Asymptotically Conical Metrics and Expanding Ricci Solitons

by

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Abstract

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In this thesis we first show, at the level of formal expansions, that any compact manifold can be the sphere at infinity of an asymptotically conical gradient expanding Ricci soliton. We then prove the existence of a smooth blowdown limit for any Ricci-DeTurck flow on $\mathbb{R}^n$, starting from possibly non-smooth data which is asymptotically conical and sufficiently L^∞-close to an expanding soliton on $\mathbb{R}^n$. Furthermore, this blowdown flow is an expanding Ricci-DeTurck soliton coming out of the asymptotic cone of the initial data.

Contents

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1 Introduction

On a fixed Riemannian manifold M, a family of metrics $(M, g(t))_{t\in[0,T)}$is said to evolve by the Ricci flow if

$$\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}(g(t)). \tag{1}$$

The Ricci flow has become an important tool in geometric analysis. When looking at the Ricci flow on noncompact manifolds, the asymptotically conical geometries are especially interesting. Before defining an asymptotically conical geometry, we recall that the cone over a manifold Y is topologically given by $Y \times (0, \infty) \cup \{\star\}$, where $\star$ denotes the cone point. The manifold Y is called the link. If the link is a compact Riemannian manifold (Y, g_Y) then the Riemannian cone over Y, denoted $C(Y)$, is a Riemannian manifold away from its cone point. Let r denote the coordinate on $(0, \infty)$, where $r = 0$ corresponds to the cone point $\star$. Then the cone metric is given by $g_{C(Y)} = dr^2 + r^2 g_Y$.

More generally, a Riemannian manifold, (M, g), is said to be asymptotically conical if there exists a cone manifold $(C(Y), g_{C(Y)})$ such that

$$\lim_{\lambda\to\infty}(M, p, \frac{1}{\lambda}g) = (C(Y), \star, g_{C(Y)})$$

in the pointed Gromov-Hausdorff sense, and there is smooth Cheeger-Gromov convergence on compact sets away from the cone point. The cone manifold $C(Y)$ is called the asymptotic cone of (M, g). This rescaling limit is closely related to another limit, called the parabolic blowdown, often studied in Ricci flow on noncompact manifolds.

The parabolic blowdown can be defined as follows. Suppose that the Ricci flow $(M, g(t))$ on a noncompact manifold M exists for all positive time, $t \geq 0$, i.e. is an immortal Ricci flow. Let $\{\lambda_i\}$ be a sequence of real numbers such that $\lambda_i \to \infty$ as $i \to \infty$. Define a sequence of flows by $g_i(t) := \lambda_i^{-1} g(\lambda_i t)$. Finally, let $\{p_i\}$ be a sequence of points on M. Then the behavior of the flow at large times and near spatial infinity can be studied by taking the pointed Hamilton-Cheeger-Gromov limit

$$\lim_{i\to\infty}\big(M, g_i(t), p_i\big) = \big(X, g_\infty(t), p_\infty\big), \tag{2}$$

called the parabolic blowdown of $(M, g(t))$. This limit flow, $g_\infty(t)$, is called the blowdown limit of $(M, g(t))$. This limit may depend on the choice of sequences $\{\lambda_i\}$ and $\{p_i\}$.

The blowdown limit is not guaranteed to exist. However, if $(M, g(t))$ is an immortal Ricci flow on a noncompact manifold, and the initial metric is asymptotically conical, then Lott and Zhang proved there is a subsequential blowdown limit flow $g_\infty(t)$ that is defined at least on the subset of $C(Y) \times [0, \infty)$ given by $\{(r, \theta, t) \in (0, \infty) \times Y \times [0, \infty) : t \leq \epsilon r^2\}$, for some ϵ ([LZ]). Furthermore, if the initial metric is asymptotically conical then each time slice of the Ricci flow is asymptotically conical with the same asymptotic cone ([LZ]).

Since the original flow is asymptotic to the same cone $C(Y)$ for all time, the simplest scenario is that $g_\infty(t)$ is a gradient expanding Ricci soliton coming out of $C(Y)$. Recall that an expanding Ricci soliton is a self-similar solution to the Ricci flow that can be written $g(t) = t\phi_t(g_0)$ where g_0 is the initial metric and ϕ_t is a family of diffeomorphisms generated by V/t. Here V is a time-independent vector field.

When this vector field can be expressed as the gradient of a function, $\nabla f = V$, the flow is called a gradient expanding soliton. The function f is called the potential function. Alternatively stated, a gradient expanding Ricci soliton is specified by a triple (M, g_0, f) such that

$$2\,\mathrm{Ric}(g_0) + 2\,\mathrm{Hess}_{g_0} f + g_0 = 0. \tag{3}$$

There are many examples of gradient expanding solitons. Schulze and Simon showed that the Ricci flow starting from a Riemannian manifold (M, g_0) with a nonnegative, bounded curvature operator, and positive asymptotic volume ratio, exists for all positive time. Further, there exists a subsequential blowdown limit, and this blowdown limit flow is a gradient expanding Ricci soliton coming out of the asymptotic cone of the initial metric (M, g_0) ([SS]).

This leads to the question of which cones admit an expanding Ricci soliton structure, i.e. which Riemannian manifolds are admissible links. Deruelle showed in [Der] that if the link (Y, g_Y) is a smooth, simply-connected, compact Riemannian manifold with strictly positive curvature operator, then there is a unique expanding Ricci soliton asymptotic to $(C(Y), dr^2 + r^2 g_Y, r\partial_r/2)$. Both of these results rely heavily on the nonnegative curvature assumption. However, recall that the Ricci curvature of a Riemannian cone is given by

$$\mathrm{Ric}(g_{C(Y)}) = \mathrm{Ric}(g_Y) - (n-1)g_Y$$

where the link is (Y, g_Y). Hence, any link with negative Ricci curvature would heuristically seem to correspond to an expanding solution of the Ricci flow. Unfortunately, negative curvature assumptions are not in general preserved by the Ricci flow. This thesis will be dedicated to trying to prove results about expanding Ricci solitons and asymptotically conical manifolds without any assumption on the curvature.

The first main result of this thesis can now be stated.

Theorem 1. *Given a compact Riemannian manifold (Y, g_Y), there is a formal solution to equation 3 on $(0, \infty) \times Y$ of the form*

$$g = dr^2 + r^2 g_Y + h_0 + r^{-2} h_2 + \cdots + r^{-2i} h_{2i} + \cdots \tag{4}$$

$$f = -\frac{1}{4} r^2 + f_0 + r^{-2} f_2 + \cdots + r^{-2i} f_{2i} + \cdots \tag{5}$$

where h_{2i} is a symmetric 2-tensor field on Y and $f_{2i} \in C^\infty(Y)$. The solution is unique up to adding a constant to f_0. [1]

In other words, on the level of formal asymptotic expansions, any compact manifold, (Y, g_Y), can be the link of the asymptotic cone of an asymptotically conical, gradient expanding Ricci soliton, i.e. the sphere at infinity. In particular, no assumptions are made about the dimension of Y or about the curvature of g_Y. This result was shown in the Kähler case by Lott and Zhang [LZ]. The same argument used to prove Theorem 1 also shows the existence of a shrinking soliton structure on the level of formal asymptotic expansions. This result will be discussed in section 2.

[1] The proof of this theorem was published by the author and Professor John Lott in PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY Volume 145, Number 8, August 2017, Pages 35253529 http://dx.doi.org/10.1090/proc/13611 Article electronically published on April 28, 2017

To state the second main result of this paper, recall that the Ricci-DeTurck flow is a parabolic flow closely related to the Ricci flow. The parabolic structure of this equation will be used in place of any curvature assumptions to guarantee uniqueness and existence or the flow. In particular, because Riemannian cones are often not C^2 at the cone point, the Ricci flow coming out of a cone manifold is not *a priori* defined.

On a fixed Riemannian manifold M, a family of metrics $(M, g(t))_{t\in[0,T)}$ is said to evolve by the Ricci-DeTurck flow if

$$\partial_t g(t) = -2\operatorname{Ric}(t) + \nabla_i V_j + \nabla_j V_i \quad \text{on } M \times (0,T) \tag{6}$$

where V is a 1-form given by $V_i = g_{il}\left({}^g\Gamma^l_{jk} - {}^{\hat{g}}\Gamma^l_{jk}\right) g^{jk}$ for some background metric, $\hat{g}$. Let $(\mathbb{R}^n, \delta)$ denote Euclidean space. On $\mathbb{R}^n$, we take $\hat{g} = \delta$ and then $V_i = g_{il}{}^g\Gamma^l_{jk} g^{jk}$.

Recent work by Koch and Lamm established the existence of Ricci-DeTurck flows on $\mathbb{R}^n$ coming out of rough initial data that is only in $L^\infty(\mathbb{R}^n)$ when the initial data is ϵ_n-close to the Euclidean metric in $L^\infty(\mathbb{R}^n)$ ([KL]). Here ϵ_n is a fixed constant depending only on the dimension. Moveover, these flows exist for all positive time and are unique in a weak Sobolev space.

This result was extended by Deruelle and Lamm in [DL] to the case when an expanding gradient Ricci soliton with positive curvature operator is used as the background metric, rather than the Euclidean metric. That is, if a metric $g_0 \in L^\infty(\mathbb{R}^n)$ is ϵ-close to an expanding gradient Ricci soliton with positive curvature operator, then there exists an immortal Ricci-DeTurck flow coming out of g_0 and the flow is unique in a small ball in a Sobolev space. Note that this ϵ is not the same as in Koch and Lamm's result and may depend on the particular soliton used.

Several of the notions from above such as an expanding soliton, a blowdown limit, and an asymptotic cone can be defined for the Ricci-DeTurck flow as well. Define an expanding Ricci-DeTurck soliton to be a Ricci-DeTurck flow, $g(t)$, that satisfies

$$g(\lambda x, \lambda^2 t) = g(x,t)$$

for all positive times, $t > 0$. Note that a Ricci-DeTurck soliton will correspond to a Ricci soliton when correctly translated back into the Ricci flow setting.

The blowdown limit of an immortal flow can also be translated into the Ricci-DeTurck flow setting. Suppose that a Ricci-DeTurck flow $g(t)$ on $\mathbb{R}^n$ exists for all positive time. Then for $\lambda \in [1,\infty)$, define the rescaled metric,

$$g_\lambda(x,t) = g(\lambda x, \lambda^2 t) \tag{7}$$

The blow-down limit of the Ricci flow, $g(t)$, is the pointed Hamilton-Cheeger-Gromov limit $\lim_{\lambda\to\infty} g_\lambda$. In general, this limit may not exist. However, the Ricci-DeTurck flows studied in this thesis will satisfy the following bounds. For every $k \in \mathbb{N}_0$ and for every multi-index $\alpha \in \mathbb{N}_0^n$, there exists a constant c, depending only on k and the magnitude of the multi-index, $|\alpha|$, such that

$$|\partial_t^k \nabla^\alpha g(x,t)| \le c t^{-k-\frac{|\alpha|}{2}}$$

These bounds are invariant under the parabolic rescaling defined by equation (7). This is enough to imply subsequential convergence (possibly to the zero metric). Unfortunately, without further assumptions, this convergence likely depends on the subsequence taken and on the choice of base point. The convergence shown in this thesis will be much stronger and the limit will be shown to be nonzero.

Define a rough cone metric on $\mathbb{R}^n$ to be a cone over $(\mathbb{S}^{n-1}, g_{n-1})$ where $g_{n-1} \in L^\infty(\mathbb{S}^{n-1})$. Furthermore, in this thesis a metric $g_0 \in L^\infty(\mathbb{R}^n)$ is said to be asymptotic to a rough cone $(C(\mathbb{S}^{n-1}), g_{C(\mathbb{S}^{n-1})})$ if there exists a non-increasing function $\eta : [0, \infty) \to [0, \infty)$ such that $\eta(r) \to 0$ as $r \to \infty$ and

$$||g_{C(\mathbb{S}^{n-1})} - g_0||_{L^\infty(\mathbb{R}^n \setminus B(0,r))} \leq \eta(r).$$

Then $(C(\mathbb{S}^{n-1}), g_{C(\mathbb{S}^{n-1})})$ is called the rough asymptotic cone of g_0. Notice that this definition only requires the cone metric to be in $L^\infty(\mathbb{R}^n)$ or, more precisely, for $g_{C(\mathbb{S}^{n-1})}$ to be in $L^\infty(\mathbb{S}^{n-1})$.

Remark. *Note that this definition of a rough asymptotic cone differs from the standard definition of an asymptotic cone given above. The more technical definition used in this thesis will be more useful for the problem at hand, but can be reconciled with the standard definition when the limit manifold is replaced by a conic metric space. Then the limit is taken only with respect to the pointed Gromov-Hausdorff metric.*

The second main result of this thesis can now be stated.

Theorem 2. *Let ϵ_n be as in Koch and Lamm's result (Theorem 4.3 of [KL], stated precisely in the next section). Let $g_0 \in L^\infty(\mathbb{R}^n)$ be a metric on $\mathbb{R}^n$ asymptotic to a rough cone, $(C(\mathbb{S}^{n-1}), g_{C(\mathbb{S}^{n-1})})$, and satisfying*

$$||g_0 - \delta||_{L^\infty(\mathbb{R}^n)} \leq \epsilon_n.$$

Koch and Lamm showed the existence of immortal Ricci-DeTurck flows $g(t)$ and $g_{sol}(t)$, with initial conditions $g(0) = g_0$ and $g_{sol}(0) = g_{C(\mathbb{S}^{n-1})}$, respectively.

Then the blowdown limit of $(\mathbb{R}^n, g(t), 0)$ is $(\mathbb{R}^n, g_{sol}(t), 0)$ with convergence in $C^\infty_{loc}(\mathbb{R}^n \times (0, \infty))$. Furthermore, g_{sol} is an expanding Ricci-DeTurck soliton.

In other words, in the setting of Ricci-DeTurck flows on $\mathbb{R}^n$, there exists an infinite class of metrics that are L^∞-close to the Euclidean metric and have smooth continues convergence to their blowdown limits, and the blowdown limit is an expanding Ricci-DeTurck solitons. In particular, this class includes some metrics with some negative sectional curvatures. This result comes from reinterpreting stability results about the Ricci-DeTurck flow proved by Kock and Lamm [KL] (and Deruelle and Lamm [DL]).

2 Formal Asymptotic Expansions

Section 2 will be organized as follows. In Section 2.1, the proof of Theorem 1 will be reduced to solving three partial differential equations (PDEs) simultaneously. Two of these PDE's will be used to define the expansions for the metric g and the potential function f, while the third equation will function as a constraint equation. The major difficulty of the proof of Theorem 1 will be to show that this constraint equation is satisfied whenever the other two are. In section 2.2, two of PDEs will be used to find the first few terms in the expansion for the metric and the potential function. In section 2.3, it will be shown explicitly that these first few terms of the expansions do satisfy the constraint equation. In section 2.4 the general argument to prove Theorem 1 will be given. In section 2.5, the calculations in section 2.2 will be compared to the expanding Bryant soliton equations [B].

2.1 Equations

First, let's fix notation. Define the following 2-tensors as

$$H(r) = r^2 g_Y + h_0 + r^{-2} h_2 + \cdots + r^{-2i} h_{2i} + \cdots \tag{8}$$

and

$$H_o(r) = h_0 + r^{-2} h_2 + \cdots + r^{-2i} h_{2i} + \cdots \tag{9}$$

Then the expansion for the metric is Theorem 1 can be written $g = dr^2 + H(r)$ and $g = dr^2 + r^2 g_Y + H_o(r)$. By orthogonality, $g^{-1} = (dr^2)^{-1} + H^{-1}$. Let ${H_{oo}}^{il} := {H_o}^{ij} g_{Y\,jk} {H_o}^{kl}$ and similarly for H_{ooo}, H_{oooo}, etc. Similarly, define $h^{il}_{02} := {h_0}^{ij} g_{Y\,jk} {h_2}^{kl}$ and similarly for h_{04}, h_{22}, etc. Then, the formal inverse of H is

$$\begin{aligned} H^{il} &= r^{-2} g^{il} - r^{-4} H^{il}_o + r^{-6} H^{il}_{oo} - r^{-8} H^{il}_{ooo} + r^{-10} H^{il}_{oooo} - \cdots \\ &= r^{-2} g^{il}_Y - r^{-4} h^{il}_0 + r^{-6} \left(-h^{il}_2 + h^{il}_{00} \right) + r^{-8} \left(-h^{il}_4 + h^{il}_{02} + h^{il}_{20} - h^{il}_{000} \right) \\ &\quad + r^{-10} \left(-h^{il}_6 + h^{il}_{04} + h^{il}_{40} + h^{il}_{22} - h^{il}_{002} - h^{il}_{020} - h^{il}_{200} + h^{il}_{0000} \right) + \cdots \end{aligned}$$

In local coordinate calculations in this thesis, commas (,) will denote partial derivatives and bars (|) denote covariant derivatives. Define, for any 2-tensor T_{ij}

$$T_{[ij,k]} := T_{ik,j} + T_{jk,i} - T_{ij,k}$$

and

$$T_{[ij|k]} := T_{ik|j} + T_{jk|i} - T_{ij|k}.$$

Throughout section 2, all covariant derivatives are with respect to the Levi-Civita connection of g_Y unless otherwise stated or denoted.

Using this notation, the Christoffel symbols of H can related to the Christoffel symbols of g_Y through the following equation.

$$2 {^H\Gamma_{jk}}^i = 2 {^Y\Gamma_{jk}}^i + H^{il} H_{o[jk|l]} \tag{10}$$

Now, we can write equation 3 in terms of H and its formal inverse. Let $x_1, \cdots, x_n$ be local coordinates for Y. Then the expander equation splits into three cases.

$$2\,\mathrm{Ric}\, g_{rr} + 2\,\mathrm{Hess}_g\, f_{rr} + 1 = 0 \tag{11}$$

$$2\,\mathrm{Ric}\, g_{rl} + 2\,\mathrm{Hess}_g\, f_{rl} = 0 \tag{12}$$

$$2\,\mathrm{Ric}\, g_{jk} + 2\,\mathrm{Hess}_g\, f_{jk} + H_{jk} = 0 \tag{13}$$

Then these three equations can be written in terms of H and dr^2. First, equation 11 simplifies to

$$-H^{il}H_{il,rr}-\frac{1}{2}H^{il}{}_{,r}H_{il,r}+2f_{,rr}+1=0 \tag{14}$$

Second, equation 12 simplifies to

$$H^{im}\left({}^{H}\nabla_i H_{ml,r}-{}^{H}\nabla_l H_{im,r}\right)+2f_{,rl}-H^{ix}H_{xl,r}f_{,i}+0=0 \tag{15}$$

where the covariant derivative with respect to the metric H is denoted, ${}^{H}\nabla$. Third, equation 13 simplifies to

$$2\,\mathrm{Ric}(H)_{jk}-H_{jk,rr}-\frac{1}{2}H^{il}H_{il,r}H_{jk,r}+H^{il}H_{ij,r}H_{kl,r}$$
$$2\,\mathrm{Hess}_Y f_{jk}+H^{il}H_{o[jk|l]}f_{,i}+H_{jk,r}f_{,r}+H_{jk}=0 \tag{16}$$

Further, the Ricci tensor of H can be written using equation 10 as

$$2\,\mathrm{Ric}(H)_{jk}=2\,\mathrm{Ric}(g_Y)_{jk}+H^{il}\left[H_{ojl|ki}+H_{okl|ji}-H_{ojk|il}-H_{oil|jk}\right]$$
$$+\ \frac{1}{2}H^{il}H^{nm}\left(H_{o[jk|n]}H_{o[il|m]}-H_{o[ik|n]}H_{o[jl|m]}\right) \tag{17}$$

and the mixed term of the Ricci tensor of g can also be made more explicit

$$2\,\mathrm{Ric}(g)_{rl}=H^{im}\left(H_{ml,r|i}-H_{im,r|l}\right)-\frac{1}{2}H^{im}H_{o[im|y]}H^{xy}H_{xl,r}-\frac{1}{2}H^{im}{}_{,r}H_{oim|l} \tag{18}$$

Equation 14 is used to define f in terms of H. Then equation 16 is used to define the metric H in terms of the original metric, g_Y. The mixed term equation 15 is a constraint equation. It must be shown that these definitions for f and H satisfy this third constraint equation. First, let's look at the first few terms of these expansions.

Solving for f in terms of H using equation 14 gives the leading term, $-\frac{1}{4}r^2$. Equation 14 does not say anything about f_0. The lower order terms are given by

$$f_2 = \frac{1}{6}\,\mathrm{tr}_Y\,h_0 \tag{19}$$
$$f_4 = \frac{1}{10}\left(3\,\mathrm{tr}_Y\,h_2-\langle h_0,h_0\rangle\right) \tag{20}$$
$$f_6 = \frac{5}{14}\,\mathrm{tr}_Y\,h_4-\frac{11}{42}\langle h_0,h_2\rangle+\frac{1}{14}\,\mathrm{tr}_Y\,h_{000} \tag{21}$$
$$f_8 = \frac{7}{18}\,\mathrm{tr}_Y\,h_6-\frac{11}{36}\langle h_0,h_4\rangle-\frac{5}{36}\langle h_2,h_2\rangle+\frac{1}{4}\,\mathrm{tr}_Y\,h_{002}+\frac{1}{18}\,\mathrm{tr}_Y\,h_{0000} \tag{22}$$

The leading term of f has the general form of

$$f_{2m+2}=\frac{2m+1}{2(2m+3)}\,\mathrm{tr}_Y\,h_{2m}+\cdots \tag{23}$$

Then, using equation 16, H can be written in terms of the link metric, g_Y, as follows

$$h_0 = -2\left(\mathrm{Ric}(g_Y)_{jk}-(n-1)g_{Y\,jk}\right)=-2\,\mathrm{Ric}(dr^2+r^2g_Y) \tag{24}$$
$$h_2 = -\Delta_L R_{jk}+\frac{1}{3}\,\mathrm{Hess}_Y\,R_{jk}-4R_{jk}+\frac{4}{3}\left[R-(n-1)(n-3)\right]g_{Y\,jk} \tag{25}$$

$$
\begin{aligned}
h_{4jk} \;=\;& -\frac{1}{3}\Delta_L^2 R_{jk} + \frac{2}{15}\operatorname{Hess}_Y(\Delta R)_{jk} - \frac{2}{3}(n+9)\Delta_L R_{jk} - \frac{2}{45}(13n-55)\operatorname{Hess}_Y R_{jk} \\
& + \frac{8}{3}(5n-13)R_{jk} - \frac{16}{3}R_{jx}R_k{}^x + \frac{28}{15}R^{xy}{}_{|j}R_{xy|k} + \frac{38}{15}R^{xy}R_{xy|jk} \\
& + \left[\frac{4}{5}\Delta R + \frac{8}{5}|\operatorname{Ric}|^2 - \frac{8}{15}(9n-25)R + \frac{8}{15}(n-1)(2n-5)(3n-11)\right] g_{Y\,jk} \\
& - \frac{1}{9}\Big(R_{|xk}R_j{}^x + R_{|xj}R_k{}^x - 2R_|{}^x R_{xj|k} - 2R_|{}^x R_{xk|j} + 3R_|{}^x R_{jk|x}\Big) \\
& - 2R^{xy}\Big(R_{jx|yk} + R_{kx|yj}\Big) - \frac{2}{3}R_j{}^x{}_|{}^y R_{[kx|y]} - \frac{2}{3}R_k{}^x{}_|{}^y R_{[jx|y]} \\
& + \frac{4}{3}R^{xy}\Big(R_{jk|xy} - R_{ja}R^a{}_{xyk} - R_{ka}R^a{}_{xyj} + 2R_y{}^z R_{xjkz}\Big)
\end{aligned}
$$

Then using these equations, f can also be written in terms of g_Y as follows.

$$
\begin{aligned}
f_2 \;=\;& -\frac{1}{3}\big[R - (n-1)n\big] \\
f_4 \;=\;& \frac{1}{5}\Big[-\Delta R - 2|\operatorname{Ric}|^2 + 2(3n-5)R - 4n(n-1)(n-2)\Big] \\
f_6 \;=\;& \frac{1}{14}\Bigg[-\Delta^2 R + \frac{4}{3}(2n-17)\Delta R - \frac{32}{9}(19n^2 - 73n + 72)R \\
& + 16|\nabla \operatorname{Ric}|^2 + 16(2n-5)|\operatorname{Ric}|^2 - \frac{5}{9}|\nabla R|^2 + \frac{88}{9}R^2 \\
& + \frac{16}{3}\langle \operatorname{Ric}, \Delta \operatorname{Ric}\rangle - \frac{40}{3}R^{ix}{}_|{}^y R_{iy|x} - 2\langle \operatorname{Ric}, \operatorname{Hess}_Y R\rangle \\
& - 8R^{xy}R_{ab}R^a{}_{xy}{}^b + \frac{8}{45}n(n-1)(118n^2 - 421n + 375)\Bigg]
\end{aligned}
$$

In the next section, these calculations will be explained in more detail.

2.2 Calculations

2.2.1 $dr \otimes dr$-terms define f

In this section, we will show in detail how to derive the equations stated in the previous section. First, look at how to derive equation 14 from equation 11. The Ricci term, $\operatorname{Ric}(g)_{rr}$, can be simplified as follows

$$\begin{aligned} 2R_{rr} &:= 2\left(\Gamma_{rr}{}^{i}{}_{,i} - \Gamma_{ir}{}^{i}{}_{,r} + \Gamma_{rr}{}^{m}\Gamma_{im}{}^{i} - \Gamma_{ir}{}^{m}\Gamma_{mr}{}^{i}\right) \\ &= -\left[H^{il}H_{il,r}\right]_{,r} + \tfrac{1}{2}H^{il}{}_{,r}H_{il,r} \\ &= -H^{il}H_{il,rr} - \tfrac{1}{2}H^{il}{}_{,r}H_{il,r} \end{aligned}$$

The Hessian of f simplifies to

$$2\,\mathrm{Hess}_g\, f_{rr} := 2f_{,rr} - 2\Gamma_{rr}{}^{i}f_{,i} = 2f_{,rr}$$

because $\Gamma_{rr}{}^{i} = 0$.

Therefore, $2f_{,rr} = -1 + H^{il}H_{il,rr} + \frac{1}{2}H^{il}{}_{,r}H_{il,r}$. Now expand each of these terms out and equate the coefficients of the same powers of r. The Hessian of f expands to

$$2\,\mathrm{Hess}_g\, f_{rr} = \sum_{m\geq 1} 2(2m)(2m+1)r^{-2m-2}f_{2m}$$

The two terms of the Ricci tensor expand out to be

$$\begin{aligned} H^{il}H_{il,rr} &= r^{-2}\Big(2n\Big) + r^{-4}\Big(-2\,\mathrm{tr}_Y\, h_0) + r^{-6}(4\,\mathrm{tr}_Y\, h_2 + 2\langle h_0, h_0\rangle\Big) \\ &+ r^{-8}\Big(18\,\mathrm{tr}_Y\, h_4 - 2\langle h_0, h_2\rangle - 2\,\mathrm{tr}_Y\, h_{000}\Big) \\ &+ r^{-10}\Big(40\,\mathrm{tr}_Y\, h_6 - 16\langle h_0, h_4\rangle - 4\langle h_2, h_2\rangle + 2\,\mathrm{tr}_Y\, h_{0000}\Big) + O(r^{-12}) \end{aligned}$$

and

$$\begin{aligned} \tfrac{1}{2}H^{il}{}_{,r}H_{il,r} &= r^{-2}\Big(-2n\Big) + r^{-4}(4\,\mathrm{tr}_Y\, h_0) + r^{-6}\Big(8\,\mathrm{tr}_Y\, h_2 - 6\langle h_0, h_0\rangle\Big) \\ &+ r^{-8}\Big(12\,\mathrm{tr}_Y\, h_4 - 20\langle h_0, h_2\rangle + 8\,\mathrm{tr}_Y\, h_{000}\Big) \\ &+ r^{-10}\Big(16\,\mathrm{tr}_Y\, h_6 - 28\langle h_0, h_4\rangle - 16\langle h_2, h_2\rangle \\ &\qquad + 36\,\mathrm{tr}_Y\, h_{002} - 10\,\mathrm{tr}_Y\, h_{0000}\Big) + O(r^{-12}) \end{aligned}$$

Hence, the Ricci tensor expands to

$$
\begin{aligned}
-2R_{rr} &= H^{il}H_{il,rr} + \tfrac{1}{2}H^{il}{}_{,r}H_{il,r} \\
&= r^{-4}(2\,\mathrm{tr}_Y\,h_0) + r^{-6}\Big(12\,\mathrm{tr}_Y\,h_2 - 4\langle h_0, h_0\rangle\Big) \\
&\qquad + r^{-8}\Big(30\,\mathrm{tr}_Y\,h_4 - 22\langle h_0, h_2\rangle + 6\,\mathrm{tr}_Y\,h_{000}\Big) \\
&\qquad + r^{-10}\Big(56\,\mathrm{tr}_Y\,h_6 - 44\langle h_0, h_4\rangle - 20\langle h_2, h_2\rangle \\
&\qquad\qquad + 36\,\mathrm{tr}_Y\,h_{002} - 8\,\mathrm{tr}_Y\,h_{0000}\Big) + O(r^{-12})
\end{aligned}
$$

Therefore, the leading order term for f in equation 5 comes from integrating the -1, while the lower order terms come from the $dr \wedge dr$ coefficients of the Ricci tensor. This gives equations 19-22. Notice that f_0 is not defined by these equations.

