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ABSTRACT

With the assumption that the unperturbed amplitude is known in the ND^{-1} representation, equations are derived for calculating the perturbed amplitude as a function of the variations of the left-hand singularities and the unitarity cuts. Exact formulas for mass and coupling-constant shifts of bound states of the unperturbed amplitude are given, and the equations are then iterated to yield a perturbation theory.

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I. INTRODUCTION

A systematic assault on the total bootstrap problem seems impossible at present. However, approximations based on the concept of supermultiplets of strongly interacting particles reduce the problem into more manageable units. After rough calculations have been performed with the supermultiplets, it is possible to induce symmetry breaking within the multiplet to obtain finer details.¹⁻³ Computations are not too difficult if the symmetry breaking is treated in a first-order S-matrix perturbation theory.² The usefulness of the linearized theory is well demonstrated in the theory of octet enhancement.³

The perturbation consists of variations of the left- and right-hand singularities of the partial-wave amplitude. The Dashen-Frautschi theory² describes the mass and coupling shifts of a dynamically bound state as a linear function of these variations. The linear theory is capable of calculating small shifts when the bound-state pole remains on a given Riemann sheet of the scattering amplitude. However, if the pole changes sheets, or if it executes a large motion on a single sheet, the linear theory is not adequate. For these and for mathematical reasons it is desirable to have a more complete theory of perturbed amplitudes and of mass and coupling shifts of bound states.

Equations for the perturbed amplitude, and for the mass and coupling shifts, have been derived by M. Kugler for single-channel potential theory⁴ where only the left-hand singularities are varied.

Using a Castillejo-Dalitz-Dyson (CDD) pole in his perturbed-amplitude equations, Kugler⁴ then recovers the potential-theory analog of the Dashen-Frautschi mass-shift formula. The inclusion of many channels in his formalism is trivial. However, the generalization to the relativistic case in which the unitarity cuts are also varied is less simple.

In Section II, we propose a set of exact nonsingular equations for the perturbed amplitude. The input into these equations is the unperturbed amplitude in the form ND^{-1} and a perturbation represented by variations of the exchange and unitarity cuts. It is clear that this is sufficient information to calculate the total amplitude. Moreover, in a two-body elastic-unitary formalism, it is consistent with the bootstrap philosophy to use the change of the unitarity cuts as input, since without three-body intermediate states it is impossible to determine whether an external particle is a bound state or "elementary." At the end of Section II we show that the perturbed amplitude equations can account for the appearance of a stable-particle pole on the physical sheet, when the perturbation supplies the necessary additional binding force.

In Section III we find the mass shifts and the coupling-constant perturbations of a bound state when the unperturbed amplitude already has a pole on the same sheet. These are given in terms of the solutions to the equations derived in Section II. To describe the mass shift of a bound state, we note that the unperturbed and total amplitudes must each have simple pole, but located at different values of s . Thus, the perturbed amplitude must have two poles: one to

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cancel the pole in the unperturbed amplitude and the other to correspond to the bound state of the total amplitude. It is possible to achieve this effect mathematically by reinterpreting the dynamical bound state of the unperturbed amplitude as an elementary particle for the calculation of the perturbed amplitude.

The most obvious application of the mass- and coupling-shift formulas of Section III is a perturbation expansion for these quantities. In Section IV, which is divided into three subsections, we derive formulas for the first-order mass shift, the first-order coupling shift, and the second-order mass shift. The first-order results are identical to those of Dashen and Frautschi.² The second-order coupling shifts and higher order terms in the perturbation expansions are easy to derive, but the complicated results are probably of little use in numerical calculations.

