UNIVERSITY OF CALIFORNIA
SANTA CRUZ

A p-ADIC ANALYTIC BRAUER GROUP

A dissertation submitted in partial satisfaction
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

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September 2025

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Abstract

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This dissertation introduces and studies the *p-adic analytic Brauer group*, a cohomological invariant that provides an analytic perspective on the classical theory of central simple algebras over p-adic fields. For a p-adic field F with absolute Galois group G_F, we define this invariant as the continuous Galois cohomology group $\mathrm{H}^2(G_F, \mathbb{C}_F^\times)$, where $\mathbb{C}_F$ is the completion of an algebraic closure of F. Our central result establishes that this analytic group is canonically isomorphic to the prime-to-p part of the classical Brauer group of F. Consequently, while division algebras of p-power index over F are trivial in $\mathrm{H}^2(G_F, \mathbb{C}_F^\times)$, we provide an analytic construction for every such division algebra by demonstrating that they arise from $\mathbb{C}_F$-semilinear representations.

To Thātha.

Thank you for helping me with my multiplication tables.

Acknowledgements

First and foremost, I wish to thank my advisor, Marty Weissman. I am deeply grateful for his generous investment of time, his guidance, and our numerous conversations that extended far beyond research into the realms of teaching and life. I especially appreciate his willingness to accept me as a student later in my graduate track.

I am also indebted to my committee members, Beren Sanders and Robert Boltje. It is fitting that they are part of my committee, as their courses profoundly shaped my mathematical development. From them, I learnt not just mathematics, but how to approach and think about the subject. I am also thankful for the many informal conversations we had, and for their willingness to entertain my thoughts and questions.

My journey as a teacher was greatly enriched by Pedro Morales-Almazan. I thank him for his guidance, support, and friendship. His availability to talk and the unwavering sense that he was always on my side provided immense comfort and confidence.

This dissertation would not have been possible, nor nearly as enjoyable, without the camaraderie and support of my friends. The bonds forged amongst graduate students, through shared struggles and triumphs, are an indelible part of my time in graduate school.

I owe a debt of gratitude to Jennifer Guerrero, my greatest cheerleader in the department. I thank her for always being there to depend on, for her unyielding positivity, and for being a constant model of perseverance and grit. I will cherish our long conversations about everything: the ones that ran too long, the passionate arguments over things which, at their core, we already agreed on, and the frequent over-communication. I am equally grateful to David Rubinstein for his unwavering support, his patient ear, and his calming presence. I thank him for all the work hangouts at breweries that invariably ended with me ranting while he listened. I also thank both David and Jen for being my partners-in-crime in all our departmental activities.

I am grateful to Brian Ma for being one of the most reliable people I know, always easy to talk to, and ever ready for our many food adventures. I must also thank the other rockstars and amazing people who have been such a vital part of my graduate school life: Ryan Pugh, Sophie Aiken, Malachi Alexander, Fulya Tastan, and Vaibhav Sutrave. I thank them for the mathematics, the friendship, and the support. I am grateful to all my other friends and peers in the graduate department, across all cohorts, whose friendship enriched my graduate experience immeasurably.

My time in graduate school, particularly my teaching experience, was made richer and more meaningful by my students and mentees over the years. Morea Lee, Miles Mitchell, Ethan Jones and Yahya Bashandy, to name a few. It warms my heart to think of the different paths they have all taken, the successes coming their way, and the resilience with which they have faced challenges.

My research life has been deeply influenced by collaborations and friendships beyond UC Santa Cruz. The Rethinking Number Theory (RNT) community has had an impact on me that cannot be overstated. I want to thank the project leaders, whose guidance was a great source of inspiration. I especially want to thank Maria Fox for being a singular source of mentorship, support, and inspiration. I thank Soumya Sankar for heavily influencing how I approach research.

A shout-out to Heidi Goodson, my collaborator and RNT co-organiser, for her support and friendship. I also extend my sincere thanks to my research collaborators, whom I am lucky to also call friends:: Santiago Arango-Piñeros, Steven Groen, Sandra Nair, Emerald Andrews, Niven Achenjang, Aashraya Jha, Caleb Ji, and Rose Lopez.

I am grateful to two of my closest friends – Pranav Sridhar and Sambhavna Biswas. They believed in my success with a certainty that I often felt I had not yet earned, and for that unflinching faith, I am eternally grateful.

I have deep love for Uma Pedda, Sharda Pinni, and Pappu Chacha, for their warmth and for having their homes be a true home away from home—a place I could always go to without a second thought. And only profound love for my Mamma and Amumma.

My baby sister's hard work and perseverance continue to be a gold standard for me. I thank her for setting the bar so high.

And to my parents, my Amma and Papa. Their unwavering support, their absolute faith in my choices, and their complete lack of doubt in my abilities are the bedrock of any confidence that I possess. They are my rocks and my constants, and none of this would have been possible without them.

Finally, in the words of Snoop Dogg, *"I want to thank me for believing in me, I want to thank me for doing all this hard work. I wanna thank me for having no days off. I wanna thank me for never quitting. I wanna thank me for always being a giver and trying to give more than I receive. I wanna thank me for trying to do more right than wrong. I wanna thank me for being me at all times..."*

Chapter 1

Introduction

The Brauer group, first defined in the 1930s, has been a cornerstone in the study of division algebras and has deep connections to class field theory. The Brauer group of a field K, denoted $\mathrm{Br}(K)$, classifies the central simple algebras over K up to an equivalence relation. This provides a meaningful invariant for fields. For a non-archimedean local field F, this theory is particularly elegant, culminating in the isomorphism $\mathrm{Br}(F) \cong \mathbb{Q}/\mathbb{Z}$.

In this dissertation, we explore an analytic counterpart to this classical algebraic structure. We introduce and study what we term the *p-adic analytic Brauer group*, defined from a cohomological perspective as the continuous Galois cohomology group $\mathrm{H}^2(G_F, \mathbb{C}_F^\times)$. Here, G_F is the absolute Galois group of a p-adic field F, and $\mathbb{C}_F$ is the field of p-adic complex numbers—the completion of an algebraic closure of F. This group arises naturally when one considers twisted forms of matrix algebras not over algebraic extensions, but over the transcendental extension $\mathbb{C}_F$.

The central investigation of this work revolves around understanding the structure of this analytic Brauer group and its relationship to the classical Brauer group. By leveraging the tools of p-adic Hodge theory, particularly Sen's theory of $\mathbb{C}_F$-semilinear representations of G_F, we establish a fundamental structural result. We show that the cohomological p-adic analytic Brauer group $H^2(G_F, \mathbb{C}_F^\times)$ is isomorphic to the prime-to-p part of the classical Brauer group.

The p-primary information is not lost in the analytic setting but rather relocated. It manifests in the Azumaya sense through $\mathbb{C}_F$-semilinear representations, where it naturally embeds within a larger p-adic family inherent to the p-adic analytic setting.

This work is structured as follows:

- **Chapter 2** provides a comprehensive review of the necessary preliminaries. This includes the theory of local fields, ramification, local class field theory, and the classical theory of Brauer groups from a cohomological perspective.

- **Chapter 3** delves into the foundational results of Sen's theory on the decompletion and descent of $\mathbb{C}_F$-semilinear representations. We extend this framework to central simple algebras.

- **Chapter 4** introduces the main object of study: the (cohomological) p-adic analytic Brauer group $\mathrm{H}^2(G_F, \mathbb{C}_F^\times)$. We show that it is isomorphic to the prime-to-p part of the classical Brauer group of F.

- **Chapter 5** applies these results to study division algebras of p-power index. We demonstrate that every such class arises from an analytic object—a $\mathbb{C}_F$-semilinear representation—despite being trivial in the (cohomological) p-adic analytic Brauer group.

Chapter 2

Preliminaries

2.1 Local Fields

Our work is in the context of local fields, specifically a p-adic field. In this section, we review some facts about p-adic fields that will be used in the sequel. Hereafter, we fix a prime p. Denote by $\mathbb{Q}_p$ the field of p-adic numbers, by $\mathbb{Z}_p$ the ring of p-adic integers, and by $\mathbb{F}_p$ the prime field of characteristic p.

Definition 2.1.1 (p-adic fields)**.** A *p-adic field* F is a finite extension of $\mathbb{Q}_p$.

The p-adic field F comes equipped with a *discrete valuation* $\nu\colon F^\times \to \mathbb{R}$, i.e., its *value group* $\nu(F^\times)$ is a discrete subgroup of $\mathbb{R}$. We extend ν to F by declaring $\nu(0) = \infty$. Moreover, F is complete with respect to ν.

Its *valuation ring* $\mathcal{O}_F = \{x \in F : \nu(x) \geqslant 0\}$, also referred to as the ring of integers of F, is a local ring. The *valuation ideal* $\mathfrak{m}_F = \{x \in F : \nu(x) > 0\}$ is its unique maximal ideal, and the group of units $\mathcal{O}_F^\times = \mathcal{O}_F \setminus \mathfrak{m}_F = \{x \in F : \nu(x) = 0\}$.

The *residue field* of $\mathcal{O}_F$ (or F) is $k_F := \mathcal{O}_F/\mathfrak{m}_F \cong \mathbb{F}_q$, where q is some power of p.

The ideal $\mathfrak{m}_F$ is principal, and any generator of $\mathfrak{m}_F$ is called a *uniformiser* of F. In particular, if $\varpi \in F$ is a uniformiser, then we may write any $x \in F^\times$ as

$x = \varpi^{\nu(x)} u$, where $u \in \mathcal{O}_F^\times$. That is, there exists a short exact sequence

$$1 \longrightarrow \mathcal{O}_F^\times \longrightarrow F^\times \xrightarrow{\nu} \nu(F^\times) \to 0$$

which splits non-canonically, depending on a choice of uniformiser.

We fix, for the remainder of Section 2.1, a p-adic field F and assume that its valuation ν is normalized, i.e., $\nu(F^\times) = \mathbb{Z}$. Let $\mathcal{O}_F$ be its valuation ring, $\mathfrak{m}_F$ its valuation ideal and k_F its residue field.

In this section, we discuss the theory of local fields that will be the most relevant, all of which can be found in [Ser79].

2.1.1 Field Extensions

Let $E|F$ be an algebraic extension. There is a unique extension of ν to a valuation $\nu_E : E \to \mathbb{R} \cup \{\infty\}$ on E. Denote by $\mathcal{O}_E = \{x \in E : \nu_E(x) \geqslant 0\}$ its valuation ring which is local, by $\mathfrak{m}_E = \{x \in E : \nu_E(x) > 0\}$ its valuation ideal, by $k_E = \mathcal{O}_E/\mathfrak{m}_E$ its residue field.

Assume $E|F$ is Galois. Then for any $\sigma \in \mathrm{Gal}(E|F)$, the valuation $\nu_E \circ \sigma$ is another extension of ν to E. Uniqueness of ν_E forces $\nu_E = \nu_E \circ \sigma$, i.e., ν_E is Galois-equivariant, where we let the value group $\nu_E(E^\times)$ be a trivial Galois module. Moreover, this implies $\mathrm{Gal}(E|F)$ acts on E isometrically, i.e., $\nu_E(\sigma(x) - \sigma(y)) = \nu_E(x - y)$ for all possibly $x, y \in E$ and $\sigma \in \mathrm{Gal}(E|F)$. Hence, in particular, $\mathrm{Gal}(E|F)$ acts on E continuously.

Definition 2.1.2 (Residue Degree & Ramification Index)**.** Assume $E|F$ is finite, and let $n = [E : F]$ be its degree. In this instance, ν_E is a discrete valuation and E is complete with respect to ν_E. Furthermore, $\mathfrak{m}_E$ is also principal.

Necessarily, $k_E|k_F$ is a finite extension and we call $f = f(E|F) = [k_E : k_F]$ the *residue degree* of $E|F$.

Let ϖ_E and ϖ_F be uniformisers of E and F respectively; the *ramification index* of $E|F$ is the unique positive integer $e = e(E|F)$ such that $\nu_E(\varpi_F) = e \cdot \nu_E(\varpi_E)$. Equivalently, it is the unique positive integer such that $\mathfrak{m}_F\mathcal{O}_E = \mathfrak{m}_E^e$. Equivalently, the value group $\nu_E(E^\times) = (1/e)\mathbb{Z}$.

One has, in general, that $n = ef$.

Definition 2.1.3 (Unramified Extensions)**.** Assume $E|F$ is finite with residue degree f and ramification index e. We say $E|F$ is *unramified* if $e = 1$. Equivalently, if we have $[E : F] = f = [k_E : k_F]$ or $\mathfrak{m}_F\mathcal{O}_E = \mathfrak{m}_E$. In particular, any uniformiser of F continues to be a uniformiser of E.

We can completely characterise finite unramified extensions of local fields.

Proposition 2.1.4. *For any $f \geqslant 1$, there exists a unique, up to isomorphism, unramified extension $E|F$ with $[E : F] = f$. Moreover, this extension is Galois and the natural map* $\mathrm{Gal}(E|F) \to \mathrm{Gal}(k_E|k_F)$ *is an isomorphism.*

Therefore $\mathrm{Gal}(E|F)$ *is cyclic, generated by a canonical element denoted* $\mathrm{Frob}_{E|F}$ *and called the (arithmetic) Frobenius element. The latter is characterized by the congruence* $\mathrm{Frob}_{E|F}(x) \equiv x^{q^f} \bmod \mathfrak{m}_E$ *for all $x \in \mathcal{O}_E$.*

In particular, $E|F$ is an abelian extension of F.

Definition 2.1.5 (Ramified Extensions)**.** Assume $E|F$ is finite with residue degree f and ramification index e. We say $E|F$ is *ramified* if $e > 1$.

A ramified extension $E|F$ is said to be *tamely ramified* if e is coprime to p, and *wildly ramified* otherwise.

We say $E|F$ is *totally ramified* if the residue degree $f = 1$, or equivalently if we have $[E : F] = e$, the ramification index. That is, the ramification index is as large as it can be. In particular, $k_E = k_F$.

Example 2.1.6. Let ζ_{p^r} denote a primitive p^r-th root of unity in an algebraic closure of $\mathbb{Q}_p$. Then the field $\mathbb{Q}_p(\zeta_{p^r})$ is a totally ramified extension of $\mathbb{Q}_p$. In general, for any p-adic field F, the extension $F(\zeta_{p^r})$ is a totally ramified extension of F.

Remark 2.1.7. An infinite extension $E|F$ is said to be unramified (resp. tamely, wildly, totally ramified) if every intermediate finite extension over F is unramified (resp. tamely, wildly, totally ramified).

Example 2.1.8. The field $\mathbb{Q}_p(\mu_{p^\infty}) = \cup_{r\geqslant 0}\mathbb{Q}_p(\zeta_{p^r})$ is an infinite totally ramified extension of $\mathbb{Q}_p$. So is $F(\mu_{p^\infty})|F$, for any p-adic field F.

Example 2.1.9. For any local field F, we may consider its maximal unramified extension F^{ur}, which is the compositum of all finite unramified extensions of F, in a fixed algebraic closure of F. By Proposition 2.1.4, we conclude that F^{ur} is an unramified Galois extension.

2.1.2 p-adic Logarithm & Exponential

We will now introduce the p-adic logarithm and exponential functions on an algebraic extension $E|F$. In this p-adic setting, the logarithm function is better behaved than the exponential function.

Definition 2.1.10 (Principal Units)**.** The subgroup of *n^{th} principal units* of $\mathcal{O}_E^\times$ is defined to be $\mathsf{U}_E^{(n)} := 1 + \mathfrak{m}_E^n$, for $n \geqslant 1$. We often refer to $\mathsf{U}_E^{(1)}$ simply as the subgroup of *principal units*. These groups provide a filtration of $\mathsf{U}_E^{(0)} := \mathcal{O}_E^\times$

$$\mathsf{U}_E^{(0)} \supseteq \mathsf{U}_E^{(1)} \supseteq \cdots$$

The canonical surjection $\mathcal{O}_E \twoheadrightarrow k_E$, gives us $\mathsf{U}_E^{(0)}/\mathsf{U}_E^{(1)} \cong k_E^\times$.

In the case when $E|F$ is finite, we also get a good description of successive quotients in the filtration above. Let ϖ_E be a uniformiser of E, then the following

$$\mathsf{U}_E^{(i)}/\mathsf{U}_E^{(i+1)} \to (k_E, +) \colon x \mapsto \frac{x-1}{\varpi_E^i} \bmod \mathfrak{m}_E$$

is an isomorphism of groups; this isomorphism is independent of ϖ_E. In particular, when $E|F$ is finite, the above filtration of $\mathsf{U}_E^{(0)} = \mathcal{O}_E^\times$ exhibits it as a profinite group.

The following can be done for any extension $E|F$, but we specialise to the case $E = \overline{F}$, an algebraic closure of F. We let $\overline{v} = v_{\overline{F}}$, and we observe that $k_{\overline{F}}$ is an algebraic closure of k_F, and therefore $k_{\overline{F}} \cong \overline{\mathbb{F}}_q$.

Remark 2.1.11. As we mentioned previously, the reduction map $\mathcal{O}_{\overline{F}} \twoheadrightarrow k_{\overline{F}}$ gives us the following short exact sequence

$$1 \longrightarrow \mathsf{U}_{\overline{F}}^{(1)} \longrightarrow \mathcal{O}_{\overline{F}}^\times \longrightarrow k_{\overline{F}}^\times \longrightarrow 1$$

This short exact sequence canonically splits by identifying $k_{\overline{F}}^\times = \overline{\mathbb{F}}_q^\times$ with $\mu_{p'}$, the subgroup of prime-to-p roots of unity in $\mathcal{O}_{\overline{F}}^\times$, using Hensel's lemma. Hence, $\mathcal{O}_{\overline{F}}^\times = \mu_{p'} \cdot \mathsf{U}_{\overline{F}}^{(1)}$. That is, we may write any $z \in \mathcal{O}_{\overline{F}}^\times$ uniquely as $z = \zeta_z(1 + m_z)$, where $\zeta_z \in \mu_{p'}$ and $m_z \in \mathfrak{m}_{\overline{F}}$.

Definition 2.1.12 (p-adic Logarithm)**.** We define the *p-adic logarithm* series formally as as follows

$$\log(1+X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{X^n}{n}$$

$\log(1+x)$ converges whenever $x \in \mathfrak{m}_{\overline{F}}$ and hence we obtain the p-adic logarithm function $\log\colon \mathrm{U}^{(1)}_{\overline{F}} \to \overline{F}$, which is continuous.
We extend the function to $\mathcal{O}^{\times}_{\overline{F}}$ by setting $\log(\zeta) = 0$ for every $\zeta \in \mu_{p'}$, and we will continue to refer to this extension also as log.

Definition 2.1.13 (p-adic Exponential)**.** We define the *p-adic exponential* series formally as

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

Proposition 2.1.14. $\exp(x)$ *converges in* $\overline{F}$ *if and only if* $\overline{v}(x) > e/(p-1)$*, where* $e = e(F|\mathbb{Q}_p)$ *is the absolute ramification index of* F*.*

Proof. Since we have assumed the valuation v on F to normalised, we have $v(p) = e = e(F|\mathbb{Q}_p)$. Let v_p be the valuation on $\mathbb{Q}_p$ such that v is its unique extension to F, in particular, one has $v(n!) = e \cdot v_p(n!)$. Legendre's formula tells us that

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor \leqslant \frac{n}{p-1}, \quad \text{therefore } v(n!) \leqslant \frac{en}{p-1}$$

A similar calculus gives you

$$v_p(n!) \geqslant \frac{n}{p-1} - \lfloor \log_p n \rfloor, \quad \text{therefore } v(n!) \geqslant \frac{en}{p-1} - e\lfloor \log_p n \rfloor$$

where $\log_p$ is the usual real logarithm in base p. Taking these two inequalities we compute the radius of convergence of $\exp(x)$, concluding that it converges exactly for those x such that $\overline{v}(x) > e/(p-1)$. $\square$

The p-adic logarithm and the exponential satisfy expected properties.

Proposition 2.1.15. *Let x, y be elements in the domain of definition of* $\log$*, and let s and t be in the domain of definition of* $\exp$*. Then* $\exp$ *and* $\log$ *are continuous and*

(a) $\exp(s+t) = \exp(s) \cdot \exp(t)$ *(b)* $\log(xy) = \log(x) + \log(y)$

(c) $\exp(\log(1+s)) = 1+s$ *and* $\log(\exp(s)) = s$.

Proof. These can all be verified formally. $\square$

Corollary 2.1.16. *For any $n > e/(p-1)$, the maps* $\log$ *and* $\exp$ *give an isomorphism of topological groups between $\mathsf{U}_{\overline{F}}^{(n)}$ and $\mathfrak{m}_{\overline{F}}^{n}$.*

Proposition 2.1.17. *There is a short exact sequence*

$$1 \longrightarrow \mu_\infty \longrightarrow \mathcal{O}_{\overline{F}}^\times \xrightarrow{\log} \overline{F} \longrightarrow 1$$

where μ_∞ denotes the subgroup of all roots of unity.

Proof. We first prove the surjectivity of log. Consider any $x \in \overline{F}$ and choose $n \gg 0$ such that $p^n x$ is in the radius of convergence of exp. Then, $z = \exp(p^n x)$ is such that $\log z = p^n x$. Since $\overline{F}$ is algebraically closed, there exists a p^n-th root $z^{1/p^n} \in \overline{F}$ of z, and necessarily $\log z^{1/p^n} = x$; showing the surjectivity of log. Let $z \in \mathsf{U}_{\overline{F}}^{(1)}$ be such that $\log z = 0$, then choose $n \gg 0$ such that $z^{p^n} \in \mathsf{U}_{\overline{F}}^{(i)}$ for some i where log is an isomorphism on $\mathsf{U}_{\overline{F}}^{(i)}$. Then, $\log z^{p^n} = 0$ gives us $z^{p^n} = 1$ using the exponential. Hence, $z \in \mu_{p^\infty}$, finishing the proof. $\square$

Remark 2.1.18. If we restrict log and exp to F itself, we obtain an isomorphism analogous to the one in Corollary 2.1.16,

$$\mathsf{U}_F^{(n)} \underset{\text{exp}}{\overset{\text{log}}{\rightleftarrows}} \mathfrak{m}_F^n$$

for $n > e/(p-1)$.

