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Journal

Annals of the American Association of Geographers, 78(2)

ISSN

2469-4452

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Publication Date

1988-06-01

DOI

10.1111/j.1467-8306.1988.tb00206.x

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Formal Analysis of Ridge and Channel Patterns in Maturely Eroded Terrain

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Abstract. The ridges delineating the drainage basins of a channel network and its sub-networks form themselves a network. This network spatially penetrates, or interlocks with, the channel network. Under fairly weak axiomatic assumptions, interlocking ridge and channel networks are of equal magnitude, and there exists a one-to-one relationship between their respective paths such that each pair of corresponding ridge and channel paths delineates a contiguous area called a drainage complex. In the limiting case that the link number of the channel path is zero, the complex coincides with the familiar concept of a drainage basin.

Data sampled from natural ridge and channel networks indicate that, for a given complex magnitude, the link numbers of the two paths follow an essentially random distribution; in contrast, the sum of the link numbers is closely dependent on the complex magnitude. Thus, while the internal connectivity of any ridge or channel network in the study area seems to be largely a matter of chance, the respective connectivities of interlocking networks exercise tight control over each other. The observed dependency is interpreted as the result of both geometric and geomorphic constraints limiting the minimum and maximum length of drainage area boundaries.

Key Words: channel networks, ridge networks, maturely eroded terrain, network axiomatization, network interdependency.

IT has been stated that "the basic business of science is to ascertain linkage relations between observables" (Simms 1983). In this paper the observables are the lines that give structure to erosional landscapes, and the link-

age relations are the principles that govern their relative positions and interconnections.

At least under environmentally uniform conditions one should expect that the pattern of ridges of a maturely eroded plateau reflects, in some rather precise fashion, the configuration of the channel pattern. As channels lengthen through headward growth, so do the ridges separating them, and as new channels develop through channel bifurcation, new ridges begin to emerge between them. Indeed, if we assume, say, a constant valley side slope V then the complete knowledge of the channel pattern C in three dimensions should permit the exact three-dimensional description of the ridge pattern R . Symbolically this statement can be expressed as $C + V = R$ where the plus sign represents the combination of two bodies of information.

But even under the far less ideal conditions of reality the equation retains some validity; a look at a topographic map will quickly reveal that the shape and location of ridges resulting from fluvial erosion seem to be significantly influenced by those of the adjacent channels. In fact, at the local level of individual channel junctions the interdependence between channel slope, valley side slope and divide angle has already been successfully modeled (Abrahams 1980).

The intimate relation between channels and ridges is not only one of form but also one of function. Like channels, ridges are lines delineating areas of overland flow in the specific sense that they will not be crossed by such flow. Just as erosional channels can be represented as lines of convergence of overland flows, so can ridges be described as lines of flow divergence. Together they constitute the classic object of fluvial morphology—the drainage basin—by defining its "rim" and "bottom." In turn,

the aggregation of ridge and channel lines into mutually penetrating networks creates the multiplicity and hierarchical nesting of drainage basins so characteristic of fluvially eroded landscapes.

This paper focuses on the formal description rather than the physical explanation of the ridge and channel patterns in maturely dissected terrain; the object of the description is the connectivity within and between these patterns rather than their geometry. My purpose is to investigate the topological relations between ridge and channel networks, with particular emphasis on the interdependencies between ridge and channel links and paths, and the drainage areas delineated by them.

Review of Previous Work

While the analysis of channels and their networks has experienced considerable and steady progress (Chorley 1972; Smart 1972; Abrahams 1984), the same does not hold for the analysis of ridge lines and the patterns they form, let alone the interdependence of these two patterns. To begin with, the problem of defining ridges and ridge lines is more difficult than the real world phenomena would suggest. Traditional definitions tend to provide broad, qualitative conceptualizations that lack in operational specificity; also, they might exclude phenomena critical for a particular research design. Examples would be Swayne's definition of a ridge as "an extended elevation of the earth's surface, long in comparison to its width" (1959), or the definition adopted by the American Geological Institute: "A relatively narrow elevation which is prominent on account of the steep angle at which it rises" (1976). Sometimes the concept of a ridge (or its crest, the ridge line) is utilized without any explicit definition (Goudie 1969; Krumbain and Shreve 1970; Werner 1972a). An operational definition will be provided in the next section.

To review alternative approaches to the concept of ridge line, we start with a related notion, the drainage divide. Divides are the lines that delineate drainage areas of particular channels or channel trees. Every link and, indeed, every channel segment has its own drainage area, and the corresponding divide lines can be determined fairly unambiguously in the field or on a topographic sheet (Schumm 1956; Werner

1972b; Woldenberg 1972; Abrahams 1980). While ridges as elevated bodies usually function as divides for certain channel segments, the reverse does not apply: divide lines may or may not have any surface expression.

Swayne's definition is probably close to the common connotation of a ridge; along the crest of such a ridge—the ridge line—contour lines show a "significant" change of direction; thus, a ridge line will appear on a topographic map as a sequence of aligned contour cusps or crenulations (Werner 1982). This concept of a ridge line is at least formally comparable to the established channel concept, as a three-dimensional map turned up-side-down will demonstrate. While a stream channel usually exhibits recognizable features or can be defined by such features (Drummond 1974), like the erosional incision into an otherwise continuous surface, such convenient indicators are not available for ridges. The most obvious feature of a ridge line is its separation of surface runoff into diverging directions. But that quality is quite impractical for the purpose of definition because it is too comprehensive: any line of steepest descent on a convex, rounded slope would meet this criterion. If, on the other hand, we require a "local" and "substantial" change of contour direction as part of a definition, then how should these requirements be quantitatively fixed without being arbitrary?

Following Cayley (1859), Maxwell (1870), and others, Warntz (1966, 1975), Warntz and Woldenberg (1967) and Woldenberg (1972) approached the definition of ridge lines as "critical" lines of natural terrains through a set of strictly formal concepts that constitute, in effect, the application of differential geometry to the study of landform patterns. While these concepts permit what appears to be an incomplete but otherwise compact and effective characterization of natural terrain surfaces, they do not readily lend themselves to a constructive methodology of fluvial landform analysis. A concise review and critique can be found in Mark (1979).

The Warntz/Woldenberg system has the advantage that its components can be located on the map or in the field with relative ease, although with results that might exhibit considerable differences (Drummond 1974; Mark 1983). In addition, they represent major landscape features and their relations to each other in a form that permits analysis by graph-

theoretical methods. Examples are Woldenberg's (1972) application of Euler's theorem for polyhedrons and Pfaltz's (1976) theorem on the contraction of ridge line/course line networks.

Mark (1979) bypassed the difficulties encountered by Warntz and Woldenberg by restricting his research to peaks interconnected by ridge lines. He simulated observed frequencies of the topology of these "ridge trees" by imposing spatial and topological constraints on the assumption of topological randomness. Significantly, Mark was able to relate his constraints to suspected or recognized geomorphic processes (Mark 1979, 1982).

Despite the sizable discrepancies between the definitions reviewed above, there is agreement among the various investigations that chance effects play a major role in the connectivity of natural ridge lines (Goudie 1969; Werner 1972b; Mark 1979). There also appears to be agreement that the geometry and topology of ridge patterns depend on the corresponding channel pattern, either as formal consequence of the definition of surface features (Woldenberg 1972; Werner 1982, 1986), or as observed regularity established through empirical investigations (Werner 1972c; Mark 1979; Abrahams 1980).

This paper is an effort to continue the investigation of channel/ridge line dependencies. It explores an alternative approach to the Warntz/Woldenberg system and adds to Mark's (1979) largely qualitative comments by formulating both logical and empirical dependencies in quantitative terms. It also adds to Abrahams's (1980) investigation by exploring the dependencies between entire patterns of ridges and channels, although only from a topological viewpoint. Specifically, it further differentiates the ridge concept so as to tie it closely to the established notion of channels; it derives several formal relationships between interlocking ridge and channel networks, and it identifies empirical dependencies between ridge and channel patterns that cannot be explained by chance effects and are the likely result of geometric and geomorphic constraints.