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II. EQUATIONS FOR THE PERTURBED AMPLITUDE

Suppose the many-channel partial-wave scattering amplitude, $A(s)$, can be written as a sum of two terms,

$$A(s) = A_0(s) + A_1(s), \quad (2.1)$$

where we shall call $A_0(s)$ the "unperturbed amplitude" and $A_1(s)$ the "perturbed amplitude." Restricting ourselves to two-body channels and two-body unitarity, we assume that A_0 is known in the form

$$A_0(s) = N(s) D^{-1}(s). \quad (2.2)$$

The discontinuity relations for $N(s)$ and $D(s)$ are the usual ones: $D(s)$ has only a right-hand cut,

$$[D(s)]_R = (2i)^{-1} [D(s + i\epsilon) - D(s - i\epsilon)] = -\rho_0(s) N(s); \quad (2.3a)$$

and $N(s)$ has only left-hand singularities,

$$[N(s)]_L = (2i)^{-1} [N(s - i\epsilon) - N(s + i\epsilon)] = V_0(s) D(s). \quad (2.3b)$$

Moreover, $A(s)$, $A_0(s)$, and $A_1(s)$ are symmetric and Hermitian analytic.

The input for the calculation of the perturbed amplitude is the change of the unitarity cuts,

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$$\delta\rho(s) = \rho(s) - \rho_0(s) , \quad (2.4a)$$

where

$$[A(s)]_R = A^\dagger(s) \rho(s) A(s) , \quad (2.4b)$$

and the change of the left-hand singularities,

$$V'(s) = V(s) - V_0(s) , \quad (2.4c)$$

with

$$[A(s)]_L = V(s) . \quad (2.4d)$$

Neither $V'(s)$ nor $\delta\rho(s)$ is assumed to be small in this section.

To derive equations for $A_1(s)$, it is convenient to define²

$$J(s) = \widetilde{D}(s) A_1(s) D(s) . \quad (2.5)$$

The tilde signifies matrix transpose. Since $J(s)$ has separate left and right singularities, it can be factored into the form

$$J(s) = n(s) d^{-1}(s) . \quad (2.6)$$

The discontinuity relation for $J(s)$ is sufficiently complicated that if all the right cuts are put into $d(s)$ and all the left cuts are put into $n(s)$, then the equations for $n(s)$ and $d(s)$ are nonlinear. To preserve the linearity of the equations, $n(s)$ must contain some of the right cuts. The equations for $n(s)$ and $d(s)$ are then singular, but they are very similar to the three-body

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ND^{-1} equations derived by Mandelstam.⁵ By following Mandelstam's analysis, we use the standard techniques⁶ for reducing the Cauchy-singular equations to nonsingular equations.

The discontinuity of $A_1(s)$ on the right is needed for deriving the discontinuity relation for $J(s)$. From Eqs. (2.1), (2.3), and (2.4), we find

$$[A_1]_R = A_0^\dagger \delta_\rho A_0 + A_0^\dagger \rho A_1 + A_1^\dagger \rho A_0 + A_1^\dagger \rho A_1. \quad (2.7)$$

In transforming Eq. (2.7) into a discontinuity relation for $J(s)$, we use the fact that J is hermitian analytic,⁷

$$J^\dagger = J^{(-)}. \quad (2.8)$$

Substitute Eqs. (2.5) and (2.8) into Eq. (2.7), multiply on the left by $D^\dagger$ and on the right by D , then use the relation,

$$D^{-1} D^* = I + 2iD^{-1} \rho_0 N. \quad (2.9)$$

The discontinuity of J across the right cuts is then

$$[J]_R = J^\dagger G J + J^\dagger B^\dagger + B J + T, \quad (2.10a)$$

where

$$\begin{aligned} G &= \tilde{D}^{-1\dagger} \rho \tilde{D}^{-1}, \\ B &= N^\dagger \delta_\rho \tilde{D}^{-1}, \\ T &= N^\dagger \delta_\rho N. \end{aligned} \quad (2.10b)$$

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The discontinuity of J across the left cuts follows from Eq. (2.4),

$$[J]_L = L \equiv \tilde{D} V' D. \quad (2.10c)$$

To obtain linear equations for n and d in Eq. (2.6), the last two terms of Eq. (2.10a) are put into the discontinuity of n .

The coupled equations for n and d are

$$d(s) = I - \frac{1}{\pi} \int_R ds' \frac{G(s') n(s')}{s' - s - i\epsilon} - \frac{1}{\pi} \int_R ds' \frac{B^\dagger(s') d(s')}{s' - s - i\epsilon}, \quad (2.11a)$$

$$n(s) = \frac{1}{\pi} \int_L ds' \frac{L(s') d(s')}{s' - s + i\epsilon} + \frac{1}{\pi} \int_R ds' \frac{T(s') d(s')}{s' - s - i\epsilon} + \frac{1}{\pi} \int_R ds' \frac{B(s') n(s')}{s' - s - i\epsilon}. \quad (2.11b)$$

The exact limits of integration are obvious from the discontinuity relations, Eq. (2.10).

