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A SEPARABLE EXPANSION FOR THE NUCLEAR FORM FACTORS

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ABSTRACT

We apply the techniques of the Fourier-Bessel series to obtain a separable expansion of the (local) nuclear form factors in the momentum space. The accuracy of the expansion is studied for a Woods-Saxon and a derivative Woods-Saxon density distributions. The technique will be useful in many nuclear scattering calculations.

I. INTRODUCTION

In this short note, we shall discuss a technique to obtain a separable expansion for the nuclear form factors. The form factor $F(\vec{k}',\vec{k})$, as a Fourier transform of the density distribution is, generally, for a local density, a function of the momentum transfer $\vec{q} = \vec{k}'-\vec{k}$, where $\vec{k}'$ and $\vec{k}$ are two momenta. We shall show that a good approximation may be devised to represent such a form factor in terms of products of functions depending on $\vec{k}$ and $\vec{k}'$ separately.

This separable expansion for the form factors will be useful in many applications associated with general scattering theory calculations. For example, the first order optical potential $U_{opt}(\vec{k},\vec{k}') = t(\vec{k},\vec{k}')\ F(\vec{q})$ will be separable when a separable form of the two-body T-matrix $t(\vec{k},\vec{k}')$ is used.[1] The technique to be discussed below has already been found useful in calculating the rescattering corrections to the two-body T-matrix in the nuclear medium.[2] Such corrections may be carried out exactly, in a closure approximation, by using a separable representation of the nuclear form factor. We have also proposed its application in solving the coupled-channel equations for pion-nucleus charge exchange reactions in a isobar-doorway approximation.[3] The method is quite general, and is applicable in many other calculations.[4] Here we shall, however, only report its numerical accuracy for a few typical nuclear form factors.

Section II contains the mathematics. The numerical results are given in Section III.

II. SEPARABLE REPRESENTATION OF THE FORM FACTORS

For a local density distribution $\rho(\vec{r})$, we may define its form factor $F(\vec{q})$ as

$$F(\vec{q}) = \int e^{-i(\vec{k}'-\vec{k}).\vec{r}} \rho(\vec{r})d\vec{r} \quad , \tag{1}$$

where $\vec{q} \equiv \vec{k}'-\vec{k}$ is the momentum transfer from momenta $\vec{k}$ to $\vec{k}'$. We are interested in a representation where $F(\vec{q})$ may consist of factors depending on k and k' separately. For density distribution of interest, we may write[5]

$$\rho(\vec{r}) = \rho_o(r) + \sum_I \rho_I(r)\ Y^*_{I0}(\hat{r}) \quad , \tag{2}$$

where $\rho_o(r)$ is the spherical density and $\rho_I(r)$ are the multiple density distributions of order I. The form factors for $\rho(\vec{r})$ of Eq. (2) may be written as

$$F(\vec{q}) = \sum_{\substack{\ell,\ell' \\ m,m' \\ I,M}} C(\ell,\ell',m,m',I,M)\ \lambda_I^{\ell\ell'}(k,k')\ Y_{\ell m}(\hat{k})\ Y^*_{\ell' m'}(\hat{k}') , \tag{3}$$

where, for simplicity, we have lumped all the angular momentum coefficients in the factor C; $\lambda_I^{\ell\ell'}(k,k')$ is defined as

$$\lambda_I^{\ell\ell'}(k,k') = \int_0^\infty j_\ell(kr) j_{\ell'}(k'r) \rho_I(r) r^2 dr \quad , \tag{4}$$

where $j_\ell(x)$ are the spherical Bessel functions of order ℓ.[6] From Eq. (3), note that the form factor will be separable in k and k', if the function $\lambda_I^{\ell\ell'}(k,k')$ may be factorized in these variables. Our main aim is, therefore, to introduce such factorization.