2.2.2 $dx^j \otimes dx^k$-terms define H_o

Look at the $dx^j \otimes dx^k$ coefficients in equation 3 to define H_o, i.e. equation 13. First, derive equation 16 from equation 13. The Hessian of f can be expressed as follows.

$$
\begin{aligned}
\mathrm{Hess}_g\, f_{jk} &= f_{,jk} - {}^g\Gamma_{jk}{}^i f_{,i} \\
&= f_{,jk} - {}^H\Gamma_{jk}{}^i f_{,i} + \frac{1}{2}H_{jk,r}f_{,r} \\
&= \mathrm{Hess}_H\, f_{jk} + \frac{1}{2}H_{jk,r}f_{,r}
\end{aligned}
$$

To write the Hessian of H in terms of the link metric, look at the Christoffel symbol of H.

$$
\begin{aligned}
2{}^H\Gamma_{jk}{}^i &= H^{il}H_{[jk,l]} \\
&= \left(r^{-2}g_Y{}^{il} - r^{-4}H_o^{il} + r^{-6}H_{oo}^{il} - \cdots\right)\left(r^2 g_{Y\,[jk,l]} + H_{o[jk,l]}\right) \\
&= 2{}^Y\Gamma_{jk}{}^i + r^{-2}(g_Y^{il}H_{o[jk,l]} - H_o{}^{il}g_{Y\,[jk,l]}) \\
&\qquad\qquad - r^{-4}(H_o{}^{il}H_{o[jk,l]} - H_{oo}{}^{il}g_{Y\,[jk,l]}) + \cdots \\
&= 2{}^Y\Gamma_{jk}{}^i + r^{-2}(g_Y^{il}H_{o[jk,l]} - 2H_o{}^i{}_x\Gamma_{jk}{}^x) \\
&\qquad\qquad - r^{-4}(H_o{}^{il}H_{o[jk,l]} - 2H_{oo}{}^i{}_x\Gamma_{jk}{}^x) + \cdots \\
&= 2{}^Y\Gamma_{jk}{}^i + r^{-2}g_Y{}^{il}H_{o[jk|l]} - r^{-4}H_o{}^{il}H_{o[jk|l]} + \cdots \\
&= 2{}^Y\Gamma_{jk}{}^i + H^{il}H_{o[jk|l]}
\end{aligned}
$$

This shows equation 10.

Hence,

$$\begin{aligned}\mathrm{Hess}_H\, f_{jk} &= f_{,jk} - {}^H\Gamma_{jk}{}^i f_{,i} \\ &= f_{,jk} - \left[{}^Y\Gamma_{jk}{}^i + \frac{1}{2} H^{il} H_{o[jk|l]} \right] f_{,i} \\ &= \mathrm{Hess}_Y\, f_{jk} - \frac{1}{2} H^{il} H_{o[jk|l]} f_{,i}\end{aligned}$$

These two equations give that the Hessian of f can be expanded to

$$2\,\mathrm{Hess}_g\, f_{jk} = 2\,\mathrm{Hess}_Y\, f_{jk} - H^{il} H_{o[jk|l]} f_{,i} + H_{jk,r} f_{,r} \tag{26}$$

as seen in equation 16. The Hessian of f can be written more explicitly as

$$\begin{aligned}2\,\mathrm{Hess}_H\, f_{jk} &= r^{-2}(2\,\mathrm{Hess}_Y\, f_{2jk}) + r^{-4}(2\,\mathrm{Hess}_Y\, f_{4jk} - g_Y^{il} h_{0[jk|l]} f_{2,i}) \\ &\quad + r^{-6}\Big(2\,\mathrm{Hess}_Y\, f_{6jk} - g_Y^{il} h_{0[jk|l]} f_{4,i} - \left[g_Y^{il} h_{2[jk|l]} - h_0^{il} h_{0[jk|l]} \right] f_{2,i}\Big) \\ &\quad + r^{-8}\Big(2\,\mathrm{Hess}_Y\, f_{8jk} - g_Y^{il} h_{0[jk|l]} f_{6,i} - \left[g_Y^{il} h_{2[jk|l]} - h_0^{il} h_{0[jk|l]} \right] f_{4,i} \\ &\qquad - \left[g_Y^{il} h_{4[jk|l]} - h_0^{il} h_{2[jk|l]} - h_2^{il} h_{0[jk|l]} + h_{00}^{il} h_{0[jk|l]} \right] f_{2,i}\Big) \\ &\quad + O(r^{-10})\end{aligned}$$

Here, we have assumed that f_0 is a constant function on Y, so it's derivatives are all zero. In fact, this is a necessary condition in order for the expansions of f and H to satisfy the constraint equation. This assumption will be justified in section 2.3 bellow.

Finally, expand out $H_{jk,r} f_{,r}$ in terms or equation 4 and 5.

$$\begin{aligned}H_{jk,r} f_{,r} &= \left(2r g_{Y\,jk} - \sum_{m=1}^{\infty} (2m) r^{-2m-1} h_{2m\,jk} \right) \cdot \left(-\frac{1}{2} r - \sum_{m=1}^{\infty} (2m) r^{-2m-1} f_{2m} \right) \\ &= r^2(-g_{Y\,jk}) + \sum_{m=1}^{\infty} r^{-2m} \big(m h_{2m\,jk} - (4m) f_{2m} g_{Y\,jk} \big) \\ &\qquad + \sum_{m=2}^{\infty} r^{-2m-2} \left(\sum_{\mu+\nu=m} (2\mu)(2\nu) f_{2\mu} h_{2\nu\,jk} \right) \\ &= r^2(-g_{Y\,jk}) + r^{-2}(h_{2jk} - 4 f_2 g_{Y\,jk}) + r^{-4}(2h_{4jk} - 8 f_4 g_{Y\,jk}) \\ &\qquad + r^{-6}(3h_{6jk} + 4 f_2 h_{2jk} - 12 f_6 g_{Y\,jk}) \\ &\qquad + r^{-8}(4h_{8jk} + 8 f_2 h_{4jk} + 8 f_4 h_{2jk} - 16 f_8 g_{Y\,jk}) + O(r^{-10})\end{aligned}$$

Combining these two equations gives the first few terms in the expansion of $2\,\mathrm{Hess}_g f_{jk}$.

Next, calculate the expansion of $\mathrm{Ric}(g)_{jk}$. From the definition in terms of Christoffel symbols,

$$\begin{aligned} 2\,\mathrm{Ric}(g)_{jk} &= 2\left(\Gamma_{jk}{}^{i}{}_{,i} - \Gamma_{ik}{}^{i}{}_{,j} + \Gamma_{jk}{}^{m}\Gamma_{im}{}^{i} - \Gamma_{ij}{}^{m}\Gamma_{mk}{}^{i}\right) \\ &= 2\left(\mathrm{Ric}(H)_{jk} + \Gamma_{jk}{}^{r}{}_{,r} + \Gamma_{jk}{}^{r}\Gamma_{ir}{}^{i} - \Gamma_{rj}{}^{m}\Gamma_{mk}{}^{r} - \Gamma_{ij}{}^{r}\Gamma_{rk}{}^{i}\right) \\ &= 2\left(\mathrm{Ric}(H)_{jk} - \frac{1}{2}H_{jk,rr} - \frac{1}{4}H^{il}H_{il,r}H_{jk,r} + \frac{1}{2}H^{il}H_{ij,r}H_{kl,r}\right) \end{aligned}$$

Thus, the Ricci tensor can be expanded into

$$2\,\mathrm{Ric}(g)_{jk} = 2\,\mathrm{Ric}(H)_{jk} - H_{jk,rr} - \frac{1}{2}H^{il}H_{il,r}H_{jk,r} + H^{il}H_{ij,r}H_{kl,r} \tag{27}$$

Combining equation 26 and equation 27 gives equation 16. The Ricci tensor of H will involve covariant derivatives, while the other terms can be expanded directly. Look at each of these lower order terms first.

$$-H_{jk,rr} = -2g_{Y\,jk} - 6r^{-4}h_{2jk} - 20r^{-6}h_{4jk} - 42r^{-8}h_{6jk} + O(r^{-10})$$

Second,

$$\begin{aligned} -\tfrac{1}{2}H^{il}H_{il,r}H_{jk,r} &= (-2ng_{Y\,jk}) + r^{-2}(2\,\mathrm{tr}_Y\,h_0)g_{Y\,jk} \\ &\quad + r^{-4}\Big(2nh_{2jk} + [4\,\mathrm{tr}_Y\,h_2 - 2\langle h_0, h_0\rangle]g_{Y\,jk}\Big) \\ &\quad + r^{-6}\Big(4nh_{4jk} - (2\,\mathrm{tr}_Y\,h_0)h_{2jk} \\ &\qquad + [6\,\mathrm{tr}_Y\,h_4 - 6\langle h_0, h_2\rangle + 2\,\mathrm{tr}_Y\,h_{000}]g_{Y\,jk}\Big) \\ &\quad + r^{-8}\Big(6nh_{6jk} - (4\,\mathrm{tr}_Y\,h_0)h_{4jk} - [4\,\mathrm{tr}_Y\,h_2 - 2\langle h_0, h_0\rangle]h_{2jk} \\ &\qquad + [8\,\mathrm{tr}_Y\,h_6 - 8\langle h_0, h_4\rangle - 4\langle h_2, h_2\rangle \\ &\qquad\quad + 8\,\mathrm{tr}_Y\,h_{002} - 2\,\mathrm{tr}_Y\,h_{0000}]g_{Y\,jk}\Big) + O(r^{-10}) \end{aligned}$$

Third,

$$H^{il}H_{ij,r}H_{kl,r} = 4g_{Y\,jk} + r^{-2}(-4h_{0jk}) + r^{-4}\Big(-12h_{2jk} + 4h_{00jk}\Big)$$
$$+ r^{-6}\Big(-20h_{4jk} + 8h_{20jk} + 8h_{02jk} - 4h_{000jk}\Big)$$
$$+ r^{-8}\Big(-28h_{6jk} + 12h_{40jk} + 12h_{04jk} + 16h_{22jk}$$
$$- 8h_{200jk} - 8h_{002jk} - 4h_{020jk} + 4h_{000jk}\Big) + O(r^{-10})$$

Thus, the lower order terms expanding out to be

$$-H_{jk,rr} - \tfrac{1}{2}H^{il}H_{il,r}H_{jk,r} + H^{il}H_{ij,r}H_{kl,r}$$
$$= -2(n-1)g_{Y\,jk} + r^{-2}\Big(\big[2\,\mathrm{tr}_Y\,h_0\big]g_{Y\,jk} - 4h_{0jk}\Big)$$
$$+ r^{-4}\Big(2(n-9)h_{2jk} + 4h_{00jk} + \big[4\,\mathrm{tr}_Y\,h_2 - 2\langle h_0, h_0\rangle\big]g_{Y\,jk}\Big)$$
$$+ r^{-6}\Bigg(4(n-10)h_{4jk} - \big[2\,\mathrm{tr}_Y\,h_0\big]h_{2jk} + 8h_{02jk} + 8h_{20jk} - 4h_{000jk}$$
$$+ \Big[6\,\mathrm{tr}_Y\,h_4 - 6\langle h_0, h_2\rangle + 2\,\mathrm{tr}_Y\,h_{000}\Big]g_{Y\,jk}\Bigg)$$
$$+ r^{-8}\Bigg((6n-70)h_{6jk} - \big[4\,\mathrm{tr}_Y\,h_0\big]h_{4jk} + \big[2\langle h_0, h_0\rangle - 4\,\mathrm{tr}_Y\,h_2\big]h_{2jk}$$
$$+ 12h_{04jk} + 12h_{40jk} + 16h_{22jk}$$
$$- 8h_{002jk} - 8h_{200jk} - 4h_{020jk} + 4h_{0000jk}$$
$$+ \Big[8\,\mathrm{tr}_Y\,h_6 - 8\langle h_0, h_4\rangle - 4\langle h_2, h_2\rangle + 8\,\mathrm{tr}_Y\,h_{002} - 2\,\mathrm{tr}_Y\,h_{0000}\Big]g_{Y\,jk}\Bigg)$$
$$+O(r^{-10})$$

Then calculate $\mathrm{Ric}(H)_{jk}$ using $2\,{}^{H}\Gamma_{jk}{}^{i} = 2\,{}^{Y}\Gamma_{jk}{}^{i} + H^{il}H_{o[jk|l]}$ where all covariant derivatives

(”|”) are with respect to the g_Y metric.

$$\begin{aligned}
2\,\mathrm{Ric}(H)_{jk} &= 2\left({}^{H}\Gamma_{jk}{}^{i}{}_{,i} - {}^{H}\Gamma_{ik}{}^{i}{}_{,j} + {}^{H}\Gamma_{jk}{}^{mH}\Gamma_{im}{}^{i} - {}^{H}\Gamma_{ik}{}^{mH}\Gamma_{mj}{}^{i}\right) \\
&= 2\,\mathrm{Ric}(g_Y)_{jk} \\
&\quad + H^{il}(H_{o[jk|l],i} - H_{o[ik|l],j}) + H^{il}{}_{,i}H_{o[jk|l]} - H^{il}{}_{,j}H_{o[ik|l]} \\
&\quad + {}^{Y}\Gamma_{jk}{}^{m}H^{il}H_{o[im|l]} + {}^{Y}\Gamma_{im}{}^{i}H^{mn}H_{o[jk|n]} \\
&\qquad - {}^{Y}\Gamma_{ik}{}^{m}H^{il}H_{o[mj|l]} - {}^{Y}\Gamma_{mj}{}^{i}H^{mn}H_{o[ik|n]} \\
&\quad + \tfrac{1}{2}H^{il}H^{nm}(H_{o[jk|n]}H_{o[im|l]} - H_{o[ik|n]}H_{o[mj|l]})
\end{aligned}$$

Notice that by definition of the covariant derivative

$$H_{o[jk|l]}(H^{in}{}_{,i} + {}^{Y}\Gamma_{im}{}^{i}H^{mn}) = H_{o[jk|l]}(H^{in}{}_{|i} - {}^{Y}\Gamma_{im}{}^{n}H^{im})$$

Then using that $H^{ab}{}_{|i}H_{bc} = -H^{ab}H_{bc|i}$

$$H^{in}{}_{|i} = (H^{il}H^{mn}H_{ml})_{|i} = -H^{il}H^{mn}H_{ml|i}$$

Hence,

$$H_{o[jk|l]}\left(\frac{1}{2}H^{il}H^{mn}H_{o[im|l]} + H^{in}{}_{|i}\right) = -\frac{1}{2}H^{il}H^{nm}H_{o[jk|n]}H_{o[il|m]}$$

Therefore,

$$\begin{aligned}
2\,\mathrm{Ric}(H)_{jk} &= 2\,\mathrm{Ric}(g_Y)_{jk} + H^{il}\left[H_{o[jk|l]|i} - H_{o[ik|l]|j}\right] \\
&\qquad + \tfrac{1}{2}H^{il}H^{nm}\left(H_{o[jk|n]}H_{o[il|m]} - H_{o[ik|n]}H_{o[jl|m]}\right) \\
&= 2\,\mathrm{Ric}(g_Y)_{jk} + H^{il}\left[H_{ojl|ki} + H_{okl|ji} - H_{ojk|il} - H_{oil|jk}\right] \\
&\qquad + \tfrac{1}{2}H^{il}H^{nm}\left(H_{o[jk|n]}H_{o[il|m]} - H_{o[ik|n]}H_{o[jl|m]}\right) \\
&= 2\,\mathrm{Ric}(g_Y)_{jk} + r^{-2}g_Y{}^{il}\left[h_{0jl|ki} + h_{0kl|ji} - h_{0jk|il} - h_{0il|jk}\right] \\
&\quad + r^{-4}\Big(g_Y{}^{il}\left[h_{2jl|ki} + h_{2kl|ji} - h_{2jk|il} - h_{2il|jk}\right] \\
&\qquad + h_0{}^{il}\left[h_{0jl|ki} + h_{0kl|ji} - h_{0jk|il} - h_{0il|jk}\right] \\
&\qquad + \tfrac{1}{2}g_Y{}^{il}g_Y{}^{nm}\left(h_{0[jk|n]}h_{0[il|m]} - h_{0[ik|n]}h_{0[jl|m]}\right)\Big) \\
&\quad + O(r^{-6})
\end{aligned}$$

This shows equation 17.

2.2.3 Calculating h_0 and h_2

Now that we have calculated the first few terms in the expansion of

$$2\,\mathrm{Ric}(g)_{jk} + 2\,\mathrm{Hess}_g\, f_{jk} + H_{jk} = 0,$$

the terms can be plugged in and we can solve for the first few terms in the expansion of H in terms of the metric g_Y.

First, look at the coefficients for r^2. Only two terms are nonzero

$$-r^2 g_{Y\,jk} + r^2 g_{Y\,jk} = 0$$

Hence, $f = -\frac{1}{4}r^2 + O(r^0)$ is the correct leading term for the potential function, f.

Second, look at the coefficients for r^0. If ϕ is a power series in r, then introduce the notation that $(\phi)^{(-2m)}$ denotes the coefficient of r^{-2m} in the expansion of ϕ. Then, using the expansions of $\mathrm{Hess}_H\, f$ and $H_{jk,r} f_{,r}$ from above gives

$$\begin{aligned}(\mathrm{Hess}_g\, f_{jk})^{(0)} &= (2\,\mathrm{Hess}_Y\, f_{jk} + H^{il} H_{o[jk|l]} f_{,i})^{(0)} + (H_{jk,r} f_{,r})^{(0)} \\ &= 0 + 0 = 0\end{aligned}$$

Using the expansion equations for $\mathrm{Ric}(g)_{jk}$ above gives

$$\begin{aligned}(\mathrm{Ric}(g)_{jk})^{(0)} &= (2\,\mathrm{Ric}(H)_{jk})^{(0)} + (-H_{jk,rr} - \frac{1}{2} H^{il} H_{il,r} H_{jk,r} + H^{il} H_{ij,r} H_{kl,r})^{(0)} \\ &= 2\,\mathrm{Ric}(g_Y)_{jk} - 2(n-1) g_{Y\,jk}\end{aligned}$$

Hence, the coefficients for r^0 give the equation

$$2\,\mathrm{Ric}(g_Y)_{jk} - 2(n-1) g_{Y\,jk} + h_0 = 0.$$

Therefore,

$$h_0 = -2\left(\mathrm{Ric}(g_Y)_{jk} - (n-1) g_{Y\,jk}\right) = -2\,\mathrm{Ric}(dr^2 + r^2 g_Y)_{jk}$$

Notice that this is the time derivative of running the Ricci flow on the cone metric $dr^2 + r^2 g_Y$. This shows equation 24.

Finally, look at the coefficients for r^{-2}. Now, the equation for h_0 in terms of g_Y can used as well.

First, look at $2\,\mathrm{Ric}(H)_{jk}$.

$$\begin{aligned}\left(2\,\mathrm{Ric}(H)_{jk}\right)^{(2)} &= {g_Y}^{il}\left[h_{0jl|ki} + h_{0kl|ji} - h_{0jk|il} - h_{0il|jk}\right] \\ &= -\Delta_L h_{0jk} + {g_Y}^{il}\left[h_{0jl|ik} + h_{0kl|ij} - h_{0il|jk}\right] \\ &= 2\Delta_L R_{jk} - 2{g_Y}^{il}\left[R_{jl|ik} + R_{kl|ij} - R_{il|jk}\right] \\ &= 2\Delta_L R_{jk} - 2\left[R_{ji|}{}^{i}{}_{k} + R_{ki|}{}^{i}{}_{j} - R_{|jk}\right] \\ &= 2\Delta_L R_{jk}\end{aligned}$$

where Δ_L denotes the Lichnerowicz Laplacian. Here the second line follows by commuting the covariant derivatives of the first two terms in the brackets on the first line. This equality is true for any 2-tensor and will be used in the next section. The last line follows from the contracted Bianchi identity

$$R_{|x} = 2R_{xi|}{}^{i}$$

Notice that this means the expansion of the Ricci tensor is $\mathrm{Ric}(H) = \mathrm{Ric}(g_Y) + r^{-2}\Delta_L R_{jk} + \cdots$ where $\Delta_L R_{jk}$ the time derivative of $\mathrm{Ric}(g_Y)$ under the Ricci flow.

Second, $\Big(- H_{jk,rr} - \frac{1}{2}H^{il}H_{il,r}H_{jk,r} + H^{il}H_{ij,r}H_{kl,r}\Big)^{(2)}$

$$\begin{aligned} &= \big[2\,\mathrm{tr}_Y\, h_0\big] g_{Y\,jk} - 4h_{0jk} \\ &= -4\big[R - (n-1)n\big] g_{Y\,jk} + 8(R_{jk} - (n-1)g_{Y\,jk}) \\ &= 8R_{jk} - 4\big[R - (n-1)(n-2)\big] g_{Y\,jk} \end{aligned}$$

Next, plug in the equation for h_0 into the equation for f_2. This gives

$$f_2 = \frac{1}{6}\,\mathrm{tr}_Y\, h_0 = -\frac{1}{3}\big[R - (n-1)n\big]$$

Third, look at the Hessian terms.

$$\begin{aligned} \Big(2\,\mathrm{Hess}_H\, f_{jk}\Big)^{(2)} &= \Big(2\,\mathrm{Hess}_Y\, f_{jk} + H^{il}H_{o[jk|l]}f_{,i}\Big)^{(2)} \\ &= 2\,\mathrm{Hess}_Y\, f_{2jk} \\ &= -\frac{2}{3}\,\mathrm{Hess}_Y\, R_{jk} \end{aligned}$$

and

$$\begin{aligned} \Big(H_{jk,r}f_{,r}\Big)^{(2)} &= h_{2jk} - 4f_2 g_{Y\,jk} \\ &= h_{2jk} + \frac{4}{3}\big[R - (n-1)n\big] g_{Y\,jk} \end{aligned}$$

Plugging these four equations in:

$$\begin{aligned} &2\Delta_L R_{jk} + 8R_{jk} - 4\big[R - (n-1)(n-2)\big] g_{Y\,jk} \\ &\qquad - \frac{2}{3}\,\mathrm{Hess}_Y\, R_{jk} + h_{2jk} + \frac{4}{3}\big[R - (n-1)n\big] g_{Y\,jk} + h_{2jk} = 0 \end{aligned}$$

Therefore, we have shown equation 25,

$$h_{2jk} = -\Delta_L R_{jk} + \frac{1}{3}\,\mathrm{Hess}_Y\, R_{jk} - 4R_{jk} + \frac{4}{3}\big[R - (n-1)(n-3)\big] g_{Y\,jk}$$

Hence,

$$\mathrm{tr}_Y\, h_2 = -\frac{2}{3}\Delta R + \frac{4}{3}(n-3)R - \frac{4}{3}n(n-1)(n-3)$$

2.2.4 Calculate h_4

Now, the equations start to become more complicated. The equation for h_{2i} will involve $2i$ covariant derivatives of $\mathrm{Ric}(g_Y)$ with respect to g_Y and has nonhomogenous terms of order i+1. Never the less, all the terms are very straight forward to calculate.

As an example, we will calculate h_4. First, look at the Ricci tensor for H. Recall that

$$\begin{aligned}\Big(2\,\mathrm{Ric}(H)_{jk}\Big)^{(4)} = {g_Y}^{il}\big[h_{2jl|ki} + h_{2kl|ji} - h_{2jk|il} - h_{2il|jk}\big] \\ + \ {h_0}^{il}\left[h_{0jl|ki} + h_{0kl|ji} - h_{0jk|il} - h_{0il|jk}\right] \\ + \ \frac{1}{2}{g_Y}^{il}{g_Y}^{nm}\big(h_{0[jk|n]}h_{0[il|m]} - h_{0[ik|n]}h_{0[jl|m]}\big)\end{aligned} \tag{28}$$

Then look at each of the lines separately. Use the metric property of g_Y,

$$\begin{aligned}&\frac{1}{2}{g_Y}^{il}{g_Y}^{nm}\big(h_{0[jk|n]}h_{0[il|m]} - h_{0[ik|n]}h_{0[jl|m]}\big) \\ &\qquad = 2{g_Y}^{il}{g_Y}^{nm}\big(R_{[jk|n]}R_{[il|m]} - R_{[ik|n]}R_{[jl|m]}\big)\end{aligned}$$

Then use the contractioned Bianchi identity, $2R_{xy|}{}^{y} = R_{|x}$, to show,

$$\begin{aligned}{g_Y}^{il}R_{[il|m]} &= {g_Y}^{il}\Big(R_{mi|l} + R_{ml|i} - R_{il|m}\Big) \\ &= \frac{1}{2}R_{|m} + \frac{1}{2}R_{|m} - R_{|m} = 0\end{aligned}$$

Hence, the last line of the equation 28 simplifies to

$$\frac{1}{2}{g_Y}^{il}{g_Y}^{nm}\big(h_{0[jk|n]}h_{0[il|m]} - h_{0[ik|n]}h_{0[jl|m]}\big) = -2{g_Y}^{il}{g_Y}^{nm}R_{[ik|n]}R_{[jl|m]}$$

Next, look at the second line of equation 28 and use the metric property again,

$$\begin{aligned}&{h_0}^{il}\left[h_{0jl|ki} + h_{0kl|ji} - h_{0jk|il} - h_{0il|jk}\right] \\ &\qquad = 4\left(R^{il} - (n-1){g_Y}^{il}\right)\left[R_{jl|ki} + R_{kl|ji} - R_{jk|il} - R_{il|jk}\right] \\ &\qquad\qquad = 4R^{il}\left[R_{jl|ki} + R_{kl|ji} - R_{jk|il} - R_{il|jk}\right] + 4(n-1)\Delta_L R_{jk}\end{aligned}$$

Now look at the first line of equation 28,

$${g_Y}^{il}\big[h_{2jl|ki} + h_{2kl|ji} - h_{2jk|il} - h_{2il|jk}\big]$$

and plug in

$$h_{2jk} = -\Delta_L R_{jk} + \frac{1}{3}\operatorname{Hess}_Y R_{jk} - 4R_{jk} + \frac{4}{3}\big[R - (n-1)(n-3)\big] g_{Y\,jk}.$$

Plug in $\frac{4}{3}\big[R - (n-1)(n-3)\big] g_{Y\,jk}$ gives

$$\frac{4}{3} g_Y{}^{il}\big[R_{|ki} g_{Y\,jl} + R_{|ji} g_{Y\,kl} - R_{|il} g_{Y\,jk} - R_{|jk} g_{Y\,il}\big]$$

$$= -\frac{4}{3}(n-2)\operatorname{Hess}_Y R_{jk} - \frac{4}{3}\Big(\Delta R\Big) g_{Y\,jk}$$

Next plugging in $-4R_{jk}$ gives $-4(-\Delta_L R_{jk}) = 4\Delta_L R_{jk}$. Then plug in $\frac{1}{3}\operatorname{Hess}_Y R_{jk}$ gives,

$$\frac{1}{3} g_Y{}^{il}\Big[R_{|jlki} + R_{|klji} - R_{|jkil} - R_{|iljk}\Big].$$

Notice that by commuting the derivatives these terms will cancel out and leave only lower order commutation terms. Calculate these commutation terms in two steps. First, turn $R_{|jik}{}^{i}$ into $R_{|jki}{}^{i}$ by exchanging the inner two covariant derivatives.

$$\begin{aligned}
R_{|jik}{}^{i} &= R_{|jki}{}^{i} + \Big[R_{|x} R_{ikj}{}^{x}\Big]_{|}{}^{i} \\
&= R_{|jki}{}^{i} + R_{|xy} R^{x}{}_{jk}{}^{y} - R_{|x}\Big(R_{ik}{}^{xi}{}_{|j} + R_{ik}{}^{i}{}_{j|}{}^{x}\Big) \\
&= R_{|jki}{}^{i} + R_{|xy} R^{x}{}_{jk}{}^{y} - R_{|x} R_{k}{}^{x}{}_{|j} + R_{|x} R_{jk|}{}^{x}
\end{aligned}$$

Second, turn $R_{|ikj}{}^{i}$ into $R_{|i}{}^{i}{}_{kj}$.

$$\begin{aligned}
R_{|ikj}{}^{i} &= R_{|ik}{}^{i}{}_{j} + \Big[R_{|xk} R_{j}{}^{i}{}_{i}{}^{x} + R_{|ix} R_{j}{}^{i}{}_{k}{}^{x}\Big] \\
&= R_{|i}{}^{i}{}_{kj} + \Big[R_{|x} R_{k}{}^{i}{}_{i}{}^{x}\Big]_{|j} + R_{|kx} R^{x}{}_{j} - R_{|xy} R^{x}{}_{jk}{}^{y} \\
&= R_{|i}{}^{i}{}_{kj} - R_{|xy} R^{x}{}_{jk}{}^{y} + R_{|x} R_{k}{}^{x}{}_{|j} + R_{|jx} R^{x}{}_{k} + R_{|kx} R^{x}{}_{j}
\end{aligned}$$