Equations (2.11a) and (2.11b) are both Cauchy singular. The reader who is uninterested in the technical details of finding equivalent nonsingular equations should now turn to the text below Eq. (2.19b). Equations (2.11a) and (2.11b) are conveniently reduced to nonsingular equations if we first remove the d term from the right-hand side of Eq. (2.11a) and the n term from the right-hand side of Eq. (2.11b). This can be done by defining

$$d(s) = m(s) f(s) , \quad (2.12a)$$

$$n(s) = t(s) u(s) , \quad (2.12b)$$

where $m(s)$ and $t(s)$ satisfy the equations,

$$m(s) = I - \frac{1}{\pi} \int_R ds' \frac{B^\dagger(s') m(s')}{s' - s - i\epsilon} , \quad (2.13a)$$

$$t(s) = I + \frac{1}{\pi} \int_R ds' \frac{B(s) t(s')}{s' - s - i\epsilon} . \quad (2.13b)$$

Equations (2.13a,b) are Cauchy singular, but are easily reduced to nonsingular equations. (We do this later). To find equations for f and u , compare the discontinuities of Eq. (2.12) and (2.11).

The formula for finding the discontinuity in f is

$$[mf]_R = m^{(-)} [f]_R + [m]_R f . \quad (2.14)$$

The integral representation of f (or u) is easily constructed.

The equations for f and u in matrix form are

$$\begin{pmatrix} f(s) \\ u(s) \end{pmatrix} = \begin{pmatrix} I \\ \frac{1}{\pi} \int_L ds' \frac{\lambda(s') f(s')}{s' - s + i\epsilon} \end{pmatrix} + \frac{1}{\pi} \int_R \frac{ds'}{s' - s - i\epsilon} \times \begin{pmatrix} 0 & -\gamma(s') \\ \tau(s') & 0 \end{pmatrix} \begin{pmatrix} f(s') \\ u(s') \end{pmatrix} , \quad (2.15a)$$

where

$$\begin{aligned}\lambda(s) &= [t^{(-)}(s)]^{-1} L(s) m(s), \\ \gamma(s) &= [m^{(-)}(s)]^{-1} G(s) t(s), \\ \tau(s) &= [t^{(-)}(s)]^{-1} T(s) m(s).\end{aligned}\tag{2.15b}$$

All four equations for m , t , f , and u are Cauchy singular. Equation (2.13a) is reduced to a nonsingular equation by operating on both sides of the equation from the left with⁵

$$I - \frac{1}{\pi} \int_R ds' \frac{B^\dagger(s')}{s' - s + i\epsilon}.$$

The nonsingular equation for $m(s)$ is

$$\begin{aligned}m(s) &= I - [I + 2i B^\dagger(s)]^{-1} \left\{ b(s) \right. \\ &\quad \left. + \frac{1}{\pi} \int_R ds' \frac{b^{(-)}(s') - b^{(-)}(s)}{s' - s} B^\dagger(s') m(s') \right\},\end{aligned}\tag{2.16a}$$

where

$$b(s) = \frac{1}{\pi} \int_R ds' \frac{B^\dagger(s')}{s' - s - i\epsilon}.\tag{2.16b}$$

The operator for reducing Eq. (2.13b) is

$$I + \frac{1}{\pi} \int_R ds' \frac{B(s')}{s' - s + i\epsilon}.$$

The nonsingular equation for $t(s)$ is

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$$t(s) = I + [I - 2i B(s)]^{-1} \left\{ b_1(s) - \frac{1}{\pi} \int_R ds' \frac{b_1^{(-)}(s') - b_1^{(-)}(s)}{s' - s} B(s') t(s') \right\}, \quad (2.17a)$$

where

$$b_1(s) = \frac{1}{\pi} \int_R ds' \frac{B(s')}{s' - s - i\epsilon}. \quad (2.17b)$$