2.1.3 Ramification Theory

Let $E|F$ be a Galois extension. The considerations below hold more generally: they do not depend on the specific choice of F as the base field, and remain true if we replace F with any finite extension.

Definition 2.1.19 (Higher Ramification Groups)**.** Assume $E|F$ is finite and let $G = \mathrm{Gal}(E|F)$ be its Galois group. For $i \geqslant -1$, define the *i^{th} ramification group* as

$$G_i = \{\sigma \in G : \sigma(x) \equiv x \bmod \mathfrak{m}_E^{i+1}, \text{ for all } x \in \mathcal{O}_E\}$$

In particular, $G_{-1} = G$ and G_0 is the *inertia subgroup* of G. We obtain the following filtration of G, called its *ramification filtration*

$$G = G_{-1} \supseteq G_0 \supseteq G_1 \supseteq \cdots \supseteq \{*\}$$

We have that $G_i = \{*\}$ is trivial for $i \gg 0$.

Remark 2.1.20. Each G_i is normal in G; in particular, we have $G/G_0 \cong \mathrm{Gal}(k_E|k_F)$. Moreover, G_0/G_1 naturally embeds into $k_E^\times$ and thus is cyclic of order prime to p, and for $i > 0$, the quotient G_i/G_{i+1} embeds into $\mathfrak{m}_E^i/\mathfrak{m}_E^{i+1}$, a k_E-vector space, thus is an abelian p-group killed by p. This exhibits G as a solvable group.

The ramification filtration detects the ramification of the extension $E|F$. In particular, one has that $G_0 = \{*\}$ if and only if $E|F$ is unramified, and $G_1 = \{*\}$ if and only if $E|F$ is tamely ramified.

The ramification filtration we have defined behaves well when restricted to subgroups of G, but does not behave well when we pass to its quotients. However, we can recover compatibility with quotients by appropriately reindexing the filtration.

Definition 2.1.21 (Upper Numbering)**.** For any $t \in \mathbb{R}_{\geqslant 0}$, let $G_t := G_{\lceil t \rceil}$. We define the *Herbrand function* $\varphi_{E|F}\colon [-1, \infty) \to [-1, \infty)$ as

$$\varphi_{E|F}(u) = \int_0^u \frac{dt}{[G_0 : G_t]}$$

This function is continuous, piecewise linear, increasing, and concave, and satisfies for $\varphi_{E|F}(u) = u$ for $u \in [-1, 0]$.

Consequently, it admits an inverse $\psi_{E|F} : [-1, \infty) \to [-1, \infty)$ which is continuous, piecewise linear, increasing, and convex.

For $u \in [-1, \infty)$, we define the ramification groups in *upper numbering* as

$$G^u := G_{\psi_{E|F}(u)}$$

Remark 2.1.22. Ramification groups in upper numbering G^u and *lower numbering* G_t dictate each other. Observe that the knowledge of the upper numbering allows us to define $\psi_{E|F}$:

$$\psi_{E|F}(t) = \int_0^t [G^0 : G^u]\, du$$

which is, in particular, continuous and increasing. This allows us to define its inverse $\varphi_{E|F}$, and obtain $G_t = G^{\varphi_{E|F}(t)}$.

Both ψ and φ enjoy pleasant composability conditions: let E_1, E_2 be Galois extensions of F with $E_1 \subseteq E_2$, then

$$\varphi_{E_2|F} = \varphi_{E_1|F} \circ \varphi_{E_2|E_1}$$

and therefore, taking inverses, we obtain

$$\psi_{E_2|F} = \psi_{E_2|E_1} \circ \psi_{E_1|F}$$

We obtain the following compatibility with quotients with upper numbering.

Proposition 2.1.23 (Herbrand)**.** *Let E_1, E_2 be Galois extensions of F with $E_1 \subseteq E_2$, then the canonical surjection $\mathrm{Gal}(E_2|F) \twoheadrightarrow \mathrm{Gal}(E_1|F)$ restricts to a surjection*

$$\mathrm{Gal}(E_2|F)^u \twoheadrightarrow \mathrm{Gal}(E_1|F)^u$$

for any $u \geqslant -1$.

Definition 2.1.24. We define the *lower* (resp. *upper*) *ramification breaks* of G as the values of u for which $G_u \neq G_{u+\varepsilon}$ (resp. $G^u \neq G^{u+\varepsilon}$) for all $\varepsilon > 0$. By definition, the lower ramification breaks are integers, while the upper ramification breaks are only guaranteed to be rational numbers.

Theorem 2.1.25 (Hasse-Arf)**.** *Let $E|F$ be an abelian extension, i.e., $G = \mathrm{Gal}(E|F)$ is an abelian group. Then the upper ramification breaks of G are integers.*

Assume now that $E|F$ is an infinite extension, then recall that

$$\mathrm{Gal}(E|F) = \varprojlim \mathrm{Gal}(E'|F)$$

where E' ranges over all finite Galois extensions of F contained in E. As a profinite group, $\mathrm{Gal}(E|F)$ is equipped with the profinite topology, also referred to as the Krull topology. In this way, we always view $\mathrm{Gal}(E|F)$ as a topological group.

Theorem 2.1.26 (Infinite Galois Correspondance)**.** *We have the following inclusion-reversing correspondence for an infinite Galois extension $E|F$. Let $G = \mathrm{Gal}(E|F)$,*

$$\{\textit{closed subgroups of } G\} \longleftrightarrow \{\textit{extensions of } F \textit{ contained in } E\}$$
$$H \longmapsto E^H$$
$$\mathrm{Gal}(E|F') \longleftarrow F'$$

In particular, open subgroups in G are precisely those closed subgroups that have finite index in G and thus correspond to finite extensions of F contained in E.

Proposition 2.1.23 allows us to define the ramification filtration in upper numbering for infinite extensions.

Definition 2.1.27 (Upper Numbering for Infinite Extensions)**.** Let $E|F$ be an infinite Galois extension, and $u \in [-1, \infty)$. Define

$$\mathrm{Gal}(E|F)^u := \varprojlim \mathrm{Gal}(E'|F)^u$$

where E' ranges over all finite Galois extensions of F contained in E.

2.1.4 Local Class Field Theory

In this section we recall statements related to Local Class Field Theory, which gives a satisfactory description of abelian extensions of F. References for this theory are [Ser79, Gui18, Mil20].

Let $G_F = \mathrm{Gal}(\overline{F}|F)$ be the absolute Galois group of G_F, and let G_F^{ab} be its maximal abelian quotient. Equivalently, $G_F^{\mathrm{ab}} = \mathrm{Gal}(F^{\mathrm{ab}}|F)$, where F^{ab} denotes the maximal abelian extension of F contained in $\overline{F}$.

Theorem 2.1.28 (Local Reciprocity Law)**.** *There exists an injective homomorphism*

$$\theta_F : F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}|F),$$

called the local reciprocity map*, such that for every finite abelian extension $E|F$ it factors as*

$$\begin{array}{ccc} F^\times & \xrightarrow{\quad\theta\quad} & \mathrm{Gal}(F^{\mathrm{ab}}|F) \\ \downarrow & & \downarrow \\ F^\times / \mathrm{N}_{E|F}(E^\times) & \xrightarrow[\theta_{E|F}]{} & \mathrm{Gal}(E|F) \end{array}$$

where $\theta_{E|F}$ is an isomorphism called the local reciprocity isomorphism *for $E|F$, and $\mathrm{N}_{E|F}$ is the field norm from E to F. In particular, $[F^\times : \mathrm{N}_{E|F}(E^\times)] = [E : F]$.*

From Theorem 2.1.28 we obtain the following fundamental correspondence. Denote $\mathrm{N}_{E|F}(E^\times)$ by $\mathrm{N}(E^\times)$.

Theorem 2.1.29 (Fundamental Correspondence & Local Existence Theorem)**.** *The map*

$$\{\textit{finite abelian extensions of } F\} \longleftrightarrow \{\textit{finite index open subgroups of } F^\times\}$$
$$E \longmapsto \mathrm{N}(E^\times)$$

is an inclusion-reversing bijection, and

$$\mathrm{N}((E \cdot E')^\times) = \mathrm{N}(E) \cap \mathrm{N}(E') \qquad \mathrm{N}(E^\times \cap E'^\times) = \mathrm{N}(E^\times) \cdot \mathrm{N}(E'^\times)$$

The local reciprocity map enjoys the following functoriality property

Theorem 2.1.30 (Functoriality of Local Reciprocity Law)**.** *Assume $E|F$ is any finite extension, then the following diagram commutes*

$$\begin{tikzcd} E^\times \arrow[r, "\theta_E"] \arrow[d, "\mathrm{N}_{E|F}"'] & \mathrm{Gal}(E^{\mathrm{ab}}|E) \arrow[d, "\mathrm{Res}"] \\ F^\times \arrow[r, "\theta_F"'] & \mathrm{Gal}(F^{\mathrm{ab}}|F) \end{tikzcd}$$

where $\mathrm{Res}\colon \mathrm{Gal}(E^{\mathrm{ab}}|E) \to \mathrm{Gal}(F^{\mathrm{ab}}|F)$ *is the restriction map.*

Proposition 2.1.31. *For any $u \in [0, \infty)$, let $\mathsf{U}_F^{(u)} = \mathsf{U}_F^{(\lceil u \rceil)}$. Then the local reciprocity map θ_F maps $\mathsf{U}_F^{(u)}$ isomorphically onto $\mathrm{Gal}(F^{\mathrm{ab}}|F)^u$, the u^{th} ramification group in upper numbering.*

In particular, given a finite abelian extension $E|F$ the local reciprocity isomorphism $\theta_{E|F}$ induces an isomorphism $\mathsf{U}_F^{(u)}/(\mathsf{U}_F^{(u)} \cap \mathrm{N}(E^\times)) \cong \mathrm{Gal}(E|F)^u$.

Remark 2.1.32. Let F^{ur} be the maximal unramified extension of F. A consequence of Proposition 2.1.4 is that it is an abelian extension with $\mathrm{Gal}(F^{\mathrm{ur}}|F) \cong \mathrm{Gal}(\bar{k}_F|k_F) \cong \widehat{\mathbb{Z}}$, and is therefore contained in F^{ab}. We have the following map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_F^\times \arrow[r] \arrow[d, "\wr"] & F^\times \arrow[r, "\nu"] \arrow[d, hook, "\theta"] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathrm{Gal}(F^{\mathrm{ab}}|F^{\mathrm{ur}}) \arrow[r] & \mathrm{Gal}(F^{\mathrm{ab}}|F) \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & 0 \end{tikzcd}$$

The Local Reciprocity Law and Existence Theorem, Theorems 2.1.28 and 2.1.29 respectively, tells us that the bottom row is the profinite completion of the top row. The short exact sequences split non-canonically, depending on the choice of a uniformiser. Observe that the vertical isomorphism on the left is the isomorphism specified in Proposition 2.1.31 for $u = 0$ as $\mathrm{Gal}(F^{\mathrm{ab}}|F^{\mathrm{ur}})$ is exactly the inertia subgroup of $\mathrm{Gal}(F^{\mathrm{ab}}|F)$.

We specialise the previous remark to the case when $F = \mathbb{Q}_p$. That is, we make the local reciprocity map explicit for $F = \mathbb{Q}_p$

Remark 2.1.33. When $F = \mathbb{Q}_p$, local Kronecker-Weber gives us $F^{\mathrm{ab}} = \mathbb{Q}_p(\mu_\infty)$, where $\mathbb{Q}_p(\mu_\infty)$ is the field obtained by adjoining all roots of unity. Moreover, by Hensel's lemma, one has $F^{\mathrm{ur}} = \mathbb{Q}_p(\mu_{p'})$, where $\mathbb{Q}_p(\mu_{p'})$ is the field obtained by adjoining all roots of unity whose order is prime-to-p. Therefore, one may write

$$F^{\mathrm{ab}} = F_p \cdot F^{\mathrm{ur}}$$

where $F_p = \mathbb{Q}_p(\mu_{p^\infty})$ is the field obtained by adjoining all roots of unity whose order is a power of p, as seen in Example 2.1.8. Therefore, one has

$$\mathrm{Gal}(F^{\mathrm{ab}}|F^{\mathrm{ur}}) \cong \mathrm{Gal}(F_p|F) \cong \mathbb{Z}_p^\times$$

Hence, the local Artin map $\theta_{\mathbb{Q}_p}$ restricted to $\mathbb{Z}_p^\times$ induces the following isomorphism, corresponding to the vertical isomorphism on the left in Remark 2.1.32,

$$\widetilde{\theta}_{\mathbb{Q}_p} : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$$

which we define as $\widetilde{\theta}_{\mathbb{Q}_p}(a) = a^{-1}$, thereby explicitly describing $\theta_{\mathbb{Q}_p}$.

2.1.5 The Cyclotomic Character

In this section, we will define a fundamental (semi-)linear algebraic object which will be an important tool in the sequel.

For our p-adic field F, consider the totally ramified extension $F(\zeta_{p^r})$, where ζ_{p^r} is a primitive p^n-th root of unity, chosen such that $\zeta_{p^r}^p = \zeta_{p^{r-1}}$. The extension $F(\zeta_{p^r})|F$ is Galois, and its Galois group $\mathrm{Gal}(F(\zeta_{p^r})|F)$ canonically embeds into $(\mathbb{Z}/p^r\mathbb{Z})^\times$ through the map

$$\chi_r \colon \mathrm{Gal}(F(\zeta_{p^r})|F) \hookrightarrow (\mathbb{Z}/p^r\mathbb{Z})^\times$$

defined by the relation $\sigma(\zeta_{p^r}) = \zeta_{p^r}^{\chi_r(\sigma)}$ for any $\sigma \in \mathrm{Gal}(F(\zeta_{p^r})|F)$. The character χ_r in general is not surjective, but it is for all r when $F = \mathbb{Q}_p$.

The field $F(\zeta_{p^\infty}) := \cup_{r \geqslant 0} F(\zeta_{p^r})$ is an infinite totally ramified Galois extension of F. We denote by $\Gamma = \Gamma_F = \mathrm{Gal}(F(\zeta_{p^\infty})|F)$.

Definition 2.1.34 (p-adic Cyclotomic Character)**.** The group Γ has a natural action on the group of p-power roots of unity, seen via each χ_r. This gives rise to the following embedding

$$\chi_\infty \colon \Gamma \hookrightarrow \mathbb{Z}_p^\times$$

defined by the relation $\sigma(\zeta_{p^m}) = \zeta_{p_m}^{\chi_\infty(\sigma)}$ for any $\sigma \in \Gamma$. The character χ_∞ is an open embedding, with respect to the natural profinite topologies on the source and target.

The character $\chi_F \colon G_F \to \mathbb{Z}_p^\times$ is obtained by inflating χ_∞, i.e., it is obtained as the following composition

$$G_F \xrightarrow{\mathrm{res}} \Gamma_F \xrightarrow{\chi_\infty} \mathbb{Z}_p^\times$$

is a continuous character of G_F called the *p-adic cyclotomic character*. We will abuse notation and conflate χ_∞ with χ_F.

Remark 2.1.35. By construction, $\ker \chi_F$ corresponds to $F(\zeta_{p^\infty})$, while the group $\ker(\chi_F \bmod p^r)$ precisely corresponds to $F(\zeta_{p^r})$.

Remark 2.1.36. We may identify G_F and Γ_F as subgroups of $G_{\mathbb{Q}_p}$ of $\Gamma_{\mathbb{Q}_p}$ compatibly. With this identification, we have the following diagram

$$\begin{tikzcd}
G_F \arrow[r, two heads] \arrow[d, hook] \arrow[rr, bend left, "\chi_F"] & \Gamma_F \arrow[r] \arrow[d, hook] & \mathbb{Z}_p^\times \arrow[d, equal] \\
G_{\mathbb{Q}_p} \arrow[r, two heads] \arrow[rr, bend right, "\chi_{\mathbb{Q}_p}"'] & \Gamma_{\mathbb{Q}_p} \arrow[r, "\sim"] & \mathbb{Z}_p^\times
\end{tikzcd}$$

giving us $\chi_F = \chi_{\mathbb{Q}_p}|_{G_F}$.

Remark 2.1.37 (Interaction with the Reciprocity Map)**.** Let $\chi_{\mathbb{Q}_p} : \Gamma_{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{Z}_p^\times$. We observed in Remark 2.1.33 that for $a \in \mathbb{Z}_p$ we have

$$(\chi_{\mathbb{Q}_p} \circ \theta_{\mathbb{Q}_p})(a) = a^{-1}$$

where $\theta_{\mathbb{Q}_p}$ is the local reciprocity map for $\mathbb{Q}_p$. Hence, for any $\gamma \in \Gamma_F$, we have

$$\theta_{\mathbb{Q}_p}(\chi_F(\gamma)) = \gamma^{-1},$$

for a p-adic field F.

2.1.6 p-adic Complex Numbers

An infinite extension $E|F$ is not complete with respect to ν_E. In our context, where E has characteristic 0, this can be shown by using Krasner's Lemma. For a proof in the case $E = \overline{F}$, an algebraic closure of F, see [Kob84, Theorem 12].

Moreover, the valuation $\overline{\nu} = \nu_{\overline{F}}$ on $\overline{F}$ is not discrete: its range is $\mathbb{Q} \cup \{\infty\}$. We also make the observation that $k_{\overline{F}}$ is an algebraic closure of k_F.

Definition 2.1.38 (The p-adic Complex Numbers)**.** The field of *p-adic complex numbers* $\mathbb{C}_F$ is the completion of $\overline{F}$ with respect to $\overline{\nu}$. There is a unique extension of $\overline{v}$ to $\mathbb{C}_F$ that we denote $\widehat{\nu}$, and $\mathbb{C}_F$ is complete with respect to $\widehat{\nu}$. The range of $\widehat{\nu}$ continues to be $\mathbb{Q} \cup \{\infty\}$.

The valuation ring of $\mathbb{C}_F$ is the subring $\mathcal{O}_{\mathbb{C}_F} = \{x \in \mathbb{C}_F : \widehat{\nu}(x) \geqslant 0\}$. It is local with the unique (non-principal) maximal ideal $\mathfrak{m}_{\mathbb{C}_F} = \{x \in \mathbb{C}_F : \widehat{\nu}(x) > 0\}$.

Remark 2.1.39. Consider any $x \in \mathcal{O}_{\mathbb{C}_F}^{\times} = \mathcal{O}_{\mathbb{C}_F} \setminus \mathfrak{m}_{\mathbb{C}_F}$. Since by construction $\overline{F}$ is dense in $\mathbb{C}_F$, we may find a $y \in \overline{F}$ such that $x - y \in \mathfrak{m}_{\mathbb{C}_F}$. Thus, we observe that $\hat{\nu}(y) = \overline{\nu}(y) = 0$. This shows that the canonical map $\mathcal{O}_{\overline{F}}/\mathfrak{m}_{\overline{F}} \to \mathcal{O}_{\mathbb{C}_F}/\mathfrak{m}_{\mathbb{C}_F}$ is surjective. Hence, the residue field $k_{\mathbb{C}_F} = \mathcal{O}_{\mathbb{C}_F}/\mathfrak{m}_{\mathbb{C}_F} \cong k_{\overline{F}} \cong \overline{k}_F$.

Theorem 2.1.40. *$\mathbb{C}_F$ is algebraically closed.*

Proof. Appropriately scaling coefficients, it suffices to prove that every non-constant monic polynomial in $\mathcal{O}_{\mathbb{C}_F}[X]$ has a root in $\mathcal{O}_{\mathbb{C}_F}$. Consider such a polynomial

$$f(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0$$

We will approximate $f(X)$ by a sequence of polynomials in $\mathcal{O}_{\overline{F}}[X]$. By density, we may find polynomials

$$f_r(X) = X^d + a_{d-1}(r)X^{d-1} + \cdots + a_1(r)X + a_0(r) \in \mathcal{O}_{\overline{F}}[X]$$

such that $f - f_r \in p^{dr}\mathcal{O}_{\mathbb{C}_F}[X]$. Since f_r's are monic, they split over $\mathcal{O}_{\overline{F}}$.

Let $\alpha_r \in \mathcal{O}_{\overline{F}}$ be a root of f_r. By construction, $f_{r+1} - f_r \in p^{dr}\mathcal{O}_{\mathbb{C}_F}[X]$ and therefore $f_{r+1}(\alpha_r) \in p^{dr}\mathcal{O}_{\mathbb{C}_F}$ for all r. Writing

$$f_{r+1}(X) = (X - \beta_{r+1,1}) \cdots (X - \beta_{r+1,d}),$$

the product $(\alpha_r - \beta_{r+1,1}) \cdots (\alpha_r - \beta_{r+1,d})$ are divisible by p^{dr}. Hence, necessarily, there exists some $1 \leqslant i \leqslant d$ such that $\alpha_r - \beta_{r+1,i}$ is divisible by p^r. Let $\alpha_{r+1} := \beta_{r+1,i}$.

In this way, proceeding by induction, we have constructed a Cauchy sequence $(\alpha_r)_r$ in $\mathcal{O}_{\overline{F}}$ and define $\alpha = \lim_{r\to\infty} \alpha_r \in \mathcal{O}_{\mathbb{C}_F}$. By construction, we observe that $f_r(\alpha) \in p^r\mathcal{O}_{\mathbb{C}_F}$, and since $f_r \to f$, we conclude that $f(\alpha) = 0$. $\square$

Remark 2.1.41. Let $G_F = \mathrm{Gal}(\overline{F}|F)$ be the absolute Galois group of F, then G_F acts continuously on $\mathbb{C}_F$. It is sufficient to observe that G_F acts isometrically on $\overline{F}$, as we did at the beginning of Section 2.1.1.

