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by

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Abstract

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The theme of this dissertation is the Brauer group of algebraic stacks. Antieau and Meier showed that if k is an algebraically closed field of $\operatorname{char} k \neq 2$, then $\operatorname{Br}(\mathscr{M}_{1,1,k}) = 0$, where $\mathscr{M}_{1,1}$ is the moduli stack of elliptic curves. We show that if $\operatorname{char} k = 2$ then $\operatorname{Br}(\mathscr{M}_{1,1,k}) = \mathbb{Z}/(2)$. In another direction, we compute the cohomological Brauer group of $\mathbb{G}_m$-gerbes; this is an analogue of a result of Gabber which computes the cohomological Brauer group of Brauer-Severi schemes. We also discuss two kinds of algebraic stacks X for which not all torsion classes in $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ are represented by Azumaya algebras on X (i.e. $\operatorname{Br} \neq \operatorname{Br}'$).

Contents

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Introduction

The Brauer group of a field k is a classical invariant which classifies central simple k-algebras, and the Brauer group of an algebraic variety X classifies Azumaya $\mathcal{O}_X$-algebras, which are "twists" of the matrix algebra $\mathrm{Mat}_{n\times n}(\mathcal{O}_X)$ over the structure sheaf $\mathcal{O}_X$. In complex geometry, the fact that the Brauer group is an invariant for birational maps can be used to determine whether a variety is rational. In number theory, the Tate conjecture for divisors for a smooth projective surface X over a finite field is known to be equivalent to the finiteness of $\mathrm{Br}\, X$.

In this dissertation, we are interested in Brauer groups of algebraic stacks. An Azumaya algebra on a moduli stack corresponds to a family of Azumaya algebras on the objects of the moduli stack compatible with all morphisms between the objects. The Brauer group of a quotient stack $[X/G]$ corresponds to Azumaya algebras on X that are equivariant with respect to the G-action. The Brauer group of stacks may sometimes be used to answer questions about algebraic varieties; for example, Lieblich [61] considered the Brauer group of classifying stacks $\mathrm{B}\mu_n$ to prove new cases of the period-index conjecture for function fields of curves over local fields.

For the moduli stack of elliptic curves $\mathscr{M}_{1,1,k}$ over an algebraically closed field k, Antieau and Meier had shown that $\mathrm{Br}(\mathscr{M}_{1,1,k}) = 0$ if $\mathrm{char}\, k \neq 2$. In Section 3 we compute $\mathrm{Br}(\mathscr{M}_{1,1,k})$ in the characteristic 2 case:

Theorem $\mathbf{A}_1$ ([4, 11.2] in $\mathrm{char}\, k \neq 2$). Let k be an algebraically closed field. Then $\mathrm{Br}\,\mathscr{M}_{1,1,k}$ is 0 unless $\mathrm{char}\, k = 2$, in which case $\mathrm{Br}\,\mathscr{M}_{1,1,k} = \mathbb{Z}/(2)$.

Theorem $\mathbf{A}_2$. Let k be a finite field of characteristic 2. Then

$$\mathrm{Br}\,\mathscr{M}_{1,1,k} = \begin{cases} \mathbb{Z}/(12) \oplus \mathbb{Z}/(2) & \text{if } x^2+x+1 \text{ has a root in } k \\ \mathbb{Z}/(24) & \text{otherwise.} \end{cases}$$

The methods of [4] do not apply to the characteristic 2 case since they rely on the finite Galois cover of $\mathscr{M}_{1,1,k}$ obtained by fixing a full level 2 structure. We study the $\mathrm{char}\, k = 2$ case by considering full level 3 structures instead, which comes at the cost of increasing the size of the group (by which $\mathscr{M}_{1,1,k}$ is a quotient stack) from $|\,\mathrm{GL}_2(\mathbb{F}_2)| = 6$ to $|\,\mathrm{GL}_2(\mathbb{F}_3)| = 48$.

Gabber proved that, given a Brauer-Severi scheme $\pi : X \to S$, the cohomological Brauer group $\mathrm{Br}'(X)$ is the quotient of $\mathrm{Br}'(S)$ by the class $[X]$. In Section 4 we prove an analogous result on the cohomological Brauer group of $\mathbb{G}_m$-gerbes:

Theorem B. Let S be a scheme and let $\pi_{\mathcal{G}} : \mathcal{G} \to S$ be a $\mathbb{G}_{m,S}$-gerbe with $[\mathcal{G}] \in \mathrm{Br}'\, S$. Then the sequence

$$\mathrm{H}^0_{\text{ét}}(S, \mathbb{Z}) \to \mathrm{Br}'\, S \xrightarrow{\pi_{\mathcal{G}}^*} \mathrm{Br}'\,\mathcal{G} \to 0$$

is exact, where the first map sends $1 \mapsto [\mathcal{G}]$.

This is one of the first computations of the Brauer group of an algebraic stack that is not a Deligne-Mumford stack. It may be viewed as a generalization of Gabber's result since the image of a Brauer-Severi scheme X under the coboundary $\mathrm{H}^1_{\text{ét}}(S, \mathrm{PGL}_n) \to \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m)$ is a torsion $\mathbb{G}_m$-gerbe. Assuming additional hypotheses on S (i.e. that it is regular and its fraction field has characteristic 0), we give another proof of Gabber's result.

We discuss two kinds examples of algebraic stacks for which $\mathrm{Br} \neq \mathrm{Br}'$, namely the classifying stack of $\mathbb{Z} \oplus \mathbb{Z}$ over a regular local ring (see Example 2.3.5), and the classifying stack of an elliptic curve (see Corollary 2.4.6).

In Section 5 we explore some questions related to the Brauer group of algebraic stacks. Whereas in Gabber's result and our Theorem B we are concerned with the cohomological Brauer group, it would also be interesting to ask whether the same statements hold with "Br′" replaced by "Br" (the Azumaya Brauer group). Partial positive results are discussed in Section 5.1 and Section 5.2.

In Section 5.3 we discuss whether the Brauer group functor is an "$\mathbb{A}^1$-homotopy invariant", namely whether $\mathrm{Br}\, S \to \mathrm{Br}\, \mathbb{A}^1_S$ is an isomorphism. This is a question that arises naturally when trying to compute the Brauer group via the descent spectral sequence associated to a smooth covering; it may be viewed as being part of a collection of questions asking when the étale cohomology functor $\mathrm{H}^i_{\text{ét}}(-, \mathbb{G}_m)$ for $i \geq 0$ is invariant with respect to polynomial extensions of the ground ring. It is known that the unit group (i.e. the $i = 0$ case) is invariant exactly when the ring is reduced, and the Picard group (i.e. the $i = 1$ case) is invariant exactly when the ring is seminormal; it would be nice to find a similar, purely ring-theoretic criterion which corresponds exactly to those cases in which the Brauer group (roughly the $i = 2$ case) is invariant. The only unknown part is concerning the ℓ-torsion for primes ℓ that are not invertible in the base. We extend a result of Knus and Ojanguren to show that the Brauer group is $\mathbb{A}^1$-homotopy invariant if the base is a monoid algebra over a small class of regular rings.

1. Generalities

In this section we discuss Azumaya algebras on locally ringed sites. Some standard references are Giraud [40], Lieblich [59], and the Stacks Project [88].

1.1. **Skolem-Noether for locally ringed sites.**

Definition 1.1.1 (Locally ringed site, locally ringed topos). [1] A locally ringed site is a ringed site $(\mathcal{C}, \mathcal{O})$ such that, for every object $U \in \mathcal{C}$ and $f \in \Gamma(U, \mathcal{O})$, there exists a covering $\{U_i \to U\}_{i \in I}$ of U such that for each $i \in I$, either $f|_{U_i}$ or $(1-f)|_{U_i}$ is a unit of $\Gamma(U_i, \mathcal{O})$.

Lemma 1.1.2. [88, 04ES] Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a locally ringed topos Definition 1.1.1. Let $U \in \mathcal{X}$ be an object and $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O}_\mathcal{X})$ elements which generate the unit ideal of $\Gamma(U, \mathcal{O}_\mathcal{X})$. Then there exists a covering $\{U_i \to U\}_{i \in I}$ such that for each $i \in I$ there exists some $\ell \in \{1, \ldots, n\}$ such that $f_\ell|_{U_i}$ is invertible in $\Gamma(U_i, \mathcal{O}_\mathcal{X})$.

Example 1.1.3. Here is an example of a ringed site which is not a locally ringed site Definition 1.1.1. Let $\mathcal{X}$ denote the Zariski site of an integral domain A, and let $\mathcal{O}_\mathcal{X}$ denote the constant sheaf associated to $\mathbb{Z}$ on $\mathcal{X}$. Since $\mathcal{X}$ is irreducible, the constant presheaf $\underline{\mathbb{Z}}^{\mathrm{pre}}$ is a sheaf, hence $\Gamma(U, \mathcal{O}_\mathcal{X}) = \mathbb{Z}$ for all $U \in \mathcal{X}$. Consider the section $3 \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$; there does not exist an object $U \in \mathcal{X}$ such that $3|_U \in \Gamma(U, \mathcal{O}_\mathcal{X})$ is invertible, and the same is true for $1-3$. Alternatively, consider any ringed (topological) space which is not a locally ringed (topological) space.

Remark 1.1.4. [88, 0409] Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a ringed site, and let $\mathcal{L}$ be an $\mathcal{O}_\mathcal{X}$-module.

(1) We say that $\mathcal{L}$ is invertible if the tensor product functor
$$- \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L} : (\mathcal{O}_\mathcal{X}\text{-mod}) \to (\mathcal{O}_\mathcal{X}\text{-mod})$$
is an equivalence of categories.

(2) We say that $\mathcal{L}$ is locally free of rank 1 if there exists a covering $\{\mathcal{X}_i \to \mathcal{X}\}_{i \in I}$ and $\mathcal{O}_{\mathcal{X}_i}$-linear isomorphisms $\mathcal{L}|_{\mathcal{X}_i} \simeq \mathcal{O}_{\mathcal{X}_i}$ for all i.

In general, locally free of rank 1 implies invertible, and the converse holds if $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is locally ringed [88, 0B8Q], [50, Exercise 19.2].

A counterexample to the converse in case $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is not locally ringed can be constructed as follows. Let A be a ring for which $\mathrm{Pic}(A) \neq 0$, and let M be a finitely generated A-module of everywhere rank 1 corresponding to a nontrivial element of $\mathrm{Pic}(A)$. Let $\mathcal{X}$ be the small Zariski site of a DVR, let $\mathcal{O}_\mathcal{X}$ be the constant sheaf $\underline{A}$ on $\mathcal{X}$, and let $\mathcal{L}$ be the constant sheaf $\underline{M}$, which is naturally an $\mathcal{O}_\mathcal{X}$-module. It may be checked that $\mathcal{L}$ is invertible but not locally free of rank 1.

Definition 1.1.5 ($\mathrm{GL}_n, \mathrm{PGL}_n$). Let $\mathcal{X}$ be a ringed site. For any quasi-coherent $\mathcal{O}_\mathcal{X}$-module $\mathcal{E}$, we denote
$$\mathrm{GL}(\mathcal{E}) := \underline{\mathrm{Aut}}_{\mathcal{O}_\mathcal{X}\text{-mod}}(\mathcal{E})$$
and denote
$$\mathrm{PGL}(\mathcal{E}) := \mathrm{GL}(\mathcal{E})/\mathbb{G}_{m,\mathcal{X}}$$
the sheaf quotient via the scalar multiplication map, and set $\mathrm{GL}_n(\mathcal{O}_\mathcal{X}) := \mathrm{GL}(\mathcal{O}_\mathcal{X}^{\oplus n})$ and $\mathrm{PGL}_n(\mathcal{O}_\mathcal{X}) := \mathrm{PGL}(\mathcal{O}_\mathcal{X}^{\oplus n})$.

[1] References: [88, 04EU, 04H8], [7, Exp. IV, Exercise 13.9], [40, V, §4], [41, §2], [37, page 153]

Definition 1.1.6 (Adjunction morphism). An $\mathcal{O}_{\mathcal{X}}$-module automorphism of $\mathcal{E}$ defines an $\mathcal{O}_{\mathcal{X}}$-algebra automorphism of $\underline{\mathrm{End}}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(\mathcal{E})$ by conjugation, so there is a morphism

$$\mathrm{GL}(\mathcal{E}) \to \underline{\mathrm{Aut}}_{\mathcal{O}_{\mathcal{X}}\text{-alg}}(\underline{\mathrm{End}}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(\mathcal{E})) \tag{1.1.6.1}$$

of sheaves of groups on $\mathcal{X}$, which descends to a morphism

$$\mathrm{PGL}(\mathcal{E}) \to \underline{\mathrm{Aut}}_{\mathcal{O}_{\mathcal{X}}\text{-alg}}(\underline{\mathrm{End}}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(\mathcal{E})) \tag{1.1.6.2}$$

of sheaves of groups on $\mathcal{X}$.

Lemma 1.1.7. The map (1.1.6.2) is injective for any ringed site.

Proof. For this it suffices to show that the kernel of (1.1.6.1) is $\mathbb{G}_{m,\mathcal{X}}$. We may reduce to the case when $\mathcal{E}$ is free. Since we may show injectivity pointwise, it suffices to show that, for any $X \in \mathcal{X}$, setting $A = \Gamma(X, \mathcal{O}_{\mathcal{X}})$, the homomorphism

$$\mathrm{GL}_n(A) \to \mathrm{Aut}_{\mathcal{O}_{\mathcal{X}}|_X\text{-alg}}(\mathrm{Mat}_{n\times n}(\mathcal{O}_{\mathcal{X}}|_X))$$

of groups has kernel $A^\times$; we have an identification

$$\mathrm{Aut}_{\mathcal{O}_{\mathcal{X}}|_X\text{-alg}}(\mathrm{Mat}_{n\times n}(\mathcal{O}_{\mathcal{X}}|_X)) \simeq \mathrm{Aut}_{A\text{-alg}}(\mathrm{Mat}_{n\times n}(A)) \tag{1.1.7.1}$$

since $\mathcal{O}_{\mathcal{X}}$-algebra automorphisms of $\mathrm{Mat}_{n\times n}(\mathcal{O}_{\mathcal{X}})$ are determined by their global sections. Now we conclude using the fact that the center of $\mathrm{Mat}_{n\times n}(A)$ is A. □

Theorem 1.1.8 (Skolem-Noether for locally ringed topoi). [40, V.4.1] [2] Let $\mathcal{X}$ be a locally ringed topos. The canonical homomorphism (1.1.6.2) is an isomorphism for any finite type locally free $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$. □

Example 1.1.9 (Skolem-Noether fails for non-locally ringed topoi). Here is an example of a ringed site $\mathcal{X}$ and a finite locally free $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$ for which (1.1.6.2) is not surjective. As in Example 1.1.3, let $\mathcal{X}$ be the Zariski site of a nonzero integral domain, let A be a ring for which $\mathrm{GL}_n(A) \to \mathrm{PGL}_n(A)$ is not surjective (e.g. the coordinate ring of PGL_n itself), let $\mathcal{O}_{\mathcal{X}}$ denote the constant sheaf associated to A, and set $\mathcal{E} := \mathcal{O}_{\mathcal{X}}^{\oplus n}$. Since the underlying site of $\mathcal{X}$ is irreducible, every constant presheaf is a sheaf. Thus $\mathrm{PGL}_{n,\mathcal{X}}$ is the constant sheaf assigning $X \mapsto \mathrm{GL}_n(A)/A^\times$ for all $X \in \mathcal{X}$, and $\underline{\mathrm{Aut}}_{\mathcal{O}_{\mathcal{X}}\text{-alg}}(\mathrm{Mat}_{n\times n}(\mathcal{O}_{\mathcal{X}}))$ is the constant sheaf assigning $X \mapsto \mathrm{Aut}_{A\text{-alg}}(\mathrm{Mat}_{n\times n}(A)) \simeq \mathrm{PGL}_n(A)$ for all $X \in \mathcal{X}$ (here the last isomorphism follows from Skolem-Noether for schemes).

Lemma 1.1.10. [3] Let X be a locally ringed site, let $\mathcal{F}, \mathcal{G}$ be finite locally free $\mathcal{O}_X$-modules of finite positive rank such that there exists an isomorphism

$$\varphi : \underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{F}) \to \underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{G})$$

---

[2] This is the generalization of the classical Skolem-Noether theorem to local rings by Auslander-Goldman [8, Thm. 3.6], to arbitrary schemes by Grothendieck [41, Thm. 5.10], and to arbitrary locally ringed topoi by Giraud [40, V.4.1]. Antieau-Williams [5, Prop. 1] have given another proof by observing that the Skolem-Noether theorem holds for the universal locally ringed topos. The result is also found in Lieblich [59, 2.1.5.3].

[3] In other words, the sequence of pointed sets

$$\mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m) \to \mathrm{H}^1_{\text{ét}}(X, \mathrm{GL}_n) \to \mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n)$$

is exact. In [88, 0A2K], there is an outline of an argument when X is a scheme; we give the details here. See also `https://mathoverflow.net/q/128364` and `http://mathoverflow.net/a/144947`. This is proved for affine schemes in [52, 2.2 Proposition].

of $\mathcal{O}_X$-algebras. Then there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and an $\mathcal{O}_X$-module isomorphism

$$\xi : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{G}$$

which induces the given isomorphism φ.

Proof. The last claim that ξ "induces the given isomorphism φ" means that φ may be factored as

$$\underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{F}) \to \underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}) \to \underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{G})$$

where the first map is the canonical isomorphism and the second map is conjugation by ξ.

Let $\mathcal{P}$ denote the presheaf of sets which assigns to every object $U \in X$ the subset

$$\Gamma(U, \mathcal{P}) \subset \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

of $\mathcal{O}_U$-linear isomorphisms $f : \mathcal{F}|_U \to \mathcal{G}|_U$ such that the conjugation-by-f map

$$c_f : \underline{\mathrm{End}}_{\mathcal{O}_U\text{-mod}}(\mathcal{G}|_U) \to \underline{\mathrm{End}}_{\mathcal{O}_U\text{-mod}}(\mathcal{F}|_U)$$

is equal to $\varphi|_U$. We check that $\mathcal{P}$ is a $\mathbb{G}_m$-torsor. By the Skolem-Noether theorem Theorem 1.1.8, the sequence

$$1 \to \mathbb{G}_{m,S} \to \mathrm{GL}(\mathcal{G}) \xrightarrow{\rho} \underline{\mathrm{Aut}}_{\mathcal{O}_X\text{-alg}}(\underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{G})) \to 1 \qquad (1.1.10.1)$$

is exact (only left exact in general), where ρ denotes the conjugation map $\alpha \mapsto \{M \mapsto \alpha M \alpha^{-1}\}$.

Given $f_1, f_2 \in \Gamma(U, \mathcal{P})$, their difference $f_2 f_1^{-1}$ is an element of $\Gamma(U, \mathrm{GL}(\mathcal{G}))$ and we have an equality

$$\rho_{f_1 f_2^{-1}} = c_{f_1} c_{f_2}^{-1} = (\varphi|_U) \circ (\varphi|_U)^{-1} = \mathrm{id}$$

of elements of $\mathrm{Aut}_{\mathcal{O}_U\text{-alg}}(\underline{\mathrm{End}}_{\mathcal{O}_U\text{-mod}}(\mathcal{G}|_U))$; this implies $f_2 f_1^{-1} \in \Gamma(U, \mathbb{G}_m)$ by left exactness of (1.1.10.1). This shows that $\mathcal{P}$ is a pseudo $\mathbb{G}_m$-torsor.

Let U be any object of X. There is a covering $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ such that there are $\mathcal{O}_{U_i}$-linear isomorphisms $f_i' : \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ for all i. On U_i, the composite $\varphi|_{U_i} \circ c_{f_i'}$ is an $\mathcal{O}_{U_i}$-algebra automorphism of $\underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{G})|_{U_i}$, so by right-exactness of (1.1.10.1), we may (after possibly refining the cover $\mathfrak{U}$) choose $\alpha_i \in \Gamma(U_i, \mathrm{GL}(\mathcal{G}))$ such that $\rho_{\alpha_i} = \varphi|_{U_i} \circ c_{f_i'}$. This implies that $\mathcal{P}(X_i) \neq \emptyset$, since it contains in particular $c_{\alpha_i f_i'}$. This shows that $\mathcal{P}$ is a $\mathbb{G}_m$-torsor, which corresponds to an invertible sheaf $\mathcal{L}$. □

1.2. **Azumaya algebras and the Brauer group.**

Definition 1.2.1 (Azumaya algebra). [41, §2], [40, V, §4], [59, 2.1.5.1] An Azumaya $\mathcal{O}_X$-algebra is a quasi-coherent (non-commutative, unital) $\mathcal{O}_X$-algebra $\mathcal{A}$ such that there exists a covering $\{X_i \to X\}_{i \in I}$, positive integers n_i, and isomorphisms

$$\mathcal{A}|_{X_i} \simeq \mathrm{Mat}_{n_i \times n_i}(\mathcal{O}_{X_i})$$

of $\mathcal{O}_{X_i}$-algebras.[4]

Remark 1.2.2. In case X is an algebraic stack, we will (unless otherwise specified) only consider Azumaya algebras on the site of X-schemes equipped with the smooth topology. If X is even a scheme, it is equivalent to consider the étale or fppf topology [41, 5.1], [69,

[4]In particular, we do not require that the rank of $\mathcal{A}$ as a quasi-coherent $\mathcal{O}_X$-module be constant (it is only locally constant).

IV, 2.1], but not the Zariski topology (see [74], [26], [31] for examples of nontrivial Azumaya algebras that are Zariski-locally trivial). □

Remark 1.2.3. We note that Hilbert's Theorem 90 does not hold for algebraic stacks, namely the canonical map

$$\mathrm{H}^1_{\mathrm{Zar}}(X, \mathrm{GL}_n) \to \mathrm{H}^1_{\text{ét}}(X, \mathrm{GL}_n)$$

is not necessarily surjective. Schröer [83] showed that there exists an algebraic space X for which $\mathrm{Pic}(X_{\mathrm{Zar}}) \to \mathrm{Pic}(X_{\text{ét}})$ is not surjective.

For another example, let k be a field, let G be a finite group, and set $X := \mathrm{B}G_k$. There is a correspondence between quasi-coherent $\mathcal{O}_X$-modules and G-representations over k, i.e. k-vector spaces V equipped with a group homomorphism $G \to \mathrm{Aut}_k(V)$, and a quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ is locally free if and only if the pullback $\xi^*\mathscr{F}$ is a locally free $\mathcal{O}_{\mathrm{Spec}\, k}$-module, where $\xi : \mathrm{Spec}\, k \to X$ is the map corresponding to the trivial G-torsor. Thus every quasi-coherent $\mathcal{O}_X$-module is étale-locally free. Then X has coarse moduli space $\pi : X \to \mathrm{Spec}\, k$. Thus the topological space associated to X is a single point, thus there is only one open substack of X. Thus Zariski-locally free $\mathcal{O}_X$-modules are in fact free. This shows that étale-locally trivial vector bundles are not necessarily Zariski-locally trivial. □

Definition 1.2.4 (Equivalence relation between Azumaya algebras). [40, §V.4, eqn (4)] Two Azumaya algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are Brauer equivalent if there exist locally free $\mathcal{O}_X$-modules $\mathcal{E}_1$ and $\mathcal{E}_2$ of finite rank and an isomorphism

$$\mathcal{A}_1 \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{E}_1) \simeq \mathcal{A}_2 \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{E}_2)$$

of $\mathcal{O}_X$-algebras.

Remark 1.2.5 (Relationship between Brauer and Morita equivalence). [59, 2.1.4] Let $\mathcal{A}$ be a unital, associative $\mathcal{O}_X$-algebra. Let $\mathrm{Mod}^{\mathrm{fp}}(\mathcal{A})$ denote the $\Gamma(X, \mathcal{O}_X)$-linear category of locally finitely presented $\mathcal{A}$-modules. The assignment $U \mapsto \mathrm{Mod}^{\mathrm{fp}}(\mathcal{A}|_U)$ defines a fibered $\mathcal{O}_X$-linear category $\mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathcal{A}} \to X$ over X. Two $\mathcal{O}_X$-algebras $\mathcal{A}_1, \mathcal{A}_2$ are fibered Morita equivalent if there is an equivalence $\mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathcal{A}_1} \simeq \mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathcal{A}_2}$ of fibered $\mathcal{O}_X$-linear categories. If $\mathcal{A}_1, \mathcal{A}_2$ are Azumaya $\mathcal{O}_X$-algebras, then $\mathcal{A}_1, \mathcal{A}_2$ are Brauer equivalent if and only if $\mathcal{A}_1, \mathcal{A}_2$ are fibered Morita equivalent [59, 2.1.5.8]. □

Definition 1.2.6 (Brauer group). The set of Brauer equivalence classes of Azumaya algebras is denoted

$$\mathrm{Br}\, X$$

and is called the (Azumaya) Brauer group of X. There is an (abelian) group structure on $\mathrm{Br}\, X$ given by tensor product of Azumaya algebras, the inverse is given by $[\mathscr{A}]^{-1} = [\mathscr{A}^{\mathrm{op}}]$, and the identity element is the class of trivial Azumaya algebras $[\underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{E})]$.

Note 1.2.7 (Determinant of Kronecker product). [5] Let A be a ring, and for $i = 1, 2$ let M_i be an $n_i \times n_i$ matrix with entries in A. The determinant of the $n_1 n_2 \times n_1 n_2$ matrix $M_1 \otimes_A M_2$ is

$$\det(M_1 \otimes_A M_2) = (\det(M_1))^{n_2} \cdot (\det(M_2))^{n_1}$$

since we have a factorization

$$M_1 \otimes_A M_2 = (M_1 \otimes_A \mathrm{id}_{n_2}) \circ (\mathrm{id}_{n_1} \otimes_A M_2)$$

[5] From `http://math.stackexchange.com/q/1316594/`.

of Kronecker products and determinant is multiplicative for products of matrices.

In particular, it may be checked that, if M is an $n \times n$ matrix, then

$$\det(M^{\otimes s}) = (\det(M))^{sn^{s-1}}$$

for all positive integers n.

