 \wedge 2\beta)} \mathbb{P}[\sum_{i=1}^n U_i > x - n]. \qquad (4.5.7)
\end{aligned}$$

Doing the same calculations as in the proof of Lemma 4.5.3, we obtain that

$$\begin{aligned}
\mathbb{P}[-\log \prod_{i=1}^k S_j(X^j_{T_i}) > x, \mathcal{F}_{\mathcal{T}_{k'}}, A_2] &\leq 2^{k'} L_j^{-\gamma k'/5} \exp(-xm_{j+1}) L_j^{-\frac{v}{300}(\gamma v_0 \wedge 2\beta)} \\
&\leq L_j^{-\gamma k'/8} \exp(-xm_{j+1}) L_{j+1}^{-10\gamma(v-1)} \qquad (4.5.8)
\end{aligned}$$

since $\gamma v_0 \wedge 2\beta > 3000\alpha\gamma$.

As before, summing over all $\mathcal{T}_{k'}$ and k' from k to ∞ gives the result. □

Lemma 4.5.5. *In the set-up of Lemma 4.5.3, we have*

$$\mathbb{P}[-\log S^*(X) > x, K_X \geq k, A_3] \leq \exp(-xm_{j+1}) L_j^{-\gamma k/10} L_{j+1}^{-10\gamma(v-1)}.$$

Proof. First fix $H' \subseteq H$ as in the definition of A_3. Fix $k' \geq k$ and $\mathcal{T}_{k'}$ as before. Now fix $h \in H'$. Set $I_h = (\cup_{i=1}^n T_i) \cap G_h$ and observe that by hypothesis $|I_h| \leq k_0$. It is not too hard to see that there exists an event S'_h such that $S_h \subseteq S'_h$ and S'_h is independent of $X^j_{I_h}$ and $P[S'_h] \leq L_j^{-10\beta}$. Indeed, that a square is an airport can be verified, even without checking a limited number of cells, and this can be established using arguments identical to the proof of Lemma 4.7.5, we omit the details.

Repeating the same calculations as in the proofs of Lemma 4.5.3 and Lemma 4.5.4 it then follows that

$$\begin{aligned}\mathbb{P}[-\log S^*(X) > x, K_X \geq k, A_3] &\leq \sum_{H'}\sum_{k'=k}^{\infty}\sum_{\mathcal{T}_{k'}} \mathbb{P}[\mathcal{F}_{\mathcal{T}_{k'}}]L_j^{-10\beta v/75} \\ &\leq 2\binom{v}{\frac{v}{75}}\exp(-xm_{j+1})L_j^{-\gamma k/10}L_j^{-10\beta v/75} \\ &\leq \exp(-xm_{j+1})L_j^{-\gamma k/10}L_{j+1}^{-10\gamma(v-1)}\end{aligned}$$

as L_j is sufficiently large and $\beta > 75\alpha\gamma$.

□

Putting together all the cases we are now ready to prove Lemma 4.5.2.

Proof of Lemma 4.5.2. Notice that for a fixed v, we have $|\mathcal{H}_v| \leq 8^v$. We now get from (4.5.1), Lemmas 4.5.3, 4.5.4, 4.5.5 by summing over all $H \in \mathcal{H}_v$ and then finally summing over all v from v' to ∞ that

$$\begin{aligned}\mathbb{P}\left[V_X \geq v', K_X \geq k, -\log S^*(X) > x\right] &\leq \sum_{v=v'}^{\infty} 8^v 50 L_j^{-\gamma k/10}\exp(-xm_{j+1})L_{j+1}^{-10\gamma(v-1)} \\ &\leq 500\exp(-xm_{j+1})L_j^{-\gamma k/10}L_{j+1}^{-9\gamma(v'-1)},\end{aligned}$$

this completes the proof of the lemma. □

We now move to the proof of Theorem 4.5.1. Our proof will be divided into four cases depending on the size of X, the total size of its bad components and how bad the bad components are. In each one we will use different α-canonical map or maps to get good lower bounds on the probability that $X = X_U^{j+1} \hookrightarrow Y_U^{j+1}$. We now present our four cases.

4.5.1 Case 1

The first case is the generic situation where the components are of small size, have small total size of bad sub-components whose embedding probabilities are not too small. For a $(j+1)$-level component X, let N_X denote the number of bad j level components contained in X and let $X_{T_1}^j, X_{T_2}^j, \ldots, X_{T_{N_X}}^j$ denote the bad subcomponents. Let $K_X = \sum_{i=1}^{N_X}|T_i|$ denote the total size of bad subcomponents in X. We define the class of blocks $\mathcal{A}_{X,j+1}^{(1)}$ as

$$\mathcal{A}_{X,j+1}^{(1)} := \left\{X : V_X \leq v_0, K_X \leq k_0 v_0, \prod_{i=1}^{N_X} S_j(X_{T_i}^j) \geq L_j^{-1/3}\right\}.$$

First we show that this case holds with extremely high probability.

Lemma 4.5.6. *The probability that* $X \in \mathcal{A}^{(1)}_{X,j+1}$ *is bounded below by*

$$\mathbb{P}[X \notin \mathcal{A}^{(1)}_{X,j+1}] \le L_{j+1}^{-3\beta} L_{j+1}^{-\gamma(v_0-1)}.$$

Proof. This follows from Lemma 4.5.2 by noting $8\gamma(v_0-1) > 3\alpha\beta$, $m \ge 9\alpha\beta + 3\alpha\gamma v_0$ and $\gamma k_0 v_0 > 300\alpha\beta + \alpha\gamma v_0$. We omit the details. □

Next we show that $S_{j+1}(X)$ is at least 1/2 for all $X \in \mathcal{A}^{(1)}_{X,j+1}$.

Lemma 4.5.7. *Condition on* $X = X^{j+1}_U \in \mathcal{A}^{(1)}_{X,j+1}$ *where* $U \subseteq \mathbb{Z}^2$ *and* $|U| \le v_0$. *Let the bad* j *level components of* X *be* $X^j_{T_1}, X^j_{T_2}, \ldots, X^j_{T_{N_X}}$ *such that* $\sum_{i=1}^{N_X} |T_i| \le v_0 k_0$. *Then we have*

$$S_{j+1}(X) \ge \frac{1}{2}.$$

Proof. Let U_X denote the domain of X, C_X denote the boundary of X, and let $\widetilde{U} \subseteq \mathbb{Z}^2$ be such that $X = X^j_{\widetilde{U}}$. By Proposition 4.4.2, there exist L_j^4 α-canonical maps at $j+1$-th level $\{\Upsilon^{j+1}_h = \Upsilon_h : h \in [L_j^2]^2\}$ from U_C to B^{j+1} with respect to $\mathcal{T} = T_1, T_2, \ldots, T_{K_X}$ and $\emptyset$ such that $\Upsilon_h(T_i)$ are different for all $h \in [L_j^2]^2$.

Clearly there exists a subset $\mathcal{H} \subset [L_j^2]^2$ with $|\mathcal{H}| = L_j < \lfloor L_j^4/100 v_0^4 k_0^4 \rfloor$ so that for all $i_1 \ne i_2$ and $h_1, h_2 \in \mathcal{H}$ we have that $\Upsilon_{h_1}(T_{i_1})$ and $\Upsilon_{h_2}(T_{i_2})$ are disjoint and non-neighbouring. We will estimate the probability that one of these maps work.

For $h \in \mathcal{H}$ and $i \in [N_X]$, let $\mathcal{D}^i_h$ denote the event

$$\mathcal{D}^i_h = \left\{ Y^j_{\Upsilon_h(T_i)} \text{ valid}, X^j_{T_i} \hookrightarrow Y^j_{\Upsilon_h(T_i)} \right\}.$$

Since we are only trying out non-neighbouring components, these events are conditionally independent given X and setting

$$\mathcal{D}_h = \bigcap_{1 \le i \le N_X} \mathcal{D}^i_h$$

we get

$$\mathbb{P}[\mathcal{D}_h \mid X] = \prod_{i=1}^{N_X} S_j(X_{T_i}) \ge L_j^{-1/3}.$$

By construction of $\mathcal{H}$, we also get that $\{\mathcal{D}_h : h \in \mathcal{H}\}$ are mutually independent given X and hence setting $\mathcal{D} = \cup_{h \in \mathcal{H}} \mathcal{D}_h$ we have

$$\mathbb{P}[\mathcal{D} \mid X] \ge 1 - (1 - L_j^{-1/3})^{L_j} \ge 1 - L_{j+1}^{-3\beta}. \tag{4.5.9}$$

Let $\mathcal{B}$ denote the event that Y^{j+1}_U is valid and has domain B^{j+1}_U. Let $\widetilde{U}_2 \subseteq \mathbb{Z}^2$ be such that $\widetilde{U}_2$ corresponds to the j-level cells contained in the blow-up of B^{j+1}_U, i.e., $B^j_{\widetilde{U}_2} = B^{j+1,\text{ext}}_U$. Let $\mathcal{J}$ denote the event

$$\mathcal{J} = \left\{ Y^j(\ell) \text{ is good for all } \ell \in \widetilde{U}_2 \right\}.$$

By Lemma 4.2.18 on $\mathcal{D}\cap\mathcal{J}\cap\mathcal{B}$, there exists an embedding and hence $S_{j+1}(X) \geq \mathbb{P}[\mathcal{D}\cap\mathcal{J}\cap\mathcal{B} \mid X]$. Using (4.3.7) at level j we get

$$\mathbb{P}[\mathcal{J} \mid X] \geq \left(1 - L_j^{-\gamma}\right)^{4v_0 L_j^{2\alpha-2}} \geq 9/10 \tag{4.5.10}$$

as $\gamma > 2\alpha$ and L_j is sufficiently large. Notice that on $\mathcal{J}$, Y_U^{j+1} is valid and B_U^{j+1} is a valid potential domain of Y_U^{j+1} and by construction

$$\mathbb{P}[\mathcal{B}\cap\mathcal{J} \mid X] \geq \mathbb{P}[\mathcal{J} \mid X]\mathbb{P}[\mathcal{B} \mid \mathcal{J}, X] \geq \frac{9}{10}(1-10^{-(j+10)})^{4v_0} \geq \frac{3}{5}. \tag{4.5.11}$$

The lemma follows from (4.5.9) and (4.5.11) since L_0 is sufficiently large. □

Now to improve upon the above estimate, we want to relax the condition that Y does not contain any bad-components by weaker conditions that define a generic block. We proceed as follows.

Let $\mathcal{S} = \mathcal{S}_{v_0}$ denote the set of all lattice animals containing the set $\{0\}$ and having size at most v_0. For $S \in \mathcal{S}_{v_0}$, let $\mathcal{U}_S$ denote the set of all potential domains for X_S^{j+1} (or Y_S^{j+1}). Also set $S_0 = \cup_{S\in\mathcal{S}} S$. Let $T_1, T_2, \ldots, T_{N_X}$ be subsets of $\mathbb{Z}^2$ such that $\{X_{T_i}^j : i \in [N_X]\}$ are the j-level bad subcomponents in $X_{S_0}^{j+1}$. Similarly let $T_1', T_2', \ldots, T_{N_Y'}'$ be subsets of $\mathbb{Z}^2$ such that $\{Y_{T_i'}^j : i \in [N_Y']\}$ denote the j-level bad subcomponents in $Y_{S_0}^{j+1}$. Fix $S \in \mathcal{S}$ and $\widetilde{U} \in \mathcal{U}_S$. Let $\widetilde{U} \subseteq \mathbb{Z}^2$ denote the set such that $X_S^{j+1} = X_{\hat{U}}^j$ on the event that $\widetilde{U}$ is the domain of X_S^{j+1}. Let $B_{\widetilde{U},X} = B_{\widetilde{U},S,X} = \{i \in [N_X] : T_i \subseteq \hat{U}\}$ and let $B_{\widetilde{U},Y}$ be defined similarly. Let

$$\mathcal{E}_{\widetilde{U},X} = \left\{ X_S^{j+1} \text{ valid}, \widetilde{U} \text{ valid}, \sum_{i\in B_{\widetilde{U},X}} |T_i| \leq k_0 v_0, \prod_{i\in B_{\widetilde{U},X}} S_j(X_{T_i}^j) \geq L_j^{-1/3} \right\}$$

and

$$\mathcal{E}_{\widetilde{U},Y} = \left\{ Y_S^{j+1} \text{ valid}, \widetilde{U} \text{ valid}, \sum_{i\in B_{\widetilde{U},Y}} |T_i'| \leq k_0 v_0, \prod_{i\in B_{\widetilde{U},Y}} S_j(Y_{T_i'}^j) \geq L_j^{-1/3} \right\}.$$

Finally let $\mathcal{B}_{\widetilde{U},X}$ (resp. $\mathcal{B}_{\widetilde{U},Y}$) denote the event that the domain of X_S^{j+1} (resp. Y_S^{j+1}) is $\widetilde{U}$. We have the following lemma.

Lemma 4.5.8. *We have that*

$$\sum_{S\in\mathcal{S}} \sum_{\widetilde{U}_1\in\mathcal{U}_S} \sum_{\widetilde{U}_2\in\mathcal{U}_S} \mathbb{P}[X_S^{j+1} \not\hookrightarrow Y_S^{j+1}, \mathcal{E}_{\widetilde{U}_1,X}, \mathcal{E}_{\widetilde{U}_2,Y}, \mathcal{B}_{\widetilde{U}_2,X}, \mathcal{B}_{\widetilde{U}_2,Y}] \leq L_{j+1}^{-3\beta} L_{j+1}^{-\gamma(v_0-1)}. \tag{4.5.12}$$

Proof. Since $|\mathcal{S}| \leq 8^{v_0}$ and for all $S \in \mathcal{S}$ we have $\mathcal{U}_S \leq (8k0)^{16k_0v_0^2}$, it suffices to prove that for each fixed S, $\widetilde{U}_1$ and $\widetilde{U}_2$ we have

$$\mathbb{P}[X_S^{j+1} \not\hookrightarrow Y_S^{j+1}, \mathcal{E}_{\widetilde{U}_1,X}, \mathcal{E}_{\widetilde{U}_2,Y}, \mathcal{B}_{\widetilde{U}_1,X}, \mathcal{B}_{\widetilde{U}_2,Y}] \leq L_{j+1}^{-4\beta} L_{j+1}^{-\gamma(v_0-1)}.$$

Now fix $S \in \mathcal{S}$ and $\widetilde{U}_1, \widetilde{U}_2 \in \mathcal{U}_S$. Notice that the total number of ways it is possible to choose disjoint sets $S_1, S_2, \ldots, S_{\ell_1} \subseteq \widetilde{U}_1$ and $S'_1, S'_2, \ldots, S'_{\ell_2} \subseteq \widetilde{U}_2$ such that $\sum_i |S_i| \leq v_0 k_0$ and $\sum_i |S'_i| \leq k_0 v_0$ is $L_j^{4\alpha k_0 v_0}$. For $\mathscr{S}_1 = \{S_1, S_2, \ldots S_{\ell_1}\}$ and $\mathscr{S}_2 = \{S'_1, S'_2, \ldots S'_{\ell_2}\}$, let

$$\mathscr{I}(\mathscr{S}_1, \mathscr{S}_2) = \left\{\mathscr{S}_1 = \{T_i : i \in B_{\widetilde{U}_1,X}\}, \mathscr{S}_2 = \{T'_i : i \in B_{\widetilde{U}_2,Y}\}\right\}.$$

Clearly then it suffices to show that for each choice of $\mathscr{S}_1$ and $\mathscr{S}_2$ as above, we have

$$\mathbb{P}[X_S^{j+1} \not\hookrightarrow Y_S^{j+1}, \mathcal{E}_{\widetilde{U}_1,X}, \mathcal{E}_{\widetilde{U}_2,Y}, \mathcal{B}_{\widetilde{U}_1,X}, \mathcal{B}_{\widetilde{U}_2,Y}, \mathscr{I}(\mathscr{S}_1, \mathscr{S}_2)] \leq L_{j+1}^{-4\beta-8k_0v_0-\gamma v_0}. \tag{4.5.13}$$

Fix $\mathscr{S}_1$ and $\mathscr{S}_2$ as above. Condition on $\{X_{S_i}^j : S_i \in \mathscr{S}_1\}$ and $\{Y_{S'_i}^j : S'_i \in \mathscr{S}_2\}$, such that they are compatible with $\mathcal{E}_{\widetilde{U}_1,X}$ and $\mathcal{E}_{\widetilde{U}_2,Y}$. Denote this conditioning by $\mathcal{F}$. Observe that, by Proposition 4.4.2, there exist L_j^4 α-canonical maps at $j+1$-th level $\{\Upsilon_h^{j+1} = \Upsilon_h : h \in [L_j^2]^2\}$ from $\widetilde{U}_1$ to $\widetilde{U}_2$ with respect to $\mathscr{S}_1$ and $\mathscr{S}_2$ satisfying the following conditions.

i. $\Upsilon_h(S_i)$ are different for all $h \in [L_j^2]^2$.

ii. $\Upsilon_h^{-1}(S'_i)$ are different for all $h \in [L_j^2]^2$.

iii. $\Upsilon_h(S_{i_1}) \neq S'_{i_2}$ for any i_1, i_2, h.

As before there exists a subset $\mathcal{H} \subset [L_j^2]^2$ with $|\mathcal{H}| = L_j < \lfloor L_j^4/100 v_0^4 k_0^4 \rfloor$ so that the sets $\{\Upsilon_h(S_{i_1}) : h \in \mathcal{H}, S_{i_1} \in \mathscr{S}_1\}$ are disjoint and non-neighbouring and also disjoint and non-neighbouring with the sets in $\mathscr{S}_2$. Also the collection of sets $\{\Upsilon_h^{-1}(S'_{i_2}) : h \in \mathcal{H}, S'_{i_2} \in \mathscr{S}_2\}$ are disjoint and non-neighbouring and also disjoint and non-neighbouring with the sets in $\mathscr{S}_1$.

For $h \in \mathcal{H}$, let $\mathcal{D}_h$ denote the event

$$\mathcal{D}_h = \left\{X_{S_{i_1}}^j \hookrightarrow Y_{\Upsilon_h(S_{i_1})}^j, X_{\Upsilon_h^{-1}(S'_{i_2}}^j \hookrightarrow Y_{S'_{i_2}}^j \ \forall S_{i_1} \in \mathscr{S}_1 \ \forall S'_{i_2} \in \mathscr{S}_2\right\}.$$

Arguing as before we have

$$\mathbb{P}[\mathcal{D}_h \mid \mathcal{F}] = \prod_{S_i \in \mathscr{S}_1} S_j^{\mathbb{X}}(X_{S_i}^j) \prod_{S'_i \in \mathscr{S}_2} S_j^{\mathbb{X}}(X_{S'_i}^j) \geq L_j^{-2/3}.$$

Since these events are independent for $h \in \mathcal{H}$ it follows that

$$\mathbb{P}[\cup_{h \in \mathcal{H}} \mathcal{D}_h \mid \mathcal{F}] \geq 1 - L_{j+1}^{-4\beta-8k_0v_0-\gamma v_0}$$

since L_0 is sufficiently large. Now observing that on

$$\mathcal{B}_{\widetilde{U}_1,X} \cap \mathcal{B}_{\widetilde{U}_2,Y} \cap \mathscr{I}(\mathscr{S}_1 \cap \mathscr{S}_2) \cap \left(\cup_{h \in \mathcal{H}} \mathcal{D}_h\right)$$

we have $X_S^{j+1} \hookrightarrow Y_S^{j+1}$ and removing the conditioning we get (4.5.13) which in turn completes the proof of the lemma. □

Lemma 4.5.9. *When* $\frac{1}{2} \leq p \leq 1 - L_{j+1}^{-1}$

$$\mathbb{P}(S_{j+1}(X) \leq p, V_X \geq v) \leq p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)}.$$

Proof. Clearly it is enough to show that

$$\mathbb{P}[S_{j+1}(X) \leq 1 - L_{j+1}^{-1}, V_X \geq v] \leq 2^{-m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)}. \tag{4.5.14}$$

For $v \geq v_0$ this follows from Lemma 4.5.2 and $8\gamma(v_0 - 1) > \beta$, so it suffices to prove that

$$\mathbb{P}[S_{j+1}(X) \leq 1 - L_{j+1}^{-1}, V_X \leq v_0] \leq 2^{-m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v_0-1)}. \tag{4.5.15}$$

Using Markov's inequality and Lemma 4.5.8 we get that

$$\mathbb{P}[S_{j+1}(X) \leq 1 - L_{j+1}^{-1}, V_X \leq v_0] \leq \mathbb{P}[X \notin \mathcal{A}_{X,j+1}^{(1)}] + L_{j+1} \left(L_{j+1}^{-3\beta} L_{j+1}^{-\gamma(v_0-1)} + \mathbb{P}[\mathcal{E}_Y] \right)$$

where

$$\mathbb{P}[\mathcal{E}_Y] = \sum_{S \in \mathcal{S}} \sum_{\widetilde{U} \in \mathcal{U}_S} \mathbb{P}[(\mathcal{E}_{C,Y})^c].$$

It can be shown as in Lemma 4.5.6 that $\mathbb{P}[(\mathcal{E}_Y)^c] \leq L_{j+1}^{-3\beta} L_{j+1}^{-\gamma(v-1)}$ and this completes the proof of the lemma. □

4.5.2 Case 2

The next case involves components which are not too large and do not contain too many bad sub-components but whose bad sub-components may have very small embedding probabilities. For a $(j+1)$-level component X, let N_X denote the number of bad j level components contained in X and let $X_{T_1}^j, X_{T_2}^j, \ldots, X_{T_{N_X}}^j$ denote the bad subcomponents. Let $K_X = \sum_{i=1}^{N_X} |T_i|$ denote the total size of bad subcomponents in X. We define the class of blocks $\mathcal{A}_{X,j+1}^{(2)}$ as

$$\mathcal{A}_{X,j+1}^{(2)} := \left\{ X : V_X \leq v_0, K_X \leq k_0 v_0, \prod_{i=1}^{N_X} S_j(X_{T_i}^j) \leq L_j^{-1/3} \right\}.$$

Lemma 4.5.10. *Condition on* $X = X_U^{j+1} \in \mathcal{A}_{X,j+1}^{(2)}$ *where* $|U| \leq v_0$. *Let the bad* j *level components of* X *be* $X_{T_1}^j, X_{T_2}^j, \ldots, X_{T_{N_X}}^j$ *such that* $\sum_{i=1}^{N_X} |T_i| \leq v_0 k_0$. *Then we have*

$$S_{j+1}(X) \geq \min \left\{ \frac{1}{2}, \frac{1}{10} L_j \prod_{i=1}^{N_X} S_j(X_{T_i}^j) \right\}.$$

Proof. Let us make some notations first. Let U_X denote the domain of X. Let $\widetilde{U} \subseteq \mathbb{Z}^2$ be such that $X = X^j_{\widetilde{U}}$. Let $\widetilde{U}_1 \subseteq \widetilde{U}_0 \subseteq \widetilde{U}_2 \subseteq \mathbb{Z}^2$ be defined as follows.

$$B^j_{\widetilde{U}_0} = B^{j+1}_U;$$

$$B^j_{\widetilde{U}_1} = B^{j+1,\text{int}}_U;$$

$$B^j_{\widetilde{U}_2} = B^{j+1,\text{ext}}_U.$$

Now by Proposition 4.4.3, there exist L_j^4 α-canonical maps at $j+1$-th level $\{\Upsilon_h^{j+1} = \Upsilon_h : h \in [L_j^2]^2\}$ from U_X to B_U^{j+1} with respect to $\mathcal{T} = \{T_1, T_2, \ldots, T_{N_X}\}$ satisfying the following conditions:

i. For each $i \in [N_X]$ such that T_i is not contained in $\widetilde{U}_2 \setminus \widetilde{U}_1$ and for all $h \in [L_j^2]^2$ we have $\Upsilon_h(T_i) \subseteq \widetilde{U}_1$.

ii. For all $h \in [L_j^2]^2$ and for all $i \in [N_X]$ we have $\Upsilon_h(T_i)$ is at least at a distance L_j^3 from the boundaries of U_0.

iii. $\Upsilon_h(T_i)$ are different for all $h \in [L_j^2]^2$.