The Concept of Interlocking Ridge and Channel Networks

We start out with an operational definition of ridge lines in general and then define inter-

locking ridge lines as a particular subset of the former.

(1) For this paper, the definition of a ridge line will be based on the information content of contour maps. In the case of a surface with sudden changes of contour line direction, a ridge line is defined as a line connecting a sequence of aligned contour cusps pointing downhill; in the case of a smooth surface, it is defined as a line connecting the points of maximum curvature in a sequence of aligned contour arcs, their convex sides facing downhill.

(2) The thresholds established for the resolution and contour curvature in the identification of ridge lines are defined by reference to the channels in the study area.

Since, in graph-theoretic terms, a channel network is a planar-rooted tree, its exterior links and its outlet link form an ordered sequence (Shreve 1967). Two exterior links (or one exterior link and the outlet link) are called neighboring if they occupy consecutive positions within that sequence. We determine the channels in the study area using the contour crenulation method described by Krumbain and Shreve (1970), but with the added restriction that exterior channel links are terminated where clearly pointed contour cusps are followed by smoothly curved contour arcs. This criterion is further sharpened by the condition that any contour line segment connecting two channels contains at least one inflection point. It guarantees that the contour lines between channels assume convex shapes downhill and thus define at least one ridge line if the required minimum curvature is set at an appropriate level. The level selected for the entire study area is set as the maximum contour curvature that will ensure the existence of at least one ridge line between any two neighboring exterior channel links.

(3) Ridge lines terminating in passes will be connected and treated as continuous lines. Ridge lines terminating on opposite sides of a channel will not be connected.

(4) Converging ridge lines give rise to ridge nodes. Often, the exact location of such a node is impossible to establish, not only on maps but in the field as well. Ridge lines frequently disappear as they converge, and instead of a node, only an undifferentiated nodal area is recognizable. In these cases node locations are established by extending ridge lines to their point of intersection.

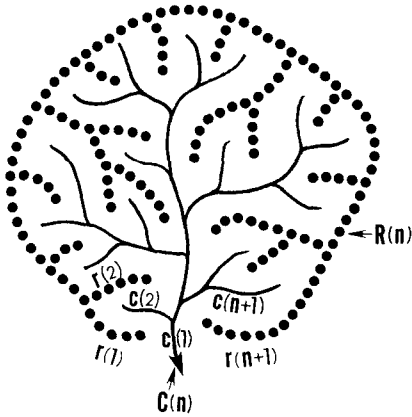


Figure 1. A pair of interlocking ridge and channel networks with alternating outer links.

(5) Ridge nodes are assumed to be always of degree three. This assumption is plausible because differential denudation between neighboring ridge lines radiating from a node will make a higher nodal degree an unstable, temporary event. Operationally the assumption does not pose a problem as a node of degree x can always be substituted by $x - 2$ nodes of degree three interconnected by links of near-zero length.

(6) Under humid conditions the ridge lines thus defined form a network that is, graph-theoretically speaking, a trivalent planar tree. This ridge tree will now be reduced by eliminating all but one ridge link between any two neighboring exterior channel links. The remaining ridge link should be the link of least gradient and must be attached to the ridge tree. We call this link an interlocking outer ridge link and the reduced ridge network an interlocking ridge network. Let C refer to a channel network and let R be the ridge network defined by the interlocking outer ridge links located between neighboring exterior channel links of C . Then R is called the interlocking network of C , and C and R are called interlocking networks (Figure 1).

The operations of extending converging ridge lines into nodal areas, locating the corresponding ridge nodes, and ensuring a nodal degree of three involve, by necessity, an element of chance that may have a randomizing effect on the emerging ridge line network. The impact

of this chance effect on the results of the paper can be disregarded because its deductive conclusions are independent of the notion of randomness and the main inductive conclusions refute it.

Using the procedure outlined above in actual map work is somewhat tedious but fairly adequate in terms of its decision rules. But what does an interlocking ridge network, once constructed, refer to in a fluvially eroded surface?

As a consequence of the definition of ridge lines, every path of the ridge network functions as a drainage divide line. By construction there exists for every channel link c two outer ridge links u, v positioned on either side of c . Thus, the ridge path f defined by u and v delineates an area that has c as its only drainage outlet. That is, the ridge path f defines the drainage basin of the channel network that has the channel link c as its outlet link. We call f the open boundary of the basin. A more formal derivation will be given in a later section.

Note that u and v might not reach the edge of the channel c ; rather, they are separated by a valley with c occupying the valley bottom. Hence, the link drainage area of c may or may not be enclosed in the basin delineated by f . But this difficulty is only a local and a geometric one; in topological terms the one-to-one relation is unaffected between the channel link c , the channel network defined by c , and the ridge path (u, v) corresponding to (but not fully congruent with) its drainage basin boundary.

To examine the graph-theoretical implications of our definition of interlocking networks, we extend the established terminology and describe channel and ridge networks in terms of

- inner nodes (nodes of degree 3, that is, all network junctions);
- outer nodes (nodes of degree 1, which in the case of a channel network consist of all sources and the outlet);
- inner links (with inner nodes as end points);
- outer links (at least one end point being an outer node; hence, in the case of a channel network, the set of outer links consists of all exterior links plus the outlet link);
- inner paths (having inner links as their end links);
- outer paths (having outer links as their end links); and

—intermediate paths (having an inner and an outer link respectively as their end links).

The number of links in a path is called its topological length or simply, its length when the context makes clear that it refers to topologic rather than geometric length.

The representation of a network by means of a binary string (Shreve 1967) shows the outer network links to be sequentially arranged. Let $c(1), c(2), \dots, c(n+1)$ be the ordered set of outer links of a given channel network $C(n)$ of magnitude n with $c(1)$ being the outlet link. Let R denote the ridge network interlocking with $C(n)$. It follows from our definition of interlocking networks that there exists between any two neighboring outer links of $C(n)$ exactly one outer ridge link of R and vice versa. Hence, the number of outer ridge links is equal to the number of outer channel links, that is, the two interlocking networks are of equal magnitude. Also the above assumption maps the ordering structure of the set of outer channel links onto the set of outer ridge links, and if we label the outer ridge link positioned between the outer channel links $c(i)$ and $c(i+1)$ with $r(i)$, then together the two sets form a sequence S consisting of alternating outer ridge and channel links:

$$S = c(1), r(1), c(2), r(2), \dots, c(i), r(i), c(i+1), r(i+1), \dots, c(n+1), r(n+1)$$

Notice that $r(1)$ and $r(n+1)$ are the outer ridge links positioned between the outlet link $c(1)$ and its neighbors $c(2)$ and $c(n+1)$, which means that the sequence S is cyclical (Figure 1).

As a consequence of our definition of interlocking ridge and channel networks, the two networks are both trivalent planar trees and are of equal magnitude; their alternating outer links establish a topologically symmetrical relationship between them. The combined properties of structural equivalence and symmetrical relationship make the two networks dual structures: whatever can be stated about one network relative to the other must also hold in reverse. The two properties constitute the axiomatic base for the following deductions; to emphasize their general applicability, we proceed to analyze networks N without specifying whether N is a ridge or channel network, and we analyze pairs of interlocking networks X, Y without specifying which one is the ridge and which one the channel network.

Selected Structural Properties of Ridge and Channel Networks

To unravel the relationships that hold between the paths of interlocking ridge and channel networks, several definitions and interdependencies of links and paths within networks will be needed. Most of the relations are obvious; where not, a short sketch of a proof is included. Unless specified otherwise, the term network represents both channel and ridge networks.