Lemma 1.2.8. Let X be a locally ringed site. Let

$$\alpha : \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}) \to \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n})$$

be an $\mathcal{O}_X$-algebra automorphism of $\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n})$. The sheaf $\mathcal{P}_\alpha$ of liftings α to GL_n is a $\mathbb{G}_m$-torsor and the class $[\mathcal{P}_\alpha]$ is n-torsion in $\mathrm{Pic}(X)$.

Proof. The first claim follows from the Skolem-Noether theorem Theorem 1.1.8. In particular, there exists a covering $\mathfrak{U} = \{X_i \to X\}_{i \in I}$ such that each restriction

$$\alpha|_{X_i} : \underline{\mathrm{End}}_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i}^{\oplus n}) \to \underline{\mathrm{End}}_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i}^{\oplus n})$$

is induced by conjugation by some element $M_i \in \mathrm{GL}_n(\mathcal{O}_{X_i})$. On the pairwise intersections $X_{i_1,i_2} := X_{i_1} \times_X X_{i_2}$, there exists a unique $\gamma_{i_1,i_2} \in \Gamma(X_{i_1,i_2}, \mathbb{G}_m)$ such that

$$(M_{i_1}|_{X_{i_1,i_2}}) \cdot (M_{i_2}|_{X_{i_1,i_2}})^{-1} = \gamma_{i_1,i_2} \, \mathrm{id}_n \tag{1.2.8.1}$$

for all i_1, i_2. Since M_i are $n \times n$ matrices, taking determinants in (1.2.8.1) implies

$$\det(M_{i_1})|_{X_{i_1,i_2}} \cdot \det(M_{i_2}^{-1})|_{X_{i_1,i_2}} = \gamma_{i_1,i_2}^n$$

in $\Gamma(X_{i_1,i_2}, \mathbb{G}_m)$ for all $i_1, i_2 \in I$. The collection $\{\gamma_{i_1,i_2}\}_{i_1,i_2 \in I}$ constitutes a 1-cocycle in $\mathbb{G}_m$ which defines a class $[\alpha] \in \mathrm{H}^1(X, \mathbb{G}_m)$; to show that $n[\alpha] = 0$ in $\mathrm{H}^1(X, \mathbb{G}_m)$, it suffices to show that the $\mathbb{G}_m$-torsor of liftings of

$$\alpha^{\otimes n} : \underline{\mathrm{End}}_{\mathcal{O}_X}((\mathcal{O}_X^{\oplus n})^{\otimes n}) \to \underline{\mathrm{End}}_{\mathcal{O}_X}((\mathcal{O}_X^{\oplus n})^{\otimes n})$$

to GL_{n^n} is trivial, i.e. has a global lifting. The same covering $\mathfrak{U}$ is a trivialization cover for $\alpha^{\otimes n}$; each restriction $\alpha^{\otimes n}|_{X_i}$ is induced by conjugation by $M_i^{\otimes n}$, and on the pairwise intersections we have

$$(M_{i_1}^{\otimes n}|_{X_{i_1,i_2}}) \cdot (M_{i_2}^{\otimes n}|_{X_{i_1,i_2}})^{-1} = \gamma_{i_1,i_2}^n \, \mathrm{id}_{n^n} \tag{1.2.8.2}$$

so $n[\alpha]$ is equivalent to the 1-coboundary $\{\det(M_i)\}_{i \in I}$. Thus the collection

$$\{\tfrac{1}{\det(M_i)} M_i^{\otimes n}\}_{i \in I}$$

agree on pairwise intersections, hence glues to give a global section $M \in \Gamma(X, \mathrm{GL}_n)$ which induces the automorphism α. We may check by Note 1.2.7 that $\det(\frac{1}{\det(M_i)} M_i^{\otimes n}) = 1$, hence also $\det(M) = 1$. □

Lemma 1.2.9 (Azumaya algebra of rank n^2 is n-torsion). [6] Let X be a locally ringed site. Let $\mathcal{A}$ be an Azumaya $\mathcal{O}_X$-algebra of constant rank n^2. There exists a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank n^n and an isomorphism

$$\mathcal{A}^{\otimes n} \simeq \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E})$$

of $\mathcal{O}_X$-algebras. Thus the class of $\mathcal{A}$ in $\mathrm{Br}\, X$ is annihilated by n.

[6] This argument is inspired by, but different from, the argument given in [88, 0A2L] (for schemes), which references [81]. The arguments of [40, §V.4.6] and [5, Thm. 3] assume that the nth power morphism $\mathbb{G}_m \to \mathbb{G}_m$ is an epimorphism, so that also $\mathrm{SL}_n \to \mathrm{PGL}_n$ is an epimorphism. For the case of fields, see [62, §30, Theorem 3] (crossed products and cocycles) and [38, 4.4.8] (finiteness of group cohomology).

Proof. Choose a covering $\mathfrak{U} = \{X_i \to X\}_{i \in I}$ for which there exist isomorphisms

$$\alpha_i : \mathcal{A}|_{X_i} \to \underline{\mathrm{End}}_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i}^{\oplus n})$$

of $\mathcal{O}_{X_i}$-algebras. On the pairwise intersections $X_{ij} := X_i \times_X X_j$, we obtain $\mathcal{O}_{X_{ij}}$-algebra automorphisms

$$\beta_{ij} := (\alpha_i|_{X_{ij}}) \circ (\alpha_j|_{X_{ij}})^{-1}$$

of $\underline{\mathrm{End}}_{\mathcal{O}_{X_{ij}}}(\mathcal{O}_{X_{ij}}^{\oplus n})$. Let

$$\mathcal{P}_{ij} \subset \underline{\mathrm{Aut}}_{\mathcal{O}_{X_{ij}}}(\mathcal{O}_{X_{ij}}^{\oplus n})$$

be the sheaf of liftings[7] of β_{ij} to GL_n; this is a pseudo-$\mathbb{G}_m$-torsor in any ringed topos and is locally nonempty, i.e. is a $\mathbb{G}_m$-torsor, by Skolem-Noether Theorem 1.1.8 since X is locally ringed. We have that $\mathcal{P}_{ij}$ is n-torsion in $\mathrm{H}^1(X_{ij}, \mathbb{G}_m)$ by Lemma 1.2.8. For any integer s, let

$$(\mathcal{P}_{ij})^{\times s} \subset \underline{\mathrm{Aut}}_{\mathcal{O}_{X_{ij}}}(\mathcal{O}_{X_{ij}}^{\oplus n})$$

denote the sheaf associated to the presheaf sending

$$U \mapsto \{\mathrm{P}_1 \cdots \mathrm{P}_s \ : \ \mathrm{P}_i \in \Gamma(U, \mathcal{P}_{ij})\}$$

for any $U \in (X/X_{ij})$. Then $(\mathcal{P}_{ij})^{\times s}$ is again a $\mathbb{G}_m$-torsor and we have

$$[(\mathcal{P}_{ij})^{\times s}] = s[\mathcal{P}_{ij}] = 0$$

in $\mathrm{H}^1(X_{ij}, \mathbb{G}_m)$. Since $(\mathcal{P}_{ij})^{\times n}$ is the trivial torsor, we may choose a global section

$$\mathrm{N}_{ij} \in \Gamma(X_{ij}, (\mathcal{P}_{ij})^{\times n})$$

which is locally on X_{ij} of the form P^n. We have

$$\beta_{jk}|_{X_{ijk}} \circ \beta_{ij}|_{X_{ijk}} = \beta_{ik}|_{X_{ijk}}$$

corresponding to an equality

$$(\mathcal{P}_{jk}|_{X_{ijk}} \cdot \mathcal{P}_{ij}|_{X_{ijk}})^{\#} = \mathcal{P}_{ik}|_{X_{ijk}}$$

of subsheaves of $\underline{\mathrm{Aut}}_{\mathcal{O}_{X_{ijk}}}(\mathcal{O}_{X_{ijk}}^{\oplus n})$, where $(-)^{\#}$ denotes sheafification. Let

$$\mathcal{R}_{ij} \subset \mathcal{P}_{ij}$$

denote the sheaf of nth roots of N_{ij}; then $\mathcal{R}_{ij}$ is a μ_n-torsor, since it is locally nonempty by assumption on N_{ij}. Let

$$\mathcal{T}_{ij} \subset \underline{\mathrm{Aut}}_{\mathcal{O}_{X_{ij}}}((\mathcal{O}_{X_{ij}}^{\oplus n})^{\otimes n})$$

be the sheaf of liftings of the $\mathcal{O}_{X_{ij}}$-algebra automorphism $\beta_{ij}^{\otimes n}$ of $\underline{\mathrm{End}}_{\mathcal{O}_{X_{ij}}}((\mathcal{O}_{X_{ij}}^{\oplus n})^{\otimes n})$ to $\underline{\mathrm{Aut}}_{\mathcal{O}_{X_{ij}}}((\mathcal{O}_{X_{ij}}^{\oplus n})^{\otimes n})$; it is a $\mathbb{G}_m$-torsor by the Skolem-Noether Theorem. Since $\mathcal{R}_{ij}$ is a μ_n-torsor, for any $U \in (X/X_{ij})$ and $\mathrm{R}_1, \mathrm{R}_2 \in \Gamma(U, \mathcal{R}_{ij})$ we have $\mathrm{R}_1^{\otimes n} = \mathrm{R}_2^{\otimes n}$ in $\Gamma(U, \mathcal{T}_{ij})$, hence glue to give some

$$\mathrm{M}_{ij} \in \Gamma(X_{ij}, \mathcal{T}_{ij})$$

which restricts to each $\mathrm{R}^{\otimes n}$; in particular the $\mathbb{G}_m$-torsor $\mathcal{T}_{ij}$ is trivial. Moreover we have

$$\beta_{jk}^{\otimes n}|_{X_{ijk}} \circ \beta_{ij}^{\otimes n}|_{X_{ijk}} = \beta_{ik}^{\otimes n}|_{X_{ijk}}$$

corresponding to an equality

$$(\mathcal{T}_{jk}|_{X_{ijk}} \cdot \mathcal{T}_{ij}|_{X_{ijk}})^{\#} = \mathcal{T}_{ik}|_{X_{ijk}}$$

[7] It is useful to consider first the case when each $\mathcal{P}_{ij}$ is trivial.

of subsheaves of $\underline{\mathrm{Aut}}_{\mathcal{O}_{X_{ijk}}}((\mathcal{O}_{X_{ijk}}^{\oplus n})^{\otimes n})$. For any $U \in (X/X_{ij})$, we have $\det(\mathrm{R}_1) = \det(\mathrm{R}_2)$ for any $\mathrm{R}_1, \mathrm{R}_2 \in \Gamma(U, \mathcal{R}_{ij})$; thus $\det(\Gamma(U, \mathcal{R}_{ij}))$ glue to give a global section

$$a_{ij} \in \Gamma(X_{ij}, \mathbb{G}_m)$$

restricting to $\det(\Gamma(U, \mathcal{R}_{ij}))$ on U and such that $a_{ij}^n = \det(\mathrm{N}_{ij})$. For any $U \in (X/X_{ijk})$ and $\mathrm{R}_{ij} \in \Gamma(U, \mathcal{R}_{ij})$, $\mathrm{R}_{ik} \in \Gamma(U, \mathcal{R}_{ik})$, $\mathrm{R}_{jk} \in \Gamma(U, \mathcal{R}_{jk})$, we have that

$$\mathrm{R}_{jk} \cdot \mathrm{R}_{ij} \cdot \mathrm{R}_{ik}^{-1}$$

is an element of $\Gamma(U, \mathbb{G}_m)$ whose nth power

$$b_{ijk} \in \Gamma(U, \mathbb{G}_m)$$

is independent of choice of $\mathrm{R}_{ij}, \mathrm{R}_{ik}, \mathrm{R}_{jk}$ since $\mathcal{R}_{ij}, \mathcal{R}_{ik}, \mathcal{R}_{jk}$ are μ_n-torsors. Moreover one may check that

$$\mathrm{M}_{jk}|_{X_{ijk}} \cdot \mathrm{M}_{ij}|_{X_{ijk}} = b_{ijk}\mathrm{M}_{ik}|_{X_{ijk}}$$

and

$$a_{jk}|_{X_{ijk}} \cdot a_{ij}|_{X_{ijk}} = b_{ijk}a_{ik}|_{X_{ijk}}$$

which implies that the collection

$$\{\tfrac{1}{a_{ij}}\mathrm{M}_{ij}\}_{i,j \in I}$$

satisfies the cocycle condition, which allows us to glue the finite free sheaves

$$\{(\mathcal{O}_{X_i}^{\oplus n})^{\otimes n}\}_{i \in I}$$

to get a finite locally free $\mathcal{E}$. □

Definition 1.2.10 (Locally nonzero). Let X be a site. Let A be a torsion free abelian group and let $\underline{\mathrm{A}}$ be the constant sheaf on X associated to A.

(1) We say that a section $n \in \Gamma(X, \underline{\mathrm{A}})$ is locally nonzero if there is a nonempty covering $\{X_i \to X\}_{i \in I}$ and nonzero integers $n_i \in \mathrm{A}$ such that for each $i \in I$ the restriction $n|_{X_i}$ is equal to the image of n_i under the sheafification map $\mathrm{A} \to \Gamma(X, \underline{\mathrm{A}})$.
(2) Given a $\Gamma(X, \underline{\mathrm{A}})$-module M, we say that an element $\alpha \in M$ is $\Gamma(X, \underline{\mathrm{A}})$-torsion if there exists a locally nonzero section $n \in \Gamma(X, \underline{\mathrm{A}})$ such that $n\alpha = 0$.

We will usually be interested in the case $\mathrm{A} = \mathbb{Z}$ and $M = \mathrm{H}^i(X, \mathbb{G}_{m,X})$. See the definition of the cohomological Brauer group Definition 1.4.2 and Appendix A.

Lemma 1.2.11. The group $\mathrm{Br}\, X$ is $\Gamma(X, \underline{\mathbb{Z}})$-torsion (see Definition 1.2.10).

Proof. Given an Azumaya $\mathcal{O}_X$-algebra $\mathcal{A}$, the assignment $U \mapsto \sqrt{\operatorname{rank} \mathcal{A}|_U}$ defines a locally nonzero section $n \in \Gamma(X, \underline{\mathbb{Z}})$ and $[\mathcal{A}]$ is n-torsion in $\mathrm{Br}\, \mathcal{X}$ by Lemma 1.2.9. □

1.3. **Gerbes and twisted sheaves.**

Definition 1.3.1. Let $\mathcal{S}$ be a site, and let $\pi : \mathcal{G} \to \mathcal{S}$ be a category fibered in groupoids. We view $\mathcal{G}$ as a site with the Grothendieck topology inherited from $\mathcal{S}$ [7, III, 3.1]. For any object $U \in \mathcal{S}$, let $\mathcal{G}(U)$ denote the fiber category of $\mathcal{G}$ over U.

The inertia stack of $\pi : \mathcal{G} \to \mathcal{S}$ is the 2-fiber product $I_{\mathcal{G}/\mathcal{S}} := \mathcal{G} \times_{\Delta_{\mathcal{G}/\mathcal{S}}, \mathcal{G} \times_{\mathcal{S}} \mathcal{G}, \Delta_{\mathcal{G}/\mathcal{S}}} \mathcal{G}$ of the diagonal $\Delta_{\mathcal{G}/\mathcal{S}} : \mathcal{G} \to \mathcal{G} \times_{\mathcal{S}} \mathcal{G}$ with itself. The inertia stack $I_{\mathcal{G}/\mathcal{S}}$ is fibered in sets over $\mathcal{G}$ via either projection $I_{\mathcal{G}/\mathcal{S}} \to \mathcal{G}$, hence we may identify $I_{\mathcal{G}/\mathcal{S}}$ with the sheaf of groups on $\mathcal{G}$ associating $x \mapsto \mathrm{Aut}_{\mathcal{G}}(x)$.

We say that π is a gerbe if the following conditions are satisfied:

(i) The fibered category $\mathcal{G}$ is a stack over $\mathcal{S}$.
(ii) For any $U \in \mathcal{S}$, there exists a covering $\{U_i \to U\}_{i \in I}$ such that $\mathcal{G}(U_i) \neq \emptyset$ for all $i \in I$.
(iii) For any $U \in \mathcal{S}$ and $x_1, x_2 \in \mathcal{G}(U)$, there exists a covering $\{U_i \to U\}_{i \in I}$ such that for all $i \in I$ there exists an isomorphism $x_1|_{U_i} \simeq x_2|_{U_i}$ in $\mathcal{G}(U_i)$.

Let $\mathbf{A}$ be an abelian sheaf on $\mathcal{S}$. We say that a gerbe π is an $\mathbf{A}$-gerbe if it is equipped with an isomorphism

$$\iota : \mathbf{A}_{\mathcal{G}} \to I_{\mathcal{G}/\mathcal{S}} \tag{1.3.1.1}$$

of sheaves of groups on $\mathcal{G}$.

If $\mathcal{S}$ is equipped with a sheaf of rings such that $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ is a locally ringed site, we set $\mathcal{O}_{\mathcal{G}} := \pi^{-1}\mathcal{O}_{\mathcal{S}}$; then the pair $(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$ is a locally ringed site.

An $\mathbf{A}$-gerbe $\mathcal{G}$ is called trivial if it has a global section, i.e. the fiber category $\mathcal{G}(\mathcal{S})$ is nonempty. In this case, for each global object $x \in \mathcal{G}(\mathcal{S})$, there is a morphism of $\mathbf{A}$-gerbes $\mathrm{B}\mathbf{A} \to \mathcal{G}$ from the classifying stack to $\mathcal{G}$, which is necessarily an isomorphism by [76, 12.2.4]. Thus for any object $x \in \mathcal{G}$ lying over $U := \pi(x)$, the restriction $\mathcal{G}_U$ of $\mathcal{G}$ to the slice category $\mathcal{S}/U$ is a trivial $\mathbf{A}_U$-gerbe. □

Theorem 1.3.2. [40], [76, 12.2.8] Let $\mathcal{S}$ be a site and let $\mathbf{A}$ be an abelian sheaf on $\mathcal{S}$. There is a bijective correspondence between isomorphism classes of $\mathbf{A}$-gerbes and classes in $\mathrm{H}^2(\mathcal{S}, \mathbf{A})$.

1.3.3 (Pullback of gerbes). Let $f : X \to Y$ be a morphism of sites, let $\mathbf{A}$ (resp. $\mathbf{B}$) be an abelian sheaf on X (resp. Y), let $\varphi : \mathbf{B} \to f_*\mathbf{A}$ be a morphism of abelian sheaves on Y, and let $\mathscr{Y}$ be a $\mathbf{B}$-gerbe on Y. Let

$$\mathscr{X} := f^{-1}\mathscr{Y}$$

denote the inverse image gerbe (see Appendix C) of $\mathscr{Y}$ by φ. It is an $\mathbf{A}$-gerbe naturally equipped with a morphism

$$F : \mathscr{X} \to \mathscr{Y}$$

of sites. Suppose P is a property of morphisms of locally ringed sites which is local on the target (e.g. flat, finite locally free). If f has P, then F has P, since locally on Y the morphism F is of the form $\mathrm{B}\mathbb{G}_{m,X} \to \mathrm{B}\mathbb{G}_{m,Y}$, which is locally on Y a pullback of f.

For the remainder of this section, we will assume the following setup:

Setup 1.3.4. Let $\mathcal{S}$ be a locally ringed site, let $\mathbf{A}$ be an abelian sheaf on $\mathcal{S}$, let $\pi : \mathcal{G} \to \mathcal{S}$ be an $\mathbf{A}$-gerbe.

Definition 1.3.5 (Inertial action, eigensheaves, twisted sheaves). [59], [60] Let $\mathscr{F}$ be an $\mathcal{O}_{\mathcal{G}}$-module. For an object $x \in \mathcal{G}$ and $a \in \Gamma(x, \mathbf{A}_{\mathcal{G}})$, let $\iota(a)^* : \Gamma(x, \mathscr{F}) \to \Gamma(x, \mathscr{F})$ be the restriction map of the sheaf $\mathscr{F}$ via the automorphism $\iota(a) : x \to x$; such x and a defines an $\mathcal{O}_{\mathcal{G}/x}$-linear automorphism of $\mathscr{F}|_{\mathcal{G}/x}$ by $\{y \to x\} \mapsto \iota(a|_y)^*$; thus we have a homomorphism $\mathbf{A}_{\mathcal{G}} \to \underline{\mathrm{Aut}}_{\mathcal{O}_{\mathcal{G}}}(\mathscr{F})$ of group sheaves on $\mathcal{G}$ corresponding to an $\mathcal{O}_{\mathcal{G}}$-linear $\mathbf{A}_{\mathcal{G}}$-action on $\mathscr{F}$, called the inertial action.

Let

$$\widehat{\mathbf{A}} := \mathrm{Hom}_{\mathrm{Ab}(\mathcal{S})}(\mathbf{A}, \mathbb{G}_{m,\mathcal{S}})$$

denote the group of characters of $\mathbf{A}$. Given an $\mathcal{O}_{\mathcal{G}}$-module $\mathscr{F}$ and a character $\chi \in \widehat{\mathbf{A}}$, the χth eigensheaf is the subsheaf

$$\mathscr{F}_\chi \subseteq \mathscr{F}$$

defined as follows: for all objects $x \in \mathcal{G}$, a section $f \in \Gamma(x, \mathscr{F})$ is contained in $\Gamma(x, \mathscr{F}_\chi)$ if for any morphism $y \to x$ and any $a \in \Gamma(y, \mathbf{A}_\mathcal{G})$ we have

$$(\iota(a)^*)(f|_y) = \chi_\mathcal{G}(a) \cdot f|_y \tag{1.3.5.1}$$

in $\Gamma(y, \mathscr{F})$. The $\mathcal{O}_\mathcal{G}$-module $\mathscr{F}$ is called χ-twisted if the inclusion $\mathscr{F}_\chi \subseteq \mathscr{F}$ is an equality. (By this definition, the zero module 0 is χ-twisted for any $\chi \in \widehat{\mathbf{A}}$.) The eigensheaf $\mathscr{F}_n$ is an $\mathcal{O}_\mathcal{G}$-submodule of $\mathscr{F}$ since $\mathrm{Aut}_{\mathcal{G}(\pi(x))}(x)$ acts trivially on $\Gamma(x, \mathcal{O}_\mathcal{G})$ by definition of $\mathcal{O}_\mathcal{G}$. We have a canonical map

$$\textstyle\bigoplus_{\chi \in \widehat{\mathbf{A}}} \mathscr{F}_\chi \to \mathscr{F} \tag{1.3.5.2}$$

which is in general neither injective nor surjective for arbitrary locally ringed sites (e.g. Example 1.3.10).

In case $\mathbf{A} = \mathbb{G}_{m,\mathcal{S}}$, for any integer $n \in \mathbb{Z}$ we have a character $\chi_n : \mathbb{G}_m \to \mathbb{G}_m$ corresponding to the nth power map, and the eigensheaf $\mathscr{F}_{\chi_n}$ is denoted $\mathscr{F}_n$ and χ_n-twisted sheaves are called n-twisted. For any $\mathcal{O}_\mathcal{S}$-module M, the pullback π^*M is 0-twisted; in particular the structure sheaf $\mathcal{O}_\mathcal{G}$ is 0-twisted. □

Lemma 1.3.6. [22, 2.11], [60, 3.1.1.7] For $i = 1, 2$, let $\mathscr{F}_i$ be a χ_i-twisted $\mathcal{O}_\mathcal{G}$-module. Then $\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{G}\text{-mod}}(\mathscr{F}_1, \mathscr{F}_2)$ is $(\chi_1^{-1} \cdot \chi_2)$-twisted and $\mathscr{F}_1 \otimes_{\mathcal{O}_\mathcal{G}} \mathscr{F}_2$ is $(\chi_1 \cdot \chi_2)$-twisted.

Proof. Let $x \in \mathcal{G}$ be an object and let

$$\varphi \in \Gamma(x, \underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{G}\text{-mod}}(\mathscr{F}_1, \mathscr{F}_2)) = \mathrm{Hom}_{\mathcal{O}_\mathcal{G}|_x}(\mathscr{F}_1|_x, \mathscr{F}_2|_x)$$

be a section. For any morphism $y \to x$ and element $a \in \Gamma(y, \mathbf{A}_\mathcal{G})$, the restriction $(\iota(a)^*)(\varphi|_y)$ sends $\chi_1(a) \cdot s \mapsto \chi_2(a) \cdot \varphi(s)$, which is the same as sending $s \mapsto (\chi_1(a)^{-1} \cdot \chi_2(a)) \cdot \varphi(s)$.

Sections of $\Gamma(x, \mathscr{F}_1 \otimes_{\mathcal{O}_\mathcal{G}} \mathscr{F}_2)$ are locally sums of elements of the form $s_1 \otimes s_2$ for $s_i \in \Gamma(x, \mathscr{F}_i)$, and we have $(\iota(a)^*)(s_1 \otimes s_2) = (\iota(a)^*)s_1 \otimes (\iota(a)^*)s_2 = (\chi_1(a) \cdot s_1) \otimes (\chi_2(a) \cdot s_2) = (\chi_1(a) \cdot \chi_2(a)) \cdot (s_1 \otimes s_2)$. □

Lemma 1.3.7 (Pushforward of twisted sheaves). Assume the setup of 1.3.3, and let $\mathscr{F}$ be a $\chi_\mathbf{A}$-twisted $\mathcal{O}_\mathscr{X}$-module. For any character $\chi_\mathbf{B} \in \widehat{\mathbf{B}}$ making the diagram

$$\begin{array}{ccc} f_*\mathbf{A} & \xrightarrow{f_*\chi_\mathbf{A}} & f_*\mathbb{G}_{m,X} \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle f^\flat} \\ \mathbf{B} & \xrightarrow[\chi_\mathbf{B}]{} & \mathbb{G}_{m,Y} \end{array}$$

commute, the pushforward $F_*\mathscr{F}$ is χ_B-twisted.

Proof. See C.0.6. □

Lemma 1.3.8 (Functoriality under localization). Assume the setup of Definition 1.3.5. Let $\mathscr{F}$ be an $\mathcal{O}_\mathcal{G}$-module, let $\chi : \mathbf{A} \to \mathbb{G}_{m,\mathcal{S}}$ be a character, and let $\{\mathcal{S}_i \to \mathcal{S}\}_{i \in I}$ be a covering. The following are equivalent:

(i) $\mathscr{F}$ is χ-twisted.
(ii) $\mathscr{F}|_{\mathcal{S}_i}$ is $\chi|_{\mathcal{S}_i}$-twisted for all i.