The reduction of the coupled equations for f and u is facilitated by the matrix notation in Eq. (2.13c). Operate on both sides of the equation from the left with

$$I + \frac{1}{\pi} \int_R \frac{ds'}{s' - s + i\epsilon} \begin{pmatrix} 0 & -\gamma(s') \\ \tau(s') & 0 \end{pmatrix}.$$

The reduced equations are

$$f(s) - I + 2i \gamma(s) u(s) = \frac{1}{\pi} \int_R ds' \frac{\gamma_I^{(-)}(s') - \gamma_I^{(-)}(s)}{s' - s} \tau(s') f(s') + \frac{1}{\pi} \int_L ds' \frac{\gamma_I^{(-)}(s') - \gamma_I^{(-)}(s)}{s' - s} \lambda(s') f(s'), \quad (2.18a)$$

$$\gamma_I(s) = \frac{1}{\pi} \int_R ds' \frac{\gamma(s')}{s' - s - i\epsilon}, \quad (2.18b)$$

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$$u(s) - 2i \tau(s) [f(s) - I] = \frac{1}{\pi} \int_L \frac{ds'}{s' - s + i\epsilon} \lambda(s') f(s') \\ + \tau_I(s) + \frac{1}{\pi} \int_R ds' \frac{\tau_I^{(-)}(s') - \tau_I^{(-)}(s)}{s' - s} \gamma(s') u(s'), \quad (2.19a)$$

$$\tau_I(s) = \frac{1}{\pi} \int_R \frac{ds'}{s' - s - i\epsilon} \tau(s'). \quad (2.19b)$$

Our final set of Cauchy-singularity-free equations for the perturbed amplitude is Eqs. (2.16) through (2.19). The integrals of Eqs. (2.18) and (2.19) are linear in the perturbation, and the kernels of Eqs. (2.16) and (2.17) are already of second order. Thus, simply expanding n and d in a Neumann series will lead to a $J(s)$ which is the ratio of two power series in a parameter describing the perturbation. It is then possible for $J(s)$ to develop poles as the perturbation is changed. Even in a first-order iteration of d , this gives a mechanism for describing the binding of a resonance into a stable-particle pole, i.e., the appearance of a pole at threshold on the physical sheet. The emergence of resonance poles onto the unphysical sheet is described in the same way, but Eq. (2.11) must be continued onto the second sheet. (The details of a similar analytic continuation are found in Sec. III.)

Kugler⁴ has studied in single-channel potential theory⁸ the appearance of poles at threshold when the perturbation supplies the final binding force. With an effective range formula, he found

that the first iteration of f correctly described the motion of the pole as it moved away from threshold. In the next section, we assume the pole is already present in the unperturbed amplitude, and that the perturbation merely shifts the position of the pole on the same sheet.

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III. MASS AND COUPLING-CONSTANT SHIFTS OF BOUND STATES

The ND^{-1} technique is frequently used in dynamical calculations to find the masses and coupling constants of the composite particles communicating with a given set of channels and generated by a certain set of input forces. The mass of the bound state satisfies

$$\det [D(s_B)] = 0. \quad (3.1)$$

Since the equations of Sec. II are readily continued to the unphysical sheet (see below), we use the term "bound state" to mean either a stable particle or a resonance pole. However, we now restrict the perturbation so that the pole will remain on the same sheet. The case in which the pole does change sheets was briefly discussed at the end of the preceding section.

We now derive exact mass- and coupling-shift formulas. Let Eq. (3.1) have only one physical solution. Since the pole in A is shifted with respect to the pole in A_0 , the amplitude A_1 must have two poles: one to cancel the pole in A_0 and the other to represent the particle in the total amplitude. From Eqs. (2.5) and (3.1), it is clear that the pole in A_0 can be cancelled only if

$$\det [J(s_B)] = 0 \quad (3.2)$$

is a simple zero. Since $\det [D(s)]$ also has a simple zero at s_B , $A_1(s)$ will have a simple pole at $s = s_B$.