We observe that the integral in Eq. (4) may be cut off at some radius, say R, where the radial density becomes negligibly small. Within such a finite domain, i.e. $r \leqslant R$, we introduce the Fourier-Bessel expansion for $j_\ell(kr)$[6]

$$j_\ell(kr) = \sum_{n=1}^{N} A_n^\ell(k)\, j_\ell(\alpha_n^\ell r) \qquad \text{for } 0 \leqslant r \leqslant R \quad , \tag{5}$$

where $\alpha_n^\ell R$ is the n-th zero of the spherical Bessel function of order ℓ.[6] In our calculation, we take N to be finite. The expansion coefficients $A_n^\ell(k)$ are given as

$$A_n^\ell(k) = \frac{2\alpha_n^\ell R\; j_\ell(kR)}{\left[(\alpha_n^\ell R)^2 - k^2R^2\right] j_{\ell+1}(\alpha_n^\ell R)} \quad . \tag{6}$$

These expansion coefficients $A_n^\ell(k)$ also satisfy the following orthogonality condition

$$\int_0^\infty k^2 dk\, A_n^\ell(k)\, A_m^\ell(k) = \frac{2\pi(\alpha_n^\ell)^2\, j_\ell(\alpha_n^\ell R)}{R\, j_{\ell+1}(\alpha_n^\ell R)}\, \delta_{mn} \tag{7}$$

Using Eq. (5), we find the following separable form for $\lambda_I^{\ell\ell'}(k,k')$:

$$\lambda_I^{\ell\ell'}(k,k') = \sum_{\substack{m=1\\n=1}}^{M,N} A_n^{\ell}(k) A_m^{\ell'}(k') K_I(\alpha_n^{\ell},\alpha_m^{\ell'}) \quad , \tag{8}$$

where

$$K_I(\alpha_n^{\ell},\alpha_m^{\ell'}) = \int_0^R j_\ell(\alpha_n^{\ell} r)\, j_{\ell'}(\alpha_m^{\ell'} r)\, \rho_I(r) r^2 dr \quad . \tag{9}$$

We note that this separable representation of Eq. (8) will be particularly useful if the number of terms needed in the sum are reasonably small. There is no apparent advantage of using Eq. (8) instead of Eq. (4) to evaluate the form factors. However, a separable form will be very convenient in the applications we described in the previous section.

In the limit of a uniform density distribution for I = 0 (e.g., a spherical ground state), only m=n terms in Eq. (8) contribute. For a surface δ-function density $\rho(r) = \delta(r-R)$, the function $\lambda_N^{\ell\ell'}(k,k')$ also factorizes as can be readily seen in Eq. (4). For a more general case, we need to include all the terms in Eq. (8). In the next section, we shall present the numerical results for the following density distributions:

(I) The spherical Woods-Saxon form

$$\rho_0(r) = \frac{\rho(0)}{1 + \exp\,[(r-r_0)/a]} \quad , \tag{10}$$

where r_0 = the half radius, a - the diffuseness, and $\rho(0)$ is the normalization.

(II) The spherical derivative Woods-Saxon form

$$\rho_{I=0}(r) = r_o \left[\frac{d\rho_o(r)}{dr}\right] , \tag{11}$$

and (III) The vector derivative Woods-Saxon form

$$\rho_{I=2}(\vec{r}) = r_o \left[\frac{d\rho_o(r)}{dr}\right] Y_{20}(\hat{r}) . \tag{12}$$

We shall take r_o = 4 fm as our example. The convergence properties do not sensitively depend on the parameters for the shape of the density distribution. We choose the above examples only to represent the general forms of the ground state density (case I) and of the inelastic transition density (cases II and III). The techniques shown here apply equally well to other shapes of the density distributions; the convergence properties will also be well represented by our numerical examples in the following section.

III. NUMERICAL RESULTS

We shall compare our results obtained with a finite number of terms in Eq. (8) to the exact results from Eq. (4). The convergence is generally quite good. We find it convenient to tabulate our results for two values of k and k'.

In Tables 1-2, we list the results for the spherical Woods-Saxon density distributions for r_o=4 fm and a=0.1 fm, 0.6 fm, and 1.0 fm. The values of k and k' are shown for each Table. We have calculated, for all cases, only the lower partial waves (small ℓ and ℓ'). However, there is no particular convergence problem associated with higher partial waves, as can be seen from these Tables. The results for a spherical derivative Woods-Saxon form are shown in Tables 3-4. The parameters are the same as in Tables 1-2. For the quadrupole form factor (I=2, in Eq. (12)),we present the results in Tables 5 and 6 for the case of a = 0.6 fm. The convergence properties are similar to the previous cases.

In general, the expansion of Eq. (8) converges rapidly for a few terms (e.g. $N \approx 6$), and then less rapidly for a few more terms. As we have shown, in many cases, the exact result is reproduced with about ten terms in Eq. (5).