Proof. (i)⇒(ii): by definition (see (1.3.5.1)).

(ii)⇒(i): Let $x \in \mathcal{G}$ be an object and let $f \in \Gamma(x, \mathscr{F})$ be a section. Let $y \to x$ be a morphism in $\mathcal{G}$; then the covering $\{\mathcal{S}_i \to \mathcal{S}\}_{i \in I}$ defines coverings $\{x_i \to x\}_{i \in I}$ and $\{y_i \to y\}_{i \in I}$ by pullback. For any section $a \in \Gamma(y, \mathbf{A}_{\mathcal{G}})$, we have

$$((\iota(a)^*)(f|_y))|_{y_i} = (\iota(a|_{y_i})^*)(f|_{y_i}) \overset{\dagger}{=} \chi_{\mathcal{G}}(a|_{y_i}) \cdot f|_{y_i} = (\chi_{\mathcal{G}}(a) \cdot f|_y)|_{y_i}$$

where equality † follows from (ii); hence $(\iota(a)^*)(f|_y) = \chi_{\mathcal{G}}(a) \cdot f|_y$. □

Proposition 1.3.9. Assume the setup of Definition 1.3.5. If $\mathcal{S}$ is an algebraic stack and $\mathscr{F}$ is quasi-coherent and $\mathbf{A}$ is a diagonalizable group scheme, the map (1.3.5.2) is an isomorphism.

Proof. This is proved in [11, 4.7]; we give the outline here. The case when $\mathcal{S}$ is a scheme is [59, 2.2.1.6]. For any scheme S admitting a map $S \to \mathcal{S}$, the restriction $\mathcal{G}|_S$ of $\mathcal{G}$ to (Sch/S) is a $\mathbf{A}_S$-gerbe, and the map (1.3.5.2) is an isomorphism for the restriction $\mathscr{F}|_{\mathcal{G}|_S}$, thus we obtain the desired result by Lemma 1.3.8. □

Example 1.3.10. Here we give examples of a locally ringed site $\mathcal{S}$, a $\mathbb{G}_{m,\mathcal{S}}$-gerbe $\mathcal{G}$, and a quasi-coherent $\mathcal{O}_{\mathcal{G}}$-module $\mathscr{F}$ for which (1.3.5.2) is not an isomorphism. Let k be a field, and let $\mathcal{S}$ be the topological space consisting of a single point $\{*\}$ and let $\mathcal{O}_{\mathcal{S}} = k$ (the small Zariski site of k). Let $\mathcal{G} := \mathrm{B}\mathbb{G}_{m,\mathcal{S}}$ be the trivial $\mathbb{G}_{m,\mathcal{S}}$-gerbe; then $\mathcal{G}$ is the groupoid consisting of a single object ξ and $\mathrm{Aut}_{\mathcal{G}}(\xi, \xi) \simeq k^\times$, equipped with the ring $\Gamma(\xi, \mathcal{O}_{\mathcal{G}}) = k$, and each element of $k^\times$ acts on $\Gamma(\xi, \mathcal{O}_{\mathcal{G}})$ by the identity. Quasi-coherent $\mathcal{O}_{\mathcal{G}}$-modules are k-vector spaces V equipped with a group homomorphism $\rho : k^\times \to \mathrm{Aut}_k(V)$. The nth eigensheaf is the subspace $V_n \subseteq V$ consisting of elements v such that $(\rho(u))(v) = u^n v$ for all $u \in k^\times$.

(i) Let $k = \mathbb{Q}$, let $V = k$, and let ρ be the map sending $u \mapsto \{v \mapsto -v\}$ if u is negative and $u \mapsto \mathrm{id}_V$ if u is positive. Suppose v is a nonzero vector contained in V_n for some n. Then $(\rho(2))(v) = v = 2^n v$, so $n = 0$ since $v \neq 0$. But we have $(\rho(-1))(v) = -v = (-1)^n v$, which is a contradiction. Hence $V_n = 0$ for all $n \in \mathbb{Z}$. Thus (1.3.5.2) is injective but not surjective.

(ii) Let $k = \mathbb{F}_p$, let $V = k$, and let ρ be the standard representation, i.e. $(\rho(u))(v) = uv$ for all $u \in k^\times$. Thus $V_1 = V$, however $V_{1+n(p-1)} = V_1$ for all $n \in \mathbb{Z}$, hence (1.3.5.2) is surjective but not injective.

Remark 1.3.11. We will most frequently apply Lemma 1.3.7 in the case $\mathbf{A} = \mathbb{G}_{m,X}$ and $\mathbf{B} = \mathbb{G}_{m,Y}$ and $\chi_{\mathbf{A}} = \mathrm{id}_{\mathbb{G}_{m,X}}$ and $\chi_{\mathbf{B}} = \mathrm{id}_{\mathbb{G}_{m,Y}}$.

Definition 1.3.12 (Category of χ-twisted modules). For a character $\chi \in \widehat{\mathbf{A}}$, let

$$\mathrm{Mod}(\mathcal{G}, \chi)$$

denote the full subcategory of $\mathrm{Mod}(\mathcal{G})$ consisting of χ-twisted $\mathcal{O}_{\mathcal{G}}$-modules. □

Remark 1.3.13. Given two $\mathcal{O}_{\mathcal{G}}$-modules $\mathscr{F}$ and $\mathscr{G}$, any $\mathcal{O}_{\mathcal{G}}$-linear morphism $\varphi : \mathscr{F} \to \mathscr{G}$ restricts to an $\mathcal{O}_{\mathcal{G}}$-linear morphism $\varphi_\chi : \mathscr{F}_\chi \to \mathscr{G}_\chi$; the assignment $\mathscr{F} \mapsto \mathscr{F}_\chi$ defines a functor $\mathrm{Mod}(\mathcal{G}) \to \mathrm{Mod}(\mathcal{G}, \chi)$ which is right adjoint (and a retraction) to the inclusion $\mathrm{Mod}(\mathcal{G}, \chi) \to \mathrm{Mod}(\mathcal{G})$.

Remark 1.3.14 (Modules on trivial gerbes). We say that an $\mathbf{A}$-gerbe $\mathcal{G}$ is trivial if there is an isomorphism $\mathcal{G} \simeq \mathrm{B}\mathbf{A}$. In this case we have the usual equivalence of categories between sheaves on $\mathcal{G}$ and sheaves on $\mathcal{S}$ equipped with an $\mathbf{A}$-action. For a sheaf $\mathscr{F} \in \mathrm{Sh}(\mathrm{B}\mathbf{A})$, the pushforward $\pi_*\mathscr{F}$ is identified with the subsheaf of $\mathscr{F}$ of sections invariant under the action of $\mathbf{A}$. For any sheaf $M \in \mathrm{Sh}(\mathcal{S})$, the inverse image $\pi^{-1}M \in \mathrm{Sh}(\mathrm{B}\mathbf{A})$ corresponds to the sheaf M equipped with the trivial $\mathbf{A}$-action. If $s : \mathcal{S} \to \mathrm{B}\mathbf{A}$ is the section of π corresponding to the trivial $\mathbf{A}$-torsor, then $s^{-1} : \mathrm{Sh}(\mathrm{B}\mathbf{A}) \to \mathrm{Sh}(\mathcal{S})$ is the functor forgetting the $\mathbf{A}$-action.

Remark 1.3.15. For any $\mathcal{O}_{\mathcal{G}}$-module $\mathscr{F}$, the counit map

$$\pi^*\pi_*\mathscr{F} \to \mathscr{F}$$

is injective and its image coincides with $\mathscr{F}_0$. Indeed, this can be checked locally on $\mathcal{S}$, in which case we may assume $\mathcal{G}$ is the trivial gerbe and use Remark 1.3.14.

Lemma 1.3.16. Let $\mathcal{S}$ be a locally ringed site, let $\mathbf{A}$ be an abelian sheaf on $\mathcal{S}$, and let $\pi : \mathcal{G} \to \mathcal{S}$ be an $\mathbf{A}$-gerbe. The pullback functor

$$\pi^* : \mathrm{Mod}(\mathcal{S}) \to \mathrm{Mod}(\mathcal{G}, 0)$$

is an equivalence of categories with quasi-inverse π_*. If P is a property of modules preserved by pullback via arbitrary morphisms of sites (e.g. quasi-coherent, flat, locally of finite type, locally of finite presentation, locally free), an $\mathcal{O}_{\mathcal{S}}$-module M has P if and only if the $\mathcal{O}_{\mathcal{G}}$-module π^*M has P.

Proof. For the first assertion, it suffices to show that for any $\mathcal{O}_{\mathcal{S}}$-module M the unit map

$$M \to \pi_*\pi^*M$$

is an isomorphism, and that for any 0-twisted $\mathcal{O}_{\mathcal{G}}$-module $\mathscr{F}$ the counit map

$$\pi^*\pi_*\mathscr{F} \to \mathscr{F}$$

is an isomorphism. Both of these claims are local on $\mathcal{S}$, hence we may assume that $\mathcal{G}$ is trivial, in which case the claims follow from Remark 1.3.14 and Remark 1.3.15. The second assertion is also local on $\mathcal{S}$, hence we may assume that $\mathcal{G}$ is trivial, in which case there is a section $s : \mathcal{S} \to \mathcal{G}$ of π. For any 0-twisted $\mathcal{O}_{\mathcal{G}}$-module $\mathscr{F}$, we have $\pi_*\mathscr{F} \simeq s^*\mathscr{F}$ by the discussion in Remark 1.3.14. □

1.4. **Brauer map.** Giraud [40] defined a functorial map from the Brauer group $\mathrm{Br}\, X$ to the étale cohomology group $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$, called the Brauer map. In cases where the Brauer map is an isomorphism, we may compute $\mathrm{Br}\, X$ using cohomological techniques. In this section we define the Brauer map and record a necessary and sufficient condition (Lemma 1.4.5) which allows us to determine when a class $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ lies in the image of the Brauer map.

Definition 1.4.1 (Gerbe of trivializations). [40, IV, §4.2], [41, §2], [76, 12.3.5, 12.3.6] There is a natural way to associate, to every Azumaya $\mathcal{O}_X$-algebra $\mathcal{A}$, a $\mathbb{G}_{m,X}$-gerbe

$$\mathcal{G}_{\mathcal{A}}$$

called the gerbe of trivializations of $\mathcal{A}$. An object of $\mathcal{G}_{\mathcal{A}}$ is a triple

$$(U, \mathcal{E}, \sigma)$$

consisting of an object $U \in X$, a finite type locally free $\mathcal{O}_U$-module $\mathcal{E}$ (necessarily everywhere positive rank), and an isomorphism $\sigma : \underline{\mathrm{End}}_{\mathcal{O}_U\text{-mod}}(\mathcal{E}) \to \mathcal{A}|_U$ of $\mathcal{O}_U$-algebras. A morphism

$$(f, f^\sharp) : (U_1, \mathcal{E}_1, \sigma_1) \to (U_2, \mathcal{E}_2, \sigma_2)$$

consists of a morphism $f \in \mathrm{Mor}_X(U_1, U_2)$ and an isomorphism $f^\sharp : f^*\mathcal{E}_2 \to \mathcal{E}_1$ of $\mathcal{O}_{U_1}$-modules such that $\sigma_2 = \sigma_1 \circ \rho_{f^\sharp}$ where $\rho_{f^\sharp}$ denotes conjugation by $f^\sharp$. The category $\mathcal{G}_{\mathcal{A}}$

comes equipped with a projection $p_{\mathcal{A}} : \mathcal{G}_{\mathcal{A}} \to X$ sending $(U, \mathcal{E}, \sigma) \mapsto U$ on objects and $(f, f^\sharp) \mapsto f$ on morphisms, and $p_{\mathcal{A}}$ presents $\mathcal{G}_{\mathcal{A}}$ as a stack in groupoids over X. For any object $(U, \mathcal{E}, \sigma) \in \mathcal{G}_{\mathcal{A}}$ there is a canonical injection

$$\iota_{(U,\mathcal{E},\sigma)} : \mathbb{G}_{m,U} \to \underline{\mathrm{Aut}}_{(U,\mathcal{E},\sigma)}$$

of sheaves on X/U, sending $u \mapsto (\mathrm{id}_U, u)$. This injection is an isomorphism, since if $(\mathrm{id}_U, f^\sharp) \in \mathrm{Aut}_{\mathcal{G}_{\mathcal{A}}(U)}((U, \mathcal{E}, \sigma))$ then $f^\sharp \in Z(\mathrm{End}_{\mathcal{O}_U\text{-mod}}(\mathcal{E}))$, which coincides with $\mathcal{O}_U$ since $Z(\mathrm{Mat}_{n\times n}(A)) = A$ for any commutative, unital ring A.

By the Skolem-Noether theorem Theorem 1.1.8, any two local trivializations of $\mathcal{A}$ are locally related by an automorphism of the trivializing vector bundle $\mathcal{E}$, i.e. any two objects of $\mathcal{G}_{\mathcal{A}}$ are locally isomorphic. Furthermore, according to the definition, an Azumaya algebra is locally trivial, i.e. for any $U \in X$ there exists a covering $\{U_i \to U\}$ such that the fiber category $\mathcal{G}_{\mathcal{A}}(U_i)$ is nonempty. The above considerations show that $\mathcal{G}_{\mathcal{A}}$ is a $\mathbb{G}_{m,X}$-gerbe.

Definition 1.4.2 (cohomological Brauer group, Brauer map). [40, V, §4] The assignment $\mathcal{A} \mapsto \mathcal{G}_{\mathcal{A}}$ of the gerbe of trivializations to an Azumaya algebra induces a group homomorphism

$$\alpha'_X : \mathrm{Br}\, X \to \mathrm{H}^2(X, \mathbb{G}_{m,X}) \tag{1.4.2.1}$$

whose image is contained in the $\Gamma(X, \underline{\mathbb{Z}})$-torsion subgroup (see Definition 1.2.10)

$$\mathrm{Br}'\, X := \mathrm{H}^2(X, \mathbb{G}_{m,X})_{\mathrm{tors}}$$

which is called the cohomological Brauer group. The restriction

$$\alpha_X : \mathrm{Br}\, X \to \mathrm{Br}'\, X \tag{1.4.2.2}$$

is called the Brauer map. The map α'_X (hence also α_X) is injective (a $\mathbb{G}_{m,X}$-gerbe is trivial if and only if it has a global object).

1.4.3 (Functoriality of the Brauer map). Let

$$(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

be a morphism of locally ringed sites. The diagram

$$\begin{array}{ccc} \mathrm{Br}\, X & \xrightarrow{\alpha'_X} & \mathrm{H}^2(X, \mathbb{G}_{m,X}) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{Br}\, Y & \xrightarrow[\alpha'_Y]{} & \mathrm{H}^2(Y, \mathbb{G}_{m,Y}) \end{array} \tag{1.4.3.1}$$

commutes, as verified in C.0.7.

1.4.4 (Tautological line bundle). On the classifying stack $\mathrm{B}\mathbb{G}_{m,X}$, there is a canonical invertible sheaf χ, called the tautological line bundle, which assigns to every $\mathbb{G}_{m,U}$-torsor $\mathscr{U} \in \mathrm{B}\mathbb{G}_{m,X}(U)$ the global sections $\Gamma(U, \mathcal{L})$ of the associated invertible $\mathcal{O}_U$-module $\mathcal{L}$.

For a general trivial gerbe $\mathscr{X}$, one can associate an invertible $\mathcal{O}_{\mathscr{X}_U}$-module $\chi_{\mathscr{U}}$ to every global object $\mathscr{U}$ by pushing forward via the induced isomorphism of $\mathbb{G}_{m,U}$-gerbes $\mathrm{B}\mathbb{G}_{m,U} \to \mathscr{X}_U$ the tautological bundle on $\mathrm{B}\mathbb{G}_{m,U}$. For any morphism $\mathscr{U}_1 \to \mathscr{U}_2$, there is an isomorphism

$$\chi_{\mathscr{U}_2}|_{\mathscr{X}_{U_1}} \to \chi_{\mathscr{U}_1} \tag{1.4.4.1}$$

which is compatible with compositions $\mathscr{U}_1 \to \mathscr{U}_2 \to \mathscr{U}_3$.

Lemma 1.4.5. [76, 12.3.11], [60, 3.1.2.1] Let $\mathscr{X}$ be a $\mathbb{G}_{m,X}$-gerbe over a locally ringed site X. The class $[\mathscr{X}] \in \mathrm{H}^2(X, \mathbb{G}_{m,X})$ is in the image of α'_X if and only if $\mathscr{X}$ admits a 1-twisted finite locally free $\mathcal{O}_{\mathscr{X}}$-module of positive rank.

Proof. Let $\mathcal{A}$ be an Azumaya $\mathcal{O}_X$-algebra and let $\mathscr{X} := \mathcal{G}_{\mathcal{A}}$ be its gerbe of trivializations Definition 1.4.1. Let $\mathscr{E}$ be the $\mathcal{O}_{\mathscr{X}}$-module assigning

$$(U, \mathcal{E}, \sigma) \mapsto \Gamma(U, \mathcal{E})$$

on objects and

$$\{(f, f^\sharp) : (U_1, \mathcal{E}_1, \sigma_1) \to (U_2, \mathcal{E}_2, \sigma_2)\} \mapsto \{f^\sharp : \Gamma(U_2, \mathcal{E}_2) \to \Gamma(U_1, \mathcal{E}_1)\}$$

on morphisms of $\mathscr{X}$. Then $\mathscr{E}$ is finite locally free sheaf of everywhere positive rank, and it is 1-twisted since an automorphism $(\mathrm{id}_U, u) \in \mathrm{Aut}_{\mathscr{X}(U)}((U, \mathcal{E}, \sigma))$ with $u \in \Gamma(U, \mathbb{G}_{m,X})$ acts on $\Gamma(U, \mathcal{E})$ by multiplication-by-u.

Conversely, suppose $\mathscr{X}$ is a $\mathbb{G}_{m,X}$-gerbe admitting a 1-twisted locally free sheaf of everywhere positive rank, say $\mathscr{E}$. The endomorphism algebra

$$\mathscr{A} := \underline{\mathrm{End}}_{\mathcal{O}_{\mathscr{X}}\text{-mod}}(\mathscr{E})$$

is an $\mathcal{O}_{\mathscr{X}}$-algebra which is 0-twisted Lemma 1.3.6 and finite locally free as an $\mathcal{O}_{\mathscr{X}}$-module; set

$$\mathcal{A} := \pi_* \mathscr{A}$$

which is an Azumaya $\mathcal{O}_X$-algebra such that the canonical map $\pi^* \mathcal{A} \to \mathscr{A}$ is an isomorphism by Lemma 1.3.16. To show that $\mathscr{X}$ is isomorphic to $\mathcal{G}_{\mathcal{A}}$, it suffices by [76, 12.2.4] to construct a morphism of $\mathbb{G}_{m,X}$-gerbes $\mathscr{X} \to \mathcal{G}_{\mathcal{A}}$ over X. Given an object $U \in X$, let $\mathscr{X}_U$ denote the restriction of $\mathscr{X}$ to the slice category X/U and let $\pi_U : \mathscr{X}_U \to X/U$ denote the restriction of π. Let $\mathscr{U} \in \mathscr{X}(U)$ be an object of the fiber category. Recall 1.4.4 that $\chi_{\mathscr{U}}$ is a 1-twisted invertible $\mathcal{O}_{\mathscr{X}_U}$-module associated to $\mathscr{U}$; then

$$\mathcal{E}_{\mathscr{U}} := \pi_{U,*}(\mathscr{E}|_{\mathscr{X}_U} \otimes_{\mathcal{O}_{\mathscr{X}_U}} \chi_{\mathscr{U}}^{-1})$$

is a finite locally free $\mathcal{O}_U$-module which trivializes $\mathcal{A}|_{X/U}$ via the isomorphism

$$\sigma_{\mathscr{U}} : \mathcal{A}|_{X/U} \simeq \pi_{U,*}(\mathscr{A}|_{\mathscr{X}_U}) \simeq \pi_{U,*}(\underline{\mathrm{End}}_{\mathcal{O}_{\mathscr{X}_U}}(\mathscr{E}|_{\mathscr{X}_U})) \simeq \underline{\mathrm{End}}_{\mathcal{O}_U}(\mathcal{E}_{\mathscr{U}})$$

of $\mathcal{O}_U$-algebras. Let $\mathscr{X} \to \mathcal{G}_{\mathcal{A}}$ be the functor sending

$$\mathscr{U} \mapsto (\pi(\mathscr{U}), \mathcal{E}_{\mathscr{U}}, \sigma_{\mathscr{U}})$$

on objects and

$$\{\mathscr{U}_1 \to \mathscr{U}_2\} \to (\{\pi(\mathscr{U}_1) \to \pi(\mathscr{U}_2)\}, \{\mathcal{E}_{\mathscr{U}_2}|_{\pi(\mathscr{U}_1)} \to \mathcal{E}_{\mathscr{U}_1}\})$$

on morphisms, where $\mathcal{E}_{\mathscr{U}_2}|_{\pi(\mathscr{U}_1)} \to \mathcal{E}_{\mathscr{U}_1}$ is the map induced by (1.4.4.1). □

1.5. **The cup product and cyclic algebras.** In this subsection, we discuss cyclic algebras on arbitrary locally ringed sites.

1.5.1 (Cyclic algebras via cocycles). Let X be a locally ringed site, let n be an integer, let $\underline{\mathbb{Z}/(n)}$ denote the constant sheaf on X associated to $\mathbb{Z}/(n)$. The natural bilinear map

$$\underline{\mathbb{Z}/(n)} \times \mu_n \to \mu_n$$

of sheaves defined by $(\alpha, \xi) \mapsto \xi^\alpha$ induces the cup product

$$\cup : \mathrm{H}^1(X, \underline{\mathbb{Z}/(n)}) \times \mathrm{H}^1(X, \mu_n) \to \mathrm{H}^2(X, \mu_n)$$

in cohomology. We can represent the classes

$$\alpha \in \mathrm{H}^1(X, \underline{\mathbb{Z}/(n)})$$

and

$$\xi \in \mathrm{H}^1(X, \mu_n)$$

as Cech 1-cocycles

$$\alpha := \{\alpha_{i_0,i_1}\}_{i_0,i_1 \in I}$$

and

$$\xi := \{\xi_{i_0,i_1}\}_{i_0,i_1 \in I}$$

for some covering

$$\mathfrak{U} := \{X_i \to X\}_{i \in I}$$

after a common refinement, if necessary; here $\alpha_{i_0,i_1} \in \Gamma(X_{i_0} \times_X X_{i_1}, \underline{\mathbb{Z}/(n)})$ and satisfies the usual cocycle condition on the triple intersections $X_{i_0} \times_X X_{i_1} \times_X X_{i_2}$, and similarly for ξ_{i_0,i_1}. Then there is a commutative diagram

$$\begin{array}{ccc} \check{\mathrm{H}}^1(\mathfrak{U}, \underline{\mathbb{Z}/(n)}) \times \mathrm{H}^1(\mathfrak{U}, \mu_n) & \xrightarrow{\cup} & \check{\mathrm{H}}^2(\mathfrak{U}, \mu_n) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(X, \underline{\mathbb{Z}/(n)}) \times \mathrm{H}^1(X, \mu_n) & \xrightarrow[\cup]{} & \mathrm{H}^2(X, \mu_n) \end{array}$$

of abelian groups and the cup product $\alpha \cup \xi$ is given by the Cech 2-cocycle

$$(\alpha \cup \xi)_{i_0,i_1,i_2,} := (\xi_{i_1,i_2})^{\alpha_{i_0,i_1}}$$

for $i_0, i_1, i_2 \in I$.

We construct the Azumaya $\mathcal{O}_X$-algebra $\mathcal{A}_{\alpha,\xi}$ by gluing the trivial Azumaya $\mathcal{O}_{X_i}$-algebras $\mathrm{Mat}_{n\times n}(\mathcal{O}_{X_i})$ on the intersections $X_{i_1} \times_X X_{i_2}$.

(1) *Definition of* $\mathcal{A}_{\alpha,\xi}|_{X_{i_0}}$: Given a ring A, the A-module $A^{\oplus n}$ is a ring with coordinatewise addition and multiplication; let $\{e_i\}_{i \in \mathbb{Z}/(n)}$ be the standard A-basis of $A^{\oplus n}$ as an A-module; the diagonal map $A \to A^{\oplus n}$ endows $A^{\oplus n}$ with the structure of A-algebra. Let

$$\mathbf{C}_n(A) := (A^{\oplus n})\langle x \rangle / (x^n = 1, \{e_i x = x e_{i+1}\}_{i \in \mathbb{Z}/(n)})$$

be the noncommutative associative A-algebra where the structure map $A \to \mathbf{C}_n(A)$ is via the A-algebra structure on $A^{\oplus n}$. We have that $\mathbf{C}_n(A)$ is a free A-module of rank n^2, and the collection

$$\{e_i x^j\}_{i,j \in \mathbb{Z}/(n)}$$

constitutes a basis for $\mathbf{C}_n(A)$ as an A-module. One may check that the multiplication structure on $\mathbf{C}_n(A)$ satisfies

$$e_{i_1} x^{j_1} \cdot e_{i_2} x^{j_2} = (e_{i_1} \cdot_{A^{\oplus n}} e_{i_2+j_1}) x^{j_1+j_2}$$

for all $i_\ell, j_\ell \in \mathbb{Z}/(n)$.

(2) *Verification that* $\mathbf{C}_n(A) \simeq \mathrm{Mat}_{n\times n}(A)$ *as* A*-algebras*: There is an A-algebra map

$$\mathbf{C}_n(A) \to \mathrm{Mat}_{n\times n}(A) \qquad (1.5.1.1)$$

sending

$$e_i \mapsto \mathrm{E}_{i,i}$$

and

$$x \mapsto \sum_{i\in\mathbb{Z}/(n)} \mathrm{E}_{i,i+1}$$

where $\mathrm{E}_{i,j} \in \mathrm{Mat}_{n\times n}(A)$ for $i,j \in \mathbb{Z}/(n)$ is the $n \times n$ matrix with 1 as the (i,j)th entry and 0s everywhere else.