Therefore,

$$\frac{1}{3}\Big(R_{|k\alpha}{}^{\alpha}{}_{j} + R_{|j\alpha}{}^{\alpha}{}_{k} - \Delta_L R_{|jk} - \operatorname{Hess}_Y(\Delta R)_{jk}\Big)$$

$$= \frac{1}{3}\Big(R_{|xk} R_{j}{}^{x} + R_{|xj} R_{k}{}^{x} + R_{|}{}^{x} R_{jk|x}\Big). \quad (29)$$

Finally, plug in $-\Delta_L R_{jk}$ to get

$$\Delta_L^2 R_{jk} + \operatorname{Hess}_Y(\Delta R)_{jk} - g_Y{}^{il}\Big[(\Delta_L R_{jl})_{|ik} + (\Delta_L R_{kl})_{|ij}\Big]$$

Notice, by commuting the derivatives of the last three terms and using the contracted Bianchi Identity, $2R_{xy|}{}^{y} = R_{|x}$, the only fourth order term is, $\Delta_L^2 R_{jk}$. To calculate the commutation terms, recall that the Lichnerowicz Laplacian of a 2−tensor, T_{il}, is defined as

$$\Delta_L T_{il} := \Delta T_{il} - R_i{}^x T_{xl} - T_i{}^x R_{xl} + 2T_{xy} R^x{}_{il}{}^y.$$

Then look at the relationship of $\nabla^i \Delta_L T_{il}$ and $\Delta \nabla^i T_{il}$.

$$\begin{aligned}
\nabla^i \Delta T_{il} &= T_{il|a}{}^{ia} + T_{xl|a} R^{ai}{}_i{}^x + T_{ix|a} R^{ai}{}_l{}^x + T_{il|x} R^{ai}{}_a{}^x \\
&= T_{il|a}{}^{ia} + T_{ix|a} R^{ai}{}_l{}^x \\
&= T_{il|ia}{}^a + T_{ix|a} R^{ai}{}_l{}^x + \Big[T_{xl} R_a{}^i{}_i{}^x + T_{ix} R_a{}^i{}_l{}^x\Big]_|{}^a \\
&= \Delta \nabla^i T_{il} - T_{xy|}{}^i R^x{}_{il}{}^y + \Big[R_i{}^x T_{xl}\Big]_|{}^i - \Big[T_{xy} R^x{}_{il}{}^y\Big]_|{}^i
\end{aligned}$$

Hence,

$$\begin{aligned}
\nabla^i \Delta_L T_{il} &= \Delta \nabla^i T_{il} - T_{xy|}{}^i R^x{}_{il}{}^y + \Big[R_i{}^x T_{xl}\Big]_|{}^i - \Big[T_{xy} R^x{}_{il}{}^y\Big]_|{}^i \\
&\quad - \Big[R_i{}^x T_{xl}\Big]_|{}^i - \Big[T_i{}^x R_{xl}\Big]_|{}^i + 2\Big[T_{xy} R^x{}_{il}{}^y\Big]_|{}^i \\
&= \Delta \nabla^i T_{il} - \Big[T_i{}^x R_{xl}\Big]_|{}^i + T_{xy} R^x{}_{il}{}^y{}_|{}^i \\
&= \Delta \nabla^i T_{il} - \Big[T_i{}^x R_{xl}\Big]_|{}^i - T_{xy}\Big[R^x{}_i{}^{yi}{}_{|l} + R^x{}_i{}^i{}_{l|}{}^y\Big] \\
&= \Delta \nabla^i T_{il} - T^{ix}{}_{|i} R_{xl} - T^{xy} R_{[xy|l]}
\end{aligned}$$

Therefore,

$$\nabla^i \Delta_L T_{il} = \Delta \nabla^i T_{il} - T^{ix}{}_{|i} R_{xl} - T^{xy} R_{[xy|l]} \tag{30}$$

Recall also that for any smooth function, ϕ

$$\Delta \nabla_i \phi = \nabla_i \Delta \phi + \phi_|{}^x R_{xi} \tag{31}$$

Now plug, $T = \mathrm{Ric}(g_Y)$, into equation 30

$$\begin{aligned}
\nabla^i \Delta_L R_{ij} &= \Delta \nabla^i R_{ij} - R^{ix}{}_{|i} R_{xj} - R^{xy} R_{[xy|j]} \\
&= \frac{1}{2}\Delta \nabla^j R - \frac{1}{2} R_|{}^x R_{xj} - R^{xy} R_{[xy|j]} \\
&= \frac{1}{2}\nabla^j \Delta R - R^{xy} R_{[xy|j]}
\end{aligned}$$

Therefore,

$$\operatorname{Hess}_Y(\Delta R)_{jk} - \Big(\Delta_L R_{ij}\Big)_{|}{}^{i}{}_{k} - \Big(\Delta_L R_{ik}\Big)_{|}{}^{i}{}_{j} = \Big[R^{xy} R_{[xy|j]}\Big]_{|k} + \Big[R^{xy} R_{[xy|k]}\Big]_{|j} \qquad (32)$$

Combining these equations and simplifying gives

$$\begin{aligned}
\Big(2\operatorname{Ric}(H)_{jk}\Big)^{(4)} &= \Delta_L^2 R_{jk} + 4n\Delta_L R_{jk} - \frac{4}{3}(n-2)\operatorname{Hess}_Y R_{jk} - \frac{4}{3}\Big(\Delta R\Big) g_{Y\,jk} \\
&\quad + \frac{1}{3}\Big(R_{|xk} R_j{}^x + R_{|xj} R_k{}^x + R_|{}^x R_{jk|x}\Big) - 4R^{xy}{}_{|j} R_{xy|k} \\
&\quad + 6R^{xy}\Big(R_{jx|yk} + R_{kx|yj} - R_{xy|jk}\Big) + 2R_j{}^x{}_|{}^y R_{[kx|y]} + 2R_k{}^x{}_|{}^y R_{[jx|y]} \\
&\quad - 4R^{xy}\Big(R_{jk|xy} - R_{ja} R^a{}_{xyk} - R_{ka} R^a{}_{xyj} + 2R_y{}^z R_{xjkz}\Big)
\end{aligned}$$

Second, look at the lower order terms of the Ricci curvature of H.

$$\begin{aligned}
&\Big(-H_{jk,rr} - \tfrac{1}{2} H^{il} H_{il,r} H_{jk,r} + H^{il} H_{ij,r} H_{kl,r} = -2(n-1) g_{Y\,jk}\Big)^{(4)} \\
&= 2(n-9) h_{2jk} + 4h_{00jk} + \Big[4\operatorname{tr}_Y h_2 - 2\langle h_0, h_0\rangle\Big] g_{Y\,jk} \\
&= -2(n-9)\Delta_L R_{jk} + \frac{2}{3}(n-9)\operatorname{Hess}_Y R_{jk} - 8(5n-13) R_{jk} + 16 R_{jx} R_k{}^x \\
&\quad + \Big[-\frac{8}{3}\Delta R - 8|\operatorname{Ric}|_Y^2 + 8(3n-7)R - 8(n-1)(2n^2-9n+11)\Big] g_{Y\,jk}
\end{aligned}$$

Next, update f_4 using the equation for h_0 and h_4,

$$\begin{aligned}
f_4 &= \frac{1}{10}\Big[3\operatorname{tr}_Y h_2 - \langle h_0, h_0\rangle\Big] \\
&= \frac{1}{10}\Big[-2\Delta R + 4(n-3)R - 4n(n-1)(n-3) \\
&\qquad\qquad -4|\operatorname{Ric}|^2 + 8(n-1)R - 4n(n-1)^2\Big] \\
&= \frac{1}{10}\Big[-2\Delta R - 4|\operatorname{Ric}|^2 + 4(3n-5)R - 8n(n-1)(n-2)\Big] \\
&= \frac{1}{5}\Big[-\Delta R - 2|\operatorname{Ric}|^2 + 2(3n-5)R - 4n(n-1)(n-2)\Big]
\end{aligned}$$

Third, look at the Hessian of f.

$$\begin{aligned}
\operatorname{Hess}_H f_{jk} &= \Big(2\operatorname{Hess}_Y f_{jk} - H^{il} H_{o[jk|l]} f_{,i}\Big)^{(4)} \\
&= 2\operatorname{Hess}_Y f_{4jk} - g_Y^{il} h_{0[jk|l]} f_{2,i} \\
&= \frac{2}{5}\operatorname{Hess}_Y\Big[-(\Delta R) - 2|\operatorname{Ric}|^2 + 2(3n-5)R\Big]_{jk} - \frac{2}{3} R_|{}^x R_{[jk|x]}
\end{aligned}$$

Then lower order term simplifies to

$$\Big(H_{jk,r}f_{,r}\Big)^{(4)} = 2h_{4jk} - 8f_4 g_{Y\,jk}$$
$$= 2h_{4jk} - \frac{8}{5}\Big[-\Delta R - 2|\operatorname{Ric}|^2 + 2(3n-5)R - 4n(n-1)(n-2)\Big]g_{Y\,jk}$$

Hence,

$$\Big(2\operatorname{Hess}_H f_{jk}\Big)^{(4)} = 2h_{4jk} - \frac{2}{3}R_{|}{}^{x}R_{[jk|x]}$$
$$-\frac{2}{5}\operatorname{Hess}_Y(\Delta R)_{jk} - \frac{4}{5}\operatorname{Hess}_Y|\operatorname{Ric}|^2{}_{jk} + \frac{4}{5}(3n-5)\operatorname{Hess}_Y R_{jk}$$
$$-\frac{8}{5}\Big[-\Delta R - 2|\operatorname{Ric}|^2 + 2(3n-5)R - 4n(n-1)(n-2)\Big]g_{Y\,jk}$$

Then combining these equations and simplifying gives

$$h_{4jk} = -\frac{1}{3}\Delta_L^2 R_{jk} + \frac{2}{15}\operatorname{Hess}_Y(\Delta R)_{jk} - \frac{2}{3}(n+9)\Delta_L R_{jk} - \frac{2}{45}(13n-55)\operatorname{Hess}_Y R_{jk}$$
$$+\frac{8}{3}(5n-13)R_{jk} - \frac{16}{3}R_{jx}R_k{}^{x} + \frac{28}{15}R^{xy}{}_{|j}R_{xy|k} + \frac{38}{15}R^{xy}R_{xy|jk}$$
$$+\left[\frac{4}{5}\Delta R + \frac{8}{5}|\operatorname{Ric}|^2 - \frac{8}{15}(9n-25)R + \frac{8}{15}(n-1)(2n-5)(3n-11)\right]g_{Y\,jk}$$
$$-\frac{1}{9}\Big(R_{|xk}R_j{}^{x} + R_{|xj}R_k{}^{x} - 2R_{|}{}^{x}R_{xj|k} - 2R_{|}{}^{x}R_{xk|j} + 3R_{|}{}^{x}R_{jk|x}\Big)$$
$$-2R^{xy}\Big(R_{jx|yk} + R_{kx|yj}\Big) - \frac{2}{3}R_j{}^{x}{}_{|}{}^{y}R_{[kx|y]} - \frac{2}{3}R_k{}^{x}{}_{|}{}^{y}R_{[jx|y]}$$
$$+\frac{4}{3}R^{xy}\Big(R_{jk|xy} - R_{ja}R^{a}{}_{xyk} - R_{ka}R^{a}{}_{xyj} + 2R_y{}^{z}R_{xjkz}\Big)$$

and

$$\operatorname{tr}_Y h_4 = -\frac{1}{5}\Delta^2 R - \frac{4}{9}(n+8)\Delta R - \frac{8}{15}(9n^2-50n+65)R$$
$$+\frac{2}{3}\langle\operatorname{Ric},\Delta_L\operatorname{Ric}\rangle + \frac{28}{15}\langle\operatorname{Ric},\Delta\operatorname{Ric}\rangle + \frac{16}{5}|\nabla\operatorname{Ric}|^2$$
$$-\frac{8}{3}R^{ix}{}_{|}{}^{y}R_{iy|x} - \frac{8}{9}\langle\operatorname{Ric},\operatorname{Hess}_Y R\rangle + \frac{8}{15}(3n-10)|\operatorname{Ric}|^2$$
$$-\frac{1}{9}|\nabla R|^2 + \frac{8}{15}n(n-1)(2n-5)(3n-11)$$

Now we can write f_6 in terms of the link metric, g_Y. Recall that

$$f_6 = \frac{1}{14}\left[5\operatorname{tr}_Y h_4 - \frac{11}{3}\langle h_0,h_2\rangle + \operatorname{tr}_Y h_{000}\right]$$

First, calculate $\langle h_0, h_2\rangle$.

$$\begin{aligned}\langle h_0, h_2\rangle &= -2\Big[\langle \mathrm{Ric}, h_2\rangle - (n-1)\,\mathrm{tr}_Y\, h_2\Big] \\ &= -2\left\langle \mathrm{Ric}, -\Delta_L \mathrm{Ric} + \frac{1}{3}\,\mathrm{Hess}_Y\, R - 4\,\mathrm{Ric} + \frac{4}{3}\Big[R-(n-1)(n-3)\Big] g_Y \right\rangle \\ &\quad + 2(n-1)\Big[-\frac{2}{3}\Delta R + \frac{4}{3}(n-3)R - \frac{4}{3}n(n-1)(n-3)\Big] \\ &= 2\langle \mathrm{Ric}, \Delta_L \mathrm{Ric}\rangle - \frac{2}{3}\langle \mathrm{Ric}, \mathrm{Hess}_Y\, R\rangle + 8|\,\mathrm{Ric}\,|^2 - \frac{8}{3}R^2 \\ &\quad - \frac{4}{3}(n-1)\Delta R + \frac{16}{3}(n-1)(n-3)R - \frac{8}{3}n(n-1)^2(n-3)\end{aligned}$$

Second, $\mathrm{tr}_Y\, h_{000}$

$$\mathrm{tr}_Y\, h_{000} = -8\Big(R^{xy}R_y{}^z R_{zx} - 3(n-1)|\,\mathrm{Ric}\,|^2 + 3(n-1)^2 R - n(n-1)^3\Big)$$

Combining these equations gives. Then combine like terms.

$$\begin{aligned} f_6 = \frac{1}{14}\Bigg[&-\Delta^2 R + \frac{4}{3}(2n-17)\Delta R - \frac{32}{9}(19n^2-73n+72)R \\ &+16|\nabla\,\mathrm{Ric}\,|^2 + 16(2n-5)|\,\mathrm{Ric}\,|^2 - \frac{5}{9}|\nabla R|^2 + \frac{88}{9}R^2 \\ &+\frac{16}{3}\langle \mathrm{Ric}, \Delta\,\mathrm{Ric}\rangle - \frac{40}{3}R^{ix}{}_{|}{}^{y}R_{iy|x} - 2\langle \mathrm{Ric}, \mathrm{Hess}_Y\, R\rangle \\ &-8R^{xy}R_{ab}R^a{}_{xy}{}^b + \frac{8}{45}n(n-1)(118n^2-421n+375)\Bigg]\end{aligned}$$

2.2.5 Highest Order Terms for General h_{2m}

From the calculations for h_4, it can be seen that as m increases, h_{2m} becomes increasingly complicated. However, it is fairly straightforward to calculate the leading order term of h_{2m} including its coefficient. First look at the coefficients of $dr \wedge dr$ in equation 3 to study the leading term of f_{2m}.

$$2f_{,rr} = -1 + H^{il}H_{il,rr} + \frac{1}{2}H^{il}{}_{,r}H_{il,r}$$

On the left hand side,

$$2f_{,rr} + 1 = 2\sum_{m=1}^{\infty}(2m)(2m+1)r^{-2m-2}f_{2m}$$

In this section, an ellipsis, $(+\cdots)$, will denote the omission of terms involving lower order derivatives of $\mathrm{Ric}(g_Y)$. These lower order terms may vary from line to line.

On the right hand side, the first term gives

$$\begin{aligned}
H^{il}H_{il,rr} &= H^{il}\left(r^2 g_Y + \sum_{m=0}^{\infty} r^{-2m}h_{2m}\right)_{il,rr} \\
&= H^{il}\left(2g_Y + \sum_{m=1}^{\infty}(2m)(2m+1)r^{-2m-2}h_{2m}\right)_{il} \\
&= \left(r^{-2}g_Y^{il} - r^{-4}H_o^{il} + \cdots\right)\left(2g_Y + \sum_{m=1}^{\infty}(2m)(2m+1)r^{-2m-2}h_{2m}\right)_{il} \\
&= (2n)r^{-2} + \sum_{m=1}^{\infty}(2m)(2m+1)r^{-2m-4}\operatorname{tr}_Y h_{2m} - 2r^{-4}\operatorname{tr}_Y H_o + \cdots \\
&= (2n)r^{-2} + \sum_{m=1}^{\infty}\left[(2m)(2m+1)-2\right]r^{-2m-4}\operatorname{tr}_Y h_{2m} + \cdots
\end{aligned}$$

The second term gives

$$\begin{aligned}
\frac{1}{2}H^{il}{}_{,r}H_{il,r} &= \frac{1}{2}\left(r^{-2}g_Y^{il} - r^{-4}H_o^{il} + \cdots\right)_{,r}\left(r^2 g_Y + \sum_{m=0}^{\infty} r^{-2m}h_{2m}\right)_{il,r} \\
&= \frac{1}{2}\left(-2r^{-3}g_Y^{il} + \sum_{m=0}^{\infty}(2m+4)r^{-2m-5}h_{2m}^{il} + \cdots\right) \\
&\qquad\qquad \cdot\left(2rg_Y - \sum_{m=0}^{\infty}(2m)r^{-2m-1}h_{2m}\right)_{il} \\
&= -(2n)r^{-2} + \sum_{m=0}^{\infty}\left[(2m+4)+2m\right]r^{-2m-4}\operatorname{tr}_Y h_{2m} + \cdots
\end{aligned}$$

Hence,

$$-2R_{rr} = \sum_{m=0}^{\infty}(2m+1)(2m+2)\operatorname{tr}_Y h_{2m}r^{-2m-4} + \cdots$$

Therefore, $2f_{,rr} + 1 = -2R_{rr}$ implies that to the leading order

$$f_{2m+2} = \frac{2m+1}{2(2m+3)}\operatorname{tr}_Y h_{2m} + \cdots$$

For future use, let

$$C_{2m} := \frac{2m+1}{2(2m+3)}$$

Notice that this agrees with the equations for f. To rewrite this in terms of the g_Y metric, further information about the leading terms of h_{2m} must be derived. Start by recalling the $dx^j \wedge dx^k$ coefficients of equation (3)

$$2\operatorname{Ric}(g)_{jk} + \operatorname{Hess}_g f_{jk} + H_{jk} = 0.$$

First, find the leading term of $2\operatorname{Ric}(g)$. In terms of H,

$$2\operatorname{Ric}(g)_{jk} = 2\operatorname{Ric}(H)_{jk} - H_{jk,rr} - \frac{1}{2}H^{il}H_{il,r}H_{jk,r} + H^{il}H_{ij,r}H_{kl,r}.$$

Taking a derivative in the r direction means less derivatives in the Y direction (or alternatively, lower r power for the same number of derivatives). Hence, the leading order term will only come from $\operatorname{Ric}(H)$. Recall the equation

$$\begin{aligned}2\operatorname{Ric}(H)_{jk} &= 2\operatorname{Ric}(g_Y)_{jk} + H^{il}\left[H_{ojl|ki} + H_{okl|ji} - H_{ojk|il} - H_{oil|jk}\right] \\ &\quad + \frac{1}{2}H^{il}H^{nm}\big(H_{o[jk|n]}H_{o[il|m]} - H_{o[ik|n]}H_{o[jl|m]}\big).\end{aligned}$$

The term on the second line of this equation will produce nonlinear terms that are lower order than the first line. This is because $H^{il}H^{nm} = r^{-4}{g_Y}^{il}{g_Y}^{nm} + O(r^{-6})$ and $H^{il} = r^{-2}{g_Y}^{il} + O(r^{-4})$. Thus, in the expansion of the second line, the least negative power of r that a derivative of h_{2m} will appear with is r^{-2m-4}. However, in the expansion of the first line the second derivative of h_{2m} will appear next to a power of r^{-2m-2}.

It is easy to see that $h_{2(m+1)}$ has a leading order term of order 2 greater than the leading order of h_{2m}. Therefore, the second line does not contribute any leading order terms. Now focus on the first line and use an inductive argument with the calculations for h_4 as the base case.

Recall that the leading order terms of $h_2 = -\Delta_L R_{jk} + \frac{1}{3}\operatorname{Hess}_Y R_{jk} + \cdots$. Plugging this into the first line initial gave

$$\Delta_L^2 R_{jk} + \operatorname{Hess}_Y(\Delta R)_{jk} - {g_Y}^{il}\Big[(\Delta_L R_{jl})_{|ik} + (\Delta_L R_{kl})_{|ij}\Big]$$

and

$$\frac{1}{3}{g_Y}^{il}\Big[R_{|jlki} + R_{|klji} - R_{|jkil} - R_{|iljk}\Big].$$

However, after commuting the derivatives, the only leading term was $\Delta_L^2 R_{jk}$. Hence, we have the base case for the following proposition.

Proposition 1. *For any $m > 0$, if h_{2m} is of the form*

$$h_{2m} = A_{2m}(\Delta_L)^m \operatorname{Ric} + B_{2m}\operatorname{Hess}_Y(\Delta)^{m-1}R + \cdots$$

for some constants A_{2m} and B_{2m}, then the leading order term of $(\operatorname{Ric}_{jk})^{(2m+2)}$ is

$$(\operatorname{Ric}(H)_{jk})^{(2m+2)} = -A_{2m}(\Delta_L)^{m+1}R_{jk} + \cdots.$$

Proof. First, plugging in $B_{2m}\operatorname{Hess}_Y(\Delta)^{m-1}R$ into the expansion for $\operatorname{Ric}(H)$ gives

$$B_{2m}{g_Y}^{il}\Big[(\Delta^{m-1}R)_{|jlki} + (\Delta^{m-1}R)_{|klji} - (\Delta^{m-1}R)_{|jkil} - (\Delta^{m-1}R)_{|iljk}\Big].$$

After rearranging the derivatives all the highest order terms cancel out. Thus, the Hessian term contributes nothing to the leading term of the $\operatorname{Ric}(H)$.

Second, plugging in $A_{2m}(\Delta_L)^m \operatorname{Ric}$ into the expansion for $\operatorname{Ric}(H)$ gives

$$-\Delta_L \left(A_{2m}\Delta_L^m \operatorname{Ric}\right)_{jk} + A_{2m}{g_Y}^{il}\left[(\Delta_L^m R_{il})_{|jk} - (\Delta_L^m R_{jl})_{|ik} - (\Delta_L^m R_{kl})_{|ij}\right]$$

The first term give

$$-A_{2m}(\Delta_L)^{m+1}R_{jk}$$

which is the only leading term that doesn't cancel out.

To simplify the second term, recall the following fact about the Lichnerowics Laplacian acting on a 2-form T_{il}.

$$\begin{aligned} \operatorname{tr}\left(\Delta_L T\right)_{il} &= \operatorname{tr}\left[\Delta T_{il} - R_i{}^x T_{xl} - T_i{}^x R_{xl} + 2T_{xy}R^x{}_{il}{}^y\right] \\ &= \Delta \operatorname{tr} T \end{aligned}$$

Then second term simplifies in the following way

$$\begin{aligned} {g_Y}^{il}(\Delta_L^m R_{il})_{|jk} &= ({g_Y}^{il}\Delta_L^m R_{il})_{|jk} \\ &= (\Delta {g_Y}^{il}\Delta_L^{m-1}R_{il})_{|jk} \\ &= (\Delta^m R)_{|jk} \end{aligned}$$

Finally, the last two terms simplify by first commuting the ith covariant derivative with the Δ_L^m and then using the contracted Bianchi identity.

$$\begin{aligned} {g_Y}^{il}\left[(\Delta_L^m R_{jl})_{|ik} + (\Delta_L^m R_{kl})_{|ij}\right] &= {g_Y}^{il}\left[(\Delta_L^m \nabla_i R_{jl})_{|k} + (\Delta_L^m \nabla_i R_{kl})_{|j} + \cdots\right] \\ &= (\Delta^m \nabla^i R_{ji})_{|k} + (\Delta^m \nabla^i R_{ki})_{|j} + \cdots \\ &= \frac{1}{2}(\Delta^m \nabla_j R)_{|k} + \frac{1}{2}(\Delta^m \nabla_k R)_{|j} + \cdots \\ &= (\Delta^m R)_{|jk} + \cdots \end{aligned}$$

Hence, to leading order,

$${g_Y}^{il}\left[(\Delta_L^m R_{il})_{|jk} - (\Delta_L^m R_{jl})_{|ik} - (\Delta_L^m R_{kl})_{|ij}\right] = 0 + \cdots$$

□

Next, look at the leading terms contributed by $2\operatorname{Hess}_g f$ in the (jk)-equation. Recall that

$$2\operatorname{Hess}_g f_{jk} = 2\operatorname{Hess}_Y f_{jk} - H^{il}H_{o[jk|l]}f_{|i} + H_{jk,r}f_{,r}$$

The second term will not contribute any leading terms because the H^{il} will bump with exponent of r down. However, the first and last terms will contribute. The last term gives

$$H_{jk,r}f_{,r} = H_{jk,r}\left(-\frac{1}{2}r + O(r^{-3})\right) = -r^2 g_{Y\,jk} + \sum_{m=1}^{\infty} m r^{-2m} h_{2m}$$

For the first term, use the definition of f and the above calculations for f using the (rr)-equation.

$$\begin{aligned} 2\,\mathrm{Hess}_Y\, f_{jk} &= 2\sum_{m=1}^{\infty} r^{-2m}(\mathrm{Hess}_Y\, f_{2m})_{jk} \\ &= 2\sum_{m=0}^{\infty} r^{-2m-2} C_{2m}(\mathrm{Hess}_Y\,\mathrm{tr}_Y\, h_{2m})_{jk} \end{aligned}$$

Hence, if h_{2m} has the form,

$$h_{2m} = A_{2m}(\Delta_L)^m \,\mathrm{Ric} + B_{2m}\,\mathrm{Hess}_Y(\Delta)^{m-1}R + \cdots,$$

then

$$(2\,\mathrm{Hess}_Y\, f_{jk})^{(2m+2)} = 2C_{2m}(A_{2m} + B_{2m})(\mathrm{Hess}_Y(\Delta)^m R)_{jk}.$$

Thus,

$$(2\,\mathrm{Hess}_g\, f_{jk})^{(2m+2)} = 2C_{2m}(A_{2m} + B_{2m})(\mathrm{Hess}_Y(\Delta)^m R)_{jk} + (m+1)(h_{2m+2})_{jk}$$

Now plug these equation into $\big(2\,\mathrm{Ric}(g) + \mathrm{Hess}_g\, f + g = 0\big)^{(2m+2)}$ assuming that h_{2m} has the form

$$h_{2m} = A_{2m}(\Delta_L)^m \,\mathrm{Ric} + B_{2m}\,\mathrm{Hess}_Y(\Delta)^{m-1}R + \cdots$$

This gives

$$(m+2)(h_{2m+2})_{jk} = A_{2m}(\Delta_L)^{m+1}R_{jk} - 2C_{2m}(A_{2m} + B_{2m})(\mathrm{Hess}_Y(\Delta)^m R)_{jk}$$

Therefore, if h_{2m} has the form

$$h_{2m} = A_{2m}(\Delta_L)^m \,\mathrm{Ric} + B_{2m}\,\mathrm{Hess}_Y(\Delta)^{m-1}R + \cdots,$$

then h_{2m+2} also has the form

$$h_{2m+2} = A_{2m+2}(\Delta_L)^{m+1} \,\mathrm{Ric} + B_{2m+2}\,\mathrm{Hess}_Y(\Delta)^{m}R + \cdots,$$

where

$$A_{2m+2} = \frac{A_{2m}}{m+2}$$

and

$$B_{2m+2} = -\frac{2}{m+2}C_{2m}(A_{2m} + B_{2m}).$$

Then these inductive equations can be made more explicit as

$$A_{2m} = \frac{-2}{(m+1)!}$$

and

$$B_{2m} = \sum_{\mu=0}^{m-1} A_{2\mu} \left(\prod_{\nu=\mu}^{m-1} \left(\frac{-2}{\nu+2} \right) C_{2\nu} \right)$$

Then simplify B_{2m}

$$\begin{aligned}
B_{2m} &= \sum_{\mu=0}^{m-1} \frac{-2}{(\mu+1)!} \left(\prod_{\nu=\mu}^{m-1} \left(\frac{-2}{\nu+2} \right) \frac{2\mu+1}{2(2\mu+3)} \right) \\
&= \sum_{\mu=0}^{m-1} \frac{2}{(\mu+1)!} \left((-1)^{m-\mu+1} \frac{(2m-1)\cdots(2\mu+3)(2\mu+1)}{(m+1)(m)\cdots(\mu+2)\cdot(2m+1)(2m-1)\cdots(2\mu+3)} \right) \\
&= \sum_{\mu=0}^{m-1} \frac{2}{(m+1)!} (-1)^{m-\mu+1} \left(\frac{2\mu+1}{2m+1} \right) \\
&= \frac{2}{(m+1)!(2m+1)} \sum_{\mu=0}^{m-1} (-1)^{m-\mu+1} (2\mu+1) \\
&= \frac{1}{(m+1)!} \frac{2m}{(2m+1)}
\end{aligned}$$