It is important to point out that the values of the $\lambda_I^{\ell\ell'}(k,k')$ are large for $k \approx k'$, and for angular momenta $\ell,\ell' \approx kr_o$. As we can see from the Tables, for the dominant $\lambda_I^{\ell\ell'}(k,k')$, the sum of Eq. (8) converges fairly rapidly with about 6 terms. The convergence for $\ell >> kr_o$ would be much slower but this feature is not detrimental since their contributions are generally negligible in actual applications. As a conclusion, we have found the techniques to be quite general and useful. Application to specific nuclear scattering calculations will be discussed elsewehre.[2,3]

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*Work performed under the auspices of the U. S. Atomic Energy Commission.

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2. M. A. Nagarajan, R. M. Thaler, and W.L. Wang, (preprint).

3. M. A. Nagarajan and W. L. Wang, Lawrence Berkeley Report, LBL-2958.

4. This method is being applied to the analysis of inelastic scattering of nucleons from nuclei (F. Petrorich, private communication).

5. See, for example, N. Austern, Direct Nuclear Reaction Theories, John Wiley and Sons (New York) 1970, p. 126.

6. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards Publication (1970); the zeros of the spherical Bessel functions used in our computation are from Royal Society Mathematical Tables, Vol. 7, Cambridge University Press, Cambridge, England (1960).

Table I. The values of $\lambda_I^{\ell\ell'}(k,k')$ for I = O, $\ell=\ell'$ for ℓ=0-4. The nuclear density distribution is the Woods-Saxon form of Eq. (10). The radius r_o is taken to be 4 fm and the three values of the diffuseness are a=0.1 fm, 0.6 fm and 1.0 fm as shown in the Table. N is the order of expansion used in Eq. (5) The values given are calculated from Eq. (8) for N=M=6, 10, and 14. The exact values are obtained by numerical integration in Eq. (4). In this Table, k=1.0 fm^{-1} and k'=1.0 fm^{-1}.

a(fm)	ℓ	N=6	N=10	N=14	Exact
	0	0.08487	0.08328	0.08250	0.08230
	1	0.09757	0.09744	0.09742	0.09743
	2	0.07721	0.07468	0.07398	0.07389
0.1	3	0.03001	0.02808	0.02751	0.02738
	4	0.006676	0.006075	0.005892	0.005839
	0	0.07493	0.07493	0.07493	0.07492
	1	0.07318	0.07306	0.07301	0.07300
	2	0.05814	0.05813	0.05812	0.05811
0.6	3	0.03235	0.03230	0.03229	0.03228
	4	0.01295	0.01281	0.01277	0.01276
	0	0.05790	0.05777	0.05774	0.05774
	1	0.05364	0.05365	0.05365	0.05365
1.0	2	0.04416	0.04406	0.04405	0.04405
	3	0.02982	0.02977	0.02976	0.02976
	4	0.01708	0.01708	0.01708	0.01708

Table II. The values of $\lambda_{\varrho}^{\ell\ell}(k,k')$ for k=1.0 fm^{-1} and k^{1}=2.0 fm^{-1}. See the description for Table I.

a(fm)	ℓ	N=6	N=10	N=14	Exact
	0	-0.008261	-0.007251	-0.006895	-0.006850
	1	-0.008083	-0.008425	-0.008488	-0.008479
	2	-0.008359	-0.007549	-0.007385	-0.007400
0.1	3	+0.003756	+0.004728	+0.005045	+0.005118
	4	+0.007110	+0.007177	+0.007251	+0.007291
	0	-0.004218	-0.004244	-0.004244	-0.004244
	1	-0.003627	-0.003745	-0.003773	-0.003783
0.6	2	-0.002471	-0.002524	-0.002529	-0.002531
	3	+0.00007531	+0.00005663	+0.00004752	+0.00004412
	4	+0.001485	+0.001442	+0.001425	+0.001420
	0	-0.001534	-0.001326	-0.001320	-0.001322
	1	-0.001188	-0.001225	-0.001229	-0.001229
	2	-0.0008856	-0.0007688	-0.0007636	-0.0007652
1.0	3	-0.0002673	-0.0002762	-0.0002846	-0.0002829
	4	+0.0001187	+0.0001237	+0.0001254	+0.0001251

Table III. The values of $\lambda_o^{\ell\ell}(k,k')$ for $k=1.0\ fm^{-1}$ and $k'=1.0\ fm^{-1}$, for the derivative Woods-Saxon density distribution given by Eq. (11). The radius r_o is 4 fm, a is the diffuseness. N is the order of expansion in Eq. (8) with M=N. The exact values are obtained from Eq. (4). The density is normalized by $4\pi\int\rho_{I=o}(r)r^2dr=1$.