(3) *Algebra automorphism on double intersections*: Suppose given $\alpha \in \underline{\mathbb{Z}/(n)}$ and $\xi \in \mu_n(A)$. After decomposing A into connected components, we may assume that α is in the image of the sheafification map $\Gamma(A, (\mathbb{Z}/(n))^{\mathrm{pre}}) \to \Gamma(A, \underline{\mathbb{Z}/(n)})$. The A-module map

$$\varphi_{\alpha,\xi} : \mathbf{C}_n(A) \to \mathbf{C}_n(A)$$

sending

$$e_i x^j \mapsto e_{i+\alpha}(\xi x)^j$$

respects the multiplication law[8] on $\mathbf{C}_n(A)$, thus $\varphi_{\alpha,\xi}$ is an A-algebra automorphism of $\mathbf{C}_n(A)$. Under the identification (1.5.1.1), this corresponds to the algebra automorphism of $\mathrm{Mat}_{n\times n}(A)$ given by the conjugation-by-$\mathrm{P}_\alpha\mathrm{D}_\xi$ map

$$s \mapsto (\mathrm{P}_\alpha\mathrm{D}_\xi)^{-1} s (\mathrm{P}_\alpha\mathrm{D}_\xi)$$

where $\mathrm{P}_\alpha := (\sum_{i\in\mathbb{Z}/(n)} \mathrm{E}_{i,i+1})^\alpha$ and $\mathrm{D}_\xi := \sum_{i\in\mathbb{Z}/(n)} \xi^i \mathrm{E}_{i,i}$. One may check that

$$\mathrm{D}_\xi\mathrm{P}_\alpha = \xi^\alpha \mathrm{P}_\alpha\mathrm{D}_\xi \tag{1.5.1.2}$$

so it does not matter whether we conjugate by $\mathrm{P}_\alpha\mathrm{D}_\xi$ or $\mathrm{D}_\xi\mathrm{P}_\alpha$.

(4) *Cocycle condition on triple intersections*: Given

$$\alpha_{12}, \alpha_{23}, \alpha_{13} \in \underline{\mathbb{Z}/(n)}$$

and

$$\xi_{12}, \xi_{23}, \xi_{13} \in \mu_n(A)$$

such that

$$\alpha_{13} = \alpha_{12} + \alpha_{23}$$

and

$$\xi_{13} = \xi_{12}\xi_{23}$$

we have

$$\varphi_{\alpha_{13},\xi_{13}} = \varphi_{\alpha_{12},\xi_{12}} \circ \varphi_{\alpha_{23},\xi_{23}} \tag{1.5.1.3}$$

since the image of $e_i x^j$ under the LHS is $\xi_{13}^j e_{i+\alpha_{13}} x^j$ and the image of $e_i x^j$ under the RHS is $\xi_{12}^j \xi_{23}^j e_{i+\alpha_{12}+\alpha_{23}} x^j$ for all $i,j \in \mathbb{Z}/(n)$.

Using the above, we obtain the desired cyclic Azumaya $\mathcal{O}_X$-algebra $\mathcal{A}_{\alpha,\xi}$.

We verify that the diagram

[8] Details: This comes down to the equality

$$(e_{i_1+\alpha}(\xi x)^{j_1}) \cdot (e_{i_2+\alpha}(\xi x)^{j_2}) = (e_{i_1+\alpha} \cdot_{A^{\oplus n}} e_{i_2+j_1+\alpha})(\xi x)^{j_1+j_2}$$

for all $i_\ell, j_\ell \in \mathbb{Z}/(n)$.

$$\begin{array}{ccc}
\mathrm{H}^1(X, \underline{\mathbb{Z}/(n)}) \times \mathrm{H}^1(X, \mu_n) & \xrightarrow{\cup} & \mathrm{H}^2(X, \mu_n) \\
\Big\downarrow {\scriptstyle f_2} & & \Big\downarrow {\scriptstyle f_3} \\
\operatorname{Br} X & \xrightarrow[f_1]{} & \mathrm{H}^2(X, \mathbb{G}_m)
\end{array}$$

commutes (here f_1 is the Brauer map and f_2 is the cyclic algebra construction as above and f_3 is the natural map induced by the inclusion $\mu_n \to \mathbb{G}_m$). Given a 1-cocycle α for $\underline{\mathbb{Z}/(n)}$ and a 1-cocycle ξ for μ_n as above, the image $f_1(f_2(\alpha, \xi))$ is the $\mathbb{G}_m$-gerbe $\mathfrak{X}_{\alpha,\xi}$ of trivializations of the algebra $\mathcal{A}_{\alpha,\xi}$ as above. We again assume that the covering $\mathfrak{U}$ trivializes both α and ξ. By (1), we have that the fiber categories $\mathfrak{X}_{\alpha,\xi}(X_i)$ are nonempty; choose trivializations of $\mathcal{A}_{\alpha,\xi}|_{X_i}$ as in (2); on double intersections $X_{i_0} \times_X X_{i_1}$, we obtain algebra automorphisms as in (3); on triple intersections $X_{i_0} \times_X X_{i_1} \times_X X_{i_2}$, we note that

$$(\mathrm{P}_{\alpha_{12}} \mathrm{D}_{\xi_{12}})(\mathrm{P}_{\alpha_{01}} \mathrm{D}_{\xi_{01}}) = \xi_{12}^{\alpha_{01}} \mathrm{P}_{\alpha_{02}} \mathrm{D}_{\xi_{02}}$$

by (1.5.1.2), where the difference $\xi_{12}^{\alpha_{01}}$ is the Cech 2-cocycle obtained via the composition $f_3 \circ \cup$. $\square$

2. Brauer groups of algebraic stacks: generalities and examples

2.1. **Surjectivity of Brauer map.** In this section we investigate the surjectivity of the Brauer map α_X (1.4.2.2).

For a scheme X, it is known that α_X is surjective (hence an isomorphism) in the following cases:

(1) if X is a 1-dimensional or 2-dimensional and regular (Grothendieck [42, Corollaire 2.2]),
(2) if X is quasi-compact and admits an ample line bundle (Gabber, see [22] and [80]),
(3) if X is the semi-separated union of two affine schemes (Gabber [37], see also [60, 3.1.4.5]),
(4) if X is a smooth toric variety over an algebraically closed field of characteristic 0 (DeMeyer-Ford [27, Theorem 1.1]),
(5) if X is a separated geometrically normal algebraic surface (Schröer [82]).

Recently S. Mathur proved a generalization of Schröer's result [82], removing the condition that X be of finite type over a field and allowing algebraic spaces:

Theorem 2.1.1 (Mathur). [65, Theorem 4.3.2] Let X be a separated, Noetherian algebraic space whose regular locus contains a dense open subset. Then for any $\alpha \in \mathrm{Br}'(X)$ there exists an open $U \subset X$ with $\mathrm{codim}(X \setminus U) \geq 3$ and $\alpha|_U \in \mathrm{Br}(U)$.

The first example of a scheme for which $\mathrm{Br} \neq \mathrm{Br}'$ was given by Edidin, Hassett, Kresch, Vistoli:

Example 2.1.2. [28, Corollary 3.11] Let X be two copies of $\operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^2)$ glued along the nonsingular locus (i.e. the origin). Then $\mathrm{Br}\, X \neq \mathrm{Br}'\, X$. (In [12] it is shown that $\mathrm{Br}\, X = 0$ and $\mathrm{Br}'\, X = \mathbb{Z}/(2)$. For the proof of [12, Lemma 4], we may also cite Gubeladze's theorem [46, Theorem 2.1].)

The following result states that a class in $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ may be represented by an Azumaya algebra after pulling back by a proper birational morphism. (In case X is finite type separated over a Noetherian affine scheme, we may also use Chow's lemma [88, 0200] and Gabber's theorem [22] to give the desired result.)

Theorem 2.1.3 (Bogomolov-Landia). [13] Let X be a Noetherian scheme, let $\gamma \in \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ be an element. There is a proper birational morphism $f : \overline{X} \to X$ such that $f^*\gamma$ is in the image of $\alpha_{\overline{X}}$.

Lemma 2.1.4. Let $f : X \to Y$ be a finitely presented, finite, flat, surjective morphism of algebraic stacks. A class $\beta \in \mathrm{H}^2(Y, \mathbb{G}_{m,Y})$ is in the image of α'_Y if and only if its pullback $f^*\beta \in \mathrm{H}^2(X, \mathbb{G}_{m,X})$ is in the image of α'_X.

Proof. (This is well-known, see [37, page 165, Lemma 4], [22, 2.14], [60, 3.1.3.5], etc.) Let $\mathscr{Y}$ be the $\mathbb{G}_{m,Y}$-gerbe corresponding to β. As in 1.3.3, let $\mathscr{X}$ be the inverse image of $\mathscr{Y}$ by f, and let $F : \mathscr{X} \to \mathscr{Y}$ be the induced morphism of algebraic stacks; here F is finite flat surjective by C.0.4. If $\mathscr{X}$ is in the image of α'_X, then by Lemma 1.4.5 it admits a 1-twisted finite locally free $\mathcal{O}_{\mathscr{X}}$-module of everywhere positive rank, say $\mathscr{E}$. The pushforward $F_*\mathscr{E}$ is a finite locally free $\mathcal{O}_{\mathscr{Y}}$-module of everywhere positive rank, which is 1-twisted by Lemma 1.3.7. Hence by Lemma 1.4.5 we have that $\mathscr{Y}$ is in the image of α'_Y.

The other direction follows from commutativity of the diagram (1.4.3.1). ☐

Corollary 2.1.5. Let X be a smooth separated generically tame Deligne-Mumford stack over a field k with quasi-projective coarse moduli space. The Brauer map α_X is surjective.

Proof. By Kresch and Vistoli [56, 2.1,2.2], such X has a finite flat surjection $Z \to X$ where Z is a quasi-projective k-scheme. By Gabber's theorem [22, 1.1], the Brauer map is surjective for Z. Thus the Brauer map is surjective for X by Lemma 2.1.4. ☐

Corollary 2.1.6. Let X be a scheme and let G be a finite discrete group with associated constant sheaf G_X. Then $\alpha_{\mathrm{B}G_X}$ is surjective if and only if α_X is surjective.

Proof. Suppose α_X is surjective. The morphism $X \to \mathrm{B}G_X$ is a finite locally free morphism so we may apply Lemma 2.1.4. Conversely, if $\alpha_{\mathrm{B}G_X}$ is surjective, then α_X is surjective by functoriality (1.4.3.1). ☐

Theorem 2.1.7. [4, 2.5 (iv)] Let X be a regular Noetherian algebraic stack, and let $U \subseteq X$ be a dense open substack. Then the restriction map

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{m,X}) \to \mathrm{H}^2_{\text{ét}}(U, \mathbb{G}_{m,U})$$

is injective.

Proof. In [4] the result is stated only for Deligne-Mumford stacks, but the proof applies more generally, the point being that reflexive sheaves of rank 1 on regular Noetherian algebraic stacks are invertible. ☐

Lemma 2.1.8. [4, 2.5 (iii)] Let X be a regular Noetherian Deligne-Mumford stack. Then $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{m,X})$ is a torsion group.

Proof. The following argument is only superficially different than that of [4]. By [58, (6.1.1)], there exists a dense open substack $\mathcal{U} \subseteq \mathcal{X}$ such that $\mathcal{U}$ is isomorphic to the quotient stack $[U/G]$ where U is an affine scheme (necessarily regular) and G is a finite group. Since the restriction $\mathrm{H}^2_{et}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}}) \to \mathrm{H}^2_{\text{ét}}(\mathcal{U}, \mathbb{G}_{m,\mathcal{U}})$ is an injection by Theorem 2.1.7, it suffices to prove the result for $\mathcal{X} = \mathcal{U}$. The cohomological descent spectral sequence (B.1.1.2) implies the desired result since $\mathrm{H}^2_{\text{ét}}(\mathcal{U}, \mathbb{G}_{m,\mathcal{U}})$ has a filtration whose successive quotients are subquotients of

$$\mathrm{H}^0(G, \mathrm{H}^2_{\text{ét}}(U, \mathbb{G}_{m,U})), \mathrm{H}^1(G, \mathrm{H}^1_{\text{ét}}(U, \mathbb{G}_{m,U})), \mathrm{H}^2(G, \mathrm{H}^0_{\text{ét}}(U, \mathbb{G}_{m,U}))$$

which are all torsion. ☐

Remark 2.1.9. In Lemma 2.1.8 we cannot replace "Deligne-Mumford stack" with "algebraic stack": there are examples showing that regular Noetherian algebraic stacks need not have torsion Brauer group. Let A be a regular local ring. The classifying stack $\mathrm{B}_A(\mathbb{Z} \oplus \mathbb{Z})$ is a regular Noetherian Deligne-Mumford stack whose diagonal is not quasi-compact, and

$$\mathrm{H}^2_{\text{ét}}(\mathrm{B}_S(\mathbb{Z} \oplus \mathbb{Z}), \mathbb{G}_m) \simeq \mathrm{Br}(S) \oplus A^\times$$

by Example 2.3.5; then we may take any A such that $A^\times$ not a torsion group. For another example, let E be an elliptic curve over a field k, and let $\mathrm{B}_k E$ be the classifying stack. We have

$$\mathrm{H}^2_{\text{ét}}(\mathrm{B}_k E, \mathbb{G}_m) = \mathrm{Br}(k) \oplus \mathrm{Pic}^0(E)$$

by Proposition 2.4.4; then we may take any E such that $\mathrm{Pic}^0(E)$ contains nontorsion elements. ☐

2.2. **Low-dimensional stacks.** Grothendieck proved [42, II.2.2] that, for a scheme X, the Brauer map α_X is an isomorphism if X has dimension 1 or has dimension 2 and is regular. Here we consider the stack-theoretic analogues of the corresponding statements.

Remark 2.2.1. The literal analogue of Tsen's theorem fails to hold. More precisely, there exists a separated Deligne-Mumford stack $\mathcal{X}$ of dimension 1 and of finite type over an algebraically closed field k such that $\mathrm{Br}'(\mathcal{X}) \neq 0$. See for example Lemma 3.2.3. See Poma's [77] which gives more detailed computations.

Question 2.2.2. Let X be a separated Deligne-Mumford stack of dimension 1. Is the Brauer map α_X surjective?

Remark 2.2.3. One approach to Question 2.2.2 would be to try to follow Lieblich's proof [60, 3.1.3.7] of the result for schemes. Let $\mathcal{X}$ be a separated Noetherian Deligne-Mumford stack of dimension 1, i.e. there is an étale cover $U \to \mathcal{X}$ where U is a scheme of dimension 1. Let $\pi : \mathcal{X} \to X$ be the coarse moduli space. Suppose $\mathcal{X}$ is integral and has the property that there is a Zariski covering $X = \bigcup X_i$ for which $\mathcal{X} \times_X X_i \simeq [U_i/G_i]$ for a (necessarily 1-dimensional) scheme U_i and a finite discrete group G_i acting on U_i (this is true in general only for the étale topology). Let $\mathcal{G} \to \mathcal{X}$ be a $\mathbb{G}_{m,\mathcal{X}}$-gerbe corresponding to a torsion class in $\mathrm{H}^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}})$. By Lemma 2.1.4, there exists a 1-twisted finite locally free $\mathcal{O}_{\mathcal{G}\times_X X_i}$-module $\mathcal{E}_i$. It suffices to glue these $\mathcal{E}_i$ together. In the scheme case, there is no problem since there exists at most one isomorphism class of 1-twisted finite locally free $\mathcal{O}_{\mathcal{G}_\eta}$-module (essentially by Wedderburn's theorem). However, Wedderburn's theorem does not generalize to stacks. Precisely, if $\mathcal{X}$ is a 0-dimensional separated Deligne-Mumford stack, then there may exist more than one isomorphism class of 1-twisted finite locally free $\mathcal{O}_{\mathcal{G}}$-modules of a given rank (for example, if $\mathcal{G}$ is trivial, then there is a correspondence between 1-twisted invertible $\mathcal{O}_{\mathcal{G}}$-modules and invertible $\mathcal{O}_{\mathcal{X}}$-modules).

Lemma 2.2.4. [9] Let $\mathcal{X}$ be a separated Noetherian Deligne-Mumford stack admitting a smooth surjection $\pi : X \to \mathcal{X}$ where X is regular Noetherian and has $\dim X \leq 2$. Then the Brauer map $\mathrm{Br}\,\mathcal{X} \to \mathrm{Br}'\,\mathcal{X}$ is surjective for $\mathcal{X}$.

Proof. By [58, (6.1.1)], there exists a dense open substack $\mathcal{U} \subseteq \mathcal{X}$ such that $\mathcal{U}$ is isomorphic to the quotient stack $[U/G]$ where U is an affine scheme and G is a finite group. Note that the Brauer map is surjective for $\mathcal{U}$ since the Brauer map is surjective for U and $U \to [U/G]$ is a finite flat cover. Let $\mathscr{G}$ be a $\mathbb{G}_m$-gerbe corresponding to a torsion class in $\mathrm{H}^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m)$. Let $j : \mathcal{U} \to \mathcal{X}$ be the inclusion and let $j' : \mathscr{G}|_{\mathcal{U}} \to \mathscr{G}$ be the pullback to $\mathscr{G}$. Since the Brauer map is surjective for $\mathcal{U}$, there exists a 1-twisted locally free sheaf $\mathcal{E}_0$ on $\mathscr{G}|_{\mathcal{U}}$. By [22, 2.12] we have that $j'_*(\mathcal{E}_0)$ is the filtered direct limit of its coherent 1-twisted subsheaves; there exists a coherent 1-twisted subsheaf $\mathcal{F} \subseteq j'_*(\mathcal{E}_0)$ such that $\mathcal{F}|_{\mathcal{U}} \simeq \mathcal{E}_0$. Let $\mathcal{E} := \mathcal{F}^{\vee\vee}$ be the double dual, which is a coherent 1-twisted reflexive sheaf on $\mathscr{G}$. It remains to check that $\mathcal{E}$ is locally free. We have that reflexive sheaves are preserved by flat pullback [67, 3.1]. The property of being locally free can be checked on a faithfully flat cover [88, 05B2]. After replacing $\mathcal{X}$ by an étale cover, we may assume that $\mathscr{G}$ is the trivial $\mathbb{G}_m$-gerbe $\mathrm{B}\mathbb{G}_{m,\mathcal{X}}$. Let $\xi : \mathcal{X} \to \mathrm{B}\mathbb{G}_{m,\mathcal{X}}$ be the canonical cover corresponding to the trivial torsor; then $\pi^*\xi^*\mathcal{E}$ is a reflexive coherent sheaf on X, which is locally free by [88, 0B3N] since X is regular Noetherian scheme of dimension ≤ 2. ∎

[9]This is claimed (without the "separated" hypothesis) in [55, Proposition 2.1 (ii)].

2.3. **Quotient stacks by discrete groups.**

Question 2.3.1. For which pairs (k, G, ρ) where k is a field and G is a discrete group and $\rho : G \times k \to k$ is an action of G on k is it true that $\operatorname{Br}\mathcal{X} = \operatorname{Br}'\mathcal{X}$ for $\mathcal{X} := [(\operatorname{Spec} k)/G]$?

Remark 2.3.2. If G is a finite group, then we can use the fact that $\operatorname{Spec} k \to [(\operatorname{Spec} k)/G]$ is a finite flat cover and apply Lemma 2.1.4. If G is infinite, then there are counterexamples, see for example Example 2.3.5.

Antieau and Meier computed the low-dimensional cohomology of classifying stacks by finite cyclic groups, and in particular proved the following:

Lemma 2.3.3 (Classifying stack of cyclic group). [4, Proposition 3.2] Let n be a positive integer, and let S be a scheme. Then we have an exact sequence

$$0 \to \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(\mathrm{B}(\mathbb{Z}/(n))_S, \mathbb{G}_m) \to \mathrm{H}^1_{\text{fppf}}(S, \mu_n) \to 0$$

which is canonically split.

2.3.4 (Group cohomology for $\mathbb{Z} \oplus \mathbb{Z}$). [10] We denote by $A := \mathbb{Z}[t_1^{\pm}, t_2^{\pm}]$ the group ring of the group $G := \mathbb{Z} \oplus \mathbb{Z}$. Then the A-module

$$\mathbb{Z} \simeq A/(t_1 - 1, t_2 - 1)A$$

has an A-module resolution

$$\cdots \to 0 \to Ae_{2,1} \xrightarrow{f_2} Ae_{1,1} \oplus Ae_{1,2} \xrightarrow{f_1} Ae_{0,1} \to \mathbb{Z} \to 0$$

where $f_2(e_{2,1}) = (t_2 - 1)e_{1,1} - (t_1 - 1)e_{1,2}$ and $f_1(e_{1,i}) = (t_i - 1)e_{0,1}$ for $i = 1, 2$. Applying $\operatorname{Hom}_A(-, M)$ to the above resolution gives a complex

$$M \xrightarrow{f_1^*} M^{\oplus 2} \xrightarrow{f_2^*} M \to 0 \to \cdots$$

of A-modules, and taking cohomology at the ith cohomological degree gives $\mathrm{H}^i(G, M)$. In particular we have

$$\mathrm{H}^2(G, M) \simeq \operatorname{coker}(M^{\oplus 2} \xrightarrow{f_2^*} M)$$

where the map $f_2^* : M^{\oplus 2} \to M$ sends $(m_1, m_2) \mapsto (t_2 - 1)m_1 - (t_1 - 1)m_2$. □

Example 2.3.5 (A special case of Question 2.3.1). Here we discuss an example of a non-separated regular Deligne-Mumford stack $\mathcal{X}$ with $\operatorname{Br}(\mathcal{X}) \to \operatorname{Br}'(\mathcal{X})$ not an isomorphism. This can be modified to have any dimension.

Let A be a ring for which all vector bundles are trivial (e.g a semi-local ring or a polynomial ring over a PID), set $S := \operatorname{Spec} A$, and let $G := \mathbb{Z} \oplus \mathbb{Z}$. We view G as acting trivially on A. Let $\mathcal{X} := [S/G] \simeq \mathrm{B}_S G$ be the classifying stack. We have the cohomological descent spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(G, \mathrm{H}^q_{\text{ét}}(S, \mathbb{G}_{m,S})) \implies \mathrm{H}^{p+q}_{\text{ét}}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}})$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. We have $\mathrm{H}^1_{\text{ét}}(S, \mathbb{G}_{m,S}) = \operatorname{Pic}(S) = 0$, and furthermore $\mathrm{E}_2^{p,q} = 0$ if $p \geq 3$ by 2.3.4, thus we have an direct sum decomposition

$$\mathrm{H}^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}}) = \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S}) \oplus \mathrm{H}^2(G, A^\times)$$

[10] There should be a way to do this using Kunneth formulas for group cohomology. This is copied from my answer at `https://math.stackexchange.com/q/2611736/`.

of abelian groups (a priori only an exact sequence but it is split as the projection $\pi : \mathcal{X} \to S$ has a section $s : S \to \mathcal{X}$). We have a direct sum decomposition $\mathrm{Br}(\mathcal{X}) = \mathrm{Br}(A) \oplus \ker(s^* : \mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(A))$.

An element of $\ker(s^* : \mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(A))$ corresponds to an Azumaya $\mathcal{O}_{\mathcal{X}}$-algebra $\mathcal{A}$ such that $s^*\mathcal{A}$ is a trivial Azumaya A-algebra; this corresponds to a group homomorphism $G \to \mathrm{PGL}_r(A)$ where $\mathcal{A}$ has rank r^2. A vector bundle on $\mathcal{X}$ of rank r corresponds to a group homomorphism $G \to \mathrm{GL}_r(A)$. Since $\mathrm{Pic}(A) = 0$, the map $\mathrm{GL}_r(A) \to \mathrm{PGL}_r(A)$ is surjective. Since G is a free abelian group, the map $\mathrm{H}^1(G, \mathrm{GL}_r(A)) \to \mathrm{H}^1(G, \mathrm{PGL}_r(A))$ is surjective. Thus such $\mathcal{A}$ is trivial, in other words the pullback $\pi^* : \mathrm{Br}(A) \to \mathrm{Br}(\mathcal{X})$ is an isomorphism.

On the other hand, we have $\mathrm{H}^2(G, A^\times) = A^\times$ by 2.3.4, thus $\mathrm{Br}'(\mathcal{X}) = \mathrm{Br}'(A) \oplus (A^\times)_{\mathrm{tors}}$. There are regular local rings A such that $A^\times$ has a lot of torsion (take a local ring of a smooth k-scheme where k is an algebraically closed field of characteristic 0, for example). □

2.4. **Classifying stack of an elliptic curve.**

Lemma 2.4.1. Let k be a field, let E be an elliptic curve over k. Let $\mathcal{L}$ be a line bundle on E of degree zero. Then for any k-point $x : \mathrm{Spec}\, k \to E$, we have $t_x^*\mathcal{L} \simeq \mathcal{L}$.

Proof. By [48, IV, Lemma 1.2], it suffices to show that $\Gamma(E, t_x^*\mathcal{L} \otimes_{\mathcal{O}_E} \mathcal{L}^\vee) \neq 0$; for this we may replace k by its algebraic closure and assume that k is algebraically closed. There exists a point $y \in E(k)$ such that $\mathcal{L} \simeq \mathcal{O}_E(e - y)$, where $e \in E$ is the identity with respect to the group law; we have $\mathcal{O}_E(e-y) \simeq \mathcal{O}_E(e) \otimes_{\mathcal{O}_E} \mathcal{O}_E(y)^\vee \simeq \mathcal{O}_E(e) \otimes_{\mathcal{O}_E} t_y^*\mathcal{O}_E(e)^\vee$. Then the statement is that

$$t_x^*\mathcal{M} \otimes_{\mathcal{O}_E} t_y^*\mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{O}_E} t_{x+y}^*\mathcal{M}$$

for $\mathcal{M} = \mathcal{O}_E(e)$, which is the Theorem of the Square [71, II, §6, Corollary 4]. □

Lemma 2.4.2. Let k be a field and let E be an elliptic curve over k. If G is a k-group scheme admitting a closed immersion $\xi : G \to E$ which is a group homomorphism, then G is either finite over k or ξ is an isomorphism.

