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The Fluid Motion Due to a Rotating Disk

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The solution of the Navier-Stokes equation for fluid motion due to a rotating disk includes characteristic parameters as presented below. We report here the most accurate values available for these parameters and compare to them values obtained by a numerical integration technique developed by Newman[3,5].

In 1921, von Karman[2] presented a separation of variable solution technique for the motion of an incompressible, Newtonian fluid which transformed the Navier-Stokes equation into a set of coupled, nonlinear, ordinary differential equations. By defining the following dimensionless variables

$$\zeta = z\sqrt{\Omega/\nu}\ , \quad P = p/\mu\Omega\ , \quad G = v_\theta/r\Omega\ ,$$
$$F = v_r/r\Omega\ , \quad \text{and} \quad H = v_z/\sqrt{\nu\Omega}\ ,$$

the transformed equations may be written as

$$2F + H' = 0, \tag{1}$$

$$F^2 - G^2 + HF' = F'', \tag{2}$$

$$2FG + HG' = G'', \tag{3}$$

$$HH' + P' = H'', \tag{4}$$

where the prime designates differentiation with respect to ζ. The boundary conditions are

$$H = F = 0,\quad G = 1 \quad \text{at} \quad \zeta = 0, \tag{5}$$

and

$$F = G = 0 \quad \text{at} \quad \zeta = \infty. \tag{6}$$

Cochran[1] solved equations 1 through 3 (subject to boundary conditions 5 and 6) by expanding the components of the velocity field first in power series in the dimensionless distance from the disk, which were assumed to be valid near the disk:

$$F = a\zeta - \frac{1}{2}\zeta^2 - \frac{b}{3}\zeta^3 + \ldots., \tag{7}$$

$$G = 1 + b\zeta + \frac{1}{3}a\zeta^3 + \ldots., \tag{8}$$

$$H = -a\zeta^2 + \frac{1}{3}\zeta^3 + \frac{b}{6}\zeta^4 + \ldots., \tag{9}$$

and second in exponential series which were assumed to be valid far from the disk:

$$F = Ae^{-\alpha\zeta} - \frac{(A^2+B^2)e^{-2\alpha\zeta}}{2\alpha^2} + \frac{A(A^2+B^2)e^{-3\alpha\zeta}}{4\alpha^4} + \ldots., \tag{10}$$

$$G = Be^{-\alpha\zeta} - \frac{B(A^2+B^2)}{12\alpha^4}e^{-3\alpha\zeta} + \ldots., \tag{11}$$

$$H = -\alpha + \frac{2A}{\alpha}e^{-\alpha\zeta} - \frac{(A^2+B^2)}{2\alpha^3}e^{-2\alpha\zeta} + \frac{A(A^2+B^2)}{6\alpha^5}e^{-3\alpha\zeta} + \ldots. \tag{12}$$

We[4] followed Cochran's suggestion and required the two sets of expansions to yield, at $\zeta=1$, the same values of the functions as well as the derivatives of F and G. In this manner we obtained the following values for the characteristic parameters:

$$a = 0.51023262,$$
$$b = -0.61592201,$$
$$\alpha = 0.88447411,$$
$$A = 0.92486353,$$
$$B = 1.20221175. \qquad (13)$$

To demonstrate the utility of Newman's[3,4,5] solution technique, estimates of the parameters a, b, α , A, and B were obtained by solving this boundary value problem numerically. The governing equations 1 through 3 were first linearized[3,4,5] about trial values and then cast in finite-difference form accurate to order h^2. The boundary conditions given by equations 5 were applied directly, whereas it was necessary to approximate those given by equations 6 at some finite value of ζ, ζ_{max}. The following expressions, derived from equations 10 and 11, were used for that purpose:

$$F' = H_\infty F - \frac{(F^2+G^2)}{2H_\infty} + \ldots., \qquad (14)$$

and

$$G' = H_\infty G + \ldots., \qquad (15)$$

where H_∞ was our estimate of $-\alpha$ according to

$$H_\infty = H + \frac{2F}{H_\infty} + \frac{(F^2+G^2)}{2H_\infty^3}\left(1 + \frac{F}{3H_\infty^2}\right) + \ldots, \qquad (16)$$

which was developed from equation 12. Equations 14, 15, and 16 were also linearized about trial values and expressed in finite-difference form accurate to order h^2. The resulting system of equations was solved by a technique developed[3,4] and extended[5] by Newman. Estimates of the five parameters obtained this way are

$$a = 0.51023262,$$
$$b = -0.61592201,$$
$$\alpha = 0.88447410,$$
$$A = 0.92486322,$$

and

$$B = 1.20221104 \qquad (17)$$

Clearly, these are very accurate estimates of the parameters given by equation 13. The poorest estimate is for B, which is in error by only seven digits in the seventh significant figure.

The very attractive feature of Newman's solution technique, in addition to its accuracy, is its suitability for solving complicated boundary value problems directly without the development of specialized techniques, such as Cochran's for the present problem.

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Nomenclature

a = characteristic parameter equal to F'(0)

b = characteristic parameter equal to G'(0)

A = characteristic parameter

B = characteristic parameter

F = dimensionless radial velocity

G = dimensionless velocity component in the tangential direction

H = dimensionless velocity component in the normal direction (from the disk)

h = dimensionless step size

p = dimensionless dynamic pressure

P = dynamic pressure, dyne/cm^2

r = radial distance from the axis of the disk, cm

v_r = velocity component in the radial direction, cm/sec

v_θ = velocity component in the tangential direction, cm/sec

v_z = velocity component in the normal direction, cm/sec

z = normal distance from the disk, cm

Greek Letters

α = characteristic parameter equal to $-H(\infty)$

ζ = dimensionless normal distance from the disk

μ = viscosity of fluid, g/cm-sec

ν = kinematic viscosity of fluid, cm^2/sec

Ω = rotation speed of the disk, sec^{-1}

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