The path connecting two neighboring outer links of a network is called a bicorn; we say that the bicorn defines, and is defined by, the two outer links. Each network has $n+1$ bicorns; we call two bicorns adjacent if they share one of their outer links in common. Since the outer links of a network form a cyclical ordered set, so do the bicorns of the network. Note in particular that the bicorn defined by the outer ridge links $r(1), r(n+1)$ of the ridge network $R(n)$ is the open boundary of the basin drained by the channel network $C(n)$ (Figure 1).

The following set of arguments will show that there exists a one-to-one relationship:

- (A) between the links of a network and the pairs of intersecting bicorns (bicorns having a link in common), and
- (B) between the inner paths of a network and the pairs of disjoint bicorns (bicorns with no links in common).

(1) Let N be a network and x be a link of it. x separates N into two subnetworks N_1, N_2 and is the root link of both. Let $j = 1, \dots, n+1$ be the cyclical sequence of outer links of N , and let $j = i+1, \dots, k$ be the sub-sequence consisting of the exterior links of N_1 . Then the exterior links of N_2 form the sub-sequence $j = k+1, \dots, i$, and the first and last links of the two subnetworks are neighboring outer links of N , defining the two bicorns $[i, i+1]$ and $[k, k+1]$ in N . Since any path connecting links of N_1 and N_2 must pass through x , it follows that the two bicorns have the link x in common (Figure 2). If the network link x is an outer link, then the two bicorns sharing it are necessarily adjacent; likewise, if two bicorns sharing a link x are not adjacent then x is necessarily an inner link.

(2) Any two bicorns have at most one link in common. That can be shown through an in-

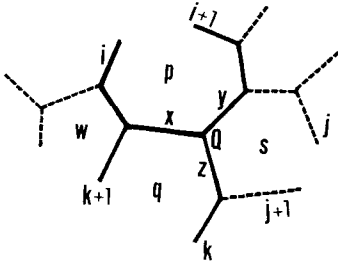


Figure 2. A network link x and the two bicorns $p = (i, i + 1)$ and $q = (k, k + 1)$ that have x in common (a bicorn is defined as a path connecting two neighboring outer links).

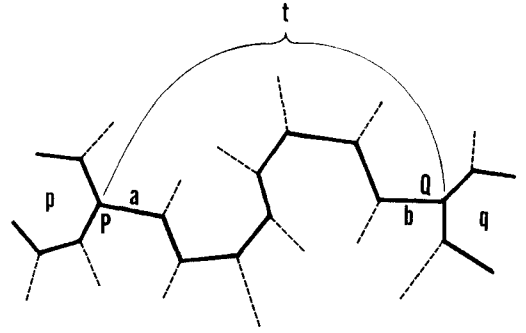


Figure 3. An inner network path $t = (P, Q)$ and the two bicorns p, q defining it.

direct proof. Consider two bicorns p, q with outer links $i, i + 1$ and $k, k + 1$. If p and q shared two or more links, these links would define a sub-path of both bicorns. The sub-path contains at least two links and therefore at least one node connecting two links; the third link incident at that node defines a subnetwork. Within the ordered sequence of the outer links of the entire network, the exterior links of this subnetwork would be positioned either between the links i and $i + 1$ or between k and $k + 1$, in violation of the assumption that p and q are bicorns, that is, that their outer links are neighbors.

(3) Let p and q be two disjoint bicorns of a given network (Figure 3). Of all the paths connecting nodes of p and q there is one path t that does not contain links of either p or q . t does exist because the network is a connected graph, and t is unique because the network is a tree. It follows from the definition of t that t is an inner network path.

(4) Since each inner node interconnects three links that are pairwise adjacent, it follows from (1) and (2) above that each inner node defines exactly three bicorns. Also for each of the three links there is exactly one among the bicorns that does not contain this link, that is, link and bicorn are disjoint. Thus, there exists a one-to-one relationship between the three links of a node and the three bicorns they define. In Figure 2 the node Q connects the three links x, y, z and defines the three bicorns $p = (i, i + 1)$, $s = (j, j + 1)$, and $q = (k, k + 1)$; the three pairs of disjoint links and bicorns are x, s ; y, q ; and z, p .

(5) Let t be an inner path connecting two

network nodes P, Q , and let a, b be its two end links (if t consists of one link only, then $a = b$) (Figure 3). Of the bicorns defined by the nodes P, Q there are two which contain neither of the end links a, b . We will label these two bicorns by p and q (Figure 3). According to (4) above p and q are uniquely identified by the links a and b and therefore by the path t ; the end nodes of t, P and Q , are nodes of the bicorns p, q respectively. Since the network is, in graph-theoretic terms, a tree, the path t does not contain any links of either p or q . Hence, t is identical with the inner path that is uniquely defined by the two bicorns p, q as shown in (3) above, thereby establishing a one-to-one correspondence between all pairs of disjoint bicorns and all inner paths of a network.

(6) Earlier we defined an intermediate path as one between an inner and an outer link. It follows from the previous discussion that there exists a one-to-one relationship between all intermediate paths of a network and all disjoint pairs consisting of a bicorn and an outer link.

A network of magnitude n has a total of $2n$ nodes, $n + 1$ of which are outer nodes, the remaining ones being inner nodes (junctions). The total number of network paths is therefore $2n(2n - 1)/2$ or $n(2n - 1)$. The outer nodes define a total of $(n + 1)n/2$ outer paths; the inner nodes define a total of $(n - 1)(n - 2)/2$ inner paths; the balance of the set of all network paths is made up by intermediate paths, their number being $(n + 1)(n - 1)$.

The network has $n + 1$ bicorns and therefore $(n + 1)n/2$ different pairs of bicorns. Of these, $2n - 1$ consist of bicorns that have a network link in common (items 1 and 2 above); the re-

maining $(n + 1)n/2 - (2n - 1) = (n - 1)(n - 2)/2$ pairs are disjoint and correspond to the inner paths of the network (item 5 above).

Since the number of outer links is $n + 1$ and the number of bicorns containing a particular outer link is 2, it follows that the number of pairs consisting of an outer link and a (disjoint) bicorn is $(n + 1)(n + 1) - 2$. This number is, of course, equal to the number of intermediate paths in the network as was already implied in item 6 above.

For the purpose of subsequent arguments, it should be stressed that the particular case of two bicorns sharing a link in common is quite different from the case of two disjoint bicorns defining an inner path consisting of one link. In Figure 2 the two disjoint bicorns s, w define the link x as an inner path while the two bicorns p, q have x as a common link. This distinction will be critical in the analysis of drainage basins and drainage complexes in later sections of the paper.

Correspondence between Ridge and Channel Paths

The definition of interlocking networks X, Y establishes a connection between the paths of the two networks. In particular, each bicorn of X defines two outer links $x(i), x(i + 1)$ which are neighbors within the sequence of outer links of X . Furthermore, the outer links of the two networks form an alternating sequence S , and within S the links $x(i), x(i + 1)$ are neighbors of a particular outer link $y(i)$ of Y , which is thereby uniquely defined. Since

- (a) each inner path of a network corresponds to two specific disjoint bicorns of that network (item 5 above), which
- (b) correspond to two particular outer links of the interlocking network, which in turn,
- (c) define an outer path of that network,

there exists for each inner path of a network a particular outer path of the interlocking network (Figure 4). In addition, since the relations (a) to (c) above hold in both directions provided the bicorns are disjoint, it follows that this relationship constitutes a one-to-one mapping between the set of all inner paths of one network and a proper subset of the outer paths of the interlocking network. Paths that are mapped

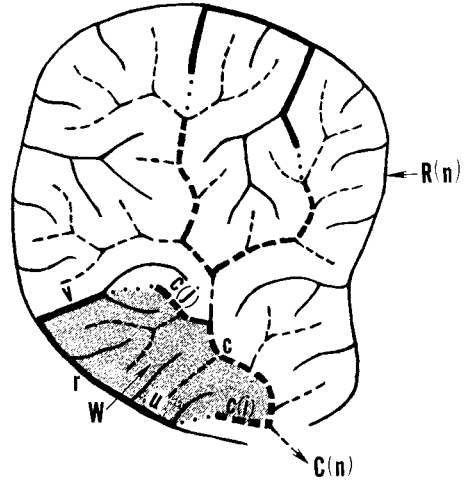


Figure 4. Two pairs of interlocking ridge and channel paths embedded in two interlocking ridge and channel networks; each pair defines a particular drainage area called a drainage complex.

onto each other in this way are called corresponding or interlocking. Two examples of channel paths (one inner and one outer path) and their interlocking (outer/inner) ridge paths are shown in Figure 4.