Formally, we can satisfy Eq. (3.2) by utilizing the CDD ambiguity in the $n d^{-1}$ equations of Section II. A CDD pole is inserted into Eq. (2.11) to force the occurrence of a zero at $s = s_B$. The $n d^{-1}$ equations for J are

$$d(s) = I - \frac{1}{\pi} \int_R ds' \frac{G(s') n(s')}{s' - s - i\epsilon} - \frac{1}{\pi} \int_R ds' \frac{B^\dagger(s') d(s')}{s' - s - i\epsilon} - \frac{\Gamma}{s - s_B}, \quad (3.3a)$$

$$n(s) = \frac{1}{\pi} \int_L ds' \frac{L(s') d(s')}{s' - s + i\epsilon} + \frac{1}{\pi} \int_R ds' \frac{T(s') d(s')}{s' - s - i\epsilon} + \frac{1}{\pi} \int_R ds' \frac{B(s') n(s')}{s' - s - i\epsilon}, \quad (3.3b)$$

where $d(s)$ now satisfies an equation with one CDD pole. The residue of the CDD pole is denoted by Γ ; the remaining terms are defined in Section II. A cancellation between the CDD pole and the remainder of the right side of Eq. (3.3a) will cause a pole in $J(s)$. The location of the new pole satisfies

$$\det [d(s_B)] = 0. \quad (3.4)$$

Thus, the zero imposed on $J(s)$ by the CDD pole can be used to cancel the pole in A_0 , and the pole generated by the CDD pole at s_B in $A_1(s)$ represents the particle when $\delta\rho$ and V' are nonzero. The restriction to one bound state in the unperturbed amplitude is trivially overcome by inserting several CDD poles into Eq. (3.3a).

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The situation for shifts of resonance poles on the second sheet is similar except that Eq. (3.3) must be analytically continued far enough onto the second sheet that the resonance region is exposed. The correct continuation consists of re-drawing the contours denoted by R so that they go below s_B . Then the position of the resonance pole in the unperturbed amplitude is explicitly exposed, and the CDD pole can be inserted as in the case of a stable bound state. The proof that this is the correct continuation is somewhat complicated by the fact that some of the right cuts begin at the unperturbed threshold and others begin at the perturbed threshold. If δp were zero, then this would certainly be the correct continuation. However, when $\delta p \neq 0$, it would appear that the s_B that solves Eq. (3.1) would not be on the sheet specified by the deformed cuts. This Oakes-Yang⁹ type dilemma is resolved by noting that there exists a family of poles on the sheets connected by the perturbed threshold. Consequently there is a resonance pole in the resonance region of the perturbed amplitude. Locating the CDD at this point would then have the same consequences as in the case of a stable bound state. Thus Eq. (3.3) is correct for the resonance case, except that R represents contours that go below s_B , rather than just along the real axis.

It is clear that the CDD pole gives the proper mathematical behavior. We now give a more physical interpretation of the CDD pole. The existence and properties of the particle were calculated

in finding A_0 . In the unperturbed amplitude, the particle-pole represents a dynamical bound state. However, the particle will be "elementary" in any further calculation. In this sense, an elementary particle is one whose existence and quantum numbers are available for further computation. Thus, the purpose of the CDD pole is to reinterpret the composite particle of the unperturbed amplitude as input (i.e., an elementary particle) for the calculation of the perturbed amplitude.

We determine Γ from the form of $A_0(s)$ near $s = s_B$,

$$A_0(s) \approx (s - s_B)^{-1} R, \quad (3.5a)$$

where R is the factorizable matrix of unperturbed channel-bound-state coupling constants. Since there exists a pole in $A_1(s)$ that cancels the pole in A_0 , $A_1(s)$ is

$$A_1(s) \approx - (s - s_B)^{-1} R \quad (3.5b)$$

for $s \approx s_B$. It follows from Eqs. (3.3), (3.5), (2.5) and (2.6) that

$$\Gamma = N^{-1}(s_B) R \tilde{N}^{-1}(s_B) n(s_B). \quad (3.6)$$

The total amplitude has a simple pole at $s = s_B'$. Since $A_0(s)$ is assumed well-behaved at $s = s_B'$,

$$A_1(s) \approx (s - s_B')^{-1} R', \quad (3.5c)$$

where R' is the new coupling matrix.