Thus, the first several leading terms for H can easily be calculated.

$$\begin{aligned}
H &= r^2 g_Y - 2r^0 \Big(\mathrm{Ric}(g_Y) - (n-1) g_Y \Big) + r^{-2} \left(-\Delta_L \mathrm{Ric}(g_Y) + \frac{1}{3} \mathrm{Hess}_Y R + \cdots \right) \\
&\quad + r^{-4} \left(-\frac{1}{3} \Delta_L^2 \mathrm{Ric}(g_Y) + \frac{2}{15} \mathrm{Hess}_Y (\Delta R) + \cdots \right) \\
&\quad + r^{-6} \left(-\frac{1}{12} \Delta_L^2 \mathrm{Ric}(g_Y) + \frac{1}{28} \mathrm{Hess}_Y (\Delta R) + \cdots \right) \\
&\quad + r^{-8} \left(-\frac{1}{60} \Delta_L^2 \mathrm{Ric}(g_Y) + \frac{1}{135} \mathrm{Hess}_Y (\Delta R) + \cdots \right) + O(r^{-10})
\end{aligned}$$

2.3 Constraint Equation Calculations

In this section, we use the first few terms in the expansions calculated above to check that the constraint equation is satisfied. That is

$$2\,\mathrm{Ric}(g)_{rl} + 2\,\mathrm{Hess}_g f_{rl} = 0$$

First, look at the Hessian of the potential function.

$$2\,\mathrm{Hess}_g f_{rl} = 2 f_{,rl} - 2\,{}^g\Gamma_{rl}{}^i f_{,i} = 2 f_{,rl} - H^{ix} H_{xl,r} f_{,i}$$

Expanding this equation out gives

$$\begin{aligned}
2\,\mathrm{Hess}_g\, f_{rl} \;&=\; r^{-1}\Big(-2f_{0|l}\Big) + r^{-3}\Big(-6f_{2|l}\Big) + r^{-5}\Big(-10f_{4|l} + 2{h_0}^i{}_l f_{2|i}\Big) \\
&\quad + r^{-7}\Big[-14f_{6|l} + 2{h_0}^i{}_l f_{4|i} + \Big(4{h_2}^i{}_l - 2{h_{00}}^i{}_l\Big) f_{2|i}\Big] \\
&\quad + r^{-9}\Big[-18f_{8|l} + 2{h_0}^i{}_l f_{6|i} + \big(4{h_2}^i{}_l - 2{h_{00}}^i{}_l\big) f_{4|i} \\
&\qquad + \Big(6{h_4}^i{}_l - 4{h_{02}}^i{}_l - 2{h_{20}}^i{}_l + 2{h_{000}}^i{}_l\Big) f_{2|i}\Big]
\end{aligned}$$

Second, look at the Ricci curvature.

$$\begin{aligned}
2\,\mathrm{Ric}(g)_{rl} \;&=\; 2\left({}^g\Gamma_{rl}{}^i{}_{,i} - {}^g\Gamma_{ir}{}^i{}_{,l} + {}^g\Gamma_{rl}{}^m \cdot {}^g\Gamma_{im}{}^i - {}^g\Gamma_{ir}{}^m \cdot {}^g\Gamma_{ml}{}^i\right) \\
&= [H^{in}H_{nl,r}]_{,i} - [H^{in}H_{in,r}]_{,l} + H^{mn}H_{nl,r}\,{}^H\Gamma_{im}{}^i - H^{mn}H_{ni,r}\,{}^H\Gamma_{ml}{}^i \\
&= {}^H\nabla_i[H^{in}H_{nl,r}] - {}^H\nabla_l[H^{in}H_{in,r}] \\
&= H^{in}\left({}^H\nabla_i H_{nl,r} - {}^H\nabla_l H_{in,r}\right)
\end{aligned}$$

Then use equation 10 to simplify this further.

$$\begin{aligned}
2\,\mathrm{Ric}(g)_{rl} \;&=\; H^{im}\left({}^H\nabla_i H_{ml,r} - {}^H\nabla_l H_{im,r}\right) \\
&= H^{im}\left(H_{ml,ri} - H_{im,rl} - {}^H\Gamma_{im}{}^x H_{xl,r} + {}^H\Gamma_{ml}{}^x H_{ix,r}\right) \\
&= H^{im}\left(H_{ml,r|i} - H_{im,r|l}\right) - \frac{1}{2}H^{im}H_{o[im|y]}H^{xy}H_{xl,r} - \frac{1}{2}H^{im}{}_{,r}H_{oim|l}
\end{aligned}$$

Here the covariant derivatives in the third line are with respect to the link metric g_Y. Then write out each term using equation 4 and equation **??**.

$$\begin{aligned}
H^{im}H_{ml,r|i} \;&=\; r^{-5}\Big(-2{g_Y}^{im}h_{2ml|i}\Big) + r^{-7}\Big(-4{g_Y}^{im}h_{4ml|i} + 2{h_0}^{im}h_{2ml|i}\Big) \\
&\quad + r^{-9}\Big(-6{g_Y}^{im}h_{6ml|i} + 4{h_0}^{im}h_{4ml|i} - 2({h_{00}}^{im} - {h_2}^{im})h_{2ml|i}\Big) + \cdots
\end{aligned}$$

Next, $-\frac{1}{2}H^{im}H_{o[im|y]}H^{xy}H_{xl,r}$

$$\begin{aligned}
&= r^{-5}\Big(- {g_Y}^{im} h_{2[im|l]} + {h_0}^{im} h_{0[im|l]}\Big) \\
&\quad + r^{-7}\Bigg(- {g_Y}^{im} h_{4[im|l]} + {h_0}^{im} h_{2[im|l]} + \Big[{h_2}^{im} - {h_{00}}^{im}\Big] h_{0[im|l]} \\
&\qquad - {h_0}^{im} h_{0[im|y]} {h_0}^{y}{}_{l} + {g_Y}^{im} h_{2[im|y]} {h_0}^{y}{}_{l}\Bigg) \\
&\quad + r^{-9}\Bigg[{g_Y}^{im} h_{6[im|l]} + {h_0}^{im} h_{4[im|l]} + \big[{h_2}^{im} - {h_{00}}^{im}\big] h_{2[im|l]} \\
&\qquad + \big[{h_4}^{im} - 2{h_{02}}^{im} + {h_{000}}^{im}\big] h_{0[im|l]} \\
&\qquad + \Big({g_Y}^{im} h_{4[im|y]} - {h_0}^{im} h_{2[im|y]} + \big[- {h_2}^{im} + {h_{00}}^{im}\big] h_{0[im|y]}\Big) {h_0}^{y}{}_{l} \\
&\qquad + \Big({g_Y}^{im} h_{2[im|y]} - {h_0}^{im} h_{0[im|y]}\Big) \big[{h_2}^{im} - {h_{00}}^{im}\big]\Bigg] + \cdots
\end{aligned}$$

Here notice that the

$${g_Y}^{im} h_{0[im|y]} = 0,$$

so the leading order term is the coefficient for r^{-5}. Finally,

$$\begin{aligned}
-\frac{1}{2} H^{im}{}_{,r} H_{oim|l} &= r^{-3}\big(\operatorname{tr}_Y h_{0|l}\big) + r^{-5}\Big(\operatorname{tr}_Y h_{2|l} - \langle h_0, h_0\rangle_{|l}\Big) \\
&\quad + r^{-7}\Bigg(\operatorname{tr}_Y h_{4|l} - 2{h_0}^{im} h_{2im|l} + 3\big[- {h_2}^{im} + {h_{00}}^{im}\big] h_{0im|l}\Bigg) \\
&\quad + r^{-9}\Bigg[\operatorname{tr}_Y h_{6|l} - 2{h_0}^{im} h_{4im|l} + 3\big[- {h_2}^{im} + {h_{00}}^{im}\big] h_{2im|l} \\
&\qquad + 4\big[- {h_4}^{im} + 2{h_{02}}^{im} - {h_{000}}^{im}\big] h_{0im|l}\Bigg] + \cdots
\end{aligned}$$

Therefore, the first few terms of the $\mathrm{Ric}(g)_{rl}$ are

$$\begin{aligned}
2\,\mathrm{Ric}(g)_{rl} &= r^{-3}\Big(\operatorname{tr}_Y h_{0|l}\Big) \\
&\quad + r^{-5}\Big(4\operatorname{tr}_Y h_{2|l} - 4h_{2il|}{}^{i} + 2h_0{}^{im}h_{0ml|i} - \frac{3}{2}\langle h_0, h_0\rangle_{|l}\Big) \\
&\quad + r^{-7}\Bigg(6\operatorname{tr}_Y h_{4|l} - 6h_{4il|}{}^{i} - h_0{}^{im}h_{2im|l} - 4\langle h_2, h_0\rangle_{|l} + \operatorname{tr}_Y h_{000|l} \\
&\qquad + h_0{}^{i}{}_{l}\Big[2h_{2im|}{}^{m} - \operatorname{tr}_Y h_{2|i} - 2h_0{}^{xy}h_{0ix|y} + \frac{1}{2}\langle h_0, h_0\rangle_{|i}\Big] \\
&\qquad + 4h_0{}^{im}h_{2ml|i} + 2h_2{}^{im}h_{0ml|i} - h_{00}{}^{im}h_{0[im|l]}\Bigg) + \cdots
\end{aligned}$$

Notice that the leading term in the expansion for $\mathrm{Ric}(g)_{rl}$ is of order r^{-3}, while the leading order term in the expansion of the $\mathrm{Hess}_g\, f_{rl}$ is of the order r^{-1}. This leads to the equation

$$0 = 0 - 2\nabla_l f_0$$

where l can be any coordinate for Y. Thus, $\nabla f_0 \equiv 0$, i.e. f_0 is a constant. Notice that while we assumed $f_0 \equiv const.$ in section 2.2 to calculate the first few terms of H, the leading order term of the $\mathrm{Ric}(g)_{rl}$ would still be of order r^{-3}. Thus, there is not circular logic here. In order for the expansions of f and H to satisfy the constraint equation, f_0 must be a constant function on the link Y.

Now we can use $\Big(2\,\mathrm{Ric}(g)_{rl} + 2\,\mathrm{Hess}_g\, f_{rl} = 0\Big)^{(3)}$ to check the equation for h_0. By the equations above this is

$$(\operatorname{tr}_Y h_0)_{|l} - 6f_{2|l} = 0$$

This is exactly the l covariant derivative of the equation derived for f_2 before, so the equation for h_0 satisfies the constraint equation to order r^{-3}.

Second, we can use $\Big(2\,\mathrm{Ric}(g)_{rl} + 2\,\mathrm{Hess}_g\, f_{rl} = 0\Big)^{(5)}$ to check the equation for h_2. From the equations above, it is equivalent to show

$$0 = 4\operatorname{tr}_Y h_{2|l} - 4h_{2il|}{}^{i} + 2h_0{}^{im}h_{0ml|i} - \frac{3}{2}\langle h_0, h_0\rangle_{|l} - 10f_{4|l} + 2h_0{}^{i}{}_{l}f_{2|i}$$

Then plug in for f_2 and f_4 in terms of H. This gives

$$0 = \operatorname{tr}_Y h_{2|l} - 4h_{2il|}{}^{i} - \frac{1}{2}\langle h_0, h_0\rangle_{|l} + 2h_0{}^{im}h_{0ml|i} + \frac{1}{3}h_0{}^{i}{}_{l}\operatorname{tr}_Y h_{0|i}$$

Look at each term:

$$\operatorname{tr}_Y h_{2|l} = -\frac{2}{3}\nabla_l \Delta R + \frac{4}{3}(n-3)R_{|l}$$

Next,

$$
\begin{aligned}
-4h_{2il|}{}^{i} &= -4\Big(-\Delta_L R_{il} + \frac{1}{3}\operatorname{Hess}_Y R_{il} - 4R_{il} + \frac{4}{3}\big[R-(n-1)(n-3)\big]g_{Y\,il}\Big)_{|}{}^{i} \\
&= 4\Delta_L R_{il|}{}^{i} - \frac{4}{3}\operatorname{Hess}_Y R_{il|}{}^{i} + 16R_{il|}{}^{i} - \frac{16}{3}\big[R_{|}{}^{i}\big]g_{Y\,il} \\
&= 4\left[\frac{1}{2}\nabla_l\Delta R + \frac{1}{2}|\operatorname{Ric}|^2 - 2R^{xy}R_{lx|y}\right] - \frac{4}{3}\left[\nabla_l\Delta R + R_{|}{}^{x}R_{xl}\right] + \left(8-\frac{16}{3}\right)R_{|l} \\
&= \frac{2}{3}\nabla_l\Delta R + 2|\operatorname{Ric}|^2{}_{|l} - 8R^{xy}R_{lx|y} - \frac{4}{3}R_{|}{}^{x}R_{xl} + \frac{8}{3}R_{|l}
\end{aligned}
$$

Third,

$$
-\frac{1}{2}\langle h_0 h_0\rangle_{|l} = -2\left[|\operatorname{Ric}|^2 - 2(n-1)R + n(n-1)^2\right]_{|l} = -2|\operatorname{Ric}|^2_{|l} + 4(n-1)R_{|l}
$$

Fourth,

$$
2h_0{}^{im}h_{0ml|i} = 8\Big(R^{im} - (n-1)g_Y{}^{im}\Big)R_{ml|i} = 8R^{im}R_{ml|i} - 4(n-1)R_{|l}
$$

Fifth,

$$
\frac{1}{3}h_0{}^{i}{}_{l}\operatorname{tr}_Y h_{0|i} = \frac{4}{3}R_{|}{}^{i}\Big(R_{il} - (n-1)g_{Y\,il}\Big) = \frac{4}{3}R_{|}{}^{i}R_{il} - \frac{4}{3}(n-1)R_{|l}
$$

Adding these five equations together gives $0=0$, so the original equation is satisfied. Thus, the equation for h_0 and h_2 satisfy the constraint to the order of r^{-5}.

Finally, we can use $\Big(2\operatorname{Ric}(g)_{rl} + 2\operatorname{Hess}_g f_{rl} = 0\Big)^{(7)}$ to check the equation for h_4. We wish to show that

$$
\begin{aligned}
0 &= (2\operatorname{Hess}_g f_{rl})^{(7)} + (2\operatorname{Ric}(g)_{rl})^{(7)} \\
&= -14f_{6|l} + 6\operatorname{tr}_Y h_{4|l} - 6h_{4il|}{}^{i} - h_0{}^{im}h_{2im|l} - 4\langle h_2, h_0\rangle_{|l} + \operatorname{tr}_Y h_{000|l} \\
&\qquad + h_0{}^{i}{}_{l}\Big[2f_{4|i} + 2h_{2im|}{}^{m} - \operatorname{tr}_Y h_{2|i} - 2h_0{}^{xy}h_{0ix|y} \\
&\qquad\qquad + \frac{1}{2}\langle h_0, h_0\rangle_{|i} - \frac{1}{3}h_0{}^{m}{}_{i}\operatorname{tr}_Y h_{0|m}\Big] \\
&\qquad + 4h_0{}^{im}h_{2ml|i} + 2h_2{}^{im}h_{0ml|i} - h_{00}{}^{im}h_{0[im|l]} + \frac{2}{3}h_2{}^{i}{}_{l}\operatorname{tr}_Y h_{0|i}
\end{aligned}
$$

Then use the equation

$$
14f_6 = 5\operatorname{tr}_Y h_4 - \frac{11}{3}\langle h_0, h_2\rangle + \operatorname{tr}_Y h_{000}
$$

to cancel out terms in the first line; and use the equation

$$0 = \operatorname{tr}_Y h_{2|l} - 4h_{2il|}{}^{i} - \frac{1}{2}\langle h_0, h_0\rangle_{|l} + 2h_0{}^{im}h_{0ml|i} + \frac{1}{3}h_0{}^{i}{}_{l}\operatorname{tr}_Y h_{0|i}$$

to simplify the second and third line.

$$\begin{aligned} 0 \;=\;& \operatorname{tr}_Y h_{4|l} - 6h_{4il|}{}^{i} - h_0{}^{im}h_{2im|l} - \frac{1}{3}\langle h_2, h_0\rangle_{|l} + 2h_0{}^{i}{}_{l}\Big[f_{4|i} - h_{2im|}{}^{m}\Big] \\ & + 4h_0{}^{im}h_{2ml|i} + 2h_2{}^{im}h_{0ml|i} - \; h_{00}{}^{im}h_{0[im|l]} + \frac{2}{3}h_2{}^{i}{}_{l}\operatorname{tr}_Y h_{0|i} \end{aligned}$$

Now split this equation into four parts and simply each part.
$-h_0{}^{im}h_{2im|l} - \frac{1}{3}\langle h_2, h_0\rangle_{|l}$

$$\begin{aligned} =\;& 2\big[\langle \operatorname{Ric}, \nabla_l h_2\rangle - (n-1)\nabla_l \operatorname{tr}_Y h_2\big] + \frac{2}{3}\nabla_l\big[\langle \operatorname{Ric}, h_2\rangle - (n-1)\operatorname{tr}_Y h_2\big] \\ =\;& 2\Bigg[-\langle \operatorname{Ric}, \nabla_l\Delta_L \operatorname{Ric}\rangle + \frac{1}{3}\langle \operatorname{Ric}, \nabla_l \operatorname{Hess}_Y R\rangle - 4\langle \operatorname{Ric}, \nabla_l \operatorname{Ric}\rangle \\ & \quad + \frac{4}{3}\langle \operatorname{Ric}, \nabla_l R g_Y\rangle - (n-1)\nabla_l\left(-\frac{2}{3}\nabla R + \frac{4}{3}(n-3)R\right)\Bigg] \\ & + \frac{2}{3}\nabla_l\Bigg[-\langle \operatorname{Ric}, \Delta_L Ric\rangle + \frac{1}{3}\langle \operatorname{Ric}, \operatorname{Hess}_Y R\rangle - 4|\operatorname{Ric}|^2 + \frac{4}{3}R^2 \\ & \quad - \frac{4}{3}(n-1)(n-3)R - (n-1)\left(-\frac{2}{3}\Delta R + \frac{4}{3}(n-3)R\right)\Bigg] \\ =\;& -2\langle \operatorname{Ric}, \nabla_l\Delta_L \operatorname{Ric}\rangle + \frac{2}{3}\langle \operatorname{Ric}, \nabla_l \operatorname{Hess}_Y R\rangle - 4\nabla_l|\operatorname{Ric}|^2 + \frac{4}{3}\nabla_l(R^2) \\ & + \nabla_l\Bigg[-\frac{2}{3}\langle \operatorname{Ric}, \Delta_L Ric\rangle + \frac{2}{9}\langle \operatorname{Ric}, \operatorname{Hess}_Y R\rangle - \frac{8}{3}|\operatorname{Ric}|^2 + \frac{8}{9}R^2 \\ & \quad - \frac{8}{9}(n-1)(n-3)R - (n-1)\frac{8}{3}\left(-\frac{2}{3}\Delta R + \frac{4}{3}(n-3)R\right)\Bigg] \\ =\;& \nabla_l\Bigg[-\frac{2}{3}\langle \operatorname{Ric}, \Delta_L Ric\rangle + \frac{2}{9}\langle \operatorname{Ric}, \operatorname{Hess}_Y R\rangle \\ & \quad + \frac{16}{9}(n-1)\Delta R - \frac{40}{9}(n-1)(n-3)R - \frac{20}{3}|\operatorname{Ric}|^2 + \frac{20}{9}R^2\Bigg] \\ & -2\langle \operatorname{Ric}, \nabla_l\Delta_L \operatorname{Ric}\rangle + \frac{2}{3}\langle \operatorname{Ric}, \nabla_l \operatorname{Hess}_Y R\rangle \end{aligned}$$

For the next terms recall that

$$h_{2im|}{}^{m} = -\frac{1}{6}\nabla_i\Delta R - \frac{1}{2}\nabla_i|\operatorname{Ric}|^2 + 2R^{xy}R_{ix|y} + \frac{1}{3}R_{|}{}^{m}R_{im} - \frac{2}{3}R_{|i}$$

$$
\begin{aligned}
&2{h_0}^i{}_l\Big[f_{4|i} - h_{2im|}{}^m\Big] \\
&= 2{h_0}^i{}_l\left[\nabla_i\left(-\frac{1}{5}\Delta R + \frac{2}{5}(3n-5)R - \frac{2}{5}|\,\mathrm{Ric}\,|^2\right)\right. \\
&\qquad\qquad \left. + \frac{1}{6}\nabla_i\Delta R + \frac{1}{2}\nabla_i|\,\mathrm{Ric}\,|^2 - 2R^{xy}R_{ix|y} - \frac{1}{3}R_|{}^m R_{im} + \frac{2}{3}R_{|i}\right] \\
&= 2{h_0}^i{}_l\left[\nabla_i\left(-\frac{1}{30}\Delta R + 2\left(\frac{3}{5}n - \frac{2}{3}\right)R + \frac{1}{10}|\,\mathrm{Ric}\,|^2\right) - 2R^{xy}R_{ix|y} - \frac{1}{3}R_|{}^m R_{im}\right] \\
&= R^i{}_l\left[\nabla_i\left(\frac{2}{15}\Delta R - 8\left(\frac{3}{5}n + \frac{16}{3}\right)R - \frac{2}{5}|\,\mathrm{Ric}\,|^2\right) + 8R^{xy}R_{ix|y} + \frac{4}{3}R_|{}^m R_{im}\right] \\
&\qquad + \nabla_l\left[-\frac{2}{15}(n-1)\Delta R + 8(n-1)\left(\frac{3}{5}n - \frac{2}{3}\right)R + \frac{2}{5}(n-1)|\,\mathrm{Ric}\,|^2\right] \\
&\qquad\qquad - 8(n-1)R_{lx|y}R^{xy} - \frac{4}{3}(n-1)R_|{}^x R_{xl} \\
&= \nabla_l\left[-\frac{2}{15}(n-1)\Delta R + 8(n-1)\left(\frac{3}{5}n - \frac{2}{3}\right)R + \frac{2}{5}(n-1)|\,\mathrm{Ric}\,|^2\right] \\
&\qquad + R^i{}_l\left[\nabla_i\left(\frac{2}{15}\Delta R - 4\left(\frac{23}{15}n - \frac{5}{3}\right)R - \frac{2}{5}|\,\mathrm{Ric}\,|^2\right) + 8R^{xy}R_{ix|y}\right] \\
&\qquad + R_{lx|y}\big[-8(n-1)R^{xy}\big] + \frac{4}{3}R_|{}^i\big[R_{ix}R^x{}_l\big]
\end{aligned}
$$

$$4{h_0}^{im}h_{2ml|i} + 2{h_2}^{im}h_{0ml|i}$$

$$\begin{aligned}
&= -8R^{im}h_{2ml|i} + 8(n-1){h_{2il|}}^{i} - 4R_{lx|y}{h_2}^{xy} \\
&= -8R^{im}\left[-\nabla_i\Delta_L R_{ml} + \frac{1}{3}\nabla_i \operatorname{Hess}_Y R_{ml} - 4R_{ml|i} + \frac{4}{3}\nabla_i R g_{Y\,ml}\right] \\
&\quad +8(n-1)\left[-\frac{1}{6}\nabla_l\Delta R - \frac{1}{2}\nabla_l|\operatorname{Ric}|^2 + 2R^{xy}R_{lx|y} + \frac{1}{3}{R_|}^{m}R_{ml} - \frac{2}{3}R_{|l}\right] \\
&\quad -4R_{lx|y}\left[-\Delta_L R^{xy} + \frac{1}{3}\operatorname{Hess}_Y R^{xy} - 4R^{xy} + \frac{4}{3}\left[R-(n-1)(n-3)\right]g_{Y\,xy}\right] \\
&= 8R^{xy}\nabla_y\Delta_L R_{xl} - \frac{8}{3}\langle \operatorname{Ric}, \nabla_l \operatorname{Hess}_Y R\rangle - \frac{8}{3}{R_|}^{i}R_{xy}{R^x}_{il}{}^{y} + R_{lx|y}\big[32R^{xy}\big] \\
&\quad +{R^i}_l\left[-\frac{32}{3}\nabla_i R\right] + 8(n-1)\nabla_l\left[-\frac{1}{6}\Delta R - \frac{1}{2}|\operatorname{Ric}|^2 - \frac{2}{3}R\right] \\
&\quad +R_{lx|y}\big[16(n-1)R^{xy}\big] + {R^i}_l\left[\frac{8}{3}(n-1)\nabla_i R\right] \\
&\quad -4R_{lx|y}\left[-\Delta_L R^{xy} + \frac{1}{3}\operatorname{Hess}_Y R^{xy} - 4R^{xy}\right] - \frac{8}{3}R_{|l}\left[R-(n-1)(n-3)\right] \\
&= \nabla_l\left[-\frac{4}{3}(n-1)\Delta R + \frac{8}{3}(n-1)(n-5)R - \frac{4}{3}R^2 - 4(n-1)|\operatorname{Ric}|^2\right] \\
&\quad +{R^i}_l\left[\frac{8}{3}(n-5)\nabla_i R\right] + R_{lx|y}\left[4\Delta_L R^{xy} - \frac{4}{3}\operatorname{Hess}_Y R^{xy} + 16(n+2)R^{xy}\right] \\
&\quad +8R^{xy}\nabla_y\Delta_L R_{xl} - \frac{8}{3}\langle \operatorname{Ric}, \nabla_l \operatorname{Hess}_Y R\rangle - \frac{8}{3}{R_|}^{i}R_{xy}{R^x}_{il}{}^{y}
\end{aligned}$$

$$-\ {h_{00}}^{im}h_{0[im|l]} + \tfrac{2}{3}{h_2}^{i}{}_{l}\operatorname{tr}_Y h_{0|i}$$

$$\begin{aligned}
&= 8({R^i}_x R^{xm} - 2(n-1)R^{im})(2R_{ml|i} - R_{im|l}) \\
&\quad -\frac{4}{3}{R_|}^{i}\left(-\Delta_L R_{il} + \frac{1}{3}\operatorname{Hess}_Y R_{il} - 4R_{il} + \frac{4}{3}\Big[R-(n-1)(n-3)\Big]g_{Y\,il}\right) \\
&= \nabla_l\left[\frac{16}{9}(n-1)(n-3)R + 8(n-1)|\operatorname{Ric}|^2 - \frac{2}{9}|\nabla R|^2 - \frac{8}{9}R^2 - \frac{8}{3}R_{xy}{R^y}_z R^{xz}\right] \\
&\quad +{R^i}_l\left[\frac{16}{3}\nabla_i R\right] + R_{lx|y}\left[16{R^x}_i R^{iy} - 32(n-1)R^{xy}\right] + \frac{4}{3}{R_|}^{i}\Delta_L R_{il}
\end{aligned}$$

Therefore, the equation becomes

$$
\begin{aligned}
0 \;=\;& \operatorname{tr}_Y h_{4|l} - 6{h_{4il|}}^{i} + 8R^{xy}\nabla_y\Delta_L R_{xl} - 2R^{xy}\nabla_l\Delta_L R_{xy} + \frac{4}{3}{R_|}^{i}\left[\Delta R_{il} - R_{ix}{R^x}_l\right] \\
&+ \nabla_l\Bigg(\frac{14}{45}(n-1)\Delta R + \frac{8}{15}(n-1)(9n-20)R + \left[\frac{22}{5}(n-1) - \frac{20}{3}\right]|\operatorname{Ric}|^2 \\
&\qquad - \frac{2}{9}|\nabla R|^2 - \frac{2}{3}\langle \operatorname{Ric}, \Delta_L \operatorname{Ric}\rangle + \frac{2}{9}\langle \operatorname{Ric}, \operatorname{Hess}_Y R\rangle - \frac{8}{3}R^{xy}{R_y}^{z}R_{zx}\Bigg) \\
&+ {R^i}_l\left(\frac{2}{15}\nabla_i\Delta R - \frac{2}{5}|\operatorname{Ric}|^2_{|i} + 8R^{xy}R_{ix|y} + \left[\frac{4}{3}(n-1) - \frac{24}{5}n\right]R_{|i}\right) \\
&+ R_{lx|y}\left(4\Delta_L R^{xy} + 16{R^x}_i R^{iy} - \frac{4}{3}{R_|}^{xy} - 24(n-3)R^{xy}\right) - 2R^{xy}R_{|xyl}
\end{aligned}
$$

Now look at the two terms involving h_4. First, plug in $\operatorname{tr}_Y h_{4|l}$ to the above equation.