It is easy to see that there exists a subset $\mathcal{H} \subset [L_j^2]^2$ with $|\mathcal{H}| = L_j^{3/2} < \lfloor L_j^4/100 v_0^4 k_0^4 \rfloor$ so that for all $i_1 \neq i_2$ and $h_1, h_2 \in \mathcal{H}$ we have that $\Upsilon_{h_1}(T_{i_1})$ and $\Upsilon_{h_2}(T_{i_2})$ are disjoint and non-neighbouring. We will estimate the probability that one of these maps work.

In trying out these $L_j^{3/2}$ different mappings there is a subtle conditioning issue since a map failing may imply that $Y_{\Upsilon_h(T_i)}$ is not good. As such we condition on an event $\mathcal{D}_h \cup \mathcal{G}_h$ which holds with high probability. For $h \in \mathcal{H}$ and $i \in [N_X]$, let $\mathcal{D}_h^i$ denote the following event. If $T_i \subseteq \widetilde{U}_2 \setminus \widetilde{U}_1$, then

$$\mathcal{D}_h^i = \left\{ X^j_{T_i} \hookrightarrow Y^j_{\Upsilon_h(T_i)}; Y^j_\ell \text{ is good for all } \ell \in T_i \right\}.$$

Otherwise set

$$\mathcal{D}_h^i = \left\{ X^j_{T_i} \hookrightarrow Y^j_{\Upsilon_h(T_i)} \right\}.$$

Define

$$\mathcal{D}_h = \bigcap_{1 \le i \le N_X} \mathcal{D}_h^i.$$

Also let

$$\mathcal{G}_h^i = \left\{ Y^j_\ell \text{ is good for all } \ell \in T_i \right\}$$

and

$$\mathcal{G}_h = \bigcap_{1 \le i \le N_X} \mathcal{G}_h^i.$$

Then using (4.3.7) at level j we get for $h \in \mathcal{H}$

$$\mathbb{P}[\mathcal{D}_h \cup \mathcal{G}_h \mid X] \geq \mathbb{P}[\mathcal{G}_h \mid X] \geq (1 - L_j^{-\gamma})^{k_0 v_0} \geq 1 - 2k_0 v_0 L_j^{-\gamma}.$$

Since $\mathcal{D}_h \cup \mathcal{G}_h, h \in \mathcal{H}$ are conditionally independent given X, we have

$$\mathbb{P}[\cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h) \mid X] \geq (1 - L_j^{-\gamma})^{v_0 k_0 L_j^{3/2}} \geq 9/10 \tag{4.5.16}$$

for L_j sufficiently large. Now

$$\mathbb{P}[\mathcal{D}_h \mid X, (\mathcal{D}_h \cup \mathcal{G}_h)] \geq \mathbb{P}[\mathcal{D}_h \mid X] = \prod_{i=1}^{N_X} \left(\frac{1}{2} \wedge S_j(X_{T_i}^j)\right).$$

Indeed, observe that if $T_i \subseteq \widetilde{U}_2 \setminus \widetilde{U}_1$, X_{T_i} is semi-bad and hence

$$\mathbb{P}[\mathcal{D}_h^i] \geq S_j(X_{T_i}^j) - v_0 L_j^{-\gamma} \geq \frac{1}{2}.$$

Also observe that since none of the multi-blocks that are tried (over all h and $i \in [N_X]$) are non-neighbouring it follows that $\{\mathcal{D}_h : h \in \mathcal{H}\}$ is independent conditionally on X and $\cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)$ and hence

$$\begin{aligned}\mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h \mid X, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)] &\geq 1 - \left(1 - (\frac{1}{2})^{k_0 v_0} \prod_{i=1}^{N_X} S_j(X_{T_i}^j)\right)^{L_j^{3/2}} \\ &\geq \frac{9}{10} \wedge \frac{1}{4} L_j \prod_{i=1}^{N_X} S_j(X_{T_i}^j)\end{aligned} \tag{4.5.17}$$

since $1 - e^{-x} \geq x/4 \wedge 9/10$ for $x \geq 0$ and $L_j^{1/2} > 2^{k_0 v_0}$ for L_j sufficiently large.

Further, set

$$\mathcal{M} = \{\exists h_1 \neq h_2 \in \mathcal{H} : \mathcal{D}_{h_1} \setminus \mathcal{G}_{h_1}, \mathcal{D}_{h_2} \setminus \mathcal{G}_{h_2}\}.$$

We then have

$$\begin{aligned}\mathbb{P}[\mathcal{M} \mid X, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)] &\leq \binom{L_j}{2} \mathbb{P}[\mathcal{D}_h \setminus \mathcal{G}_h \mid X, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)]^2 \\ &\leq \binom{L_j}{2} 2 \left(\prod_{i=1}^{N_X} S_j(X_{T_i}^j) \wedge 2v_0 k_0 L_j^{-\gamma}\right)^2 \\ &\leq 2k_0 v_0 L_j^{-(\gamma-2)} \prod_{i=1}^{N_X} S_j(X_{T_i}^j).\end{aligned} \tag{4.5.18}$$

Finally let $\mathcal{J}$ denote the event

$$\mathcal{J} = \left\{ Y_k^j \text{ is good for all } k \in \widetilde{U}_2 \setminus \cup_{h\in\mathcal{H}, 1\le i\le N_X} \{\Upsilon_h(T_i)\} \right\}.$$

Then using (4.3.7) again

$$\mathbb{P}[\mathcal{J} \mid X, \cup_{h\in\mathcal{H}}\mathcal{D}_h, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h), \neg\mathcal{M}] \ge \left(1 - L_j^{-\gamma}\right)^{4v_0 L_j^{2\alpha-2}} \ge 9/10. \tag{4.5.19}$$

Now let $\mathcal{B}$ denote the event that none of the external buffer zones of Y_U^{j+1} are conjoined and B_U^{j+1} is the domain of Y_U^{j+1}. Observe that on $\cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h) \cap \mathcal{J}$, B_U^{j+1} is a valid potential domain for Y_U^{j+1} and hence we have that

$$\mathbb{P}[\mathcal{B} \mid X, \cup_{h\in\mathcal{H}}\mathcal{D}_h, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h), \mathcal{J}, \neg\mathcal{M}] \ge (1 - 10^{-(j+10)})^{4v_0} \ge 9/10. \tag{4.5.20}$$

If $\mathcal{B}, \mathcal{J}, \cup_{h\in\mathcal{H}}\mathcal{D}_h$ and $\cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)$ all hold and $\mathcal{M}$ does not hold then by definition Y_U^{j+1} is valid and there is $h_0 \in \mathcal{H}$ such that $\mathcal{D}_{h_0}$ holds and $\mathcal{G}_{h'}$ holds for all $h' \in \mathcal{H} \setminus \{h_0\}$. The α-canonical map Υ_{h_0} then gives rise to an embedding of X into $Y = Y_U^{j+1}$. It follows from (4.5.16), (4.5.17), (4.5.18), (4.5.19) and (4.5.20) that

$$\begin{aligned}
S_{j+1}(X) &\ge \mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h), \mathcal{J}, \mathcal{B}, \neg\mathcal{M} \mid X] \\
&= \mathbb{P}[\mathcal{J} \cap \mathcal{B} \mid X, \cup_{h\in\mathcal{H}}\mathcal{D}_h, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h), \mathcal{J}, \neg\mathcal{M}] \\
&\quad \mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h, \neg\mathcal{M} \mid X, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)]\mathbb{P}[\cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h) \mid X] \quad (4.5.21)\\
&\ge \frac{7}{10}\left[\left(\frac{9}{10} \wedge \frac{1}{4} L_j \prod_{i=1}^{N_X} S_j(X_{\ell_i})\right) - 2v_0 k_0 L_j^{-(\gamma-2)} \prod_{i=1}^{N_X} S_j(X_{\ell_i})\right] \quad (4.5.22)\\
&\ge \frac{1}{2} \wedge \frac{1}{10} L_j \prod_{i=1}^{N_X} S_j(X_{\ell_i}). \quad (4.5.23)
\end{aligned}$$

This completes the proof. □

Lemma 4.5.11. *When $0 < p < \frac{1}{2}$ and $v \ge 1$,*

$$\mathbb{P}(X \in \mathcal{A}_{X,j+1}^{(2)}, S_{j+1}(X) \le p, V_X \ge v) \le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)}.$$

Proof. We have that

$$\begin{aligned}
\mathbb{P}(X \in \mathcal{A}_{X,j+1}^{(2)}, S_{j+1}(X) \le p, V_X \ge v) &\le \mathbb{P}\left[\frac{1}{10} L_j \prod_{i=1}^{N_X} S_j(X_{T_i}) \le p, V_X \ge v\right] \\
&\le 500 \left(\frac{10p}{L_j}\right)^{m_{j+1}} L_{j+1}^{-\gamma(v-1)} \le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)}
\end{aligned} \tag{4.5.24}$$

where the first inequality holds by Lemma 4.5.10, the second by Lemma 4.5.2 and the third holds for large enough L_0 since $m_{j+1} > m > \alpha\beta$. □

4.5.3 Case 3

Case 3 involves components with very large size. The class of components $\mathcal{A}^{(3)}_{X,j+1}$ is defined as

$$\mathcal{A}^{(3)}_{X,j+1} := \{X : V_X > v_0\}.$$

Lemma 4.5.12. *Condition on $X = X^{j+1}_U \in \mathcal{A}^{(3)}_{X,j+1}$ with $|U| = v > v_0$. Let the bad j level components of X be $X^j_{T_1}, X^j_{T_2}, \ldots, X^j_{T_{N_X}}$. Then we have*

$$S_{j+1}(X) \geq (8k_0)^{-16vk_0^2} 100^{-4(j+10)v} 2^{-v} 2^{-4k_0 v} \prod_{i=1}^{N_X} S_j(X^j_{T_i})$$

Proof. Let $\hat{U} = U_X \subseteq \mathbb{R}^2$, $\widetilde{U} \subseteq \mathbb{Z}^2$, $\widetilde{U}_1 \subseteq \widetilde{U}_0 \subseteq \widetilde{U}_2 \subseteq \mathbb{Z}^2$, be defined as in the proof of Lemma 4.5.10. For $i \in [N_X]$, let $\mathcal{D}_i$ denote the following event.

If $T_i \subseteq \widetilde{U}_2 \setminus \widetilde{U}_1$, then

$$\mathcal{D}_i = \left\{X^j_{T_i} \hookrightarrow Y^j_{T_i}; Y^j_\ell \text{ is good for all } \ell \in T_i\right\}.$$

Otherwise set

$$\mathcal{D}_i = \left\{X^j_{T_i} \hookrightarrow Y^j_{v_h(T_i)}\right\}.$$

Let

$$\mathcal{D} = \cap_{i=1}^{N_X} \mathcal{D}_i.$$

For $\ell \in \widetilde{U}_2 \setminus (\cup_{i=1}^{N_X} T_i)$ let

$$\mathcal{G}_\ell = \{Y^j_\ell \text{ is good}\}$$

and set

$$\mathcal{G} = \cap_{\ell \in \widetilde{U}_2 \setminus (\cup_{i=1}^{N_X} T_i)} \mathcal{G}_\ell.$$

Observe that $|\widetilde{U}_2| \leq 4vL_j^{2\alpha}$. Finally let $\mathcal{B}$ denote the event that $Y = Y^{j+1}_U$ is valid and $\hat{U}$ is the domain of Y^{j+1}_U. Let Υ be the $*$-canonical map from $\hat{U}$ to itself with respect to $\mathcal{T} = \{T_1, T_2, \ldots, T_{N_X}\}$ which exists by Proposition 4.4.1. On $\mathcal{D} \cap \mathcal{G} \cap \mathcal{B}$, we get an embedding of X into $Y = Y^{j+1}_U$ given by Υ. Hence it follows that

$$S_{j+1}(X) \geq \mathbb{P}[\mathcal{D}]\mathbb{P}[\mathcal{G} \mid \mathcal{D}]\mathbb{P}[\mathcal{B} \mid \mathcal{G}, \mathcal{D}].$$

Now observe that the total size of T_i's contained in $\widetilde{U}_2 \setminus \widetilde{U}_1$ must be at most $4k_0 v$ and hence arguing as in the proof of Lemma 4.5.10 we get that

$$\mathbb{P}[\mathcal{D}] = \left(\frac{1}{2}\right)^{4k_0 v} \prod_{i=1}^{N_X} S_j(X^j_{T_i}).$$

Also using the recursive hypothesis (4.3.7) we get that

$$\mathbb{P}[\mathcal{G} \mid \mathcal{D}] \geq (1 - L_j^{-\gamma})^{4vL_j^{2\alpha}} \geq 2^{-v}$$

as $\gamma > \alpha$ and L_0 is sufficiently large. Finally observe that on $\mathcal{D} \cap \mathcal{G}$, the curve corresponding to the boundary of $\hat{U}$ is a valid level $(j+1)$ boundary curve for Y and hence,

$$\mathbb{P}[\mathcal{B} \mid \mathcal{G}, \mathcal{D}] \geq (8k_0)^{-16vk_0^2} 100^{-4(j+10)v}.$$

Putting all these together we get the lemma. ☐

Lemma 4.5.13. *When* $0 < p \leq \frac{1}{2}$ *and* $v \geq 1$*,*

$$\mathbb{P}(X \in \mathcal{A}^{(3)}_{X,j+1}, S_{j+1}(X) \leq p, V_X \geq v) \leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)}.$$

Proof. Without loss of generality we can take $v \geq v_0$. Then we have using Lemma 4.5.12 and Lemma 4.5.2

$$\begin{aligned}
\mathbb{P}(X \in \mathcal{A}^{(3)}_{X,j+1}, S_{j+1}(X) \leq p, V_X \geq v) &= \sum_{v'=v}^{\infty} \mathbb{P}[S_{j+1}(X) \leq p, V_X = v'] \\
&\leq \sum_{v'=v}^{\infty} \mathbb{P}\Big[(8k_0)^{-16v'k_0^2} 100^{-4(j+10)v'} 2^{-v'-4k_0 v'} \\
&\qquad \prod_{i=1}^{N_X} S_j(X^j_{T_i}) \leq p, V_X = v'\Big] \\
&\leq \sum_{v'=v}^{\infty} 500 p^{m_{j+1}} \times (2000k_0)^{64k_0^2 v'(j+10)m_{j+1}} L_{j+1}^{-9\gamma(v'-1)} \\
&\leq 500 p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)} \\
&\qquad \times \left(\sum_{v'=v}^{\infty} (2000k_0)^{64k_0^2(j+10)m_{j+1}} L_{j+1}^{-5\gamma v'}\right) \\
&\leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)} \qquad (4.5.25)
\end{aligned}$$

where the penultimate inequality follows from $\gamma(v_0 - 1) > \beta$ and $v_0 > 5$ and the last inequality follows by taking L_0 sufficiently large. ☐

4.5.4 Case 4

The final case is the case of components of size not too large, but with a large size of bad subcomponents. The class of blocks $\mathcal{A}^{(4)}_{X,j+1}$ is defined as

$$\mathcal{A}^{(4)}_{X,j+1} := \{X : K_X \geq V_X k_0, V_X \leq v_0\}.$$

Lemma 4.5.14. *Condition on a $(j+1)$ level component $X = X_U^{j+1} \in \mathcal{A}^{(5)}_{X,j+1}$ with $|U| = v \leq v_0$. Let the bad j level components of X be $X^j_{T_1}, X^j_{T_2}, \ldots, X^j_{T_{N_X}}$ such that $\sum_{i=1}^{N_X} |T_i| \geq vk_0$. Then we have*

$$S_{j+1}(X) \geq (8k_0)^{-16vk_0^2} 100^{-4(j+10)v} 2^{-v} 2^{-4k_0 v} \prod_{i=1}^{N_X} S_j(X^j_{T_i})$$

Proof of Lemma 4.5.14 is identical to the proof of Lemma 4.5.12, i.e. we once again get the result by considering the $*$-canonical map from the domain of X to itself and asking tor X and Y to have the same domain. We omit the details.

To complete the analysis of this case we have the following lemma.

Lemma 4.5.15. *When $0 < p \leq \frac{1}{2}$ and $v \geq 1$,*

$$\mathbb{P}(X \in \mathcal{A}^{(4)}_{X,j+1}, S_{j+1}(X) \leq p, V_X \geq v) \leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)}.$$

Proof. Fix $p \leq \frac{1}{2}$ and $v \geq 1$. By definition of $\mathcal{A}^{(4)}_{X,j+1}$ and using Lemma 4.5.14 and Lemma 4.5.2, we get that

$$\begin{aligned}
\mathbb{P}(X \in \mathcal{A}^{(4)}_{X,j+1}, S_{j+1}(X) \leq p, V_X \geq v) &= \sum_{v'=v}^{v_0} \mathbb{P}[X \in \mathcal{A}^{(5)}_{X,j+1}, S_{j+1}(X) \leq p, V_X = v'] \\
&\leq \sum_{v'=v}^{v_0} \mathbb{P}\Bigg[\prod_{i=1}^{N_X} S_j(X^j_{T_i}) \leq (2000k_0)^{64k_0^2 v'(j+10)} p, \\
&\qquad V_X = v', K_X \geq vk_0\Bigg] \\
&\leq \sum_{v'=v}^{v_0} 500 p^{m_{j+1}} (2000k_0)^{64k_0^2 v'(j+10) m_{j+1}} L_j^{-\gamma v' k_0/10} \\
&\leq 500 p^{m_{j+1}} (2000k_0)^{64k_0^2 v_0 (j+10) m_{j+1}} \sum_{v'=v}^{v_0} L_j^{-\gamma v' k_0/10} \\
&\leq 2000 p^{m_{j+1}} (2000k_0)^{64k_0^2 v_0 (j+10) m_{j+1}} L_j^{-\gamma v k_0/10} \\
&\leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)} \qquad (4.5.26)
\end{aligned}$$

where the final inequality follows because $\gamma k_0 > 10\alpha\beta$ and $k_0 > 10\alpha\gamma$ and taking L_0 sufficiently large. This completes the proof. □

4.5.5 Proof of Theorem 4.5.1

We now put together the four cases to establish the tail bounds.

Proof of Theorem 4.5.1. The case of $\frac{1}{2} \leq p \leq 1 - L_{j+1}^{-1}$ is established in Lemma 4.5.9. By Lemma 4.5.7 and Lemma 4.5.6 we have that $S_{j+1}(X) \geq \frac{1}{2}$ for all $X \in \mathcal{A}_{X,j+1}^{(1)}$ since L_0 is sufficiently large. Hence we need only consider $0 < p < \frac{1}{2}$ and cases 2 to 4. By Lemmas 4.5.11, 4.5.13 and 4.5.15 then

$$\mathbb{P}(S_{j+1}(X) \leq p) \leq \sum_{l=2}^{4} \mathbb{P}(X \in \mathcal{A}_{X,j+1}^{(l)}, S_{j+1}(X) \leq p) \leq p^{m_{j+1}} L_{j+1}^{-\beta} L_{j+1}^{-\gamma(v-1)}.$$

The bound for $S_{j+1}^{\mathbb{Y}}$ follows similarly. □

4.6 Estimates on Sizes of Components

Our objective here is to bound the probability the $(j+1)$-level components have large size, i.e., we want to prove recursive estimates (4.3.3) and (4.3.4) at level $(j+1)$. We only prove the following theorem, the corresponding bound for $\mathbb{Y}$ components is identical.