In order to find a convenient formula for the mass shift, we

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define

$$F(s) = I - \frac{1}{\pi} \int_R ds' \frac{G(s') n(s')}{s' - s - i\epsilon} \quad (3.7)$$

$$- \frac{1}{\pi} \int_R ds' \frac{B^+(s) d(s)}{s' - s - i\epsilon} .$$

$F(s)$ is not simply related to $d(s)$ in Eq. (3.3) since the solution to the nd^{-1} equations without CDD poles is not simply related to the solution of the equations with CDD poles. To evaluate $F(s)$, Eq. (3.3) must first be solved, then the solution inserted into the defining equation for $F(s)$.

Equation (3.4) is equivalent to

$$\det [\delta s_B - F^{-1}(s_B') \Gamma] = 0 , \quad (3.8)$$

where

$$\delta s_B = s_B' - s_B$$

is just the mass shift of the bound state. Equation (3.8) has the appearance of an eigenvalue equation except that $F^{-1}(s)$ is evaluated at the unknown value of $s = s_B'$. However, it is extremely convenient to analyze Eq. (3.8) formally as an eigenvalue equation. Then there are n values (n is the number of channels) of δs_B which solve Eq. (3.8). Thus, $n-1$ of these must be zero for δs_B to be unique. The derivation of this result relies on the pole-factorization theorem for the bound-state pole residues,

$$R_{ij} = -g_i g_j . \quad (3.9)$$

From the Laplace expansion of Eq. (3.8), it follows by induction that

$$\det(\delta s_B - F^{-1}(s_B') \Gamma) = (\delta s_B)^{n-1} [\delta s_B - \text{tr}(F^{-1}(s_B') \Gamma)] . \quad (3.10)$$

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We emphasize that no approximation has been made. Equation (3.10) follows directly from the factorizability of $F^{-1}(s_B')$. Consequently,

$$\delta s_B = \text{tr}[F^{-1}(s_B') \Gamma] . \quad (3.11)$$

Equation (3.9) aids in writing another expression for δs_B ,

$$\delta s_B = - \underline{g}^T \tilde{N}^{-1}(s_B) n(s_B) F^{-1}(s_B') N^{-1}(s_B) \underline{g} \quad (3.12)$$

where $\underline{g} (\underline{g}^T)$ is a column (row) vector whose elements are g_i ; g_i is the coupling constant of the bound state to channel i .

Equation (3.12) can be rewritten in terms of

$$\Delta = \lim_{s \rightarrow s_B} (s - s_B) D^{-1}(s) . \quad (3.13a)$$

Then we find that

$$\underline{g} = - (\underline{g}^T \underline{g})^{-1} N(s_B) \Delta \underline{g} , \quad (3.13b)$$

so that substituting Eq. (3.13b) into Eq. (3.12) yields a formula for δs_B of similar form to the first-order mass shift formula derived by Dashen and Frautschi.²

We find R' by calculating the residue of $A_1(s)$ at $s = s_B'$,

$$R' = \tilde{D}^{-1}(s_B') n(s_B') K D^{-1}(s_B') , \quad (3.14)$$

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with

$$K = \lim_{s \rightarrow s_B'} (s - s_B') [F(s) - (s - s_B')^{-1} \Gamma]^{-1} \quad (3.15a)$$

$$= F^{-1}(s_B') \Gamma F^{-1}(s_B') . \quad (3.15b)$$

The pole-factorization theorem has again been used in obtaining Eq. (3.15b) from Eq. (3.15a). Equation (3.7) relates $F(s)$ to $n(s)$ and $d(s)$.

The factorization theorem satisfied by R' is

$$R_{ij}' = -G_i G_j . \quad (3.16)$$

The coupling shift is $\underline{h}$ if $\underline{G} = \underline{g} + \underline{h}$. Setting R' equal to the right side of Eq. (3.14) yields an equation that contains zeroth-order terms. It is difficult to solve for $\underline{h}$ since the equation is not linear. However, a more useful equation for application to a perturbation expansion of the coupling shifts can be obtained by substituting the mass-shift formula into Eqs. (3.14) and (3.15).