$$
\begin{aligned}
0 \;=\;& -6{h_{4il|}}^{i} + 8R^{xy}\nabla_y\Delta_L R_{xl} - 2R^{xy}\nabla_l\Delta_L R_{xy} + \frac{4}{3}{R_|}^{i}\left[\Delta R_{il} - R_{ix}{R^x}_l\right] \\
&+ \nabla_l\Bigg(-\frac{1}{5}\Delta^2 R + \frac{16}{5}|\nabla \operatorname{Ric}|^2 - \frac{8}{3}{{R^{ix}}_|}^{y}R_{iy|x} + \frac{28}{15}\langle \operatorname{Ric}, \Delta \operatorname{Ric}\rangle \\
&\qquad - \frac{2}{15}(n+29)\Delta R - \frac{8}{5}(7n-15)R - \frac{1}{3}|\nabla R|^2 \\
&\qquad + \left[6n - 12 - \frac{22}{5}\right]|\operatorname{Ric}|^2 - \frac{2}{3}\langle \operatorname{Ric}, \operatorname{Hess}_Y R\rangle - \frac{8}{3}R^{xy}{R_y}^{z}R_{zx}\Bigg) \\
&+ {R^i}_l\left(\frac{2}{15}\nabla_i\Delta R - \frac{2}{5}|\operatorname{Ric}|^2_{|i} + 8R^{xy}R_{ix|y} + \left[\frac{4}{3}(n-1) - \frac{24}{5}n\right]R_{|i}\right) \\
&+ R_{lx|y}\left(4\Delta_L R^{xy} + 16{R^x}_i R^{iy} - \frac{4}{3}{R_|}^{xy} - 24(n-3)R^{xy}\right) - 2R^{xy}R_{|xyl}
\end{aligned}
$$

Finally, calculate $-6h_{4il|}{}^{i}$.

$$
\begin{aligned}
-6h_{4il|}{}^{i} \quad = \quad & \Bigg[2\Delta_L^2 R_{il} - \frac{4}{5}(\Delta R)_{|il} + 4(n+9)\Delta_L R_{il} + \frac{4}{15}(13n-55)R_{|il} \\
& -16(5n-13)R_{il} + 32R_{ix}R^{x}{}_{l} - \frac{56}{5}R^{xy}{}_{|i}R_{xy|l} - \frac{76}{5}R^{xy}R_{xy|il} \\
& + \frac{2}{3}\Big(R_{|xl}R_{i}{}^{x} + R_{|xi}R_{l}{}^{x} - 2R_{|}{}^{x}R_{xi|l} - 2R_{|}{}^{x}R_{xl|i} + 3R_{|}{}^{x}R_{il|x}\Big) \\
& +12R^{xy}\Big(R_{ix|yl} + R_{lx|yi}\Big) + 4R_{i}{}^{x}{}_{|}{}^{y}R_{[lx|y]} + 4R_{l}{}^{x}{}_{|}{}^{y}R_{[ix|y]} \\
& -8R^{xy}\Big(R_{il|xy} - R_{ia}R^{a}{}_{xyl} - R_{la}R^{a}{}_{xyi} + 2R_{y}{}^{z}R_{xilz}\Big)\Bigg]_{|}{}^{i} \\
& + \nabla_l\Big[-\frac{24}{5}\Delta R - \frac{48}{5}|\,\mathrm{Ric}\,|^2 + \frac{16}{5}(9n-25)R\Big]
\end{aligned}
$$

Now calculate $\nabla^i \Delta_L^2 R_{il}$ by using equation 30. First, use $T_{il} = \Delta_L R_{il}$ and then use $T_{il} = R_{il}$.

$$
\begin{aligned}
2\nabla^i\Delta_L^2 R_{il} &= 2\Delta\nabla^i\Big(\Delta_L R_{il}\Big) - 2\nabla_i\Big(\Delta_L R^{ix}\Big)R_{xl} - 2\Big(\Delta_L R^{xy}\Big)R_{[xy|l]} \\
&= 2\Delta\Big[\Delta\nabla^i R_{il} - \nabla_i R^{ix}R_{xl} - R^{xy}R_{[xy|l]}\Big] \\
&\quad -2\Big[\Delta\nabla^i R_{ix} - \nabla_i R^{iy}R_{yx} - R^{ab}R_{[ab|x]}\Big]R^x{}_l \\
&\quad +R_{lx|y}[-4\Delta_L R^{xy}] + 2R_{xy|l}\Delta_L R^{xy} \\
&= \Delta\Big[\Delta\nabla_l R - R_|{}^x R_{xl} - 4R_{lx|y}R^{xy} + \nabla_l|\,\mathrm{Ric}\,|^2\Big] \\
&\quad -\Big[\Delta\nabla_x R - R_|{}^y R_{xy} - 4R_{xa|b}R^{ab} + \nabla_x|\,\mathrm{Ric}\,|^2\Big]R^x{}_l \\
&\quad +R_{lx|y}[-4\Delta_L R^{xy}] + 2R_{xy|l}\Delta_L R^{xy} \\
&= \Delta\Big[\nabla_l\Delta R - 4R_{lx|y}R^{xy} + \nabla_l|\,\mathrm{Ric}\,|^2\Big] \\
&\quad -\Big[\nabla_x\Delta R - 4R_{xa|b}R^{ab} + \nabla_x|\,\mathrm{Ric}\,|^2\Big]R^x{}_l \\
&\quad +R_{lx|y}[-4\Delta_L R^{xy}] + 2R_{xy|l}\Delta_L R^{xy} \\
&= \nabla_l\big[\Delta^2 R + \Delta|\,\mathrm{Ric}\,|^2\big] + R^i{}_l\big[4R^{xy}R_{ix|y}\big] + R_{lx|y}\big[-4\Delta_L R^{xy} - 4\Delta R^{xy}\big] \\
&\quad +2R_{xy|l}\Delta_L R^{xy} - 4R^{xy}\Delta\nabla_y R_{xl} - 8R^{xy}{}_|{}^i R_{lx|yi}
\end{aligned}
$$

Then rewrite the term $-4R^{xy}\Delta\nabla_y R_{xl}$ as follows.

$$
\begin{aligned}
-4R^{xy}\Delta\nabla_y R_{xl} &= -4R^{xy}\Big[R_{xl|iy} + R_{al}R_{yix}{}^a + R_{xa}R_{yil}{}^a\Big]_{|}{}^{i} \\
&= -4R^{xy}\Big[\nabla_y \Delta R_{xl} + R_{al|i}R_y{}^i{}_x{}^a + R_{xa|i}R_y{}^i{}_l{}^a + R_{xl|a}R_y{}^i{}_i{}^a\Big] \\
&\quad -4R^{xy}\big(R_{al|}{}^i R_{yix}{}^a + R_{xa|}{}^i R_{yil}{}^a\big) \\
&\quad +4R^{xy}\bigg(R_{al}\Big[R_{yi}{}^i{}_{x|}{}^a + R_{yi}{}^{ai}{}_{|x}\Big] + R_{xa}\Big[R_{yi}{}^i{}_{l|}{}^a + R_{yi}{}^{ai}{}_{|l}\Big]\bigg) \\
&= -4R^{xy}\nabla_y\Delta R_{xl} + R^i{}_l\Big[2|\operatorname{Ric}|^2_{|i} - 4R^{xy}R_{ix|y}\Big] \\
&\quad -\frac{4}{3}\nabla_l\big(R^{xy}R_y{}^z R_{zx}\big) + 8R_{lx|y}R^{ab}R^x{}_{ab}{}^y - 8R_x{}^a R_{ay|}{}^i R^x{}_{il}{}^y
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2\nabla^i\Delta_L^2 R_{il} &= \nabla_l\bigg(\Delta^2 R + 2\langle \operatorname{Ric}, \Delta \operatorname{Ric}\rangle + 2\big|\nabla \operatorname{Ric}\big|^2 - \frac{4}{3}R^{xy}R_y{}^z R_{zx}\bigg) \\
&\quad +R^i{}_l\Big[2|\operatorname{Ric}|^2_{|i}\Big] + R_{lx|y}\Big[-8\Delta R^{xy} + 8R^x{}_i R^{iy}\Big] + 2R_{xy|l}\Delta_L R^{xy} \\
&\quad -4R^{xy}\nabla_y\Delta R_{xl} - 8R^{ix}{}_|{}^y R_{il|xy} - 8R_x{}^a R_{ay|}{}^i R^x{}_{il}{}^y
\end{aligned}
$$

Next, look at the other 4th order term, $\frac{4}{5}\nabla^i \operatorname{Hess}_Y(\Delta R)_{il}$.

$$
\begin{aligned}
-\frac{4}{5}\nabla^i \operatorname{Hess}_Y(\Delta R)_{il} &= -\frac{4}{5}\Delta\nabla_l\Delta R \\
&= -\frac{4}{5}\nabla_l\Delta^2 R - \frac{4}{5}R^i{}_l\nabla_i\Delta R
\end{aligned}
$$

Next look at the second order terms.

$$
\begin{aligned}
4(n+9)\nabla^i\Delta_L R_{il} &= 4(n+9)\Big[\Delta\nabla^i R_{il} - \nabla_i R^{ix}R_{xl} - R^{xy}R_{[xy|l]}\Big] \\
&= 2(n+9)\nabla_l\Big[\Delta R + |\operatorname{Ric}|^2\Big] - 8(n+9)R^{xy}R_{lx|y}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{4}{15}(13n-55)\nabla^i \operatorname{Hess}_Y R_{il} &= \frac{4}{15}(13n-55)\Delta\nabla_l R \\
&= \frac{4}{15}(13n-55)\nabla_l\Delta R + \frac{4}{15}(13n-55)R^i{}_l R_{|i}
\end{aligned}
$$

Thus, the first line simplifies to.

$$\left[2\Delta_L^2 R_{il} - \tfrac{4}{5}(\Delta R)_{|il} + 4(n+9)\Delta_L R_{il} + \tfrac{4}{15}(13n-55)R_{|il}\right]_{|}^{\;i}$$

$$\begin{aligned}
&= \nabla_l\left[\frac{1}{5}\Delta^2 R + \left(\frac{82}{15}n + \frac{10}{3}\right)\Delta R + 2(n+9)|\operatorname{Ric}|^2 + \Delta|\operatorname{Ric}|^2\right. \\
&\quad \left. -\frac{4}{3}R^{xy}R_y{}^z R_{zx}\right] + R^i{}_l\left[-\frac{4}{5}\nabla_i\Delta R + 2|\operatorname{Ric}|^2_{|i} + \frac{4}{15}(13n-55)R_{|i}\right] \\
&\quad + R_{lx|y}\left[-8\Delta R^{xy} + 8R^x{}_i R^{iy} - 8(n+9)R^{xy}\right] + 2R_{xy|l}\Delta_L R^{xy} \\
&\quad - 4R^{xy}\nabla_y\Delta R_{xl} - 8R^{ix}{}_|{}^y R_{il|xy} - 8R_x{}^a R_{ay|}{}^i R^x{}_{il}{}^y
\end{aligned}$$

Next look at the second line.

$$\begin{aligned}
-16(5n-13)R_{il|}{}^i &= -8(5n-13)\nabla_l R \\
32\nabla^i\left[R_{ix}R^x{}_l\right] &= 16R^i{}_l R_{|i} + 32R_{lx|y}R^{xy} \\
-\frac{56}{5}\nabla^i\left[R^{xy}{}_{|i}R_{xy|l}\right] &= -\frac{56}{5}\left[R_{xy|l}\Delta R^{xy} + R^{xyi}{}_| R_{xy|li}\right] \\
&= -\frac{56}{5}\left[R_{xy|l}\Delta R^{xy} + \frac{1}{2}\nabla_l\left|\nabla\operatorname{Ric}\right|^2 + 2R^{xy}{}_|{}^i R_y{}^a R_{lixa}\right] \\
-\frac{76}{5}\nabla^i\left[R^{xy}R_{xy|il}\right] &= -\frac{76}{5}\left[\frac{1}{2}\nabla_l\left|\nabla\operatorname{Ric}\right|^2 + R^{xy}R_{xy|il}{}^i\right] \\
&= -\frac{38}{5}\nabla_l\left|\nabla\operatorname{Ric}\right|^2 - \frac{76}{5}R^{xy}\nabla_l\Delta R_{xy} \\
&\qquad -\frac{152}{5}R^{xy}R_y{}^a{}_|{}^i R_{lixa} - \frac{38}{5}R^i{}_l|\operatorname{Ric}|^2_{|i}
\end{aligned}$$

Hence, the second line simplify to

$$\left[-16(5n-13)R_{il} + 32R_{ix}R^x{}_l - \tfrac{56}{5}R^{xy}{}_{|i}R_{xy|l} - \tfrac{76}{5}R^{xy}R_{xy|il}\right]_{|}^{\;i}$$

$$\begin{aligned}
&= \nabla_l\left[-\frac{56}{5}\langle\operatorname{Ric},\Delta\operatorname{Ric}\rangle - \frac{66}{5}|\operatorname{Ric}|^2 - 8(5n-13)R\right] \\
&\quad + R^i{}_l\left[-\frac{38}{5}|\operatorname{Ric}|^2_{|i} + 16R_{|i}\right] + 32R_{lx|y}R^{xy} - 4R^{xy}\nabla_l\Delta R_{xy}
\end{aligned}$$

Then look at the third line.

$$\tfrac{2}{3}\Big(R_i{}^xR_{|xl}+R_l{}^xR_{|xi}-2R_|{}^xR_{xi|l}-2R_|{}^xR_{xl|i}+3R_|{}^xR_{il|x}\Big)_|{}^i$$

$$\begin{aligned}
&= \frac{2}{3}\Big(\frac{1}{2}R_|{}^xR_{|xl}+R^{xy}R_{|xly}+R^i{}_l\Delta\nabla_iR+R_{lx|y}R_|{}^{xy}-2R_{xy|l}R_|{}^{xy} \\
&\qquad -2R_|{}^xR_{ix|l}{}^i-2R_{lx|y}R_|{}^{xy}-2R_|{}^x\Delta R_{xl}+3R_|{}^xR_{il|x}{}^i+3R_{lx|y}R_|{}^{xy}\Big) \\
&= \frac{2}{3}\Big(\frac{1}{4}\nabla_l\big|\nabla R\big|^2+2R_{lx|y}R_|{}^{xy}-2R_{xy|l}R_|{}^{xy}+R^{xy}\Big[R_{|xyl}+R_{|a}R_{lyx}{}^a\Big] \\
&\qquad +R^i{}_l\Big[\nabla_i\Delta R+R^x{}_iR_{|x}\Big]+R_|{}^x\Big[-2\Big(R_{ix|}{}^i{}_l+R_{ai}R_l{}^i{}_x{}^a+R_{xa}R_l{}^i{}_i{}^a\Big) \\
&\qquad\qquad -2\Delta R_{xl}+3\Big(R_{il|}{}^i{}_x+R_{al}R_x{}^i{}_i{}^a+R_{ia}R_x{}^i{}_l{}^a\Big)\Big]\Big) \\
&= \frac{2}{3}\Big(\frac{1}{4}\nabla_l\big|\nabla R\big|^2+2R_{lx|y}R_|{}^{xy}-2R_{xy|l}R_|{}^{xy}+R^{xy}R_{|xyl}+R_|{}^xR_{ab}R^a{}_{xl}{}^b \\
&\qquad +R^i{}_l\Big[\nabla_i\Delta R+R^x{}_iR_{|x}\Big]+R_|{}^x\Big[-2\Big(\frac{1}{2}R_{|xl}-R_{ab}R^a{}_{xl}{}^b+R_x{}^aR_{al}\Big) \\
&\qquad\qquad -2\Delta R_{xl}+3\Big(\frac{1}{2}R_{|xl}+R_x{}^aR_{al}-R_{ab}R^a{}_{xl}{}^b\Big)\Big]\Big) \\
&= \nabla_l\left[\frac{1}{3}|\nabla R|^2+\frac{2}{3}\langle \mathrm{Ric},\mathrm{Hess}_Y\,R\rangle\right]+R^i{}_l\left[\frac{2}{3}\nabla_i\Delta R\right]+R_{lx|y}\left[\frac{4}{3}R_|{}^{xy}\right] \\
&\quad -2R_|{}^{xy}R_{xy|l}-\frac{4}{3}R_|{}^i\big[\Delta R_{il}-R_i{}^xR_{xl}\big]
\end{aligned}$$

Next look at $\left[12R^{xy}\Big(R_{ix|yl}+R_{lx|yi}\Big)\right]_|{}^i$

$$\begin{aligned}
12R^{xy}R_{ix|yl}{}^i &= 12R^{xy}\Big[R_{ix|y}{}^i{}_l+R_{ax|y}R_l{}^i{}_i{}^a+R_{ia|y}R_l{}^i{}_x{}^a+R_{ix|a}R_l{}^i{}_y{}^a\Big] \\
&= 6R^{xy}R_{|xyl}+12R^{xy}\Big[R_{ax}R_y{}^i{}_i{}^a+R_{ia}R_y{}^i{}_x{}^a\Big]_{|l} \\
&\quad +12R^i{}_lR^{xy}R_{ix|y}-12R^{xy}R_{ab|y}R^a{}_{xl}{}^b-12R^{xy}R_{xa|b}R^b{}_{yl}{}^a \\
&= 6R^{xy}R_{|xyl}+12R^{xy}\Big[R_x{}^iR_{iy}-R_{ab}R^a{}_{xy}{}^b\Big]_{|l} \\
&\quad +12R^i{}_lR^{xy}R_{ix|y}-12R^{xy}R_{ab|x}R^a{}_{yl}{}^b+12R_x{}^aR_{ay|}{}^iR^x{}_{il}{}^y
\end{aligned}$$

and

$$
\begin{aligned}
12R^{xy}R_{lx|yi}{}^{i} &= 12R^{xy}\Delta\nabla_y R_{xl} \\
&= 12R^{xy}\nabla_y\Delta R_{xl} + R^{i}{}_{l}\Big[-6|\operatorname{Ric}|^2_{|i} + 12R^{xy}R_{ix|y}\Big] \\
&\quad + 4\nabla_l\big(R^{xy}R_y{}^{z}R_{zx}\big) - 24R_{lx|y}R^{ab}R^{x}{}_{ab}{}^{y} + 24R_x{}^{a}R_{ay|}{}^{i}R^{x}{}_{il}{}^{y}
\end{aligned}
$$

Thus, $\left[12R^{xy}\Big(R_{ix|yl} + R_{lx|yi}\Big)\right]_{|}{}^{i}$

$$
\begin{aligned}
&= 12\Big[R^{xy}{}_{|}{}^{i}\Big(R_{ix|yl} + R_{lx|yi}\Big) + R^{xy}\Big(R_{ix|yl}{}^{i} + R_{lx|yi}{}^{i}\Big)\Big] \\
&= \nabla_l\Big[6R^{ix}{}_{|}{}^{y}R_{iy|x} + 4R^{xy}R_y{}^{z}R_{zx}\Big] + \ R^{i}{}_{l}\Big[-6|\operatorname{Ric}|^2_{|i} + 24R^{xy}R_{ix|y}\Big] \\
&\quad + R_{lx|y}\big[-24R^{ab}R^{x}{}_{ab}{}^{y}\big] + 6R^{xy}R_{|xyl} + 12R^{ix}{}_{|}{}^{y}R_{il|xy} + 12R^{xy}\nabla_y\Delta R_{xl} \\
&\quad + 12R^{xy}\Big[R_x{}^{i}R_{iy} - R_{ab}R^{a}{}_{xy}{}^{b}\Big]_{|l} - 12R^{xy}R_{ab|x}R^{a}{}_{yl}{}^{b} + 36R_x{}^{a}R_{ay|}{}^{i}R^{x}{}_{il}{}^{y}
\end{aligned}
$$

Next look at

$$
\begin{aligned}
&4\Big[R_i{}^{x}{}_{|}{}^{y}R_{[lx|y]} + R_l{}^{x}{}_{|}{}^{y}R_{[ix|y]}\Big]_{|}{}^{i} \\
&\qquad = 4R_i{}^{x}{}_{|}{}^{yi}R_{[lx|y]} + 4R_i{}^{x}{}_{|}{}^{y}R_{[lx|y]}{}^{i} + 4R_l{}^{x}{}_{|}{}^{yi}R_{[ix|y]} + 4R_l{}^{x}{}_{|}{}^{y}R_{[ix|y]}{}^{i}
\end{aligned}
$$

Then simplify each term separately. First,

$$
\begin{aligned}
4R_i{}^{x}{}_{|}{}^{yi}R_{[lx|y]} &= 4\left[\frac{1}{2}R_{|}{}^{xy} + R_a{}^{x}R^{yi}{}_{i}{}^{a} + R_{ia}R^{yixa}\right]R_{[lx|y]} \\
&= R_{xy|l}\Big[2R_{|}{}^{xy} + 4R^{x}{}_{i}R^{iy} - 4R^{ab}R^{x}{}_{ab}{}^{y}\Big]
\end{aligned}
$$

Second,

$$
\begin{aligned}
4R_i{}^x{}_|{}^y R_{[lx|y]}{}^i &= 4R_i{}^x{}_|{}^y\Big[R_{xy|l}{}^i + R_{ly|x}{}^i - R_{lx|y}{}^i\Big] \\
&= 4R^{ix}{}_|{}^y R_{iy|lx} + 4R^{xy}{}_|{}^i R_{il|xy} - 4R^{ix}{}_|{}^y R_{il|yx} \\
&= 2\nabla_l\Big(R^{ix}{}_|{}^y R_{iy|x}\Big) + 4R^{ix}{}_|{}^y\Big[R_{ay}R_{lxi}{}^a + R_{ia}R_{lxy}{}^a\Big] \\
&\quad +4R^{xy}{}_|{}^i R_{il|xy} - 4R^{ix}{}_|{}^y\Big[R_{il|xy} + R_{al}R_{yxi}{}^a + R_{ia}R_{yxl}{}^a\Big] \\
&= 2\nabla_l\Big(R^{ix}{}_|{}^y R_{iy|x}\Big) + 4R^{xy}{}_|{}^i R_{il|xy} - 4R^{ix}{}_|{}^y R_{il|xy} \\
&\quad +4R^{ix}{}_|{}^y\Big[R_{ay}R_{lxi}{}^a + R_{ia}R_{lxy}{}^a - R_{al}R_{yxi}{}^a - R_{ia}R_{yxl}{}^a\Big]
\end{aligned}
$$

Third,

$$
\begin{aligned}
4R_l{}^x{}_|{}^{yi} R_{[ix|y]} &= 4R_l{}^x{}_|{}^{yi}\Big[R_{xy|i} + R_{iy|x} - R_{ix|y}\Big] \\
&= 4R_{xy|}{}^i\Big[R_{il|}{}^{xy} + R^x{}_{l|}{}^y{}_i - R^x{}_{l|i}{}^y\Big] \\
&= 4R_{xy|}{}^i\Big[R_{il|}{}^{xy} + R^a{}_l R^y{}_i{}^x{}_a + R^x{}_a R^y{}_{il}{}^a\Big]
\end{aligned}
$$

Forth,

$$
\begin{aligned}
4R_l{}^x{}_|{}^y R_{[ix|y]}{}^i &= 4R_l{}^x{}_|{}^y\Big[\Delta R_{xy} + R_{iy|x}{}^i - R_{ix|y}{}^i\Big] \\
&= 4R_l{}^x{}_|{}^y\Big[\Delta R_{xy} + R_{ay}R_x{}^i{}_i{}^a + R_{ia}R_x{}^i{}_y{}^a - R_{ax}R_y{}^i{}_i{}^a - R_{ia}R_y{}^i{}_x{}^a\Big] \\
&= 4R_{lx|y}\Delta R^{xy}
\end{aligned}
$$

Hence, $4\Big[R_i{}^x{}_|{}^y R_{[lx|y]} + R_l{}^x{}_|{}^y R_{[ix|y]}\Big]_|{}^i$

$$
\begin{aligned}
&= 2\nabla_l\Big(R^{ix}{}_|{}^y R_{iy|x}\Big) + 8R^{xy}{}_|{}^i R_{il|xy} - 4R^{ix}{}_|{}^y R_{il|xy} + 4R_{lx|y}\Delta R^{xy} \\
&\quad + R_{xy|l}\Big[2R_|{}^{xy} + 4R^x{}_i R^{iy} - 4R^{ab}R^x{}_{ab}{}^y\Big] - 8R^i{}_l\Big[R_{xy|}{}^a R^x{}_{ai}{}^y\Big] \\
&\quad + 4R^{xy}R_{ab|x}R^a{}_{yl}{}^b - 4R_x{}^a R_{ay|}{}^i R^x{}_{il}{}^y + 8R_{xa|}{}^i R_{ay}R^x{}_{il}{}^y
\end{aligned}
$$

Therefore, line four simplifies to

$$\Big[12R^{xy}\Big(R_{ix|yl}+R_{lx|yi}\Big)+4R_i{}^x{}_|{}^yR_{[lx|y]}+4R_l{}^x{}_|{}^yR_{[ix|y]}\Big]_|{}^i$$

$$\begin{aligned}
&= \nabla_l\Big[2\langle \mathrm{Ric}, \mathrm{Hess}_Y R\rangle + 8R^{ix}{}_|{}^yR_{iy|x}+4R^{xy}R_y{}^zR_{zx}\Big]\\
&\quad +R^i{}_l\Big[-6|\,\mathrm{Ric}\,|^2_{|i}+24R^{xy}R_{ix|y}-8R_{xy|}{}^aR^x{}_{ai}{}^y\Big]+4R^{xy}R_{|xyl}\\
&\quad +R_{lx|y}\big[4\Delta R^{xy}-24R^{ab}R^x{}_{ab}{}^y\big]+R_{xy|l}\Big[4R^x{}_iR^{iy}-4R^{ab}R^x{}_{ab}{}^y\Big]\\
&\quad +8R^{ix}{}_|{}^yR_{il|xy}+8R^{xy}{}_|{}^iR_{il|xy}+12R^{xy}\nabla_y\Delta R_{xl}+12R^{xy}\Big[R_x{}^iR_{iy}-R_{ab}R^a{}_{xy}{}^b\Big]_{|l}\\
&\quad -8R^{xy}R_{ab|x}R^a{}_{yl}{}^b+32R_x{}^aR_{ay|}{}^iR^x{}_{il}{}^y+8R_{xa|}{}^iR_{ay}R^x{}_{il}{}^y
\end{aligned}$$

Then look at the fifth line.

$$\Big[-8R^{xy}\Big(R_{il|xy}-R_{ia}R^a{}_{xyl}-R_{la}R^a{}_{xyi}+2R_y{}^zR_{xilz}\Big)\Big]_|{}^i$$

$$\begin{aligned}
-8\Big[R^{xy}R_{il|xy}\Big]_|{}^i &= -8R^{xy}{}_|{}^iR_{il|xy}-8R^{xy}R_{il|xy}{}^i\\
&= -8R^{xy}{}_|{}^iR_{il|xy}-8R^{xy}R_{il|x}{}^i{}_y\\
&\quad -8R^{xy}\Big[R_{al|x}R_y{}^i{}_i{}^a+R_{ia|x}R_y{}^i{}_l{}^a+R_{il|a}R_y{}^i{}_x{}^a\Big]\\
&= -8R^{xy}{}_|{}^iR_{il|xy}-4R^{xy}R_{|lxy}\\
&\quad -8R^{xy}\Big[R_{al|x}R_y{}^a-R_{ia|x}R^i{}_{yl}{}^a-R_{il|a}R^i{}_{xy}{}^a\Big]\\
&\quad -8R^{xy}\Big[R_{al}R_x{}^a-R_{ia}R^i{}_{xl}{}^a\Big]_{|y}\\
&= -8R^{xy}{}_|{}^iR_{il|xy}-4R^{xy}\Big[R_{|xyl}+R_{|a}R_{lyx}{}^a\Big]\\
&\quad -8R^{xy}\Big[R_{al|x}R_y{}^a-R_{ia|x}R^i{}_{yl}{}^a-R_{il|a}R^i{}_{xy}{}^a\Big]\\
&\quad -8R^{xy}\Big[R_{al}R_x{}^a-R_{ia}R^i{}_{xl}{}^a\Big]_{|y}
\end{aligned}$$

$$8\Big[R_i{}^aR_{xy}R^x{}_{al}{}^y\Big]_|{}^i = 4R_|{}^iR_{xy}R^x{}_{il}{}^y+8R^{xy}\Big[R_{ab}R^a{}_{xl}{}^b\Big]_{|y}$$

$$8\Big[R_l{}^a R_{xy} R^x{}_{ia}{}^y\Big]_|{}^i = 8R_{lx|y}R^{ab}R^x{}_{ab}{}^y + 8R^i{}_l R_{xy|}{}^a R^x{}_{ai}{}^y - 8R_l{}^a R_{xy}\Big[R^x{}_{ia|}{}^{i}{}^y + R^x{}_{i|a}{}^{yi}\Big]$$

$$= 8R_{lx|y}R^{ab}R^x{}_{ab}{}^y + R^i{}_l\Big[8R_{xy|}{}^a R^x{}_{ai}{}^y - 8R^{xy}R_{ix|y} + 4|\operatorname{Ric}|^2_{|i}\Big]$$

$$-16\Big[R_{xa}R^a{}_y R^x{}_{il}{}^y\Big]_|{}^i = -16\Big[R_{xa}R^a{}_y\Big]_|{}^i R^x{}_{il}{}^y + 16R_{xa}R^a{}_y\Big[R^x{}_i{}^{yi}{}_{|l} + R^x{}_i{}^i{}_{l|}{}^y\Big]$$

$$= -16\Big[R_{xa}R^a{}_y\Big]_|{}^i R^x{}_{il}{}^y + 16R_{lx|y}R^{xi}R_i{}^y - \frac{16}{3}\nabla_l(R^{xy}R_y{}^z R_{zx})$$

Therefore, line 5 simplifies to

$$\Big[-8R^{xy}\Big(R_{il|xy} - R_{ia}R^a{}_{xyl} - R_{la}R^a{}_{xyi} + 2R_y{}^z R_{xilz}\Big)\Big]_|{}^i$$

$$= -\frac{16}{3}\nabla_l(R^{xy}R_y{}^z R_{zx}) + R^i{}_l\Big[4|\operatorname{Ric}|^2_{|i} + 8R_{xy|}{}^a R^x{}_{ai}{}^y\Big]$$

$$+ R_{lx|y}\Big[16R^{xi}R_i{}^y + 16R^{ab}R^x{}_{ab}{}^y\Big] - 4R^{xy}R_{|xyl} - 8R^{xy}{}_|{}^i R_{il|xy}$$

$$+ 8R^{xy}\Big[-2R_{xi}R^i{}_l + 2R_{ab}R^a{}_{xl}{}^b\Big]_{|y} + 8R^{xy}R_{ab|x}R^a{}_{yl}{}^b - 16\Big[R_{xa}R^a{}_y\Big]_|{}^i R^x{}_{il}{}^y$$

Combining these equations shows that the equation is satisfied by h_4.