The following theorem can be seen as a transcendental Galois correspondance

Theorem 2.1.42 (Ax–Sen–Tate). *Let $H \leqslant G_F$ be a closed subgroup corresponding to an extension $E|F$ contained in $\overline{F}$. Then $\mathbb{C}_F^H = \widehat{E}$. In particular, $\mathbb{C}_F^{G_F} = F$.*

Proof. Let H be as in the statement of the theorem and $E = \overline{F}^H$. It is clear from density of E in $\widehat{E}$ that $\widehat{E} \subseteq \mathbb{C}_F^H$. Consider any $x \in \mathbb{C}_F^H$. We will show that it a limit of some sequence in E. By density, we may consider a sequence $(x_n)_n$ in $\overline{F}$ such that $x_n \to x$ and $\hat{v}(x_n - x) \geqslant n$.

For any $\sigma \in H$, observe that

$$\begin{aligned}\widehat{v}(\sigma(x_n) - x_n) &= \widehat{v}(\sigma(x_n - x) - (x_n - x)) \\ &\geqslant \min\{\widehat{v}(\sigma(x_n - x)), \widehat{v}(x_n - x)\} = \widehat{v}(x_n - x) \geqslant n\end{aligned}$$

By [LB10, Théorème 1.7], which improves [Ax70, Proposition 1], for every n, there exists a $y_n \in E = \overline{F}^H$ such that

$$\widehat{v}(x_n - y_n) > n - \frac{1}{p-1}$$

Since $x_n \to x$, we necessarily obtain that $y_n \to x$. Thus, $x \in \widehat{E}$. $\square$

We end the section by remarking that since the p-adic logarithm and exponential are continuous functions, we may extend them to $\mathbb{C}_F$. In particular, all results (including definitions) in Section 2.1.2 remain true when $\overline{F}$ is replaced by $\mathbb{C}_F$ and $\overline{v}$ by $\widehat{v}$, since $\mathbb{C}_F$ is algebraically closed.

2.2 Semilinear Algebra & Cohomological Descent

2.2.1 Semilinear Representations

The reference for this section is [Car19]. Throughout this section, fix a topological group G and B a topological field. Assume B is equipped with a *continuous* action of G, i.e., the map $G \times B \to B\colon (\sigma, b) \mapsto \sigma \cdot b$ is continuous where $G \times B$ has the product topology. Additionally, we require the action to commute with the addition and multiplication structures of B.

Definition 2.2.1 (Semilinear Representation)**.** A B-*semilinear representation* of G is a topological vector space W over B equipped with a continuous action of G such that

$$\sigma(x+y) = \sigma(x) + \sigma(y) \quad \text{and} \quad \sigma(\lambda \cdot x) = \sigma(\lambda) \cdot \sigma(x)$$

for all $\sigma \in G$, $\lambda \in B$ and $x, y \in W$.

Observe that if G acts trivially on B, then a B-semilinear representation of G is equivalent to the usual notion of a B-linear representation of G.

Definition 2.2.2 (Trivial Semilinear Representation)**.** By definition, B, endowed with its G-action, is a B-semilinear representation of G itself. We can similarly turn B^m into a B-semilinear representation by letting G act coordinate-wise. This is called the *trivial (semilinear) representation* of dimension m.

Notation 2.2.3 (Category of Semilinear Representations)**.** We introduce the category $\mathsf{Rep}_B(G)$ which has as objects B-semilinear representations of G. Morphisms in $\mathsf{Rep}_B(G)$ are B-linear maps which are G-equivariant.

$\mathsf{Rep}_B(G)$ is straightforwardly seen to be an abelian category, and moreover it comes with a notion of tensor product and internal hom: if W_1 and W_2 are objects in $\mathsf{Rep}_B(G)$, then $W_1 \otimes_B W_2$ is an object in $\mathsf{Rep}_B(G)$ equipped with the diagonal action $\sigma \cdot (x \otimes y) = \sigma \cdot x \otimes \sigma \cdot y$, and $\mathrm{Hom}_B(W_1, W_2)$ is an object in $\mathsf{Rep}_B(G)$ equipped with the action $(\sigma \cdot \varphi)(x) = \sigma \cdot \varphi(\sigma^{-1} \cdot x)$. Let $\mathsf{Rep}_B^{\mathsf{f}}(G)$ be the full subcategory consisting of objects that are finite dimensional B-vector spaces; in particular, this category contains the trivial semilinear representations.

Remark 2.2.4 (Scalar Extension)**.** Let C be a closed subfield of B which stable under the action of G. Therefore the notion of C-semilinear representations of G makes sense and we have the following functor

$$B \otimes_C - \colon \mathsf{Rep}_C(G) \to \mathsf{Rep}_B(G)$$

In particular, assume that we are given a (topological) field F and we have G and B such that B is an F-algebra and the action of G on F is trivial. Then

$\mathsf{Rep}_F(G)$ is just the category of F-linear representations of G, and given any such representation V, then $W = B \otimes_F V$ is a B-semilinear representation of G.

Specializing to the case of linear characters $\chi\colon G \to F^\times$, we will denote by $B(\chi) = B \otimes_F \chi$ the 1-dimensional semilinear representation generated by a vector e_χ on which G acts as $\sigma \cdot e_\chi = \chi(\sigma) \cdot e_\chi$ for all $\sigma \in G$. By semilinearity, we then have

$$\sigma(be_\chi) = (\sigma \cdot b)(\chi(\sigma) \cdot e_\chi)$$

for all $\sigma \in G$ and $b \in B$.

We give the following example, which is situated in the exact context in which we wish to apply the theory in this section.

Example 2.2.5 (Tate Twists & Hodge-Tate Representations)**.** Assume F is a p-adic field and G_F its absolute Galois group. Then $\mathsf{Rep}_F(G_F)$ is the the category of continuous F-linear representations. We have already introduced a fundamental object in this category: the p-adic cyclotomic character $\chi = \chi_F$ on F; see Definition 2.1.34.

For any integer n, by χ^n we mean the character defined as $\chi^n(\sigma) := \chi(\sigma)^n$. For any field extension $E|F$, we define the *n^{th}-Tate twist* of E as

$$E(n) := E(\chi^n) \in \mathsf{Rep}_E(G_F).$$

These objects have the following properties:

$$E(n)^\vee = E(-n) \quad \text{and} \quad E(m) \otimes_E E(n) = E(m+n)$$

where $(-)^\vee = \mathrm{Hom}_E(-, E)$, the internal hom in $\mathsf{Rep}_E(G_F)$.

The representation $\oplus_{i=1}^m E(i)^{n_i}$ is called a *Hodge-Tate representation* of E of weight

$\underline{n} = (n_1, \ldots, n_m)$. In particular, the n^{th}-Tate of twist $E(n)$ of E is a Hodge-Tate representation of weight n.

2.2.2 Continuous Cohomology

Throughout this section, fix a topological group G and X a topological group, not necessarily abelian, on which G acts continuously. For $\sigma \in G$ and $a \in X$, denote by ${}^{\sigma}a \in X$ the element obtained by the left-action of σ on a.

Definition 2.2.6 (Continuous Cohomology)**.** We define $\mathrm{H}^0(G, X) := X^G$.

A *(continuous) 1-cocycle* of G valued in X is a continuous function $G \to X \colon \sigma \mapsto a_\sigma$ such that $a_{\sigma\tau} = a_\sigma \cdot {}^{\sigma}a_\tau$. Letting $\sigma = 1_G$, we have that $a_{1_G} = 1_X$.

Let $\mathrm{Z}^1(G, X)$ be the set of all 1-cocycles. A pair $c, c' \in \mathrm{Z}^1(G, X)$ is considered *cohomologous*, denoted $c \sim c'$, if there exists an $x \in X$ such that

$$c(\sigma) = x^{-1} \cdot c'(\sigma) \cdot {}^{\sigma}x, \quad \text{for all } \sigma \in G$$

This defines an equivalence relation on $\mathrm{Z}^1(G, X)$, and

$$\mathrm{H}^1(G, X) := \mathrm{Z}^1(G, X)/{\sim}$$

is the *first cohomology set* of G with coefficients in X. It is a *pointed set* as it comes equipped with a distinguished *base point*: the class of the trivial cocycle $\sigma \mapsto 1_X$, denoted $\mathrm{B}^1(G, X)$. Elements of this class are called *1-coboundaries* and are 1-cocycles of the form $\sigma \mapsto x^{-1} \cdot {}^{\sigma}x$, for some $x \in X$.

Assume for the next definition that X is abelian, written additively for notational clarity. A *(continuous) 2-cocycle* is a continuous function $c \colon G \times G \to X$

such that

$$c(\sigma_1\sigma_2, \sigma_3) - c(\sigma_1, \sigma_2\sigma_3) = c(\sigma_1, \sigma_2) - \sigma_1 \cdot c(\sigma_2, \sigma_3)$$

for any $\sigma_1, \sigma_2, \sigma_3 \in G$. A 2-cocyle is a *2-coboundary* if

$$c(\sigma_1, \sigma_2) = a_{\sigma_1\sigma_2} - a_{\sigma_1} - {}^{\sigma_1}a_{\sigma_2}, \quad \text{for all } \sigma_1, \sigma_2 \in X,$$

for some continuous function $\sigma \mapsto a_\sigma$. The collection of 2-cocycles is denoted as $\mathrm{Z}^2(G, X)$ and is an abelian group under pointwise addition. The collection of 2-coboundaries forms a subgroup and is denoted $\mathrm{B}^2(G, X)$. We define the *second cohomology group*, an abelian group, as

$$\mathrm{H}^2(G, X) := \mathrm{Z}^2(G, X)/\mathrm{B}^2(G, X)$$

Remark 2.2.7. When X is abelian, then $\mathrm{Z}^1(G, X)$ is an abelian group and $\mathrm{B}^1(G, X)$ is a subgroup. In particular, $\mathrm{H}^1(G, X) = \mathrm{Z}^1(G, X)/\mathrm{B}^1(G, X)$, an abelian group.

Definition 2.2.8. Let X, Y, Z be topological groups where X is abelian and Y and Z are not necessarily abelian. Then the following sequence of topological groups with continuous group homomorphisms

$$1 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 1$$

is called *short exact* if

- f is a closed embedding.
- $\operatorname{im} f$ is a normal subgroup of Y.
- $\operatorname{im} f = \ker g$
- g is surjective, and induces a homeomorphism $\overline{g} : Y/\operatorname{im} f \xrightarrow{\sim} Z$.

Furthermore, suppose X, Y and Z come equipped with a continuous action of a topological group G. Then a short exact sequence will always have G-equivariant continuous group homomorphisms.

Proposition 2.2.9. *Let G be a topological group and*

$$1 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 1$$

a short exact sequence of topological groups equipped with a continuous G-action. Then, there exists an exact sequence of pointed sets

$$1 \to X^G \to Y^G \to Z^G \xrightarrow{\delta} \mathrm{H}^1(G, X) \to \mathrm{H}^1(G, Y) \to \mathrm{H}^1(G, Z) \xrightarrow{\delta} \mathrm{H}^2(G, X)$$

Proof. The proof in [GS17, Proposition 4.4.1] goes through without change. In particular, the construction of the connecting maps δ goes without change, while the other morphisms are induced by f and g. □

Remark 2.2.10. If Y and Z are abelian, we can extend the exact sequence given in Proposition 2.2.9 using f and g since $\mathrm{H}^2(G, Y)$ and $\mathrm{H}^2(G, Z)$ is defined.

$$\cdots \to \mathrm{H}^1(G, Z) \xrightarrow{\delta} \mathrm{H}^2(G, X) \to \mathrm{H}^2(G, Y) \to \mathrm{H}^2(G, Z)$$

Throughout this dissertation, we employ this concrete version of group cohomology as that is enough for our purposes.

Assume now that we are given a closed normal subgroup N of G, then the subgroup of N-fixed points X^N of X naturally has a continuous action of G/N, given the quotient topology. Then, we have the following operations on 1-cocycles.

Definition 2.2.11 (Inflation & Restriction)**.** Consider a 1-cocycle $c\colon G/N \to X^N$. Then the *inflation* of c is the function $\mathrm{Inf}(c)\colon G \to X$ given by

$$G \twoheadrightarrow G/N \xrightarrow{c} X^N \hookrightarrow X$$

which is straightforwardly seen to be a 1-cocycle. Now Suppose we have a 1-cocycle $c'\colon G \to X$. Then the *restriction* of c' is the 1-cocycle $c'|_N\colon N \to X$. These maps give a well-defined sequence

$$\mathrm{H}^1(G/N, X^N) \xrightarrow{\mathrm{Inf}} \mathrm{H}^1(G, X) \xrightarrow{\mathrm{Res}} \mathrm{H}^1(N, X)$$

Proposition 2.2.12 (Inflation-Restriction Sequence)**.** *There is an exact sequence*

$$1 \longrightarrow \mathrm{H}^1(G/N, X^N) \xrightarrow{\mathrm{Inf}} \mathrm{H}^1(G, X) \xrightarrow{\mathrm{Res}} \mathrm{H}^1(N, X)$$

of pointed sets. That is, Inf *is injective and* $\ker \mathrm{Res} = \operatorname{im} \mathrm{Inf}$*. By* $\ker \mathrm{Res}$ *we mean the classes that get mapped to the class of the distinguished base point under* Res*.*

Proof. Consider 1-cocycles $c_1, c_2\colon G/N \to X^N$ such that $\mathrm{Inf}(c_1)$ and $\mathrm{Inf}(c_2)$ are cohomologous in $\mathrm{H}^1(G, X)$. That is,

$$c_1(\sigma N) = \mathrm{Inf}(c_1)(\sigma) = x^{-1} \cdot \mathrm{Inf}(c_2)(\sigma) \cdot {}^{\sigma}x = x^{-1} \cdot c_2(\sigma N) \cdot {}^{\sigma}x \qquad (*)$$

for every $\sigma \in G$, for some $x \in X$. As a cocycle, since N is the unit of G/N, we have that $c_1(N) = c_2(N) = 1_X$. Therefore, for any $\eta \in N$, we have

$$1_X = x^{-1} \cdot {}^{\eta}x$$

That is, x is fixed by the action of N, and therefore we have $x \in X^N$ and hence $(*)$ shows that c_1 and c_2 themselves are cohomologous and thus Inf is injective.

It is straightforward to see that $\operatorname{im} \mathrm{Inf} \subseteq \ker \mathrm{Res}$. Now, consider a 1-cocycle

$c\colon G \to X$ such that $\mathrm{Res}(x)$ is a 1-coboundary in $\mathrm{H}^1(N, X)$. So, $c(\eta) = y^{-1} \cdot {}^{\eta}y$ for any $\eta \in N$, for some $y \in X$. Let $x = y^{-1}$, and define a 1-cocycle $c' : G \to X$ as

$$c'(\sigma) = x^{-1} \cdot c(\sigma) \cdot {}^{\sigma}x$$

By definition, we have $[c] = [c']$, and $c'(\eta) = 1_X$ for all $\eta \in N$. Consider any $\eta \in N$ and $\sigma \in G$, then by normality of N, we observe that $\eta\sigma = \sigma\eta'$ for some $\eta' \in N$. Then, we observe that

$$c'(\eta\sigma) = c'(\eta) \cdot {}^{\eta}c'(\sigma) = {}^{\eta}c'(\sigma)$$

and

$$c'(\eta\sigma) = c'(\sigma\eta') = c'(\sigma) \cdot {}^{\sigma}c'(\eta') = c'(\sigma)$$

From this, we obtain that $c'(\sigma)$ is fixed by the action of N and hence $c'(\sigma) \in X^N$ for any $\sigma \in G$. Moreover, since $c'(\sigma\eta') = c'(\sigma)$ we also obtain that c' is constant on the coset σN. Thus, let us define

$$\tilde{c}\colon G/N \to X^N,\ \tilde{c}(\sigma N) := c'(\sigma).$$

One verifies that $\tilde{c}$ is a 1-cocycle and therefore we have $[\mathrm{Inf}(\tilde{c})] = [c'] = [c]$, and hence $\ker \mathrm{Res} \subseteq \mathrm{im}\, \mathrm{Inf}$. □

When X is abelian, a similar argument gives us the following

Proposition 2.2.13 (Inflation-Restriction Sequence)**.** *There is an exact sequence*

$$1 \longrightarrow \mathrm{H}^2(G/N, X^N) \xrightarrow{\mathrm{Inf}} \mathrm{H}^2(G, X) \xrightarrow{\mathrm{Res}} \mathrm{H}^2(N, X)$$

of abelian groups.

2.2.3 Cohomological Descent

We formalise a connection between semilinear representations and continuous cohomology. Let G and B be as in Section 2.2.1. The reference for this section is [BC09, Exercise 14.4.1]

Remark 2.2.14. There is a natural topology on $\mathrm{GL}_m(B)$ that makes it a topological group. For our purposes, we describe this topology in a hands-on manner; for a more functorial approach, see [Con12, §2].

Topologise $\mathrm{GL}_m(B)$ using the subspace topology from $\mathrm{M}_m(B)^2$ via the anti-diagonal map $g \mapsto (g, g^{-1})$. One quickly verifies that this makes $\mathrm{GL}_m(B)$ into a topological group, as the continuity of the inversion map $g \mapsto g^{-1}$ is clear from definition. In particular, the topology of $B^\times$ is the one on $\mathrm{GL}_1(B)$.

The continuous action of G on B extends to an action of G on $\mathrm{GL}_m(B)$ where for $A \in \mathrm{GL}_m(B)$, one has $({}^\sigma A)_{ij} := \sigma(A_{ij})$. To see that the action of G on $\mathrm{GL}_m(B)$ is continuous, it is enough to conclude that the following maps are continuous

$$(\sigma, A) \mapsto \sigma(A_{ij}) \quad \text{and} \quad (\sigma, A) \mapsto \sigma(A_{ij}^{-1}),$$

which they are since they are compositions of certain projections and the continuous G-action on B.

Let $V \in \mathsf{Rep}_B^{\mathrm{f}}(G)$ be free of rank m. Choosing a B-basis $\mathbf{e} = (e_1, \dots, e_m)$ of V, we topologise V via the isomorphism $V = \oplus_i Be_i \cong B^m$. We note that this topology is independent of the choice of basis. We define a function $U_{\mathbf{e}} : G \to \mathrm{GL}_m(B)$, where $U_{\mathbf{e}}(\sigma) = (b_{i,j}^{\mathbf{e}}(\sigma))$ is the matrix such that

$$\sigma(e_j) = \sum_{i=1}^{m} b_{i,j}^{\mathbf{e}}(\sigma) \cdot e_i$$

Lemma 2.2.15. *$U_{\mathbf{e}}$ is a continuous 1-cocycle and if $\mathbf{f}$ is another B-basis of V, then $U_{\mathbf{f}}$ is cohomologous to $U_{\mathbf{e}}$.*

Proof. Let $\sigma, \tau \in G$. One has $\tau(e_j) = \sum_{k=1}^{\prime} U_{\mathbf{e}}(\tau)_{kj} \cdot e_k$. By semilinearity,

$$\begin{aligned}(\sigma\tau)(e_j) &= \sum_{k=1}^{m} \sigma(U_{\mathbf{e}}(\tau)_{kj}) \cdot \sigma(e_k) \\ &= \sum_{i=1}^{m} \left(\sum_{k=1}^{m} U_{\mathbf{e}}(\sigma)_{ik} \cdot \sigma(U_{\mathbf{e}}(\tau)_{kj}) \right) \cdot e_i\end{aligned}$$

Therefore, by definition, we obtain

$$U_{\mathbf{e}}(\sigma\tau)_{ij} = \sum_{k=1}^{m} U_{\mathbf{e}}(\sigma)_{ik} \cdot \sigma(U_{\mathbf{e}}(\tau)_{kj})$$

That is, $U_{\mathbf{e}}(\sigma\tau) = U_{\mathbf{e}}(\sigma) \cdot {}^{\sigma}U_{\mathbf{e}}(\tau)$, and hence $U_{\mathbf{e}}$ is a 1-cocycle. The continuity of $U_{\mathbf{e}}$ follows from an observation similar to the one made towards the end of Remark 2.2.14.

Let $\mathbf{f}$ be another B-basis of V and let $P \in \mathrm{GL}_m(B)$ be the change-of-basis matrix from $\mathbf{f}$ to $\mathbf{e}$, then one shows directly that

$$U_{\mathbf{f}}(\sigma) = P^{-1} \cdot U_{\mathbf{e}}(\sigma) \cdot {}^{\sigma}P, \quad \text{for every } \sigma \in G$$

Hence $U_{\mathbf{e}}$ and $U_{\mathbf{f}}$ are cohomologous. ◻

Let $\mathrm{Rep}_B^m(G)$ be the set of B-linear isomorphism classes of objects in $\mathrm{Rep}_B^{\mathrm{f}}(G)$ of rank m. This is a pointed set, with the base point given by the class of the trivial semilinear representation B^m. Observe that, the trivial semilinear representation with the standard basis $\mathbf{1}$ produces the trivial cocycle $U_{\mathbf{1}} : \sigma \mapsto \mathrm{Id}_m$; here Id_m is the $m \times m$ identity matrix.