Theorem 4.6.1. *Let X be a component of $\mathbb{X}$ at level $(j+1)$ having law $\mu_j^{\mathbb{X}}$. Let V_X denote the size of $\mathbb{X}$. Then we have for all $v \geq 1$,*

$$\mathbb{P}[V_X \geq v] \leq L_{j+1}^{-\gamma(v-1)}.$$

Proof. This follows immediately from Lemma 4.5.2 and observing that L_0 and hence L_j is sufficiently large. □

4.7 Estimates for Good Blocks

4.7.1 Most Blocks are Good

First we prove the recursive estimates (4.3.6) and (4.3.7) at level $(j+1)$. We shall only prove the estimate (4.3.6) as the other one follows in a similar manner.

Theorem 4.7.1. *For $u \in \mathbb{Z}^2$, let $X = X_u^{j+1}$ denote the corresponding $\mathbb{X}$-block at level $(j+1)$. For $V \subseteq \mathbb{Z}^2 \setminus \{u\}$, let $\mathcal{F}_V = \mathcal{F}_V^{\mathbb{X}}$ be as defined in § 4.3 (at level $j+1$). Then we have*

$$\mathbb{P}[X \text{ is good} \mid \mathcal{F}_V] \geq 1 - L_{j+1}^{-\gamma}.$$

Let us first set-up some notation before we move towards proving Theorem 4.7.1. Let $\mathcal{C}_u$ denote the set of all potential boundary curves of X provided $V_X = 1$, i.e., $\mathcal{C}_u$ denotes the set of potential boundary curves through the buffer zone of $B^{j+1}(u)$. Conditional on $\mathcal{F}_V$, let $\mathcal{C}_{u,V}^* \subseteq \mathcal{C}_u$ denote the set of all potential boundary curves that are compatible with $\mathcal{F}_V$. By the assumption on $\mathcal{F}_V$, we must have that $\mathcal{C}_{u,V}^*$ is non-empty, e.g., if V contains all the vertices surrounding u, then $\mathcal{C}_{u,V}^*$ will be a singleton.

Now let us fix $C \in \mathcal{C}_u$. Let $\hat{U} = \hat{U}(C)$ denote the domain having boundary C Let $\mathcal{E}_{\hat{U}}$ denote the event that $\hat{U}$ is the domain of X. On $\mathcal{E}_{\hat{U}}$, let $U \subseteq \mathbb{Z}^2$ be such that $X = X_U^j$, i.e., the $(j+1)$ level block X consists of the j level blocks corresponding to U. Let ∂U denote the the vertices on the boundary of U (i.e. the vertices in U that have neighbours outside U) and $U^* = U \setminus \partial U$. Let $V^* \subseteq \mathbb{Z}^2$ be such that $X_{V^*}^j = X_V^{j+1}$. Let $\mathcal{F}_{V^*}$ denote the conditioning on $X_{V^*}^j$ being valid, i.e., $X_{V^*}^j$ being a union of j-level blocks.

Fix $C \in \mathcal{C}_u$. Let U_X^C denote the total size of bad components in U^* and let W_X^C denote the number of really bad components in U^*. We have the following lemma.

Lemma 4.7.2. *In the above set-up* $\mathbb{P}[\{U_X^C \geq k_0\} \cup \{W_X^C \geq 1\}] \leq L_j^{-\beta/2}$.

Proof. This follows from the arguments in Lemma 4.5.2 and using that $\beta > 4\alpha + 2\gamma$, $\gamma(v_0 - 1) > 2\beta$ and $\gamma k_0 > 10\beta$ are sufficiently large. □

Next define the following event about a stronger notion of airport. A $(L_j^{3/2}-1) \times (L_j^{3/2}-1)$ square S of j cells contained in $X_{U^*}^j$ is called a strong airport if any $L_j^{3/2} \times L_j^{3/2}$ square $\widetilde{S}$ of j-level blocks containing $\widetilde{S}$ is an airport. Let $\mathcal{E}_{\hat{U}}^*$ denote the event that all $(L_j^{3/2}-1) \times (L_j^{3/2}-1)$ square of j level cells contained in $X_{U^*}^j$ are strong airports.

Now we have the following Lemma about airports. First observe the following. Fix a square S_1 of size $L_j^{3/2}$ and a square S_2 of size $L_j^{3/2} - 1$. Further fix a lattice animal S of size at most v_0. Let $N(S, S_1)$ (resp. $N(S, S_2)$) denote the number of subsets of S_1 (resp. S_2) that are translates of S. It is easy to see that $|N(S, S_2)| \geq (1 - L_j^{-1})|N(S, S_1)|$.

Lemma 4.7.3. *Let $S \subseteq \mathbb{Z}^2$ be a fixed square of size $(L_j^{3/2} - 1)$. Consider the set of blocks X_S^j. Fix a j-level semi-bad $\mathbb{Y}$-component $Y = Y_{S'}^j$. Let $\mathcal{S}$ denote the set of subsets of S that are translates of S'. Let H denote the event that*

$$\#\{\widetilde{S} \in \mathcal{S}; A_{valid}^{X_{\widetilde{S}}^j}, X_{\widetilde{S}}^j \hookrightarrow Y\} \geq (1 - v_0^{-3} k_0^{-4} 100^{-j})|\mathcal{S}|.$$

Then we have $\mathbb{P}[H \mid Y] \geq 1 - e^{-cL_j^{5/2}}$ for some constant c not depending on L_j.

Proof. Observe the following. As $|S'| \leq v_0$ it follows that $\mathcal{S}$ can be partitioned into $4v_0^2$ subsets $\mathcal{S}_i$, $i \in [4v_0^2]$ such that $\widetilde{S}_1, \widetilde{S}_2 \in \mathcal{S}_i$ for some i implies that $\widetilde{S}_1$ and $\widetilde{S}_2$ are non neighbouring. By using a Chernoff bound and $S_j(Y) \geq (1 - v_0^{-5} k_0^{-4} 100^{-j})$ it follows that for each i,

$$\mathbb{P}[\#\{\widetilde{S} \in \mathcal{S}_i; \neg A_{\text{valid}}^{X_{\widetilde{S}}^j} \text{ or } X_{\widetilde{S}}^j \not\hookrightarrow Y\} \geq v_0^{-3} k_0^{-4} 100^{-j} |\mathcal{S}|] \leq e^{-cL_j^{5/2}}$$

for some constant c not depending on L_j. Taking a union bound over all i we get the lemma. □

Lemma 4.7.4. *In the set-up of Lemma 4.7.3, we have*

$$\mathbb{P}[X_S^j \textit{ is a strong airport}] \geq 1 - e^{-c' L_j^{9/4}}$$

for some constant $c' > 0$ not depending on L_j.

Proof. Observe that, conditioned on $X^j_{\widetilde{S}}$ the event $X^j_{\widetilde{S}} \hookrightarrow Y$ is determined by the 0 level structure of Y, i.e., by looking at whether each 0 level block contained in Y is $\mathbf{0}, \mathbf{1}$ or good. Hence for our purposes, the different number of semi-bad Y at level j is at most $8^{v_0}3^{4v_0L_j^2}$. The lemma now follows from Lemma 4.7.3 by taking a union bound over all semi-bad Y as L_j is sufficiently large, and from the observation immediately preceding Lemma 4.7.3. □

Lemma 4.7.5. *Fix $C \in \mathcal{C}_u$ and let $\hat{U}$ denote the domain enclosed by C. Then we have $\mathbb{P}[\mathcal{E}^*_{\hat{U}}] \geq 1 - e^{-c'L_j^{9/4}}$ for some constant $c' > 0$ not depending on L_j.*

Proof. Follows in a similar manner to Lemma 4.7.4 and noting that the number of $L_j^{3/2} \times L_j^{3/2}$ squares in U^* are $O(L_j^{2\alpha})$ and taking a union bound over all of them. □

Lemma 4.7.6. *On $\mathcal{E}_{\hat{U}} \cap \mathcal{E}^*_{\hat{U}} \cap \{U^C_X < k_0\} \cap \{W^C_X = 0\}$, we have that X is good.*

Proof. Noticing that on $\mathcal{E}_{\hat{U}}$, the j level $\mathbb{X}$-blocks corresponding to ∂U are all good, so this lemma follows immediately from the definition of good blocks. □

Now we are ready to prove Theorem 4.7.1.

Proof of Theorem 4.7.1. For $C \in \mathcal{C}^*_u$, define $U = U_C, \hat{U} = \hat{U}_C$ and $U^* = U^*_C$ as above. Set $\widetilde{V} = \mathbb{Z}^2 \setminus U$. Let $\mathcal{F}^*_C$ denote the event that C is a valid level $(j+1)$ boundary.

Observe that

$$\mathbb{P}[X \text{ is bad} \mid \mathcal{F}_V] \leq \max_{C \in \mathcal{C}^*_u} \mathbb{P}[X \text{ is bad} \mid \mathcal{F}_V, \mathcal{E}_{\hat{U_C}}].$$

Now fix $C \in \mathcal{C}^*_u$. We have

$$\begin{aligned}\mathbb{P}[X \text{ is bad} \mid \mathcal{F}_V, X^j_{\widetilde{V}}, \mathcal{E}_C, \mathcal{F}^*_C] &= \frac{\mathbb{P}[X \text{is bad}, \mathcal{E}_C \mid X^j_{\widetilde{V}}, \mathcal{F}^*_C]}{\mathbb{P}[\mathcal{E}_{\hat{U}} \mid X^j_{\widetilde{V}}, \mathcal{F}^*_C]} \\ &\leq (8k_0)^{16k_0^2} 100^{j+10} L_{j+1}^{-2\gamma}\end{aligned}$$

where the last inequality follows from Lemma 4.7.7 below and the construction of boundaries at level $j+1$. The theorem follows by averaging over the distribution of j level blocks outside V^*. □

It remains to prove the following lemma.

Lemma 4.7.7. *Fix $C \in \mathcal{C}^*_u$ and let $\hat{U}$ be as above. Set $I^C_{\text{bad}} = (\neg\mathcal{E}^*_{\hat{U}}) \cup \{U^C_X \geq k_0\} \cap \{W^C_X \geq 1\}$. Consider the above set-up where we condition on $X^j_{\widetilde{V}}$ such that it is compatible with $\mathcal{E}_{\hat{U}}$. Then we have*

$$\mathbb{P}[X \textit{ is bad}, \mathcal{E}_{\hat{U}} \mid X^j_{\widetilde{V}}, \mathcal{F}^*_C] \leq 2\mathbb{P}[I^C_{\text{bad}} \mid X^j_{\widetilde{V}}] \leq 2L_{j+1}^{-2\gamma}.$$

Proof. It follows from Lemma 4.7.6 that

$$\begin{aligned}\mathbb{P}[X \text{ is bad}, \mathcal{E}_{\hat{U}} \mid X^j_{\tilde{V}}, \mathcal{F}^*_C] &= \frac{\mathbb{P}[X\text{is bad}, \mathcal{E}_{\hat{U}}, \mathcal{F}^*_C \mid X^j_{\tilde{V}}]}{\mathbb{P}[\mathcal{F}^*_C \mid X^j_{\tilde{V}}]} \\ &\leq \frac{\mathbb{P}[I^C_{\text{bad}} \mid X^j_{\tilde{V}}]}{\mathbb{P}[\mathcal{F}^*_C \mid X^j_{\tilde{V}}]}.\end{aligned}$$

Let $\mathcal{G}_U$ denote the event that j level blocks $X^j(u')$ for all $u' \in U$ are good. Observe that since $X^j_{\tilde{V}}$ is such that it is compatible with $\mathcal{E}_{\hat{U}}$, it follows that on $X^j_{\tilde{V}} \cap \mathcal{G}_U$ we have $\mathcal{F}^*_C$. Hence we have using the recursive estimate (4.3.6) at level j that

$$\mathbb{P}[\mathcal{F}^*_C \mid X^j_{\tilde{V}}] \geq \mathbb{P}[\mathcal{G}_U \mid X^j_{\tilde{V}}] \geq 1 - 4L_j^{\alpha-\gamma} \geq \frac{1}{2}.$$

Observe that I^C_{bad} only depends on the blocks corresponding to the set U^*_C and hence is independent of $X^j_{\tilde{V}}$. The lemma now follows from Lemma 4.7.2 and Lemma 4.7.5 since $\beta > 4\alpha\gamma$ and L_j is sufficiently large. □

4.7.2 Good Blocks Embed into Good Blocks

Theorem 4.7.8. *Let $X = X^{j+1}_u$ and $Y = Y^{j+1}_u$ be level $(j+1)$ good $\mathbb{X}$ and $\mathbb{Y}$ blocks respectively. Then we have $X \hookrightarrow Y$.*

Proof. Let C_X and C_Y denote the boundary curves of X and Y respectively. Let $\widetilde{U}_X$ and $\widetilde{U}_Y$ denote the domains bounded by these curves. Let $\hat{U}_X, \hat{U}_Y \subseteq \mathbb{Z}^2$, such that $X = X^j_{\hat{U}_X}$ and $Y = Y^j_{\hat{U}_Y}$. Let $\mathcal{T} = \{T_1, T_2, \ldots, T_{N_X}\}$ (resp. $\mathcal{T}' = \{T'_1, T'_2, \ldots T'_{N_Y}\}$) be the set of subsets of $\hat{U}_X$ (resp. $\hat{U}_Y$) such that $X^j_{T_i}$ (resp. $Y^j_{T'_i}$) are the j-level bad subcomponents of X (resp. Y).

By Proposition 4.4.2 there exists canonical maps $\Upsilon^{j+1}_{h_1,h_2}$, $(h_1, h_2) \in [L_j^2] \times [L_j^2]$ from $\widetilde{U}_X$ to $\widetilde{U}_Y$ with respect to $\mathcal{T}$ and $\mathcal{T}'$ which are bi-Lipschitz with Lipschitz constant $(1 + 10^{-(j+5)})$ such that for each $i \in [N_X]$ we have $\Upsilon_{h_1,h_2}(T_i) = (h_1 - 1, h_2 - 1) + \Upsilon_{1,1}(T_i)$ and for all $i \in [N_Y]$ we have $\Upsilon^{-1}_{h_1,h_2}(T_i) = -(h_1 - 1, h_2 - 1) + \Upsilon^{-1}_{1,1}(T'_i)$. Now since all rectangles of $L_j^{3/2} \times L_j^{3/2}$ sub-blocks are airports, it follows that for all $i \in [N_X]$

$$\#\{(h_1, h_2) \in [L_j^2] \times [L_j^2] : X^j_{T_i} \not\hookrightarrow Y^j_{\Upsilon_{h_1,h_2}(T_i)}\} \leq v_0^{-2} k_0^{-4} 100^{-(j-1)} L_j^4$$

and for all $i \in [N'_Y]$

$$\#\{(h_1, h_2) \in [L_j^2] \times [L_j^2] : X^j_{\Upsilon^{-1}_{h_1,h_2}(T'_i)} \not\hookrightarrow Y^j_{T'_i}\} \leq v_0^{-2} k_0^{-4} 100^{-(j-1)} L_j^4.$$

By taking a union bound it follows that there exists a canonical map $\Upsilon = \Upsilon^{j+1}$ from $\widetilde{U}_X$ to $\widetilde{U}_Y$ with respect to $\mathcal{T}$ and $\mathcal{T}'$ which are bi-Lipschitz with Lipschitz constant $(1 + 10^{-(j+5)})$, such that for all $i \in [N_X]$ for all $i' \in [N'_Y]$ we have $X^j_{T_i} \hookrightarrow Y^j_{\Upsilon_{h_1,h_2}(T_i)}$. The theorem now follows from definition that $X \hookrightarrow Y$. □

Chapter 5

Bi-Lipschitz Expansion of Measurable Sets

Our main result in this chapter concerns bijections from the unit square on the plane to itself that are identity on the boundary. Here we provide an affirmative answer to Question 1.3.3. Recall the set-up of Question 1.3.3. For any set A of a fixed Lebesgue measure γ, we are interested in constructing such a function, which additionally is bi-Lipschitz, and maps A to a set with Lebesgue measure above a fixed threshold $1-\gamma'$. For a fixed choice of γ and γ' (with $1-\gamma' > \gamma$) we want to maintain a uniform control over the bi-Lipschitz constants of the such functions. The main result in this chapter Theorem 5.1, establishes that it is possible to do so.

5.1 Statement of the Results

We denote the sigma algebra of all Borel-measurable subsets of $[0,1]^2$ by $\mathcal{B}([0,1]^2)$ and let λ denote the Lebesgue measure on $\mathbb{R}^2$. Our main result in this chapter is the following.

Theorem 5.1. *For each $\gamma, \gamma' \in (0,1)$, $\gamma + \gamma' < 1$, there exists $C_0 = C_0(\gamma, \gamma') > 0$ such that for all $A \in \mathcal{B}([0,1]^2)$ with $\lambda(A) = \gamma$, there exists a bijection $\phi_0 : [0,1]^2 \to [0,1]^2$ such that*

1. *ϕ_0 is C_0-bi-Lipschitz, i.e.*

$$\frac{1}{C_0}|x-y| \leq |\phi_0(x) - \phi_0(y)| \leq C_0|x-y| \quad \forall x, y \in [0,1]^2.$$

2. *ϕ_0 is identity on the boundary, i.e., $\phi_0(x) = x$, for all $x \in \partial[0,1]^2$.*

3. *$\lambda(\phi_0(A)) \geq 1 - \gamma'$.*

As mentioned in § 1.3.1 Theroem 5.1 plays a crucial role in the study of rough isometries of i.i.d. copies of 2-dimensional Poisson point process. Recall the definition of rough-isometry between two metric spaces, Definition 1.1.2 from Chapter 1. The importance of Theorem 5.1 in proving rough isometry of Poisson processes is illustrated by our next result.

Let X and Y be two independent Poisson point processes on $\mathbb{R}^2$. Let X_n (resp. Y_n) denote the random metric space formed by points of X (resp. Y) within $[0,n]^2$ along with the boundary of $[0,n]^2$. Let $X_n \hookrightarrow_{(M,D,C)} Y_n$ denote the event that there exists a rough isometry with parameters (M,D,C) between X_n and Y_n which is identity on the boundary of $[0,n]^2$. Let $k_X(n)$ denote the number of unit squares in $[0,n]^2$ which contains at least one point of X. We have the following theorem as a consequence of Theorem 5.1.

Theorem 5.2. *Fix $\epsilon,\delta>0$. Then there exist positive constants M,D,C depending on ϵ and δ (not depending on n) such that we have that for all n sufficiently large and for all X with $k_X(n)\geq \delta n^2$, $\mathbb{P}[X_n \hookrightarrow_{(M,D,C)} Y_n \mid X_n] \geq e^{-\epsilon n^2}$.*

By way of proving Theorem 5.1 we also establish the following result which shows that it is possible to increase the measure of a set by an arbitrarily small amount in a bi-Lipschitz manner.

Theorem 5.3. *Fix $0<\gamma<1-\gamma'<1$, and $\eta>0$. Then there exists $\varepsilon=\varepsilon(\gamma,\gamma',\eta)>0$ such that the following holds. For every Borel set $A\subseteq[0,1]^2$ with $\lambda(A)\in[\gamma,1-\gamma']$, there exists a bijection $\phi=\phi_A:[0,1]^2\to[0,1]^2$ such that*

1. *ϕ is $(1+\eta)$-bi-Lipschitz.*
2. *ϕ is identity on the boundary of $[0,1]^2$.*
3. *$\lambda(\phi(A))\geq\lambda(A)+\varepsilon$.*

5.1.1 Related Works

The question of existence of bi-Lipschitz homeomorphisms between different subsets of $\mathbb{R}^n$, satisfying certain conditions, has been studied classically, notably by McMullen [33]. It is shown in that paper that there exists a positive function f on $\mathbb{R}^n$, $n\geq 2$, such that there does not exist a bi-Lipschitz homeomorphism $\phi:\mathbb{R}^n\to\mathbb{R}^n$ with $\det D\phi=f$. It is also proved that there exists separated nets in $\mathbb{R}^n$ ($n\geq 2$) which cannot be mapped into $\mathbb{Z}^n$ in a bi-Lipschitz way.

Shortly before posting this work [5] on arXiv we came across a very recent paper [14] on arXiv which investigates related questions and proves a variant of Theorem 5.1 for sets A of sufficiently small measure ([14, Proposition 11]), as well as much more. Their proof uses a covering lemma based on a result of [2], which is no longer true in higher dimensions. Our proof is different and we believe can be adapted to work in higher dimensions as well. Moreover, the stretching result only for sufficiently small sets is not sufficient for our purposes in proving rough isometry. Our work was independent of [14].

The question considered in this chapter has connections to several different problems in analysis, an interested reader is referred to [14] and the references therein for more details.

5.1.2 Proofs of Theorem 5.1 and Theorem 5.2

In this subsection we establish Theorem 5.1 from Theorem 5.3 and Theorem 5.2 as a consequence of Theorem 5.1. We start with the easy proof of Theorem 5.1 assuming Theroem 5.3.

Proof of Theorem 5.1. Fix $0 < \gamma < 1 - \gamma' < 1$. Fix a set A with $\lambda(A) = \gamma$. Define a sequence of bijections ψ_i on $[0, 1]^2$ and a sequence of Borel sets A_i as follows. Set $A = A_1$. If $\lambda(A_i) < 1 - \gamma'$, then define $\psi_i = \phi_{A_i}$ where ϕ_{A_i} is given by Theorem 5.3. If $\lambda(A_i) \geq 1 - \gamma'$, set ψ_i to be the identity map. Set $A_{i+1} = \psi_i(A_i)$. Fix $\eta > 0$. Set $n_0 = \lceil \frac{1-\gamma'-\gamma}{\varepsilon(\gamma,\gamma',\eta)} + 1 \rceil$. It follows from Theorem 5.3 that the function $\phi_0 = \psi_{n_0} \circ \cdots \circ \psi_1$ satisfies all the conditions of Theorem 5.1 with $C_0 = (1 + \eta)^{n_0}$. This completes the proof. □

Next we show how Theorem 5.2 follows from Theorem 5.1.