We also set

$$D^{-1}(s_B') = N^{-1}(s_B') R(s_B' - s_B)^{-1} + N^{-1}(s_B') \beta(s_B') , \quad (3.17)$$

where $\beta(s)$ is the background term, i.e., it is the unperturbed amplitude minus the pole at s_B . It is of order R . The zeroth-order terms cancel and after some manipulations, we find

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$$\begin{aligned}
g_i h_j + h_i g_j + h_i h_j &= (X_1 X_2 - X_1 - X_2) g_i g_j + (1 - X_2) g_i b_j \\
&+ (1 - X_1) b_i' g_j + b_i' b_j,
\end{aligned}
\tag{3.18a}$$

where

$$X_1 = \underline{g}^T \tilde{N}^{-1}(s_B) n(s_B) F^{-1}(s_B') [N^{-1}(s_B') - N^{-1}(s_B)] (s_B' - s_B)^{-1} \underline{g},
\tag{3.18b}$$

$$\begin{aligned}
X_2 &= \underline{g}^T [\tilde{N}^{-1}(s_B') n(s_B') - \tilde{N}^{-1}(s_B) n(s_B)] (s_B' - s_B)^{-1} \\
&\times F^{-1}(s_B') N^{-1}(s_B) \underline{g},
\end{aligned}
\tag{3.18c}$$

$$\underline{b}^T = \underline{g}^T \tilde{N}^{-1}(s_B) n(s_B) F^{-1}(s_B') N^{-1}(s_B') \beta(s_B'),
\tag{3.18d}$$

$$\underline{b}' = \tilde{\beta}(s_B') \tilde{N}^{-1}(s_B') n(s_B') F^{-1}(s_B') N^{-1}(s_B) \underline{g}.
\tag{3.18e}$$

The leading terms of X_1 , X_2 , $\underline{b}$, and $\underline{b}'$ are first order in the perturbations.

Equations (3.12) and (3.18) are the exact mass- and coupling-shift formulas when the pole position does not change sheets under the influence of the perturbation.

IV. PERTURBATION THEORY

Equations (3.12) and (3.18), along with the equations of Section II, are sufficient to generate a perturbation expansion for the mass and coupling shifts. The zeroth-order iteration of the perturbed amplitude equations leads to results which are identical to the first order Dashen-Frautschi formulas.² The first order iteration of the perturbed amplitude equations is used in the second-order mass and coupling shifts. We only display the second-order mass-shift formula since it is straightforward to derive other terms in the expansion. The higher-order terms in the expansions are probably of little use in numerical calculations.

The application of the reduction procedure of Section II to Eq. (3.3) yields a set of non-Cauchy-singular equations that are identical to Eqs. (2.16) through (2.19), except that the term,

$$- [m^{(-)}(s_B)]^{-1} \Gamma(s - s_B)^{-1}, \quad (4.1a)$$

must be added to the right side of Eq. (2.18a), and the term,

$$- \frac{1}{\pi} \int_R ds' \frac{\tau(s') [m^{(-)}(s_B)]^{-1} \Gamma}{(s' - s + i\epsilon)(s' - s_B)}, \quad (4.1b)$$

must be added to the right side of Eq. (2.19a).

To generate a perturbation series for δs_B , we need the first several iterations of Eq. (2.18a) and (2.19b),

$$f(s) = I + f^{(1)}(s) + \dots \quad (4.2a)$$

$$u(s) = w(s) + u^{(2)}(s) + \dots \quad (4.2b)$$

where $f^{(j)}(s)$ and $u^{(j)}(s)$ are of j -the order in the perturbations $\delta\rho$ and V' , and $u^{(1)}(s)$ is denoted by $w(s)$.

1. First-Order Mass Shift.

Since Eq. (2.19) is already first order in the perturbation, the first-order mass shift is simply

$$\delta s_B = -\underline{g}^T \hat{N}^{-1}(s_B) w(s_B) N^{-1}(s_B) \underline{g}, \quad (4.3)$$

where

$$\begin{aligned} w(s) = & \frac{1}{\pi} \int_L ds' \frac{\tilde{D}(s') V'(s') D(s')}{s' - s + i\epsilon} \\ & + \frac{1}{\pi} \int_R ds' \frac{\tilde{N}(s') \delta\rho(s') N(s')}{s' - s - i\epsilon}. \end{aligned} \quad (4.4)$$

The Dashen-Frautschi result² is recovered by inserting Eq. (3.13b) for g and g^T .