Theorem 2.2.16. *The map* $(V, \mathbf{e}) \mapsto U_{\mathbf{e}}$ *yields a isomorphism of pointed sets*

$$\Phi_{G,B} \colon \operatorname{Rep}_B^m(G) \xrightarrow{\sim} \mathrm{H}^1(G, \mathrm{GL}_m(B))$$

Proof. Lemma 2.2.15 tells us that the map $(V, \mathbf{e}) \mapsto U_{\mathbf{e}}$ gives a well-defined map of pointed sets

$$\Phi : \operatorname{Rep}_B^m(G) \to \mathrm{H}^1(G, \mathrm{GL}_m(B))$$

We sketch the inverse. Let $U \colon G \to \mathrm{GL}_m(B)$ be a 1-cocycle, and let $V_U = B^m$ be equipped with the following action of G: for $\sigma \in G$, define

$$\sigma \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} := U(\sigma) \begin{pmatrix} \sigma(v_1) \\ \vdots \\ \sigma(v_m) \end{pmatrix}.$$

That this is a semilinear action is clear from the definition, and its continuity follows from the continuity of U and the action of G on B. A straightfoward computation shows that this map is well-defined and descends to a map of pointed sets $\Psi : \mathrm{H}^1(G, \mathrm{GL}_m(B)) \to \operatorname{Rep}_B^m(G)$, and that it is the inverse of Φ. □

Remark 2.2.17. The isomorphism in Theorem 2.2.16 is natural in G and B. Let G' be another topological group and C a topological ring with a continuous action of G'. Assume that there exist continuous homomorphisms $\varphi : G \to G'$ and $\psi : C \to B$. These morphisms induce a functor

$$\mathrm{Asc} \colon \mathsf{Rep}_C(G') \to \mathsf{Rep}_B(G),\ W \mapsto B \otimes_C W.$$

We describe how one views $B \otimes_C W$ as a B-semilinear representation of G: let $\sigma \in G$, then $\sigma(w \otimes b) = \varphi(\sigma)(w) \otimes \sigma(b)$. This restricts to a functor and map of pointed sets

$$\mathrm{Asc}\colon \mathsf{Rep}^{\mathrm{f}}_C(G') \to \mathsf{Rep}^{\mathrm{f}}_B(G) \quad \text{and} \quad \mathrm{Asc}_m\colon \mathsf{Rep}^m_C(G') \to \mathsf{Rep}^m_B(G)$$

respectively. This generalises the early discussions in Remark 2.2.4.

These morphisms also induce a map on cohomology sets

$$\mathrm{Asc}_{G,G',C,B}\colon \mathrm{H}^1(G',\mathrm{GL}_m(C)) \to \mathrm{H}^1(G,\mathrm{GL}_m(B)),\ [U] \mapsto [\mathrm{GL}_m(\psi)\circ U \circ \varphi]$$

Since Φ is defined explicitly in Theorem 2.2.16, it is easily shown that the following diagram of pointed sets commutes

$$\begin{tikzcd}
\mathsf{Rep}^m_C(G') \arrow[r, "\Phi_{G',C}"] \arrow[d, "\mathrm{Asc}_m"'] & \mathrm{H}^1(G',\mathrm{GL}_m(C)) \arrow[d, "\mathrm{Asc}'_{G,G',C,B}"] \\
\mathsf{Rep}^m_B(G) \arrow[r, "\Phi_{G,B}"'] & \mathrm{H}^1(G,\mathrm{GL}_m(B))
\end{tikzcd}$$

If N is a closed normal subgroup of G (with the subspace topology), then the subring of N-fixed points B^N naturally has a continuous action of G/N. Using Remark 2.2.17 with $N \hookrightarrow G \twoheadrightarrow G/N$, where G/N is given the quotient topology, and $B^N \hookrightarrow B$ we obtain the following sequence of cohomology sets

$$\mathrm{H}^1(G/N,\mathrm{GL}_m(B^N)) \xrightarrow{\mathrm{Asc}_N} \mathrm{H}^1(G,\mathrm{GL}_m(B)) \xrightarrow{\mathrm{Asc}_H} \mathrm{H}^1(H,\mathrm{GL}_m(B))$$

where $\mathrm{Asc}_N = \mathrm{Asc}_{G,G/N,B^N,B}$ and $\mathrm{Asc}_H = \mathrm{Asc}_{H,G,B,B}$.

We can get rid of this clunky notation by observing that since $\mathrm{GL}_m(B^N) = \mathrm{GL}_m(B)^N$, the above sequence is precisely the inflation-restriction sequence from Proposition 2.2.12 with $\mathrm{Asc}_N = \mathrm{Inf}$ and $\mathrm{Asc}_H = \mathrm{Res}$. This gives us

Corollary 2.2.18 (Inflation-Restriction Sequence)**.** *There is an exact sequence*

$$1 \longrightarrow \mathrm{H}^1(G/N, \mathrm{GL}_m(B^N)) \xrightarrow{\mathrm{Inf}} \mathrm{H}^1(G, \mathrm{GL}_m(B)) \xrightarrow{\mathrm{Res}} \mathrm{H}^1(N, \mathrm{GL}_m(B))$$

We have the following functors

$$(-)^N \colon \mathsf{Rep}_B(G) \to \mathsf{Rep}_{B^N}(G/N)$$

$$B \otimes_{B^N} - \colon \mathsf{Rep}_{B^N}(G/N) \to \mathsf{Rep}_B(G)$$

Lemma 2.2.19. *For any $W \in \mathsf{Rep}^{\mathrm{f}}_{B^N}(G/N)$, the canonical map*

$$\iota \colon W \to (W \otimes_{B^N} B)^N,\ w \mapsto w \otimes 1$$

is an isomorphism.

Proof. Easily follows once a basis has been chosen. □

On the other hand, for any $V \in \mathsf{Rep}_B(G)$, one has the map

$$\alpha_V : B \otimes_{B^N} V^N \to V,\ b \otimes v \mapsto bv$$

Assuming V has rank m, it determines a cohomology class $\xi_V \in \mathrm{H}^1(G, \mathrm{GL}_m(B))$.

Proposition 2.2.20 (Cohomological Descent)**.** *The following are equivalent:*

(a) α_V *is an isomorphism;*

(b) $\mathrm{Res}(\xi_V) = [1]$, *the class of the trivial cocycle;*

(c) ξ_V *is in the image of* Inf.

If any of these conditions are satisfied, one says that V descends to B^N.

Proof. Observe that equivalence of (b) and (c) follows from Proposition 2.2.12, the inflation-restriction sequence. Assume that (a) holds, therefore V has an N-invariant basis, say $\boldsymbol{e}$, and hence ξ_V is represented by the cocycle $U_{\boldsymbol{e}}$, which by construction is such that $U_{\boldsymbol{e}}|_N = 1$, the trivial cocycle. Thus, $\mathrm{Res}(\xi_V) = [1]$.

Assume now that (b) holds, then $\mathrm{Res}(\xi_V)$ is represented by a 1-coboundary

$$U'_{\boldsymbol{e}} \colon N \to \mathrm{GL}_m(B),\ \eta \mapsto P^{-1} \cdot {}^{\eta}P,$$

for some $P \in \mathrm{GL}_m(B)$, where $\boldsymbol{e}$ is a basis of W. The basis $\boldsymbol{e}' := P\boldsymbol{e}$ is such that $U'_{\boldsymbol{e}'} = 1$ is trivial, and therefore the basis $\boldsymbol{e}'$ is N-invariant, and hence we use it to conclude α_V is an isomorphism. $\square$

Corollary 2.2.21 (Cohomological Descent)**.** $B \otimes_{B^N} - \colon \mathsf{Rep}^{\mathsf{f}}_{B^N}(G/N) \to \mathsf{Rep}^{\mathsf{f}}_{B}(G)$ *is an equivalence with quasi-inverse* $(-)^N \colon \mathsf{Rep}^{\mathsf{f}}_{B}(G) \to \mathsf{Rep}^{\mathsf{f}}_{B^N}(G/N)$ *if and only if the map* $\mathrm{Inf} \colon \mathrm{H}^1(G/N, \mathrm{GL}_m(B^N)) \to \mathrm{H}^1(G, \mathrm{GL}_m(B))$ *is an isomorphism.*

2.3 Brauer Groups

In this section we fix a perfect field K. We do so to conflate algebraic and separable extensions to simplify the exposition, although all results in this section hold for general fields after adding some separability assumptions.

Assume throughout that all K-algebras under consideration are non-zero and finite dimensional over K. We will often refer to K-algebras as *algebras over* K. References for this section include [GS17, Gui18].

2.3.1 Central Simple Algebras

Definition 2.3.1 (Central Simple Algebra)**.** A K-algebra A is *simple* if it has no non-trivial two-sided ideals. We call a K-algebra *central* if its centre $Z(A) = K$.

Example 2.3.2. A division algebra D over K is clearly simple, and since its centre $Z(D)$ is necessarily a field, it is central over $Z(D) \supseteq K$.

Example 2.3.3. If D is a division algebra over K, then the algebra $\mathrm{M}_n(D)$ of $n \times n$ matrices over D is simple for all $n \geqslant 1$. Similarly, one may also verify that $Z(\mathrm{M}_n(D)) = Z(D)$, i.e., the centre consists only of scalar multiples of the identity with scalars in $Z(D)$. Therefore, $\mathrm{M}_n(D)$ is central over $Z(D)$.

The following classical theorem states that our examples above are essentially the only examples there are.

Theorem 2.3.4 (Wedderburn)**.** *Let A be a finite dimensional simple algebra over K. Then there exists a unique positive integer n and a unique, up to K-isomorphism, division algebra D over K such that $A \cong \mathrm{M}_n(D)$ as K-algebras.*

Definition 2.3.5 (Splitting Fields)**.** Given a central simple K-algebra A, we say a finite extension $L|K$ is a *splitting field* of A if $A \otimes_K L \cong \mathrm{M}_n(L)$, as L-algebras, for some n. In particular, K is a splitting field of A if and only if $A \cong \mathrm{M}_n(K)$, as K-algebras, for some n; we call such central simple algebras *split*.

The following theorem tells us that splitting fields exist and we can take them to be Galois.

Theorem 2.3.6. *A K-algebra A is central simple if and only if there exists an $n \geqslant 1$ and finite Galois extension $L|K$ such that $A \otimes_K L \cong \mathrm{M}_n(L)$ as L-algebras.*

Proof. [GS17, Theorem 2.2.1] shows that a splitting field for A exists, this follows from Theorem 2.3.4 and the fact that A is finite-dimensional over K. A result of Noether and Köthe [GS17, Proposition 2.5] tells us that a splitting field for A which is Galois exists. □

Remark 2.3.7. Let $\overline{K}$ be an algebraic closure of K. Then Theorem 2.3.6 tells us that, in particular, we have $A \otimes_K \overline{K} \cong \mathrm{M}_n(\overline{K})$, as $\overline{K}$-algebras.

Corollary 2.3.8. *If A is a central simple algebra over K, its K-dimension is a square.*

Definition 2.3.9 (Degree & Index of a CSA)**.** For a central simple K-algebra A, its *degree* is the number $\deg_K A = \sqrt{\dim_K A}$.

By Theorem 2.3.4, one has $A \cong \mathrm{M}_n(D)$, for some n and some central simple K-division algebra D. Then $\mathrm{ind}_K(A)$, the *(Schur) index* of A, is $\deg_K D$.

Definition 2.3.10 (Brauer equivalence)**.** Let A and B be central simple algebras over K. We say A and B are *Brauer equivalent* if $A \otimes_K \mathrm{M}_r(K) \cong B \otimes_K \mathrm{M}_s(K)$ for some integers $r, s \geqslant 1$.

Remark 2.3.11. Theorem 2.3.4 tells us that any central simple algebra over K is Brauer equivalent to a division algebra over K. Moreover, since $\mathrm{M}_n(D) \otimes_K L \cong \mathrm{M}_n(D \otimes_K L)$, we conclude that splitting fields only depend on the Brauer class.

Definition 2.3.12 (Brauer group)**.** The *Brauer group* of K is the set of all Brauer classes of all central simple algebras over K, which is denoted as $\mathrm{Br}(K)$. This is indeed a group, where the group operation is given by $\otimes_K$ and the unit is the class $[K]$, which consists of all split central simple algebras. The inverse of a class $[A]$ is the class $[A^{\mathrm{op}}]$ represented by the opposite algebra of A.

The fact that for a class $[A] \in \mathrm{Br}(K)$, its inverse is given by the class $[A^{\mathrm{op}}]$ is a consequence of the following result which we record for future use.

Proposition 2.3.13. *Let A be a finite dimensional K-algebra. Then A is a central simple algebra over K if and only if the map*

$$\varphi\colon A \otimes_K A^{\mathrm{op}} \to \mathrm{End}_K(A),\ a \otimes b \mapsto (\varphi_{a,b} : x \mapsto axb)$$

is an isomorphism.

Proof. For the forward direction, see [GS17, Proposition 2.4.8]. For the converse, it suffices to assume φ is surjective and we observe then that $\operatorname{im}\varphi = \mathrm{End}_K(A) \cong \mathrm{M}_{\deg A}(K)$, a central simple algebra over K.

Let I be a non-trivial two-sided ideal of A, then $I \otimes_K A^{\mathrm{op}}$ is non-trivial two-sided ideal in $A \otimes_K A^{\mathrm{op}}$, and therefore so is $J = \varphi(I \otimes_K A^{\mathrm{op}})$ in the image of φ. Since $\operatorname{im}\varphi$ is simple, we conclude that $J = \operatorname{im}\varphi = \mathrm{End}_K(A)$ by surjectivity of φ. Hence, $\mathrm{id}_A = \varphi(\sum_{i=1}^n y_i \otimes b_i) = \sum_{i=1}^n \varphi_{y_i,b_i}$, where $y_i \in I$ and $b_i \in A^{\mathrm{op}}$. Therefore, for any $a \in A$, we have

$$a = \sum_{i=1}^n \varphi_{y_i,b_i}(a) = \sum_{i=1}^n y_i a b_i \in I$$

since I is a two-sided ideal of A. Hence $I = A$, and thus A is simple.

Let $z \in Z(A)$, then $z \otimes 1 \in Z(A \otimes_K A^{\mathrm{op}})$ and hence $\varphi(z \otimes 1) \in Z(\operatorname{im}\varphi) =$

$K \cdot \mathrm{id}_A$, by assumption. Let $\varphi(z \otimes 1) = \lambda \,\mathrm{id}_A$, for some $\lambda \in K$, and we note that $z = \varphi(z \otimes 1)(1) = \lambda$. Thus, $Z(A) \subseteq K$, and thus A is central. □

Remark 2.3.14. By Remark 2.3.11, we can identify $\mathrm{Br}(K)$, as a set, with the set of isomorphism classes of division algebras that are K-central. Moreover, we observe that the index (Definition 2.3.9) is a well-defined function on $\mathrm{Br}(K)$.

Since splitting fields only depend on the Brauer class of a central simple algebra, we give the following result on splitting fields of division algebras. For a central simple algebra, we may replace degree with index.

Proposition 2.3.15. *Assume D is a division algebra of degree n over K, then D has a splitting field of dimension n over K, and the dimension of any splitting field of D over K is a multiple of n.*

Proof. See [Gui18, Corollary 6.39]. □

Example 2.3.16. We note that Theorem 2.3.6 tells us that $\mathrm{Br}(K)$ is trivial if K is algebraically (or separably) closed. This can also be shown directly by using Remark 2.3.14 and showing that the only division algebra that is central over an algebraically closed field K is K itself.

Definition 2.3.17 (Relative Brauer group)**.** For any extension $L|K$, the map

$$\mathrm{Br}(K) \to \mathrm{Br}(L)\colon [A] \mapsto [A \otimes_K L]$$

is well-defined and a group homomorphism. The kernel of this map is called the *relative Brauer group* of $E|K$ and denoted $\mathrm{Br}(L|K)$.

Observe that $[A] \in \mathrm{Br}(L|K)$ if and only if L is a splitting field of A.

In particular, one has $\mathrm{Br}(K) = \mathrm{Br}(\overline{K}|K)$.

Remark 2.3.18. Combining Definition 2.3.17 with Theorem 2.3.6 we observe that the Brauer group $\mathrm{Br}(K)$ is the union of all its subgroups of the form $\mathrm{Br}(L|K)$ with $L|K$ finite and Galois. In particular, if $L'|K$ is a Galois extension containing L, then $i_{L'L} = (-) \otimes_L L' \colon \mathrm{Br}(L|K) \to \mathrm{Br}(L'|K)$ is injective and all such $i_{L'L}$ form a directed system that $\mathrm{Br}(K)$ is a colimit of.

2.3.2 Cohomological Descent Revisited

We recall definitions and results from Sections 2.2.2 and 2.2.3 and apply them to the case where $G = G_K$ is the absolute Galois group of K equipped with its profinite (Krull) topology, while $B = \overline{K}$, an algebraic closure of K, equipped with the *discrete topology*. In this set-up, the cohomology groups and sets we consider are termed *Galois cohomology*. A standard reference is [Ser02].

Below we give a useful characterisation of cohomology in this context.

Proposition 2.3.19. *Let G be a profinite group and X a discrete group, not necessarily abelian. Write $G = \varprojlim G_\alpha$, where G_α is finite. Then we consider $U_\alpha = \ker \pi_\alpha$, where $\pi_\alpha \colon G \twoheadrightarrow G_\alpha$ is the canonical projection; U_α is an open normal subgroup. Define $X_\alpha := X^{U_\alpha}$, which is naturally equipped with a continuous G_α-action, where G_α is given the discrete module. Then,*

$$\mathrm{H}^1(G, X) = \varinjlim \mathrm{H}^1(G_\alpha, X_\alpha)$$

where the directed system is given by inflation maps. Moreover, if X is abelian, we also have

$$\mathrm{H}^2(G, X) = \varinjlim \mathrm{H}^2(G_\alpha, X_\alpha)$$

where once again the directed system is given by inflation maps.

Proof. This is well-known, see [Sta25, Tag 0DVG]. $\square$

Remark 2.3.20. Applying Proposition 2.3.19 to G_K tells us that given a discrete group X with a continuous action of G_K, we have

$$\mathrm{H}^i(G_K, X) = \varinjlim \mathrm{H}^i(G_{L|K}, X^{G_L})$$

(for $i = 1, 2$, where for $i = 2$ we assume X is abelian)

where the limit is over finite Galois extensions $L|K$ contained in $\overline{K}$. Here we identify G_L with the subgroup $\mathrm{Gal}(\overline{L}|K)$ in G_K, and we denote $G_{L|K} = \mathrm{Gal}(L|K)$, the cokernel of $G_L \hookrightarrow G_K$.

This tells us that we can restrict our focus on finite Galois extensions of K, and cohomology of finite groups.

The following example illustrates Remark 2.3.20 where $X = \mathrm{GL}_m(\overline{K})$.

Proposition 2.3.21 (Hilbert's Theorem 90)**.** *Let $L|K$ be a Galois extension, then*

$$\mathrm{H}^1(G_K, \mathrm{GL}_m(\overline{K})) = \{*\}$$

Proof. We show that for any finite Galois extension $L|K$, one has

$$\mathrm{H}^1(G_{L|K}, \mathrm{GL}_m(L)) = \{*\}\,.$$

We invoke Proposition 2.2.20. Let V be a finite-dimensional L-vector space with a semilinear action of $G = G_{L|K}$, and consider the canonical map

$$\alpha_V \colon L \otimes_K V^G \to V$$

We show that α_V is an isomorphism. The proof of surjectivity does not require V to be finite-dimensional, only that $[L : K] < \infty$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be a K-basis of

L, then by Artin's independence of characters there exist $\lambda_1, \ldots, \lambda_n \in L$ such that $\sum_{i=1}^n \lambda_i \sigma(\varepsilon_i) = 1$ if $\sigma \in G$ is the identity and 0 otherwise. Consider the trace function $T \colon V \to V,\ x \mapsto \sum_{\sigma \in G} \sigma(x)$, and observe that $\operatorname{im} T \subseteq V^G$. Finally, we observe that for any $v \in V$, we have

$$\sum_{i=1}^n \lambda_i T(\varepsilon_i v) = \sum_{\sigma \in G} \sum_{i=1}^n \lambda_i \sigma(\varepsilon_i) \sigma(v) = v,$$

showing the surjectivity of α_V. The proof of injectivity is similar to the proof of Artin's independence of characters. Observe that it is enough to show that every finite family of vectors in V^G, linearly independent over K, remain linearly independent over L. Assume $\{v_1, \ldots, v_r\}$ is one such family in V^G and assume the following is a non-trivial linear dependence relation over L

$$\lambda_1 v_1 + \cdots + \lambda_r v_r = 0 \tag{$*$}$$

with $\lambda_i \in L$. Assume that this is a relation such that the number of λ_i's which are non-zero is minimal. We may assume, upto reindexing and rescaling, that $\lambda_1 = 1$. Then, applying $\sigma - 1_G$ to the relation above, for any $\sigma \in G$, gives us

$$(\sigma(\lambda_2) - \lambda_2) v_2 + \cdots + (\sigma(\lambda_r) - \lambda_r) v_r = 0$$

Our assumption of minimality gives us that this relation is trivial, and therefore $\sigma(\lambda_i) = \lambda_i$ for $i \geqslant 2$. Since σ was arbitrary, we conclude $\lambda_i \in K$ and hence ($*$) is a non-trivial independence relation over K, contradicting our assumption. Thus, $\{v_1, \ldots, v_r\}$ is K-linear independent and therefore α_V is injective. $\square$

We resume our focus on central simple algebras over K. In what follows all algebras over K are finite-dimensional and all field extensions $L|K$ are assumed to be Galois.

Definition 2.3.22 (Twisted Forms)**.** Given a pair of K-algebras A and A' and a field extension $L|K$, we say B is a *$(L|K)$-twisted form* of A if $A \otimes_K L \cong A' \otimes_K L$ as L-algebras.

Remark 2.3.23. Remark 2.3.7 tells us that central simple algebras are precisely the $(\overline{K}|K)$-twisted forms of $\mathrm{M}_n(\overline{K})$ for some n.

We relate twisted forms to cohomology.

Example 2.3.24 (Twisted Forms give Cocycles)**.** Consider A a K-algebra, $L|K$ a field extension and let $A_L = A \otimes_K L$. Then any $\sigma \in G_{L|K}$ induces an L-algebra automorphism $A_L \to A_L \colon a \otimes x \mapsto a \otimes \sigma(x)$ which we also denote as σ. Moreover, if A' is another K-algebra and we have an L-algebra isomorphism $f \colon A_L \to A'_L$, then we obtain an L-algebra isomorphism ${}^{\sigma}f \colon A_L \to A'_L$. One sees that $(\sigma, f) \mapsto {}^{\sigma}f$ describes a left action of $G_{L|K}$ on $\mathrm{Aut}_L(A_L)$, the L-algebra automorphism group of A_L.