Proof of Theorem 5.2. Fix X such that $k_X(n) \geq \delta n^2$. Let A denote the union of unit squares in $[0, n]^2$ that contain points of X. Now fix $\kappa > 0$ such that $(1 - e^{-\kappa^2})^{1/\kappa^2} > e^{-\epsilon}$ and $\delta' > 0$ such that $(1-e^{-\kappa^2})^{(1-\delta')\kappa^{-2}} e^{-\delta'} \geq e^{-\epsilon}$. Since A has measure at least δn^2, by Theorem 5.1 there exists a $C(\delta, \delta')$ bi-Lipschitz map ϕ from $[0, n]^2$ to $[0, n]^2$ which is identity on the boundary and such that $\lambda(\phi(A)) \geq (1 - \delta')n^2$ (if $\lambda(A) \geq (1 - \delta')n^2$, we take ϕ to be the identity map). Now let B denote the union of squares in $[0, n]^2$ of the form $[\kappa j, \kappa(j + 1)] \times [\kappa\ell, \kappa(\ell + 1)]$ that intersect $\phi(A)$, and let $k'_Y(n)$ denote their numbers. Clearly $k'_Y(n) \geq (1 - \delta')n^2\kappa^{-2}$. Now let $\mathcal{E}$ denote the event that each square of the form $[\kappa j, \kappa(j + 1)] \times [\kappa\ell, \kappa(\ell + 1)]$ contained in B contains at least one point of Y and there are no points of Y in $[0, n]^2$ outside B. It is easy to see that on $\mathcal{E}$, there exists $M = M(\delta, \epsilon)$, $D = D(\delta, \epsilon)$, $C = C(\delta, \epsilon)$ such that we have $X_n \hookrightarrow_{(M,D,C)} Y_n$. Hence

$$\mathbb{P}[X_n \hookrightarrow_{(M,D,C)} Y_n \mid X_n] \geq \mathbb{P}[\mathcal{E} \mid X] \geq (1 - e^{-\kappa^2})^{(1-\delta')n^2\kappa^{-2}} e^{-\delta' n^2} \geq e^{-\epsilon n^2}.$$

This completes the proof. □

Rest of this chapter is devoted to proving Theorem 5.3. From now on, γ and γ' and η will be fixed positive numbers such that $0 < \gamma < 1 - \gamma' < 1$. Also we shall fix a Borel set $A \subseteq [0, 1]^2$ with $\lambda(A) \in [\gamma, 1 - \gamma']$.

5.1.3 An Overview of the Proof of Theorem 5.3

To prove Theorem 5.3 one needs to construct a map satisfying the required conditions which expands the regions where the set A has higher density and compress the regions where set A has lower density. For example it is not hard to see that one can construct such a function if the set A is contained in, say, the left half of the unit square (i.e., $[0, \frac{1}{2}] \times [0, 1]$) then the conclusion of Theorem 5.3 holds. Similar construction works for sets which have different densities in the left half and the right half of $[0, 1]^2$, (see § 5.2). For more complicated sets we use the same idea recursively at different scales.

In § 5.2, we construct an auxiliary family of bijections $\{\Psi_\delta\}_{\delta\in(-1,1)}$ from $[0,1]^2$ onto itself which are identity on the boundary and which stretches all regions within the left half of $[0,1]^2$ by the same factor $1+\delta$. The functions Ψ_δ also satisfy certain regularity conditions, in particular these are bi-Lipschitz functions with Lipschitz constant $1+O(\delta)$. We divide the unit square into dyadic squares and rectangles at different scales recursively such that each square (rectangle) at a level is divided in two halves by the rectangles (squares) in the next level (see § 5.3). We use the auxiliary functions Ψ_δ to construct bijections on the dyadic squares (rectangles) at different levels which stretches each half of these dyadic squares (which are dyadic rectangles at the next scale) proportionally to the density of A in these dyadic rectangles. Finally we compose these functions (see § 5.4) up to a large number of levels (with the number of levels depending on the set A and the location of the dyadic square) to obtain the required stretching function. However, the control on the Lipschitz constant worsens with the number of compositions and it is necessary to establish that one can maintain a uniform control over the Lipschitz constant. While the construction is deterministic, our proof makes use of a probabilistic analysis.

We let X denote a uniformly chosen point in $[0,1]^2$. Let us reveal sequentially for $n\geq 1$, which dyadic square at level n it belongs to. The expected density of X is then a martingale. We make use of this martingale to analyse the function described above. Roughly we show that there exists a stopping time τ such that if we compose the stretching functions up to level τ, then the area of A is increased by ϵ, however the bi-Lipschitz constant is still controlled by $1+\eta$. This argument is spanned over § 5.5, § 5.6, § 5.7 and finally we complete the proof of Theorem 5.3 in § 5.8. The proofs of a few technical estimates used in § 5.2 is postponed to § 5.9.

A word about notation: parameters and constants In the course of the proof of Theorem 5.3 over the next few sections, we shall have occasion of using many constants and parameters. By an absolute constant we shall mean a constant that depends only on γ and γ'. We shall denote by C, c absolute constants whose values may vary through the proof while numbered constants $C_1, C_2, \ldots$, and ε_1, $\varepsilon_2, \ldots$, will denote fixed constants whose values are fixed throughout the chapter and in particular are independent of the set A. When we use a matrix norm for a matrix $\mathbf{M}$, unless otherwise stated, $||\mathbf{M}||$ will denote its ℓ_∞ norm, i.e., the maximum of absolute values of its entries.

5.2 The Stretching Function

To construct the map ϕ we shall need an auxiliary stretching map Ψ_δ where $\delta\in(-1,1)$ is a stretching parameter. Ψ_δ will be a bijection on $[0,1]^2$ which is identity on the boundary of $[0,1]^2$. We now move towards the construction of Ψ_δ.

5.2.1 Construction of Ψ_δ

For the rest of this subsection, fix $\delta \in (-1, 1)$. We first construction the following parametrisation.

Parametrisation

Let $h : [0, \frac{1}{2}) \to [0, \infty)$ be a function with the following properties.

1. $h(0) = 0$, and $h(r) = 0$ for all $r \leq \frac{1}{10}$.
2. $h(r) \to \infty$ as $r \to \frac{1}{2}$.
3. h is (weakly) increasing.
4. h is thrice continuously differentiable with $(h'(r))^2 = O((r + h(r))^3)$, $h''(r)h'(r) = O((r + h(r))^3)$ and $h^{(3)}(r) = O((r + h(r))^2)$.

It is easy to see that such an h exists, e.g., we could take a function that behaves like $e^{-1/(r-1/10)}$ near $\frac{1}{10}$ and like $(r - \frac{1}{2})^{-6}$ near $\frac{1}{2}$. Fix such an h for the rest of this section.

Clearly, there is a unique $r_0 \in (0, \frac{1}{2})$ such that

$$2r_0 h(r_0) + r_0^2 = \frac{1}{4}.$$

Define $\Theta(r)$ as follows.

$$\Theta(r) = \begin{cases} \arccos \frac{h(r)}{h(r)+r} & \text{if } r \leq r_0, \\ \arcsin \frac{1}{2(h(r)+r)} & \text{if } r > r_0. \end{cases}$$

The $r - \theta$ parametrisation

Consider the bijection $K : [0, 1] \times [0, \frac{1}{2}) \setminus \{(1/2, 0)\} \to (0, \frac{1}{2}) \times (-1, 1)$ defined as follows. We have $(x_1, x_2) \mapsto (r, \theta)$ defined by

$$x_1 = \frac{1}{2} + (r + h(r)) \sin(\theta\Theta(r)), x_2 = (r + h(r)) \cos(\theta\Theta(r)) - h(r).$$

We shall work with this parametrisation in the lower half of the unit square. The level lines of the function r and their reflections about the line $x_2 = \frac{1}{2}$ is shown in Figure 5.1.

Notice that this transformation is C^1 except on $\{r = r_0\}$ and the Jacobian matrix $J(r, \theta)$ for the transformation K^{-1} is given by

$$J(r, \theta) = \begin{bmatrix} J_{1,1}(r, \theta) & J_{1,2}(r, \theta) \\ J_{2,1}(r, \theta) & J_{2,2}(r, \theta) \end{bmatrix}$$

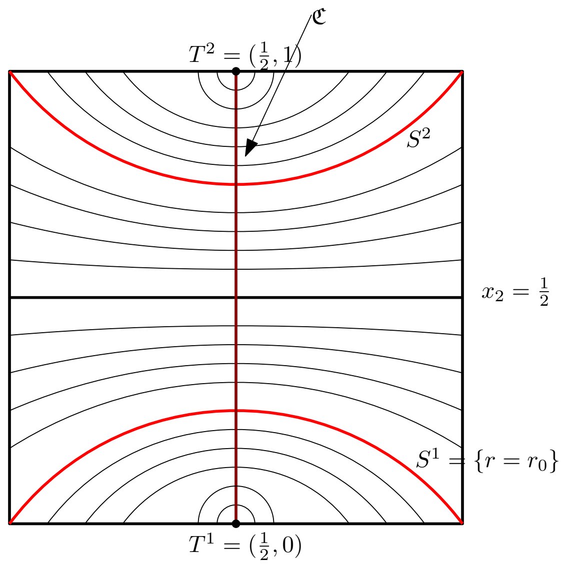


Figure 5.1: Level lines of the function r in the $r-\theta$ parametrisation and their reflections about the line $x_2 = \frac{1}{2}$

where

$$\begin{aligned}
J_{1,1}(r,\theta) &= (1+h'(r))\sin(\theta\Theta(r)) + (r+h(r))\Theta'(r)\theta\cos(\theta\Theta(r)) \\
J_{1,2}(r,\theta) &= (r+h(r))\Theta(r)\cos(\theta\Theta(r)) \\
J_{2,1}(r,\theta) &= (1+h'(r))\cos(\theta\Theta(r)) - (r+h(r))\Theta'(r)\theta\sin(\theta\Theta(r)) - h'(r) \\
J_{2,2}(r,\theta) &= -(r+h(r))\Theta(r)\sin(\theta\Theta(r)).
\end{aligned}$$

The (absolute value of) the determinant of $J(r,\theta)$ is given by

$$\Theta(r)(r+h(r))(1+h'(r)-h'(r)\cos(\theta\Theta(r))).$$

Constructing Ψ_δ

Let $\delta \in (-1,1)$ be fixed. For $r \in (0, \frac{1}{2})$, define the function $g_{r,\delta} = g_r : [-1,1] \to [-1,1]$ with the following properties.

1. g_r is an increasing bijection with $g(-1) = -1$ and $g(1) = 1$.

2. For each $\ell \in [-1,0]$, we have

$$(1+\delta)\int_{-1}^{\ell}\left(1+h'(r)-h'(r)\cos(\theta\Theta(r))\right)\,d\theta=\int_{-1}^{g_r(\ell)}\left(1+h'(r)-h'(r)\cos(\theta\Theta(r))\right)\,d\theta. \tag{5.2.1}$$

3. For each $\ell\in[0,1]$ we have

$$(1-\delta)\int_{\ell}^{1}\left(1+h'(r)-h'(r)\cos(\theta\Theta(r))\right)\,d\theta=\int_{g_r(\ell)}^{1}\left(1+h'(r)-h'(r)\cos(\theta\Theta(r))\right)d\theta. \tag{5.2.2}$$

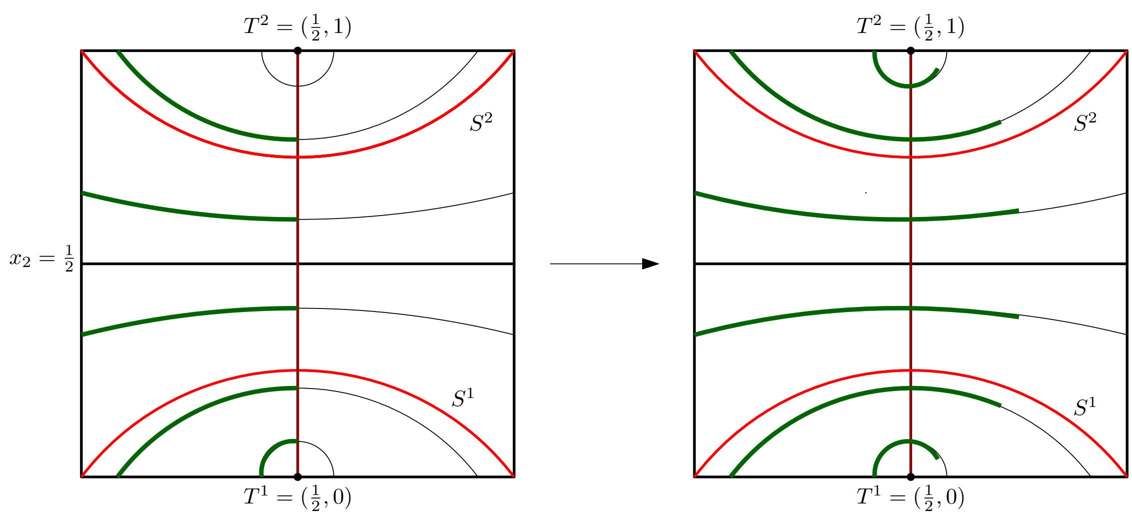


Figure 5.2: For $\delta>0$, the functions g_r stretch the left half of the level lines of the function r

That such a function exists and is unique follows from the facts that for each $r\in(0,\frac{1}{2})$ the integrand $1+h'(r)-h'(r)\cos(\theta\Theta(r))$ is strictly positive on $[-1,1]$, is invariant under the transformation $\theta\mapsto-\theta$ and the hypothesis that $\delta\in(-1,1)$. Now define the bijection $H_\delta=H:(0,\frac{1}{2})\times[-1,1]\to[0,\frac{1}{2})\times[-1,1]$ defined by $(r,\theta)\mapsto(r,g_r(\theta))$. Define Ψ_δ on $[0,1]\times[0,\frac{1}{2})\setminus\{(1/2,0)\}$ by

$$\Psi_\delta(x_1,x_2)=(\Psi_\delta^1(x_1,x_2),\Psi_\delta^2(x_1,x_2))=K^{-1}(H_\delta(K(x_1,x_2))).$$

We define $\Psi_\delta(1/2,0)=(1/2,0)$ and extend Ψ_δ to $[0,1]^2$ in the following way. For $(x_1,x_2)\in[0,1]^2$ with $x_2>\frac{1}{2}$ define

$$\Psi_\delta(x_1,x_2)=(\Psi_\delta^1(x_1,x_2),\Psi_\delta^2(x_1,x_2))=(\Psi_\delta^1(x_1,1-x_2),1-\Psi_\delta^2(x_1,1-x_2)).$$

On the line $x_2 = \frac{1}{2}$, we set

$$\Psi_\delta(x_1, \frac{1}{2}) = (\Psi_\delta^1(x_1, \frac{1}{2}), \Psi_\delta^2(x_1, \frac{1}{2})) = \begin{cases} (x_1(1+\delta), \frac{1}{2}) \text{ for } x_1 \le \frac{1}{2}, \\ (\frac{1+\delta}{2} + (1-\delta)(x_1 - \frac{1}{2}), \frac{1}{2}) \text{ for } x_1 \ge \frac{1}{2}. \end{cases}$$

5.2.2 Basic properties of Ψ_δ

Over the next few lemmas we list useful properties of the function Ψ_δ as constructed above. The next lemma is immediate and we omit the proof.

Lemma 5.2.1. *For each $\delta \in (-1, 1)$, Ψ_δ as constructed above is a bijection from $[0,1]^2$ onto itself and $\Psi_\delta(x) = x$ for all $x \in \partial[0,1]^2$,*

Lemma 5.2.2. *Ψ_δ as defined above is continuous on $[0,1]^2$.*

Proof of this lemma is deferred to § 5.9. The next lemma shows that the left and rights sides are stretched uniformly by ratios of $1+\delta$ and $1-\delta$ respectively.

Lemma 5.2.3. *For each $\delta \in (-1, 1)$, Ψ_δ defined as above it satisfies the following properties.*

(i) For $\Lambda_L = [0, \frac{1}{2}] \times [0, 1]$, we have $\lambda(\Psi_\delta(\Lambda_L)) = (1+\delta)\lambda(\Lambda_L)$.

(ii) For Λ_L as above and $\Lambda_R = [\frac{1}{2}, 1] \times [0, 1]$, we have for $i = L, R$, and for $B \subseteq \Lambda_i$, B measurable,

$$\lambda(B) = (1+\delta)\lambda(B \cap \Lambda_L) + (1-\delta)\lambda(B \cap \Lambda_R).$$

Proof. We first prove that for all $B \subseteq \Lambda_L$, $\lambda(\Psi_\delta(B)) = (1+\delta)\lambda(B)$. Fix $B \subseteq \Lambda_L$. Without loss of generality assume $B \subseteq \widetilde{\Lambda}^1$ as well. We have using (5.2.1),

$$\begin{aligned}
(1+\delta)\lambda(B) &= (1+\delta)\int_0^{1/2}\int_0^{1/2} 1_B \, dx \, dy \\
&= (1+\delta)\int_{K(B)} (r+h(r))(1+h'(r) - h'(r)\cos(\theta\Theta(r)))\Theta(r) \, dr \, d\theta \\
&= (1+\delta)\int_0^{1/2} (r+h(r))\Theta(r) \left(\int_{K(B)_r} (1+h'(r) - h'(r)\cos(\theta\Theta(r))) \, d\theta\right) dr \\
&= \int_0^{1/2} (r+h(r))\Theta(r) \left(\int_{g_r(K(B)_r)} (1+h'(r) - h'(r)\cos(\theta\Theta(r))) \, d\theta\right) dr \\
&= \int_{H(K(B))} (r+h(r))(1+h'(r) - h'(r)\cos(\theta\Theta(r)))\Theta(r) \, dr \, d\theta \\
&= \lambda(\Psi_\delta(B)).
\end{aligned}$$

Similarly it can be shown using (5.2.2) that for all $B \subseteq \Lambda_R$, we have $\lambda(\Psi_\delta(B)) = (1-\delta)\lambda(B)$. This completes the proof of the lemma. □

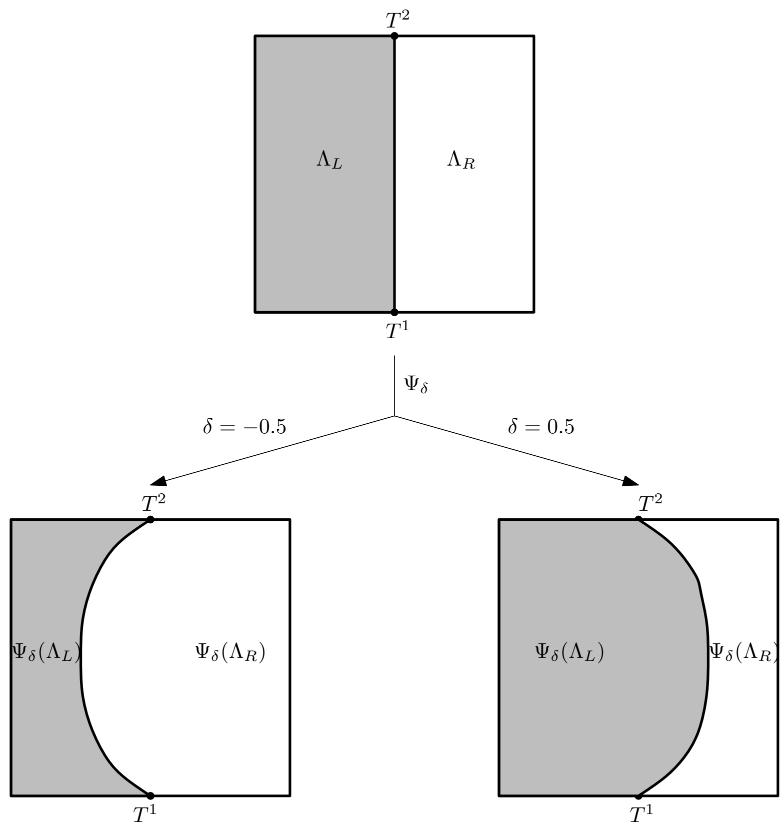


Figure 5.3: Ψ_δ for different values of δ

5.2.3 Smoothness of Ψ_δ

Now we need to establish that Ψ_δ has certain smoothness properties.

Crack, Twists and Seams: Geometric definitions

We introduce the following geometric definitions for $[0,1]^2$.

Definition 5.2.4 (Crack and Twists)**.** *The line* $x_1 = \frac{1}{2}$ *is called the* crack $\mathfrak{C}$ *in the unit square* $[0,1]^2$*. Let* T^1 *and* T^2 *denote the points where the* crack *intersects the boundary,* T^1 *and* T^2 *are called* twists *in* $[0,1]^2$*. Often we shall call* $T = T^1 \cup T^2$ *as* twists *in* $[0,1]^2$*. For* $r_1 < \frac{1}{10}$ *small, define* $T_r^1 = K^{-1}(\{r < r_1\})$ *and define* T_r^2 *to be the reflection of* T_r^1 *on the line* $x_2 = \frac{1}{2}$*. We call* $T_r = T_r^1 \cup T_r^2$ *as the* blown up twists *of* $[0,1]^2$.

Definition 5.2.5 (Seams)**.** *Let $S^1 = K^{-1}(\{r = r_0\})$ in the parametrisation described above. Let S^2 be the reflection of S_1 on the line $x_2 = 1/2$. We shall call S^1 and S^2 (or, $S = S^1 \cup S^2$)* seams *of* $[0,1]^2$.

Estimates for Ψ'_δ

Proposition 5.2.6. *Ψ_δ is differentiable at all points in $[0,1]^2$ except possibly on the* crack *$\mathfrak{C}$ and the* seams *S. For $(x_1, x_2) \in [0,1]^2 \setminus (\mathfrak{C} \cup S)$, let $\mathbf{J}_{\Psi_\delta}(x)$ denote the Jacobian matrix of the transformation Ψ_δ evaluated at $x = (x_1, x_2)$. Then $\mathbf{J}_{\Psi_\delta}$ is continuous on $[0,1]^2 \setminus (\mathfrak{C} \cup S$ and there exists an absolute constant C_1 (not depending on δ, x, possibly depending on h) such that*

$$||\mathbf{J}_{\Psi_\delta}(x) - \mathbf{I}|| \leq C_1 \delta.$$

To prove this proposition, we shall need a few lemmas, dealing with the functions $g_r(\theta)$ and $J(r,\theta)$. These lemmas will be proved in § 5.9.