Proof. This follows from the following more general fact: if E/k is a finite type k-scheme which is dimension 1 and geometrically integral and $\xi : G \to E$ is a closed immersion, then either G is finite over k or ξ is an isomorphism. If the underlying map $|\xi| : |G| \to |E|$ is not surjective, then G is 0-dimensional and finite type over k, hence finite. If $|\xi|$ is surjective, then it must be an isomorphism since E is reduced. □

Lemma 2.4.3. [11] Let k be a field, let E be an elliptic curve over k. Let $m : E \times_k E \to E$ be the group law and let $p_i : E \times_k E \to E$ be the ith projection. Let us denote by

$$\xi := m^* - p_1^* - p_2^* : \mathrm{Pic}(E) \to \mathrm{Pic}(E \times_k E)$$

the map sending a line bundle to its associated Mumford bundle. Then $\ker \xi = \mathrm{Pic}^0(E)$, the subgroup of degree zero line bundles on E.

Proof. Let $\mathcal{L}$ be a line bundle on E. Let $K_{(E,\mathcal{L})} \subset E$ be the subgroup scheme representing the functor $(\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Set})$ sending S to $*$ if $\mathcal{L}|_{E_S}$ is the pullback of a line bundle on S, and $\emptyset$ otherwise. Then $K_{(E,\mathcal{L})}$ is representable by a subgroup scheme of X [29, (2.18) Proposition].

[11] This is proved here: see [71, IV, §8, (i) and (iv)] and `https://math.stackexchange.com/q/2446400` and `https://mathoverflow.net/q/282435` and Lemma 1 (a) of `https://www2.mathematik.hu-berlin.de/~bakkerbe/Abelian9.pdf`.

We have that $\mathcal{L} \in \ker \xi$ if and only if $K_{(E,\mathcal{L})} \to E$ is an isomorphism. If $\deg \mathcal{L} > 0$, then $\mathcal{L}$ is ample by [48, IV, Corollary 3.3]; then $K_{(E,\mathcal{L})}$ is finite over k by [29, (2.19) Lemma]. If $\deg \mathcal{L} = 0$, then $K_{(E,\mathcal{L})} \to E$ is an isomorphism by Lemma 2.4.1 and Lemma 2.4.2. □

Proposition 2.4.4. Let k be a field and let E be an elliptic curve over k. We have an isomorphism

$$\mathrm{H}^2_{\text{ét}}(\mathrm{B}E, \mathbb{G}_m) \simeq \mathrm{Br}(k) \oplus \mathrm{Pic}^0(E)$$

of groups.

Proof. We compute $\mathrm{H}^2_{\text{ét}}(\mathrm{B}E, \mathbb{G}_m)$ using the cohomological descent spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q_{\text{ét}}(E^{\times p}, \mathbb{G}_m) \implies \mathrm{H}^{p+q}_{\text{ét}}(\mathrm{B}E, \mathbb{G}_m)$$

with differentials $\mathrm{E}_1^{p,q} \to \mathrm{E}_1^{p+1,q}$. We have $\mathrm{H}^0_{\text{ét}}(E^{\times p}, \mathbb{G}_m) = k$ for all p, and the complex $\mathrm{H}^0_{\text{ét}}(E^{\times \bullet}, \mathbb{G}_m)$ is acyclic except at $p = 0$. The map $\mathrm{E}^2_\infty \to \mathrm{E}_1^{0,2}$ corresponds to the pullback $\mathrm{H}^2_{\text{ét}}(\mathrm{B}E, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m)$, which is a split surjection since the composite $\operatorname{Spec} k \to \mathrm{B}E \to \operatorname{Spec} k$ is the identity. We have $\mathrm{E}_2^{1,1} \simeq \mathrm{Pic}^0(E)$ by Lemma 2.4.3. □

Proposition 2.4.5. Let k be a field and let E be an elliptic curve over k. The Azumaya Brauer group of $\mathrm{B}E$ is $\mathrm{Br}(\mathrm{B}E) = \mathrm{Br}(k)$.

Proof. Let $\operatorname{Spec} k \to \mathrm{B}E$ be the morphism corresponding to the trivial E-torsor. There is a direct sum decomposition $\mathrm{Br}(\mathrm{B}E) = \mathrm{Br}(k) \oplus \ker(\xi^*)$ where $\xi^* : \mathrm{Br}(\mathrm{B}E) \to \mathrm{Br}(k)$ is the pullback map. A class in $\ker(\xi^*)$ corresponds to an Azumaya $\mathcal{O}_{\mathrm{B}E}$-algebra $\mathcal{A}$ which is trivialized after pullback by ξ; this is the data of a positive integer n and an element $\varphi \in \mathrm{PGL}_n(E)$ which satisfies the cocycle condition on $E \times_k E$, more precisely $m^*\varphi = p_1^*\varphi \cdot p_2^*\varphi$ where m, p_1, p_2 are as in Lemma 2.4.3. Since PGL_n is affine, the pullback $\mathrm{PGL}_n(\Gamma(E, \mathcal{O}_E)) \to \mathrm{PGL}_n(E)$ is an isomorphism; similarly $\mathrm{PGL}_n(\Gamma(E, \mathcal{O}_E)) \to \mathrm{PGL}_n(E \times_k E)$ is an isomorphism as well. Since E is geometrically integral, we have $k \to \Gamma(E, \mathcal{O}_E)$ and $k \to \Gamma(E \times_k E, \mathcal{O}_{E \times_k E})$ are isomorphisms. Thus φ is an element of $\mathrm{PGL}_n(k)$ which satisfies $\varphi = \varphi \cdot \varphi$, in other words $\varphi = \mathrm{id}$. Thus $\mathcal{A}$ is isomorphic to $\mathrm{Mat}_{n \times n}(\mathcal{O}_{\mathrm{B}E})$. □

Corollary 2.4.6. Let k be a field and let E be an elliptic curve over k. Then the Brauer map $\alpha_{\mathrm{B}E} : \mathrm{Br}(\mathrm{B}E) \to \mathrm{Br}'(\mathrm{B}E)$ is an isomorphism if and only if $\mathrm{Pic}^0(E)$ is torsion-free.

Proof. This follows from Proposition 2.4.4 and Proposition 2.4.5. □

Remark 2.4.7. Heinloth and Schröer [49, §3] show that there exists a $\mathbb{G}_m$-gerbe over $\mathrm{B}E$ which does not lie in the image of the so-called "bigger Brauer group" but this gerbe corresponds to a nontorsion class in $\mathrm{H}^2_{\text{ét}}(\mathrm{B}E, \mathbb{G}_m)$ so it does not provide a counterexample to $\mathrm{Br} = \mathrm{Br}'$. □

2.5. **Classifying stack of diagonalizable groups.** We consider the Brauer groups of classifying stacks by diagonalizable groups associated to finite cyclic groups and finitely generated torsion-free abelian groups.

Lemma 2.5.1. Let $(A, \mathfrak{m})$ be a local ring and let G be a finite subgroup of the group of units $A^\times$. If $|G|$ is invertible in A, then G is a cyclic group.

Proof. This is a generalization of the case when A is a field. Let $k := A/\mathfrak{m}$ be the residue field. If G is not a cyclic group, then there exists an integer n dividing $|G|$ and such that the polynomial $X^n - 1$ has more than n roots in A. Since n is invertible in k, the polynomial

$X^n - 1$ is separable over k. By [88, 06RR], the polynomial $X^n - 1$ has more than n roots in k, contradiction. $\square$

Lemma 2.5.2. [12] Let A be a strictly henselian local ring, let n be an integer invertible in A. Then $\mathrm{H}^2(\mathrm{B}\mu_{n,A}, \mathbb{G}_m) = 0$.

Proof. There exist pairwise distinct $\xi_1, \dots, \xi_n \in A^\times$ such that we have the factorization

$$t^n - 1 = (t - \xi_1) \cdots (t - \xi_n)$$

in $A[t]$; by Lemma 2.5.1, there exists some $\xi \in A^\times$ such that $t^n - 1 = \prod_{i=0}^{n-1}(t - \xi^i)$; thus μ_n is (noncanonically) isomorphic to $\mathbb{Z}/(n)$ as abelian sheaves; thus we conclude using Lemma 2.3.3. $\square$

Remark 2.5.3. It would be nice to remove the "strictly henselian" hypothesis in Lemma 2.5.2. For this, we would have to compute the unit groups of $A[T_1, \dots, T_p]/(T_1^n - 1, \dots, T_p^n - 1)$ for $p = 1, 2, 3$ in case the polynomial $T^n - 1$ does not split completely over A.

Lemma 2.5.4 (Sheaf of units on $\mathbb{G}_m$ over an integral scheme). Let X be an integral scheme. Then the canonical map

$$\Gamma(X, \mathbb{G}_{m,X}) \oplus \mathbb{Z}^{\oplus n} \to \Gamma(X \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}^{\times n}, \mathbb{G}_m)$$

is an isomorphism.

Proof. We have the result when X is affine. In general, let $X = \bigcup_{i \in I} X_i$ be an affine open cover of X. We have a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Gamma(X, \mathbb{G}_m) \oplus \mathbb{Z}^{\oplus n} & \xrightarrow{f_1} & \Gamma(X \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}^{\times n}, \mathbb{G}_m) \\
\downarrow & & \downarrow \\
\prod_{i \in I} \Gamma(X_i, \mathbb{G}_m) \oplus \mathbb{Z}^{\oplus n} & \xrightarrow{f_2} & \prod_{i \in I} \Gamma(X_i \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}^{\times n}, \mathbb{G}_m) \\
\downarrow & & \downarrow \\
\prod_{i_1, i_2 \in I} \Gamma(X_{i_1,i_2}, \mathbb{G}_m) \oplus \mathbb{Z}^{\oplus n} & \xrightarrow{f_3} & \prod_{i_1, i_2 \in I} \Gamma(X_{i_1,i_2} \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}^{\times n}, \mathbb{G}_m)
\end{array}$$

where the columns are equalizer sequences. By the affine case, we have that f_2 is an isomorphism; hence f_1 is an injection. Applying this argument to X_{i_1,i_2}, we have that f_3 is an injection. Thus f_1 is an isomorphism by a diagram chase. $\square$

Lemma 2.5.5. Let M be a finitely generated torsion-free abelian group, let

$$\mathbf{T} := \mathrm{D}(\mathrm{M})_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[\mathrm{M}]$$

[12] This is a generalization of [61]; the proof there (which is ring-theoretic) works if one replaces "separably closed field" by "strictly henselian local ring". Lieblich only considers Brauer classes of order prime to the characteristic but this restriction is unnecessary. See also [63, VI.6], "Cohomology of free abelian groups", regarding the Koszul complex associated to the regular sequence $\{t_1 - 1, \dots, t_n - 1\}$ in $\mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}]$.

be the associated $\mathbb{Z}$-group scheme. Let X be an integral scheme, let $\mathrm{B}\mathbf{T}_X$ be the classifying stack, and let $\xi : X \to \mathrm{B}\mathbf{T}_X$ be the morphism corresponding to the trivial $\mathbf{T}$-torsor. For $p \geq 0$, let

$$X^p := X \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} X$$

be the $(p+1)$-fold fiber product of X over $\mathscr{X}$. The bottom row of the cohomological descent spectral sequence gives a complex

$$\Gamma(X^0, \mathbb{G}_m) \xrightarrow{\mathrm{d}^0} \Gamma(X^1, \mathbb{G}_m) \xrightarrow{\mathrm{d}^1} \Gamma(X^2, \mathbb{G}_m) \xrightarrow{\mathrm{d}^2} \Gamma(X^3, \mathbb{G}_m) \to \cdots \tag{2.5.5.1}$$

of abelian groups. Then (2.5.5.1) is acyclic in degrees $p \geq 1$.

Proof. We have $X^p \simeq X \times \mathbf{T}^p$ for all p. Since X is an integral scheme, by Lemma 2.5.4 the map

$$\Gamma(X, \mathbb{G}_m) \oplus \mathrm{M}^{\oplus p} \to \Gamma(X \times \mathbf{T}^p, \mathbb{G}_m) \tag{2.5.5.2}$$

is an isomorphism. With the identification (2.5.5.2), the differential d^p is the alternating sum of $p+2$ maps, each of which is the identity on the $\Gamma(X, \mathbb{G}_m)$ summand; the map $\mathrm{M}^{\oplus p} \to \mathrm{M}^{\oplus (p+1)}$ is given by the formula

$$\mathrm{d}^p([a_1, \ldots, a_p]) = [0, a_1, \ldots, a_p] - (\textstyle\sum_{i=1}^p (-1)^i [a_1, \ldots, a_i, a_i, \ldots, a_p]) + (-1)^{p+1}[a_1, \ldots, a_p, 0]$$

where "$[a_1, \ldots, a_i, a_i, \ldots, a_p]$" is the vector obtained by replacing "a_i" with "a_i, a_i" in $[a_1, \ldots, a_p]$. A computation shows that if p is odd, then the image of $[a_1, \ldots, a_p]$ under d^p is given by

$$[0, a_2, a_2, a_4, a_4, \ldots, a_{p-1}, a_{p-1}, 0]$$

and if p is even, then the image of d^p is given by

$$[-a_1, 0, a_2 - a_3, 0, a_4 - a_6, 0, \ldots, 0, a_{p-2} - a_{p-1}, 0, a_p]$$

which gives exactness for $p \geq 1$. □

Lemma 2.5.6. Let A be a Noetherian normal ring. Then the pullback

$$\mathrm{Pic}(A) \to \mathrm{Pic}(A[t^{\pm}])$$

is an isomorphism.

Proof. After taking connected components, we may assume that A is a Noetherian normal domain. We have an exact sequence

$$0 \to \mathrm{Pic}(A) \to \mathrm{Pic}(A[t]) \oplus \mathrm{Pic}(A[t^{-1}]) \to \mathrm{Pic}(A[t^{\pm}]) \to \mathrm{LPic}(A) \to 0$$

by [93, Lemma 1.5.1], and an isomorphism $\mathrm{LPic}(A) \simeq \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\, A, \mathbb{Z})$ by [93, Theorem 5.5]; we have $\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\, A, \mathbb{Z}) = 0$ by [45, Exp. VIII, Prop. 5.1] since A is geometrically unibranch. □

Lemma 2.5.7. Let S be a locally Noetherian, integral scheme such that, for every point $s \in S$ of codimension 1, the local ring $\mathcal{O}_{S,s}$ is regular. Set $\mathbf{T} := \mathrm{Spec}\, \mathbb{Z}[t^{\pm}]$ and $\mathbf{T}_S := S \times_{\mathrm{Spec}\, \mathbb{Z}} \mathbf{T}$, and let $\pi : \mathbf{T}_S \to S$ be the projection. Then the pullback map

$$\pi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(\mathbf{T}_S)$$

is an isomorphism.

Proof 1. [13] We check the conditions of [44, IV_4, (21.4.9)]. The projection π is faithfully flat and has a section, hence π^* is injective; the map π is both quasi-compact and open. Given a codimension 1 point $s \in S$, set $A := \mathcal{O}_{S,s}$; since A is seminormal, the pullback $\mathrm{Pic}(A) \to \mathrm{Pic}(\mathbb{A}^1_A)$ is an isomorphism by Traverso's theorem [91, Theorem 3.6]; since A is regular, for any open subscheme $U \subseteq \mathbb{A}^1_A$ we have an isomorphism $\mathrm{Pic}(U) \simeq \mathrm{Cl}(U)$; the restriction map $\mathrm{Cl}(\mathbb{A}^1_A) \to \mathrm{Cl}(U)$ is surjective; we take $U := \operatorname{Spec} A[t^{\pm}]$. □

Proof 2, if S is normal and quasi-compact. After taking connected components, we may assume that S is a Noetherian normal integral scheme. Since the projection π has a section, it is clear that π^* is injective. For any quasi-compact scheme Y, let $n(Y)$ be the minimal size of an affine open covering of Y.

We proceed by induction on $n(S)$. The case $n(S) = 1$ (in other words S is affine) is Lemma 2.5.6.

In general, suppose $S = S_1 \cup S_2$ where S_1, S_2 are open subschemes of S such that $n(S_i) < n(S)$. Let $\pi_i : \mathbf{T}_{S_i} \to S_i$ be the projections. Suppose $\mathcal{L}$ is an invertible sheaf on $\mathbf{T}_S$; by the induction hypothesis, there exist invertible $\mathcal{O}_{S_i}$-modules $\mathcal{M}_i$ and isomorphisms

$$\varphi_i : \mathcal{L}|_{\mathbf{T}_{S_i}} \to \pi_i^* \mathcal{M}_i$$

of $\mathcal{O}_{\mathbf{T}_{S_i}}$-modules. Set $S_{12} := S_1 \cap S_2$ and $\pi_{12} : \mathbf{T}_{S_{12}} \to S_{12}$ the projection; since $\mathrm{Pic}(S_{12}) \to \mathrm{Pic}(\mathbf{T}_{S_{12}})$ is injective, there is an isomorphism

$$\alpha : \mathcal{M}_1|_{S_{12}} \to \mathcal{M}_2|_{S_{12}}$$

of $\mathcal{O}_{S_{12}}$-modules; moreover, since the inclusion

$$\Gamma(S_{12}, \mathbb{G}_m) \times t^{\mathbb{Z}} \to \Gamma(\mathbf{T}_{S_{12}}, \mathbb{G}_m)$$

is an isomorphism (by Lemma 2.5.4), we may multiply α by a unit in $\mathbb{G}_m(S_{12})$ and multiply $\mathcal{M}_1$ by a character t^n so that $\pi_{12}^* \alpha$ corresponds to $(\varphi_2|_{\mathbf{T}_{S_{12}}}) \circ (\varphi_1|_{\mathbf{T}_{S_{12}}})^{-1}$. Thus the invertible $\mathcal{O}_S$-module obtained by gluing $\mathcal{M}_1, \mathcal{M}_2$ along α gives the desired element of $\mathrm{Pic}(S)$ whose image in $\mathrm{Pic}(\mathbf{T}_S)$ is $\mathcal{L}$. □

Proposition 2.5.8. Let X be a Noetherian normal scheme. Let M be a finitely generated torsion-free abelian group, and let

$$\mathbf{T} := \mathrm{D}(\mathrm{M})_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[\mathrm{M}]$$

be the associated diagonalizable $\mathbb{Z}$-group scheme. Let $\mathrm{B}\mathbf{T}_X$ be the classifying stack and let $\xi : X \to \mathrm{B}\mathbf{T}_X$ be the morphism corresponding to the trivial $\mathbf{T}$-torsor. Then the pullback map

$$\xi^* : \mathrm{H}^2_{\text{ét}}(\mathrm{B}\mathbf{T}_X, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$$

is an isomorphism.

Proof. The cohomological descent spectral sequence associated to the covering ξ gives a spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q_{\text{ét}}(\mathbf{T}^p_X, \mathbb{G}_m) \implies \mathrm{H}^{p+q}_{\text{ét}}(\mathrm{B}\mathbf{T}_X, \mathbb{G}_m) \tag{2.5.8.1}$$

with differentials $\mathrm{d}_1^{p,q} : \mathrm{E}_1^{p,q} \to \mathrm{E}_1^{p+1,q}$.

[13]Following comments by user “Heer” in `https://mathoverflow.net/q/84414`.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\mathrm{H}^3_{\text{ét}}(\mathbf{T}^0_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(\mathbf{T}^1_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(\mathbf{T}^2_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(\mathbf{T}^3_X, \mathbb{G}_m) & \longrightarrow & \cdots \\
\mathrm{H}^2_{\text{ét}}(\mathbf{T}^0_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbf{T}^1_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbf{T}^2_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbf{T}^3_X, \mathbb{G}_m) & \longrightarrow & \cdots \\
\mathrm{H}^1_{\text{ét}}(\mathbf{T}^0_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(\mathbf{T}^1_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(\mathbf{T}^2_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(\mathbf{T}^3_X, \mathbb{G}_m) & \longrightarrow & \cdots \\
\mathrm{H}^0_{\text{ét}}(\mathbf{T}^0_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^0_{\text{ét}}(\mathbf{T}^1_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^0_{\text{ét}}(\mathbf{T}^2_X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^0_{\text{ét}}(\mathbf{T}^3_X, \mathbb{G}_m) & \longrightarrow & \cdots
\end{array}$$

Note that each differential $\mathrm{d}_1^{0,q} : \mathrm{E}_1^{0,q} \to \mathrm{E}_1^{1,q}$ is the 0 map since the two projection maps $\mathbf{T}^1_X \rightrightarrows X$ are equal (since $\mathbf{T}$ acts trivially on X). By Lemma 2.5.4, the map

$$\Gamma(X, \mathbb{G}_m) \oplus \mathrm{M}^{\oplus p} \to \Gamma(\mathbf{T}^p_X, \mathbb{G}_m) \tag{2.5.8.2}$$

is an isomorphism. Note that there are $p+2$ projection maps $\mathbf{T}^{p+1} \to \mathbf{T}^p$. Since X is Noetherian normal, by Lemma 2.5.7 we have that $\mathrm{d}_1^{p,1} : \mathrm{E}_1^{p,1} \to \mathrm{E}_1^{p+1,1}$ is 0 if p is even and an isomorphism if p is odd; thus $\mathrm{E}_2^{p,1} = 0$ for $p \geq 1$.

Via the identification (2.5.8.2), we obtain a complex $\mathrm{E}_1^{\bullet,0}$ which is exact in degrees $p \geq 1$ by Lemma 2.5.5.

The above considerations show that the desired map ξ^* is an isomorphism. ☐

Remark 2.5.9. In Section 4 we will remove the "normality" hypothesis in the case $\mathrm{M} = \mathbb{Z}$. The difficulty in working with torsion-free abelian groups M of higher rank is that $\mathrm{Pic}(A[\mathrm{M}])$ can be large (see Weibel [93]). ☐

Lemma 2.5.10. In the setup of Proposition 2.5.8, if X is quasi-compact and admits an ample line bundle, then the Brauer map $\alpha_{\mathrm{B}\mathbf{T}_X} : \mathrm{Br}\,\mathrm{B}\mathbf{T}_X \to \mathrm{Br}'\,\mathrm{B}\mathbf{T}_X$ is an isomorphism.

Proof. The projection $\pi : \mathrm{B}\mathbf{T}_X \to X$ has a section $\xi : X \to \mathrm{B}\mathbf{T}_X$. We have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Br}\,X & \xrightarrow{\pi^*} & \mathrm{Br}\,\mathrm{B}\mathbf{T}_X & \xrightarrow{\xi^*} & \mathrm{Br}\,X \\
\downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_{\mathrm{B}\mathbf{T}_X}} & & \downarrow{\scriptstyle \alpha_X} \\
\mathrm{Br}'\,X & \xrightarrow[\pi^*]{} & \mathrm{Br}'\,\mathrm{B}\mathbf{T}_X & \xrightarrow[\xi^*]{} & \mathrm{Br}'\,X
\end{array}$$

where the morphisms on the bottom row are isomorphisms by Proposition 2.5.8. We have that α_X is an isomorphism by [22]. Thus $\alpha_{\mathrm{B}\mathbf{T}_X}$ is an isomorphism. ☐

2.6. **Classifying stack of GL_n.**

Setup 2.6.1. Let A be a ring, let

$$\mathbf{X} := \{\mathrm{X}_{i,j}\}_{i,j=1,\dots,n}$$

be n^2 variables, let $A[\mathbf{X}]$ be the polynomial ring. The localization $A[\mathbf{X}, \frac{1}{\det}]$ is the coordinate ring of $\mathrm{GL}_{n,A}$.

There is a map

$$\Phi_A : A^\times \oplus \Gamma(\operatorname{Spec} A, \underline{\mathbb{Z}}) \to (A[\mathbf{X}, \tfrac{1}{\det}])^\times \tag{2.6.1.1}$$

sending $(a, n) \mapsto a \det^n$. □

Lemma 2.6.2. Assume the notation of Setup 2.6.1. The map

$$A \to A[\mathbf{X}]/(\det)$$

is faithfully flat.

Proof 1. It suffices to take the case $A = \mathbb{Z}$. Then it suffices to show that $\mathbb{Z}[\mathbf{X}]/(\det)$ is torsion-free. Suppose $\ell \in \mathbb{Z}$ and $a \in \mathbb{Z}[\mathbf{X}]$ such that $\ell a \in (\det)$; since $\mathbb{Z}[\mathbf{X}]/(\det)$ is an integral domain [17, (2.10) Theorem], either $\ell \in (\det)$ or $a \in (\det)$, but it is not possible that $\ell \in \det$ since ℓ has degree 0 whereas det has degree n. □

Proof 2. We can make a change of coordinates $\mathrm{X}_{i,i} \mapsto \mathrm{X}_{i,i} + \mathrm{X}_{1,1}$ for $i \geq 2$. Let f be the polynomial that det gets sent to; then f is monic of degree n in the variable $\mathrm{X}_{1,1}$, hence $A[\mathbf{X}, \frac{1}{\det}]$ is finite locally free over $A[\mathbf{X} \setminus \{\mathrm{X}_{1,1}\}]$, which is smooth over A. □

Lemma 2.6.3. Assume the notation of Setup 2.6.1. The element det is a nonzerodivisor of $A[\mathbf{X}]$.

Proof. Since $\mathbb{Z}[\mathbf{X}]$ is an integral domain, we have that det is irreducible element of the UFD $\mathbb{Z}[\mathbf{X}]$; hence it is a nonzerodivisor on $\mathbb{Z}[\mathbf{X}]$; hence the sequence

$$0 \to \mathbb{Z}[\mathbf{X}] \to \mathbb{Z}[\mathbf{X}] \to \mathbb{Z}[\mathbf{X}]/(\det) \to 0 \tag{2.6.3.1}$$

is exact; here $\mathbb{Z}[\mathbf{X}]/(\det)$ is flat over $\mathbb{Z}$ by Lemma 2.6.2; tensoring (2.6.3.1) with $- \otimes_{\mathbb{Z}} A$ gives

$$0 \to A[\mathbf{X}] \xrightarrow{*} A[\mathbf{X}] \to A[\mathbf{X}]/(\det) \to 0$$

where the map $*$ is injective by e.g. [88, 00HL]. □

Lemma 2.6.4. The map (2.6.1.1) is injective.