2.4 Proof of Theorem 1

Now that the first few terms in the expansions of f and H have been calculated, we are ready to prove Theorem 1. As the calculations in section 2.2 show, for $i \geq 1$, we can determine f_{2i} in terms of the quantities $\{g_Y, h_0, \ldots, h_{2i-2}\}$ using equation 14; and we can determine h_{2i} in terms of the quantities $\{g_Y, h_0, \ldots, h_{2i-2}, f_0, \ldots, f_{2i}\}$ using equation 16. Notice that the $H_{jk,r}f_{,r}$-term and the H_{jk}-term in equation 16 will combine to give a factor of $(i+1)r^{-2i}(h_{2i})_{jk}$, so in particular, it will always be nonzero. By this construction, the expansions for f and H satisfy equation 14 and 16. Further, the calculations in section 2.3 show that equation 15 is of order no more that $O(r^{-9})$. It remains to show that equation 15 is satisfied to all orders. First we will derive another formula involving the potential function and the Ricci tensor. Then we will show how to argue using this formula that the mixed term equation is satisfied to every order.

To derive this new equation, consider a general Riemannian manifold (M, g) and a smooth function $f \in C^\infty(M)$. Then consider the metric-measure space $(M, g, e^{-f}\mathrm{dvol}_g)$. The analog of the Ricci tensor for a metric-measure space is the Bakey-Emery-Ricci tensor $\operatorname{Ric} + \operatorname{Hess}(f)$. Looking for weighted analog of the contracted Bianchi identity $\nabla^a R_{ab} = \frac{1}{2}\nabla_b R$ leads to the following equation

$$\nabla^a(R_{ab} + \nabla_a\nabla_b f) - (\nabla^a f)(R_{ab} + \nabla_a\nabla_b f) = \frac{1}{2}\nabla_b(R + 2\Delta f - |\nabla f|^2)\ . \tag{33}$$

Notice, while this equation is motivated by the measure structure of $(M, g, e^{-f}\mathrm{dvol}_g)$, this equation does not use the measure. One recognizes the right hand of this equation as Perelman's weighted scalar curvature, $R + 2\Delta f - |\nabla f|^2$.

This equation can be rewritten as

$$\nabla^a\left(\mathrm{Ric}(g)_{ab} + \mathrm{Hess}_g f_{ab} + \frac{1}{2}g_{ab}\right) - (\nabla^a f)\left(\mathrm{Ric}(g)_{ab} + \mathrm{Hess}_g f_{ab} + \frac{1}{2}g_{ab}\right) = \frac{1}{2}\nabla_b(R + 2\Delta f - |\nabla f|^2 - f). \quad (34)$$

From this equation one can see the known fact that if (M, g, f) is a gradient expanding Ricci soliton then $R + 2\Delta f - |\nabla f|^2 - f$ is a constant. After modifying f by this constant, we can assume that the soliton satisfies $R + 2\Delta f - |\nabla f|^2 - f = 0$.

Returning to the setting of Theorem 1, equation 34 can be simplified. Recall that we use coordinates $\{r, x^1, \ldots, x^n\}$. By construction, the expansions in equations 4 and 5 satisfy

$$\mathrm{Ric}(g)_{jk} + \mathrm{Hess}_g f_{jk} + \frac{1}{2}g_{jk} = 0$$

and

$$\mathrm{Ric}(g)_{rr} + \mathrm{Hess}_g f_{rr} + \frac{1}{2}g_{rr} = 0.$$

Let $X_{ir} = R_{ir} + \nabla_i\nabla_r f$ and $S = R + 2\Delta f - |\nabla f|^2 - f$. Then substituting equation 4 and 5 into equation 34 gives

$$\nabla^i X_{ir} - (\nabla^i f)X_{ir} = \frac{1}{2}\partial_r S \quad (35)$$

when the b coordinate is replaced with r; and

$$\nabla_r X_{ir} - (\partial_r f)X_{ir} = \frac{1}{2}\partial_i S \quad (36)$$

when the b coordinate is replaced with one of the coordinates on Y. We rewrite these equations in terms of $H(r)$ and the Levi-Civita connection of $H(r)$. This gives

$$H^{ij}\left[\nabla_j X_{ir} - (\nabla_j f)X_{ir}\right] = \frac{1}{2}\partial_r S \quad (37)$$

and

$$\partial_r X_{ir} - \frac{1}{2}H^{jk}H_{ki,r}X_{jr} - (\partial_r f)X_{ir} = \frac{1}{2}\partial_i S \quad (38)$$

Now that we have these equations we can prove the following lemma

Lemma 3. *If S vanishes to all orders in r^{-1}, then X_{ir} vanishes to all orders in r^{-1}.*

Proof. We suppose not and show a contradiction. If X_{ir} does not vanish to all orders in r^{-1}, then $X_{ir} = r^{-N}\phi + O(r^{-N-1})$ for some integer $N \geq 1$ and some nonzero $\phi \in \Omega^1(Y)$. Using the leading order terms form the asymptotic expansions for f and H, the left-hand side of equation 38 becomes

$$\frac{1}{2}r^{-N+1}\phi_i + O(r^{-N})$$

However, the right-hand side of equation 38 vanishes to all orders, so ϕ must be zero. This is a contradiction as we assumed ϕ was the leading nonzero term in the expansion of X_{ir}. □

Now we can proof Theorem 1. It is enough to show that X_{ir} vanishes to all orders. Suppose, by way of contradiction, that $X_{ir} = r^{-N}\phi + O(r^{-N-1})$ for some $N \geq 1$ and some nonzero $\phi \in \Omega^1(Y)$. From Lemma 3, S does cannot vanish to all orders in r^{-1}. Thus, $S = r^{-M}\psi + O(r^{-M-1})$ for some $M \geq 1$ and some nonzero $\psi \in C^\infty(Y)$. Using the leading order terms of the asymptotic expansions for f and H, the left-handed side of 38 is

$$\frac{1}{2}r^{-N+1}\phi_i + O(r^{-N}).$$

The right-hand side of 38 is

$$\frac{1}{2}r^{-M}\partial_i\psi + O(r^{-M-1}).$$

Because ϕ is nonzero, this implies $M \leq N-1$.

Next, look at equation 37 and plug in the leading order terms for f and H. The left-hand side of 37 is

$$r^{-N-2}h^{ij}\nabla_j\phi_i + O(r^{-N-3}),$$

while the right-hand of equation 37 becomes

$$-\frac{1}{2}Mr^{-M-1}\psi + O(r^{-M-2})$$

Because ψ is nonzero, this implies $M \geq N+1$. This is a contradiction and thus

$$X_{ir} = R_{ir} + \nabla_i\nabla_r f = 0$$

which proves Theorem 1.

This argument can also be used to construct asymptotic expansions for conical gradient shrinking solitons. The leading order term for f becomes $\frac{1}{4}r^2$.

2.5 Bryant Soliton Example

The Bryant Soliton is one example of an asymptotically conical expanding Ricci soliton. Thus, if the above formal calculations are reflective of actual asymptotic behavior, then the asymptotic behavior of the Bryant soliton should agree with the above, general calculations. In this section, we will first recall the equations satisfied by the Bryant soliton and then show that specializing equations 14, 15 and 16 from section 2.1 give the same system of equations. Lastly, the asymptotic expansion of the Bryant soliton will be calculated and the above calculations for H_o in this thesis will be specialized to the Bryant soliton case to show the two agree.

Start by recall the Bryant Soliton metric is of the form:

$$g_{BS} = dr^2 + a(r)d\sigma^2$$

where σ^2 is the standard round metric on $\mathbb{S}^2$ and $a(t)$ is only a function of the radial parameter r. (note in his paper the radial parameter is denoted as t.) Further, Bryant showed that there exists a smooth function, $a(r)$, that is odd in r and satisfies the Ricci soliton equation:

$$\text{Ric}(g_{BS}) = \text{Hess}_{g_{BS}}(f) - \lambda g_{BS}.$$

Notice that the expander equation considered in the rest of this thesis uses $\lambda = 1/2$ and the potential function has the opposite sign.

Bryant shows the metric, g_{BS}, satisfies the Ricci soliton equation by showing that $a(r)$ and $f(r)$ satisfy the system of ODEs (equation (2.4) in [B]).

$$-2a(r)a''(r) = a(r)^2\left[f''(r) - \lambda\right] \tag{39}$$

$$1 - a'(r)^2 - a(r)a''(r) = a(r)a'(r)f'(r) - \lambda a(r)^2 \tag{40}$$

To compare these equations to equations 14, 15 and 16 from section 2.1, i.e. let

$$H = a(r)^2 d\sigma^2$$

as in the Bryant soliton case. Then clearly, $H^{-1} = a(r)^{-2} d\sigma^\sharp \otimes d\sigma^\sharp$. Hence, equation 14 becomes

$$2a(r)^2\left[f''(r) - \frac{1}{2}\right] = 2na(r)a''(r)$$

Setting $\lambda = 1/2$, $n = 2$, and changing the sign of f gives the first equation in Bryant's system of equations. Notice that this equation shows that f is a function of r and not of any coordinates on the sphere. Hence, the partials of f are all zero unless they are of the form $\partial r^k f$ for some $k \in \mathbb{N}$. Next, the mixed term equation vanishes identically as follows.

$$2\,\mathrm{Hess}_g\, f_{rl} = 2f_{,rl} - H^{ix}H_{xl,r}f_{,i} = 0$$

by the statement above. Then

$${}^{H}\nabla_i H_{ml,r} = [a(r)^2]' \cdot {}^{d\sigma^2}\nabla_i d\sigma^2_{ml} = 0$$

by the metric property. Finally,

$$\mathrm{Ric}(H) = \mathrm{Ric}(a(r)^2 d\sigma^2) = \mathrm{Ric}(d\sigma^2) = (n-1)d\sigma^2$$

Plugging this into equation 16 gives

$$(n-1) - \left[a(r)^2\right]'' + a(r)a'(r)f'(r) + \frac{1}{2}a(r)^2 = 0.$$

Setting $\lambda = 1/2$, $n = 2$, and changing the sign of f gives the second equation in Bryant's system of equations.

Now, calculate the first few terms in the asymptotic expansion of $a(r)$. To get an equation for $a(r)$ without f, take the derivative of equation 40 and then plug in equation 39 for the $f''(r)$ term.

The derivative equation is

$$0 = a(r)a'(r)\left[f'(r) - 2\lambda\right] + \left[a(r)a'(r)\right]' f'(r) + \left[a(r)a'(r)\right]''$$

Then plugging in for $f'(r)$ and $f''(r)$.

$$\begin{aligned} 0 = {} & a'(r)^2 + a(r)a''(r) + \lambda a(r)^3 a''(r) \\ & - a'(r)^2 a(r)a''(r) - a'(r)^4 - a(r)^2 a''(r)^2 + a(r)^2 a'(r)a'''(r) \end{aligned} \tag{41}$$

Then, Bryant showed that $a(r)$ is an analytic, odd function of r such that there exists a constant $c \in \mathbb{R}$ satisfying

$$\lim_{r\to\infty} \left|a(r) - cr\right| = 0$$

Hence, the asymptotic expansion of $a(r)$ should be of the form

$$a(r) \sim cr + c_1 r^{-1} + c_3 r^{-3} + c_5 r^{-5} + \cdots$$

for some constants c_i. Notice, if $c = 1$, then g_{BS} is asymptotic to the standard flat metric on $\mathbb{R}^2$ and the lower order terms are clearly zero. In the following, assume that $c \neq 1$. The following calculations show that the lower order terms do in fact cancel out when $c = 1$.

Then, look at the coefficients for r^0.

$$0 = c^2 + 2\lambda c^3 c_1 - c^4$$

which implies

$$c_1 = \frac{1}{2\lambda}\left(c - \frac{1}{c}\right)$$

Next, look at the coefficients for r^{-2}.

$$0 = 6\lambda c^2 c_1^2 + 12\lambda c^3 c_3 - 4c^3 c_1$$

which implies

$$c_3 = \frac{4c^3 c_1 - 6\lambda c^2 c_1^2}{12\lambda c^3} = \frac{1}{8\lambda^2}\left(\frac{c}{3} + \frac{2}{3c} - \frac{1}{c^3}\right)$$

Then, look at the coefficients for r^{-4}.

$$0 = 3c_1^2 + 6cc_3 + 6\lambda c c_1^3 + 42\lambda c^2 c_1 c_3 + 30\lambda c^3 c_5 - 14c^2 c_1^2 - 60c^3 c_3$$

which implies

$$c_5 = \frac{1}{240\lambda^3}\left(35c - \frac{13}{c} - \frac{7}{c^3} - \frac{15}{c^5}\right)$$

Then calculate $a(r)^2$

$$a(r)^2 \sim c^2 r^2 + \frac{1}{\lambda}(c^2 - 1) + \frac{1}{3\lambda^2}(c^2 - 1)r^{-2} + \frac{1}{3\lambda^3}\left(c^2 - \frac{1}{5} - \frac{4}{5c^2}\right) r^{-4} + O(r^{-6}) \quad (42)$$

Hence, the metric becomes

$$\begin{aligned} g_{BS} &= dr^2 + a(r)^2 d\sigma^2 \\ &\sim dr^2 + \left[r^2 + 2\left(1 - \frac{1}{c^2}\right) + \frac{4}{3}\left(1 - \frac{1}{c^2}\right) r^{-2} \right. \\ &\qquad \left. + \frac{8}{3}\left(1 - \frac{1}{5c^2} - \frac{4}{5c^4}\right) r^{-4} + O(r^{-4})\right] c^2 d\sigma^2 \end{aligned}$$

In particular, notice that plugging in $c = 1$ gives exactly $g_{BS} \sim dr^2 + r^2 d\sigma^2$.

Then the expansion of $a(r)$ can be used to calculate the expansion of $f(r)$. Recall that

$$a(r)f''(r) = \lambda a(r) - 2a''(r)$$

It is clear from this equation that f is an even function of r. Hence, f has the form

$$f(r) \sim br^2 + b_0 + b_2 r^{-2} + b_4 r^{-4} + O(r^{-6})$$

or

$$f''(r) \sim 2b + 6b_2 r^{-4} + 20b_4 r^{-6} + O(r^{-8})$$

These terms expand to

$$\begin{aligned} a(r)f''(r) &\sim 2bcr + 2bc_1 r^{-1} + (2bc_3 + 6b_2 c)r^{-3} + \big(20b_4 c + 6b_2 c_1 + 2bc_5\big)r^{-5} \\ &\quad + \big(42b_6 c + 20b_4 c_1 + 6b_2 c_3 + 2bc_7\big)r^{-7} + O(r^{-9}) \\ \lambda a(r) &\sim \lambda cr + \lambda c_1 r^{-1} + \lambda c_3 r^{-3} + \lambda c_5 r^{-5} + \lambda c_7 r^{-7} \\ -2a''(r) &\sim -4c_1 r^{-3} - 24c_3 r^{-5} - 60c_5 r^{-7} \end{aligned}$$

Now look at the coefficients for r^1. This gives

$$2bc = \lambda c$$

which implies

$$b = \frac{\lambda}{2}$$

Next look at the coefficients for r^{-1}. This gives

$$2bc_1 = \lambda c_1$$

which also implies

$$b = \frac{\lambda}{2}$$

Then look at the coefficients for r^{-3}. This gives

$$2bc_3 + 6b_2 c = \lambda c_3 - 4c_1$$

which implies

$$b_2 = \frac{-2c_1}{3c} = \frac{-1}{3\lambda}\left(1 - \frac{1}{c^2}\right)$$

Then, look at the coefficients for r^{-5}. This gives

$$20b_4 c + 6b_2 c_1 + 2bc_5 = \lambda c_5 - 24c_3$$

which implies

$$b_4 = \frac{-6b_2c_1 - 24c_3}{20c} = \frac{-1}{5\lambda^2}\left(\frac{1}{c^2} - \frac{1}{c^4}\right)$$

Finally, look at the coefficients for r^{-7}. This gives

$$42cb_6 + 20b_4c_1 + 6b_2c_3 = -60c_5$$

which implies

$$b_6 = \frac{1}{21\lambda^3}\left(\frac{-13}{3} + \frac{8}{3c^2} - \frac{4}{3c^4} + \frac{3}{c^6}\right) = \frac{8}{21}\left(\frac{-13}{3} + \frac{8}{3c^2} - \frac{4}{3c^4} + \frac{3}{c^6}\right)$$

Then plug in $\lambda = 1/2$, and the expansion for f becomes

$$f(r) = \frac{1}{4}r^2 + const. - \frac{2}{3}\left(1 - \frac{1}{c^2}\right)r^{-2} - \frac{4}{5}\left(\frac{1}{c^2} - \frac{1}{c^4}\right)r^{-4} + O(r^{-6})$$

Second, plug in $g_Y = c^2 d\sigma^2$ into the equations for h_0, h_2, h_4, and f to check that they coincide. Recall that

$$\begin{aligned}
Rm(c^2d\sigma^2)_{xyzw} &= c^2 Rm(d\sigma^2)_{xyzw} &&= c^2 d\sigma^2(\partial_x \wedge \partial_y, \partial_w \wedge \partial_z) \\
\mathrm{Ric}(c^2d\sigma^2) &= \mathrm{Ric}(d\sigma^2) &&= (n-1)d\sigma^2 \\
R(c^2d\sigma^2) &= c^{-2}R(d\sigma^2) &&= n(n-1)c^{-2}
\end{aligned}$$

In particular, all derivatives of the curvatures are zero by the metric property. First, calculate h_0.

$$\begin{aligned}
h_0 &= -2\big[\mathrm{Ric}(c^2d\sigma^2) - (n-1)c^2d\sigma^2\big] \\
&= -2\big[(n-1)d\sigma^2 - (n-1)c^2d\sigma^2\big] \\
&= 2(n-1)(c^2-1)d\sigma^2 \\
&= 2(c^2-1)d\sigma^2
\end{aligned}$$

Next calculate h_2.

$$\begin{aligned}
h_2 &= -4\,\mathrm{Ric}(c^2d\sigma^2) + \frac{4}{3}\big[R(c^2d\sigma^2) - (n-1)(n-3)\big](c^2d\sigma^2) \\
&= -4(n-1)d\sigma^2 + \frac{4}{3}\big[n(n-1)c^{-2} - (n-1)(n-3)\big](c^2d\sigma^2) \\
&= \frac{4}{3}(n-1)(n-3)(1-c^2)d\sigma^2 \\
&= \frac{4}{3}(c^2-1)d\sigma^2
\end{aligned}$$

Third, calculate h_4.

$$
\begin{aligned}
h_4 &= \frac{8}{3}(5n-13)R_{jk} - \frac{16}{3}R_{jx}R^x{}_k \\
&\quad + \left[\frac{8}{5}|\operatorname{Ric}|^2 - \frac{8}{15}(9n-25)R + \frac{8}{15}(n-1)(2n-5)(3n-11)\right] g_{Y\,jk} \\
&= \frac{8}{3}(5n-13)\operatorname{Ric}(c^2 d\sigma^2)_{jk} - \frac{16}{3}(c^{-2}d\sigma^{xy})\operatorname{Ric}(c^2 d\sigma^2)_{jx}\operatorname{Ric}(c^2 d\sigma^2)_{yk} \\
&\quad + \left[\frac{8}{5}(c^{-2}d\sigma^{ab})(c^{-2}d\sigma^{xy})\operatorname{Ric}(c^2 d\sigma^2)_{ax}\operatorname{Ric}(c^2 d\sigma^2)_{by}\right](c^2 d\sigma^2) \\
&\quad + \frac{8}{15}\left[-(9n-25)R(c^2 d\sigma^2) + (n-1)(2n-5)(3n-11)\right](c^2 d\sigma^2) \\
&= \frac{8}{3}(n-1)(5n-13)d\sigma^2_{jk} - \frac{16}{3}(n-1)^2 c^{-2} d\sigma^2_{jk} + \frac{8}{5}n(n-1)^2 c^{-2} d\sigma^2 \\
&\quad + \frac{8}{15}\Big[-n(n-1)(9n-25)c^{-2} + (n-1)(2n-5)(3n-11)\Big](c^2 d\sigma^2) \\
&= \frac{8}{3}\left[1 - \frac{1}{5c^2} - \frac{4}{5c^4}\right] c^2 d\sigma^2
\end{aligned}
$$

Therefore, the Bryant soliton metric (n=2) has the same asymptotic expansion as the formal calculation.

Finally, calculation the potential function's expansion in the Bryant soliton case. Recall

$$
f(r) = -\frac{1}{4}r^2 + const. + f_2 r^{-2} + f_4 r^{-4} + f_6 r^{-6}
$$

Then

$$
\begin{aligned}
f_2 &= -\frac{1}{3}\left[R(c^2 d\sigma^2) - n(n-1)\right] \\
&= \frac{1}{3}n(n-1)\left(1 - \frac{1}{c^2}\right) \\
&= \frac{2}{3}\left(1 - \frac{1}{c^2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
f_4 &= \frac{1}{5}\left[-2|\operatorname{Ric}|^2 + 2(3n-5)R - 4n(n-1)(n-2)\right] \\
&= \frac{1}{5}\left[-2\left(c^{-2}d\sigma^{ab}\right)\left(c^{-2}d\sigma^{xy}\right)((n-1)d\sigma_{ax})((n-1)d\sigma_{by})\right. \\
&\qquad \left. +2(3n-5)n(n-1)c^{-2} - 4n(n-1)(n-2)\right] \\
&= \frac{1}{5}\left[-2n(n-1)^2c^{-4} + 2n(n-1)(2n-5)c^{-2} - 4n(n-1)(n-2)\right] \\
&= \frac{4}{5}\left[\frac{1}{c^2} - \frac{1}{c^4}\right]
\end{aligned}
$$

Therefore, the Bryant Soliton example agrees with the calculates done in section 2.1. In particular, this provides an example where the formal expansion corresponds to a well defined metric on the manifold.

3 Asymptotic stability for Ricci-DeTurck flows with rough initial data and nice asymptotic cones

The layout of this section will be as follows. In section 3.2, the notation and background needed in the proof of Theorem 2 will be introduced. In section 3.3, the structure of g_{sol} will be discussed. In section 3.4, convergence will be shown in a weak Sobolev norm. In section 3.5, the full regularity of the convergence will be shown.

3.1 Background

The intent of this section is to fix notation and review several ideas and facts from [KL] needed in this paper. Throughout this paper, all derivatives and distance-balls will be taken with respect to the Euclidean metric unless otherwise stated. More precisely,

$$B(x,r) := \{y \in \mathbb{R}^n \mid d_\delta(x,y) = |x-y| < r\}$$

The main result needed from [KL] is the following, which is Theorem 4.3 in [KL].

Theorem 4. *There exists $\epsilon_n > 0$, $C_n > 0$ such that for every metric $g_0 \in L^\infty(\mathbb{R}^n)$ satisfying $||g_0 - \delta||_{L^\infty(\mathbb{R}^n)} < \epsilon_n$ there exists a global analytic solution $g \in \delta + X_\infty$ of the Ricci-DeTurck flow with $g(\cdot,0) = g_0$ and $||g-\delta||_{X_\infty} \leq C_n||g_0-\delta||_{L^\infty(\mathbb{R}^n)}$. The solution is unique in the ball $B^{X_\infty}(\delta, C_n\epsilon_n) = \{g| \ ||g-\delta||_{X_\infty} \leq C_n\epsilon_n\}$. (The space X_∞ will be defined in the proof below.)*

More precisely, there exists $R > 0$, $c > 0$ such that for every $k \in \mathbb{N}_0$ and every multi-index $\alpha \in \mathbb{N}_0^n$ we have the estimate

$$\sup_{x\in\mathbb{R}^n}\sup_{t>0} |(t^{\frac{1}{2}}\nabla)^\alpha (t\partial_t)^k (g-\delta)(x,t)| \leq c||g_0-\delta||_{L^\infty(\mathbb{R}^n)} R^{(|\alpha|+k)}(|\alpha|+k)!.$$

Moreover, the solution, g, depends analytically on g_0.

Proof. This will only be an outline of the proof highlighting elements needed in the argument of this paper. For a full proof of this theorem, refer to [KL].

First, perform a series of formal manipulations to reformulate the problem as follows. Suppose that a family of metrics, $g(t)$, solves the Ricci-DeTurck flow. Then define the related symmetric 2-tensor $h(t) := g(t) - \delta$ and write down the Ricci-DeTurk flow equation in terms of $h(t)$. Through out the paper $h(t)$ will be referred to as the related 2-tensor to $g(t)$. This equation can be manipulated to look like a non-homogeneous heat equation in Euclidean coordinates.

$$(\partial_t - \Delta)h = R[h] \tag{43}$$

The differential operator R acts on a one-parameter family of symmetric 2-tensors and has the form: $R[u] := \nabla R_1[u] + R_0[u]$

$$\begin{aligned} \nabla R_1[u]_{ij} &:= \nabla_a\Big(\big((\delta+u)^{ab} - \delta^{ab}\big)\nabla_b u_{ij}\Big) \\ R_0[u]_{ij} &:= \frac{1}{2}(\delta+u)^{ab}(\delta+u)^{pq}\Big(\nabla_i u_{pa}\nabla_j u_{qb} + 2\nabla_a u_{jp}\nabla_q u_{ib} - 2\nabla_a u_{jp}\nabla_b u_{iq} \\ &\qquad -2\nabla_j u_{pa}\nabla_b u_{iq} - 2\nabla_i u_{pa}\nabla_b u_{jq} + 2\nabla_a u_{bq}\nabla_p u_{ij}\Big) \end{aligned}$$

where all derivatives are taken with respect to the Euclidean metric.

Continuing these formal manipulations, let K denote the standard heat kernel on Euclidean space

$$K(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}. \tag{44}$$

Then the standard solution to the non-homogeneous heat equation with initial data, h_0, is of the form

$$h(x,t) := S[h_0](x,t) + VR[h](x,t) \tag{45}$$

where for $u_0 \in L^\infty(\mathbb{R}^n)$ and $u \in C^2(\mathbb{R}^n \times (0,\infty))$.

$$S[u_0](x,t) := \int_{\mathbb{R}^n} K(x-y,t)u_0(y)dy$$

$$VR[u](x,t) := \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)R[u](y,s)dyds.$$

Thus, a solution to this equation will solve the Ricci-DeTurck flow. For fixed initial condition, h_0, define the operator

$$\Phi[u](x,t) := S[h_0](x,t) + VR[u](x,t).$$

Then a fixed point of Φ satisfies equation 43, and thus gives a solution to the Ricci-DeTurck flow. However, the function space $C^2(\mathbb{R}^n \times (0,\infty))$ is too restrictive to find a fixed point, so expand the domain of definition of Φ as follows.

For a family of symmetric two-tensors parametrized by $t \in [0,\infty)$, defined a norm, X_∞, by

$$||u||_{X_\infty} := \sup_{t>0} ||u(t)||_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n}\sup_{R>0} R^{-\frac{n}{2}}||\nabla u||_{L^2(B(x,R)\times(0,R^2))} + \sup_{x\in\mathbb{R}^n}\sup_{R>0} R^{\frac{2}{n+4}}||\nabla u||_{L^{n+4}(B(x,R)\times(R^2/2,R^2))}$$

Denote the space of such families with finite X_∞-norm as X_∞. This norm makes X_∞ a Banach space. Then define two norms Y^0_∞ and Y^1_∞ so that R_0 and R_1 naturally map X_∞ into a function space with finite Y^0_∞ and Y^1_∞ norm respectively. The precise definitions are

$$||u||_{Y^0_\infty} := \sup_{x\in\mathbb{R}^n}\sup_{R>0}\left(R^{-n}||u||_{L^1(B(x,R)\times(0,R^2))} + R^{\frac{4}{n+4}}||u||_{L^{\frac{n+4}{2}}(B(x,R)\times(\frac{R^2}{2},R^2))}\right)$$

$$||u||_{Y^1_\infty} := \sup_{x\in\mathbb{R}^n}\sup_{R>0}\left(R^{-\frac{n}{2}}||u||_{L^2(B(x,R)\times(0,R^2))} + R^{\frac{2}{n+4}}||u||_{L^{n+4}(B(x,R)\times(\frac{R^2}{2},R^2))}\right).$$

Let $Y_\infty = Y^0_\infty + Y^1_\infty$. The precise meaning will be clear in context. Then Koch and Lamm show that if $R[h] \in Y_\infty$, then V maps $R[h]$ back into X_∞.