a(fm)	ℓ	N=6	N=10	N=14	Exact
	0	0.08924	0.08905	0.08905	0.08905
	1	0.08305	0.08354	0.08348	0.08368
0.6	2	0.1402	0.1404	0.1403	0.1404
	3	0.1281	0.1280	0.1280	0.1280
	4	0.06622	0.06601	0.06600	0.06627
	0	0.07625	0.07664	0.07663	0.07680
	1	0.08221	0.08227	0.08227	0.08229
	2	0.09879	0.09867	0.09866	0.09879
1.0	3	0.09310	0.09309	0.09310	0.09322
	4	0.06748	0.06746	0.06746	0.06747

Table IV. The values of $\lambda_o^{\ell\ell}(k,k')$ for $k=1.0$ fm^{-1} and $k'=2.0$ fm^{-1}, for derivative Woods-Saxon density. See the description for Table V.

a(fm)	ℓ	N=6	N=10	N=14	Exact
	0	-0.01882	-0.01885	-0.01884	-0.01883
	1	-0.01690	-0.01809	-0.01813	-0.01803
0.6	2	-0.02679	-0.02711	-0.02710	-0.02707
	3	-0.02476	-0.02482	-0.02484	-0.02480
	4	-0.01124	-0.01150	-0.01151	-0.01140
	0	-0.005080	-0.006686	-0.006719	-0.006750
	1	-0.007349	-0.007609	-0.007611	-0.007581
1.0	2	-0.008387	-0.008004	-0.008005	-01008044
	3	-0.007685	-0.007745	-0.007748	-0.007694
	4	-0.005139	-0.005102	-0.005100	-0.005110

Table V. The values of $\lambda^{\ell\ell'}_{I=2}(k,k')$ for k=0.5 fm^{-1} and k'=1.5 fm^{-1}, for I=2 (Quadrupole) derivative Woods-Saxon density of Eq. (11) with r_o = 4 fm and a=0.6 fm. The density is not normalized, but is taken to be $\rho_{I=2}(r) = (r_o/\rho_o)\frac{d\rho_o(r)}{dr}$. The values of N are the order of expansion in Eq. (8) with M=N. The exact values are obtained from Eq. (4).

ℓ	ℓ'	N=6	N=10	N=14	Exact
0	2	0.1999	0.1999	0.1998	0.1995
	1	-1.842	-1.857	-1.859	-1.853
1	3	1.537	1.531	1.531	1.526
	0	0.3552	0.3761	0.3756	0.3738
2	2	-0.7736	-0.7919	-0.7921	-0.7888
	4	1.513	1.514	1.513	1.505
	1	-0.2331	-0.2268	-0.2265	-0.2220
3	3	0.01569	0.01116	0.01148	0.009036
	5	0.6825	0.6822	0.6820	0.6809
	2	-0.1071	-0.1069	-0.1068	-0.1059
4	4	0.1072	0.1064	0.1064	0.1050
	3	-0.01179	-0.01200	-0.01204	-0.01243
5	5	0.04218	0.04195	0.04193	0.04182

Table VI. The values of $\lambda_{I=2}^{\ell\ell'}(k,k')$ for k=1.0 fm^{-1} and k'=2.0 fm^{-1} for I=2 derivative Woods-Saxon density. See the description for Table IX.

ℓ	ℓ'	N=6	N=10	N=14	Exact
0	2	0.2989	0.2987	0.2985	0.2982
1	1	-0.3847	-0.4110	-0.4118	-0.4097
	3	0.1306	0.1475	0.1480	0.1457
2	0	0.5886	0.5955	0.5950	0.5944
	2	-0.6087	-0.6160	-0.6157	-0.6150
	4	0.3183	0.3244	0.3242	0.3225
3	1	0.3419	0.3424	0.3425	0.3411
	3	-0.5626	-0.5640	-0.5644	-0.5635
	5	0.6075	0.6044	0.6045	0.6044
4	2	-0.01551	-0.005989	-0.006226	-0.007999
	4	-0.2553	-0.2612	-0.2615	-0.2590
5	3	-0.1305	-0.1255	-0.1255	-0.1231
	5	-0.01458	-0.01743	-0.01729	-0.01756

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