Proof. The element det is a nonzerodivisor on $A[\mathbf{X}]$ by Lemma 2.6.3. □

Lemma 2.6.5. Assume the notation of Setup 2.6.1. If A is an integral domain, the map (2.6.1.1) is an isomorphism.

Proof. Suppose $\frac{a_1}{\det^{f_1}}$ is a unit of $A[\mathbf{X}, \frac{1}{\det}]$, with inverse $\frac{a_2}{\det^{f_2}}$. Then $a_1 a_2 = \det^{f_1+f_2}$ since det is a nonzerodivisor Lemma 2.6.3. Since A is an integral domain, we may assume that a_1, a_2 are homogeneous. We have that det is a prime element of $A[\mathbf{X}]$ by [17, (2.10) Theorem]. □

Lemma 2.6.6 (Units of GL_n). [14] Assume the notation of Setup 2.6.1. The map Φ_A (2.6.1.1) is an isomorphism if and only if A is reduced.

Proof. If A is an integral domain, we have that Φ_A is an isomorphism by Lemma 2.6.5. More generally, if A is the finite product of integral domains, then Φ_A is an isomorphism.

[14]Broughton [16] shows that the units of the coordinate ring of an algebraic group over any algebraically closed field are given by characters.

Suppose that A is reduced. By limit arguments, we may assume that A is (reduced and) a finite type $\mathbb{Z}$-algebra. We may assume that $\operatorname{Spec} A$ is connected. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal primes of A. Then the total ring of fractions of A is

$$\mathrm{Q}(A) = k(\mathfrak{p}_1) \oplus \cdots \oplus k(\mathfrak{p}_r)$$

by e.g. [88, 02LX]. Let

$$\frac{\beta_1}{\det^{f_1}}, \frac{\beta_2}{\det^{f_2}}$$

be two elements of $A[\mathbf{X}, \frac{1}{\det}]$ with $\beta_i \in A[\mathbf{X}]$ and $f_1, f_2 \in \mathbb{Z}_{\geq 0}$ such that

$$\frac{\beta_1\beta_2}{\det^{f_1+f_2}} = 1$$

in $A[\mathbf{X}, \frac{1}{\det}]$. Then $\beta_1\beta_2 \det^{f_3} = \det^{f_1+f_2+f_3}$ in $A[\mathbf{X}]$ for some f_3, but det is a nonzerodivisor in $A[\mathbf{X}]$ (by Lemma 2.6.3) so

$$\beta_1\beta_2 = \det^{f_1+f_2} \tag{2.6.6.1}$$

in $A[\mathbf{X}]$. Plugging in $\mathbf{X} = T \cdot \mathrm{id}_n$ for a variable T into (2.6.6.1) gives

$$\beta_1|_{T\cdot\mathrm{id}_n} \cdot \beta_2|_{T\cdot\mathrm{id}_n} = T^{n(f_1+f_2)}$$

so $\beta_1|_{T\cdot\mathrm{id}_n}, \beta_2|_{T\cdot\mathrm{id}_n}$ are units of $A[T^{\pm}]$; thus (since A is connected and reduced) we have by [73, Corollary 6] that $\beta_1|_{T\cdot\mathrm{id}_n}, \beta_2|_{T\cdot\mathrm{id}_n}$ are homogeneous.

The image of β_i in $(\mathrm{Q}(A)[\mathbf{X}, \frac{1}{\det}])^\times$ is contained in the image of $\Phi_{\mathrm{Q}(A)}$ so by limit arguments there exists a nonzerodivisor $s_i \in A$ such that the image of β_i in $(A[\frac{1}{s_i}][\mathbf{X}, \frac{1}{\det}])^\times$ is contained in the image of $\Phi_{A[\frac{1}{s_i}]}$; in other words, there exist $a_{i,1}, \dots, a_{i,m_i} \in A[\frac{1}{s_i}]$ (say $a_{i,1} \neq 0$) and integers $0 \leq e_{i,1} < \cdots < e_{i,m_i}$ such that

$$\beta_i = \textstyle\sum_{\ell=1}^{m_i} a_{i,\ell}\det^{e_{i,\ell}}$$

in $A[\frac{1}{s_i}][\mathbf{X}]$; here $\beta_i \in A[\mathbf{X}]$ implies $a_{i,\ell} \in A$ for all ℓ. Since $\beta_i|_{T\cdot\mathrm{id}_n} = \sum_{\ell=1}^{m_i} a_\ell T^{ne_{i,\ell}}$ is homogeneous in $A[\frac{1}{s_i}][\mathbf{X}]$, all but one $a_{i,\ell}$ is nonzero, in other words $\beta_i = a_{i,1}\det^{e_{i,1}}$. This means

$$a_{1,1}a_{2,1}\det^{e_{1,1}+e_{2,1}} = \det^{f_1+f_2}$$

in $A[\frac{1}{s_1 s_2}][\mathbf{X}]$; thus $a_{1,1}a_{2,1} = 1$ in A, so $a_{1,1}, a_{2,1}$ are units of A.

(Thanks to Justin Chen for pointing out the following.) If $a \in A$ is nonzero and satisfies $a^2 = 0$, then

$$(\det + a)(\det - a) = \det^2$$

so $\frac{\det + a}{\det}$ is a unit of $A[\mathbf{X}, \frac{1}{\det}]$ which is not in the image of (2.6.1.1). □

Lemma 2.6.7. Let S be a locally Noetherian, integral scheme such that, for every point $s \in S$ of codimension 1, the local ring $\mathcal{O}_{S,s}$ is regular. For any positive integer p, the pullback

$$\mathrm{Pic}(S) \to \mathrm{Pic}(S \times_{\operatorname{Spec}\mathbb{Z}} (\mathrm{GL}_{n,\mathbb{Z}})^{\times p}) \tag{2.6.7.1}$$

is an isomorphism.

Proof. We check the conditions of [44, IV$_4$, (21.4.9)]. Let $\pi : S \times_{\operatorname{Spec}\mathbb{Z}} (\mathrm{GL}_{n,\mathbb{Z}})^{\times p} \to S$ be the projection; it is faithfully flat and has a section, hence (2.6.7.1) is injective; the map π is both quasi-compact and open. Given a codimension 1 point $s \in S$, set $A := \mathcal{O}_{S,s}$; since A is seminormal, the pullback $\mathrm{Pic}(A) \to \mathrm{Pic}(\mathbb{A}_A^{pn^2})$ is an isomorphism; since A is regular, for any open subscheme $U \subseteq \mathbb{A}_A^{pn^2}$ we have an isomorphism $\mathrm{Pic}(U) \simeq \mathrm{Cl}(U)$; the restriction map $\mathrm{Cl}(\mathbb{A}_A^{pn^2}) \to \mathrm{Cl}(U)$ is surjective; we take $U := \operatorname{Spec} A \times_{\operatorname{Spec}\mathbb{Z}} (\mathrm{GL}_{n,\mathbb{Z}})^{\times p}$. ☐

Proposition 2.6.8 (Brauer group of classifying stack BGL_n). Let S be a locally Noetherian, integral scheme such that, for every point $s \in S$ of codimension 1, the local ring $\mathcal{O}_{S,s}$ is regular. Let $\xi : S \to \mathrm{B}_S \mathrm{GL}_n$ be the morphism corresponding to the trivial GL_n-torsor. Then the pullback map

$$\xi^* : \mathrm{H}^2_{\text{ét}}(\mathrm{B}_S \mathrm{GL}_n, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m)$$

is an isomorphism.

Proof. We set $\mathbf{G} := \mathrm{GL}_{n,S}$ for convenience. The cohomological descent spectral sequence associated to the covering $\xi : S \to \mathrm{B}\mathbf{G}$ gives a spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q_{\text{ét}}(\mathbf{G}^p, \mathbb{G}_m) \implies \mathrm{H}^{p+q}_{\text{ét}}(\mathrm{B}\mathbf{G}, \mathbb{G}_m) \tag{2.6.8.1}$$

with differentials $\mathrm{d}_1^{p,q} : \mathrm{E}_1^{p,q} \to \mathrm{E}_1^{p+1,q}$.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\mathrm{H}^3_{\text{ét}}(\mathbf{G}^0, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(\mathbf{G}^1, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(\mathbf{G}^2, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(\mathbf{G}^3, \mathbb{G}_m) & \longrightarrow & \cdots \\
\mathrm{H}^2_{\text{ét}}(\mathbf{G}^0, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbf{G}^1, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbf{G}^2, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbf{G}^3, \mathbb{G}_m) & \longrightarrow & \cdots \\
\mathrm{H}^1_{\text{ét}}(\mathbf{G}^0, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(\mathbf{G}^1, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(\mathbf{G}^2, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(\mathbf{G}^3, \mathbb{G}_m) & \longrightarrow & \cdots \\
\mathrm{H}^0_{\text{ét}}(\mathbf{G}^0, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^0_{\text{ét}}(\mathbf{G}^1, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^0_{\text{ét}}(\mathbf{G}^2, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^0_{\text{ét}}(\mathbf{G}^3, \mathbb{G}_m) & \longrightarrow & \cdots
\end{array}$$

Note that each differential $\mathrm{d}_1^{0,q} : \mathrm{E}_1^{0,q} \to \mathrm{E}_1^{1,q}$ is the 0 map since the two projection maps $\mathbf{G} \rightrightarrows S$ are equal (since $\mathbf{G}$ acts trivially on S). By Lemma 2.6.5, there is an isomorphism

$$A^\times \oplus \mathbb{Z}^{\oplus p} \to \Gamma(\mathbf{G}^p, \mathbb{G}_m) \tag{2.6.8.2}$$

sending the ith generator to the determinant of the ith component of $\mathbf{G}^p$, and furthermore the complex $\mathrm{E}_1^{\bullet,0}$ becomes identified with the corresponding complex for $\mathrm{B}\mathbb{G}_m$ via the map $\mathrm{B}\mathbf{G} \to \mathrm{B}\mathbb{G}_m$ defined by the determinant, hence is acyclic in degrees $p \geq 1$ by the argument in Lemma 2.5.5. Moreover we have $\mathrm{E}_1^{p,1} = \mathrm{H}^1_{\text{ét}}(\mathbf{G}^p, \mathbb{G}_m) = \mathrm{Pic}(\mathbf{G}^p)$ and the pullback $\mathrm{Pic}(S) \to \mathrm{Pic}(\mathbf{G}^p)$ is an isomorphism for all $p \geq 0$ by Lemma 2.6.7. The above considerations show that the canonical pullback map

$$\mathrm{H}^2_{\text{ét}}(\mathrm{B}\mathbf{G}, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m)$$

is an isomorphism. ☐

Corollary 2.6.9. In the setup of Proposition 2.6.8, if S is quasi-compact and admits an ample line bundle, then the Brauer map $\alpha_{\mathrm{B}_S \mathrm{GL}_n}$ is an isomorphism.

Proof. The argument of Lemma 2.5.10 applies, using Proposition 2.6.8. ☐

3. The Brauer group of the moduli stack of elliptic curves

The material in this section is more or less the same as in [84].

3.1. **Introduction.** The moduli stack of elliptic curves $\mathscr{M}_{1,1,\mathbb{Z}}$ is a Deligne-Mumford stack which parametrizes elliptic curves over arbitrary base schemes (not necessarily fields). For any scheme S, we may study its restriction $\mathscr{M}_{1,1,S} := \mathscr{M}_{1,1,\mathbb{Z}} \times_{\mathbb{Z}} S$ to the category of S-schemes. Mumford [70, §6] showed that $\mathrm{Pic}(\mathscr{M}_{1,1,k}) = \mathbb{Z}/(12)$ for an arbitrary algebraically closed field k with $\mathrm{char}\, k \neq 2, 3$. Fulton and Olsson [36] generalized this computation to schemes S that are either reduced or on which 2 is invertible.

This section is devoted to the computation of the (cohomological) Brauer group of the moduli stack of elliptic curves $\mathscr{M}_{1,1,S}$ over various base schemes S. Antieau and Meier [4, 11.2] computed the Brauer group $\mathrm{Br}\,\mathscr{M}_{1,1,S}$ for various base schemes S, and in particular proved that for any algebraically closed field k of characteristic not 2 the Brauer group $\mathrm{Br}\,\mathscr{M}_{1,1,k}$ is trivial. We compute $\mathrm{Br}\,\mathscr{M}_{1,1,k}$ in the characteristic 2 case. This then completes the calculation of $\mathrm{Br}\,\mathscr{M}_{1,1,k}$ over algebraically closed fields k. We summarize the result in the following theorem.

Theorem 3.1.1 ([4, 11.2] in $\mathrm{char}\, k \neq 2$). Let k be an algebraically closed field. Then $\mathrm{Br}\,\mathscr{M}_{1,1,k}$ is 0 unless $\mathrm{char}\, k = 2$, in which case $\mathrm{Br}\,\mathscr{M}_{1,1,k} = \mathbb{Z}/(2)$.

To prove the theorem, we calculate the cohomology groups $\mathrm{H}^2_{\text{ét}}(\mathscr{M}_{1,1,k}, \mu_n)$ for varying n. There are essentially two ways to approach this calculation: (1) using the coarse moduli space, and (2) using a presentation of $\mathscr{M}_{1,1,k}$ as a quotient stack. In this paper we give a new proof of the Antieau-Meier result using approach (1), and calculate in characteristic 2 using approach (2).

We also compute the Brauer group of $\mathscr{M}_{1,1,k}$ where k is a finite field of characteristic 2:

Theorem 3.1.2. Let k be a finite field of characteristic 2. Then

$$\mathrm{Br}\,\mathscr{M}_{1,1,k} = \begin{cases} \mathbb{Z}/(12) \oplus \mathbb{Z}/(2) & \text{if } x^2+x+1 \text{ has a root in } k \\ \mathbb{Z}/(24) & \text{otherwise.} \end{cases}$$

Remark 3.1.3. In [4] it is shown that $\mathrm{Br}(\mathscr{M}_{1,1,\mathbb{Z}}) = 0$ by first showing that $\mathrm{Br}(\mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{2}]}) = \mathrm{Br}(\mathbb{Z}[\frac{1}{2}]) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(4)$ then showing that these classes do not extend to $\mathscr{M}_{1,1,\mathbb{Z}}$. The computation of $\mathrm{Br}(\mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{2}]})$ is achieved via a description of $\mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{2}]}$ as a "two-fold quotient stack"; more precisely, they show (in our notation D.1.1) that $\mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{2}]} \simeq [[\Gamma(2)]/S_3]$ and that $[\Gamma(2)] \simeq \mathrm{B}_{Y(2)}(\mathbb{Z}/(2))$ where $Y(2) := \mathrm{Spec}\,\mathbb{Z}[\frac{1}{2}, t^{\pm}, (t-1)^{-1}]$. Thus it is natural to ask whether $\mathscr{M}_{1,1,\mathbb{Z}}$ has a presentation as a quotient stack by a finite group, namely whether $\mathscr{M}_{1,1,\mathbb{Z}} \simeq [X/G]$ where X is a scheme and G is a finite discrete group acting on X. As Will Chen explained to me in [19], the answer is "no", using the fact that the étale fundamental group of $\mathscr{M}_{1,1,\mathbb{Z}}$ is trivial (see [92]).

3.2. **Preliminary observations.** Let $\mathscr{M}_{1,1,\mathbb{Z}}$ be the stack of (relative) elliptic curves. For any scheme S, let $\mathscr{M}_{1,1,S} := \mathscr{M}_{1,1,\mathbb{Z}} \times_{\mathbb{Z}} S$ denote the restriction to the category of S-schemes. The stack $\mathscr{M}_{1,1,\mathbb{Z}}$ is a Deligne-Mumford stack smooth and separated over $\mathbb{Z}$ [76, 13.1.2]; hence if S is a regular Noetherian scheme then $\mathscr{M}_{1,1,S}$ is a regular Noetherian stack. If S is a locally Noetherian scheme, the morphism

$$\pi : \mathscr{M}_{1,1,S} \to \mathbb{A}^1_S$$

sending an elliptic curve to its j-invariant identifies $\mathbb{A}^1_S$ with the coarse moduli space of $\mathscr{M}_{1,1,S}$ [36, 4.4].

In general, if $\mathscr{X}$ is a separated Deligne-Mumford stack and $\pi : \mathscr{X} \to X$ is its coarse moduli space, then π is initial among maps from $\mathscr{X}$ to an algebraic space, so the map $\mathcal{G}(X) \to \mathcal{G}(\mathscr{X})$ is an isomorphism for any group scheme $\mathcal{G}$; moreover if $U \to X$ is an étale morphism, then $\pi_U : \mathscr{X} \times_X U \to U$ is a coarse moduli space. Applying these observations to $\mathcal{G} = \mathbb{G}_a, \mathbb{G}_m, \mu_n$ implies that the canonical maps $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathscr{X}}$, $\mathbb{G}_{m,X} \to \pi_*\mathbb{G}_{m,\mathscr{X}}$, $\mu_{n,X} \to \pi_*\mu_{n,\mathscr{X}}$ are isomorphisms; thus we will omit subscripts and denote $\mu_n, \mathbb{G}_m$ for the corresponding sheaves on either $\mathscr{M}_{1,1,S}$ or $\mathbb{A}^1_S$.

Lemma 3.2.1. Let S be a quasi-compact scheme admitting an ample line bundle. Then the Brauer map $\alpha_{\mathscr{M}_{1,1,S}} : \operatorname{Br} \mathscr{M}_{1,1,S} \to \operatorname{Br}' \mathscr{M}_{1,1,S}$ is an isomorphism.

Proof. By Lemma 2.1.4, it suffices to show that there is an affine scheme Z with a finite flat surjection $Z \to \mathscr{M}_{1,1,\mathbb{Z}}$. Indeed, in this case the base change $Z_S := Z \times_{\mathbb{Z}} S$ is quasi-compact and admits an ample line bundle [88, 0892], hence the Brauer map α_{Z_S} is surjective by Gabber's theorem (see [22]). Since the map $Z_S \to \mathscr{M}_{1,1,Z}$ is finite locally free, we have by Corollary 2.1.5 that $\alpha_{\mathscr{M}_{1,1,S}}$ is surjective.

Since $\mathscr{M}_{1,1,\mathbb{Z}}$ is a separated Deligne-Mumford stack, its diagonal is finite; thus by [28, Theorem 2.7] (or [58, (16.6)]) there exists a scheme Z and a finite surjection $f : Z \to \mathscr{M}_{1,1,\mathbb{Z}}$. Here the composite $Z \to \mathscr{M}_{1,1,\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$ is a proper and quasi-finite morphism between schemes, hence finite; hence Z is an affine scheme of finite type over $\mathbb{Z}$. We may replace Z by its reduction Z_{red} and assume that Z itself is reduced. Since Z is finite type over $\mathbb{Z}$, the normalization map $\overline{Z} \to Z$ is finite; here $\overline{Z}$ decomposes as a disjoint union of finitely many affine normal integral schemes $\overline{Z} = Z_1 \sqcup \cdots \sqcup Z_r$; by replacing Z by some Z_i for which the composite $Z_i \to \overline{Z} \to Z \to \mathbb{A}^1_{\mathbb{Z}}$ is surjective (e.g. some Z_i which intersects the fiber over the generic point of $\mathbb{A}^1_{\mathbb{Z}}$), we may assume that Z is an affine normal integral scheme. Since $Z \to \mathbb{A}^1_{\mathbb{Z}}$ is finite surjective, we have $\dim Z = 2$ and hence Z is Cohen-Macaulay (e.g. [66, Exercise 17.3]). Since $f : Z \to \mathscr{M}_{1,1,\mathbb{Z}}$ is a finite map where Z is integral Cohen-Macaulay and $\mathscr{M}_{1,1,\mathbb{Z}}$ is a regular Deligne-Mumford stack, we have that f is flat (reduce to the case of schemes by taking a smooth cover of $\mathscr{M}_{1,1,\mathbb{Z}}$ by a scheme smooth over $\mathbb{Z}$; then use [66, Theorem 23.1]).

One can give an alternate argument in case at least one prime is invertible on S. By [51, 4.7.2], for $N \geq 3$ the moduli stack of full level N structures is representable by an affine $\mathbb{Z}[\frac{1}{N}]$-scheme $Y(N)$, and we proceed as above. $\square$

Lemma 3.2.2. Let $U := \operatorname{Spec} \mathbb{Z}[t, (t(t-1728))^{-1}] \subset \mathbb{A}^1_{\mathbb{Z}}$ and let $\mathscr{M}^{\circ}_{1,1,\mathbb{Z}} := U \times_{\mathbb{A}^1_{\mathbb{Z}}} \mathscr{M}_{1,1,\mathbb{Z}}$. Then the restriction $\pi^{\circ} : \mathscr{M}^{\circ}_{1,1,\mathbb{Z}} \to U$ of π to U is a trivial $\mathbb{Z}/(2)$-gerbe, i.e. $\mathscr{M}^{\circ}_{1,1,\mathbb{Z}} \simeq \mathrm{B}(\mathbb{Z}/(2))_U$.

Proof. Let S be a scheme and let E_1, E_2 be two elliptic curves over S. If $j(E_1) = j(E_2) \in \Gamma(S, \mathcal{O}_S)$ and $j(E_i), j(E_i) - 1728$ are units of $\Gamma(S, \mathcal{O}_S)$, then by [23, 5.3] one can find a finite étale cover $S' \to S$ such that there is an isomorphism $S' \times_S E_1 \simeq S' \times_S E_2$ of elliptic curves over S'. For any connected scheme S and an elliptic curve E/S for which $j(E)$ and $j(E) - 1728$ are invertible, we have $\operatorname{Aut}(E/S) \simeq \mathbb{Z}/(2)$ by [51, (8.4.2)]. It suffices now to show that there is an elliptic curve E_U over U with j-invariant t. For this we may take the elliptic curve E_U defined by the Weierstrass equation

$$Y^2Z + XYZ = X^3 - \tfrac{36}{t-1728}XZ^2 - \tfrac{1}{t-1728}Z^3$$

which satisfies $\Delta(E_U) = \frac{t^2}{(t-1728)^3}$ and $j(E_U) = t$ (see [86, Proposition III.1.4(c)]). □

Lemma 3.2.3. Let k be an algebraically closed field and let U be a smooth curve over k. If $\mathrm{Pic}(U) = 0$, then $\mathrm{Br}'\,\mathrm{B}(\mathbb{Z}/(2))_U \simeq (\mathbb{G}_m(U))/(2)$.

Proof. The cohomological descent spectral sequence associated to the cover $U \to \mathrm{B}(\mathbb{Z}/(2))_U$ is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathbb{Z}/(2), \mathrm{H}^q_{\text{ét}}(U, \mathbb{G}_m)) \implies \mathrm{H}^{p+q}_{\text{ét}}(\mathrm{B}(\mathbb{Z}/(2))_U, \mathbb{G}_m) \qquad (3.2.3.1)$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. We have by [69, III.2.22 (d)] that $\mathrm{H}^q_{\text{ét}}(U, \mathbb{G}_m) = 0$ for all $q \geq 2$. Moreover, we have $\mathrm{H}^1_{\text{ét}}(U, \mathbb{G}_m) = \mathrm{Pic}(U) = 0$ by assumption. Thus the only row of the E_2-page of (3.2.3.1) containing nonzero entries is $q = 0$, which gives an isomorphism

$$\mathrm{H}^2_{\text{ét}}(\mathrm{B}(\mathbb{Z}/(2))_U, \mathbb{G}_m) \simeq \mathrm{H}^2(\mathbb{Z}/(2), \mathrm{H}^0_{\text{ét}}(U, \mathbb{G}_m)) \simeq (\mathbb{G}_m(U))/(2)$$

of abelian groups. □

Lemma 3.2.4. Let k be an algebraically closed field. If $\mathrm{char}\, k \neq 2, 3$, then $\mathrm{Br}'\,\mathscr{M}_{1,1,k}$ is a subgroup of $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$. If $\mathrm{char}\, k$ is 2 or 3, then $\mathrm{Br}'\,\mathscr{M}_{1,1,k}$ is a subgroup of $\mathbb{Z}/(2)$.

Proof. We have that $\mathscr{M}_{1,1,k}$ is regular Noetherian and that $\mathscr{M}^\circ_{1,1,k} := \mathscr{M}^\circ_{1,1,\mathbb{Z}} \times_{\mathbb{Z}} k$ is a dense open substack; thus by Theorem 2.1.7 the map

$$\mathrm{Br}'\,\mathscr{M}_{1,1,k} \to \mathrm{Br}'\,\mathscr{M}^\circ_{1,1,k}$$

induced by restriction is an injection. Here Lemma 3.2.2 implies $\mathrm{Br}'\,\mathscr{M}^\circ_{1,1,k} = \mathrm{Br}'\,\mathrm{B}(\mathbb{Z}/(2))_U$ for $U = \mathrm{Spec}\, k[t, (t(t-1728))^{-1}]$, and Lemma 3.2.3 implies $\mathrm{Br}'\,\mathrm{B}(\mathbb{Z}/(2))_U$ is $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ if $\mathrm{char}\, k \neq 2, 3$ and $\mathbb{Z}/(2)$ otherwise (here we use that $k^\times = (k^\times)^2$ since k is algebraically closed). □

3.3. **The case** $\mathrm{char}\, k$ **is not** 2. Antieau and Meier [4] compute the Brauer group $\mathrm{Br}\,\mathscr{M}_{1,1,S}$ for various base schemes S, including algebraically closed fields k of odd characteristic [4, 11.2] (the case $\mathrm{char}\, k \neq 2$ in Theorem 3.1.1). In this section we give a proof via a dévissage argument, using the fact that the coarse moduli space morphism $\pi : \mathscr{M} \to \mathbb{A}^1_k$ is a trivial $\mathbb{Z}/(2)$-gerbe away from $0, 1728 \in \mathbb{A}^1_k$ (see Lemma 3.2.2). Our proof is divided into two cases, depending on whether $\mathrm{char}\, k = 3$ or $\mathrm{char}\, k \neq 3$ (this will determine whether we puncture $\mathbb{A}^1_k$ at one or two points, respectively). We first fix notation and record some observations that apply to both cases.