Once again, assume we are given an isomorphism $\alpha \colon A_L \xrightarrow{\sim} A'_L$, then we get a function $c_\alpha \colon G_{L|K} \to \mathrm{Aut}_L(A_L)$, $\sigma \mapsto a_\sigma = \alpha^{-1} \circ {}^{\sigma}\alpha$, then one may verify that

$$a_{\sigma\tau} = a_\sigma \circ {}^{\sigma}a_\tau$$

Therefore $c_\alpha \in \mathrm{Z}^1(G_{L|K}, \mathrm{Aut}_L(A_L))$. Moreover, if $\beta \colon A_L \xrightarrow{\sim} A'_L$ is another isomorphism giving the 1-cocycle $c_\beta \colon \sigma \mapsto b_\sigma = \beta^{-1} \circ {}^{\sigma}\beta$, then

$$a_\sigma = x^{-1} \circ b_\sigma \circ {}^{\sigma}x$$

where $x = \beta^{-1} \circ \alpha$, i.e., c_α and c_β are cohomologous.

Theorem 2.3.25. *For a K-algebra A, consider the pointed set $\mathrm{TF}_L(A)$ of twisted $(L|K)$-forms of A, where the base point is given by (A, id_A). Then the map $(B, \alpha) \mapsto [c_\alpha]$, defined in Example 2.3.24 yields a isomorphism of pointed sets*

$$\mathrm{TF}_L(A) \cong \mathrm{H}^1(G_{L|K}, \mathrm{Aut}_L(A_L)).$$

Proof. We sketch the inverse. A similar argument to the one given in Theorem 2.2.16 tells us that given a cocycle $c\colon G_{L|K} \to \mathrm{Aut}_L(A_L)$, we can equip $\mathrm{Aut}_L(A_L)$ with a semilinear action of $G_{L|K}$. Define $A' = A_L^{G_{L|K}}$ to be the $G_{L|K}$-fixed points of A under this semilinear action. We now prove that this is indeed a form of A, but this follows from the fact that the extension of scalars $\alpha\colon A' \otimes_K L \to A_L$ is an isomorphism of vector spaces by the proof of Proposition 2.3.21 (Hilbert's Theorem 90), but α is an algebra map, finishing the proof. □

Let $L|K$ be a Galois extension. Observe that $\mathrm{M}_n(L) = \mathrm{M}_n(K) \otimes_K L$ and that $\mathrm{TF}_L(\mathrm{M}_n(L))$ is the set of K-central simple algebras of degree n split by L. To emphasise that, let $\mathrm{CSA}_K(n, L) = \mathrm{TF}_L(\mathrm{M}_n(L))$.

Proposition 2.3.26 (Skolem-Noether)**.** *The K-algebra automorphism group of a K-central simple algebra A is naturally isomorphic to $A^\times/K^\times$.*

Proof. Let L be a splitting field of A, then $A \otimes_K L = \mathrm{M}_n(L)$ and $(A \otimes_K L)^\times = \mathrm{GL}_n(L)$ has as its centre $L^\times$. It is a general fact that we have the following short exact sequence

$$1 \to L^\times \to \mathrm{GL}_n(L^\times) \to \mathrm{Inn}(\mathrm{M}_n(L)) \to 1$$

where $\mathrm{Inn}(\mathrm{M}_n(L))$ is the subgroup of inner automorphisms of $\mathrm{M}_n(L)$ and the latter map is $P \mapsto (M \mapsto PMP^{-1})$. See [GS17, Lemma 2.4.1] for a proof that all automorphisms of $\mathrm{M}_n(L)$ are inner, giving us $\mathrm{Aut}_L(\mathrm{M}_n(L)) = \mathrm{Inn}(\mathrm{M}_n(L))$. The following short exact sequence of discrete groups is $G_{L|K}$-equivariant

$$1 \to L^\times \to (A \otimes_K L)^\times \to \mathrm{Aut}_L(A \otimes_K L) \to 1$$

then by Proposition 2.2.9 we get

$$1 \to K^\times \to A^\times \to \mathrm{Aut}_K(A) \to \mathrm{H}^1(G_{L|K}, L^\times)$$

Since $\mathrm{H}^1(G_{L|K}, L^\times)$ is trivial by Proposition 2.3.21 (Hilbert's Theorem 90), we obtain that $\mathrm{Aut}_K(A) \cong A^\times / K^\times$. □

Corollary 2.3.27. *The L-algebra automorphism group of* $\mathrm{M}_n(L)$ *is the projective general linear group* $\mathrm{PGL}_n(L) := \mathrm{GL}_n(L)/L^\times$*, where* $L^\times$ *is identified with the subgroup of scalar matrices in* $\mathrm{GL}_n(L)$ *which is its centre. Moreover,* $\mathrm{PGL}_n(L)$ *has an action of* $G_{L|K}$ *as the automorphism group of* $\mathrm{M}_n(L)$*, then* $\mathrm{PGL}_n(L)^{G_{L|K}} = \mathrm{PGL}_n(K)$*.*

Proof. Apply the proof of Proposition 2.3.26 to $A = \mathrm{M}_n(K)$. □

Corollary 2.3.27 and Theorem 2.3.25 give us

Corollary 2.3.28. *There is an isomorphism of pointed sets*

$$\mathrm{CSA}_K(n, L) \cong \mathrm{H}^1(G_{L|K}, \mathrm{PGL}_n(L)).$$

We use Corollary 2.3.28 to give an interesting class of central simple algebras.

Definition 2.3.29. Let $L|K$ be a cyclic Galois extension of degree m. We fix an isomorphism $\chi \colon G_{L|K} \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$, which is a character of $G_{L|K}$, and some $a \in K^\times$

be given. We associate to χ and a a central simple algebra over K which is a $(L|K)$-twisted form of $\mathrm{M}_m(K)$ called a *cyclic algebra*, which is denoted as (χ, a). Consider the matrix,

$$\widetilde{U}_a = \begin{pmatrix} 0 & a \\ \mathrm{Id}_{m-1} & 0 \end{pmatrix} \in \mathrm{GL}_m(K)$$

Let its image in $\mathrm{PGL}_m(K)$ be denoted as U_a. A straightforward computation shows that U_a has exact order m in $\mathrm{PGL}_m(K)$, in particular, $\widetilde{U}_a^m = a \cdot \mathrm{Id}_m$. The following,

$$\begin{array}{ccccccc} G_{L|K} & \xrightarrow{\chi} & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \mathrm{PGL}_m(K) & \hookrightarrow & \mathrm{PGL}_m(L) \\ & & 1 & \longmapsto & U_a & & \end{array}$$

gives a 1-cocycle in $\mathrm{Z}^1(G_{L|K}, \mathrm{PGL}_m(L))$. Via the isomorphism in Corollary 2.3.28, this cocycle corresponds to a K-central simple algebra of degree m split by L, denoted (χ, a).

Let $L|K$ be as in Definition 2.3.29.

Proposition 2.3.30. *The class of the cyclic algebra (χ, a) is trivial in $\mathrm{Br}(L|K)$ if and only if a is a norm from L, i.e., $a \in \mathrm{N}_{L|K}(L^\times)$.*

Proof. See [GS17, Corollary 4.7.5]. □

Moreover, cyclic algebras exhaust $\mathrm{Br}(L|K)$ for cyclic Galois extensions $L|K$. Once again, let $L|K$ be as in Definition 2.3.29.

Proposition 2.3.31. *Let $[A] \in \mathrm{Br}(L|K)$, then*

- *There exist $\chi\colon G_{L|K} \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$ and $a \in K$ such that $[A] = [(\chi, a)]$.*

- *Moreover, if A has degree m, then $A \cong (\chi, a)$.*

Proof. See [GS17, Corollary 4.7.6]. $\square$

2.3.3 Cohomological Brauer group

In this section we give two cohomological interpretations of the Brauer group. We let $L|K$ continue to be a Galois extension.

Remark 2.3.32. The following natural maps

$$\mathrm{GL}_m(L) \times \mathrm{GL}_n(L) \to \mathrm{GL}_{mn}(L), \qquad (\varphi, \psi) \mapsto \varphi \otimes \psi$$

$$\mathrm{GL}_m(L) \hookrightarrow \mathrm{GL}_{mn}(L), \qquad \psi \mapsto \psi \otimes \mathrm{Id}_n$$

descend to maps on PGL, giving the following maps of cohomology sets

$$\mathrm{H}^1(G, \mathrm{PGL}_m(L)) \times \mathrm{H}^1(G, \mathrm{PGL}_n(L)) \to \mathrm{H}^1(G, \mathrm{PGL}_{mn}(L)) \tag{2.1}$$

$$\lambda_{mn} \colon \mathrm{H}^1(G, \mathrm{PGL}_n(L)) \to \mathrm{H}^1(G, \mathrm{PGL}_{mn}(L)) \tag{2.2}$$

These maps correspond to the following maps

$$- \otimes - \colon \mathrm{CSA}_K(n, L) \times \mathrm{CSA}_K(m, L) \to \mathrm{CSA}_K(m, L)$$

$$- \otimes \mathrm{M}_n(K) \colon \mathrm{CSA}_K(m, L) \to \mathrm{CSA}_K(mn, L)$$

The maps λ_{mn} in (2.2) form a directed system, and we define

$$\mathrm{H}^1(G_{L|K}, \mathrm{PGL}_\infty(L)) := \varinjlim \mathrm{H}^1(G_{L|K}, \mathrm{PGL}_n(L))$$

This is an abelian group equipped with the multiplication described in (2.1).

Inflation gives us

$$\mathrm{H}^1(G_K, \mathrm{PGL}_\infty(\overline{K})) = \varinjlim \mathrm{H}^1(G_{L|K}, \mathrm{PGL}_\infty(L))$$

as L ranges throught finite Galois extensions of K.

Proposition 2.3.33. *There are natural group isomorphisms*

$$\mathrm{H}^1(G_{L|K}, \mathrm{PGL}_\infty(L)) \cong \mathrm{Br}(L|K) \quad \textit{and} \quad \mathrm{H}^1(G_K, \mathrm{PGL}_\infty(\overline{K})) \cong \mathrm{Br}(K).$$

We now give the second interpretation, identifying the Brauer group with a more familiar cohomology group.

Recall the following exact sequence of groups equipped with a $G_{L|K}$-action

$$1 \to L^\times \to \mathrm{GL}_m(L) \to \mathrm{PGL}_m(L) \to 1$$

Applying Proposition 2.2.9 we get an exact sequence of pointed sets

$$\mathrm{H}^1(G_{L|K}, \mathrm{GL}_m(L)) \to \mathrm{H}^1(G_{L|K}, \mathrm{PGL}_m(L)) \xrightarrow{\delta_{m,L}} \mathrm{H}^2(G_{L|K}, L^\times)$$

Triviality of the set $\mathrm{H}^1(G_{L|K}, \mathrm{GL}_m(L))$ by Proposition 2.3.21 (Hilbert's Theorem 90) means $\delta_{m,L}$ is injective. Moreover, $\delta_{m,L}$ induces the following isomorphism [GS17, Theorem 4.4.5].

Theorem 2.3.34. *There is an isomorphism of abelian group*

$$\mathrm{H}^1(G_{L|K}, \mathrm{PGL}_\infty(L)) \cong \mathrm{H}^2(G_{L|K}, L^\times).$$

Furthermore, $\delta_{m,L}$, as L ranges through finite Galois extensions, collect to form a system, inducing the following isomorphisms [GS17, Theorem 4.4.7].

Theorem 2.3.35. *Let K be a field, $L|K$ a Galois extension and G_K the absolute Galois group of K. Then there exist natural isomorphisms of abelian groups*

$$\mathrm{Br}(L|K) \cong \mathrm{H}^2(G_{L|K}, L^\times) \quad \textit{and} \quad \mathrm{Br}(K) \cong \mathrm{H}^2(G_K, \overline{K}^\times).$$

Describing the Brauer group as a cohomology group allows us to use cohomological machinery to give structural results about the Brauer group.

Theorem 2.3.36. *Let $L|K$ be a finite Galois extension. Then each element of the relative Brauer group $\mathrm{Br}(L|K)$ has order dividing the degree $[L : K]$. Consequently, the Brauer group $\mathrm{Br}(K)$ is a torsion abelian group.*

Proof. This follows from general facts about group cohomology, see [GS17, Corollary 3.3.8]. □

Definition 2.3.37. For a central simple K-algebra A, its period $\mathrm{per}_K(A)$ is the order of its Brauer class $[A]$ in $\mathrm{Br}(K)$. Recall $\mathrm{ind}_K(A)$, the Schur index of A (Definition 2.3.9). It is known that $\mathrm{per}_K(A) \mid \mathrm{ind}_K(A)$ and that $\mathrm{per}_K(A)$ and $\mathrm{ind}_K(A)$ share the same prime factors. When K is a local or global field, we know that $\mathrm{per}_K(A) = \mathrm{ind}_K(A)$ by the work of Albert, Brauer, Hasse, and Noether; see [GS17, Remark 6.5.6].

Observe that if one considers $A \in \mathrm{CSA}_K(n, \overline{K})$, then necessarily $\mathrm{ind}_K(A) \mid n$. Since $\mathrm{per}_K(A) \mid \mathrm{ind}_K(A)$, we conclude that $[A] \in \mathrm{Br}(K)[n]$, the subgroup of n-torsion of the Brauer group, i.e., elements whose order divides n. That is, the inclusion induced by the connecting map $\mathrm{H}^1(G_K, \mathrm{PGL}_n(\overline{K})) \hookrightarrow \mathrm{H}^2(G_K, \overline{K}^\times) = \mathrm{Br}(K)$ factors through an inclusion $\mathrm{H}^1(G_K, \mathrm{PGL}_n(\overline{K})) \hookrightarrow \mathrm{Br}(K)[n]$.

In particular, if $\mathrm{per}_K(A) = \mathrm{ind}_K(A)$, as is the case when K is a local or global field, then one can argue that $\mathrm{H}^1(G_K, \mathrm{PGL}_n(\overline{K})) \cong \mathrm{Br}(K)[n]$.

We now give a cohomological argument for $\mathrm{H}^1(G_K, \mathrm{PGL}_n(\overline{K})) \hookrightarrow \mathrm{Br}(K)[n]$ in the case n is invertible in K.

Remark 2.3.38. For a positive integer $n \in K^\times$, consider

$$1 \longrightarrow \mu_n \longrightarrow \overline{K}^\times \xrightarrow{(\cdot)^n} \overline{K}^\times \longrightarrow 1$$

where μ_n is the group of n^{th}-roots of unity, which is G_K-equivariant. Then, by Proposition 2.2.9 and Remark 2.2.10 we get the exact sequence

$$\mathrm{H}^1(G_K, \overline{K}^\times) \longrightarrow \mathrm{H}^2(G_K, \mu_n) \longrightarrow \mathrm{H}^2(G_K, \overline{K}^\times) \xrightarrow{[n]} \mathrm{H}^2(G_K, \overline{K}^\times)$$

where $[n]$ denotes the multiplication-by-n map (we think of $\mathrm{Br}(K) = \mathrm{H}^2(G_K, \overline{K}^\times)$ as an additive group). Since $\mathrm{H}^1(G_K, \overline{K}^\times)$ is trivial by Proposition 2.3.21 (Hilbert's Theorem 90) we get that the subgroup of n-torsion

$$\mathrm{Br}(K)[n] = \mathrm{H}^2(G_K, \mu_n).$$

Consider $\mathrm{SL}_n(\overline{K})$, whose centre is $\mu_n \cdot \mathrm{Id}_n$, define the *projective special linear group* as $\mathrm{PSL}_n(\overline{K}) := \mathrm{SL}_n(\overline{K})/\mu_n$. Since $\overline{K}$ is algebraically closed, we observe that $\mathrm{PGL}_n(\overline{K}) = \mathrm{PSL}_n(\overline{K})$. Therefore, we may consider the following (map of) short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mu_n \arrow[r] \arrow[d, hook] & \mathrm{SL}_n(\overline{K}) \arrow[r] \arrow[d, hook] & \mathrm{PGL}_n(\overline{K}) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \overline{K}^\times \arrow[r] & \mathrm{GL}_n(\overline{K}) \arrow[r] & \mathrm{PGL}_n(\overline{K}) \arrow[r] & 1
\end{tikzcd} \tag{2.3}$$

which induces the following (Proposition 2.2.9)

$$\begin{tikzcd}
\mathrm{H}^1(G_K, \mathrm{PSL}_n(\overline{K})) \arrow[r] \arrow[d, equal] & \mathrm{H}^2(G_K, \mu_n) = \mathrm{Br}(K)[n] \arrow[d, hook] \\
\mathrm{H}^1(G_K, \mathrm{PGL}_n(\overline{K})) \arrow[r, hook] & \mathrm{H}^2(G_K, \overline{K}^\times) = \mathrm{Br}(K)
\end{tikzcd} \tag{2.4}$$

allowing us to conclude that the bottom map in (2.4) factors through the top map and is necessarily injective.

2.3.4 Brauer Group of a Local Field

Theorem 2.3.39. *Let F be a local field. Then the* Hasse invariant *is a canonical embedding*

$$\mathsf{Inv}\colon \mathrm{Br}(F) \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

where

- *If $F = \mathbb{C}$, then $\mathsf{Inv} = 0$, since $\mathbb{C}$ is algebraically closed.*
- *If $F = \mathbb{R}$, then Inv induces $\mathrm{Br}(F) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, where the non-trivial class in $\mathrm{Br}(\mathbb{R})$ is represented by Hamiltonian quaternions $\mathbb{H}$.*
- *If F is non-archimedean, then Inv is an isomorphism.*

While the results below are true for any local field F, we restrict our attention to the case when F is a p-adic field.

The Hasse invariant enjoys the following functoriality property.

Proposition 2.3.40. *Let $E|F$ be a finite extension, then there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Br}(F) \arrow[r, "\mathsf{Inv}"] \arrow[d, "-\otimes_F E"'] & \mathbb{Q}/\mathbb{Z} \arrow[d, "{[E:F]\times -}"] \\
\mathrm{Br}(E) \arrow[r, "\mathsf{Inv}"'] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Proof. This is [Gui18, Theorem 8.10] □

Corollary 2.3.41. *Let $E|F$ be a finite extension, then* $\mathrm{Br}(E|F)$ *is cyclic of order* $[E : F]$.

Assume $E|F$ is a cyclic Galois extension with the character

$$\chi\colon G_{E|F} \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z} \cong \frac{1}{m}\mathbb{Z}/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z},$$

and some $a \in F^{\times}$. Then, we obtain the cyclic algebra $(\chi, a) \in \mathrm{CSA}_F(m, E)$, see Definition 2.3.29.

Recall the local reciprocity isomorphism $\theta_{E|F}$ (Theorem 2.1.28).

Proposition 2.3.42. $\mathsf{Inv}([(\chi, a)]) = \chi(\theta_{E|F}(a))$

Proof. [GS17, Proposition 4.7.3] tells us that our (χ, a) coincides with the one in [Ser79, Chapter XIV, §1]. Now the result follows from Proposition 3 in [Ser79, Chapter XIV, §1]. □

Let $\sigma \in G_{E|K}$ be such that $\chi(\sigma) = \dfrac{1}{m} \bmod 1$ and k is such that $\theta_{E|K}(a) = \sigma^k$. Then

$$\mathsf{Inv}([(\chi, a)]) = \frac{k}{m} \bmod 1$$

Chapter 3

Descent & Decompletion

Throughout this chapter F denotes a p-adic field and G_F is its absolute Galois group. $\mathbb{C}_F$ is the completion of an algebraic closure of F, and $\chi = \chi_F \colon G_F \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character (Definition 2.1.34).

The group $H = \ker \chi$ is a closed subgroup and cuts out the extension $F(\zeta_{p^\infty})$, an infinitely ramified extension obtained by adjoining all p-power roots of unity to F. Recall $\Gamma = G_F/H = \mathrm{Gal}(F(\zeta_{p^\infty})|F)$ is the Galois group $F_\infty|F$. Therefore it contains a subgroup of finite index $\Gamma_0 \cong (\mathbb{Z}_p, +)$ such that $\Gamma = \Sigma \times \Gamma_0$, where Σ is a finite abelian group with order $= [\Gamma : \Gamma_0]$ prime to p.

Let γ_0 be a topological generator of Γ_0, and let $\Gamma_r := \Gamma_0^{p^r}$ be the subgroup topologically generated by $\gamma_r = \gamma_0^{p^r}$. The intermediate extension F_∞ cut out by Σ is such that $F_\infty|F$ is totally ramified, $\mathrm{Gal}(F(\zeta_{p^\infty})|F_\infty) = \Sigma$ and $\mathrm{Gal}(F_\infty|F) = \Gamma_0$, i.e., F_∞ is a $\mathbb{Z}_p$-extension. Also define $F_r := F_\infty^{\Gamma_r}$, which are cyclic extensions of F of order p^r where $\mathrm{Gal}(F_r|F) = \Gamma_0/\Gamma_r \cong \mathbb{Z}/p^r\mathbb{Z}$.

Let $\widetilde{H} \leqslant G_F$ be such that $\Gamma_0 = G_F/\widetilde{H}$. We also observe that $\mathbb{C}_F^H = \widehat{F(\zeta_{p^\infty})}$ and $\mathbb{C}_F^{\widetilde{H}} = \widehat{F}_\infty$, by Theorem 2.1.42 (Ax–Sen–Tate).

Notation 3.0.1. The absolute Galois group G_F will always be considered as a topological group, equipped with its profinite topology. Note that any extension $E|F$ can be equipped with two topologies – the natural metric topology induced by its valuation or the discrete topology, and this distinction will matter when we consider the cohomology of G_F valued in these fields. We recall that G_F acts continuously on E with its valuation topology.

Let X_E denote either E, $E^\times$, $\mathrm{GL}_n(E)$ or $\mathrm{PGL}_n(E)$, which are groups that our cohomology will be valued in. Once we fix a topology on E, the remaining X_E's inherit an induced topology: see Remark 2.2.14 for the induced topology on $E^\times$ and $\mathrm{GL}_n(E)$, and $\mathrm{PGL}_n(E)$ is given the quotient topology. When E has the discrete topology, so does any X_E.