Lemma 5.2.7. *There exists an some absolute constant $C > 0$ such that the following hold.*

(i) $(g_r(\ell) - \ell) \leq C\delta(\ell + 1)$ *for all* $\ell \leq 0$.

(ii) We have

$$\sup_{0<r<\frac{1}{2}, \theta \in [-1,0) \cup (0,1]} \left|\frac{\partial g_r(\theta)}{\partial \theta} - 1\right| \leq C\delta.$$

(iii) $(g_r(\ell) - \ell) = O(\frac{h'(r)}{(r+h(r))^2})$ *as* $r \to \frac{1}{2}$ *uniformly for all* $\ell \leq 0$.

Proof of this lemma is provided in § 5.9.

Lemma 5.2.8. *There exists an absolute constant $C > 0$ such that we have the following.*

(i) We have

$$\sup_{0<r<\frac{1}{2}, r \neq r_0, \theta \in [-1,0) \cup (0,1]} \left|\frac{\partial g_r(\theta)}{\partial r}\right| \leq C\delta.$$

(ii) We have $|\frac{\partial g_r(\theta)}{\partial r}| \leq \delta O\left(\frac{h''(r)}{(r+h(r))^2} + \frac{h'(r)^3}{(r+h(r))^5}\right)$ *as* $r \to \frac{1}{2}$.

The proof is deferred to § 5.9.

Lemma 5.2.9. *Let $\mathbf{M}(r,\theta) = J(r, g_r(\theta)) - J(r,\theta)$. Then there exists an absolute constant C such that $||\mathbf{M}(r,\theta)|| \leq C\delta$.*

Proof. This follows immediately from the formula for $J(r,\theta)$ and part (i) of Lemma 5.2.7. □

Now we are ready to prove Proposition 5.2.6.

Proof of Proposition 5.2.6. By symmetry it is enough to consider $x = (x_1, x_2)$ such that $x_2 \le \frac{1}{2}$. To start with, we assume $x = (x_1, x_2)$ with $x_2 < \frac{1}{2}$. The differentiability is easy to establish. Let $(r, \theta) = K(x_1, x_2)$. Let $\mathbf{J}_H(r, \theta)$ denote the Jacobian matrix of the transformation H evaluated at the point (r, θ). By Chain rule, it follows that

$$\mathbf{J}_{\Psi_\delta}(x) = J(r,\theta)^{-1}\mathbf{J}_H(r,\theta)J(r, g_r(\theta)).$$

It follows that

$$\mathbf{J}_{\Psi_\delta}(x) - \mathbf{I} = J(r,\theta)^{-1}(\mathbf{J}_H(r,\theta) - \mathbf{I})J(r, g_r(\theta)) + J(r,\theta)^{-1}\mathbf{M}(r,\theta). \tag{5.2.3}$$

It follows from the definition of H that

$$\mathbf{J}_H(r,\theta) = \begin{bmatrix} 1 & 0 \\ \frac{\partial g_r(\theta)}{\partial r} & \frac{\partial g_r(\theta)}{\partial \theta} \end{bmatrix}.$$

It follows from Lemma 5.2.7 and Lemma 5.2.8 that there exists an absolute constant C that

$$||\mathbf{J}_H(r,\theta) - \mathbf{I}|| \le C\delta.$$

Observe the following. Fix $r_0 > \frac{1}{10} > r_1 > 0$. Observe that there exists a constant C such that

$$\sup_{r_1 < r < \frac{1}{2}, \theta} ||J(r,\theta)|| \vee ||J(r,\theta)^{-1}|| \le C. \tag{5.2.4}$$

Hence it follows from Lemma 5.2.9 and (5.2.3) that there exists and absolute constant C such that for all $x \in K^{-1}(\{r_1 < r < \frac{1}{2}\})$ we have

$$||\mathbf{J}_{\Psi_\delta}(x) - \mathbf{I}|| \le C\delta.$$

Since $h(r) = 0$ for each $r \le r_1$ it follows that for each $r < r_1$, and $\theta \le 0$, we have $g_r(\theta) = -1 + (1+\delta)(\theta + 1)$ and it can be verified by direct computation that there exists $C > 0$ such that

$$\sup_{0 < r \le r_1} ||J(r,\theta)^{-1}(\mathbf{J}_H(r,\theta) - \mathbf{I})J(r, g_r(\theta)) + J(r,\theta)^{-1}\mathbf{M}(r,\theta)|| \le C\delta.$$

It is also easy to see using symmetry of construction and Lemma 5.2.8 that $\mathbf{J}_{\Psi_\delta}$ is continuous on $\{x_2 = \frac{1}{2}\} \setminus \{(1/2, 1/2)\}$.

This completes the proof of the proposition by choosing C_1 appropriately. □

Finally we have the following.

Proposition 5.2.10. *Let* $1 > \chi > 0$ *be fixed. Then for all* $\delta < \delta_0 = \frac{\chi}{100(C_1+1)}$, Ψ_δ *is bi-Lipschitz with Lipschitz constant* $(1 + \chi)$.

Proof. For $\delta < \frac{\chi}{100C_1} \wedge \frac{1}{100}$, it follows from Proposition 5.2.6 that

$$\max\{||\mathbf{J}_{\Psi_\delta} - \mathbf{I}||, ||\mathbf{J}_{\Psi_\delta}^{-1} - \mathbf{I}||\} \leq \frac{\chi}{2}.$$

The proposition follows. □

Proposition 5.2.11. *For $x \in [0,1]^2$, let $\widetilde{d}(x)$ denotes its distance from the corners of $[0,1]^2$. Then there exists an absolute constant $C > 0$ such that for all $x \in [0,1]^2$ we have*

$$|\Psi_\delta(x) - x| \leq C\delta\widetilde{d}(x).$$

Proof. This follows easily from the definition of Ψ_δ and part (i) of Lemma 5.2.7. □

Estimates for Ψ''_δ

We want to show that the second derivative of Ψ_δ remains bounded away from the *Twists*.

Proposition 5.2.12. *Ψ_δ is twice differentiable at all points in $[0,1]^2$ except possibly on the* crack $\mathfrak{C}$ *and the* seams S*. Then there exists an absolute constant $C_2 > 0$ (not depending on δ, x, possibly depending on h) such that for $(x_1, x_2) \in [0,1]^2 \setminus (\mathfrak{C} \cup S \cup T_{r_1/2})$, we have*

$$||\Psi''_\delta(x_1, x_2)|| \leq C_2\delta.$$

Proof. Without loss of generality consider $x = (x_1, x_2)$ with $x_1 < \frac{1}{2}$ and $x_2 < \frac{1}{2}$. Let $(r,\theta) = K(x_1, x_2)$. Let us denote

$$\mathbf{H}(r,\theta) = J(r,\theta)^{-1}(\mathbf{J}_H(r,\theta) - \mathbf{I})J(r, g_r(\theta)).$$

For the rest of this subsection, let us introduce the following piece of notation. For a matrix $\mathbf{A}$, $\mathbf{A}_r$ (resp. $\mathbf{A}_\theta$) shall denote the entrywise derivative w.r.t. r (resp. θ) of the matrix A. Because of (5.2.4), it suffices to prove that for some absolute constant C we have

$$||\mathbf{H}_r(r,\theta)|| \leq C\delta, ||\mathbf{H}_\theta(r,\theta)|| \leq C\delta.$$

It follows now from Lemma 5.2.9 and Propostion 5.2.6 that it suffices to prove the following.

(i) Let us denote $\widetilde{\mathbf{J}}(r,\theta) = J(r,\theta)^{-1}$. There exists an absolute constant $C > 0$ such that $||\widetilde{\mathbf{J}}_r(r,\theta)|| \leq C, ||\widetilde{\mathbf{J}}_\theta(r,\theta)|| \leq C$.

(ii) Let $\mathbf{J}^0(r,\theta) = (\mathbf{J}_H(r,\theta) - \mathbf{I})$. Then there exists an absolute constant C such that $||\mathbf{J}^0_r(r,\theta)|| \leq C\delta, ||\mathbf{J}^0_\theta(r,\theta)|| \leq C\delta$.

(iii) There exists a constant $C > 0$ such that $||J_r(r, g_r(\theta))|| \leq C, ||J_\theta(r,\theta)|| \leq C\delta$.

The above three assertions are proved below in Lemma 5.2.13, Lemma 5.2.14 and Lemma 5.2.15 respectively. This completes the proof of the proposition. □

Lemma 5.2.13. *Let $\widetilde{\mathbf{J}}_r(r,\theta)$ be defined as in the proof of Proposition 5.2.12. Then there exists an absolute constant C such that for $r \in (r_1, \frac{1}{2})$ and $\theta \le 0$, we have $||\widetilde{\mathbf{J}}_r(r,\theta)|| \le C, ||\widetilde{\mathbf{J}}_\theta(r,\theta)|| \le C$.*

Lemma 5.2.14. *Let $\mathbf{J}^0$ be defined as in the proof of Proposition 5.2.12. Then there exists an absolute constant C such that $||\mathbf{J}_r^0(r,\theta)|| \le C\delta, ||\mathbf{J}_\theta^0(r,\theta)|| \le C\delta$.*

Lemma 5.2.15. *There exists a constant $C > 0$ such that $||J_r(r, g_r(\theta))|| \le C, ||J_\theta(r,\theta)|| \le C\delta$.*

Proofs of the above three lemmas are deferred to § 5.9.

Finally we have the following proposition.

Proposition 5.2.16. *Let $x, x' \in [0,1]^2$ be such that Ψ_δ is differentiable at both x and x'. Let $S_{x,x'}$ denote the event that the line joining x and x' intersects S. Also set*

$$g(x,x') = \min\{|x-(1/2,0)|, |x'-(1/2,0)|, |x-(1/2,1)|, |x'-(1/2,1)|\}.$$

Then there exists an absolute constant $C_3 > 0$ such that we have

$$||\Psi'_\delta(x) - \Psi'_\delta(x')|| \le C_3\delta\left(\frac{|x-x'|}{g(x,x')} \wedge 1 + 1_{S_{x,x'}}\right).$$

Proof. Clearly it follows from Proposition 5.2.6 that for all x, x', we have $||\Psi'_\delta(x) - \Psi'_\delta(x')|| \le 2C_1\delta$. For $x \in T_{r_1}$ it follows from explicit computations that $||\Psi''_\delta(x)|| \le \frac{C}{g(x,x)}$ for some absolute constant $C > 0$. It follows from mean value theorem that for $x, x' \in K^{-1}(T_{r_1})$ we have

$$||\Psi'_\delta(x) - \Psi'_\delta(x')|| \le C\delta\left(\frac{|x-x'|}{g(x,x')}\right)$$

for some absolute constant C. Notice that the same also follows for $x, x' \in [0,1]^2 \setminus T_{r_1}$ if $S_{x,x'}$ does not hold using Proposition 5.2.12. Now consider the case $x \in T_{r_1}, x' \notin T_{r_1}$, $S_{x,x'}$ does not hold. If $x \notin T_{r_1/2}$, $g(x,x') > r_1/2$ and mean value theorem once again gives the result using Proposition 5.2.12. In the only remaining case, $\frac{|x-x'|}{g(x,x')} > 1$ and so there is nothing to check. All these combined proves the lemma for an appropriate choice of C_3. □

5.2.4 Stretching Rectangles

For the proof of Theorem 5.1, we shall need to stretch not only the unit square but also squares and rectangles of different sizes. Also we shall need to stretch rectangles not only along its length (x_1-direction) but also along its height (x_2-direction). We can do these by using Ψ_δ composed with some linear functions as follows.

For $u = (u_1, u_2) \in \mathbb{R}^2$, and $a, b > 0$, let $D_{1,u,a,b} : [0,1]^2 \to u + [0,a] \times [0,b]$ be the bijection given by $(x_1, x_2) \mapsto u + (ax_1, bx_2)$. Similarly, let $D_{2,u,a,b} : [0,1]^2 \to u + [0,a] \times [0,b]$ be the bijection given by $(x_1, x_2) \mapsto u + (ax_2, bx_1)$. When u, a, b are clear from the context we shall suppress the subscript (u, a, b) and write D_1 or D_2 only.

Consider the rectangle $R = u + [0, a] \times [0, b]$. For $\Psi_\delta = (\Psi_\delta^1, \Psi_\delta^2)$ constructed as above, the function $\Psi_{\delta,R,\to} : R \to R$ is a bijection defined by $\Psi_{\delta,R,\to} = D_1 \circ \Psi_\delta \circ D_1^{-1}$. Similarly, the function $\Psi_{\delta,R,\uparrow} : R \to R$ is a bijection defined by $\Psi_{\delta,R,\uparrow} = D_2 \circ \Psi_\delta \circ D_2^{-1}$. Note that $\Psi_{\delta,a,b,\to}$ stretches the rectangle in a left-right direction whereas $\Psi_{\delta,a,b,\uparrow}$ stretches the rectangle in an up-down direction.

5.3 Dyadic Squares

For a rectangle R in $\mathbb{R}^2$, whose sides are aligned with the coordinate axes (i.e., of the for $[x, x + a] \times [y, y + b]$), we call a to be the *length* of R, and b to be the height of R. At each level $n \geq 0$ we write $[0, 1]^2$ as a union of (not necessarily disjoint) rectangles aligned with the co-ordinate axes $\{\Lambda_n^j\}_{j=1}^{2^{n-1}}$ satisfying the following properties.

1. $\Lambda_1 := \Lambda_1^1 = [0, 1]^2$.
2. For a fixed n, for each $j \in [2^{n-1}]$, area of $\Lambda_n^j = 2^{-(n-1)}$.
3. If n is odd, then each Λ_n^j is a square, i.e., has length and height equal. If n is even, then for each $j \in [2^{n-1}]$, the height of Λ_n^j is twice the length of Λ_n^j.
4. For each n and j, $\Lambda_n^j = \Lambda_{n+1}^{2j-1} \cup \Lambda_{n+1}^{2j}$. For n odd, Λ_n^j and Λ_{n+1}^{2j-1} has same height. For n even, Λ_n^j and Λ_{n+1}^{2j-1} have same length.

It is clear that there is a way to partition Λ_1 into rectangles at each level in such a manner, see Figure 5.4. Suppose $u_{n,j}$ denote the top right corner of $\Lambda_{n,j}$. For definiteness we shall adopt the following convention. For each j, $u_{n,2j} \succeq u_{n,2j-1}$ where $\succeq$ denotes the lexicographic ordering on $\mathbb{R}^2$. It is clear that under such a convention there is a unique way to construct Λ_n^js. We shall call the $\Lambda_{n,j}$ *dyadic boxes* at level n.

Let
$$B_n = \{u \in \Lambda_1 : u \in \Lambda_n^j \cap \Lambda_n^{j'} \text{ for some } j \neq j'\}.$$
It is then clear from the construction that we have for $B := \cup_n B_n$, $\lambda(B) = 0$.

For $x \in \Lambda_1 \setminus B$, let us define $\Lambda_{n,x} = \Lambda_n^j$ where j is such that $x \in \Lambda_n^j$. Let $\rho_{n,x}$ denote the density of A in $\Lambda_{n,x}$, i.e.,
$$\rho_{n,x} = \frac{\lambda(A \cap \Lambda_{n,x})}{\lambda(\Lambda_{n,x})}.$$
Also, define $\Delta_{n,x} = \rho_{n+1,x} - \rho_{n,x}$. Notice that $|\Delta_{n,x}|$ is constant on $\Lambda_{n,x} \setminus B$. We shall let $v_{n,x}$ denote the bottom left corner of $\Lambda_{n,x}$. We shall denote by $L_1(n, x)$ and $L_2(n, x)$, the length and height of $\Lambda_{n,x}$ respectively. Also $\delta(n, x) = \frac{\Delta_{n,y}}{\rho_{n-1,x}}$ for $y \in \Lambda_{n,x}$ which is very close to $v_{n,x}$.

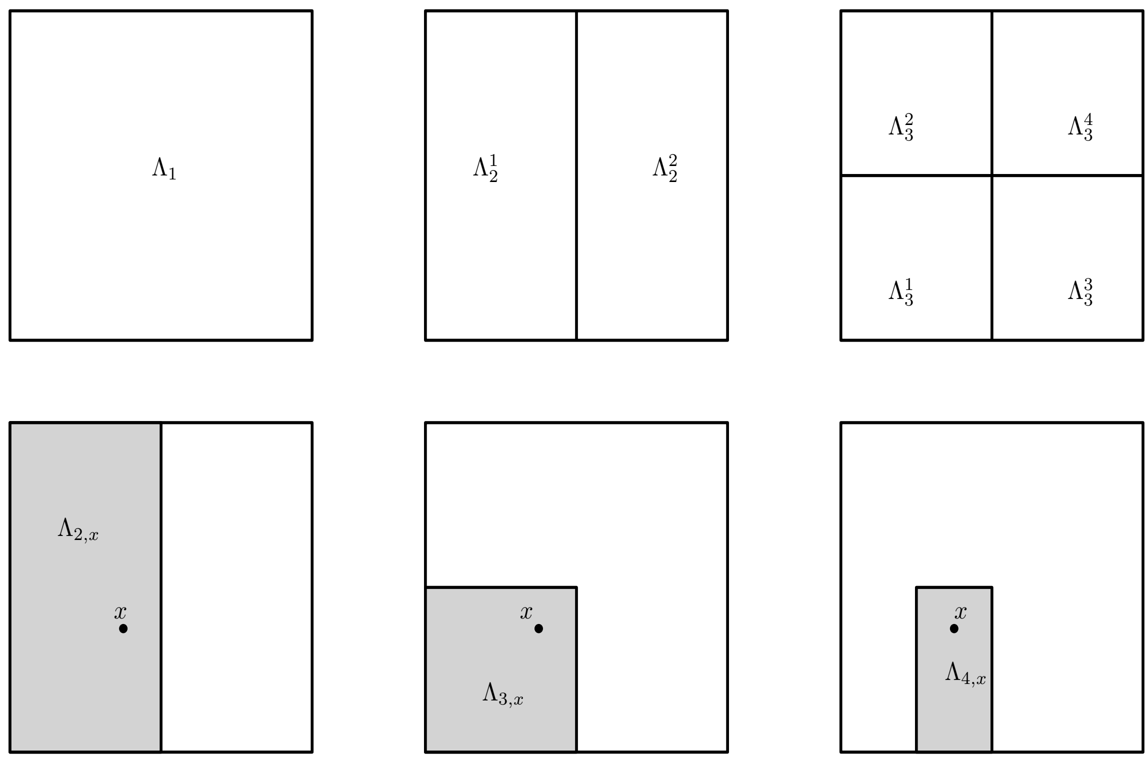


Figure 5.4: Dyadic squares at different levels

5.3.1 Crack, Seams and Twists on Dyadic Boxes

For $x \in [0,1]^2$, consider $\Lambda_{i,x}$, the i-th level dyadic box of x. If i is even, then define $S_{i,x} = D_{2,v_{i,x},L_1(i,x),L_2(i,x)}(S)$ to be the *seams* of $\Lambda_{i,x}$, $\mathfrak{C}_{i,x} = D_2(\mathfrak{C})$ to be the *crack* in $\Lambda_{i,x}$, and $T_{i,x} = D_2(T)$ to be the *twists* in $\Lambda_{i,x}$. If i is odd, we have same definitions except D_2 is replaced by D_1. Notice that, with these definition, it is clear that dyadic boxes at level $(i+1)$ are created by splitting level i dyadic boxes in half along the cracks at level i.

5.4 Stretching

5.4.1 Martingales

Now let X be a random variable which is uniformly distributed on Λ_1. Notice that $\Lambda_{n,X}$, $\rho_{n,X}$, $\Delta_{n,X}$ are almost surely well-defined. Let $\mathcal{F}_n$ denote the σ-algebra generated by $\Lambda_{n,X}$.

The following observation is trivial.

Observation 5.4.1. *We have* $\rho_{n,X} = \mathbb{P}[X \in A \mid \mathcal{F}_n]$ *is a martingale with respect the filtration* $\{\mathcal{F}_n\}_{n\geq 1}$*. Furthermore,* $\rho_{n,X} \to I(X \in A)$ *a.s. as* $n \to \infty$*.*

Clearly it follows from definitions that $|\Delta_{n,X}|$ is $\mathcal{F}_n$ measurable. This leads to following easy and useful observation.

Observation 5.4.2. *Now consider a random time* τ *which is* $\mathcal{F}_n$*-measurable. Then we have using the Optional Stopping Theorem*

$$Var[\rho_{\tau,X}] = \mathbb{E}\left[\sum_{i=2}^{\tau} \mathbb{E}[(\rho_{i,X} - \rho_{i-1,X})^2 \mid \mathcal{F}_{i-1}]\right] = \mathbb{E}\left(\sum_{i=1}^{\tau-1} \Delta_{i,X}^2\right). \tag{5.4.1}$$

5.4.2 Stretching at Different Scales

For $n = 1, 2, \ldots,$, define the function φ_n on Λ_1 as follows. If n is odd and $y \in \Lambda_{n,x}$, we define

$$\varphi_n(y) = \Psi_{\delta(n,x),\Lambda_{n,x},\to}(y).$$

If n is even and $y \in \Lambda_{n,x}$, we define

$$\varphi_n(y) = \Psi_{\delta(n,x),\Lambda_{n,x},\uparrow}(y).$$

Clearly, each φ_n is a bijection on Λ_1 that is identity on the boundary.

Define

$$\Phi_n(x) = \varphi_1 \circ \ldots \circ \varphi_n(x). \tag{5.4.2}$$

Clearly, Φ_n is also a bijection from Λ_1 to itself which is identity on the boundary. Also, set Φ_0 to be the identity. In Figure 5.5, we illustrate the sequence of functions Φ_0, Φ_1, Φ_2, where $\delta(1, x) = 0.5$ and $\delta(2, x) = -0.5$ on Λ_2^1.