The number of outer paths that do not have corresponding inner paths in the interlocking network is simply the numerical difference between the two sets, or $(n + 1)n/2 - (n - 1)(n - 2)/2 = 2n - 1$. The outer links of these $2n - 1$ outer paths correspond to the $2n - 1$ pairs of bicorns of the interlocking network that share a link in common. Thus, there exists a one-to-one relationship between a particular subset of $2n - 1$ outer paths of one network and the $2n - 1$ links of the interlocking network. We will now examine this relationship in detail.

Ridge Paths as Basin Boundaries of Channel Networks

We will demonstrate that there exists a unique and reversible mapping between (A) each link of a channel network, (B) the subnetwork defined by the link, and (C) the ridge path that forms the open boundary of the basin drained by the subnetwork.

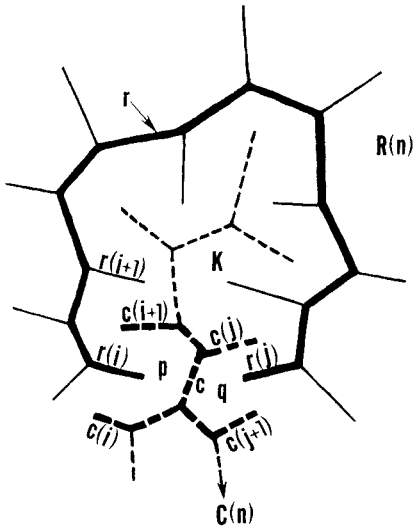


Figure 5. A channel link c defining a channel subnetwork K with exterior links $c(i+1)$, $c(i+2)$, ..., $c(j)$. K drains the area delineated by the ridge path r where r is defined by the outer ridge links $r(i)$, $r(j)$; they correspond to the bicorns p , q which have the channel link c in common.

Let r be one of the $2n - 1$ outer paths in the ridge network $R(n)$ that does not correspond to an inner path in the interlocking channel network $C(n)$; let $r(i)$ and $r(j)$ be the two outer ridge links of r ; and let p and q be the two bicorns of $C(n)$ defined by $r(i)$ and $r(j)$ (Figure 5). Since p and q do not define an inner path, they must intersect in one and only one link ([2] and [5] above) which we will label c . Let $c(i)$, $c(i+1)$ and $c(j)$, $c(j+1)$ be the outer channel links defined by the bicorns p and q . The link c defines a subnetwork K of $C(n)$; the exterior links of this subnetwork are the channel links $c(i+1)$, $c(i+2)$, ..., $c(j)$ and the magnitude of K is therefore $j - i$. Since, by construction, the outer ridge links $r(i)$, $r(j)$ are positioned "within" the bicorns p and q , that is, on either side of the channel link c , it follows that the corresponding outer ridge path r constitutes the open boundary of the area drained by the subnetwork K .

To summarize: there are exactly $2n - 1$ outer ridge paths of $R(n)$ that do not correspond to inner channel paths of $C(n)$; instead, they display a one-to-one relationship with the $2n - 1$ channel links. Each ridge path r that corre-

sponds to a particular channel link c functions as the open boundary of the basin that is drained by the subnetwork having c as its outlet link.

Drainage Areas Defined by Interlocking Ridge and Channel Paths

Interlocking ridge and channel paths possess a peculiar property that is, at least at first glance, rather surprising. Their respective end points are spatially positioned in such a way that in combination the paths approximate a simply-closed curve that defines a particular drainage area.

Without any loss in generality, we will assume that c is an outer channel path of $C(n)$ and r is c 's interlocking inner ridge path of $R(n)$ (Figure 4). Let u , v be the two (inner) end links of the path r , and $c(i)$, $c(j)$ the (outer) end links of the path c . According to item (5) above (p. 258), u defines a ridge bicorn; the channel link positioned between the outer links of the bicorn is one of the two end links of c , say $c(i)$. A similar statement holds for the end links v and $c(j)$. If we connect u and $c(i)$ as well as v and $c(j)$ by following the lines of steepest descent from the bicorns to the end points of $c(i)$, $c(j)$ (see dotted lines in Figure 4), we create a singly closed line (r, c) . This line delineates an area W that we call the drainage complex defined by the paths c , r ; we call c and r the channel and ridge boundaries of W , or simply its boundary paths, and use (c, r) as the symbol for the (open) complex boundary.

Since the number of outer ridge links in a drainage complex is either equal to the number of outer channel links in the complex or else differs by one (item 6 below), we define the magnitude of a complex as the number of outer channel links it contains. This number is of course equal to the sum of the magnitudes of the channel networks draining the complex.

The following statements describe selected properties of drainage complexes and their boundaries that are embedded within a given pair $C(n)$, $R(n)$ of interlocking ridge and channel networks:

1. Each drainage complex is a closed catchment area in the specific sense that overland flow does not cross its boundary.
2. Drainage basins constitute special cases of

drainage complexes. Let $s(x)$ designate the link number of the path x , then the subset of all complexes W with boundary paths c, r for which $s(c) = 0$, is the set of all basins and subbasins of the channel network $C(n)$.

3. The intersection of two drainage complexes is itself a complex.

4. For a given pair of interlocking networks of magnitude n , the number of drainage complexes is equal to the number of paths of each network, $(2n)(2n - 1)/2$, plus the link number $(2n - 1)$ of each network, or $(n + 1)(2n - 1)$ in total. Hence, the ratio of drainage complexes to drainage basins is $n + 1$.

5. The channel networks draining the area of a complex W with boundary paths c and r form an ordered sequence and consist of all and only the networks which enter the channel path c from the side bordering W . An equivalent statement applies to the ridge networks in W and the ridge path r .

6. The number of outer ridge links inside a drainage complex W with boundary paths c and r is equal to the number of outer channel links in W if r and c are intermediate paths; they differ by one if r and c are a pair of inner and outer paths.

The Topological Length of Drainage Complex Boundaries

In this and the following section we will:

- (1) assume that the topologies of ridge and channel networks are random in the specific sense that all topologically different network configurations of equal magnitude are equally likely to occur;
- (2) assume that the topology of any network is independent of the topology of its interlocking network (except, of course, for their equal magnitude);
- (3) calculate the expected distributions of both the channel and ridge boundary link numbers of magnitude k complexes; and
- (4) present empirical evidence that supports the assumption of the topological randomness of interlocking networks but is incompatible with the assumption of their topological independence.

Let $W(k)$ be a drainage complex of magnitude k embedded in a pair of interlocking networks

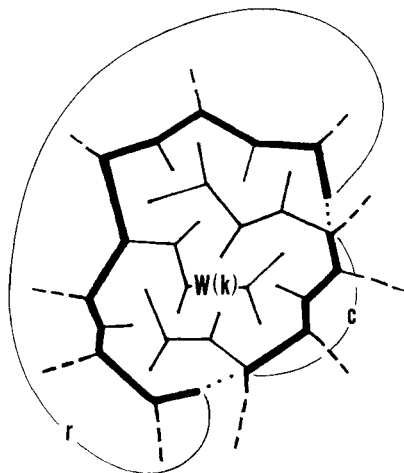


Figure 6. A pair of channel and ridge paths c, r delineating a drainage area (a "drainage complex") of magnitude k .