The undecorated $\mathrm{H}^i(G_F, X_E)$ will denote cohomology where X_E has the (induced) valuation topology and we will use results from Sections 2.2.2 and 2.2.3. On the other hand, $\mathrm{H}^i_{\mathrm{disc}}(G_F, X_E)$ will denote cohomology where X_E has the discrete topology, where results from Sections 2.3.2 and 2.3.3 will come in handy.

Finally, observe that if G' is a finite subgroup (resp. quotient) of G_F, then its subspace (resp. quotient) topology is the discrete topology. Therefore, if this G' acts on E, then we have $\mathrm{H}^i(G', X_E) = \mathrm{H}^i_{\mathrm{disc}}(G', X_E)$.

The distinction in Notation 3.0.1 is necessary. For example, consider the groups $\mathrm{H}^1(G_F, \overline{F}^\times)$ and $\mathrm{H}^1_{\mathrm{disc}}(G_F, \overline{F}^\times)$. The latter is trivial by Proposition 2.3.21 (Hilbert's Theorem 90), but the former is not as it contains a non-trivial cocycle represented by the p-adic cyclotomic character χ.

Lemma 3.0.2. *The p-adic cyclotomic character $\chi\colon G_F \to \overline{F}^\times$ is not a* 1*-coboundary*

Proof. Assume χ is a 1-coboundary, and therefore

$$\chi(\sigma) = x^{-1} \cdot {}^{\sigma}x$$

for all $\sigma \in G_F$, for some $x \in \overline{F}^\times$. Necessarily x belongs to a finite extension $E|F$, and we may apply the field norm $\mathrm{N}_{E|F}$ to the above expression

$$\chi(\sigma)^{[E:F]} = \mathrm{N}_{E|F}(\chi(\sigma)) = \mathrm{N}_{E|F}(x)^{-1} \cdot \mathrm{N}_{E|F}({}^{\sigma}x) = 1$$

for all $\sigma \in G_F$, a contradiction since χ does not have finite order. $\square$

Our main object of study is the following short exact sequence of topological groups equipped with a continuous action of G_F

$$1 \longrightarrow \mathbb{C}_F^\times \longrightarrow \mathrm{GL}_m(\mathbb{C}_F) \longrightarrow \mathrm{PGL}_m(\mathbb{C}_F) \longrightarrow 1 \tag{3.1}$$

which gives the following sequence in cohomology by Proposition 2.2.9

$$\begin{array}{ccccc} \mathrm{H}^1(G_F, \mathbb{C}_F^\times) & \rightarrow & \mathrm{H}^1(G_F, \mathrm{GL}_m(\mathbb{C}_F)) & \rightarrow & \mathrm{H}^1(G_F, \mathrm{PGL}_m(\mathbb{C}_F)) \\ & & & & \downarrow \\ & & & & \mathrm{H}^2(G_F, \mathbb{C}_F^\times) \end{array} \tag{3.2}$$

We recall from Theorem 2.2.16 that the set $\mathrm{H}^1(G_F, \mathrm{GL}_m(\mathbb{C}_F))$ classifies $\mathbb{C}_F$-linear isomorphism classes of objects in $\mathsf{Rep}^{\mathrm{f}}_{\mathbb{C}_F}(G_F)$ of dimension m. Similarly, by following the proof of Theorem 2.2.16, we can show that $\mathrm{H}^1(G_F, \mathrm{PGL}_m(\mathbb{C}_F))$ classifies the possible continuous semilinear actions of G_F on the matrix algebra $\mathrm{M}_m(\mathbb{C}_F)$. On this level, the maps in the sequence (3.2) correspond to

$$L \mapsto L^{\oplus m} \quad \text{and} \quad V \mapsto \mathrm{End}(V)$$

where L is a one-dimensional $\mathbb{C}_F$-semilinear representation of G_F, and V is an m-dimensional $\mathbb{C}_F$-semilinear representation of G_F. Moreover, the $\mathrm{H}^1(G_F, \mathbb{C}_F^\times)$-action on the pointed set $\mathrm{H}^1(G_F, \mathrm{GL}_m(\mathbb{C}_F))$ corresponds to $(L, V) \mapsto L \otimes_{\mathbb{C}_F} V$.

3.1 Sen Theory

In this section, we exposit the theory of Sen on $\mathbb{C}_F$-semilinear representations of G_F. An excellent reference is Sen's original paper [Sen81], and much of our exposition is based on [BC09].

Sen, in [Sen81], establishes an equivalence between the categories $\mathsf{Rep}^{\mathrm{f}}_{\mathbb{C}_F}(G_F)$ and $\mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma)$, which we recall are rigid (closed) symmetric monoidal abelian categories. We record this equivalence in the following theorem, a reference for which is [BC09, Theorem 15.1.2, Corollary 15.1.6]

Theorem 3.1.1. *There exists a quasi-inverse* $\mathrm{D}_{\mathsf{Sen}} \colon \mathsf{Rep}^{\mathrm{f}}_{\mathbb{C}_F}(G_F) \to \mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma_0)$ *to the extension of scalars functor* $(-) \otimes_{F_\infty} \mathbb{C}_F$*. Moreover, for any* $V \in \mathsf{Rep}^{\mathrm{f}}_{\mathbb{C}_F}(G_F)$*, there exists some* r *such that* $\mathrm{D}_{\mathsf{Sen}}(V) \in \mathsf{Rep}^{\mathrm{f}}_{F_r}(\Gamma_0)$*. The functor* $\mathrm{D}_{\mathsf{Sen}}$ *is a composition of two equivalences:*

- (Descent) *The functor* $(\cdot)^{\widetilde{H}} \colon \mathsf{Rep}^{\mathrm{f}}_{\mathbb{C}_F}(G_F) \to \mathsf{Rep}^{\mathrm{f}}_{\widehat{F}_\infty}(\Gamma_0)$ *of* $\widetilde{H}$*-invariance is an equivalence, quasi-inverse to extension of scalars* $(-) \otimes_{\widehat{F}_\infty} \mathbb{C}_F$*.*
- (Decompletion) *There exists a quasi-inverse* $\mathrm{D}'_{\mathsf{Sen}} \colon \mathsf{Rep}^{\mathrm{f}}_{\widehat{F}_\infty}(\Gamma_0) \to \mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma_0)$ *to extension of scalars* $(-) \otimes_{F_\infty} \widehat{F}_\infty$*. Moreover, for any* $W \in \mathsf{Rep}^{\mathrm{f}}_{\widehat{F}_\infty}(\Gamma_0)$*, there exists some* r *such that* $\mathrm{D}'_{\mathsf{Sen}}(W) \in \mathsf{Rep}^{\mathrm{f}}_{F_r}(\Gamma_0)$*.*

Moreover,

- $\mathrm{D}_{\mathsf{Sen}}(V_1 \oplus V_2) \cong \mathrm{D}_{\mathsf{Sen}}(V_1) \oplus \mathrm{D}_{\mathsf{Sen}}(V_2)$
- $\mathrm{D}_{\mathsf{Sen}}(V_1 \otimes_{\mathbb{C}_F} V_2) \cong \mathrm{D}_{\mathsf{Sen}}(V_1) \otimes_{F_\infty} \mathrm{D}_{\mathsf{Sen}}(V_2)$
- $\mathrm{D}_{\mathsf{Sen}}(\mathrm{Hom}_{\mathbb{C}_F}(V_1, V_2)) \cong \mathrm{Hom}_{F_\infty}(\mathrm{D}_{\mathsf{Sen}}(V_1), \mathrm{D}_{\mathsf{Sen}}(V_2))$

For any $W \in \mathsf{Rep}^{\mathrm{f}}_{\widehat{F}_\infty}(\Gamma_0)$, we sketch the construction of $\mathrm{D}'_{\mathsf{Sen}}(W)$. Define a vector $w \in W$ to be *F-finite*, if its translates by Γ_0 generate an F-vector space of finite dimension. Then $\mathrm{D}'_{\mathsf{Sen}}(W)$ is defined to be the subspace of F-finite vectors; it is a F_∞-subspace of W on which Γ_0 acts.

We translate Theorem 3.1.1 into a cohomological statement.

Corollary 3.1.2. *The natural map* $\mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(F_\infty)) \to \mathrm{H}^1(G_F, \mathrm{GL}_m(\mathbb{C}_F))$*, induced by* $G_F \twoheadrightarrow \Gamma_0$ *and* $\mathrm{GL}_m(F_\infty) \hookrightarrow \mathrm{GL}_m(\mathbb{C}_F)$ *is an isomorphism for all* $m \geqslant 1$*. It is a composition of two isomorphisms*

- (Descent) Inf: $\mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(\widehat{F}_\infty)) \to \mathrm{H}^1(G_F, \mathrm{GL}_m(\mathbb{C}_F))$ *is an isomorphism.*
- (Decompletion) *There natural map* $\mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(F_\infty)) \to \mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(\widehat{F}_\infty))$ *induced by* $\mathrm{GL}_m(F_\infty) \hookrightarrow \mathrm{GL}_m(\widehat{F}_\infty)$ *is an isomorphism.*

Moreover, if $U \colon \Gamma_0 \to \mathrm{GL}_m(F_\infty)$ *is a* 1*-cocycle, then for large enough* r*, one has* $U(\sigma) = U_\sigma \in \mathrm{GL}_m(F_r)$ *for every* $\sigma \in \Gamma_0$*.*

Remark 3.1.3. In particular, one can even show that

$$\mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(F_\infty)) = \varinjlim \mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(F_r)).$$

Since we will not use this fact, we do not provide a proof.

Remark 3.1.4. Sen in [Sen81] proves his results for the case where F_∞ is replaced by $F(\zeta_{p^\infty})$. We justify why we can state our results with F_∞. Observe that the inflation-restriction sequence (Proposition 2.2.12) gives us

$$1 \longrightarrow \mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(F_\infty)) \xrightarrow{\mathrm{Inf}} \mathrm{H}^1(\Gamma, \mathrm{GL}_m(F(\zeta_{p^\infty}))) \xrightarrow{\mathrm{Res}} \mathrm{H}^1(\Sigma, \mathrm{GL}_m(F(\zeta_{p^\infty})))$$

where we recall that $\Gamma = \Sigma \times \Gamma_0$. Since Σ is finite, we have $\mathrm{H}^1(\Sigma, \mathrm{GL}_m(F(\zeta_{p^\infty}))) = \mathrm{H}^1_{\mathrm{disc}}(\Sigma, \mathrm{GL}_m(F(\zeta_{p^\infty})))$, which is trivial by Proposition 2.3.21 (Hilbert's Theorem 90). Hence Inf is an isomorphism, which tells us that Σ-invariance $(\cdot)^\Sigma$ is quasi-inverse to $(-) \otimes_{F_\infty} F(\zeta_{p^\infty})$.

To understand objects $D \in \mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma_0)$, Sen shows that one can differentiate the semilinear action of Γ_0 on D and obtain a F_∞-linear endomorphism called the *Sen operator*.

Theorem 3.1.5 (Sen81, Theorem 4)**.** *For each $D \in \mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma_0)$ there is a unique F_∞-linear endomorphism Θ_D on D such that for all $x \in D$ we have*

$$\sigma(x) = \exp(\log(\chi(\gamma)) \cdot \Theta_D)(x)$$

for all γ in an open subgroup Γ_x of Γ.

Proof. We sketch the proof of existence. Let $\boldsymbol{e} = (e_1, \ldots, e_m)$ be an F_∞-basis of D, and using this we may define the corresponding cocycle $U = U_{\boldsymbol{e}} \colon \Gamma_0 \to \mathrm{GL}_m(F_\infty)$. We know that we have $U_\gamma = U(\gamma) \in \mathrm{GL}_m(F_r)$, for some r, for all $\gamma \in \Gamma_0$. Restricting U to Γ_r, which fixes F_r, we have that $U : \Gamma_r \to \mathrm{GL}_m(F_r)$ is a continuous homomorphism, and thus describes a continuous linear representation of Γ_r over F_r. We define,

$$\Theta_D = \frac{\log(U_\gamma)}{\log(\chi(\gamma))} \in \mathrm{M}_m(F_r) \subseteq \mathrm{M}_m(F_\infty)$$

for all $\gamma \in \Gamma_r \setminus \{1\}$. Shrinking Γ_r so that exp is well-defined, we obtain the expression in the statement of the theorem. □

We have the following facts about the Sen operator.

Lemma 3.1.6. *For $D_1, D_2 \in \mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma_0)$, the Sen operators on $D_1 \oplus D_2$, $D_1 \otimes_{F_\infty} D_2$ and $\mathrm{Hom}_{F_\infty}(D_1, D_2)$ are respectively given by*

- $\Theta_{D_1 \oplus D_2} = \Theta_{D_1} \oplus \Theta_{D_2}$
- $\Theta_{D_1 \otimes_{F_\infty} D_2} = \Theta_{D_1} \otimes 1 + 1 \otimes \Theta_{D_2}$
- $\Theta_{\mathrm{Hom}_{F_\infty}(D_1, D_2)}(T) = \Theta_{D_2} \circ T - T \circ \Theta_{D_1}$

Let $\mathcal{S}_{F_\infty}$ be the category of finite-dimensional F_∞-vector spaces equipped with a linear endomorphism. Then, we have the functor

$$\mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma_0) \to \mathcal{S}_{F_\infty} \colon D \mapsto (D, \Theta_D).$$

This functor is not an equivalence and neither is it fully faithful, but it detects isomorphisms.

Proposition 3.1.7. *Objects $D_1, D_2 \in \mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma_0)$ are isomorphic if and only if (D_1, Θ_{D_1}) and (D_2, Θ_{D_2}) are isomorphic in $\mathcal{S}_{F_\infty}$.*

Example 3.1.8. Let $D = F_\infty(n)$ be the Hodge-Tate representation of weight n, then one sees that Θ_D is the multiplication-by-n operator. More generally, D is the Hodge-Tate representation of weight $\underline{n} = (n_1, \ldots, n_m)$, i.e., isomorphic to $\oplus_{i=1}^m F_\infty(i)^{n_i}$, if and only if Θ_D is diagonalisable with eigenvalues $(n_1, \ldots, n_m)$.

In light of this example, for any object $D \in \mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma_0)$, the eigenvalues of Θ_D are called *generalised Hodge-Tate weights* of D.

3.2 Descent & Decompletion of CSAs

Sen proves that we have an isomorphism

$$\mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(F_\infty)) \to \mathrm{H}^1(G_F, \mathrm{GL}_m(\mathbb{C}_F)).$$

In this section we extend Sen's results to central simple algebras.

3.2.1 Azumaya Monoids

We introduce the general theory of Azumaya objects in symmetric monoidal category. The original reference for this theory is [Par76]. [San14] is an excellent source on the foundations of symmetric monoidal categories.

Definition 3.2.1 (Symmetric Monoidal Categories)**.** A *symmetric monoidal category* is a category $\mathscr{V}$ equipped with a bifunctor $- \otimes - : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$, called the tensor product, and object $\mathbb{1} \in \mathscr{V}$, called the unit, along with natural isomorphisms $(\alpha, \lambda, \rho, \tau)$ which are subject to certain conditions and commutative diagrams as laid out in [San14, Definitions 2.2.1, 2.2.12]. Roughly speaking, α exhibits associativity of $\otimes$, while λ and ρ exhibit $\mathbb{1}$ as a two-sided unit with respect to $\otimes$ and τ exhibits the commutativity of $\otimes$.

Definition 3.2.2 (Lax & Strong Monoidal Functor)**.** Let $(\mathscr{V}, \otimes, \mathbb{1})$ and $(\mathscr{V}', \otimes', \mathbb{1}')$ be symmetric monoidal categories. A functor $F : \mathscr{V} \to \mathscr{V}'$ is said to be *(symmetric) lax monoidal* if it is equipped with morphisms $\varepsilon : \mathbb{1}' \to F(\mathbb{1})$ and $m_{X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y)$ for all $X, Y \in \mathscr{V}$, such that, roughly speaking, F is compatible with the datum $(\alpha, \lambda, \rho, \tau)$ of $\mathscr{V}$ and $(\alpha', \lambda', \rho', \tau')$ of $\mathscr{V}'$. See [San14, Definition 2.2.27] for the appropriate commutative diagrams. F is said to be *strong monoidal* if ε and $m_{X,Y}$ is an isomorphism.

Definition 3.2.3 (Equivalences)**.** An *equivalence of monoidal categories* is a strong monoidal functor that is an equivalence of the underlying categories.

Definition 3.2.4 (Monoids)**.** A *monoid (object)* in a symmetric monoidal category $\mathscr{V}$ is an object $A \in \mathscr{V}$ equipped with morphisms $\mu : A \otimes A \to A$ and $\eta : \mathbb{1} \to A$ in $\mathscr{V}$ such that, roughly speaking, μ is associative and η is a monoid unit for A. See [San14, Definition 2.2.18] for the appropriate commutative diagrams.

Definition 3.2.5 (Morphisms of Monoids)**.** Given two monoids (A, μ, η) and (A', μ', η'), then a morphism $f : A \to A'$ is a morphism of monoids if the following diagrams commutes

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\mu"'] & A' \otimes A' \arrow[d, "\mu'"] \\ A \arrow[r, "f"'] & A' \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{1} \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\ A \arrow[rr, "f"'] & & A' \end{tikzcd}$$

Definition 3.2.6 (Category of $\mathscr{V}$-Monoids)**.** We may define the following category of monoids of a symmetric monoidal category $\mathscr{V}$, denoted $\mathsf{Mon}_{\mathscr{V}}$ where the objects are monoid objects in $\mathscr{V}$ and morphisms are morphism of monoids. It is itself a monoidal category with $\otimes$, but not necessarily symmetric as τ does not restrict to this category. This is exhibited by the following construction.

Definition 3.2.7 (Opposite Monoid)**.** Given a monoid $A = (A, \mu, \eta)$, we define its *opposite monoid* $A^{\mathrm{op}} = (A, \mu', \eta)$, where $\mu' = \mu \circ \tau$. We observe that $A \cong A^{\mathrm{op}}$ in $\mathsf{Mon}_{\mathscr{V}}$ if and only if $\mu = \mu \circ \tau$, in which case such a monoid is said to be *commutative*. The possible existence of non-commutative monoids makes $\mathsf{Mon}_{\mathscr{V}}$ a monoidal category that is not necessarily symmetric.

Lemma 3.2.8. *Let $F : \mathscr{V} \to \mathscr{V}'$ be a lax monoidal functor between two symmetric monoidal categories $\mathscr{V}$ and $\mathscr{V}'$, then there is an induced functor $F : \mathsf{Mon}_{\mathscr{V}} \to \mathsf{Mon}_{\mathscr{V}'}$. In particular, if F is an equivalence, the induced functor on the category of monoids is an equivalence as well.*

Proof. The fact that the functor induces a functor on the category of monoids is clear from definitions. While the statement about equivalence follows from the fact that any quasi-inverse of F may be given the structure of a lax monoidal functor; see [San14, Lemma 2.2.34, Proposition 2.2.42]. □

Definition 3.2.9 (Closedness)**.** A symmetric monoidal category $\mathscr{V}$ is said to be *closed* if it is equipped with a functor

$$[-,-] : \mathscr{V} \times \mathscr{V}^{\mathrm{op}} \to \mathscr{V}$$

called the *internal hom* and an isomorphism

$$\mathrm{Hom}_{\mathscr{V}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathscr{V}}(X, [Y, Z])$$

for any $X, Y, Z \in \mathscr{V}$ and natural in all three.

In particular, for any object Y, one has the adjunction $- \otimes Y \dashv [Y, -]$. The unit and counit of this adjunction are denoted as

$$\mathrm{coev}\colon X \to [Y, X \otimes Y] \quad \text{and} \quad \mathrm{ev}\colon [Y, X] \otimes Y \to X$$

and called the *coevaluation* and *evaluation* maps respectively.

Remark 3.2.10. We may define lax and strong closed functors, similar to lax and strong monoidal functors Definition 3.2.2. One can show that for functors between closed symmetric monoidal categories a lax monoidal functor is equivalent to a lax closed functor. Therefore one can adapt the definition of an equivalence Definition 3.2.3 and Lemma 3.2.8 to closed symmetric monoidal categories by asking the equivalence to be induced by a closed monoidal functor, i.e., a functor that is both strong monoidal and strong closed.

Remark 3.2.11. For any object X in a closed symmetric monoidal category $\mathscr{V}$, then the object $[X, X]$ has the structure of a monoid $([X, X], \circ_X, I_X)$, where

$$\circ_X\colon [X, X] \otimes [X, X] \to [X, X]$$

is the morphism corresponding to

$$([X, X] \otimes [X, X]) \otimes X \xrightarrow{\alpha} [X, X] \otimes ([X, X] \otimes X) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} [X, X] \otimes X \xrightarrow{\mathrm{ev}} X$$

under the tensor-internal hom adjunction. The monoid unit $I_X\colon \mathbb{1} \to [X, X]$ is the coevaluation. $[X, X] = ([X, X], \circ_X, I_X)$ is called the *internal endomorphism monoid* of X.

Definition 3.2.12 (Duals)**.** Let $\mathscr{V}$ be a symmetric monoidal category, then an object $X \in \mathscr{V}$ is said to be *dualisable* if there exists an object $X^\vee$ such that $X^\vee \otimes -$ is right adjoint to $- \otimes X$. Let η and ε be the unit and counit of this adjunction. The object $X^\vee$ is called the *dual* of X.

Note that if $\mathscr{V}$ is closed and we have a dualisable X, then there is a natural isomorphism $[X, -] \cong X^\vee \otimes -$ as they are both right adjoints of $- \otimes X$. Therefore, in particular, $[X, \mathbb{1}] \cong X^\vee$.