Our primary objective will be to control the derivative of Φ_n which we can write as

$$\Phi_n'(x) = \prod_{i=1}^{n} \varphi_i'(\varphi_{i+1} \circ \ldots \circ \varphi_n(x)). \tag{5.4.3}$$

Notice that the product in the above equation is a product of matrices. We will anaylse the following $\mathcal{F}_n$-measurable approximation to Φ_n',

$$\mathbf{Y}_n = \prod_{i=1}^{n} \mathbb{E}\left[\varphi_i'(\varphi_{i+1} \circ \ldots \circ \varphi_n(X)) \mid \mathcal{F}_n\right]. \tag{5.4.4}$$

We further define the quantities

$$\mathbf{W}_{i,n} = \mathbb{E}\left[\varphi_i'(\varphi_{i+1} \circ \ldots \circ \varphi_n(X)) \mid \mathcal{F}_n\right] - \mathbb{E}\left[\varphi_i'(\varphi_{i+1} \circ \ldots \circ \varphi_{n-1}(X)) \mid \mathcal{F}_{n-1}\right]; \tag{5.4.5}$$

$$\mathbf{V}_{i,n} = \mathbb{E}\left[\varphi_i'(\varphi_{i+1} \circ \ldots \circ \varphi_{n-1}(X)) \mid \mathcal{F}_n\right] - \mathbb{E}\left[\varphi_i'(\varphi_{i+1} \circ \ldots \circ \varphi_{n-1}(X)) \mid \mathcal{F}_{n-1}\right]; \tag{5.4.6}$$

$$\mathbf{U}_{i,n} = \mathbb{E}\left[\varphi_i'(\varphi_{i+1} \circ \ldots \circ \varphi_n(X)) \mid \mathcal{F}_n\right] - \mathbb{E}\left[\varphi_i'(\varphi_{i+1} \circ \ldots \circ \varphi_{n-1}(X)) \mid \mathcal{F}_n\right]. \tag{5.4.7}$$

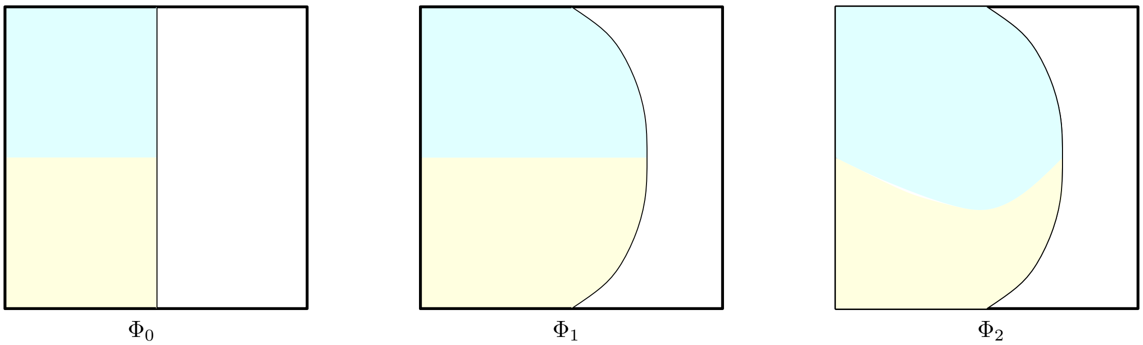


Figure 5.5: $\delta(1, x) = \frac{1}{2}$ and $\delta(2, x) = -0.5$ on Λ_2^1

It is clear that $\mathbf{W}_{i,n} = \mathbf{U}_{i,n} + \mathbf{V}_{i,n}$. Observe that

$$\mathbb{E}\left[\varphi_i'(\varphi_{i+1} \circ \ldots \circ \varphi_n(X)) \mid \mathcal{F}_n\right] - \sum_{j=i+1}^{n} \mathbf{W}_{i,j} = \mathbb{E}[\varphi_i'(X) \mid \mathcal{F}_i].$$

Since $\varphi_i(X)$ is identity on the boundary of $\Lambda_{i,X}$, Green's Theorem implies that integral of φ_i' over $\Lambda_{i,X}$ is equal to the integral of φ_i over the boundary of $\Lambda_{i,X}$ and hence

$$\mathbb{E}[\varphi_i'(X) \mid \mathcal{F}_i] = \mathbf{I}.$$

It follows that

$$\mathbf{Y}_n = \prod_{i=1}^{n} \left(\mathbf{I} + \sum_{j=i+1}^{n} \mathbf{W}_{i,j} \right). \tag{5.4.8}$$

Dealing with Twists and Seams:

While the derivative of Ψ is well behaved in most regions, observe that Ψ_δ is not differentiable on the *seams* S, and the second derivative is unbounded on the *twists* T. These shortcomings in smoothness are inherited by the functions ϕ_i. To deal with these issues we shall need the following notations.

For a fixed x, let

$$J_{i,x} = \max\{\ell : \Lambda_{i+\ell,x} \text{ intersects } T_{i,x}\}.$$

Clearly $J_{i,x}$ is almost surely well defined. It measures for how long the n-th level dyadic box containing x intersected the *twists* in $\Lambda_{i,x}$.

For a fixed x and i, and for $n > i$, let $A_{i,n,x}$ denote the event that $\varphi_{i+1} \circ \varphi_{i+2} \circ \ldots \circ \varphi_n(\Lambda_{n-1,x})$ intersects $S_{i,x}$. Let

$$\alpha_{i,n,x} = (n-i)1_{A_{i,n,x}}.$$

By construction the sets $A_{i,n,x}$ is decreasing in n. Now let $\alpha_{i,x} = \max\{n : \alpha_{i,n,x}\}$ and define $\beta_{i,x} = J_{i,x} \vee \alpha_{i,x}$. We set $\widetilde{\Delta}_{i,x} = |\Delta_{i,x}| 2^{\beta_{i,x}/10}$, re weighting $\Delta_{i,x}$ when it is close to a twist or seam. Finally for $n > i$, we define

$$\widetilde{\Delta}_{i,x,n} = |\Delta_{i,x}| 2^{(\beta_{i,x} \wedge (n-i))/10}.$$

Notice that $\widetilde{\Delta}_{i,x,n}$ is increasing in n and $\widetilde{\Delta}_{i,x,n} \leq \widetilde{\Delta}_{i,x}$. Also notice that $\widetilde{\Delta}_{i,x,n}$ is $\mathcal{F}_n$ measurable.

5.5 Stopping Times

The primary philosophy of our proof is to keep stretching A on smaller scales but stopping before it violates the Lipschitz constant. To implement this approach we define a series of stopping times. Let $\varepsilon_1, \varepsilon_2$ be small positive constants that will be chosen later in the proof and set $\varepsilon_3 = \frac{1}{2}\min\{\gamma, 1-\gamma'\}, \varepsilon_4 = \frac{\eta}{200}$. We set

$$\tau_1 := \tau_1(\varepsilon_1) = \inf\{n : \sum_{i=1}^{n-1} \widetilde{\Delta}^2_{i,X,n} > \varepsilon_1\}, \tag{5.5.1}$$

$$\tau_2 := \tau_2(\varepsilon_2) = \inf\{n : \Delta^2_{n,X} > \varepsilon_2\}, \tag{5.5.2}$$

$$\tau_3 := \tau_3(\varepsilon_3) = \inf\{n : |\rho_{n,X} - \lambda(A)| > \varepsilon_3\}, \tag{5.5.3}$$

$$\tau_4 := \tau_4(\varepsilon_4) = \inf\{n : ||Y_n(X) - I||_\infty > \varepsilon_4\}. \tag{5.5.4}$$

Also we define

$$\tau := \tau_1 \wedge \tau_2 \wedge \tau_3 \wedge \tau_4.$$

It is clear from Observation 5.4.1 that τ is finite almost surely. We primary work of the next two sections is to prove the following theorem.

Theorem 5.5.1. *There exists ε_1 and $\varepsilon_2 > 0$ such that for the stopping times defined above, we have*

$$\mathbb{P}[\tau_4 = \tau] < \frac{1}{3}.$$

5.6 Estimates on U and V

In this section we show that for a fixed i, on $\{n < \tau\}$, $||\mathbf{U}_{i,n}||$ and $||\mathbf{V}_{i,n}||$ cannot be too large and decays exponentially with $(n-i)$. We start with the estimate on $\mathbf{V}_{i,n}$.

Proposition 5.6.1. *There exists some absolute constant $C_4 > 0$ such that for each $i \geq 1$, and $n > i$, we have*

$$||\mathbf{V}_{i,n}(X)||1_{\{n<\tau\}} \leq C_4\widetilde{\Delta}_{i,X,n}2^{-(n-i)/20} \leq C_4\widetilde{\Delta}_{i,X}2^{-(n-i)/20}. \tag{5.6.1}$$

Proof. Observe that,

$$||\mathbf{V}_{i,n}(X)|| \leq \max_{x,x'\in\Lambda_{n-1,X}} ||\varphi_i'(\varphi_{i+1}\circ\ldots\circ\varphi_{n-1}(x)) - \varphi_i'(\varphi_{i+1}\circ\ldots\circ\varphi_{n-1}(x'))||. \tag{5.6.2}$$

Observe that, we have on $\{n < \tau\}$, $\delta(n,X) \leq \frac{2}{\gamma}\Delta_{n,X}$. Set $C_5 = \frac{2C_1}{\gamma}$. Now we need to consider two different cases.

Case 1: $i + \beta_{i,X} < n$. In this case, we have that the line joining x, x' does not intersect $S_{i,X}$. Fix a constant $\varepsilon_5 > 0$ sufficiently small. Notice that we have by Proposition 5.2.10 that if $\varepsilon_2 < \frac{\varepsilon_5}{100(C_5+1)}$, then on $\{n < \tau\}$, $\varphi_{i+1}\circ\ldots\circ\varphi_{n-1}$ is a bi-Lipschitz continuous function on $\Lambda_{n-1,X}$ with Lipschitz constant at most $(1+\varepsilon_5)^{n-i}$. Further observe the following. Since $i + \beta_{i,X} < n$, for any point $x \in \Lambda_{n-1,X}$, we have $d(x, T_{i,X}) \geq \frac{1}{4}2^{-(i+\beta_{i,X})/2}$. By bi-Lipschitz continuity of $\varphi_{i+1}\circ\ldots\circ\varphi_{n-1}$ it follows that $d(\varphi_{i+1}\circ\ldots\circ\varphi_{n-1}(\Lambda_{n-1,X}), T_{i,X}) \geq \frac{1}{4}(1+\varepsilon_5)^{-(n-i)}2^{-(i+\beta_{i,X})/2}$.

Hence it follows from (5.6.2) and Proposition 5.2.16 that on $\{n \leq \tau\}$ for some absolute constants C and ε_5 sufficiently small

$$\begin{aligned} ||\mathbf{V}_{i,n}(X)|| &\leq C\Delta_{i,X}\frac{\max_{x,x'\in\Lambda_{n-1,X}}|x-x'|||\varphi_{i+1}\circ\ldots\circ\varphi_{n-1}||_{lip}}{d(\varphi_{i+1}\circ\ldots\circ\varphi_{n-1}(\Lambda_{n-1,X}), T_{i,X})} \\ &\leq C\frac{2^{-n/2}(1+\varepsilon_5)^{n-i}}{(1+10\varepsilon_4)^{-(n-i)}2^{-(i+\beta_{i,X})/2}} \\ &\leq C\Delta_{i,X}(1+\varepsilon_5)^{2(n-i)}2^{-(n-i-\beta_{i,X})/2} \\ &\leq C\Delta_{i,X}2^{\beta_{i,X}/10}2^{-(n-i)/20} \end{aligned} \tag{5.6.3}$$

where the final inequality follows by taking ε_5 sufficiently small.

Case 2: $i + \beta_{i,X} \geq n$.

In this case it follows from Proposition 5.2.16 that we have on $\{i < \tau\}$

$$\max_{x,x'\in\Lambda_{i,X}} ||\varphi_i'(x) - \varphi_i'(x')|| \leq C\Delta_{i,X}.$$

It follows now from (5.6.2) that on $\{n < \tau\}$,

$$||\mathbf{V}_{i,n}(X)|| \leq C\Delta_{i,X} \leq C\Delta_{i,X}2^{(n-i)/10}2^{-(n-i)/20}. \tag{5.6.4}$$

The proposition now follows from (5.6.3) and (5.6.4) by choosing C_4 appropriately. □

We have a similar result for $\mathbf{U}_{i,n}$ where we get a $\widetilde{\Delta}_{i,X}\Delta_{n,X}$ term instead of the $\Delta_{i,X}$ term in the above proposition.

Proposition 5.6.2. *There exists some absolute constant $C_6 > 0$ such that for each $i \geq 1$, and $n > i$, we have*

$$||\mathbf{U}_{i,n}(X)||1_{\{n<\tau\}} \leq C_6\widetilde{\Delta}_{i,X,n}\Delta_{n,X}2^{-(n-i)/20}. \tag{5.6.5}$$

To prove Proposition 5.6.2 we need some additional work to deal with the possibility that $\varphi_{i+1} \circ \cdots \varphi_{n-1}(\Lambda_{n-1,X})$ might intersect $S_{i,X}$. To this end we make the following definition. For $x \in \Lambda_{n,X}$, let $A_{i,n,x}$ denote the event that the line segment joining $\varphi_{i+1} \circ \ldots \circ \varphi_{n-1}(x)$ and $\varphi_{i+1} \circ \ldots \circ \varphi_n(x)$ intersects $S_{i,X}$. Let

$$\mathcal{A}_{i,n,X} = \{x \in \Lambda_{n,X} : 1_{\{A_{i,n,x}\}} > 0\}.$$

We have the following lemma bounding the measure of the set $\mathcal{A}_{i,n,X}$.

Lemma 5.6.3. *For some absolute constant $C > 0$, we have on $\{n < \tau\}$*

$$\lambda(\mathcal{A}_{i,n,X}) \leq C\Delta_{n,X}2^{(n-i)/20}2^{-n}. \tag{5.6.6}$$

Proof. Denote the two seams in $\Lambda_{i,X}$ by $S^1_{i,X}$ and $S^2_{i,X}$ respectively. For $x \in \Lambda_{n,X}$, let $A^1_{i,n,x}$ denote the event that the line segment joining $\varphi_{i+1} \circ \ldots \circ \varphi_{n-1}(x)$ and $\varphi_{i+1} \circ \ldots \circ \varphi_n(x)$ intersects $S^1_{i,X}$. Let

$$\mathcal{A}^1_{i,n,X} = \{x \in \Lambda_{n,X} : 1_{\{A_{i,n,x}\}} > 0\}.$$

By symmetry, it suffices to prove that for some absolute constant $C > 0$, we have on $\{n < \tau\}$

$$\lambda(\mathcal{A}^1_{i,n,X}) \leq C\Delta_{n,X}2^{(n-i)/20}2^{-n}. \tag{5.6.7}$$

Interpreting $S^1_{i,X}$ as a directed simple curve there exist a *first point* $y \in S^1_{i,X}$ where $S^1_{i,X}$ enters $\varphi_{i+1} \circ \ldots \circ \varphi_{n-1}(\Lambda_{n,X})$ and a *last point* $y' \in S^1_{i,X}$ where $S^1_{i,X}$ exits $\varphi_{i+1} \circ \ldots \circ \varphi_{n-1}(\Lambda_{n,X})$. If ε_5 is such that the bi-Lipschitz constant of $\varphi_{i+1} \circ \ldots \circ \varphi_{n-1}$ is at most $(1+\varepsilon_5)^{n-i}$ then we get that $|y - y'| \leq C(1+\varepsilon_5)^{(n-i)}2^{-n/2}$ for some absolute constant $C > 0$. Let $S^1_{i,X}(y, y')$ denote the curve segment $S^1_{i,X}$ from y to y'. Let $\ell(y, y')$ denote the length $S^1_{i,X}$ from y to y'. It then follows that $\ell(y, y') \leq C(1+\varepsilon_5)^{(n-i)}2^{-n/2}$ for some absolute constant $C > 0$. Now define

$$A_{y,y',C'} = \{x \in \Lambda_{i,X} : \exists z \in S^1_{i,X}(y, y') \text{ such that} |x - z| \leq C'\Delta_{n,X}(1+\varepsilon_5)^{(n-i)}2^{-n/2}\}.$$

Clearly $x \in \mathcal{A}^1_{i,n,X}$ implies $\varphi_{i+1} \circ \ldots \circ \varphi_{n-1}(x) \in A_{y,y',C'}$ if $|\varphi_n(x) - x| \leq C'\Delta_{n,X}2^{-n/2}$. It follows that

$$\lambda(\mathcal{A}^1_{i,n,X}) \leq (1+\varepsilon_5)^{(n-i)}\lambda(A_{y,y',C'}).$$

Clearly for some constant $C > 0$,

$$\lambda(A_{y,y',C'}) \leq C\ell(y,y')\Delta_{n,X}(1+\varepsilon_5)^{(n-i)}2^{-n/2} \leq C\Delta_{n,X}(1+\varepsilon_5)^{2(n-i)}2^{-n}$$

and hence we have

$$\lambda(\mathcal{A}^1_{i,n,X}) \leq C\Delta_{n,X}(1+\varepsilon_5)^{3(n-i)}2^{-n}.$$

By taking ε_5 sufficiently small we establish (5.6.7) and the lemma is proved. □

We also need the following lemma.

Lemma 5.6.4. *For some absolute constant $C > 0$, we have for each $x \in \Lambda_{n,x}$, on $\{n < \tau\}$*

$$\frac{|\varphi_{i+1}\circ\ldots\circ\varphi_n(x) - \varphi_{i+1}\circ\ldots\circ\varphi_{n-1}(x)|}{d(\varphi_{i+1}\circ\ldots\circ\varphi_n(x), T_{i,X})} \leq C\Delta_{n,X}2^{(n-i)/20}. \tag{5.6.8}$$

Proof. It follows from Proposition 5.2.11 that for some absolute constant C for all $x \in \Lambda_{n,X}$ we have $|\varphi_n(x) - x| \leq C\Delta_{n,X}d(x, T_{i,X})$. Now for ε_5 as in the previous case, we have

$$|\varphi_{i+1}\circ\ldots\circ\varphi_n(x) - \varphi_{i+1}\circ\ldots\circ\varphi_{n-1}(x)| \leq (1+\varepsilon_5)^{(n-i)}d(x, T_{i,X})$$

and also

$$d(\varphi_{i+1}\circ\ldots\circ\varphi_n(x), T_{i,X}) \geq (1+\varepsilon_5)^{-(n-i)}d(x, T_{i,X}).$$

Taking ε_5 sufficiently small, (5.6.8) follows from the above two equations. □

Now we are ready to prove Proposition 5.6.2.

Proof of Proposition 5.6.2. For the proof of this proposition also we need to consider two cases.

Case 1: $i + \beta_{i,X} < n$.

In this case, we have that $\varphi_{i+1}\circ\cdots\varphi_{n-1}(\Lambda_{n-1,X})$ does not intersect $S_{i,X}$. Hence it follows that by arguments similar to those in the proof of Proposition 5.6.1 that on $\{n < \tau\}$ we have using Proposition 5.2.16

$$||\mathbf{U}_{i,n}(X)|| \leq \frac{2C_3}{\gamma}\Delta_{i,X}\max_{x\in\Lambda_{n,X}}\frac{|\varphi_{i+1}\circ\ldots\circ\varphi_n(x) - \varphi_{i+1}\circ\ldots\circ\varphi_{n-1}(x)|}{d(\varphi_{i+1}\circ\ldots\circ\varphi_n(x), T_{i,X})}. \tag{5.6.9}$$

Now observe that since $i+\beta_{i,X} < n$, we have that $d(\Lambda_{n,X}, T_{i,X}) \geq \frac{1}{4}2^{-(i+\beta_{i,X})/2}$. Choosing ε_5 and ε_2 as in the proof of Proposition 5.6.1 such that the bi-Lipschitz constant of $\varphi_{i+1}\circ\ldots\circ\varphi_n$ is at most $(1+\varepsilon_5)^{n-i}$ we get that $d(\varphi_{i+1}\circ\ldots\circ\varphi_n(\Lambda_{n,X}), T_{i,X}) \geq \frac{1}{4}(1+\varepsilon_5)^{-(n-i)}2^{-(i+\beta_{i,X})/2}$. Also observe that it follows from Proposition 5.2.11 that for some absolute constant $C > 0$ we have

$$\max_{x\in\Lambda_{n,X}}|\varphi_n(x) - x| \leq C\Delta_{n,X}2^{-n/2}$$

It follows now from (5.6.9) that on $\{n < \tau\}$ for some absolute constant C

$$\begin{aligned} ||\mathbf{U}_{i,n}(X)|| &\leq C\Delta_{i,X}\frac{||\varphi_{i+1}\circ\ldots\circ\varphi_{n-1}||_{lip}C\Delta_{n,X}}{(1+\varepsilon_5)^{-(n-i)}2^{-(i+\beta_{i,X})/2}} \\ &\leq C\Delta_{i,X}\Delta_{n,X}(1+\varepsilon_5)^{2(n-i)}2^{-(n-i-\beta_{i,X})/2} \\ &\leq C\Delta_{i,X}\Delta_{n,X}2^{\beta_{i,X}/10}2^{-(n-i)/20}. \end{aligned} \tag{5.6.10}$$

Case 2: $i + \beta_{i,X} \geq n$. It follows from Proposition 5.2.16 that

$$||\mathbf{U}_{i,n}(X)|| \leq C\Delta_{i,X}\left(2^n\lambda(\mathcal{A}_{i,n,X}) + \max_{x\in\Lambda_{n,X}\setminus\mathcal{A}_{i,n,X}}\frac{|\varphi_{i+1}\circ\ldots\circ\varphi_n(x) - \varphi_{i+1}\circ\ldots\circ\varphi_{n-1}(x)|}{d(\varphi_{i+1}\circ\ldots\circ\varphi_n(x), T_{i,X})}\right) \tag{5.6.11}$$

and using (5.6.11), Lemma 5.6.3 and Lemma 5.6.4 we get for some absolute constant $C > 0$

$$||\mathbf{U}_{i,n}(X)|| \leq C\Delta_{i,X}\Delta_{n,X}2^{(n-i)/20} \leq C\Delta_{i,X}\Delta_{n,X}2^{(n-i)/10}2^{-(n-i)/20} \leq C\widetilde{\Delta}_{i,X,n}\Delta_{n,X}2^{-(n-i)/20}.$$

Proof of the proposition is completed by choosing C_6 appropriately. □

5.7 Bounding $\mathbf{Y}_n$

Our next step is to prove Theorem 5.5.1. That is, we need to show that it is unlikely that $||\mathbf{Y}_n - \mathbf{I}||$ becomes large before either $\rho_{n,X}$ deviates significantly from $\lambda(A)$, $\Delta_{n,X}$ becomes sufficiently large or $\sum_{k\leq n}\widetilde{\Delta}^2_{k,X,n}$ becomes sufficiently large. For this purpose we construct a matrix-valued martingale $\mathbf{M}_n$.