3.3.1. We abbreviate $\mathscr{M} := \mathscr{M}_{1,1,k}$. By Lemma 3.2.1, the Brauer map $\alpha_{\mathscr{M}} : \mathrm{Br}\,\mathscr{M} \to \mathrm{Br}'\,\mathscr{M}$ is an isomorphism. By Lemma 3.2.4, the main task is to show that the 2-torsion in $\mathrm{Br}\,\mathscr{M}$ is 0.

For any integer $n \geq 1$, the étale Kummer sequence

$$1 \to \mu_{2^n} \to \mathbb{G}_m \overset{\times 2^n}{\to} \mathbb{G}_m \to 1$$

gives an exact sequence

$$0 \to (\mathrm{Pic}\,\mathscr{M})/(2^n) \to \mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \to \mathrm{H}^2(\mathscr{M}, \mathbb{G}_m)[2^n] \to 0 \qquad (3.3.1.1)$$

of abelian groups. Since we have $\mathrm{Pic}\,\mathscr{M} \simeq \mathbb{Z}/(12)$ by [36], we wish to compute $\mathrm{H}^2(\mathscr{M}, \mu_{2^n})$.

Set

$$U := \mathrm{Spec}\, k[t, (t(t-1728))^{-1}] = \mathbb{A}^1_k \setminus \{0, 1728\}$$

with inclusion $j : U \to \mathbb{A}^1_k$ and let $i : Z \to \mathbb{A}^1_k$ be the complement with reduced induced closed subscheme structure. (Thus, if $\operatorname{char} k$ is 2 or 3 then $Z \simeq \operatorname{Spec} k$, otherwise $Z \simeq \operatorname{Spec} k \amalg \operatorname{Spec} k$.) Set

$$\mathscr{M}^\circ := U \times_{\mathbb{A}^1_k} \mathscr{M}$$
$$\mathscr{M}_Z := Z \times_{\mathbb{A}^1_k} \mathscr{M}$$

with projections $\pi^\circ : \mathscr{M}^\circ \to U$ and $\pi_Z : \mathscr{M}_Z \to Z$. We have a commutative diagram

$$\begin{array}{ccccc} \mathscr{M}^\circ & \longrightarrow & \mathscr{M} & \longleftarrow & \mathscr{M}_Z \\ {\scriptstyle \pi^\circ}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Z} \\ U & \xrightarrow[j]{} & \mathbb{A}^1_k & \xleftarrow[i]{} & Z \end{array} \tag{3.3.1.2}$$

with cartesian squares.

We have a distinguished triangle

$$j_! j^* \mathbf{R}\pi_* \mu_{2^n} \to \mathbf{R}\pi_* \mu_{2^n} \to i_* i^* \mathbf{R}\pi_* \mu_{2^n} \xrightarrow{+1} \tag{3.3.1.3}$$

in the derived category of bounded-below complexes of abelian sheaves on the étale site of $\mathbb{A}^1_k$, whose associated long exact sequence has the form

$$\begin{array}{l} \mathrm{H}^0(\mathbb{A}^1_k, j_! \mathbf{R}\pi^\circ_* \mu_{2^n}) \to \mathrm{H}^0(\mathscr{M}, \mu_{2^n}) \longrightarrow \mathrm{H}^0(Z, i^* \mathbf{R}\pi_* \mu_{2^n}) \\ \to \mathrm{H}^1(\mathbb{A}^1_k, j_! \mathbf{R}\pi^\circ_* \mu_{2^n}) \to \mathrm{H}^1(\mathscr{M}, \mu_{2^n}) \longrightarrow \mathrm{H}^1(Z, i^* \mathbf{R}\pi_* \mu_{2^n}) \\ \to \mathrm{H}^2(\mathbb{A}^1_k, j_! \mathbf{R}\pi^\circ_* \mu_{2^n}) \to \mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \longrightarrow \mathrm{H}^2(Z, i^* \mathbf{R}\pi_* \mu_{2^n}) \end{array} \tag{3.3.1.4}$$

since $j^* \mathbf{R}\pi_* \mu_{2^n} \simeq \mathbf{R}\pi^\circ_* \mu_{2^n}$ and

$$\mathrm{H}^s(\mathbb{A}^1_k, \mathbf{R}\pi_* \mu_{2^n}) \simeq \mathrm{H}^s(\mathscr{M}, \mu_{2^n})$$
$$\mathrm{H}^s(\mathbb{A}^1_k, i_* i^* \mathbf{R}\pi_* \mu_{2^n}) \simeq \mathrm{H}^s(Z, i^* \mathbf{R}\pi_* \mu_{2^n})$$

for all s. We will first compute the groups $\mathrm{H}^s(\mathbb{A}^1_k, j_! j^* \mathbf{R}\pi_* \mu_{2^n})$ in the left column of (3.3.1.4).

Lemma 3.3.2. Let k be an algebraically closed field, let $x_1, \ldots, x_r \in \mathbb{A}^1_k$ be r distinct k-points, set

$$Z := \operatorname{Spec} k(x_1) \amalg \cdots \amalg \operatorname{Spec} k(x_r)$$

and let $U = \mathbb{A}^1_k \setminus Z$ be the complement with inclusion $j : U \to \mathbb{A}^1_k$. For any positive integer ℓ invertible in k, we have

$$\mathrm{H}^s(\mathbb{A}^1_k, j_! \mu_\ell) = \begin{cases} 0 & s \neq 1 \\ (\mu_\ell(k))^{\oplus(r-1)} & s = 1 \end{cases}.$$

Proof. Let $i : Z \to \mathbb{A}^1_k$ be the inclusion. We have a distinguished triangle

$$j_! \mu_\ell|_U \to \mu_\ell \to i_* i^* \mu_\ell \xrightarrow{+1}$$

in the derived category of bounded-below complexes of abelian sheaves on the big étale site of $\mathbb{A}^1_k$, which gives a long exact sequence

$$\begin{array}{l} \mathrm{H}^0(\mathbb{A}^1_k, j_!\mu_\ell|_U) \longrightarrow \mathrm{H}^0(\mathbb{A}^1_k, \mu_\ell) \longrightarrow \mathrm{H}^0(Z, \mu_\ell) \\ \hookrightarrow \mathrm{H}^1(\mathbb{A}^1_k, j_!\mu_\ell|_U) \longrightarrow \mathrm{H}^1(\mathbb{A}^1_k, \mu_\ell) \longrightarrow \mathrm{H}^1(Z, \mu_\ell) \\ \hookrightarrow \mathrm{H}^2(\mathbb{A}^1_k, j_!\mu_\ell|_U) \longrightarrow \mathrm{H}^2(\mathbb{A}^1_k, \mu_\ell) \longrightarrow \mathrm{H}^2(Z, \mu_\ell) \\ \hookrightarrow \mathrm{H}^3(\mathbb{A}^1_k, j_!\mu_\ell|_U) \longrightarrow \cdots \end{array}$$

in cohomology. The map $\mathrm{H}^0(\mathbb{A}^1_k, \mu_\ell) \to \mathrm{H}^0(Z, \mu_\ell)$ is identified with the diagonal map $\mu_\ell(k) \to (\mu_\ell(k))^{\oplus r}$. Since k is algebraically closed, the étale site of Z is trivial, hence $\mathrm{H}^s(Z, \mu_\ell) = 0$ for $s \geq 1$. By [24, Exp. 1, III, (3.6)] we have $\mathrm{H}^s(\mathbb{A}^1_k, \mu_\ell) = 0$ for $s \geq 2$. We have $\mathbb{G}_m(\mathbb{A}^1_k) \simeq \mathbb{G}_m(k)$ and the multiplication-by-ℓ map $\times\ell : \mathbb{G}_m(k) \to \mathbb{G}_m(k)$ is surjective; thus $\mathrm{H}^1(\mathbb{A}^1_k, \mu_\ell) = \mathrm{H}^1(\mathbb{A}^1_k, \mathbb{G}_m)[\ell] = (\operatorname{Pic} \mathbb{A}^1_k)[\ell] = 0$ by the Kummer sequence. □

Lemma 3.3.3. In the setup of Lemma 3.3.2, let n be any positive integer and let $\pi^\circ : \mathrm{B}(\mathbb{Z}/(n))_U \to U$ be the trivial $\mathbb{Z}/(n)$-gerbe over U. Then

$$\mathrm{H}^s(\mathbb{A}^1_k, j_!\mathbf{R}\pi^\circ_*\mu_\ell) = \begin{cases} 0 & \text{if } s = 0\ , \\ (\mu_\ell(k))^{\oplus(r-1)} & \text{if } s = 1\ , \\ (\mu_{\gcd(n,\ell)}(k))^{\oplus(r-1)} & \text{if } s = 2\ . \end{cases}$$

Proof. We set

$$\mathcal{C} := j_!\mathbf{R}\pi^\circ_*\mu_\ell$$

for convenience. We will compute the groups $\mathrm{H}^s(\mathbb{A}^1_k, \mathcal{C})$ using the fact that the canonical truncations $\tau_{\leq s}\mathcal{C}$ satisfy

$$\mathrm{H}^s(\mathbb{A}^1_k, \tau_{\leq t}\mathcal{C}) \simeq \mathrm{H}^s(\mathbb{A}^1_k, \mathcal{C}) \tag{3.3.3.1}$$

for $s \leq t$. For any $s \in \mathbb{Z}$, the distinguished triangle

$$\tau_{\leq s-1}\mathcal{C} \to \tau_{\leq s}\mathcal{C} \to (h^s\mathcal{C})[-s] \xrightarrow{+1} \tag{3.3.3.2}$$

gives a long exact sequence

$$\begin{array}{l} \mathrm{H}^0(\mathbb{A}^1_k, \tau_{\leq s-1}\mathcal{C}) \longrightarrow \mathrm{H}^0(\mathbb{A}^1_k, \tau_{\leq s}\mathcal{C}) \longrightarrow \mathrm{H}^{0-s}(\mathbb{A}^1_k, j_!\mathbf{R}^s\pi^\circ_*\mu_\ell) \\ \hookrightarrow \mathrm{H}^1(\mathbb{A}^1_k, \tau_{\leq s-1}\mathcal{C}) \longrightarrow \mathrm{H}^1(\mathbb{A}^1_k, \tau_{\leq s}\mathcal{C}) \longrightarrow \mathrm{H}^{1-s}(\mathbb{A}^1_k, j_!\mathbf{R}^s\pi^\circ_*\mu_\ell) \\ \hookrightarrow \mathrm{H}^2(\mathbb{A}^1_k, \tau_{\leq s-1}\mathcal{C}) \longrightarrow \mathrm{H}^2(\mathbb{A}^1_k, \tau_{\leq s}\mathcal{C}) \longrightarrow \mathrm{H}^{2-s}(\mathbb{A}^1_k, j_!\mathbf{R}^s\pi^\circ_*\mu_\ell) \end{array} \tag{3.3.3.3}$$

where

$$h^s\mathcal{C} \simeq j_!\mathbf{R}^s\pi^\circ_*\mu_\ell$$

since $j_!$ is exact.

Since $\pi^\circ : \mathrm{B}(\mathbb{Z}/(n))_U \to U$ is a trivial $\mathbb{Z}/(n)$-gerbe, by Lemma B.2.2 we have

$$\mathbf{R}^s\pi^\circ_*\mu_\ell \simeq \begin{cases} \mu_\ell & s = 0 \\ \mu_\ell[n] & s = 1, 3, 5, \ldots \\ \mu_\ell/(n) & s = 2, 4, 6, \ldots \end{cases} \tag{3.3.3.4}$$

where $\mu_\ell[n]$ and $\mu_\ell/(n)$ are defined by the exact sequence

$$1 \to \mu_\ell[n] \to \mu_\ell \xrightarrow{\times n} \mu_\ell \to \mu_\ell/(n) \to 1$$

of abelian sheaves. Since k is algebraically closed of characteristic prime to ℓ, the sheaves $\mu_\ell[n]$ and $\mu_\ell/(n)$ are both isomorphic to $\mu_{\gcd(n,\ell)}$, but for us the difference is important for reasons of functoriality (as ℓ is allowed to vary). More precisely, if ℓ_1 divides ℓ_2, then the inclusion $\mu_{\ell_1} \to \mu_{\ell_2}$ induces an inclusion

$$\mu_{\ell_1}[n] \to \mu_{\ell_2}[n]$$

whereas

$$\mu_{\ell_1}/(n) \to \mu_{\ell_2}/(n) \qquad (3.3.3.5)$$

is not necessarily injective since an element $x \in \mu_{\ell_1}$ which is not an nth power of any $y_1 \in \mu_{\ell_1}$ may be an nth power of some $y_2 \in \mu_{\ell_2}$ (in particular, if $\ell_2 = n\ell_1$, then (3.3.3.5) is the zero morphism).

We have

$$\tau_{\leq 0}\mathcal{C} \simeq h^0\mathcal{C} \simeq j_!\mathbf{R}^0\pi^\circ_*\mu_\ell \simeq j_!\pi^\circ_*\mu_\ell \simeq j_!\mu_\ell$$

since π° is a coarse moduli space morphism and $\mathbf{R}^1\pi^\circ_*\mu_\ell \simeq \mu_{\gcd(n,\ell)}$ by (3.3.3.4). Applying Lemma 3.3.2 to the case $s = 1$ in (3.3.3.3) implies $\mathrm{H}^0(\mathbb{A}^1_k, \tau_{\leq 1}\mathcal{C}) = 0$ and gives isomorphisms $\mathrm{H}^1(\mathbb{A}^1_k, j_!\mu_\ell) \simeq \mathrm{H}^1(\mathbb{A}^1_k, \tau_{\leq 1}\mathcal{C})$ and $\mathrm{H}^2(\mathbb{A}^1_k, \tau_{\leq 1}\mathcal{C}) \simeq \mathrm{H}^1(\mathbb{A}^1_k, j_!\mu_{\gcd(n,\ell)})$.

Since $\mathbf{R}^2\pi^\circ_*\mu_\ell \simeq \mu_{\gcd(n,\ell)}$ by (3.3.3.4) and $\mathrm{H}^s(\mathbb{A}^1_k, j_!\mu_{\gcd(n,\ell)}) = 0$ for $s = -2, -1, 0$, the case $s = 2$ in (3.3.3.3) gives isomorphisms $\mathrm{H}^s(\mathbb{A}^1_k, \tau_{\leq 1}\mathcal{C}) \simeq \mathrm{H}^s(\mathbb{A}^1_k, \tau_{\leq 2}\mathcal{C})$ for $s = 0, 1, 2$, which implies the desired result. □

3.3.4 (Proof of Theorem 3.1.1 for $\operatorname{char} k = 3$). If $\operatorname{char} k = 3$, then Z consists of one point, so taking $r = 1$ in Lemma 3.3.3 implies

$$\mathrm{H}^s(\mathbb{A}^1_k, j_!\mathbf{R}\pi^\circ_*\mu_{2^n}) = 0 \qquad (3.3.4.1)$$

for $s = 0, 1, 2$. Therefore, to compute $\mathrm{H}^2(\mathscr{M}, \mu_{2^n})$, it now remains to compute $\mathrm{H}^2(Z, i^*\mathbf{R}\pi_*\mu_{2^n})$ in (3.3.1.4). The stabilizer of any object of $\mathscr{M}$ lying over $i : Z \to \mathbb{A}^1_k$ is the automorphism group of an elliptic curve with j-invariant 0, which is the semidirect product $\Gamma = \mathbb{Z}/(3) \rtimes \mathbb{Z}/(4)$ since k has characteristic 3. The underlying reduced stack $(\mathscr{M}_Z)_{\mathrm{red}}$ is the residual gerbe associated to the unique point of $|\mathscr{M}_Z|$ and is isomorphic to the classifying stack $\mathrm{B}\Gamma_k$. We have natural isomorphisms

$$\mathrm{H}^2(Z, i^*\mathbf{R}\pi_*\mu_{2^n}) \simeq i^*\mathbf{R}^2\pi_*\mu_{2^n} \overset{1}{\simeq} \mathrm{H}^2(\mathscr{M}_Z, \mu_{2^n}) \overset{2}{\simeq} \mathrm{H}^2(\mathrm{B}\Gamma_k, \mu_{2^n}) \overset{3}{\simeq} \mathrm{H}^2(\Gamma, \mu_{2^n}(k))$$

where isomorphism 1 follows from proper base change [75, 1.3], isomorphism 2 is by Lemma 3.3.5, and isomorphism 3 is by the cohomological descent spectral sequence for the covering $\operatorname{Spec} k \to \mathrm{B}\Gamma_k$ (and the fact that $\mathrm{H}^i(\operatorname{Spec} k, \mu_{2^n}) = 0$ for $i > 0$ since k is algebraically closed). The Hochschild-Serre spectral sequence for the exact sequence

$$1 \to \mathbb{Z}/(3) \to \Gamma \to \mathbb{Z}/(4) \to 1$$

gives an isomorphism

$$\mathrm{H}^2(\Gamma, \mu_{2^n}(k)) \simeq \mathrm{H}^2(\mathbb{Z}/(4), \mu_{2^n}(k)) \simeq \mu_{2^n}(k)/(4)$$

where $\mathrm{H}^i(\mathbb{Z}/(3), \mu_{2^n}(k)) = 0$ for $i > 0$ since 3 is coprime to the order of $\mu_{2^n}(k)$. Since the first term in the last row of the diagram (3.3.1.4) is zero by (3.3.4.1), the above observations imply that we have natural inclusions

$$\mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \to \mu_{2^n}(k)/(4)$$

compatible with the inclusions $\mu_{2^n} \subset \mu_{2^{n+1}}$ for all n. The inclusion $\mu_{2^n} \subset \mu_{2^{n+2}}$ induces the zero map $\mu_{2^n}(k)/(4) \to \mu_{2^{n+2}}(k)/(4)$, so $\mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \to \mathrm{H}^2(\mathscr{M}, \mu_{2^{n+2}})$ is the zero map as well, hence

$$\varinjlim_{n\in\mathbb{N}} \mathrm{H}^2(\mathscr{M}, \mu_{2^n}) = 0$$

which by (3.3.1.1) gives $\mathrm{H}^2(\mathscr{M}, \mathbb{G}_m)[2^n] = 0$ for all n.

Lemma 3.3.5. Let X be a Deligne-Mumford stack, let $\mathcal{I} \subseteq \mathcal{O}_X$ be a square-zero ideal, and let $\pi : X_0 \to X$ denote the closed immersion corresponding to $\mathcal{I}$. If n is an integer which is invertible on X, then for all $i \geq 1$ the reduction map

$$\mathrm{H}^i_{\text{ét}}(X, \mu_n) \to \mathrm{H}^i_{\text{ét}}(X_0, \mu_n)$$

is an isomorphism.

Proof. We have an exact sequence

$$1 \to 1 + \mathcal{I} \to \mathbb{G}_{m,X} \to \pi_*(\mathbb{G}_{m,X_0}) \to 1$$

of abelian sheaves on the (small) étale site of X. Since n is invertible on X, we have by the snake lemma that $\mu_{n,X} \to \pi_*(\mu_{n,X_0})$ is an isomorphism, thus it remains to show that the map

$$\mathrm{H}^i_{\text{ét}}(X, \pi_*(\mu_{n,X_0})) \to \mathrm{H}^i_{\text{ét}}(X_0, \mu_{n,X_0})$$

is an isomorphism. For this it suffices to show that $\mathbf{R}^i\pi_*(\mu_{n,X_0}) = 0$ for $i \geq 1$. The stalk of $\mathbf{R}^i\pi_*(\mu_{n,X_0}) = 0$ at a geometric point $\overline{x} \in X$ is the cohomology $\mathrm{H}^i_{\text{ét}}(\operatorname{Spec} A, \mu_{n,A_{\text{red}}})$ of the reduction of some strictly henselian local ring A, which is 0 if $i \geq 1$. ∎

3.3.6 (Proof of Theorem 3.1.1, for $\operatorname{char} k \neq 2, 3$). We describe the terms in (3.3.1.4). For the right column, we have

$$\mathrm{H}^s(Z, i^*\mathbf{R}\pi_*\mu_{2^n}) \simeq \mathrm{H}^s(\mathbb{Z}/(4), \mu_{2^n}(k)) \oplus \mathrm{H}^s(\mathbb{Z}/(6), \mu_{2^n}(k))$$

by [1, A.0.7]. For the middle column, we have

$$\mathrm{H}^0(\mathscr{M}, \mu_{2^n}) \simeq \mathrm{H}^0(\mathbb{A}^1_k, \mu_{2^n}) \simeq \mu_{2^n}(k)$$

since $\mathbb{A}^1_k$ is the coarse moduli space of $\mathscr{M}$, and we have

$$\mathrm{H}^1(\mathscr{M}, \mu_{2^n}) \overset{1}{\simeq} \mathrm{H}^1(\mathscr{M}, \mathbb{G}_m)[2^n] \overset{2}{\simeq} (\mathbb{Z}/(12))[2^n] \overset{3}{\simeq} \mathbb{Z}/(4)$$

where isomorphism 1 follows since $k^\times = (k^\times)^{2^n}$, isomorphism 2 is by [70], and isomorphism 3 holds for $n \gg 0$. For the left column, we have

$$\mathrm{H}^s(\tau_{\leq 1} j_! \mathbf{R}\pi^\circ_* \mu_{2^n}) = \begin{cases} 0 & s = 0 \\ \mu_{2^n} & s = 1 \\ \mu_2 & s = 2 \end{cases}$$

by Lemma 3.3.3.

To summarize, (3.3.1.4) simplifies to

$$\begin{array}{l} 0 \longrightarrow \mu_{2^n} \longrightarrow \mu_{2^n} \oplus \mu_{2^n} \longrightarrow \\ \mu_{2^n} \longrightarrow \mathbb{Z}/(4) \longrightarrow \mu_4 \oplus \mu_2 \longrightarrow \\ \mu_2 \longrightarrow \mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \longrightarrow \mu_{2^n}/(4) \oplus \mu_{2^n}/(6) \end{array} \tag{3.3.6.1}$$

for $n \gg 0$, and counting the number of elements in each group in (3.3.6.1) implies that the last morphism

$$\mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \to \mu_{2^n}/(4) \oplus \mu_{2^n}/(6)$$

is injective. Furthermore, the inclusion

$$\mu_{2^n} \subset \mu_{2^{n+2}}$$

induces the zero map

$$\mu_{2^n}/(4) \oplus \mu_{2^n}/(6) \to \mu_{2^{n+2}}/(4) \oplus \mu_{2^{n+2}}/(6)$$

so the map $\mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \to \mathrm{H}^2(\mathscr{M}, \mu_{2^{n+2}})$ is the zero map as well, hence

$$\varinjlim_{n \in \mathbb{N}} \mathrm{H}^2(\mathscr{M}, \mu_{2^n}) = 0$$

which by (3.3.1.1) gives $\mathrm{H}^2(\mathscr{M}, \mathbb{G}_m)[2^n] = 0$ for all n.

3.4. **The case** $\operatorname{char} k$ **is** 2**.** In this section we prove Theorem 3.1.1 (in case $\operatorname{char} k = 2$) and Theorem 3.1.2. For convenience, we denote $\mathrm{GL}_{n,p} := \mathrm{GL}_n(\mathbb{Z}/(p))$ and $\mathrm{SL}_{n,p} := \mathrm{SL}_n(\mathbb{Z}/(p))$. We denote by e the identity element of $\mathrm{GL}_{n,p}$.

3.4.1 (Hesse presentation of $\mathscr{M}_{1,1,k}$). By [36, 6.2] (and explained in more detail in D.2.3), there is a left action of $\mathrm{GL}_{2,3}$ on the $\mathbb{Z}[\frac{1}{3}]$-algebra

$$A_{\mathrm{H}} := \mathbb{Z}[\tfrac{1}{3}, \mu, \omega, \tfrac{1}{\mu^3-1}]/(\omega^2 + \omega + 1)$$

sending

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} * (\mu, \omega) &= (\mu, \omega^2) \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} * (\mu, \omega) &= (\omega\mu, \omega) \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} * (\mu, \omega) &= (\tfrac{\mu+2}{\mu-1}, \omega) \end{aligned} \tag{3.4.1.1}$$

for which the corresponding right action of $\mathrm{GL}_{2,3}$ on the $\mathbb{Z}[\frac{1}{3}]$-scheme

$$S_{\mathrm{H}} := \operatorname{Spec} A_{\mathrm{H}}$$

gives a presentation

$$\mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{3}]} \simeq [S_{\mathrm{H}}/\mathrm{GL}_{2,3}] \tag{3.4.1.2}$$

of $\mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{3}]}$ as a global quotient stack. The morphism

$$S_{\mathrm{H}} \to \mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{3}]} \tag{3.4.1.3}$$

is given by the elliptic curve

$$X^3 + Y^3 + Z^3 = 3\mu XYZ$$

over S_{H}.

3.4.2 (Cohomological descent). Let k be an algebraically closed field of characteristic 2. The Brauer map $\alpha_{\mathscr{M}_{1,1,k}} : \mathrm{Br}\,\mathscr{M}_{1,1,k} \to \mathrm{Br}'\,\mathscr{M}_{1,1,k}$ is an isomorphism by Lemma 3.2.1. By Lemma 3.2.4, there is only 2-torsion in $\mathrm{Br}\,\mathscr{M}_{1,1,k}$. By Grothendieck's fppf-étale comparison theorem for smooth commutative group schemes [43, (11.7)], it suffices to compute the 2-torsion in $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mathbb{G}_m)$. Since $\mathrm{Spec}\,k$ is a reduced scheme, we have

$$\mathrm{H}^1_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mathbb{G}_m) = \mathrm{Pic}(\mathscr{M}_{1,1,k}) = \mathbb{Z}/(12)$$

by [36, 1.1]. Thus, for any integer n, the fppf Kummer sequence

$$1 \to \mu_2 \to \mathbb{G}_m \stackrel{\times 2}{\to} \mathbb{G}_m \to 1 \tag{3.4.2.1}$$

gives an exact sequence

$$1 \to \mathbb{Z}/(2) \stackrel{\partial}{\to} \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mu_2) \to \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mathbb{G}_m)[2] \to 1 \tag{3.4.2.2}$$

of abelian groups. It remains to compute the middle term $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mu_2)$.

The cohomological descent spectral sequence associated to the cover (3.4.1.3) is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{GL}_{2,3}, \mathrm{H}^q_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2)) \implies \mathrm{H}^{p+q}_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mu_2) \tag{3.4.2.3}$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$.