Lemma 3.2.13. *For a closed symmetric monoidal category $\mathscr{V}$, an object X dualizable if and only if the natural map*

$$[X, \mathbb{1}] \otimes X \to [X, X],$$

adjoint to $\mathrm{ev} \otimes \mathrm{id} \colon [X, \mathbb{1}] \otimes X \otimes X \to X \otimes X$, *is an isomorphism.*

Proof. See [San14, Lemma 2.2.61]. $\square$

Remark 3.2.14. If X dualisable in a symmetric monoidal category $\mathscr{V}$, then the maps $\eta \colon \mathbb{1} \to X^\vee \otimes X$ and

$$(X^\vee \otimes X) \otimes (X^\vee \otimes X) \xrightarrow{\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}} X^\vee \otimes X$$

provide $X^\vee \otimes X$ with the structure of a monoid in $\mathscr{V}$. If $\mathscr{V}$ is closed then this monoid structure on $X^\vee \otimes X \cong [X, \mathbb{1}] \cong X \cong [X, X]$ corresponds to the internal endomorphism monoid structure on $[X, X]$ described in Remark 3.2.11.

Definition 3.2.15 (Rigidity)**.** A symmetric monoidal category $\mathscr{V}$ is said to be *rigid* if there exists a functor $(\cdot)^\vee \colon \mathscr{V}^{op} \to \mathscr{V}$ and an isomorphism

$$\mathrm{Hom}_{\mathscr{V}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathscr{V}}(X, Y^\vee \otimes Z)$$

for any $X, Y, Z \in \mathscr{V}$ and natural in all three. In particular, for any object Y, one has the adjunction $- \otimes Y \dashv Y^\vee \otimes -$, i.e., every object is dualisable.

The functor $(\cdot)^\vee$ is unique up to isomorphism, is an equivalence of categories and $(X \otimes Y)^\vee \cong X^\vee \otimes Y^\vee$ and $\mathbb{1}^\vee \cong \mathbb{1}$.

Remark 3.2.16. A rigid symmetric monoidal category is always closed, where the internal hom is defined as $[X, Y] := X^\vee \otimes Y$.

Definition 3.2.17 (Modules over Monoids)**.** Let $\mathscr{V}$ be a symmetric monoidal category, and A a monoid in $\mathscr{V}$. Then an object $M \in \mathscr{V}$ equipped with a map $\mathsf{L}_M\colon A \otimes M \to M$, such that the commutative diagrams in [San14, Definition 2.2.26] hold, is called a *left A-module*. One similarly define a *right A-module*.

Let N be a right A-module with structure map R_N and M a left A-module with structure map L_M, then one has

$$N \otimes A \otimes M \underset{\mathrm{id} \otimes \mathsf{L}_M}{\overset{\mathsf{R}_N \otimes \mathrm{id}}{\rightrightarrows}} N \otimes M$$

The coequaliser of this diagram, if it exists, is denoted $N \otimes_A M$.

Assume that P is a dualisable and consider the monoid object $A = P^\vee \otimes P$. Then $P^\vee$ (resp. P) is a right (resp. left) A-module via the evaluation

$$P^\vee \otimes (P^\vee \otimes P) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} P^\vee \qquad\qquad (P^\vee \otimes P) \otimes P \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} P$$

Moreover, if $P^\vee \otimes_A P$ exists, then the evaluation map gives an induced map

$$\mathrm{tr}_P\colon P^\vee \otimes_A P \to \mathbb{1}$$

called the *trace map*.

Definition 3.2.18 (Faithfully Projective)**.** Let $\mathscr{V}$ be a symmetric monoidal category, and let P be a dualisable object such that for the monoid $A = P^\vee \otimes P$, the object $P^\vee \otimes_A P$ exists. Then P is said to be *faithfully projective* if the trace map, defined above,

$$\mathrm{tr}_P\colon P^\vee \otimes_A P \to \mathbb{1}$$

is an isomorphism.

Definition 3.2.19 (Azumaya Monoid)**.** Let $\mathscr{A}$ be a faithfully projective monoid, in particular dualisable, object in a closed symmetric monoidal category $\mathscr{V}$.

Recall that for the monoid $(\mathscr{A}, \mu)$, we have $\mathscr{A}^{\text{op}} = (\mathscr{A}, \mu \circ \tau)$. Consider the following morphism

$$(\mathscr{A} \otimes \mathscr{A}) \otimes \mathscr{A} \xrightarrow{\alpha} \mathscr{A}^{\text{op}} \otimes (\mathscr{A} \otimes \mathscr{A}) \xrightarrow{\text{id} \otimes \mu} \mathscr{A} \otimes \mathscr{A} \xrightarrow{\mu \circ \tau} \mathscr{A}$$

This corresponds uniquely to a morphism of monoids via adjunction

$$\varphi_{\mathscr{A}} \colon \mathscr{A}^{\text{op}} \otimes \mathscr{A} \to [\mathscr{A}, \mathscr{A}]$$

We call $\mathscr{A}$ an *Azumaya monoid* if $\varphi_{\mathscr{A}}$ is an isomorphism of monoids.

Fix a closed symmetric monoidal category $\mathscr{V}$ for the following results.

Proposition 3.2.20. *If $\mathscr{A}$ and $\mathscr{B}$ are Azumaya monoids in $\mathscr{V}$, then so is $\mathscr{A} \otimes \mathscr{B}$.*

Proof. This is [Par76, Proposition 3]. □

Proposition 3.2.21. *If P is faithfully projective, then $[P, P]$ is an Azumaya monoid.*

Proof. This is [Par76, Proposition 4]. □

Definition 3.2.22 (Categorical Brauer Group)**.** Let $\mathscr{V}$ be a closed symmetric monoidal category, and consider two Azumaya monoids $\mathscr{A}, \mathscr{B}$ in $\mathscr{V}$. We say $\mathscr{A} \sim \mathscr{B}$ if $\mathscr{A} \otimes [P, P] \cong \mathscr{B} \otimes [Q, Q]$, for some faithfully projective objects P, Q in $\mathscr{V}$. This is an equivalence relation, and the set of equivalence classes $\text{Br}(\mathscr{V})$ is an abelian group with product $[\mathscr{A}][\mathscr{B}] = [\mathscr{A} \otimes \mathscr{B}]$, with neutral element $[\mathbb{1}]$ and inverse $[\mathscr{A}^{\text{op}}]$ for $[\mathscr{A}]$. We call $\text{Br}(\mathscr{V})$ the *Brauer group* of $\mathscr{V}$.

Remark 3.2.23. In [Par76], two Brauer groups of a (closed) symmetric monoidal category $\mathscr{V}$ are defined; our Definition 3.2.22 corresponds to the first Brauer group. The two Brauer groups coincide when the unit $\mathbb{1}$ is a projective object in $\mathscr{V}$, which will be the case for the choices of $\mathscr{V}$ we are interested in.

Combining Lemma 3.2.8 and Remark 3.2.10, we have

Proposition 3.2.24. *Let* $\mathsf{Az}_{\mathscr{V}}$ *be the full subcategory of* $\mathsf{Mon}_{\mathscr{V}}$ *consisting of Azumaya monoids as objects. Consider a closed monoidal functor* $F : \mathscr{V} \to \mathscr{V}'$ *between closed symmetric monoidal category. Then* F *restricts to functors* $\mathsf{Mon}_{\mathscr{V}} \to \mathsf{Mon}_{\mathscr{V}'}$ *and* $\mathsf{Az}_{\mathscr{V}} \to \mathsf{Az}_{\mathscr{V}'}$*. Moreover, if* F *is an equivalence of monoidal categories, then it induces an equivalence on the categories of (Azumaya) monoids.*

3.3 Analytic Forms of the Matrix Algebra

Using the notation from Section 2.2.1, consider the category $\mathsf{Rep}_B(G)$. This is a closed symmetric monoidal abelian category, where the relevant structure is described in Notation 2.2.3. Moreover, the category $\mathsf{Rep}^{\mathsf{f}}_B(G)$ is a rigid (closed) symmetric monoidal abelian category. Moreover, for $\mathscr{V} = \mathsf{Rep}_B(G)$, the category $\mathsf{Alg}_B(G) := \mathsf{Mon}_{\mathscr{V}}$ of monoid objects precisely consists of B-algebras equipped with a continuous semilinear action of G that respects the addition and multiplication of the B-algebra. For $\mathscr{V} = \mathsf{Rep}^{\mathsf{f}}_B(G)$, the category $\mathsf{Alg}^{\mathsf{f}}_B(G) := \mathsf{Mon}_{\mathscr{V}}$ of monoid objects is the subcategory of $\mathsf{Alg}_B(G)$ where the underlying module of the B-algebra is finite dimensional over B.

Specialising to the case where $(G, B) = (G_F, \mathbb{C}_F)$ or (Γ_0, F_∞), for $\mathscr{V} = \mathsf{Rep}_B(G)$, the full subcategory $\mathsf{CSA}_B(G) = \mathsf{Az}_{\mathscr{V}}$ of $\mathsf{Alg}^{\mathsf{f}}_B(G)$ precisely consists of B-central simple algebras with a continuous semilinear action of G. We observe this by noting that the condition in Definition 3.2.19 is proved in Proposition 2.3.13.

We investigate how the subcategory $\mathsf{CSA}_B(G)$ interacts with the equivalence in Theorem 3.1.1.

Theorem 3.3.1. *The equivalence* $(-)\otimes_{F_\infty}\mathbb{C}_F\colon \mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma_0)\to\mathsf{Rep}^{\mathrm{f}}_{\mathbb{C}_F}(G_F)$ *restricts to an equivalence* $(-)\otimes_{F_\infty}\colon \mathsf{CSA}_{F_\infty}(\Gamma_0)\to\mathsf{CSA}_{\mathbb{C}_F}(G_F)$*. Moreover, for any object* $A\in\mathsf{CSA}_{F_\infty}(\Gamma_0)$ *of degree* m*, the natural map*

$$\mathrm{H}^1(\Gamma_0, A^\times/F_\infty^\times)\to\mathrm{H}^1(G_F,\mathrm{PGL}_m(\mathbb{C}_F))$$

is injective and induces an isomorphism

$$\coprod_{A\in\mathrm{CSA}^m_{F_\infty}(\Gamma_0)}\mathrm{H}^1(\Gamma_0, A^\times/F_\infty^\times)\cong\mathrm{H}^1(G_F,\mathrm{PGL}_m(\mathbb{C}_F))$$

where $\mathrm{CSA}^m_{F_\infty}(\Gamma_0)$ *is the set of representatives of isomorphism classes of objects in* $\mathsf{CSA}_{F_\infty}(\Gamma_0)$ *with degree* m*. In particular, the natural map*

$$\mathrm{H}^1(\Gamma_0,\mathrm{PGL}_m(F_\infty))\hookrightarrow\mathrm{H}^1(G_F,\mathrm{PGL}_m(\mathbb{C}_F))$$

is injective.

Proof. For any $A\in\mathsf{CSA}_{F_\infty}(\Gamma_0)$, by Proposition 2.3.26, $A^\times/F_\infty^\times\cong\mathrm{Aut}_{F_\infty}(A)$ and $\mathrm{PGL}_m(\mathbb{C}_F)\cong\mathrm{Aut}_{\mathbb{C}_F}(A\otimes_{F_\infty}\mathbb{C}_F)$. The natural map in the statement

$$\mathrm{H}^1(\Gamma_0, A^\times/F_\infty^\times)\to\mathrm{H}^1(G_F,\mathrm{PGL}_m(\mathbb{C}_F))$$

is induced by the natural map $\mathrm{Aut}_{F_\infty}(A)\to\mathrm{Aut}_{\mathbb{C}_F}(A\otimes_{F_\infty}\mathbb{C}_F)$.

The extension of scalars $(-)\otimes_{F_\infty}\mathbb{C}_F\colon \mathsf{Rep}^{\mathrm{f}}_{F_\infty}(\Gamma_0)\to\mathsf{Rep}^{\mathrm{f}}_{\mathbb{C}_F}(G_F)$ and its quasi-inverse $\mathrm{D}_{\mathsf{Sen}}$ are closed monoidal functors by Theorem 3.1.1. Therefore, by Proposition 3.2.24, we have an equivalence

$$(-)\otimes_{F_\infty}\mathbb{C}_F\colon \mathsf{CSA}_{F_\infty}(\Gamma_0)\leftrightarrows\mathsf{CSA}_{\mathbb{C}_F}(G_F):\mathrm{D}_{\mathsf{Sen}}$$

Therefore the result follows from this along with the fact that since $\mathbb{C}_F$ is algebraically closed, the only $\mathbb{C}_F$-central simple algebras are matrix algebras. □

Chapter 4

The p-adic analytic Brauer group

The set-up of this section is the same as Chapter 3, and we are interested in the following continuous cohomology group

$$\mathrm{H}^2(G_F, \mathbb{C}_F^\times)$$

that we dub the (cohomological) p-adic analytic Brauer group. Our main result is that this group sees less than the traditional Brauer group $\mathrm{Br}(F)$.

4.1 Set-up

Recall that for a field extension $E|F$, one has the the E-semilinear representation of G_F, $E(n) = E \otimes_F \chi^n$, the n^{th} Tate twist of E. In particular, we may consider $\mathbb{C}_F(n)$. We begin with the following result [Tat67].

Theorem 4.1.1 (Tate, Springer)**.** *One has* $\mathrm{H}^i(G_F, \mathbb{C}_F(n)) = 0$ *whenever* $i = 0, 1$ *and* $n \neq 0$*. While,* $\mathrm{H}^0(G_F, \mathbb{C}_F) = F$ *and* $\mathrm{H}^1(G_F, \mathbb{C}_F)$ *is a one-dimensional vector space over* F*. One may take the class of* $\log \chi$ *as an* F*-basis for* $\mathrm{H}^1(G_F, \mathbb{C}_F)$*.*

We indulge in some diagram chasing towards our goal of computing $\mathrm{H}^2(G_F, \mathbb{C}_F^\times)$.

The valuation $\hat{v}$ on $\mathbb{C}_F^\times$ determines the following short exact sequence of groups with a continuous G_F-action

$$1 \longrightarrow \mathcal{O}_{\mathbb{C}_F}^\times \xrightarrow{\iota} \mathbb{C}_F^\times \xrightarrow{\hat{v}} \mathbb{Q} \longrightarrow 1 \tag{4.1}$$

By Theorem 2.1.42 (Ax–Sen–Tate), one has $\mathrm{H}^0(G_F, \mathbb{C}_F^\times) = F^\times$; moreover since since G_F is compact and $\mathbb{Q}$ is discrete we obtain $\mathrm{H}^1(G_F, \mathbb{Q}) = 0$. Hence the exact sequence in cohomology induced by (4.2) gives us

$$F^\times \xrightarrow{\text{val}} \mathbb{Q} \longrightarrow \mathrm{H}^1(G_F, \mathcal{O}_{\mathbb{C}_F}^\times) \xrightarrow{i} \mathrm{H}^1(G_F, \mathbb{C}_F^\times) \longrightarrow 0 \tag{4.2}$$

Since F has a discrete valuation, this is equivalent to

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\delta} \mathrm{H}^1(G_F, \mathcal{O}_{\mathbb{C}_F}^\times) \xrightarrow{i} \mathrm{H}^1(G_F, \mathbb{C}_F^\times) \longrightarrow 0 \tag{4.3}$$

Recall now that the p-adic logarithm determines the following short exact sequence (see Proposition 2.1.17)

$$1 \longrightarrow \mu_\infty \hookrightarrow \mathcal{O}_{\mathbb{C}_F}^\times \xrightarrow{\log} \mathbb{C}_F \longrightarrow 1 \tag{4.4}$$

which is G_F-equivariant with the natural continuous G_F-action on these groups. This induces the following exact sequence in cohomology

$$\mathrm{H}^1(G_F, \mu_\infty) \xrightarrow{j} \mathrm{H}^1(G_F, \mathcal{O}_{\mathbb{C}_F}^\times) \xrightarrow{\lambda} \mathrm{H}^1(G_F, \mathbb{C}_F) \tag{4.5}$$

Putting together (4.3) and (4.4), we have the following

$$\begin{tikzcd}
\mathbb{Q}/\mathbb{Z} \arrow[dr, hook, "\delta"] & & \mathrm{H}^1(G_F, \mathbb{C}_F) \\
& \mathrm{H}^1(G_F, \mathcal{O}_{\mathbb{C}_F}^\times) \arrow[ur, two heads, "\lambda"] \arrow[dr, "i"'] & \\
\mathrm{H}^1(G_F, \mu_\infty) \arrow[ur, "j"'] & & \mathrm{H}^1(G_F, \mathbb{C}_F^\times)
\end{tikzcd} \tag{4.6}$$

By Theorem 4.1.1, we know that $\mathrm{H}^1(G_F, \mathbb{C}_F)$ is a (one-dimensional) F-vector space, and therefore $\lambda \circ \delta = 0$. Hence there is a unique map

$$L : \mathrm{H}^1(G_F, \mathbb{C}_F^\times) \to \mathrm{H}^1(G_F, \mathbb{C}_F) \tag{4.7}$$

such that $L \circ i = \lambda$.

4.2 The map L

Proposition 4.2.1. *The map $L : \mathrm{H}^1(G_F, \mathbb{C}_F^\times) \hookrightarrow \mathrm{H}^1(G_F, \mathbb{C}_F)$ is injective.*

Proof. This is [Ser98, §III, A2, Proposition 2]. Using the exact sequences in (4.6), in particular the surjectivity of i, one sees that a sufficient condition to prove the injectivity of L is that the composition $i \circ j : \mathrm{H}^1(G_F, \mu_\infty) \to \mathrm{H}^1(G_F, \mathbb{C}_F^\times)$ is 0. Observe that μ_∞ is a discrete subgroup of $\overline{F}^\times$, hence $i \circ j$ factors through $\mathrm{H}^1(G_F, \overline{F}^\times)$, where $\overline{F}^\times$ has the discrete topology. By Hilbert's theorem 90, the group $\mathrm{H}^1(G_F, \overline{F}^\times)$ vanishes, and hence $i \circ j = 0$. Thus, L is injective. $\square$

Recall that $\mathrm{H}^1(G_F, \mathbb{C}_F)$ is a one-dimensional F-vector space, where we choose as a basis the class represented by $\log \chi$, where χ is the cyclotomic character.

Theorem 4.2.2. *The map $L : \mathrm{H}^1(G_F, \mathbb{C}_F^\times) \hookrightarrow \mathrm{H}^1(G_F, \mathbb{C}_F)$ has an open image in $F \cong \mathrm{H}^1(G_F, \mathbb{C}_F)$, where the topology on the latter is the one on F.*

Proof. Recall that L is defined as follows: given a cocycle $c : G_F \to \mathbb{C}_F^\times$, this cocycle factors through $\mathcal{O}_{\mathbb{C}_F}^\times$ – we continue to refer to this factored cocycle as $c : G_F \to \mathcal{O}_{\mathbb{C}_F}^\times$. Then $L([c]) = [\log c]$. Taking $[\log \chi]$ as a basis for $\mathrm{H}^1(G_F, \mathbb{C}_F)$, we write $[\log c] = \theta(c)[\log \chi]$, for some $\theta(c) \in F$.

We wish to determine that the subgroup $\operatorname{im} L = \{\theta(c) : c \in \mathrm{H}^1(G_F, \mathbb{C}_F^\times)\}$ is open in F. Let triv be the trivial cocycle, and consider $0 = \theta(\mathrm{triv}) \in \operatorname{im} L$. Choose $a \in \mathbb{Z}_{\geqslant 0}$ such that $\exp(\varpi^a \log \chi)$ is well-defined, where ϖ is a uniformiser of F. Then $U_a = \{\exp(\varpi^a \beta \log \chi) : \beta \in \mathcal{O}_F\}$ consists of cocycles such that $\theta(U_a) = \varpi^a \mathcal{O}_F$. Thus, $0 + \varpi^a \mathcal{O}_F \in \operatorname{im} L$ and we may conclude that $\operatorname{im} L$ is open. $\square$

Proposition 4.2.3. *The map $L : \mathrm{H}^1(G_F, \mathbb{C}_F^\times) \hookrightarrow \mathrm{H}^1(G_F, \mathbb{C}_F)$ is not an isomorphism.*

The proof of this proposition relies on the ramification theory of $\mathbb{Z}_p$-extensions. Our $\mathbb{Z}_p$-extension is $F_\infty|F$ and $\Gamma_0 = \mathrm{Gal}(F_\infty|F)$, with topological generator γ_0 and we have let $\gamma_r = \gamma^{p^r}$. Recall the ramification groups in upper numbering of infinite Galois groups from Definition 2.1.27. Moreover, let $e = e(F|\mathbb{Q}_p)$ be the absolute ramification index of $F|\mathbb{Q}_p$.

Proposition 4.2.4. *There exists an $a \in \mathbb{Z}$ such that Γ_0^u for $u > e/(p-1)$ is the closed subgroup generated by $\gamma_{\lceil (u-a)/e \rceil}$.*

Proof. See [Car19, Proposition 2.1.1]. $\square$

The lemma below is inspired by a remark in [Sen81].

Lemma 4.2.5. *There exists a $\Theta \in F$ and $r_0 \in \mathbb{Z}$ such that the element* $\exp(p^r \Theta \log \chi(\gamma_0))$ *is not an element of $\mathrm{N}_{E_r|F}(\mathcal{O}_{E_r}^\times)$, for every $r \geqslant r_0$.*

Proof. For any $u > e/(p-1)$, let $r_0 = \lceil (u-a)/e \rceil + 1$, where $a \in \mathbb{Z}$ is as above in Proposition 4.2.4. Since upper numbering is compatible with quotients (Proposition 2.1.23), we deduce that $\mathrm{Gal}(F_{r_0}|F)^u$ is non-trivial. The local reciprocity isomorphism gives the following isomorphism (Proposition 2.1.31)

$$\mathrm{Gal}(F_{r_0}|F)^u \cong \mathrm{U}_F^u / (\mathrm{U}_F^u \cap \mathrm{N}_{F_{r_0}|E}(F_{r_0}^\times))$$

Since $F_{r_0}|F$ is a cyclic extension, the group $\mathrm{Gal}(F_{r_0}|F)^u$ is necessarily cyclic as well. Let λ represent a class of a generator of $\mathsf{U}_F^u/(\mathsf{U}_F^u \cap N_{F_{r_0}|E}(F_{r_0}^\times))$. Define,

$$\Theta = \frac{\log \lambda}{p^{r_0} \log \chi(\gamma_0)} \in F$$

Since $\lambda \in \mathsf{U}_F^u$, we may write $\lambda = \exp(p^{r_0}\Theta \log \chi(\gamma))$ and this is not a norm by construction.