5.7.1 Constructing $\mathbf{M}_n$

Define a sequence $\{\mathbf{M}_n\}_{n\geq 1}$ of matrix-valued random objects as follows.

1. Set $\mathbf{M}_1 = \mathbf{I}$.
2. For $n \geq 1$, set

$$\mathbf{M}_{n+1} - \mathbf{M}_n = \sum_{k=1}^{n}\prod_{i=1}^{k}\left(\mathbf{I} + \sum_{j=i+1}^{n}\mathbf{W}_{i,j}\right)\mathbf{V}_{k,n+1}\prod_{i=k+1}^{n}\left(\mathbf{I} + \sum_{j=i+1}^{n}\mathbf{W}_{i,j}\right). \tag{5.7.1}$$

Clearly, it follows that $\mathbf{M}_n$ is $\mathcal{F}_n$ measurable and since $\mathbb{E}[\mathbf{V}_{k,n+1} \mid \mathcal{F}_n] = 0$ it follows that $\mathbf{M}_n$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}$.

Let τ_1, τ_2, τ_3 be defined by (5.5.1), (5.5.2) and (5.5.3) respectively. Define the stopping times τ_5 and τ_6 by

$$\tau_5 = \{\inf n : ||Y_n|| \geq 2\}. \tag{5.7.2}$$

$$\tau_6 = \tau_6(\varepsilon_6) = \{\inf n : ||\mathbf{M}_n - \mathbf{I}|| \geq \varepsilon_6\}. \tag{5.7.3}$$

Let us define

$$\tau' = \tau_1 \wedge \tau_2 \wedge \tau_3 \wedge \tau_5$$

Observe that $\tau \leq \tau'$.
We shall prove the following theorem.

Theorem 5.7.1. *Set $\varepsilon_6 = \eta/800$. Then there exists ε_1, $\varepsilon_2 > 0$ such that we have*

$$\mathbb{P}[\tau_6 < \tau'] \leq \frac{1}{5}. \tag{5.7.4}$$

Before proving Theorem 5.7.1, observe the following. It follows from (5.4.8) that

$$\mathbf{M}_{n+1} - \mathbf{M}_n = \sum_{k=1}^{n} \mathbf{Y}_n \left(\prod_{i=k+1}^{k} \left(\mathbf{I} + \sum_{j=i+1}^{n} \mathbf{W}_{i,j} \right) \right)^{-1} \mathbf{V}_{k,n+1} \prod_{i=k+1}^{n} \left(\mathbf{I} + \sum_{j=i+1}^{n} \mathbf{W}_{i,j} \right). \tag{5.7.5}$$

Choose ε_5 sufficiently small and set $\varepsilon_2 = \frac{\varepsilon_5}{100(C_5+1)}$. By choosing ε_5 sufficiently small we have for each $i \in [k+1, n]$ we have $||\mathbf{I} + \sum_{j=i+1}^{n} \mathbf{W}_{i,j}|| \leq (1+\varepsilon_5)$, $|| \left(\mathbf{I} + \sum_{j=i+1}^{n} \mathbf{W}_{i,j} \right)^{-1} || \leq (1+\varepsilon_5)$ using Proposition 5.2.6. It follows now using Proposition 5.6.1 that on $\{n < \tau'\}$ we have for ε_5 small enough and for some absolute constant $C > 0$

$$\begin{aligned}
||\mathbf{M}_{n+1} - \mathbf{M}_n||^2 &\leq C||\mathbf{Y}_n||^2 \left(\sum_{k=1}^{n} 2^{-(n+1-k)/20} (1+\varepsilon_5)^{2(n+1-k)} \widetilde{\Delta}_{k,X,n} \right)^2 \\
&\leq C||\mathbf{Y}_n||^2 \left(\sum_{k=1}^{n} 2^{-(n+1-k)/50} \widetilde{\Delta}_{k,X,n} \right)^2 \\
&\leq C||\mathbf{Y}_n||^2 \left(\sum_{k=1}^{n} 2^{-(n+1-k)/100} \right) \left(\sum_{k=1}^{n} 2^{-(n+1-k)/100} \widetilde{\Delta}^2_{k,X,n} \right) \\
&\leq C \sum_{k=1}^{n} 2^{-(n+1-k)/100} \widetilde{\Delta}^2_{k,X,n}.
\end{aligned} \tag{5.7.6}$$

where the third inequality is by the Cauchy-Schwartz Inequality. Hence on the event $\{n+1 < \tau'\}$ we have

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{E}[||\mathbf{M}_{i+1} - \mathbf{M}_i||^2 \mid \mathcal{F}_i] &\leq C \sum_{i=1}^{n} \sum_{k=1}^{i} 2^{-(i+1-k)/100} \widetilde{\Delta}_{k,X}^2 \\ &\leq C \sum_{i=1}^{n} \widetilde{\Delta}_{i,X,n}^2 \leq C_8 \varepsilon_1. \end{aligned} \tag{5.7.7}$$

Similarly to (5.7.6), we have that on the event $\{n < \tau'\}$ for some absolute constant $C > 0$

$$\begin{aligned}||\mathbf{M}_{n+1} - \mathbf{M}_n|| &\leq C||Y_n|| \sum_{k=1}^{n} 2^{-(n+1-k)/50} \widetilde{\Delta}_{k,X,n} \\ &\leq C \sum_{k=1}^{n} 2^{-(n+1-k)/50} \left(2^{(n-k)/10} \sqrt{\varepsilon_2} \wedge \sqrt{\varepsilon_1}\right) \\ &\leq 3\varepsilon_1^{1/2} \end{aligned} \tag{5.7.8}$$

for ε_2 sufficiently small.

Having bounded the quadratic variation and increment size of $\mathbf{M}_{n+1}$ we use the following inequality for tail probability of a martingale, which is a generalisation of Bernstein's inequality.

Theorem 5.7.2 (Freedman, 1975 [15])**.** *Let $\{X_n\}_{n\geq 1}$ be a martingale with respect to the filtration $\{\mathcal{F}_n\}$, with $|X_{k+1} - X_k| \leq R$ almost surely. Let $Y_n = \sum_{i=1}^{n-1} \mathbb{E}\left[(X_{i+1} - X_i)^2 \mid \mathcal{F}_i\right]$. Then for each t*

$$\mathbb{P}[\exists n : X_n - X_1 > t, Y_n \leq \sigma^2] \leq \exp\left\{-\frac{t^2}{2(\sigma^2 + Rt/3)}\right\}. \tag{5.7.9}$$

Now we are ready to prove Theorem 5.7.1.

Proof of Theorem 5.7.1. Choose ε_1 sufficiently small so that $\varepsilon_6 \geq \left(\sqrt{20C_8} \vee \sqrt{20}\right) \varepsilon_1^{1/2}$. For $i, j = 1, 2$ let the (i, j)-th entry of $\mathbf{M}_n$ be $M_n^{i,j}$. Consider the martingales $\{X_n^{i,j}\} = \{M_{n\wedge\tau'}^{i,j}\}$. It follows from Theorem 5.7.2 using (5.7.7) and (5.7.8) that

$$\mathbb{P}[\exists n < \tau' : |X_n^{i,j} - \delta_{i,j}| \geq \varepsilon_6] \leq 2e^{-5}.$$

Taking a union bound over different values of i and j it follows that

$$\mathbb{P}[\exists n < \tau' : ||\mathbf{M}_n - \mathbf{I}|| \geq \varepsilon_6] \leq 8e^{-5} < \frac{1}{5}.$$

This finishes the proof of the theorem. ∎

5.7.2 Bounding $\mathbf{Y}_n - \mathbf{M}_n$

We define $\mathbf{D}_n = \mathbf{Y}_n - \mathbf{M}_n$. Note that $\mathbf{D}_1 = \mathbf{0}$. We have the following lemma.

Lemma 5.7.3. *Set $\varepsilon_8 = \eta/800$. There exists $\varepsilon_1, \varepsilon_2 > 0$, satisfying the conclusion of Theorem 5.7.1 such that we have*

$$||\mathbf{D}_t||1_{\{t<\tau'\}} \leq \varepsilon_8. \tag{5.7.10}$$

Proof. Observe that on $\{t < \tau'\}$, we have

$$||\mathbf{D}_t|| \leq \sum_{n=1}^{t-1} ||\mathbf{D}_{n+1} - \mathbf{D}_n||.$$

Expanding out D_n we have

$$\begin{aligned} \mathbf{D}_{n+1} - \mathbf{D}_n &= \sum_{k=1}^{n} \left(\prod_{i=1}^{k} \left(\mathbf{I} + \sum_{j=i+1}^{n} \mathbf{W}_{i,j} \right) \mathbf{U}_{k,n+1} \prod_{i=k+1}^{n} \left(\mathbf{I} + \sum_{j=i+1}^{n} \mathbf{W}_{i,j} \right) \right) \\ &+ \sum_{S \subseteq [n], |S| \geq 2} \left(\prod_{i=1}^{n} \left(1_{\{i \in S\}} \mathbf{W}_{i,n+1} + 1_{\{i \notin S\}} \left(\mathbf{I} + \sum_{j=i+1}^{n} \mathbf{W}_{i,j} \right) \right) \right). \end{aligned} \tag{5.7.11}$$

Call the first term on the right hand side of the above equation $\mathbf{A}_n$, call the second term $\mathbf{B}_n$. Choosing $\varepsilon_5 = 100(C_5 + 1)\varepsilon_2$ and arguing as in (5.7.6) we get that for some absolute constant $C > 0$ we get that on $\{n + 1 < \tau'\}$,

$$\begin{aligned} ||\mathbf{A}_n|| &\leq C \sum_{k=1}^{n} 2^{-(n+1-k)/20} (1 + \varepsilon_5)^{2(n+1-k)} \widetilde{\Delta}_{k,X,n+1} \Delta_{n+1,X} \\ &\leq C \sum_{k=1}^{n} 2^{-(n+1-k)/50} \left(\widetilde{\Delta}^2_{k,X,n+1} + \Delta^2_{n+1,X} \right) \\ &\leq C \Delta^2_{n+1,X} + C \sum_{k=1}^{n} 2^{-(n+1-k)/50} \widetilde{\Delta}^2_{k,X,n+1}. \end{aligned} \tag{5.7.12}$$

It follows that on $\{t < \tau'\}$ for some absolute constants $C, C_9 > 0$

$$\begin{aligned} \sum_{n=1}^{t-1} ||\mathbf{A}_n|| &\leq C \sum_{n=1}^{t-1} \Delta^2_{n+1,X} + C \sum_{n=1}^{t-1} \sum_{k=1}^{n} 2^{-(n+1-k)/50} \widetilde{\Delta}^2_{k,X,t} \\ &\leq C \sum_{i=1}^{t} \widetilde{\Delta}^2_{i,x,t} \leq C_9(\varepsilon_1 + \varepsilon_2). \end{aligned} \tag{5.7.13}$$

For obtaining a bound on $\mathbf{B}_n$, observe the following. Fix $S \subseteq [n]$ with $|S| \geq 2$. It follows from Proposition 5.6.1 and Proposition 5.6.2 that for some absolute constant $C >$

0 we have $|\mathbf{W}_{i,n+1}| \leq C\widetilde{\Delta}_{i,X}2^{-(n+1-i)/20}$ for each $i \in S$. Write $F_{i,n} = C\widetilde{\Delta}_{i,X,n+1}(1+\varepsilon_5)^{2(n+1-i)}2^{-(n+1-i)/20}$. Arguing as in (5.7.6) it follows by taking ε_2 sufficiently small, on $\{n+1<\tau'\}$, we have

$$\begin{aligned}\prod_{i=1}^{n}\left(1_{\{i\in S\}}\mathbf{W}_{i,n+1} + 1_{\{i\notin S\}}\left(\mathbf{I}+\sum_{j=i+1}^{n}\mathbf{W}_{i,j}\right)\right) &\leq ||Y_n||\prod_{i\in S}F_{i,n}\\ &\leq \prod_{i\in S}C\widetilde{\Delta}_{i,X,n+1}2^{-(n+1-i)/50}.\end{aligned} \tag{5.7.14}$$

Summing over all $S\subseteq[n]$ with $|S|\geq 2$ we get that

$$||\mathbf{B}_n|| \leq \prod_{i=1}^{n}\left(1+C\widetilde{\Delta}_{i,X,n+1}2^{-(n+1-i)/50}\right) - 1 - \sum_{i=1}^{n}C\widetilde{\Delta}_{i,X,n+1}2^{-(n+1-i)/50}. \tag{5.7.15}$$

Now observe that on $\{n+1<\tau'\}$, by choosing ε_2 sufficiently small we have

$$\sum_{i=1}^{n}C\widetilde{\Delta}_{i,X,n+1}2^{-(n+1-i)/50} \leq \frac{1}{10}$$

by the argument used in (5.7.8). It then follows that

$$||\mathbf{B}_n|| \leq 2C\sum_{i=1}^{n}\widetilde{\Delta}_{i,X,n+1}2^{-(n+1-i)/50} \leq 10C\sum_{i=1}^{n}\widetilde{\Delta}^2_{i,X,n+1}2^{-(n+1-i)/200}, \tag{5.7.16}$$

where the final step follows from the Cauchy-Schwarz inequality.

Summing over n we get on $\{t<\tau'\}$, for some absolute constant $C_{10}>0$

$$\sum_{n=1}^{t-1}||\mathbf{B}_n|| \leq C_{10}\sum_{i=1}^{t}\widetilde{\Delta}^2_{i,X,t} \leq C_{10}(\varepsilon_1+\varepsilon_2), \tag{5.7.17}$$

It now follows from (5.7.11), (5.7.13) and (5.7.17) that we have

$$||\mathbf{D}_t||1_{\{t<\tau'\}} \leq (C_9+C_{10})(\varepsilon_1+\varepsilon_2).$$

Choosing ε_1 and ε_2 sufficiently small such that $(C_9+C_{10})(\varepsilon_1+\varepsilon_2)\leq\varepsilon_8$, we complete the proof of the lemma. □

5.7.3 Proof of Theorem 5.5.1

Now we are ready to prove Theorem 5.5.1.

Proof. Observe that we have $\varepsilon_6+\varepsilon_8<\varepsilon_4$. Fix ε_1, $\varepsilon_2>0$ such that Theorem 5.7.1 and Lemma 5.7.3 holds. It follows from and Lemma 5.7.3, that on $\{\tau_6\geq\tau'\}$, for all $n<\tau'$ we have

$$||\mathbf{Y}_n-\mathbf{I}|| \leq \varepsilon_6+\varepsilon_8<\varepsilon_4.$$

Proof of Theorem 5.5.1 is then completed using Theorem 5.7.1. □

5.8 Proof of Theorem 5.3

We complete the proof of Theorem 5.3 in this section. We first need the following lemmas.

Lemma 5.8.1. *Set $\varepsilon_9 = \eta/400$. Then we can choose ε_1, $\varepsilon_2 > 0$, in Theorem 5.5.1 so that we have on $\{n < \tau\}$, $||\Phi'_n - \mathbf{Y}_n|| \leq \varepsilon_9$.*

Proof. Write
$$\xi_{i,n} = \varphi'_i(\varphi_{i+1} \circ \ldots \circ \varphi_n) - \mathbb{E}[\varphi'_i(\varphi_{i+1} \circ \ldots \circ \varphi_n) \mid \mathcal{F}_n].$$
Observe that $||\xi_{i,n}(X)||$ is upper bounded by the right hand side of (5.6.2) with n there replaced by $(n+1)$. Now arguing as in the proof of Proposition 5.6.1 it follows that for some absolute constant $C > 0$ we have

$$||\xi_{i,n}(X)|| \leq C\widetilde{\Delta}_{i,X,n}2^{-(n-i)/20}. \tag{5.8.1}$$

Notice that

$$\Phi'_n = \prod_{i=1}^{n}\left(\mathbf{I} + \sum_{j=i+1}^{n}\mathbf{W}_{i,j} + \xi_{i,n}\right). \tag{5.8.2}$$

It now follows that

$$\Phi'_n - \mathbf{Y}_n = \sum_{\emptyset \neq S \subseteq [n]}\prod_{i=1}^{n}\left(1_{i\notin S}\left(\mathbf{I} + \sum_{j=i+1}^{n}\mathbf{W}_{i,j}\right) + 1_{i\in S}\xi_{i,n}\right). \tag{5.8.3}$$

An argument similar to the one used in (5.7.15) gives that for ε_2 sufficiently small, we have on $\{n < \tau\}$

$$||\Phi'_n - \mathbf{Y}_n|| \leq \prod_{i=1}^{n}\left(1 + C\widetilde{\Delta}_{i,x,n}2^{-(n-i)/100}\right) - 1 \leq 3C\sum_{i=1}^{n}\widetilde{\Delta}_{i,x,n}2^{-(n-i)/100}. \tag{5.8.4}$$

Using the Cauchy-Schwarz inequality as in (5.7.16) we get that for some absolute constant $C > 0$ we have
$$||\Phi'_n - \mathbf{Y}_n|| \leq C(\varepsilon_1 + \varepsilon_2).$$
The lemma follows by taking ε_1 and ε_2 sufficiently small. □

Lemma 5.8.2. *We can choose ε_1, $\varepsilon_2 > 0$ in Theorem 5.5.1 such that for some absolute constant $C_{11} > 0$ we have*
$$\mathbb{E}[\widetilde{\Delta}^2_{i,X,\tau} \mid \mathcal{F}_{i+1}] \leq C_{11}\Delta^2_{i,X}. \tag{5.8.5}$$

Proof. The above lemma is an immediate consequence of the following lemma. □

Lemma 5.8.3. *We can choose ε_1, $\varepsilon_2 > 0$ in Theorem 5.5.1 such that we have $\mathbb{P}[\beta_{i,X} \geq (n-i), \tau \geq n \mid \mathcal{F}_{i+1}] \leq 10 \times 2^{-(n-i)/3}$, for n sufficiently large.*

Proof. It is clear from definition that $\mathbb{P}[J_{i,X} \geq (n-i) \mid \mathcal{F}_{i+1}] \leq 2\times 2^{-(n-i)/3}$. Hence it suffices to show that $\mathbb{P}[\alpha_{i,X} \geq (n-i), \tau > n \mid \mathcal{F}_{i+1}] \leq 8 \times 2^{-(n-i)/3}$. It is easy to see that it suffices to show $\mathbb{P}[\alpha_{i,X} \geq (n-i), \tau \geq n \mid \mathcal{F}_i] \leq 4 \times 2^{-(n-i)/3}$.

From definition, if $\{\alpha_{i,X} \geq (n-i), n \leq \tau\}$, then $\varphi_{i+1}\circ \cdots \varphi_n(\Lambda_{n,X}) = \varphi_{i+1}\circ\cdots\varphi_{n-1}(\Lambda_{n,X})$ intersects $S_{i,X}$. Notice that the total length of the curve(s) $S_{i,X}$ is at most $C_{12}2^{-i/2}$ for some absolute constant C_{12}. Let ε_5 be a constant such that on $\{n < \tau\}$, $\varphi_{i+1}\circ\cdots\varphi_n$ is bi-Lipschitz with Lipschitz constant at most $(1+\varepsilon_5)^{n-i}$. It follows that $\varphi_{i+1}\circ\cdots\varphi_{n\wedge\tau-1}$ is also bi-Lipschitz with Lipschitz constant at most $(1+\varepsilon_5)^{n-i}$. Note that, as before, ε_5 can be made arbitrarily small by taking ε_2 small. Hence it follows that there exists a set $\mathcal{M}$ of $N = 8C_{12}(1+\varepsilon_5)^{n-i}2^{(n-i)/2}$ points on $S_{i,X}$ such that any point on $S_{i,X}$ is at most distance $\frac{1}{4}(1+\varepsilon_5)^{(n-i)}2^{-n/2}$ from some point in $\mathcal{M}$. Let $\mathcal{M} = \{x_1, x_2, \ldots, x_N\}$. It follows that any point on $(\varphi_{i+1}\circ\cdots\circ\varphi_{n\wedge\tau-1})^{-1}(S_{i,X})$ is at most at distance $\frac{1}{4}2^{-n/2}$ from $(\varphi_{i+1}\circ\cdots\circ\varphi_{n\wedge\tau-1})^{-1}(x_k)$ for some k. It follows that for

$$\mathbb{P}[\Lambda_{n,X} \text{ intersects } \varphi_{i+1}\circ\cdots\circ\varphi_{n-1})^{-1}(S_{i,X}), \tau \geq n \mid \mathcal{F}_i] \leq 32C_{12}(1+\varepsilon_5)^{n-i}2^{-(n-i)/2} \leq 2^{-(n-i)/3}$$

by taking ε_5 sufficiently small completing the lemma. □

Lemma 5.8.4. *We have for the stopping time τ w.r.t. the filtration $\mathcal{F}_i$,*

$$\lambda(\Phi_{\tau-1}(A)) = \frac{\mathbb{E}\rho^2_{\tau,X}}{\mathbb{E}\rho_{\tau,X}} = \lambda(A) + \frac{1}{\lambda(A)}Var(\rho_{\tau,X}). \tag{5.8.6}$$

Proof. Let $W_1, W_2, \ldots$ be the disjoint (except may be at the boundary) τ level dyadic boxes (i.e., $W_k = \Lambda_{\tau,X}$ on $\{X \in W_k\}$) such that $\cup_k W_k = \Lambda_1$. It follows from the definition od $\varphi_{\tau-1}$ that for each k we have that subsets of W_k are expanded uniformly

$$\frac{\lambda(\Phi_{\tau-1}(A\cap W_k))}{\lambda(\Phi_{\tau-1}(W_k))} = \frac{\lambda(A\cap W_k)}{\lambda(W_k)}, \tag{5.8.7}$$

and that the measure of the image of a box is proportion to its density

$$\frac{\lambda(\Phi_{\tau-1}(W_k))}{\lambda(W_k)} = \frac{\lambda(A\cap W_k)}{\lambda(W_k)\lambda(A)}. \tag{5.8.8}$$

Combining (5.8.7) and (5.8.8) we get

$$\begin{aligned}\lambda(\Phi_{\tau-1}(A)) &= \sum_k \lambda(\Phi_{\tau-1}(A\cap W_k)) = \sum_k \frac{\lambda(A\cap W_k)^2}{\lambda(A)\lambda(W_k)} \\ &= \frac{1}{\lambda(A)}\sum_k \frac{\lambda(A\cap W_k)^2}{\lambda(W_k)^2}\lambda(W_k) = \frac{1}{\lambda(A)}\mathbb{E}[\rho^2_{\tau,X}],\end{aligned} \tag{5.8.9}$$

which completes the proof of the lemma. □

Now we are ready to prove Theorem 5.3.