$C(n), R(n)$ of magnitude n ; let c and r denote the channel and ridge boundaries of W , and let $L(c), L(r)$ be their respective topological lengths. We further define $E(u, k, n)$ and $E(v, k, n)$ to be the probabilities that the channel and ridge boundaries of the complex $W(k)$ have, respectively, u and v links, or $L(c) = u$ and $L(r) = v$. Without loss in generality we will assume that c is an inner channel path and r therefore an outer ridge path. A sketch of the complex $W(k)$ is portrayed in Figure 6; for simplicity the parts of the networks C and R located outside of W have been omitted.

As an inner path with u links, the channel boundary c contains a total of $u + 1$ inner nodes in which the remainder of the channel network $C(n)$ is attached to it in the form of $u + 3$ subnetworks. x of these enter c from the left side and constitute the set of all channel networks draining the complex W while the other $u + 3 - x$ subnetworks enter c from the right side and constitute the remainder of the channel network. The sequence in which the subnetworks of the left and right enter the channel path c is, of course, not fixed but has to allow for all possible permutations; the number of different arrangements is given by the binomial coefficient $(u - 1)$ over $(x - 2)$. The value $(u - 1)$ takes into account that of the total of $u + 3$ subnetworks entering c , four join it in its two

end nodes, with two merging from the left and two from the right side of c ; for the same reason only $x - 2$ of the x subnetworks on the left of c are subject to permutation. In Figure 6 the magnitude k of the complex W is 8, the link numbers u, v of the boundary paths of W are 4 and 11 respectively, and the number x of subnetworks entering the channel path from the left side is 3.

The x subnetworks draining the complex $W(k)$ and entering c from the left have a combined magnitude k , whereas the combined magnitude of the $u + 3 - x$ subnetworks on the right is $n + 1 - k$ (notice that in this graph-theoretical analysis the outlet link of $C(n)$ is now one of the exterior links of the subnetworks joining c from the right). The number $Z(i, n)$ of different ways in which n exterior links can combine so as to form i different networks is

$$Z(i, n) = i \binom{(2n - i)}{2n - i} \quad (1)$$

(Werner 1972b); in particular, for the special case of $i = 1$ we get

$$Z(1, n) = \frac{(2n - 1)}{2n - 1} \quad (2)$$

a result reported earlier by Shreve (1966). Applied to our case, it means that the channel network $C(n)$ and the interlocking ridge network $R(n)$ can both assume any of $Z(1, n)$ different topological configurations.

Finally, it should be clear that the k exterior channel links of W can at most generate k different subnetworks and that they must not generate more than $u + 1$ as that is the number of nodes of the channel path c . In addition they must generate at least two networks because c is an inner path.

Aggregating these individual results, we can now give a mathematical formulation of the probability $E(u, k, n)$ that the link number of the channel boundary of a complex of magnitude k embedded in interlocking networks of magnitude n and random topology is:

$$E(u, k, n) = \frac{1}{Z(1, n)} \sum_{x=2}^{\text{Min}(u+1, k)} Z(x, k) \cdot Z(u + 3 - x, n + 1 - k) \cdot \binom{u - 1}{x - 2} \quad (3)$$

(Slightly different formulations apply if c is an intermediate or outer path; the derivation of the function E for the latter is sketched out later in this section.)

Many empirical studies of natural drainage networks actually deal with subnetworks of much larger networks whose magnitude, for a variety of research purposes (including those of the last part of this paper), can be assumed to approach infinity. It is for this reason that we need to establish the mathematical expression of $E(u, k, n)$ where $n \rightarrow \infty$. Whereas the derivation of (3) above provides at least some insight into the mathematical analysis of drainage networks, the case for networks of infinite magnitude relies essentially on abstract combinatorial arguments; thus, for this paper, presentation of the key results together with sketches of some of the proofs may be sufficient.

Using equation (1) it is easy to verify that $Z(i, n)$ satisfies the recursive relationship

$$Z(i, n) = Z(i - 1, n) - Z(i - 2, n - 1) \quad (4)$$

Repeated application of (4) allows us to express $Z(i, n)$ in terms of $Z(1, x)$:

$$Z(i, n) = \sum_{x=0}^{\substack{i \text{ even: } (i-2)/2 \\ i \text{ odd: } (i-1)/2}} (-1)^x \cdot Z(1, n - x) \cdot \binom{i - 1 - x}{x} \quad (5)$$

Applying Stirling's approximation for binomial coefficients to the ratio $Z(1, n - x)/Z(1, n)$ yields

$$\frac{Z(1, n - x)}{Z(1, n)} \underset{n \rightarrow \infty}{=} \frac{1}{2^{2x}} \quad (6)$$

Substituting (6) into (5) and dividing by $Z(1, n)$ results in

$$\frac{Z(i, n)}{Z(1, n)} \underset{n \rightarrow \infty}{=} S(i) = \sum_{x=0}^{\substack{i \text{ even: } (i-2)/2 \\ i \text{ odd: } (i-1)/2}} (-1)^x \cdot \frac{\binom{i - 1 - x}{x}}{2^{2x}} \quad (7)$$

Notice that the function $S(i)$ is recursive:

$$S(i) = \frac{S(i-1)}{2} + \frac{1}{2^{i-1}} \quad (8)$$

so that, through mathematical induction, we obtain for n approaching infinity,

$$Z(i, n)/Z(1, n) = \frac{i}{2^{i-1}} \quad (9)$$

Combining equations (3) and (9) gives us the desired result (desired in as much as it permits the calculation of numerical values of E for n approaching infinity; we have not been successful in transforming the right side of (10) into a closed expression):

$$E(u, k, n) = \sum_{x=2}^{\text{Min}(u+1, k)} Z(x, k) \cdot \frac{u+3-x}{2^{2k+u-x}} \binom{u-1}{x-2} \quad (10)$$

A line of reasoning parallel to the derivation of the probability $E(u, k, n)$ of equation (3) generates the probability $E(v, k, n)$ that the link number of the (outer) ridge boundary r of the drainage complex $W(k)$ is v :

$$E(v, k, n) = \frac{1}{Z(1, n)} \sum_{x=1}^{\text{Min}(v-2, k)} Z(x, k) \cdot Z(v-1-x, n-1-k) \binom{v-1}{x-2} \quad (11)$$

and therefore

$$E(v, k, n) = \sum_{x=1}^{\text{Min}(v-2, k)} Z(x, k) \cdot \frac{v-1-x}{2^{2k+v-x}} \binom{v-1}{x} \quad (12)$$

By assumption the topologies of interlocking networks are independent. Consequently, the probability $E(u, v, k, \infty)$ that a randomly chosen drainage complex of magnitude k embedded in a pair of interlocking networks of infinite magnitude has channel and ridge boundaries of u and v links respectively, is the product of the individual probabilities;

$$E(u, v, k, \infty) = E(u, k, \infty)E(v, k, \infty) \quad (13)$$

The field of probabilities $E(u, v, k, \infty)$ for variable values u and v is shown in Figure 7; the

magnitude of the drainage complex W is $k = 25$ and the magnitudes of the interlocking networks C and R approach infinity.

Strictly speaking the probability field is, of course, discrete, with positive values for pairs of integers (u, v) where $u \geq 0$ and $v \geq 3$, and zero everywhere else. The special case of $u = 0$ refers to the common drainage basin delineated by a ridge path; the special case of $v = 3$ refers to a drainage complex whose ridge boundary (here an outer path) is of minimum topological length. Each of the two sets of ridges (those within and those without the complex) merge into a single subnetwork which enters the ridge boundary of the complex in a single node, thus producing an outer ridge boundary of three links.

The reasons that (u, v) pairs of equal probability approximate circles are the assumed independence and the mathematical equivalence of the distributions $E(u, k, \infty)$ and $E(v, k, \infty)$; the skewness of these two distributions (resulting from the non-negativity of their respective domains) explains why the circles are not quite concentric. It follows that for large values of k the probability distribution $E(u, v, k, \infty)$ will essentially be bell-shaped.