Consider any $r \geqslant r_0$. Then $\exp(p^r \Theta \log(\chi(\gamma_0))) = \lambda^{p^{r-r_0}}$. Assume that

$$\lambda^{p^{r-r_0}} \in N_{F_r|F}(F_r^\times)$$

Since we are in a $\mathbb{Z}_p$-extension, we obtain using local class field theory that

$$N_{F_{r_0}|F}(F_{r_0}^\times)/N_{F_r|F}(F_r^\times) \cong \mathrm{Gal}(F_r|F_{r_0}) \cong \mathbb{Z}/p^{r-r_0}\mathbb{Z}$$

and hence $N_{F_r|F}(F_r^\times) = N_{F_{r_0}|F}(F_{r_0}^\times)^{p^{r-r_0}}$. We obtain from this that $\lambda = \zeta_{r-r_0}\beta$ for $\beta \in U_E^u \cap N_{E_{r_0}|E}(E_{r_0}^\times)$, where ζ_{r-r_0} is a p^{r-r_0}-th root of unity. Assume ζ_{r-r_0} is a primitive p^s-th root of unity, for $s > 0$, then

$$v(\zeta_{r-r_0} - 1) = \frac{e}{p^{s-1}(p-1)} \leqslant \frac{e}{p-1} < u$$

and so $\lambda/\beta = \zeta_{r-r_0} \notin \mathsf{U}_F^u$ for $s > 0$. Hence, necessarily, $\zeta_{r-r_0} = 1$ and $\lambda = \beta$, a norm. This is a contradiction. Thus, $\exp(p^r\Theta \log(\chi(\gamma_0))$ is not a norm for all $r \geqslant r_0$. $\square$

Proof of Proposition 4.2.3. We wish to prove L is not surjective. Let $\Theta \in F$ be as in Lemma 4.2.5 and consider the element $\Theta[\log \chi] \in \mathrm{H}^1(G_F, \mathbb{C}_F)$. Then we claim that this element is not in the image of L. Using Corollary 3.1.2, it is enough to show that no cocycle $c : \Gamma_0 \to \mathcal{O}_{F_t}^\times$, for some t, is such that $\log c = \theta \log \chi$. Assume such a cocycle c existed, then $c(\gamma_0) \in \mathcal{O}_{F_r}^\times$, for $r = \max\{t, r_0\}$, where r_0

is as in Lemma 4.2.5. We note that

$$\log c(\gamma_0^{p^r}) = \Theta \log \chi(\gamma_0^{p^r}) = p^r \Theta \log \chi(\gamma_0).$$

Observe that $c(\gamma_0^{p^r}) = \mathrm{N}_{F_r|F}(c(\gamma_0))$, contradicting Lemma 4.2.5. Hence, L is not surjective. $\square$

In the following section, we will compute the cokernel of L.

4.3 p-adic Analytic Brauer Group

We focus now on another part of the exact sequence induced by (4.4): consider the following exact sequence

$$\begin{array}{ccccccc} \mathrm{H}^1(G_F, \mathcal{O}_{\mathbb{C}_F}^\times) & \xrightarrow{\lambda} & \mathrm{H}^1(G_F, \mathbb{C}_F) & \longrightarrow & \mathrm{H}^2(G_F, \mu_\infty) & \longrightarrow & \mathrm{H}^2(G_F, \mathcal{O}_{\mathbb{C}_F}^\times) \\ & & & & & & \downarrow \\ & & & & & & \mathrm{H}^2(G_F, \mathbb{C}_F) \end{array} \tag{4.8}$$

By a remark of Fontaine [Fon04] we have $\mathrm{H}^2(G_F, \mathbb{C}_F) = 0$ and we know from Proposition 4.2.1 that λ factors through L, giving us the following exact sequence

$$\mathrm{H}^1(G_F, \mathbb{C}_F^\times) \xrightarrow{L} \mathrm{H}^1(G_F, \mathbb{C}_F) \longrightarrow \mathrm{H}^2(G_F, \mu_\infty) \longrightarrow \mathrm{H}^2(G_F, \mathcal{O}_{\mathbb{C}_F}^\times) \longrightarrow 0 \tag{4.9}$$

Since G_F is compact and $\mathbb{Q}$ discrete, observe that we have $\mathrm{H}^i(G_F, \mathbb{Q}) = 0$, and therefore (4.1) induces the isomorphism $\mathrm{H}^2(G_F, \mathcal{O}_{\mathbb{C}_F}^\times) \cong \mathrm{H}^2(G_F, \mathbb{C}_F^\times)$. Hence (4.9) is equivalent to

$$\mathrm{H}^1(G_F, \mathbb{C}_F^\times) \xrightarrow{L} \mathrm{H}^1(G_F, \mathbb{C}_F) \longrightarrow \mathrm{H}^2(G_F, \mu_\infty) \longrightarrow \mathrm{H}^2(G_F, \mathbb{C}_F^\times) \longrightarrow 0 \quad (4.10)$$

Lemma 4.3.1. *The subspace topology on $\mu_\infty \hookrightarrow \overline{\mathbb{Q}}_p^\times$ is the discrete topology.*

Proof. Let ν_p also denote the unique extension of the valuation on $\mathbb{Q}_p$ to $\overline{\mathbb{Q}}_p^\times$. Since $\nu_p(\zeta) = 0$ for any $\zeta \in \mu_\infty$, we observe that $\nu_p(\zeta_1 - \zeta_2) = \nu_p(1 - \zeta_1\zeta_2^{-1})$ for distinct $\zeta_1, \zeta_2 \in \mu_\infty$. Thus, it suffices to consider $\nu(1-\zeta)$ for a non-trivial $\zeta \in \mu_\infty$. Assume ζ has order m, and let $E_m = \mathbb{Q}_p(\zeta)$ be the corresponding cyclotomic extension. Recall that we have

$$\nu_p(1-\zeta) = \frac{\nu_p(\mathrm{N}_{E_m/\mathbb{Q}_p}(1-\zeta))}{[E_m : \mathbb{Q}_p]} = \frac{\nu_p(\Phi_m(1))}{\varphi(m)}$$

where Φ_m is the m^{th} cyclotomic polynomial, and φ the Euler totient function. It is well-known that $\Phi_m(1) = \ell$ if m is a power of a prime ℓ or $\Phi_m(1) = 1$ if m has at least two distinct prime factors. Therefore, if $m = p^s$ for some integer $s \geqslant 1$, we have

$$\nu_p(1-\zeta) = \frac{1}{p^{s-1}(p-1)}$$

and $\nu_p(1-\zeta) = 0$ otherwise. In particular, for any $\zeta \in \mu_\infty$, one has

$$\nu_p(1-\zeta) \leqslant \frac{1}{p-1} =: d_{\min}$$

Therefore the open neighbourhood $B_{d_{\min}}(\zeta) = \left\{x \in \overline{\mathbb{Q}}_p : \nu_p(x-\zeta) > d_{\min}\right\}$ of ζ is such that $B_{d_{\min}}(\zeta) \cap \mu_\infty = \{\zeta\}$. Thus, μ_∞ is discrete. $\square$

Lemma 4.3.2. $\mathrm{H}^2(G_F, \mu_\infty) \cong \mathrm{Br}(F)$

Proof. By Lemma 4.3.1, the subspace topology on $\mu_\infty \hookrightarrow \mathbb{C}_F^\times$ is the discrete topology and therefore $\mathrm{H}^2(G_F, \mu_\infty) = \mathrm{H}^2_{\mathrm{disc}}(G_F, \mu_\infty)$.

Let $(-)_{\mathbb{Q}} := (-) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $\overline{F}$ is algebraically closed, $\overline{F}^{\times}$ is a divisible abelian group, Therefore the canonical map $\overline{F}^{\times} \to \overline{F}^{\times}_{\mathbb{Q}}$ is surjective with kernel equalling the torsion subgroup of $\overline{F}^{\times}$, i.e., μ_∞. This gives us the following short exact sequence

$$1 \longrightarrow \mu_\infty \longrightarrow \overline{F}^{\times} \longrightarrow \overline{F}^{\times}_{\mathbb{Q}} \longrightarrow 1$$

Taking the exact sequence in cohomology we obtain the following exact sequence

$$\mathrm{H}^1_{\mathrm{disc}}(G_F, \overline{F}^{\times}_{\mathbb{Q}}) \longrightarrow \mathrm{H}^2(G_F, \mu_\infty) \longrightarrow \mathrm{H}^2_{\mathrm{disc}}(G_F, \overline{F}^{\times}) \longrightarrow \mathrm{H}^2_{\mathrm{disc}}(G_F, \overline{F}^{\times}_{\mathbb{Q}})$$

Since $\mathbb{Q}$ is a flat abelian group, one has $\mathrm{H}^i_{\mathrm{disc}}(G_F, \overline{F}^{\times}_{\mathbb{Q}}) = \mathrm{H}^i_{\mathrm{disc}}(G_F, \overline{F}^{\times})_{\mathbb{Q}}$. By Hilbert's 90, one has $\mathrm{H}^1_{\mathrm{disc}}(G_F, \overline{F}^{\times}) = 0$, and therefore $\mathrm{H}^1_{\mathrm{disc}}(G_F, \overline{F}^{\times}_{\mathbb{Q}}) = 0$. Moreover, by local class field theory, we have $\mathrm{H}^2_{\mathrm{disc}}(G_F, \overline{F}^{\times}) = \mathrm{Br}(F) = \mathbb{Q}/\mathbb{Z}$, a torsion abelian group, and therefore $\mathrm{H}^2_{\mathrm{disc}}(G_F, \overline{F}^{\times}_{\mathbb{Q}}) = 0$. Hence, the exact sequence above induces the isomorphism $\mathrm{H}^2(G_F, \mu_\infty) \cong \mathrm{H}^2_{\mathrm{disc}}(G_F, \overline{F}^{\times}) = \mathrm{Br}(F)$. □

Thus, (4.10) is equivalent to

$$0 \longrightarrow \operatorname{coker} L \xrightarrow{\tau} \mathrm{Br}(F) \longrightarrow \mathrm{H}^2(G_F, \mathbb{C}^{\times}_F) \longrightarrow 0 \tag{4.11}$$

Definition 4.3.3. For a prime ℓ, a group is called ℓ-primary if all its elements have order a power of ℓ. We let the trivial group be ℓ-primary for every ℓ.

Any torsion abelian group A is a direct sum of its ℓ-primary subgroups, as ℓ ranges through all primes; see [Rot95, Theorem 10.7]. These groups are called its ℓ-primary components and we denote them as A_ℓ. For a fixed prime ℓ, the complement of A_ℓ in A will be called the ℓ'-primary component of A and will be denoted as $A_{\ell'}$.

Example 4.3.4. For any prime ℓ, the group $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ is ℓ-primary. Moreover, it is (isomorphic to) the ℓ-primary component of the divisible abelian group $\mathbb{Q}/\mathbb{Z}$.

Theorem 4.3.5. $\tau(\operatorname{coker} L) = \operatorname{Br}(F)_p$*, the p-primary part of* $\operatorname{Br}(F)$.

Lemma 4.3.6. $\operatorname{coker} L$ *is infinite.*

Proof. By Proposition 4.2.3, $\operatorname{im} L$ is an open subgroup of $\mathrm{H}^1(G_F, \mathbb{C}_F) \cong F$. Identify $\mathrm{H}^1(G_F, \mathbb{C}_F)$ as $\mathbb{Q}_p^d$, where $d = [F : \mathbb{Q}_p]$, and identify $\operatorname{im} L$ as an open subgroup of $\mathbb{Q}_p^d$. Since $\operatorname{im} L$ is open, necessarily $\mathbb{Z}_p^m \subseteq \operatorname{im} L$ for some $m \leqslant d$. Hence, $\operatorname{coker} L$ can be seen as a quotient of $\mathbb{Q}_p^d/\mathbb{Z}_p^m$. Recall that quotients of divisible abelian groups are also divisible. Thus we conclude that $\operatorname{coker} L$ is a divisible abelian group, as $\mathbb{Q}_p^d$ is divisible. The short exact sequence (4.11) tells us that $\operatorname{coker} L$ is a torsion group. Hence we may deduce that $m = d$ and thus $\operatorname{coker} L$ is a quotient of $(\mathbb{Q}_p/\mathbb{Z}_p)^d$. Furthermore, if $\operatorname{coker} L$ was finite, then it would be trivial being a finite divisible abelian group, contradicting the non-surjectivity of L. Hence $\operatorname{coker} L$ is infinite. $\square$

Proof of Theorem 4.3.5. Let $X = \tau(\operatorname{coker} L)$. By Lemma 4.3.6, $\operatorname{coker} L$ is an infinite p-primary divisible abelian group, and therefore so is X. As a p-primary subgroup of $\operatorname{Br}(F) \cong \mathbb{Q}/\mathbb{Z}$, X must be a subgroup of its p-primary component, $\operatorname{Br}(F)_p = (\mathbb{Q}/\mathbb{Z})_p \cong \mathbb{Z}(p^\infty)$, where $\mathbb{Z}(p^\infty)$ is the Prüfer p-group. By [Rot95, Theorem 10.13], all proper subgroups of $\mathbb{Z}(p^\infty)$ are finite. Thus, necessarily, $X = \operatorname{Br}(F)_p$. $\square$

Theorem 4.3.7. $\mathrm{H}^2(G_F, \mathbb{C}_F^\times) \cong \operatorname{Br}(F)_{p'}$*, the p'-primary part of* $\operatorname{Br}(F)$.

Proof. The short exact sequence (4.11) tells us that $\mathrm{H}^2(G_F, \mathbb{C}_F)$ is the cokernel of the injective map $\tau : \operatorname{coker} L \to \mathrm{Br}(F)$, and $\tau(\operatorname{coker} L) = \mathrm{Br}(F)_p$ by Theorem 4.3.5. Since $\mathrm{Br}(F)_{p'}$ is precisely the complement of $\mathrm{Br}(F)_p$ in $\mathrm{Br}(F)$, we necessarily conclude that $\mathrm{H}^2(G_F, \mathbb{C}_F^\times) \cong \mathrm{Br}(F)_{p'}$. $\square$

The authors in [AHLB22] also compute coker L, using different methods, where L is referred to as HT log and they characterise this isomorphism in terms of the absolute trace map $\mathrm{Tr}_{F|\mathbb{Q}_p}$.

Chapter 5

Division Algebras of p-power Index

Let the set-up be as in Chapters 3 and 4. Fix $m = p^r$, where $r > 1$ is arbitrary. We have the following exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \mu_m \arrow[r] \arrow[d, hook] & \mathrm{SL}_m(\mathbb{C}_p) \arrow[r] \arrow[d, hook] & \mathrm{PGL}_m(\mathbb{C}_p) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{C}_p^\times \arrow[r] & \mathrm{GL}_m(\mathbb{C}_p) \arrow[r] & \mathrm{PGL}_m(\mathbb{C}_p) \arrow[r] & 1 \end{tikzcd} \tag{5.1}$$

which gives us

$$\begin{tikzcd} \mathrm{H}^1(G_F, \mathrm{SL}_m(\mathbb{C}_F)) \arrow[r] \arrow[d] & \mathrm{H}^1(G_F, \mathrm{PGL}_m(\mathbb{C}_F)) \arrow[r] \arrow[d, equal] & \mathrm{H}^2(G_F, \mu_m) \arrow[d] \\ \mathrm{H}^1(G_F, \mathrm{GL}_m(\mathbb{C}_F)) \arrow[r] & \mathrm{H}^1(G_F, \mathrm{PGL}_m(\mathbb{C}_F)) \arrow[r] & \mathrm{H}^2(G_F, \mathbb{C}_F^\times) \end{tikzcd} \tag{5.2}$$

Since we have shown that $\mathrm{H}^2(G_F, \mathbb{C}_F^\times)$ is p-torsion free by Theorem 4.3.7, the last vertical map is the trivial map. Giving us that

$$\mathrm{H}^1(G_F, \mathrm{GL}_m(\mathbb{C}_F)) \longrightarrow \mathrm{H}^1(G_F, \mathrm{PGL}_m(\mathbb{C}_F)) \longrightarrow * \tag{5.3}$$

Therefore the natural map $\mathrm{H}^1(G_F, \mathrm{GL}_m(\mathbb{C}_F)) \to \mathrm{H}^1(G_F, \mathrm{PGL}_m(\mathbb{C}_F))$ is surjective.

Consider now the following diagram

$$\begin{tikzcd}
\mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(F_\infty)) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^1(\Gamma_0, \mathrm{PGL}_m(F_\infty)) \arrow[d, hook] \\
\mathrm{H}^1(G_F, \mathrm{GL}_m(\mathbb{C}_F)) \arrow[r, two heads] & \mathrm{H}^1(G_F, \mathrm{PGL}_m(\mathbb{C}_F))
\end{tikzcd} \tag{5.4}$$

where the first vertical isomorphism is by Corollary 3.1.2, and the the second vertical inclusion is by Theorem 3.3.1. This gives us the following

Theorem 5.0.1. *The natural maps* $\mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(F_\infty)) \to \mathrm{H}^1(G_F, \mathrm{PGL}_m(\mathbb{C}_F))$ *and* $\mathrm{H}^1(\Gamma_0, \mathrm{GL}_m(F_\infty)) \to \mathrm{H}^1(\Gamma_0, \mathrm{PGL}_m(F_\infty))$ *are an isomorphism and surjective respectively for* $m = p^r$*, for any* $r \geqslant 1$.

We complement this result with the following construction.

Proposition 5.0.2. *For any* $r \geqslant 1$*, there exists a non-trivial* $\mathbb{C}_F$*-semilinear representation* V *such that* $\mathrm{End}(V)^{G_F}$ *represents an element of* $\mathrm{Br}(F)$ *with Hasse invariant* $1/p^r \bmod 1$.

Proof. Let $v = \nu_p(d)$, and therefore $p^{-v}d$ is coprime to p. Let $s = r + v$, and consider the following representation of Γ_0

$$\rho : G_F \twoheadrightarrow \Gamma_0 \to \mathrm{GL}_{p^s}(\mathbb{Z}_p) \hookrightarrow \mathrm{GL}_{p^s}(\mathbb{C}_F)$$

$$\gamma_0 \mapsto \begin{pmatrix} 0 & \chi(\gamma_0) \\ \mathrm{Id}_{p^s-1} & 0 \end{pmatrix}$$

Let $U_0 = \rho(\gamma_0)$, and we observe that $U_0^{p^s} = \chi(\gamma_0)\,\mathrm{Id}_{p^s}$. We can conclude from here that ρ is continuous, since χ is. Let V be the corresponding $\mathbb{C}_F$-semilinear representation. We furthermore observe that the Sen operator of ρ (or V) is

$$\Theta_\rho = \frac{\log U_0}{\log \chi(\gamma_0)} = \frac{1}{p^s}\,\mathrm{Id}_{p^s}$$

V has generalised Hodge-Tate weight $(1/p^s, \dots, 1/p^s)$.

Now, observe that ρ gives rise to a homomorphism

$$\overline{\rho} : G_F \xrightarrow{\rho} \mathrm{GL}_{p^s}(\mathbb{C}_F) \twoheadrightarrow \mathrm{PGL}_{p^r}(\mathbb{C}_F)$$

such that $\rho(\gamma_0)^{p^s} = [\mathrm{Id}_{p^s}]$, and hence we may use $\rho(\gamma_0)$ to identify $\mathrm{End}(V)^{G_F} = \mathrm{End}(\mathrm{D}_{\mathrm{Sen}}(V))^{\Gamma_0}$ with the cyclic algebra $D_s = (\chi_s, \chi(\gamma_0))$ representing a class in $\mathrm{Br}(F_s|F)$, where $\chi_s = \chi \bmod p^s$ is the character

$$\chi_s \colon \mathrm{Gal}(F_s|F) \xrightarrow{\sim} \Gamma_0/\Gamma_s \cong \mathbb{Z}/p^s\mathbb{Z}$$

All that remains is the computation of the Hasse invariant of D_s.

Lemma 5.0.3. $\mathsf{Inv}([D_s]) = -\dfrac{d}{p^s} \bmod 1 = -\dfrac{p^{-v}d}{p^r} \bmod 1$

Proof. We recall from Proposition 2.3.42 that we equivalently wish to compute $\theta_{F_s|F}(\chi(\gamma_0))$. Since $\chi(\gamma_0) \in \mathbb{Q}_p^\times$, we use the following diagram from Theorem 2.1.30 restricted to Γ_F

$$\begin{tikzcd} F^\times \arrow[r, "\theta_F"] \arrow[d, "\mathrm{N}_{F|\mathbb{Q}_p}"'] & \Gamma_F \arrow[d, hook, "\mathrm{Res}"] \\ \mathbb{Q}_p^\times \arrow[r, "\theta_{\mathbb{Q}_p}"'] & \Gamma_{\mathbb{Q}_p} \end{tikzcd}$$

In Remark 2.1.37, we comuted $\theta_{\mathbb{Q}_p}(\chi(\gamma_0)) = \gamma_0^{-1}$. Therefore, using the diagram we get $\theta_{F_s|F}(\chi(\gamma_0)) = \gamma_0^{-d}$, where $d = [F : \mathbb{Q}_p]$, proving our claim. □

Having computed $\mathsf{Inv}([D_s])$, where $p^{-v}d$ is coprime to p, we see that an appropriate power of D_s represents a class in $\mathrm{Br}(F_s|F)$ which has Hasse invariant $1/p^r \bmod 1$. □

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