Proof of Theorem 5.3. Consider τ_1, τ_2, τ_3, τ_4 as in Theorem 5.5.1. It follows from Theorem 5.5.1 that one of the following three cases must hold.

(i) $\mathbb{P}[\tau = \tau_1] \geq \frac{1}{6}$.

(ii) $\mathbb{P}[\tau = \tau_2] \geq \frac{1}{3}$.

(iii) $\mathbb{P}[\tau = \tau_3] \geq \frac{1}{6}$. We treat each of these cases separately.

Case 1: $\mathbb{P}[\tau = \tau_1] \geq \frac{1}{6}$.

In this case it follows that

$$\mathbb{E}[\sum_{i=1}^{\tau-1} \widetilde{\Delta}^2_{i,X}] \geq \frac{\varepsilon_1}{6}.$$

Now observe that using Lemma 5.8.2 we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{\tau-1} \widetilde{\Delta}^2_{i,X}\right] &= \mathbb{E}\left[\sum_{i} \widetilde{\Delta}^2_{i,X,\tau} 1_{\{\tau \geq i+1\}}\right] \\
&= \mathbb{E}\left(\sum_{i} \mathbb{E}[\widetilde{\Delta}^2_{i,X,\tau} 1_{\{\tau \geq i+1\}} \mid \mathcal{F}_{i+1}]\right) \\
&\leq C_{11}\mathbb{E}\left[\sum_{i} \Delta^2_{i,X} 1_{\{\tau \geq i+1\}}\right] \\
&= C_{11}\mathbb{E}\left[\sum_{i=1}^{\tau-1} \Delta^2_{i,X}\right]. \qquad (5.8.10)
\end{aligned}$$

It follows using Obervation 5.4.2 that

$$\mathrm{Var}(\rho_{\tau,X}) \geq \frac{\varepsilon_1}{6C_{11}}.$$

It follows now from Lemma 5.8.4 that

$$\lambda(\Phi_{\tau-1}(A)) \geq \lambda(A) + 2\varepsilon$$

where ε is a fixed constant smaller than $\frac{\varepsilon_1}{12\gamma' C_{11}}$. Choose m sufficiently large so that $\mathbb{P}[\tau < m/2] < \varepsilon$. It then follows that

$$\lambda(\Phi_{(\tau-1)\wedge m}(A)) \geq \lambda(A) + \varepsilon.$$

Also, it follows from Lemma 5.8.1 and the definition of τ_4 that $||\Phi'_{(\tau-1)\wedge m} - \mathbf{I}|| < \eta/100$. Since $\Phi_{(\tau-1)\wedge m}$ is continuously differentiable except on finitely many curves, it follows that we have $\phi := \Phi_{(\tau-1)\wedge m}$ is bi-Lipschitz with Lipschitz constant $1+\eta$. So the proof of Theorem 5.3 is finished in this case.

Case 2: $\mathbb{P}[\tau = \tau_2] \geq \frac{1}{3}$.

In this case we have $\mathbb{P}[\tau_2 < \infty] \geq \frac{1}{3}$. Hence it follows that there exists $x_1, x_2, \ldots x_n \in \Lambda_1$ and $i_1, i_2, \ldots, i_n > 0$ such that Λ_{i_k,x_k} are disjoint (except may be at the boundary), $|\Delta_{i_k,x_k}| \geq \sqrt{\varepsilon_2}$, for each k and $\sum_{k=1}^{n} \lambda(\Lambda_{i_k,x_k}) \geq \frac{1}{12}$. Define the function ϕ as follows. Set $\phi = \Psi_{\sqrt{\varepsilon_2},\Lambda_{i_k,x_k},\rightarrow}$ on Λ_{i_k,x_k} if i_k is odd and $\phi = \Psi_{\sqrt{\varepsilon_2},\Lambda_{i_k,x_k},\uparrow}$ on Λ_{i_k,x_k} if i_k is even. Set ϕ to be identity on $\Lambda_1 \setminus (\cup_k \Lambda_{i_k,x_k})$. It is clear that such a ϕ is well-defined, identity on the boundary of Λ_1 and is bi-Lipschitz with Lipschitz constant $(1+\eta)$ by choosing ε_2 sufficiently small. Now observe that

$$\lambda(\phi(A \cap \Lambda_{x_k,i_k})) - \lambda(A \cap \Lambda_{i_k,x_k})) \geq \lambda(\Lambda_{i_k,x_k}))\varepsilon_2.$$

Summing over k we get that

$$\lambda(\phi(A)) \geq \lambda(A) + \frac{\varepsilon_2}{12}.$$

So the conclusion of Theorem 5.3 holds for $\varepsilon < \frac{\varepsilon_2}{12}$.

Case 3: $\mathbb{P}[\tau = \tau_3] \geq \frac{1}{6}$.

In this case also it follows that $\mathrm{Var}(\rho_{\tau,X}) \geq \frac{\varepsilon_3^2}{6}$. Arguing as in case 1, it follows that in this case also there is a bi-Lipschitz bijection ϕ with Lipschitz constant $1+\eta$ such that

$$\lambda(\phi(A)) = \lambda(A) + \varepsilon$$

where ε is a constant smaller than $\frac{\varepsilon_3^2}{12}$.

This completes the proof of Theorem 5.3. □

5.9 Estimates for g_r and Ψ_δ

In this section we provide the proofs of Lemma 5.2.2, Lemma 5.2.7, Lemma 5.2.8, Lemma 5.2.13, Lemma 5.2.14 and Lemma 5.2.15.

Proof of Lemma 5.2.2. Let $\widetilde{\Lambda}^1 = [0,1] \times [0, \frac{1}{2})$ and $\widetilde{\Lambda}^2 = [0,1] \times (\frac{1}{2}, 1]$.

Step 1: Ψ_δ is continuous on $\widetilde{\Lambda}^1 \cup \widetilde{\Lambda}^2$.

Notice that it is clear that Ψ_δ is continuous at $(1/2, 0)$. Hence it suffices to prove that for g_r defined by (5.2.1) and (5.2.2) we have $(r, \ell) \to g_r(\ell)$ is continuous. Without loss of generality assume $l \leq 0$. Define

$$H_r(\ell) = (1 + h'(r))\ell - \frac{h'(r)\sin(\ell\Theta(r))}{\Theta(r)}. \tag{5.9.1}$$

Notice that it follows from (5.2.1) that

$$(1+\delta)(H_r(\ell) - H_r(-1)) = H_r(g_r(\ell)) - H_r(-1) \tag{5.9.2}$$

and the assertion follows from the continuity of H_r.

Step 2: Let $x = (x_1, \frac{1}{2})$ with $x \in [0, \frac{1}{2}]$. Then Ψ_δ is continuous at x.

Without loss of generality assume $x_1 \le \frac{1}{2}$. Take $u_n = (u_1^n, u_2^n) \in [0,1]^2$ converging to x. Without loss of generality assume $\{u_n\} \subseteq \widetilde{\Lambda}^1$. Let $(r_n, \ell_n) = K(u_n)$. Then $(r_n, \ell_n) \to (\frac{1}{2}, 2x_1 - 1)$. To prove that $\ell_n \to \ell = 2x_1 - 1$, observe that,

$$u_2^n = \frac{1}{2} + (r_n + h(r_n)) \sin(\ell\Theta(r_n)).$$

Taking limit as $r_n \to \frac{1}{2}$ we get the result. Now taking limit as $r_n \to \frac{1}{2}$, and $\ell_n \to \ell$ in (5.9.1) we get $(1+\delta)(\ell+1) = \lim(g_r(\ell)+1)$, which proves Step 2. □

Proof of Lemma 5.2.7. Without loss of generality fix $\ell \in [-1, 0)$. Observe that

$$\delta \int_{-1}^{\ell} (1 + h'(r) - h'(r)\cos(\theta\Theta(r)))\, d\theta = \int_{\ell}^{g_r(\ell)} (1 + h'(r) - h'(r)\cos(\theta\Theta(r)))\, d\theta. \quad (5.9.3)$$

Using mean value theorem it follows that there exists $\theta^* \in (-1, \ell), \theta^{**} \in (\ell, g_r(\ell))$ such that

$$\delta(\ell+1)(1 + h'(r) - h'(r)\cos(\theta^*\Theta(r))) = (g_r(\ell) - \ell)(1 + h'(r) - h'(r)\cos(\theta\Theta^{**}(r))).$$

It follows that there exists a constant $C > 0$, such that we have for all r, θ,

$$(g_r(\ell) - \ell) \le C\delta(\ell+1) \quad (5.9.4)$$

since $h'(r) = O(h(r)^2)$ as $r \to \frac{1}{2}$. Moreover, since $h'(r) = o(h(r)^2)$ as $r \to \frac{1}{2}$, we have

$$(g_r(\ell) - \ell) = \delta(\ell+1)(1 + o(1)) \quad (5.9.5)$$

as $r \to \frac{1}{2}$.

Differentiating the integral equation (5.9.3), we get

$$(1+\delta)(1 + h'(r) - h'(r)\cos(\ell\Theta(r))) = \frac{\partial g_r}{\partial \ell}(1 + h'(r) - h'(r)\cos(g_r(\ell)\Theta(r))).$$

It follows that

$$(\frac{\partial g_r}{\partial \ell} - 1) = \delta \frac{(1 + h'(r) - h'(r)\cos(\ell\Theta(r))) + \delta^{-1} h'(r)(\cos(g_r(\ell)\Theta(r)) - \cos(\ell\Theta(r)))}{(1 + h'(r) - h'(r)\cos(g_r(\ell)\Theta(r)))}. \quad (5.9.6)$$

It follows now using (5.9.4) that since $h'(r) = O(h(r))^2$ as $r \to \frac{1}{2}$, there is a constant $C > 0$ such that $\sup_{r,\theta} |\frac{\partial g_r(\theta)}{\partial \theta} - 1| \le C\delta$. □

Proof of Lemma 5.2.8. Without loss of generality we again assume that $\theta \leq 0$. Observe that by differentiating both sides of (5.9.2) w.r.t. r we get that

$$\frac{\partial g_r(\theta)}{\partial r}\left[\frac{\partial H_r(\ell)}{\partial \ell}\right]_{\ell=g_r(\theta)} + \left[\frac{\partial H_r(\ell)}{\partial r}\right]_{\ell=g_r(\theta)} - \frac{\partial H_r(\theta)}{\partial r} = \delta\left(\frac{\partial H_r(\theta)}{\partial r} - \frac{\partial H_r(-1)}{\partial r}\right). \quad (5.9.7)$$

It follows that there exists $\theta^* \in (\theta, g_r(\theta))$ such that the left hand side of (5.9.7) reduces to

$$\frac{\partial g_r(\theta)}{\partial r}\left(1 + h'(r) - h'(r)\cos(g_r(\theta)\Theta(r))\right) +$$
$$(g_r(\theta) - \theta)(h''(r) - h''(r)\cos(\theta^*\Theta(r)) + h'(r)\Theta'(r)\theta\sin(\theta^*\Theta(r))).$$

Similarly there exists $\theta^{**} \in (-1, \theta)$ such that the right hand side of (5.9.7) is equal to

$$\delta(\theta + 1)(h''(r) - h''(r)\cos(\theta^{**}\Theta(r)) + h'(r)\Theta'(r)\theta\sin(\theta^{**}\Theta(r))).$$

Now observe that

$$\Theta'(r) = -\frac{1 + h'(r)}{(r + h(r))\sqrt{4((r + h(r)))^2 - 1}}$$

for $r > r_0$ and $\Theta'(r)$ is bounded away from infinity if $r < r_0$.

Hence it follows from the above equations that as long as $h'(r) = o(h(r)^2)$ and $h''(r) = O(h(r)^3)$ as $r \to \frac{1}{2}$, there exists an absolute constant $C > 0$ such that

$$\left|\frac{\partial g_r(\theta)}{\partial r}\right|\left(1 + h'(r) - h'(r)\cos(g_r(\theta)\Theta(r))\right) \leq C\delta.$$

The final assertion follows using (5.9.5). ☐

Proof of Lemma 5.2.13. Let $D(r, \theta)$ denote the determinant of the $J(r, \theta)$. Clearly since $D(r, \theta)$ is bounded away from 0 and ∞ it suffices to prove the following two statements.

(i) $|\frac{\partial D}{\partial r}| \leq C$, $|\frac{\partial D}{\partial \theta}| \leq C$.

(ii) $||J_r(r, \theta)|| \leq C, ||J_\theta(r, \theta)|| \leq C$.

The first assertion follows directly by differentiating D since $(h'(r))^2 = O(h(r)^3)$ and $h''(r) = O(h(r)^2)$ as $r \to \frac{1}{2}$. The second assertion follows directly by differentiating $J(r, \theta)$ entrywise (w.r.t. r and θ) since $(h'(r))^2 = O(h(r)^3)$ and $h''(r) = O(h(r)^2)$ as $r \to \frac{1}{2}$. ☐

Proof of Lemma 5.2.14. Without loss of generality, we can assume $\theta \leq 0$. It suffices to prove that there exists an absolute constant C such that

(i) $|\frac{\partial^2 g_r}{\partial \theta^2}| \leq C\delta$.

(ii) $|\frac{\partial^2 g_r}{\partial\theta\partial r}| \leq C\delta$.

(iii) $|\frac{\partial^2 g_r}{\partial r^2}| \leq C\delta$.

Since the functions are sufficiently smooth the mixed partial derivatives will be equal.

For the proof of (i) and (ii), consider (5.9.6). Call the numerator $A(r,\theta)$ and the denominator $B(r,\theta)$. Since $B(r,\theta)$ is bounded away from 0 and ∞, it suffices to prove that for some absolute constant $C > 0$, we have $|\frac{\partial A}{\partial r}| \leq C\delta, |\frac{\partial A}{\partial \theta}| \leq C\delta, |\frac{\partial B}{\partial r}| \leq C, |\frac{\partial B}{\partial \theta}| \leq C$.

We have from (5.9.6) that

$$\frac{\partial B}{\partial r} = h''(r)(1-\cos(g_r(\theta)\Theta(r))) + h'(r)\sin(g_r(\theta)\Theta(r))[\frac{\partial g_r}{\partial r}\Theta(r) + g_r(\theta)\Theta'(r)].$$

Since the functions above are bounded if r is bounded away from $\frac{1}{2}$ and as $r \to \frac{1}{2}$ we have $h''(r) = O(h(r)^2)$, and $(h'(r))^2 = O(h(r)^3)$ it follows using Lemma 5.2.8 that $|\frac{\partial B}{\partial r}| \leq C$ for some absolute constant C.

We also have

$$\frac{\partial B}{\partial \theta} = h'(r)\sin(g_r(\theta)\Theta(r))\Theta(r)\frac{\partial g_r}{\partial \theta}.$$

It follows that $|\frac{\partial B}{\partial \theta}| \leq C$ for some absolute constant C.

Next observe that

$$\begin{aligned} \frac{\partial A}{\partial r} &= \delta[h''(r)(1-\cos(\theta\Theta(r))) + h'(r)\theta\Theta'(r)\sin(\theta\Theta(r))] \\ &+ h''(r)(\cos(g_r(\theta)\Theta(r)) - \cos(\theta\Theta(r))) \\ &+ h'(r)\Theta'(r)\Big((\theta - g_r(\theta))\sin(g_r(\theta)\Theta(r)) - \theta(\sin(\theta\Theta(r))\sin(g_r(\theta)\Theta(r))\Big) \\ &- h'(r)\frac{\partial g_r}{\partial r}\Theta(r)\sin(g_r(\theta)\Theta(r)). \end{aligned}$$

As before, notice that everything is bounded if r is bounded away from 0 and $\frac{1}{2}$. It follows using Lemma 5.2.7, Lemma 5.2.8 and $h'(r)^2 = O(h(r)^3)$, $h''(r) = O(h(r)^2)$ as $r \to \frac{1}{2}$ that $|\frac{\partial A}{\partial r}| \leq C\delta$ for some absolute constant $C > 0$.

Finally observe that

$$\begin{aligned} \frac{\partial A}{\partial \theta} &= \delta h'(r)\Theta(r)\sin(\theta\Theta(r)) + h'(r)\Theta(r) \\ &\times \Big(\sin(\theta\Theta(r)) - \sin(g_r(\theta)\Theta(r)) - (\frac{\partial g_r}{\partial \theta} - 1)\sin(g_r(\theta)\Theta(r))\Big) \end{aligned}$$

Arguing as before it follows from Lemma 5.2.7 and $h'(r) = O(h(r)^2)$ as $r \to \frac{1}{2}$ that $|\frac{\partial A}{\partial \theta}| \leq C\delta$ for some absolute constant $C > 0$. This completes the proof of (i) and (ii) above.

For proof of (iii), consider (5.9.7), let us denote

$$A_1(r,\theta) = \left[\frac{\partial H_r(\ell)}{\partial r}\right]_{\ell=g_r(\theta)} - \frac{\partial H_r(\theta)}{\partial r},$$

$$A_2(r,\theta) = \left(\frac{\partial H_r(\theta)}{\partial r} - \frac{\partial H_r(-1)}{\partial r}\right)$$

and

$$A_3(r,\theta) = (1 + h'(r) - h'(r)\cos(g_r(\theta)\Theta(r))).$$

Observe that A_3 is bounded away from 0 and ∞, and there exists a constant C such that $|\frac{\partial A_3}{\partial r}| \leq C$ since $h''(r) = O(h(r)^2)$ and $h'(r)^2 = O((h(r))^3)$ as $r \to \frac{1}{2}$. Hence using (5.9.7), it suffices to show that for some absolute constant C such that $|\frac{\partial A_1}{\partial r}| \leq C\delta$ and $|\frac{\partial A_2}{\partial r}| \leq C$.

Observe that

$$\frac{\partial A_2}{\partial r} = \int_{-1}^{\theta} \frac{\partial^2}{\partial r^2}(1 + h'(r) - h'(r)\cos(\theta\Theta(r)))\ d\theta$$

and

$$\begin{aligned}\frac{\partial A_1}{\partial r} &= \int_{\theta}^{g_r(\theta)} \frac{\partial^2}{\partial r^2}(1 + h'(r) - h'(r)\cos(\theta\Theta(r)))\ d\theta \\ &+ \frac{\partial g_r(\theta)}{\partial r}\left(h''(r)(1 - \cos(g_r(\theta)\Theta(r))) + h'(r)\Theta'(r)g_r(\theta)\sin(g_r(\theta)\Theta(r))\right).\end{aligned}$$

Arguing as before, using Lemma 5.2.8, $h''(r) = O(h(r)^2)$, $h'(r)^2 = O(h(r))^3$ as $r \to \frac{1}{2}$ it follows that it is in fact enough to show that $\frac{\partial^2}{\partial r^2}(1 + h'(r) - h'(r)\cos(\theta\Theta(r)))$ is bounded. This follows directly by differentiating since $h^{(3)}(r) = O(h(r)^2)$ and $h''(r)h'(r) = O(h(r)^3)$ as $r \to \frac{1}{2}$ (the second derivative remains bounded if r is bounded away from 0 and $\frac{1}{2}$).

This completes the proof of the Lemma. □

Proof of Lemma 5.2.15. Notice that any real valued smooth function $F = F(r,\theta)$, let $\widetilde{F}$ denote the function $\widetilde{F}(r,\theta) = F(r, g_r(\theta))$. Then we have

$$\frac{\partial \widetilde{F}}{\partial r} = \left[\frac{\partial F}{\partial r}\right]_{r,g_r(\theta)} + \left[\frac{\partial F}{\partial \theta}\right]_{r,g_r(\theta)} \frac{\partial g_r}{\partial r}.$$

$$\frac{\partial \widetilde{F}}{\partial \theta} = \left[\frac{\partial F}{\partial \theta}\right]_{r,g_r(\theta)} \frac{\partial g_r}{\partial \theta}.$$

Now the lemma follows from Lemma 5.2.7, Lemma 5.2.8 and the fact that $||J_r(r,\theta)|| \leq C, ||J_\theta(r,\theta)|| \leq C$ which was established in the proof of Lemma 5.2.13. □

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