Test Results

To reduce the influence of external determinants on the topology of natural interlocking networks, data were sampled from an area in Eastern Kentucky that shows a relatively constant distribution of major geologic and climatic parameters (U.S.G.S. 1:24,000 topographic map, Kermit Quadrangle; for details see Krumbein and Shreve 1970).

Natural channel and ridge network data were sampled to test two hypotheses:

- (1) that the observed frequency distributions of channel and ridge link numbers in interlocking network paths are compatible with Shreve's random topology model (1966); and
- (2) that the link numbers of the two boundary paths of a drainage complex depend only on the complex magnitude and are otherwise independent of each other.

With regard to channels, hypothesis (1) constitutes a special case of the more comprehen-

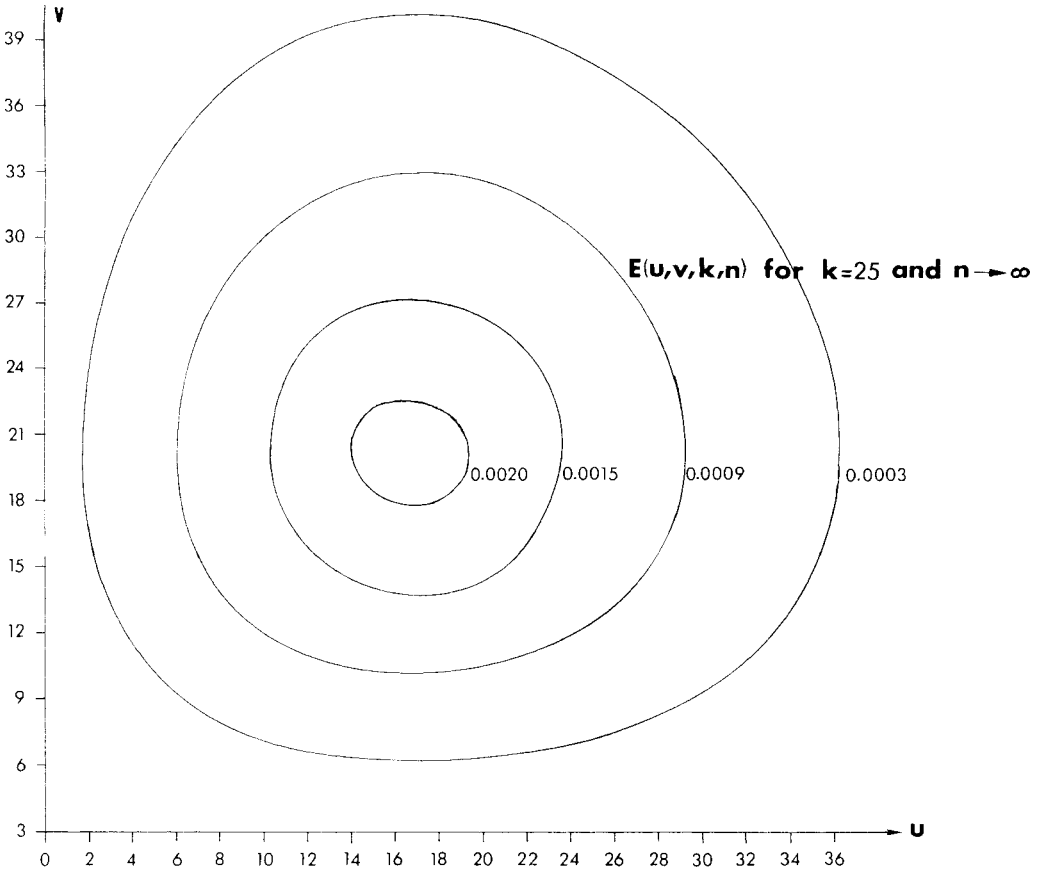


Figure 7. Probability field for pairs of interlocking channel and ridge paths with u and v links respectively; the paths delineate drainage complexes of magnitude $k = 25$ embedded in interlocking networks of random topology and infinite magnitude.

sive hypothesis that natural channel networks possess a random topology. While there is a large body of empirical tests that tends to support this hypothesis at least at the aggregate level, systematic deviations were discovered early on (e.g., James and Krumbein 1969; Howard 1971; for a review and critique see Abrahams 1984 and Abrahams and Mark 1986). For our purpose the network parameters derived from the concept of randomness will serve as a benchmark against which sample data can be numerically compared.

The frequency distribution $F(u)$ of fifty observed inner channel paths, by number of links u , that were sampled at random, are shown in Figure 8a. Each path forms the channel boundary of a drainage complex of magnitude $k = 15$.

Superimposed is the distribution as expected on the basis of random network topology (equation 10). With chi square = 9.15 and 10 degrees of freedom, the data are quite compatible with the hypothesis.

There are only a few bodies of data on the topology of interlocking ridge networks and, in particular, the link number distributions of their paths (Werner 1972a, 1982). They all indicate that, in the absence of environmental controls, the topological behavior of ridge networks is principally the same as that of channel networks, viz., that the random model constitutes a good approximation, at least for the parameters sampled. The ridge data sampled for this paper consist of the link numbers v of the outer ridge paths that interlock with the chan-

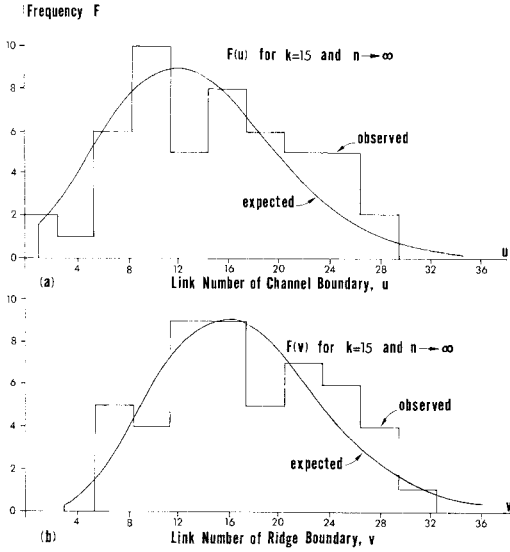


Figure 8. Expected and observed link number distributions of channel and ridge paths delineating drainage complexes of magnitude $k = 15$ in interlocking networks of infinite magnitude.

nel paths used for testing the first part of hypothesis (1). Their frequency distribution $F(v)$ is shown in Figure 8b, together with the corresponding theoretical distribution of expected values (equation 12). Once again, with chi square = 8.02 and 10 degrees of freedom, the data are clearly compatible with the second part of the hypothesis. Caution must be exercised in accepting this statement because the sampling procedure itself might contain a random bias (see above, p. 256).

Hypothesis (2) which implies the topologic independence of interlocking networks is firmly rejected by the data. Scatters of the topological lengths of channel and ridge boundaries of 50 randomly selected drainage complexes of magnitude 5 and 15 respectively are shown in Figures 9 and 10.

“Regular” drainage basins ($u = 0$) are shown as circles; their positions in the scatter provide empirical support for the theoretical argument presented earlier that drainage basins constitute the limiting case for the “continuum” of drainage complexes.