Hence, letting $B^{X_\infty}(0,\gamma) := \{u \in X_\infty |\ ||u||_{X_\infty} \leq \gamma\}$, [KL] shows that for every $\gamma \in (0,1)$, Φ maps $B^{X_\infty}(0,\gamma)$ into X_∞.

Further, for any two families of metrics $h_1, h_2 \in B^{X_\infty}(0,\gamma)$,

$$||\Phi[h_1] - \Phi[h_2]||_{X_\infty} \leq c||h_0||_{L^\infty(\mathbb{R}^n)}\ ||h_1 - h_2||_{X_\infty}$$

where c is a constant that depends only on the dimension coming from heat kernel estimates and integral bound constants.

Hence, there exist constants $\epsilon_n > 0$ and $C_n > 0$ such that if $||h_0||_{L^\infty(\mathbb{R}^n)} < \epsilon_n$, then the operator, Φ, is a contraction on a ball $B^{X_\infty}(0, C_n\epsilon_n)$. Note in particular that neither $C_n > 0$ nor $\epsilon_n > 0$ depend on the initial data. (Note that this is the same ϵ_n as in the main result of this paper.)

Thus, the Banach fixed point theorem implies there exists a unique fixed point $h(t) \in B^{X_\infty}(0, C_n\epsilon_n)$ such that

$$h(t) = \Phi[h(t)] = S[h_0] + VR[h(t)].$$

Hence, $g(t) = h(t) + \delta$ is a solution to the Ricci-DeTurk flow for all positive time and is unique in the ball $B^{X_\infty}(\delta, C_n\epsilon_n) := \{u \in X_\infty |\ ||u - \delta||_{X_\infty} \leq C_n\epsilon_n\}$. Standard theory gives that $g(t)$ is in fact smooth for positive time. This completes the proof in [KL]. □

3.2 Structure of the Asymptotic Cones

The results in this section are straight forward applications of Koch and Lamm's work. It is unknown to the author if these propositions have been stated explicitly in the literature before. However, it should be noted that Deruelle has shown stronger results under a positive curvature assumption [Der].

Consider the cone metric, $g_{C(\mathbb{S}^{n-1})}$, from theorem 2. The cone metric, $g_{C(\mathbb{S}^{n-1})}$, is ϵ_n-close to δ in $L^\infty(\mathbb{R}^n)$ because g_0 of theorem 2 is. Hence, Theorem 4.3 of [KL] applies to $g_{C(\mathbb{S}^{n-1})}$ and there exists a flow, $g_{sol}(0)$ coming out of this cone.

Further, rescaling the initial data by $g_{C(\mathbb{S}^{n-1})}(\lambda x)$ corresponds to parabolic rescaling of the flow metrics. Hence, if the initial metric is invariant under rescaling, i.e. a cone, then the flow is invariant under parabolic rescaling. This is collected in the following two propositions.

Proposition 2. *For initial data, $g_0 \in L^\infty(\mathbb{R}^n)$, such that $||g_0 - \delta||_{L^\infty(\mathbb{R}^n)} < \epsilon_n$ define $h_0 = g_0 - \delta$ and the rescaling $h^\lambda_0(x) := h_0(\lambda x)$. Then define the operators $\Phi[u](x,t) = S[h_0](x,t) + VR[u](x,t)$ and $\Phi_\lambda[u](x,t) = S[h^\lambda_0](x,t) + VR[u](x,t)$. If $h(x,t)$ is the unique fixed point of Φ as in [KL], then $h_\lambda(x,t) := h(\lambda x, \lambda^2 t)$ is the unique fixed point of Φ_λ.*

Proof. Notice that convergence back to the initial data would make this result trivial. By the parabolic invariance of the X_∞ norm,

$$||h_\lambda||_{X_\infty} = ||h||_{X_\infty} \leq C_n \epsilon_n.$$

Hence, it is enough to show that $h_\lambda \in B^{X_\infty}(0, C_n\epsilon_n)$ is a fixed point of the operator Φ_λ. This is a straight forward calculation. First, let $\bar{y} = y/\lambda$. Then

$$\begin{aligned}
S[h_0](\lambda x, \lambda^2 t) &= \int_{\mathbb{R}^n} K(\lambda x - y, \lambda^2 t) h_0(y) dy \\
&= \int_{\mathbb{R}^n} (4\pi\lambda^2 t)^{-\frac{n}{2}} e^{-\frac{|\lambda x - y|^2}{4\lambda^2 t}} h_0(y) dy \\
&= \int_{\mathbb{R}^n} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x - y/\lambda|^2}{4t}} h_0(y) \lambda^{-n} dy \\
&= \int_{\mathbb{R}^n} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x - \bar{y}|^2}{4t}} h_0(\lambda \bar{y}) d\bar{y} \\
&= S[h_0^\lambda](x, t)
\end{aligned}$$

Second, let $\bar{s} = s/\lambda^2$. Then, using that R involves two spacial derivatives in all of its terms,

$$\begin{aligned}
VR[h](\lambda x, \lambda^2 t) &= \int_0^{\lambda^2 t} \int_{\mathbb{R}^n} K(\lambda x - y, \lambda^2 t) R[h](y, s) dy ds \\
&= \int_0^{\lambda^2 t} \int_{\mathbb{R}^n} (4\pi\lambda^2 t)^{-\frac{n}{2}} e^{-\frac{|\lambda x - y|^2}{4(\lambda^2 t - s)}} R[h](y, s) dy ds \\
&= \int_0^{\lambda^2 t} \int_{\mathbb{R}^n} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x - y/\lambda|^2}{4(t - s/\lambda^2)}} R[h](\lambda \bar{y}, \lambda^2 \bar{s}) d(y/\lambda) d(s/\lambda^2) \\
&= \int_0^{t} \int_{\mathbb{R}^n} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x - \bar{y}|^2}{4(t - \bar{s})}} R[h_\lambda](\bar{y}, \bar{s}) d\bar{y} d\bar{s} \\
&= VR[h_\lambda](x, t)
\end{aligned}$$

Hence,

$$\Phi[u](\lambda x, \lambda^2 t) = \Phi_\lambda[u_\lambda](x, t)$$

where $u_\lambda(x, t) = u(\lambda x, \lambda^2 t)$. Then by the fixed point property of h

$$h_\lambda(x, t) = h(\lambda x, \lambda^2 t) = \Phi[h](\lambda x, \lambda^2 t) = \Phi_\lambda[h_\lambda]$$

Thus, h_λ is the unique fixed point with initial data h_0^λ. □

Proposition 3. *If the initial metric as in Koch and Lamm's theorem 4.3 is a rough cone, $g_{C(\mathbb{S}^{n-1})}$, then the flow, g_{sol}, coming out of this rough initial data is invariant under the parabolic rescaling defined in equation 7, i.e.*

$$g_{sol,\lambda}(x, t)_{ij} := g_{sol}(\lambda x, \lambda^2 t)_{ij} = g_{sol}(x, t)_{ij} \tag{46}$$

Further, this flow is an expanding Ricci-DeTurck soliton.

Proof. The initial data is a rough cone metric, i.e. $g_0 = g_{C(\mathbb{S}^{n-1})} := dr^2 + r^2 g_{n-1}$ for some rough metric $g_{n-1} \in L^\infty(\mathbb{S}^{n-1})$. Let $h_{sol}(x,t)_{ij} := g_{sol}(x,t) - \delta$. The goal is to show that

$$g_{sol}(\lambda x, \lambda^2 t)_{ij} = g_{sol}(x,t)_{ij}.$$

Because $\delta(x,t) = \delta(\lambda x, \lambda^2 t)$, this is equivalent to showing that

$$h_{sol,\lambda}(x,t) := h_{sol}(\lambda x, \lambda^2 t)_{ij} = h_{sol}(x,t)_{ij}.$$

First, look at the rescaled cone metric. Using polar coordinates $(r,\theta) \in (0,\infty) \times \mathbb{S}^{n-1}$, the rescaling $g_0^\lambda(x) := g_0(\lambda x)$ can be written $g^\lambda_{C(\mathbb{S}^{n-1})}(r,\theta) = g_{C(\mathbb{S}^{n-1})}(\lambda r, \theta)$. Then

$$g_{C(\mathbb{S}^{n-1})}(\lambda r, \theta) = d(\lambda r)^2 + (\lambda r)^2 g_{n-1} = \lambda^2 g_{C(\mathbb{S}^{n-1})}(r,\theta).$$

Thus, $g^\lambda_{C(\mathbb{S}^{n-1})}(r,\theta) = \lambda^2 g_{C(\mathbb{S}^{n-1})}(r,\theta)$. Further, let $h_{C(\mathbb{S}^{n-1})} = g_{C(\mathbb{S}^{n-1})} - \delta$. Then, because δ is also a cone metric,

$$h^\lambda_{C(\mathbb{S}^{n-1})}(r,\theta) := h_{C(\mathbb{S}^{n-1})}(\lambda r, \theta) = \lambda^2 h_{C(\mathbb{S}^{n-1})}(r,\theta).$$

Second, turn attention back to the flows h_{sol} and $h_{sol,\lambda}$. By assumption, h_{sol} is the unique fixed point of

$$\Phi_{sol}[u](x,t) = S[h_{C(\mathbb{S}^{n-1})}](x,t) + VR[u](x,t).$$

Then, by the previous proposition, the rescaled flow, $h_{sol,\lambda}$, is the unique fixed points for the operator

$$\Phi_{sol,\lambda}[u](x,t) = S[h^\lambda_{C(\mathbb{S}^{n-1})}](x,t) + VR[u](x,t).$$

By the above calculation for the initial cone metric,

$$S[h^\lambda_{C(\mathbb{S}^{n-1})}](x,t) = \lambda^2 S[h_{C(\mathbb{S}^{n-1})}](x,t)$$

Then by the fixed point property of $h_{sol,\lambda}$,

$$h_{sol,\lambda}(x,t) = S[h^\lambda_{C(\mathbb{S}^{n-1})}](x,t) + VR[h_{sol,\lambda}](x,t)$$

By the equation above,

$$h_{sol,\lambda}(x,t) = \lambda^2 S[h_{C(\mathbb{S}^{n-1})}](x,t) + VR[h_{sol,\lambda}](x,t)$$

Then divide by λ^2,

$$\lambda^{-2} h_{sol,\lambda}(x,t) = S[h_{C(\mathbb{S}^{n-1})}](x,t) + VR[\lambda^{-2} h_{sol,\lambda}](x,t) = \Phi_{sol}[\lambda^{-2} h_{sol,\lambda}](x,t)$$

Hence, $\lambda^{-2} h_{sol,\lambda}(x,t)$ is a fixed point on Φ_{sol}. Further,

$$||\lambda^{-2} h_{sol,\lambda}||_{X_\infty} = \lambda^{-2} ||h_{sol,\lambda}||_{X_\infty} = \lambda^{-2} ||h_{sol}||_{X_\infty}.$$

Thus, for all $\lambda \geq 1$, if $h_{sol} \in B^{X_\infty}(0, C_n \epsilon_n)$, then $\lambda^{-2} h_{sol,\lambda} \in B^{X_\infty}(0, C_n \epsilon_n)$. On the other hand, h_{sol} is the unique fixed point of Φ_{sol} in $B^{X_\infty}(0, C_n \epsilon_n)$. Therefore,

$$h_{sol}(x,t) = \lambda^{-2} h_{sol}(\lambda x, \lambda^2 t)$$

and hence,

$$g_{sol}(x,t) = \lambda^{-2} g_{sol}(\lambda x, \lambda^2 t).$$

Rewriting this for $t = 1$ and letting λ^2 become the time parameter gives

$$g_{sol}(t) = t g_{sol}(1)$$

for all times $t > 0$. Thus, this solution is an expanding Ricci-DeTurck soliton.

This may seem to contradict equation 46. However, notice that when the two flows are calculated using the same basis of 1-forms, they are the same. Let $\bar{x} := \lambda x$. Then

$$\begin{aligned} h_{sol}(x,t)_{ij} dx^i \otimes dx^j &= \lambda^{-2} h_{sol}(\lambda x, \lambda^2 t)_{ij} d\bar{x}^i \otimes d\bar{x}^j \\ &= h_{sol}(\lambda x, \lambda^2 t)_{ij} dx^i \otimes dx^j \end{aligned}$$

Subscripts will be used to denote that the same basis is used on both sides of an equation. Hence, the above equation will be written

$$h_{sol}(x,t)_{ij} = h_{sol}(\lambda x, \lambda^2 t)_{ij}. \tag{47}$$

This equation implies equation 46 directly. □

Remark. *One would like to translate this result back into the Ricci flow setting as follows. Take a cone metric, g_0, on $\mathbb{R}^n$ sufficiently close to the Euclidean metric but with angle greater than $\pi/2$, i.e. with negative curvature. This metric will not be smooth at the cone point, so the Ricci flow coming out of this metric is not a priori defined. However, the result of Koch and Lamm says that there is an immortal Ricci-DeTurck flow, $g(t)$, stating from $g(0) = g_0$. By the argument above it is an expanding Ricci-DeTurck soliton. Take the time slice at $t = 1$ of this Ricci-DeTurck flow, $(\mathbb{R}^n, g(1))$. This will be a smooth manifold, so it can be translated back to the Ricci flow setting and will be an expanding Ricci soliton, $(\mathbb{R}^n, \tilde{g}(t))_{t\geq 1}$. Then taking the flow back in time is just a matter of rescaling the metric. The limiting metric at time $t = 0$ is just the asymptotic cone of a later time slice (see [Der]). By Lott and Zhang's work, this asymptotic cone must be the same cone as the initial metric for the Ricci-DeTurck flow, $\tilde{g}(0) = g_0$.*

Unfortunately, it is not clear to the author how to make this argument rigorous. In particular, it is not clear that the Ricci-DeTurck flow nor the diffeomorphism that translates between the Ricci flow and Ricci-DeTurck flow preserve the asymptotic cone.

3.3 Proof of Convergence

With the preliminaries set, turn back to proving theorem 2 of this paper. In this section, let $g(t)$ be the metric from theorem 2, i.e. $g(0) = g_0$ is ϵ_n-close to the Euclidean metric and asymptotic to a rough cone, $g_{C(\mathbb{S}^{n-1})}$. Let g_{sol} be the Ricci-DeTurck flow coming out of $g_{C(\mathbb{S}^{n-1})}$ and define the 2-tensor, $h_{C(\mathbb{S}^{n-1})} := g_{C(\mathbb{S}^{n-1})} - \delta$ and the family of 2-tensors $h_{sol}(t) := g_{sol}(t) - \delta$. (Recall that Koch and Lamm's result is need to ensure that g_{sol} exists because $g_{C(\mathbb{S}^{n-1})}$ is not C^2 at the cone point unless it is the flat metric.) To study the blow down of g, define the family of flows

$$g_\lambda(x,t) := g(\lambda x, \lambda^2 t) \tag{48}$$

and the related families of symmetric 2-tensors $h(t) := g(t) - \delta$ and $h_\lambda(t) := g_\lambda(t) - \delta$. The goal of this section will be to show that $h_\lambda \to h_{sol}$ as $\lambda \to \infty$ on $\mathbb{R}^n \times (0, \infty)$ in a weak sense.

3.3.1 Reduction of Problem

To begin, establish two basic properties of the rescaled flows, $g_\lambda(t)$. First, the Ricci-DeTurck flow is invariant under the parabolic rescaling defined by equation 7. Hence, for each $\lambda \in (0,\infty)$, the family of metrics, $g_\lambda(t)$, is also a solution to the Ricci-DeTurck flow.

Second, the X_∞-norm is also invariant under parabolic rescaling, so the following lemma holds.

Lemma 5. *Let h_{sol} be the related 2-tensor of a Ricci-DeTurck flow coming out of a rough cone as described in section 3.2. Let $h(t)$ be the related 2-tensor of a Ricci-DeTurck flow as in Koch and Lamm's result. Then, for every $1 \leq \lambda < \infty$,*

$$||h_\lambda - h_{sol}||_{X_\infty} = ||h - h_{sol}||_{X_\infty}$$

and in particular,

$$||h_\lambda||_{X_\infty} = ||h||_{X_\infty}.$$

Proof. Notice, the asymptotic assumptions are not used in this lemma at all. Hence, the second equation is just a special case of the first equation when the rough cone metric is the Euclidean metric, $g_{C(\mathbb{S}^{n-1})} = \delta$, i.e. when $h_{sol} \equiv 0$. The proof is a simple calculation.

Recall that by definition,

$$\begin{aligned} ||h_\lambda - h_{sol}||_{X_\infty} &= \sup_{t>0} ||\left(h_\lambda - h_{sol}\right)(t)||_{L^\infty(\mathbb{R}^n)} \\ &\quad + \sup_{x\in\mathbb{R}^n} \sup_{R>0} R^{-\frac{n}{2}} ||\nabla(h_\lambda - h_{sol})||_{L^2(B(x,R)\times(0,R^2))} \\ &\quad + \sup_{x\in\mathbb{R}^n} \sup_{R>0} R^{\frac{2}{n+4}} ||\nabla(h_\lambda - h_{sol})||_{L^{n+4}(B(x,R)\times(R^2/2,R^2))} \end{aligned}$$

Further, recall that $h_\lambda(y,s) = h(\lambda y, \lambda^2 s)$ by definition and $h_{sol}(y,s) = h_{sol}(\lambda y, \lambda^2 s)$ by proposition 2 in section 3.2 (equation 46). Define the coordinates $\bar{y} := \lambda y$, $\bar{s} := \lambda^2 s$, $\bar{x} := \lambda x$, and $\bar{t} := \lambda^2 t$. Also let $\bar{R} := \lambda R$. Then look at each term in the X_∞-norm.

First,

$$\begin{aligned} \sup_{t>0} ||\left(h_\lambda - h_{sol}\right)(t)||_{L^\infty(\mathbb{R}^n)} &= \sup_{t>0} ||\left(h - h_{sol}\right)(\lambda^2 t)||_{L^\infty(\mathbb{R}^n)} \\ &= \sup_{\lambda^2 t>0} ||\left(h - h_{sol}\right)(\lambda^2 t)||_{L^\infty(\mathbb{R}^n)} \\ &= \sup_{\bar{t}>0} ||\left(h - h_{sol}\right)(\bar{t})||_{L^\infty(\mathbb{R}^n)} \end{aligned}$$

Second,

$$\begin{aligned}
||\nabla(h_\lambda - h_{sol})||_{L^2(B(x,R)\times(0,R^2))} &= \left(\int_0^{R^2}\int_{B(x,R)} \left(\nabla(h_\lambda - h_{sol})(y,s)\right)^2 dyds\right)^{\frac{1}{2}}\\
&= \left(\int_0^{R^2}\int_{B(x,R)} \left(\lambda\nabla(h - h_{sol})(\lambda y,\lambda^2 s)\right)^2 dyds\right)^{\frac{1}{2}}\\
&= \left(\int_0^{R^2}\int_{B(x,R)} \left(\nabla(h - h_{sol})(\bar{y},\bar{s})\right)^2 \lambda^2 dyds\right)^{\frac{1}{2}}\\
&= \left(\int_0^{\lambda^2R^2}\int_{B(\lambda x,\lambda R)} \left(\nabla(h - h_{sol})(\bar{y},\bar{s})\right)^2 \lambda^{-n} d(\lambda y)d(\lambda^2 s)\right)^{\frac{1}{2}}\\
&= \lambda^{-\frac{n}{2}}\left(\int_0^{\bar{R}^2}\int_{B(\bar{x},\bar{R})} \left(\nabla(h - h_{sol})(\bar{y},\bar{s})\right)^2 d\bar{y}d\bar{s}\right)^{\frac{1}{2}}\\
&= \lambda^{-\frac{n}{2}}||\nabla(h - h_{sol})||_{L^2(B(\bar{x},\bar{R})\times(0,\bar{R}^2))}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n}\sup_{R>0} R^{-\frac{n}{2}}||\nabla(h_\lambda - h_{sol})||_{L^2(B(x,R)\times(0,R^2))}\\
&\qquad = \sup_{\bar{x}\in\mathbb{R}^n}\sup_{\bar{R}>0} \bar{R}^{-\frac{n}{2}}||\nabla(h - h_{sol})||_{L^2(B(\bar{x},\bar{R})\times(0,\bar{R}^2))}
\end{aligned}$$

Third,

$$\begin{aligned}
||\nabla(h_\lambda - h_{sol})||_{L^{n+4}(B(x,R)\times(0,R^2))} &= \left(\int_0^{R^2}\int_{B(x,R)} \left(\nabla(h_\lambda - h_{sol})(y,s)\right)^{n+4} dyds\right)^{\frac{1}{n+4}}\\
&= \left(\int_0^{R^2}\int_{B(x,R)} \left(\lambda\nabla(h - h_{sol})(\lambda y,\lambda^2 s)\right)^{n+4} dyds\right)^{\frac{1}{n+4}}\\
&= \left(\int_0^{R^2}\int_{B(x,R)} \left(\nabla(h - h_{sol})(\bar{y},\bar{s})\right)^{n+4} \lambda^{n+4} dyds\right)^{\frac{1}{n+4}}\\
&= \left(\int_0^{\bar{R}^2}\int_{B(\bar{x},\bar{R})} \left(\nabla(h - h_{sol})(\bar{y},\bar{s})\right)^{n+4} \lambda^2 d(\lambda y)d(\lambda^2 s)\right)^{\frac{1}{n+4}}\\
&= \lambda^{\frac{2}{n+4}}\left(\int_0^{\bar{R}^2}\int_{B(\bar{x},\bar{R})} \left(\nabla(h - h_{sol})(\bar{y},\bar{s})\right)^{n+4} d\bar{y}d\bar{s}\right)^{\frac{1}{n+4}}\\
&= \lambda^{\frac{2}{n+4}}||\nabla(h - h_{sol})||_{L^{n+4}(B(\bar{x},\bar{R})\times(0,\bar{R}^2))}
\end{aligned}$$

Hence,

$$\sup_{x\in\mathbb{R}^n}\sup_{R>0} R^{\frac{2}{n+4}}||\nabla(h_\lambda - h_{sol})||_{L^{n+4}(B(x,R)\times(R^2/2,R^2))}$$
$$= \sup_{\bar{x}\in\mathbb{R}^n}\sup_{\bar{R}>0} \bar{R}^{\frac{2}{n+4}}||\nabla(h - h_{sol})||_{L^{n+4}(B(\bar{x},\bar{R})\times(\bar{R}^2/2,\bar{R}^2))}$$

Therefore,

$$\begin{aligned}
||h_\lambda - h_{sol}||_{X_\infty} &= \sup_{t>0}||\left(h_\lambda - h_{sol}\right)(t)||_{L^\infty(\mathbb{R}^n)} \\
&\quad + \sup_{x\in\mathbb{R}^n}\sup_{R>0} R^{-\frac{n}{2}}||\nabla(h_\lambda - h_{sol})||_{L^2(B(x,R)\times(0,R^2))} \\
&\quad + \sup_{x\in\mathbb{R}^n}\sup_{R>0} R^{\frac{2}{n+4}}||\nabla(h_\lambda - h_{sol})||_{L^{n+4}(B(x,R)\times(R^2/2,R^2))} \\
&= \sup_{\bar{t}>0}||\left(h - h_{sol}\right)(\bar{t})||_{L^\infty(\mathbb{R}^n)} \\
&\quad + \sup_{\bar{x}\in\mathbb{R}^n}\sup_{\bar{R}>0} \bar{R}^{-\frac{n}{2}}||\nabla(h - h_{sol})||_{L^2(B(\bar{x},\bar{R})\times(0,\bar{R}^2))} \\
&\quad + \sup_{\bar{x}\in\mathbb{R}^n}\sup_{\bar{R}>0} \bar{R}^{\frac{2}{n+4}}||\nabla(h - h_{sol})||_{L^{n+4}(B(\bar{x},\bar{R})\times(\bar{R}^2/2,\bar{R}^2))} \\
&= ||h - h_{sol}||_{X_\infty}
\end{aligned}$$

Note, while these two properties are true for all positive λ, the only important values of λ when studying the blowdown are $\lambda \geq 1$. □

Next, define the operator Φ_{sol} on X_∞ such that for $u \in X_\infty$

$$\Phi_{sol}[u] := S[h_{C(\mathbb{S}^{n-1})}] + VR[u].$$

Then lemma 5 can be used to show the following.

Lemma 6. *$\Phi^m_{sol}[h_\lambda] \to h_{sol}$ in X_∞ as $m \to \infty$ uniformly for all $1 \leq \lambda < \infty$. (Here the superscript m denotes applying the map m times.)*

Proof. By [KL], Φ_{sol} is a contraction on $B^{X_\infty}(0, C_n\epsilon_n)$ which has a unique fixed point h_{sol} (see section 3.1). Hence, for any $u \in B^{X_\infty}(0, C_n\epsilon_n)$, $\Phi^m_{sol}[u] \to h_{sol}$ in X_∞ as $m \to \infty$. This convergence is allowed to depend on u.

More precisely, Φ_{sol} is a contraction on $B^{X_\infty}(0, C_n\epsilon_n)$, so there exists a constant $0 < \gamma < 1$ such that for any $u_1, u_2 \in B^{X_\infty}(0, C_n\epsilon_n)$

$$||\Phi_{sol}[u_1] - \Phi_{sol}[u_2]||_{X_\infty} \leq \gamma||u_1 - u_2||_{X_\infty}$$

and hence,

$$||\Phi^m_{sol}[u_1] - \Phi^m_{sol}[u_2]||_{X_\infty} \leq \gamma^m||u_1 - u_2||_{X_\infty}.$$

Then because $0 < \gamma < 1$, $\gamma^m \to 0$ as $m \to \infty$. Let $u_2 = h_{sol}$ and use that $\Phi_{sol}[h_{sol}] = h_{sol}$. Hence,

$$||\Phi^m_{sol}[u_1] - h_{sol}||_{X_\infty} \leq \gamma^m||u_1 - h_{sol}||_{X_\infty}.$$

By the assumptions on the initial metric, $||g_0 - \delta||_{L^\infty(\mathbb{R}^n)} \le \epsilon_n$, and Theorem 4.3 of [KL],

$$||h_\lambda(t)||_{X_\infty} = ||h(t)||_{X_\infty} \le C_n ||g_0 - \delta||_{L^\infty(\mathbb{R}^n)} \le C_n \epsilon_n.$$

Thus, for any fixed λ, plugging in $u_1 = h_\lambda$ gives

$$||\Phi^m_{sol}[h_\lambda] - h_{sol}||_{X_\infty} \le \gamma^m ||h_\lambda - h_{sol}||_{X_\infty}.$$

Moreover, by lemma 5

$$||h_\lambda - h_{sol}||_{X_\infty} = ||h - h_{sol}||_{X_\infty}$$

Therefore,

$$||\Phi^m_{sol}[h_\lambda] - h_{sol}||_{X_\infty} \le \gamma^m ||h - h_{sol}||_{X_\infty}$$

Thus, there exists a bound independent of λ that goes to zero as $m \to \infty$. □

Remark. *Lemma 5 and lemma 6 do not make use of the asymptotic assumptions on g_0, and all the statements above hold for any flow coming out of initial data, g_0, that is ϵ_n-close to δ in $L^\infty(\mathbb{R}^n)$ and any flow coming, g_{sol}, coming out of a rough cone, $g_{C(\mathbb{S}^{n-1})}$.*

Unfortunately, lemma 5 also shows that h_λ does not converge to h_{sol} in the X_∞-norm. Thus, define a weaker norm as follows

$$||u||_{X'_\infty(r)} := \sup_{(x,t)\notin Q(r)} |u| + \sup_{(x,t)\notin Q(r)} t^{-\frac{n}{4}} ||\nabla u||_{L^2(B(x,\sqrt{t})\times(0,t))} + \sup_{(x,t)\notin Q(r)} t^{\frac{1}{n+4}} ||\nabla u||_{L^{n+4}(B(x,\sqrt{t})\times(t/2,t))}$$

where $Q(r) := B(0,r) \times (0, r^2) \subset \mathbb{R}^n \times (0,\infty)$. Notice in particular, that convergence in X_∞ implies convergence in $X'_\infty(r)$ for every $r > 0$. (see section 3.3.3 for a more detailed discussion of this bound.) Then the following will be shown.