2. First-Order Coupling Shift.

As in Reference 2, we define

$$\Delta' = \frac{d}{ds} N^{-1}(s_B) R + N^{-1}(s_B) \beta(s_B) \quad (4.5a)$$

$$= \frac{d}{ds} [(s - s_B) D^{-1}(s)]_{s=s_B}. \quad (4.5b)$$

The first order, Eq. (3.18) becomes

$$g_i h_j + g_j h_i = -(\tilde{\Delta}' w(s_B) \Delta + \tilde{\Delta} w'(s_B) \Delta + \tilde{\Delta} w(s_B) \Delta')_{ij}. \quad (4.6)$$

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The first-order Dashen-Frautschi result follows from Eq. (4.6) as shown in footnote 8 of Ref. 2,

$$\underline{h} = - (\underline{g}^T \underline{g})^{-1} (\tilde{\Delta}' w(s_B) \Delta + \frac{1}{2} \tilde{\Delta} w'(s_B) \Delta) \underline{g} . \quad (4.7)$$

3. Second-Order Mass Shift.

The second-order mass shift is

$$\begin{aligned} \delta_{(2)} s_B &= \underline{g}^T \tilde{N}^{-1}(s_B) \left\{ \frac{1}{\pi} \int_L ds' L(s') \left[\frac{U(s') - U(s_B)}{s' - s_B} \right. \right. \\ &+ \left. \left. \frac{\Gamma(1)}{(s' - s_B - i\epsilon)(s' - s_B)} \right] + \frac{1}{\pi} \int_R ds' T(s') \right. \\ &\times \left. \left[\frac{U(s') - U(s_B)}{s' - s_B} + \frac{\Gamma(1)}{(s' - s_B - i\epsilon)(s' - s_B)} \right] \right. \\ &- \left. \frac{1}{\pi} \int_R ds' \frac{\tilde{N}(s') \delta\rho(s') \tilde{D}^{-1}(s') w(s')}{s' - s_B - i\epsilon} \right\} N^{-1}(s_B) \underline{g} \quad (4.8) \end{aligned}$$

where

$$U(s) = \frac{1}{\pi} \int_R \frac{ds' \tilde{D}^{-1\dagger}(s')}{s' - s - i\epsilon} [\rho(s') \tilde{D}^{-1}(s') w(s') + \delta\rho(s') N(s')] , \quad (4.9a)$$

$$\Gamma(1) = N^{-1}(s_B) R \tilde{N}^{-1}(s_B) w(s_B) , \quad (4.9b)$$

$$L(s) = \tilde{D}(s) V'(s) D(s) , \quad (4.9c)$$

$$T(s) = \tilde{N}(s) \delta\rho(s) N(s) . \quad (4.9d)$$

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Equation (4.8) is the second-order mass shift for both stable particles and resonances. In the case of a resonance shift, R is a set of contours that goes below s_B .

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FOOTNOTES AND REFERENCES

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 4. M. Kugler, Perturbation Methods in Dispersion Theory, Phys. Rev. (to be published), (Columbia preprint).
 5. S. Mandelstam, Phys. Rev. 140, B375 (1965). A list of theorems on singular integral equations is given in the appendix.
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 7. For notational convenience, the argument s , of all functions will be omitted. We follow the convention $J = J(s + i\epsilon)$ and $J^{(-)} = J(s - i\epsilon)$, etc. Thus, Hermitian analyticity and the symmetry of A_0 imply that $D^{(-)} = D^*$. The longer notation would read $D(s - i\epsilon) = D^*(s + i\epsilon)$.
 8. For potential theory, $\delta_0 = 0$. Then $t(s) = m(s) = I$, or $f(s) = d(s)$ and $n(s) = u(s)$. The nd^{-1} equations for $J(s)$ are $d(s) = I - \pi^{-1} \int_R ds' G(s') n(s') (s' - s - i\epsilon)^{-1}$ and $n(s) = \pi^{-1} \int_L ds' L(s') d(s') (s' - s + i\epsilon)^{-1}$. This is the multichannel generalization of Kugler's equations.
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