Superimposed are the corresponding probability distributions based on the assumptions of topological randomness and independence

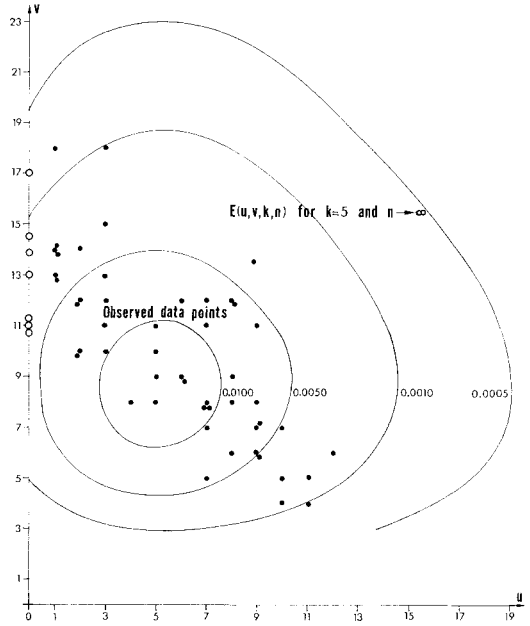


Figure 9. Probability field for pairs of interlocking channel and ridge paths with u and v links respectively; the paths delineate drainage complexes of magnitude $k = 5$ that are embedded in interlocking networks of random topology and infinite magnitude. Superimposed are the link numbers (u, v) of the boundary paths of 50 observed drainage complexes of magnitude 5.

of the networks in which the complexes are embedded (equation 13, which is the product of equations 10 and 12). Unlike these nearly bell-shaped fields of theoretical probabilities, the observed data display a pronounced linear alignment along the line $v = b - u$, b being some constant. Apparently, for any given magnitude k the sum of the boundary ridge and channel links scatter fairly tightly around a mean value b . This point is clearly reiterated by Figure 11 which compares the observed total link numbers ($u + v$) of drainage complex boundaries with their theoretically expected distribution; all data refer to complexes of magnitude 15.

To examine the dependency of the approximate relationship $u + v = b$ on the complex magnitude k , (u, v, k), triples of 40 drainage basin complexes were sampled at random, except for k which was chosen so as to provide representative coverage of the values between 1 and 100. The plot of the number of boundary

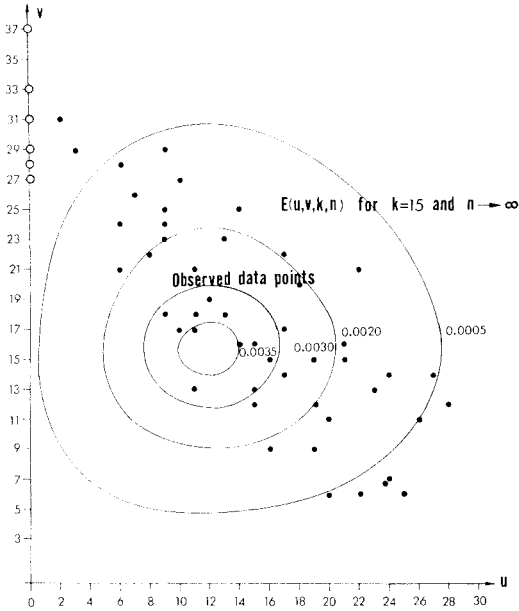


Figure 10. Probability field for pairs of interlocking channel and ridge paths with u and v links respectively; the paths delineate drainage complexes of magnitude $k = 15$ that are embedded in interlocking networks of random topology and infinite magnitude. Superimposed are the link numbers (u, v) of the boundary paths of 50 observed drainage complexes of magnitude 15.

links, $u + v$, against the magnitude k is portrayed in Figure 12. Superimposed are selected percentile zones of the expected distribution based on the assumption of topological independence between interlocking networks. Clearly, neither the median nor the percentile distribution of the observed values are compatible with this assumption. At least for the sampling domain the data demonstrate that, for any given complex magnitude k , the number of boundary links tend to approximate a constant value b ; that b increases monotonically with k , and that, except for small values of k , this increase is distinctly steeper than the median of the expected distribution.

The following are possible causes for the discrepancies between the expected and observed distributions:

(1) While $u + v$ represents the link number of the boundary of a drainage complex, the magnitude k is positively correlated with the number of channel links enclosed and there-

fore with the area of the complex. Even under very weak assumptions about the geometric length distributions of the ridge and channel links in the study area, the minimum number of boundary links required to encircle a given area must increase with area size. Thus, as the magnitude k increases, an increasing number of theoretically possible $u + v$ values are impossible in natural terrains because the minimum possible length of a boundary is dependent on the size of the area enclosed.

(2) For equivalent reasons, large values of ($u + v$) for given k imply drainage complex areas with wildly meandering boundaries. Assuming again environmentally uniform conditions, the competition between adjacent drainage networks for a fixed amount of space available and the basin area adjustments brought about by stream capture preclude such extreme basin shapes.

(3) The cause for the discrepancy between observed values of ($u + v$) and the expected median for other than small k values might lie in a combination of (1) and (2). The first may have a larger impact than the second, thus pushing the observed median above the expected value.

Summary, Concluding Remarks

What can geographic research tell us about ridge patterns when the corresponding patterns of channels are known? Or, to phrase it differently:

How much of the geometric and topologic properties of ridge networks can be accounted for by the properties of the channels separating and surrounding them?

Even though the formal and functional ridge/channel dependency is obvious, there is presently no methodological framework capable of expressing one in terms of the other (one exception being Abrahams's [1980] work referred to earlier). This paper constitutes an effort to provide such a methodology and to demonstrate its capability by deriving several relations and functions that tie parameters of ridge networks to those of channels. A number of graph-theoretic results are simply mathematical consequences of the way ridges and channels are defined in this paper. Substantive test results include the rejection of the hypothesis that interlocking ridge and channel networks are topologically independent and the demonstra-

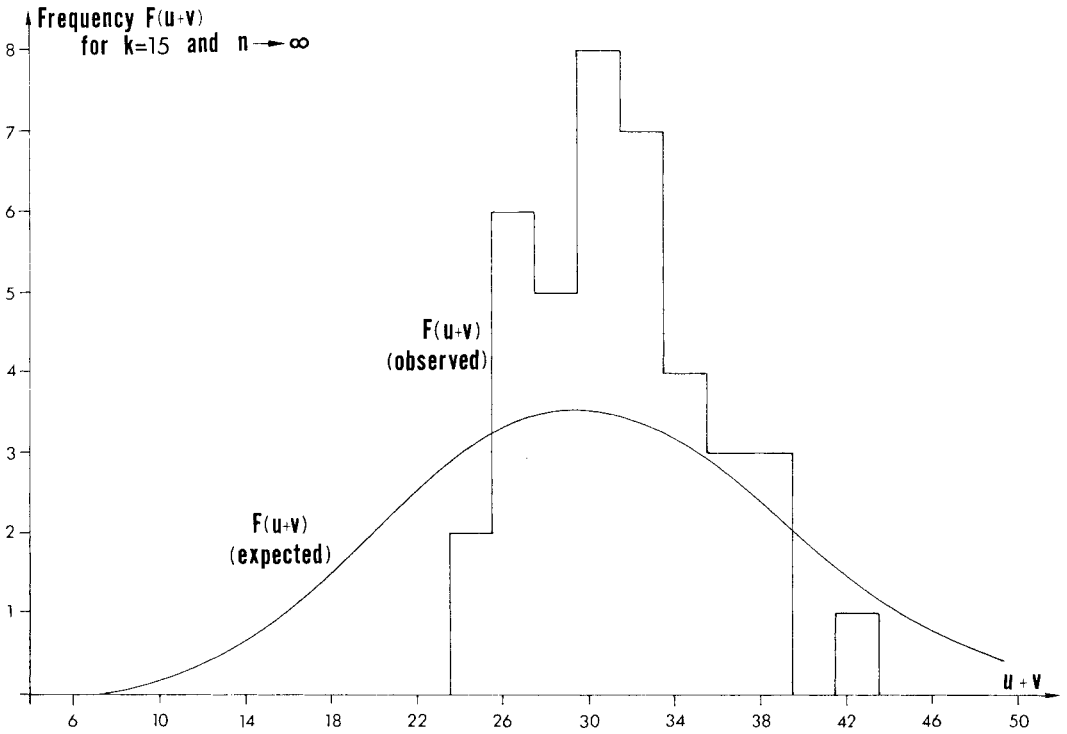


Figure 11. Observed and expected distributions of the combined link numbers of the channel and ridge paths delineating drainage complexes of magnitude $k = 15$.

tion of a close functional dependence between the link numbers of ridge and channel paths. The conceptual approach and the two types of results are summarized below.