Lemma 7. *For every $\epsilon' > 0$, every $r > 0$ and each fixed $m \in \mathbb{N}$, there exists a constant $\Lambda \in \mathbb{R}$ such that for all $\lambda > \Lambda$*

$$||h_\lambda - \Phi^m_{sol}[h_\lambda]||_{X'_\infty(r)} < \frac{\epsilon'}{2}.$$

To see why this lemma is enough to imply convergence, argue as follows. First, by lemma 6, for every $\epsilon' > 0$, there exists $M \in \mathbb{N}$ (independent of λ) such that for all $m > M$,

$$||\Phi^m_{sol}[h_\lambda] - h_{sol}||_{X'_\infty(r)} \le ||\Phi^m_{sol}[h_\lambda] - h_{sol}||_{X_\infty} < \frac{\epsilon'}{2}$$

Pick $m = 2M$ and fix this choice of m. Second, by lemma 7, for every $r > 0$, there exists $\Lambda \in \mathbb{R}$ such that for all $\lambda > \Lambda$

$$||h_\lambda - \Phi^m_{sol}[h_\lambda]||_{X'_\infty(r)} < \frac{\epsilon'}{2}$$

Therefore, by the triangle inequality, for any $\epsilon' > 0$ and any $r > 0$, there exists Λ such that for all $\lambda > \Lambda$

$$||h_\lambda - h_{sol}||_{X'_\infty(r)} \le ||h_\lambda - \Phi^m_{sol}[h_\lambda]||_{X'_\infty(r)} + ||\Phi^m_{sol}[h_\lambda] - h_{sol}||_{X'_\infty(r)} < \epsilon'.$$

Therefore, $h_\lambda \to h_{sol}$ in $X'_\infty(r)$ for each $r > 0$.

The proof of Lemma 7 will be given in the next three subsections. The proof will follow by induction on m.

3.3.2 Base Case

For the base case, let $m = 1$ and study $h_\lambda - \Phi_{sol}[h_\lambda]$. Recall that h_λ can be written

$$\begin{aligned} h_\lambda(x,t) &= S[h_\lambda(\cdot,0)](x,t) + VR[h_\lambda](x,t) \\ &= \int_{\mathbb{R}^n} K(x-y,t)h_\lambda(y,0)dy + \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)R[h_\lambda](y,s)dyds \quad (49) \end{aligned}$$

Then from the definition of Φ_{sol},

$$h_\lambda - \Phi_{sol}[h_\lambda] = S[g_\lambda - g_{sol}] := \int_{\mathbb{R}^n} K(x-y,t)(g_\lambda - g_{sol})(y,0)dy$$

which is the standard solution to the heat equation on $\mathbb{R}^n$ with initial data $(g_\lambda - g_{sol})(\cdot,0)$. Recall the following bounds on the Euclidean heat kernel. For any integer $k \in \mathbb{N}_0$, multi-index $\alpha \in \mathbb{N}_0^n$, and $t > 0$, there exists a constant, c, such that

$$||\partial_t^k \nabla^\alpha K(\cdot,t)||_{L^1(\mathbb{R}^n)} \leq ct^{-k-|\alpha|/2}$$

Moreover, for any $x \in \mathbb{R}^n$

$$|\partial_t^k \nabla^\alpha K(x,t)| \leq c\left(t^{\frac{1}{2}} + |x|\right)^{-n-2k-|\alpha|}$$

Then, for any time $t > 0$, any integer $k \in \mathbb{N}_0$, any multi-index $\alpha \in \mathbb{N}_0^n$ and any real number $r > 0$,

$$\begin{aligned} &\nabla^\alpha \partial_t^k \big|h_\lambda - \Phi_{sol}[h_\lambda]\big|(x,t) \\ &= \int_{\mathbb{R}^n} (\nabla^\alpha \partial_t^k K)(x-y,t)(g_\lambda - g_{sol})(y,0)dy \\ &\leq ||(g_\lambda - g_{sol})(\cdot,0)||_{L^\infty(\mathbb{R}^n \setminus B(0,r))} \cdot ||\nabla^\alpha \partial_t^k K(\cdot,t)||_{L^1(\mathbb{R}^n \setminus B(0,r))} \\ &\quad + ||(g_\lambda - g_{sol})(\cdot,0)||_{L^\infty(B(0,r))} \cdot ||\nabla^\alpha \partial_t^k K(t)||_{L^\infty(B(0,r))} \cdot \mathrm{vol}(B(0,r)) \\ &\leq \eta(\lambda r) \cdot c_1 t^{-|\alpha|+k} + 2\epsilon_n \cdot c_2 \left(t^{\frac{1}{2}} + \inf_{y \in B(x,r)} |y|\right)^{-n-|\alpha|-2k} \cdot c_3 r^n \end{aligned}$$

where here c_1 and c_2 are from the above heat kernel bounds and c_3 is a geometric constant that only depends on the dimension, n. Take $r = \lambda^{-1/2}$. Then for each fixed pair $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$, there exists a neighborhood, U around (x_0,t_0) such that this bound goes uniformly to zero in U as $\lambda \to \infty$. Thus, as $\lambda \to \infty$, $|h_\lambda - \Phi_{sol}[h_\lambda]| \to 0$ in $C^\infty_{loc}(\mathbb{R}^n \times (0,\infty))$.

Further, let $Q(r_0) := B(0,r_0) \times (0, r_0^2)$ and consider $h_\lambda - \Phi_{sol}[h_\lambda]$, i.e. let $k = 0$ and $\alpha = \vec{0}$. Then note that for any fixed $r_0 > 0$, this bound converges uniformly to zero for every $(x,t) \in \mathbb{R}^n \times (0,\infty) \setminus Q(r_0)$, i.e. for all $r_0 > 0$, $\exists\, c$ depending only on the dimension such that

$$\sup_{(x,t) \notin Q(r_0)} \big|h_\lambda - \Phi_{sol}[h_\lambda]\big|(x,t) \leq c\left(\eta\left(\lambda^{1/2}\right) + \lambda^{-n/2}\right)$$

which converges to zero as $\lambda \to \infty$. In particular, $|h_\lambda - \Phi_{sol}[h_\lambda]| \to 0$ in $X'_\infty(r)$ for every $r > 0$. However, notice that h_λ does not approach $\Phi_{sol}[h_\lambda]$ in the X_∞-norm precisely because

X_∞ is invariant under parabolic rescaling and the bad behavior near $(0,0)$. More precisely, the $L^\infty(\mathbb{R}^n)$-norm does not improve under rescaling even though it may be zero in the limit, so

$$||h_\lambda - \Phi_{sol}[h_\lambda]||_{X_\infty} \geq ||h(0) - \Phi_{sol}[h](0)||_{L^\infty(\mathbb{R}^n)}.$$

For example, take g_0 to be $g_{sol}(0)$ perturbed by a smooth bump function with compact support around 0. The next section will that the $X'_\infty(r)$-norm fixes this problem.

3.3.3 Modified Norms

To motivate this section, notice that for any $i > 1$,

$$\begin{aligned}\Phi^{i-1}_{sol}[h_\lambda] - \Phi^i_{sol}[h_\lambda] &= V\left(R[\Phi^{i-2}_{sol}[h_\lambda]] - R[\Phi^{i-1}_{sol}[h_\lambda]]\right)\\ &:= \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)\left(R[\Phi^{i-2}_{sol}[h_\lambda]] - R[\Phi^{i-1}_{sol}[h_\lambda]]\right)(y,s)dyds\end{aligned}$$

is the standard solution for the non-homogeneous heat equation with initial data identically zero. From [KL], the composition of the operators V and R are a contraction on X_∞. Hence,

$$||h_\lambda - \Phi^m_{sol}[h_\lambda]||_{X_\infty} \leq m||h_\lambda - \Phi_{sol}[h_\lambda]||_{X_\infty}$$

However, as seen in the last section, the X_∞-norm of the initial step does not improve as $\lambda \to \infty$. Nevertheless, it was enough to have convergence on the complement of any open set containing $(0,0)$. This section will be devoted to replacing the X_∞-norm with the $X'_\infty(r)$-norm in the above bound. Then the next section will show that this bound goes to zero as $\lambda \to \infty$ and implies convergence on the complement of any open set containing $(0,0)$.

Define the following norm (first seen in section 3.3.1)

$$\begin{aligned}||u||_{X'_\infty(r)} := \sup_{(x,t)\notin Q(r)} |u| + \sup_{(x,t)\notin Q(r)} t^{-\frac{n}{4}}||\nabla u||_{L^2(B(x,\sqrt{t})\times(0,t))}\\ + \sup_{(x,t)\notin Q(r)} t^{\frac{1}{n+4}}||\nabla u||_{L^{n+4}(B(x,\sqrt{t})\times(t/2,t))}\end{aligned}$$

where $Q(r) := B(0,r) \times (0,r^2) \subset \mathbb{R}^n \times (0,\infty)$. To clarify, the supremums are taken over pairs where $t > 0$ because the initial data only has finite essential supremum. Compare this to the X_∞-norm stated in section 3.1 with $t = R^2$. Notice in particular, that convergence in X_∞ implies convergence in $X'_\infty(r)$ for every $r > 0$. However, the converse is false as seen in the previous section where $h_\lambda \to \Phi_{sol}[h_\lambda]$ in $X'_\infty(r)$ for every $r > 0$ but not in X_∞.

Similarly, define the norms

$$||u||_{Y'_{\infty,0}(r)} := \sup_{(x,t)\notin Q(r)} \left(t^{-\frac{n}{2}}||u||_{L^1(B(x,\sqrt{t})\times(0,t))} + t^{\frac{2}{n+4}}||u||_{L^{\frac{n+4}{2}}(B(x,\sqrt{t})\times(t/2,t))}\right)$$

$$||u||_{Y'_{\infty,1}(r)} := \sup_{(x,t)\notin Q(r)} \left(t^{-\frac{n}{4}}||u||_{L^2(B(x,\sqrt{t})\times(0,t))} + t^{\frac{1}{n+4}}||u||_{L^{n+4}(B(x,\sqrt{t})\times(t/2,t))}\right)$$

which correspond to the Y-norms in [KL]. Following Koch and Lamm's notation, denote $Y'_\infty(r) = Y'_{\infty,0}(r) + Y'_{\infty,1}(r)$. The precise meaning will be clear in context. The following is a simple adaptation of the arguments in [KL] Lemma 4.1 and Lemma 4.2. By the above definitions for $X'_\infty(r)$ and $Y'_\infty(r)$ and the definition of the differential operator $R = R_0 + \nabla R_1$ from section 3.1, we have the following.

Lemma 8. *For every* $0 < \gamma \leq \sigma < 1$*, the operators* $\nabla R_1[\cdot] : X'_\infty(r) \to Y'_{\infty,1}(r)$ *and* $R_0[\cdot] : X'_\infty(r) \to Y'_{\infty,0}(r)$ *are analytic with the estimates*

$$||R_0[h] + \nabla R_1[h]||_{Y'_\infty(r)} \leq c(\gamma, \sigma)||h||^2_{X'_\infty(r)}$$

for all $h \in B^{X'_\infty(r)}(0, \gamma) \cap B^{X_\infty}(0, \sigma)$ *and*

$$||R[h_1] - R[h_2]||_{Y'_\infty(r)} \leq c(\gamma, \sigma)\Big(||h_1||_{X'_\infty(r)} + ||h_2||_{X'_\infty(r)}\Big)||h_1 - h_2||_{X'_\infty(r)}$$

for all $h_1, h_2 \in B^{X'_\infty(r)}(0, \gamma) \cap B^{X_\infty}(0, \sigma)$.

Then use the heat kernel estimates from section 3.3.2 to prove the following.

Lemma 9. *Let* $R = R_0 + \nabla R_1 \in Y'_\infty(r_1) \cap Y_\infty$*. Then for any* $r_0 > 2r_1$*,* VR *is in* $X'_\infty(r_0)$ *and we have the estimate*

$$||VR||_{X'_\infty(r_0)} \leq C||R||_{Y'_\infty(r_1)} + C||R||_{Y_\infty}\left(\frac{r_1}{r_0}\right)^n$$

Proof. Divide up the space $\mathbb{R}^n \times (0, t)$ over which we will integrate as follows. Let $Q_1 := B(0, r_1) \times (0, r_1^2)$ be a small cylinder around $(0, 0)$ where we only have Y_∞ bounds for R. Let $Q_2 := B(x, r_2) \times (t - r_2^2, t)$ be a small cylinder around (x, t) where K becomes unbounded. Finally, by definition of $X'_\infty(r_0)$, we only need to consider $(x, t) \notin Q_0 := B(0, r_0) \times (0, r_0^2)$. Look at the integral over Q_1, Q_2, and $\mathbb{R}^n \times (0, t)\backslash(Q_1 \cup Q_2)$ separately.

First, consider the integral over Q_1. Let $Q'_1 := B(x, r_1) \times (t - r_1^2, t]$ and let c_i be constants depending only on n. Then
$\int_{Q_1} K \cdot R_0 + \nabla K \cdot R_1 d\delta$

$$\begin{aligned}
&\leq ||K||_{L^\infty(Q'_1)} \cdot ||R_0||_{L^1(Q_1)} + ||\nabla K||_{L^\infty(Q'_1)} \cdot ||1||_{L^2(Q_1)} \cdot ||R_1||_{L^2(Q_1)} \\
&\leq c_1 \left(\sqrt{(t - r_1^2)_+} + (|x| - r_1)_+\right)^{-n} \cdot r_1^n ||R_0||_{Y^0_\infty} \\
&\quad + c_2 \left(\sqrt{(t - r_1^2)_+} + (|x| - r_1)_+\right)^{-n-1} \cdot (r_1^2)^{1/2} r_1^{n/2} \cdot r_1^{n/2} ||R_1||_{Y^1_\infty} \\
&\leq c_3 ||R||_{Y_\infty} \left(\frac{r_1}{r_0}\right)^n \left(1 + \frac{r_1}{r_0}\right) \\
&\leq c_4 ||R||_{Y_\infty} \left(\frac{r_1}{r_0}\right)^n
\end{aligned}$$

Second, Q_2 corresponds to Q in Koch and Lamm's paper, and the integral over Q_2 is bounded using the same argument as they use for I. Third, the integral over the region $\mathbb{R}^n \times (0, t)\backslash(Q_1 \cup Q_2)$ corresponds to II in [KL] and also follows the same argument. Similarly, the L^2 and L^{n+4} bounds on the derivatives follow from the argument in Koch and Lamm as well. This completes the proof of Lemma 9. □

3.3.4 Induction Step

Now that the correct norms has been chosen, the induction step can be carried out. Applying the two lemmas and the modified norms for $i > 1$

$$\begin{aligned} &||\Phi^{i-1}_{sol}(h_\lambda) - \Phi^{i}_{sol}(h_\lambda)||_{X'_\infty(r_0)} \\ &\quad = ||V\left(R[\Phi^{i-2}_{sol}(h_\lambda)] - R[\Phi^{i-1}_{sol}(h_\lambda)]\right)||_{X'_\infty(r_0)} \\ &\quad \leq c_1||R[\Phi^{i-2}_{sol}(h_\lambda)] - R[\Phi^{i-1}_{sol}(h_\lambda)]||_{Y'_\infty(r_1)} + c_2\epsilon_n\left(\frac{r_1}{r_0}\right)^n \\ &\quad \leq c_3||\Phi^{i-2}_{sol}(h_\lambda) - \Phi^{i-1}_{sol}(h_\lambda)||_{X'_\infty(r_1)} + c_2\epsilon_n\left(\frac{r_1}{r_0}\right)^n \end{aligned}$$

Here the third line follows from Lemma 9 and the bound

$$||R[\Phi^{i-2}_{sol}(h_\lambda)] - R[\Phi^{i-1}_{sol}(h_\lambda)]||_{Y_\infty} \leq c_5||\Phi^{i-2}_{sol}(h_\lambda) - \Phi^{i-1}_{sol}(h_\lambda)||_{X_\infty} \leq c_5 \cdot 2C_n\epsilon_n.$$

The fourth line follows from Lemma 8.

This completes the induction for an appropriate choice of r_0 and r_1. The only complication is that at each step r_1 becomes r_0. To show this isn't a problem, fix $m \in \mathbb{N}$ and $r_0 > 0$. Let $a \in \mathbb{R}_{>2}$ be a constant to be determined later. For each integer, i from 1 to m, let $r_i = r_0/a^i$. Then the following holds.

$$\begin{aligned} &||(h_\lambda) - \Phi^{m}_{sol}(h_\lambda)||_{X'_\infty(r_0)} \\ &\quad \leq \sum_{i=1}^{m}||\Phi^{i-1}_{sol}(h_\lambda) - \Phi^{i}_{sol}(h_\lambda)||_{X'_\infty(r_0)} \\ &\quad \leq c_1\sum_{i=0}^{m-1}||(h_\lambda) - \Phi_{sol}(h_\lambda)||_{X'_\infty(r_i)} + c_2\epsilon_n\sum_{i=1}^{m-1}(m-i)\left(\frac{r_i}{r_{i-1}}\right)^n \\ &\quad \leq c_3\sum_{i=1}^{m}\eta(\lambda r_i) + c_2\epsilon_n\sum_{i=1}^{m}(m-i+1)\left(\frac{r_i}{r_{i-1}}\right)^n \\ &\quad \leq c_4 m\eta(\lambda r_m) + c_5\epsilon_n m^2 a^{-n} \end{aligned}$$

where the constants c_i depend on the dimension, n, and the constants m, and r_0. Then there exist a and λ large enough so that $c_4 m\eta(\lambda r_m) + c_5\epsilon_n m^2 a^{-n}$ is arbitrarily small.

More precisely, fix $r_0 > 0$. Then for every $\epsilon' > 0$ and for each $m \in \mathbb{N}$, there exists $A \in \mathbb{R}$ such that for all $a > A$,

$$c_5\epsilon_n m^2 a^{-n} < \frac{\epsilon'}{4}$$

Pick $a = 2A$ and fix this constant. Then there exists $\Lambda \in \mathbb{R}$ such that for all $\lambda > \Lambda$,

$$c_4 m\ \eta\left(\frac{r\lambda}{a^m}\right) < \frac{\epsilon'}{4}$$

Thus, for every $r_0, \epsilon' > 0$ and for each $m \in \mathbb{N}$, there exists $\Lambda \in \mathbb{R}$ such that for all $\lambda > \Lambda$,

$$||(h_\lambda) - \Phi^{m}_{sol}(h_\lambda)||_{X'_\infty(r_0)} \leq c_4 m\eta(\lambda r_m) + c_5\epsilon_n m^2 a^{-n} \leq \frac{\epsilon'}{2}.$$

This proves lemma 7. Therefore, $h_\lambda \to h_{sol}$ in $X'_\infty(r)$ for each $r > 0$. The next section will be devoted to increasing the regularity of this convergence.

3.4 Regularity of Convergence

This section will focus on replacing convergence in $X'_\infty(r)$ for each $r > 0$ with convergence in $C^\infty_{loc}(\mathbb{R}^n \times (0,\infty))$. First, notice that convergence in $X'_\infty(r)$ for each $r > 0$ directly implies $C^0_{loc}(\mathbb{R}^n \times (0,\infty))$. To see this take $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$. Let $r_0 = t_0/2 > 0$ and $Q(r_0) = B(0, r_0) \times (0, r_0^2)$. Then

$$\sup_{(x,t)\notin Q(r_0)} |h_\lambda - h_{sol}| \leq ||h_\lambda - h_{sol}||_{X'_\infty(r_0)}$$

which converges to zero as $\lambda \to \infty$.

Note that both g_λ and g_{sol} are smooth for all $t > 0$, with scale invariant bounds given by

$$\sup_{x\in\mathbb{R}^n} \sup_{t>0} |(t^{\frac{1}{2}}\nabla)^\alpha (t\partial_t)^k (g - \delta)(x,t)| \leq C\epsilon_n \tag{50}$$

where C is a constant depending only on α, k, and n (see section 3.1). In particular, this derivative bound on g_λ does not depend on λ. Hence, it make sense to ask if their derivatives converge as $\lambda \to \infty$. Notice, this is the same as asking about the derivatives of h_λ and h_{sol}.

Define the two-parameter family of symmetric two-tensors

$$u_\lambda(x,t) := h_\lambda(x,t) - h_{sol}(x,t) = g_\lambda(x,t) - g_{sol}(x,t) \tag{51}$$

for $t > 0$ and $\lambda \geq 1$. Similarly, define

$$f_\lambda(x,t) = R[h_\lambda](x,t) - R[h_{sol}](x,t). \tag{52}$$

Note that f_λ is a perfectly well-defined smooth function for all λ because h_λ and h_{sol} are both smooth functions. Hence, u_λ and f_λ satisfy

$$(\partial_t - \Delta)u_\lambda(x,t) = f_\lambda(x,t)$$

for the initial condition $u_\lambda(\cdot,0) = (g_\lambda - g_{sol})(\cdot,0)$. Then, u_λ can be written

$$\begin{aligned} u_\lambda(x,t) &= S[u_\lambda(\cdot,0)](x,t) + V[f_\lambda](x,t) \\ &= \int_{\mathbb{R}^n} K(x-y,t)\,(g_\lambda - g_{sol})\,(y,0)dy \\ &\qquad + \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s) f_\lambda(y,s)dyds \end{aligned}$$

Now we can state the following lemma.

Lemma 10. *With u_λ and f_λ as above, if equation 51 holds for u_λ and u_λ converges to zero in $X'_\infty(r)$ for all $r > 0$, then u_λ and f_λ converge to zero in $C^\infty_{loc}(\mathbb{R}^n \times (0,\infty))$.*

Proof. The idea is then to show $S[u_\lambda(\cdot,0)](x,t)$ goes to zero smoothly because the initial condition goes to zero; and then $V[f_\lambda](x,t)$ goes to zero smoothly because f_λ goes to zero. It is not yet clear that f_λ goes to zero because it requires C^2_{loc} convergence of u_λ.

Look at $S[u_\lambda(\cdot,0)](x,t)$. Recall, from section 3.3.2 that h_λ converges to $\Phi_{sol}[h_\lambda]$ in $C^\infty_{loc}(\mathbb{R}^n \times (0,\infty))$. It is worth noting that a stronger induction argument is probably possible. However, because $\Phi^m_{sol}[h_\lambda]$ only converges to h_{sol} in X_∞, no stronger statement could be made about the convergence of h_λ to h_{sol} directly.

Nevertheless, notice

$$S[u_\lambda(\cdot,0)](x,t) = \int_{\mathbb{R}^n} K(x-y,t)\left(g_\lambda - g_{sol}\right)(y,0)dy = \big(h_\lambda - \Phi_{sol}[h_\lambda]\big)(x,t)$$

Hence, by the argument of section 3.3.2, for all $t > 0$, for all integers $k \in \mathbb{N}_0$ and all multi-indexes $\alpha \in \mathbb{N}_0^n$,

$$\left|\partial_t^k \nabla^\alpha S[u_\lambda(\cdot,0)](x,t)\right| \to 0 \text{ as } \lambda \to \infty$$

Therefore, $S[u_\lambda(\cdot,0)] \to 0$ in $C^\infty_{loc}(\mathbb{R}^n \times (0,\infty))$.

Now focus on the non-homogeneous part, Vf_λ. To bound the space derivatives of Vf_λ, divide up the integral and change coordinates. Then take the spacial derivative. For $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned}
\nabla^\alpha V f_\lambda(x,t) &= \int_{\mathbb{R}^n\times(0,t)\setminus(B(x,r)\times(t-r^2,t))} \nabla^\alpha K(x-y,t-s) f_\lambda(y,s)dyds \\
&\quad + \int_{B(0,r)\times(0,r^2)} K(y,s)\nabla^\alpha f_\lambda(x-y,t-s)dyds \\
&=: I + II
\end{aligned}$$

Look at each of the integrals separately. The derivatives of the heat kernel act similar to the heat kernel away from the point $(0,0)$. Using this similarity, the same argument as in Lemma 8 above shows that for any fix $r > 0$, the integral I goes to zero as $\lambda \to \infty$. However, the same argument does not work for integral II. The change of variables avoids the bad behavior of the derivatives of the heat kernel near $(0,0)$.

Fortunately, the derivative bounds for u_λ are enough to gain control in this region.

$$\begin{aligned}
II \le \int_0^{r^2} ||K(\cdot,s)||_{L^1(\mathbb{R}^n)}ds \cdot \sup_{(y,s)\in B(x,r)\times(t-r^2,t)} |\nabla^\alpha f_\lambda(y,s)| \\
\le r^2 \cdot c\epsilon_n (t-r^2)^{-\frac{|\alpha|+2}{2}}
\end{aligned}$$

For every $t > 0$, this goes to zero as r goes to zero. Thus, by first picking r small enough and then picking λ big enough, both I and II can be made arbitrarily small. Thus, all the spacial derivatives of Vf_λ, and hence all the spacial derivatives of u_λ, converge to zero as $\lambda \to \infty$. In particular, f_λ converges to zero because it only involves space derivatives.

Finally, the time derivatives of Vf_λ will be bounded using the same method as the space integral. However, the boundary terms of V involve t, so the time derivatives add extra terms. Consider the first time derivative.

$$\begin{aligned}
\partial_t[Vf_\lambda(x,t)] &= \int_{\mathbb{R}^n\times(0,t)\setminus(B(0,r)\times(t-r^2,t))} \partial_t K(x-y,t-s)\cdot f_\lambda(y,s)dyds \\
&\quad + \int_0^{r^2}\int_{B(0,r)} K(y,s)\cdot\partial_t f_\lambda(x-y,t-s)dyds \\
&\quad + \int_{B(x,r)} K(x-y,r^2)\cdot f_\lambda(y,t-r^2)dy \\
&=: I_1 + II_1 + III_1
\end{aligned}$$

Here, the terms I_1 and II_1 are controlled in the exact same way as the spacial derivatives. The only new term is III_1. However, III_1 goes to zero as λ goes to infinity because f_λ does.

Taking the spacial derivative of I_1, II_1, and III_1 shows that all spacial derivatives with one time derivative converge to zero. For I_1 and II_1 use the same arguments as in the case of no time derivative. For III, use that $r > 0$, so the L^1-norm of $\nabla^\alpha K(\cdot, r^2)$ is bounded and f_λ converge to zero as before. In particular, $\partial_t f_\lambda$ converges continuously to zero.

Now the rest of the time derivatives are shown through induction. To illustrate, look at ∂_t^2

$$\begin{aligned}
\partial_t^2[Vf_\lambda(x,t)] &= \int_{\mathbb{R}^n\times(0,t)\setminus(B(0,r)\times(t-r^2,t))} \partial_t^2 K(x-y,t-s)\cdot f_\lambda(y,s)dyds \\
&\quad + \int_0^{r^2}\int_{B(0,r)} K(y,s)\cdot \partial_t^2 f_\lambda(x-y,t-s)dyds \\
&\quad + \int_{B(x,r)} K(x-y,r^2)\cdot \partial_t f_\lambda(y,t-r^2)dy \\
&\quad + \int_{B(x,r)} \partial_t K(x-y,r^2)\cdot f_\lambda(y,t-r^2)dy \\
&=: I_2 + II_2 + III_2 + IV_2
\end{aligned}$$

The first three terms are all bound exactly as in the first case. Notice that the first two bounds work for all derivatives of time, while the third uses the induction hypothesis, i.e. $\partial_t f_\lambda(x,t) \to 0$ for all $t > 0$ as $\lambda \to \infty$. All that remains is the derivative of the boundary term from I_1 in the previous derivative, integral IV_2. Here use that for all $r^2 > 0$, the $L^1(\mathbb{R}^n)$-norm of the all space and time derivatives of the heat kernel are bounded at time r^2. Then use that f_λ goes to zero as $\lambda \to \infty$ as before. Lastly, take the spacial derivatives of each term and use the same argument to show they all converge to zero. In particular, this shows that $\partial_t^2 f_\lambda$ goes to zero as λ goes to infinity.

Continuing this process shows that all time and space derivatives converge in a locally controllable way. Therefore, h_λ converges to h_{sol} in $C^\infty_{loc}(\mathbb{R}^n \times (0,\infty))$. Equivalently, g_λ converges to g_{sol} in $C^\infty_{loc}(\mathbb{R}^n \times (0,\infty))$. □

Remark. *The argument in this paper can be used to show a similar result in the setting of [DL]. The only major difference would be in the regularity argument.*

References

[B] Robert Bryant. Ricci flow solitons in dimension three with SO(3)-symmetries, available at www.math.duke.edu/∼bryant/3DRotSymRicciSolitons.pdf

[Der] Alix Deruelle. Smoothing out positively curved metric cones by Ricci expanders. ArXiv e-prints, February 2015. https://arxiv.org/pdf/1502.07921.pdf

[DL] Alix Deruelle and Tobias Lamm. Weak stability of Ricci expanders with positive curvature operator. *ArXiv e-prints*, January, 2016 https://arxiv.org/pdf/1601.00325.pdf

[KL] Herbert Koch and Tobias Lamm. Geometric flows with rough initial data. *Asian J. Math*, 16(2): 209-235, 2012 https://arxiv.org/abs/0902.1488

[LZ] Lott and Zhang. Quasiprojective Manifolds II. J. of the Eur. Math. Soc. 18, p. 1813-1854. https://arxiv.org/pdf/1309.6252v2.pdf

[SS] Felix Schulze and Miles Simon. Expanding Solitons with Non-negative Curvature Operator Coming Out of Cones. Math. Z., 275(1-2):625639, 2013. https://arxiv.org/pdf/1008.1408.pdf

[SSS] Oliver C. Schnrer, Felix Schulze, Miles Simon. Stability of Euclidean space under Ricci flow. Comm. Anal. Geom., 19(5):10231047, 2011. https://arxiv.org/pdf/0706.0421.pdf