A. The formal definition of interlocking ridges identifies them as a function of channels, thus providing an axiomatic base for interlocking ridge and channel networks. As defined here the network of interlocking ridges is embedded in the network of all ridges. While the latter can be established without reference to channels, the former associates a single outer ridge link with each pair of neighboring outer channel links, that is, with each undissected hilltop area. The particular definition of interlocking ridge and channel networks makes them trivalent planar trees of equal magnitude and linked by a symmetrical relation (and, very importantly, not in approximation but exactly). Hence, all theorems derived for interlocking networks apply to those delineated on maps completely and unequivocally without the need of empirical tests (just as one would not test

the statement that the magnitude of a channel network is the sum of the magnitudes of its subnetworks). Note that the deduced interrelationships between channels and ridges hold only in approximation if we study corresponding ridge and channel networks in general (Werner 1972c), that is, without the assumption of equal magnitude and alternating outer links.

The close interdependency between interlocking networks becomes apparent from the following relations:

- with each channel link there is associated exactly one outer ridge path; this path corresponds to the boundary of the drainage area of the channel subnetwork defined by the channel link;
- to each inner (outer) channel path there corresponds exactly one outer (inner) ridge path; together the two paths define an area called a drainage complex. This area is hydrologically closed in the specific sense that the only channel subnetworks contained in it are those

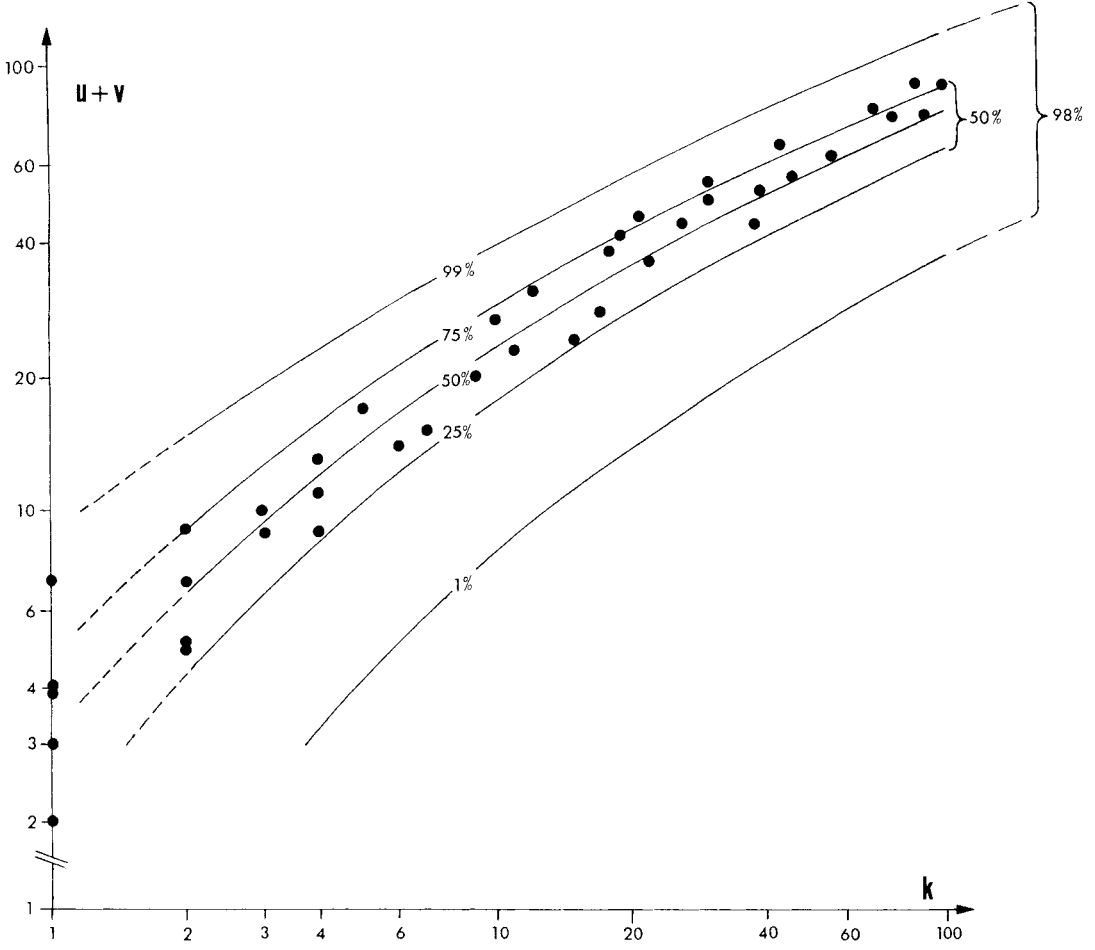


Figure 12. Observed link number ($u + v$) of ridge and channel paths delineating a drainage complex, as a function of complex magnitude k . Superimposed are median and percentile zones of expected distribution assuming topologic independence of interlocking networks.

entering the channel path from its side facing the ridge path;
 —interlocking networks are dual structures. Thus, if the dual concepts of channel and ridge are exchanged in the preceding statements, the result will be new (dual) statements of equal validity.

B. Analyzed separately, the observed link numbers of interlocking ridge and channel paths exhibit distributions for which the assumption of randomness provides a fairly good description. But analyzed together, they do not follow a chance distribution; rather, their sum approximates a constant value if the magnitude

of the complexes they define is kept constant. If complex magnitude is treated as an independent variable, then the sum of the link numbers of interlocking paths approximate a monotonic function of the associated complex magnitudes. At least within the domain examined here, the function matches the median of the expected distribution only for small values of complex magnitude, displaying a growing positive deviation as the magnitude increases.

If channel links are used as a surrogate for channel link areas and path link numbers for geometric path lengths, then the observed relations can be tentatively interpreted as the re-

sult of geometric and geomorphic constraints, the first imposing minimum requirements on the perimeter of any given area, and the second referring to the competition for space between channel networks under environmentally uniform conditions.

Further comments:

(1) Limitations of the present approach: Keeping in mind that the definition of interlocking ridges corresponds only to a particular subset of all ridges that produce contour crenulations on topographic maps, it permits the precise topologic description and analysis of interlocking networks. Also, for geometric description and analysis the approach is suitable only at the macro-level, that is, above the level of undissected terrain between neighboring outer channel links. The definition of interlocking ridge networks correctly identifies the ridge paths forming the drainage basin boundaries of a given channel network and its sub-networks, except for the end links of these paths. At the level of interbasin areas and their delineation, the definition either fails to recognize ridges or misidentifies them.

(2) The duality between ridge and channel patterns puts the particular physical phenomenon of a ridge line at least formally on an equal footing with the drainage channel. From an applied standpoint, it is reasonable that channels have been the subject of many more investigations than ridges; as a consequence, their form and their change over time is much better understood. From a purely academic viewpoint, ridges should command a similar level of attention. Frequently, they are equally prominent landscape features; they have their own longitudinal and cross-sectional profiles, and they change over time as a consequence of specific geomorphic processes.

(3) The development of ridge lines is intimately linked to that of the adjacent drainage channels, but in ways that are not yet understood in any detail. This paper has been an effort to identify some of the formal linkages that exist between the connectivity patterns of channels and ridges. It will be the subject of a future study to establish, in a parallel fashion, spatial linkages that pertain between the two patterns.

The accurate description of ridges in three dimensions as a function of the neighboring channels will no doubt be a matter of approximation at best, and then only under rather

restrictive conditions. There is every reason to believe that the residuals of such modeling efforts in particular localities will help in the identification of local geomorphic factors that control slope and ridge development beyond that exercised by channel erosion.

Acknowledgments

The various corrections and improvements of the reviewers, both in form and substance, are herewith gratefully acknowledged.

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