Let

$$\xi \in k$$

be a fixed primitive 3rd root of unity. By the Chinese Remainder Theorem, there is a k-algebra isomorphism

$$A_{\mathrm{H},k} = k[\mu, \omega, \tfrac{1}{\mu^3-1}]/(\omega^2+\omega+1) \to k[\nu_1, \tfrac{1}{\nu_1^3-1}] \times k[\nu_2, \tfrac{1}{\nu_2^3-1}] \tag{3.4.2.4}$$

sending $\mu \mapsto (\nu_1, \nu_2)$ and $\omega \mapsto (\xi, \xi^2)$. Since $S_{\mathrm{H},k}$ is a smooth curve over an algebraically closed field, we have by [69, III.2.22 (d)] that $\mathrm{H}^q_{\text{ét}}(S_{\mathrm{H},k}, \mathbb{G}_m) = 0$ for all $q \geq 2$; since $S_{\mathrm{H},k}$ is a disjoint union of two copies of a distinguished affine open subset of $\mathbb{A}^1_k$, we have $\mathrm{H}^1_{\text{ét}}(S_{\mathrm{H},k}, \mathbb{G}_m) = \mathrm{Pic}(S_{\mathrm{H},k}) = 0$. By [43, (11.7)] we have $\mathrm{H}^q_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mathbb{G}_m) = \mathrm{H}^q_{\text{ét}}(S_{\mathrm{H},k}, \mathbb{G}_m)$ for all $q \geq 0$; thus the fppf Kummer sequence implies $\mathrm{H}^q_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2) = 0$ for all $q \geq 2$. Furthermore, we have $\mathrm{H}^0_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2) = 0$ since $S_{\mathrm{H},k}$ is the product of two integral domains of characteristic 2. Thus the only nonzero terms on the E_2-page of (3.4.2.3) occur on the $q = 1$ row, so we have an isomorphism

$$\mathrm{H}^{p+1}_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mu_2) \simeq \mathrm{H}^p(\mathrm{GL}_{2,3}, \mathrm{H}^1_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2)) \tag{3.4.2.5}$$

for all $p \geq 0$. We are interested in the case $p = 1$.

3.4.3 (Description of the $\mathrm{GL}_{2,3}$-action on $\mathrm{H}^1_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2)$). We describe the abelian group

$$M := \mathrm{H}^1_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2)$$

and the left $\mathrm{GL}_{2,3}$-module structure it inherits from (3.4.1.1). Since $k[\mu, (\mu^3-1)^{-1}]$ is a principal localization of the polynomial ring $k[\mu]$ by a polynomial $\mu^3-1 = (\mu-1)(\mu-\xi)(\mu-\xi^2)$ splitting into three distinct irreducible factors, we have an isomorphism

$$(k[\mu, \tfrac{1}{\mu^3-1}])^\times \simeq k^\times \cdot (\mu-1)^{\mathbb{Z}} \cdot (\mu-\xi)^{\mathbb{Z}} \cdot (\mu-\xi^2)^{\mathbb{Z}} \tag{3.4.3.1}$$

of abelian groups. Thus (3.4.2.4) and the Kummer sequence (3.4.2.1) gives an isomorphism

$$M \simeq (\mathbb{Z}/(2))^{\oplus 6} \tag{3.4.3.2}$$

of abelian groups, with generators given by the classes of $\nu_i - \xi^j$ for $i = 1, 2$ and $j = 0, 1, 2$.

The isomorphism (3.4.2.4) is given by the map

$$s_1(\mu)\omega + s_0(\mu) \mapsto \big(s_1(\nu_1)\xi + s_0(\nu_1), s_1(\nu_2)\xi^2 + s_0(\nu_2)\big) \tag{3.4.3.3}$$

for $s_0, s_1 \in k[\mu, \frac{1}{\mu^3-1}]$. The inverse of (3.4.2.4) is given by the map

$$(f_1(\nu_1), f_2(\nu_2)) \mapsto f_1(\mu)\left(\frac{\omega}{\xi - \xi^2} + \frac{\xi}{\xi - 1}\right) + f_2(\mu)\left(\frac{-\omega}{\xi - \xi^2} + \frac{-1}{\xi - 1}\right) \tag{3.4.3.4}$$

where $f_i(\nu_i) \in k[\nu_i, \frac{1}{\nu_i^3-1}]$. (Note that, if we set $\mathrm{A}_1(t) := \frac{t}{\xi-\xi^2} + \frac{\xi}{\xi-1}$ and $\mathrm{A}_2(t) := \frac{-t}{\xi-\xi^2} + \frac{-1}{\xi-1}$, then $\mathrm{A}_1(t) + \mathrm{A}_2(t) = 1$ and $\mathrm{A}_i(\xi^j)$ is the Kronecker delta function.)

A computation with (3.4.1.1), (3.4.3.3), (3.4.3.4) shows that the action of $\mathrm{GL}_{2,3}$ on the right hand side of (3.4.2.4) is given by

$$\begin{aligned}
\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} * (f_1(\nu_1), f_2(\nu_2)) &= (f_2(\nu_1), f_1(\nu_2))\\
\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} * (f_1(\nu_1), f_2(\nu_2)) &= (f_1(\xi\nu_1), f_2(\xi^2\nu_2))\\
\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} * (f_1(\nu_1), f_2(\nu_2)) &= (f_1(\tfrac{\nu_1+2}{\nu_1-1}), f_2(\tfrac{\nu_2+2}{\nu_2-1}))
\end{aligned} \tag{3.4.3.5}$$

for $f_i(\nu_i) \in k[\nu_i, \frac{1}{\nu_i^3-1}]$. A computation with (3.4.3.5) (and using that $\mathrm{char}\, k = 2$) shows that the action of $\mathrm{GL}_{2,3}$ on (3.4.3.2) is given by (3.4.3.6), where every element is considered up to multiplication by $k^\times$.

(3.4.3.6)

	$\nu_1 - 1$	$\nu_1 - \xi$	$\nu_1 - \xi^2$	$\nu_2 - 1$	$\nu_2 - \xi$	$\nu_2 - \xi^2$
$\mathsf{M}_1 := \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$	$\nu_2 - 1$	$\nu_2 - \xi$	$\nu_2 - \xi^2$	$\nu_1 - 1$	$\nu_1 - \xi$	$\nu_1 - \xi^2$
$\mathsf{M}_2 := \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$	$\nu_1 - \xi^2$	$\nu_1 - 1$	$\nu_1 - \xi$	$\nu_2 - \xi$	$\nu_2 - \xi^2$	$\nu_2 - 1$
$\mathsf{i} := \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$	$\frac{1}{\nu_1 - 1}$	$\frac{\nu_1 - \xi^2}{\nu_1 - 1}$	$\frac{\nu_1 - \xi}{\nu_1 - 1}$	$\frac{1}{\nu_2 - 1}$	$\frac{\nu_2 - \xi^2}{\nu_2 - 1}$	$\frac{\nu_2 - \xi}{\nu_2 - 1}$

3.4.4. We compute $\mathrm{H}^1(\mathrm{GL}_{2,3}, M)$. (In Appendix E we provide Magma code that can be used to verify this computation, using (3.4.3.6).) We have a filtration of groups

$$\mathrm{Q}_8 \trianglelefteq \mathrm{SL}_{2,3} \trianglelefteq \mathrm{GL}_{2,3} \tag{3.4.4.1}$$

where each is a normal subgroup of the next. Here Q_8 denotes the quaternion group

$$\mathrm{Q}_8 = \{\pm\mathsf{e}, \pm\mathsf{i}, \pm\mathsf{j}, \pm\mathsf{k} \;:\; \mathsf{ijk} = \mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -\mathsf{e}\}$$

and is identified with the subgroup of $\mathrm{GL}_{2,3}$ as follows:

$$\mathsf{i} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \qquad \mathsf{j} = \begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix} \qquad \mathsf{k} = \begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}$$

The quotient $\mathrm{GL}_{2,3}/\mathrm{SL}_{2,3}$ is cyclic of order 2 and is generated by M_1 in (3.4.3.6). The quotient $\mathrm{SL}_{2,3}/\mathrm{Q}_8$ is cyclic of order 3 and is generated by M_2 in (3.4.3.6). For $i = 1, 2$, let $\langle \mathsf{M}_i \rangle$ denote the subgroup of $\mathrm{GL}_{2,3}$ generated by M_i. We note that $\mathrm{SL}_{2,3}$ is generated by i and M_2.

Let

$$F : (\mathbb{Z}[\mathrm{GL}_{2,3}]\text{-Mod}) \to (\mathbb{Z}[\mathrm{SL}_{2,3}]\text{-Mod})$$

be the forgetful functor. An inspection of (3.4.3.6) implies that $F(M)$ is the direct sum $N_1 \oplus N_2$ where N_i is the $\mathrm{SL}_{2,3}$-submodule of $F(M)$ generated by the classes of $\nu_i - 1, \nu_i - \xi, \nu_i - \xi^2$, and moreover M_1 switches the summands N_1 and N_2. Under the adjunction

$$\mathrm{Hom}_{\mathrm{SL}_{2,3}}(F(M), N_1) \simeq \mathrm{Hom}_{\mathrm{GL}_{2,3}}(M, \mathrm{Ind}_{\mathrm{SL}_{2,3}}^{\mathrm{GL}_{2,3}}(N_1))$$

the projection map $F(M) \simeq N_1 \oplus N_2 \to N_1$ onto the first factor corresponds to a morphism

$$M \to \mathrm{Ind}_{\mathrm{SL}_{2,3}}^{\mathrm{GL}_{2,3}}(N_1) \tag{3.4.4.2}$$

of $\mathrm{GL}_{2,3}$-modules. Given $m \in M$, write $m = n_1 + n_2$ for $n_i \in N_i$; then the image of m under (3.4.4.2) is the function $\varphi_m \in \mathrm{Hom}_{\mathbb{Z}[\mathrm{SL}_{2,3}]}(\mathbb{Z}[\mathrm{GL}_{2,3}], N_1)$ such that $\varphi_m([\mathsf{e}]) = n_1$ and $\varphi_m([\mathsf{M}_1]) = \mathsf{M}_1 \cdot n_2$; thus (3.4.4.2) is an isomorphism.

A computation using (3.4.3.6) and the identities

$$\begin{aligned} \mathsf{k} &= \mathsf{M}_2^{-1} \cdot \mathsf{i} \cdot \mathsf{M}_2 \\ \mathsf{i} &= \mathsf{M}_2^{-1} \cdot \mathsf{j} \cdot \mathsf{M}_2 \\ \mathsf{j} &= \mathsf{M}_2^{-1} \cdot \mathsf{k} \cdot \mathsf{M}_2 \end{aligned} \tag{3.4.4.3}$$

shows that the action of an element $\mathsf{g} \in \mathrm{SL}_{2,3}$ on N_1 is by left multiplication by the matrix T_{g} as in (3.4.4.4), with elements of N_1 being viewed as vertical vectors. We note $T_{-\mathsf{e}} = T_{\mathsf{i}}^2 = T_{\mathsf{j}}^2 = T_{\mathsf{k}}^2 = \mathrm{id}_{N_1}$, i.e. $-\mathsf{e}$ acts trivially on N_1.

$$\begin{array}{c|cccc} \mathsf{g} & \mathsf{M}_2 & \mathsf{i} & \mathsf{j} & \mathsf{k} \\ T_{\mathsf{g}} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \tag{3.4.4.4}$$

Since M is an induced module, the restriction map

$$\mathrm{H}^1(\mathrm{GL}_{2,3}, M) \to \mathrm{H}^1(\mathrm{SL}_{2,3}, N_1) \tag{3.4.4.5}$$

is an isomorphism so we reduce to computing $\mathrm{H}^1(\mathrm{SL}_{2,3}, N_1)$.

The Hochschild-Serre spectral sequence for the inclusion $\mathrm{Q}_8 \trianglelefteq \mathrm{SL}_{2,3}$ degenerates on the E_2 page since the order of the quotient group $\langle \mathsf{M}_2 \rangle$ is coprime to the order of N_1. In particular the restriction map

$$\mathrm{H}^1(\mathrm{SL}_{2,3}, N_1) \to \mathrm{H}^0(\langle \mathsf{M}_2 \rangle, \mathrm{H}^1(\mathrm{Q}_8, N_1)) \tag{3.4.4.6}$$

is an isomorphism.

Let $\mathrm{C}^i(\mathrm{Q}_8, N_1) := \mathrm{Fun}((\mathrm{Q}_8)^i, N_1)$ denote the group of inhomogeneous i-cochains. By Remark 3.4.5, the group $\mathrm{SL}_{2,3}$ has a natural left action on $\mathrm{C}^i(\mathrm{Q}_8, N_1)$ (by entrywise conjugation on the source $(\mathrm{Q}_8)^i$ and by its usual action on N_1) such that the differentials in the inhomogeneous cochain complex

$$\mathrm{C}^0(\mathrm{Q}_8, N_1) \xrightarrow{d_0} \mathrm{C}^1(\mathrm{Q}_8, N_1) \xrightarrow{d_1} \mathrm{C}^2(\mathrm{Q}_8, N_1) \to \cdots$$

are $\mathrm{SL}_{2,3}$-linear. Since the order of the subgroup $\langle \mathsf{M}_2 \rangle$ is coprime to the orders of $\mathrm{C}^i(\mathrm{Q}_8, N_1)$, we have that $\mathrm{H}^0(\langle \mathsf{M}_2 \rangle, \mathrm{H}^1(\mathrm{Q}_8, N_1)) \simeq (\mathrm{H}^1(\mathrm{Q}_8, N_1))^{\mathsf{M}_2}$ is isomorphic to the middle cohomology of the sequence

$$(\mathrm{C}^0(\mathrm{Q}_8, N_1))^{\mathsf{M}_2} \overset{(d_0)^{\mathsf{M}_2}}{\to} (\mathrm{C}^1(\mathrm{Q}_8, N_1))^{\mathsf{M}_2} \overset{(d_1)^{\mathsf{M}_2}}{\to} (\mathrm{C}^2(\mathrm{Q}_8, N_1))^{\mathsf{M}_2}$$

i.e. cohomology commutes with taking M_2-invariants.

We now describe $\ker((d_1)^{\mathsf{M}_2})$ and $\mathrm{im}((d_0)^{\mathsf{M}_2})$.

An element $f \in (\mathrm{C}^1(\mathrm{Q}_8, N_1))^{\mathsf{M}_2}$ is a function $f : \mathrm{Q}_8 \to N_1$ satisfying

$$f(\mathsf{g}) = \mathsf{M}_2 \cdot f(\mathsf{M}_2^{-1}\mathsf{g}\mathsf{M}_2) \tag{3.4.4.7}$$

for all $\mathsf{g} \in \mathrm{Q}_8$. We have that $f \in \ker d_1$ if

$$f(\mathsf{g}_1 \cdot \mathsf{g}_2) = \mathsf{g}_1 \cdot f(\mathsf{g}_2) + f(\mathsf{g}_1) \tag{3.4.4.8}$$

for all $\mathsf{g}_1, \mathsf{g}_2 \in \mathrm{Q}_8$.

Suppose $f \in \ker((d_1)^{\mathsf{M}_2}) = (\ker d_1) \cap (\mathrm{C}^1(\mathrm{Q}_8, N_1))^{\mathsf{M}_2}$; taking $(\mathsf{g}_1, \mathsf{g}_2) = (\mathsf{e}, \mathsf{e})$ in (3.4.4.8) implies $f(\mathsf{e}) = 0$; taking $\mathsf{g} = -\mathsf{e}$ in (3.4.4.7) and using (3.4.4.4) implies that

$$f(-\mathsf{e}) = (s, s, s)$$

for some $s \in \mathbb{Z}/(2)$; taking $(\mathsf{g}_1, \mathsf{g}_2) = (-\mathsf{e}, -\mathsf{e})$ in (3.4.4.8) and using the fact that $-\mathsf{e}$ acts trivially on N_1 implies that $2f(-\mathsf{e}) = 0$, which imposes no condition on s. We note that

$$\mathsf{g} \cdot f(-\mathsf{e}) = f(-\mathsf{e})$$

for any $\mathsf{g} \in \mathrm{SL}_{2,3}$.

Setting $\mathsf{g} = \mathsf{i}, \mathsf{j}, \mathsf{k}$ in (3.4.4.7) and using (3.4.4.3) gives

$$\begin{aligned} f(\mathsf{i}) &= \mathsf{M}_2 \cdot f(\mathsf{k}) \\ f(\mathsf{j}) &= \mathsf{M}_2 \cdot f(\mathsf{i}) \\ f(\mathsf{k}) &= \mathsf{M}_2 \cdot f(\mathsf{j}) \end{aligned} \tag{3.4.4.9}$$

respectively; thus we have

$$\begin{aligned} f(\mathsf{i}) &= (s_1, s_2, s_3) \\ f(\mathsf{j}) &= (s_2, s_3, s_1) \\ f(\mathsf{k}) &= (s_3, s_1, s_2) \end{aligned}$$

for some $s_1, s_2, s_3 \in \mathbb{Z}/(2)$.

Setting either $\mathsf{g}_1 = -\mathsf{e}$ or $\mathsf{g}_2 = -\mathsf{e}$ in (3.4.4.8) implies

$$f(-\mathsf{g}) = f(\mathsf{g}) + f(-\mathsf{e}) \tag{3.4.4.10}$$

for any $\mathsf{g} \in \mathrm{Q}_8$.

Setting $(\mathsf{g}_1, \mathsf{g}_2) = (\pm\mathsf{i}, \pm\mathsf{j}), (\pm\mathsf{j}, \pm\mathsf{k}), (\pm\mathsf{k}, \pm\mathsf{i})$ in (3.4.4.8) (where the signs can vary independently of each other) all impose the condition

$$s_2 = 0 \tag{3.4.4.11}$$

for s, s_2 (check the case $(\mathsf{g}_1, \mathsf{g}_2) = (\mathsf{i}, \mathsf{j})$, then use (3.4.4.10) to show that changing the signs don't give new relations, then use (3.4.4.9) to show that one can permute using left multiplication by M_2).

Setting $(\mathsf{g}_1,\mathsf{g}_2) = (\pm\mathsf{j},\pm\mathsf{i}),(\pm\mathsf{k},\pm\mathsf{j}),(\pm\mathsf{i},\pm\mathsf{k})$ in (3.4.4.8) (where the signs can vary independently of each other) all impose the condition

$$s = s_3 \tag{3.4.4.12}$$

for s_3 (check the case $(\mathsf{g}_1,\mathsf{g}_2) = (\mathsf{j},\mathsf{i})$, then use (3.4.4.10) to show that changing the signs don't give new relations, then use (3.4.4.9) to show that one can permute using left multiplication by M_2).

Setting $(\mathsf{g}_1,\mathsf{g}_2) = (\pm\mathsf{g},\pm\mathsf{g})$ for $\mathsf{g} = \mathsf{i},\mathsf{j},\mathsf{k}$ (where the signs can vary independently of each other) all impose the condition

$$s = s_2 + s_3 \tag{3.4.4.13}$$

on s, s_2, s_3 (check the case $\mathsf{g} = \mathsf{i}$, then use (3.4.4.10) to show that changing the signs don't give new relations, then use (3.4.4.9) to show that one can permute using left multiplication by M_2), but (3.4.4.13) is implied by (3.4.4.11) and (3.4.4.12).

These are the only relations satisfied by the s, s_1, s_2, s_3. Thus we have

$$\ker((d_1)^{\mathsf{M}_2}) \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$$

since there are no relations on $s, s_1 \in \mathbb{Z}/(2)$.

An element of $(\mathrm{C}^0(\mathrm{Q}_8, N_1))^{\mathsf{M}_2}$ corresponds to an element $(t,t,t) \in N_1$; since every element of $\mathrm{SL}_{2,3}$ fixes elements of this form (see (3.4.4.4)), the image of (t,t,t) under $(d_0)^{\mathsf{M}_2}$ corresponds to the function $f : \mathrm{Q}_8 \to N_1$ sending every element to $(0,0,0)$, in other words

$$\mathrm{im}((d_1)^{\mathsf{M}_2}) = 0$$

which implies

$$\mathrm{H}^0(\langle \mathsf{M}_2 \rangle, \mathrm{H}^1(\mathrm{Q}_8, N_1)) \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \tag{3.4.4.14}$$

and so

$$\mathrm{Br}\, \mathscr{M}_{1,1,k} = \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mathbb{G}_m)[2] = \mathbb{Z}/(2) \tag{3.4.4.15}$$

by combining (3.4.4.14) with (3.4.4.6), (3.4.4.5), (3.4.2.5), and (3.4.2.2). □

Remark 3.4.5 (The inhomogeneous cochain complex admits a left G-action). Let G be a group, let $H \trianglelefteq G$ be a normal subgroup, and let M be a left G-module. Set $P_i := \mathbb{Z}[H^{i+1}]$; we denote by $[\mathsf{h}_0,\ldots,\mathsf{h}_i]$ the canonical $\mathbb{Z}$-basis of P_i. We view P_i as a left H-module via the diagonal action $\mathsf{h}\cdot[\mathsf{h}_0,\ldots,\mathsf{h}_i] = [\mathsf{h}\mathsf{h}_0,\ldots,\mathsf{h}\mathsf{h}_i]$; then P_i is a free left $\mathbb{Z}[H]$-module with basis consisting of elements of the form $[\mathsf{e},\mathsf{h}_1,\ldots,\mathsf{h}_i]$. Applying the functor $\mathrm{Hom}_H(-,M)$ to the bar resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

gives the usual homogeneous cochain complex

$$\mathrm{Hom}_{\mathbb{Z}[H]}(P_0, M) \xrightarrow{\delta_0} \mathrm{Hom}_{\mathbb{Z}[H]}(P_1, M) \xrightarrow{\delta_1} \mathrm{Hom}_{\mathbb{Z}[H]}(P_2, M) \to \cdots$$

whose cohomology gives $\mathrm{H}^i(H,M)$.

We note that there is a natural left G-action on $\mathrm{Hom}_{\mathbb{Z}[H]}(P_i, M)$ for which the differential $\delta_i : \mathrm{Hom}_{\mathbb{Z}[H]}(P_i, M) \to \mathrm{Hom}_{\mathbb{Z}[H]}(P_{i+1}, M)$ is G-linear. Namely, the action of $\mathsf{g} \in G$ on $\varphi_i \in \mathrm{Hom}_{\mathbb{Z}[H]}(P_i, M)$ is described by

$$(\mathsf{g}\varphi_i)([\mathsf{h}_0,\ldots,\mathsf{h}_i]) := \mathsf{g}\cdot(\varphi_i([\mathsf{g}^{-1}\mathsf{h}_0\mathsf{g},\ldots,\mathsf{g}^{-1}\mathsf{h}_i\mathsf{g}]))$$

for all $\mathsf{h}_0, \ldots, \mathsf{h}_i \in H$. Let

$$\mathrm{C}^i(H, M) := \mathrm{Fun}(H^i, M)$$

denote the abelian group of functions $H^i \to M$. Via the usual abelian group isomorphism

$$\mathrm{Hom}_{\mathbb{Z}[H]}(P_i, M) \simeq \mathrm{C}^i(H, M)$$

sending $\varphi_i \mapsto \{(\mathsf{h}_1, \ldots, \mathsf{h}_i) \mapsto \varphi_i(\mathsf{e}, \mathsf{h}_1, \mathsf{h}_1\mathsf{h}_2, \ldots, \mathsf{h}_1 \cdots \mathsf{h}_i)\}$, the abelian group $\mathrm{C}^i(H, M)$ inherits a left action of G described by

$$(\mathsf{g} f_i)(\mathsf{h}_1, \ldots, \mathsf{h}_i) = \mathsf{g} \cdot (f_i(\mathsf{g}^{-1}\mathsf{h}_1\mathsf{g}, \ldots, \mathsf{g}^{-1}\mathsf{h}_i\mathsf{g})) \tag{3.4.5.1}$$

for $\mathsf{g} \in G$ and $f_i \in \mathrm{C}^i(H, M)$. The inhomogeneous cochain complex

$$\mathrm{C}^0(H, M) \xrightarrow{d_0} \mathrm{C}^1(H, M) \xrightarrow{d_1} \mathrm{C}^2(H, M) \to \cdots$$

is G-linear as well.

For $f_0 \in \mathrm{C}^0(H, M)$, we have $(d_0 f_0)(\mathsf{h}_1) = \mathsf{h}_1 \cdot f_0(\mathsf{e}) - f_0(\mathsf{e})$.

For $f_1 \in \mathrm{C}^1(H, M)$, we have $(d_1 f_1)(\mathsf{h}_1, \mathsf{h}_2) = \mathsf{h}_1 \cdot f_1(\mathsf{h}_2) - f_1(\mathsf{h}_1\mathsf{h}_2) + f_1(\mathsf{h}_1)$.

Let $\Sigma := G/H$ be the quotient; then there is an induced left action of Σ on the cohomology $\mathrm{h}^i(\mathrm{C}^\bullet(H, M))$. In case $G \to \Sigma$ has a section, in which case G is the semi-direct product $G \simeq H \rtimes \Sigma$, then this Σ-action coincides with the one obtained by restricting the G-action on $\mathrm{C}^\bullet(H, M)$ to Σ.

Remark 3.4.6. The arguments used in 3.4.3 and 3.4.4 are similar to those of Mathew and Stojanoska [64, Appendix B], who show $\mathrm{H}^1(\mathrm{GL}_{2,3}, (TMF(3)_0)^\times) = \mathbb{Z}/(12)$ where $\mathrm{GL}_{2,3}$ acts on

$$TMF(3)_0 = \mathbb{Z}[\tfrac{1}{3}, \zeta, t, \tfrac{1}{t}, \tfrac{1}{1-\zeta t}, \tfrac{1}{1+\zeta^2 t}]/(\zeta^2 + \zeta + 1) \tag{3.4.6.1}$$

as in [89, §4.3].

Note 3.4.7 (Explicit description of inhomogeneous 1-cocycles). We describe the 1-cocycles $\mathrm{GL}_{2,3} \to M$ obtained via the compositions (3.4.4.6) and (3.4.4.5). By our computation in 3.4.4, the 1-cocycles

$$f_{\mathrm{Q}_8} : \mathrm{Q}_8 \to N_1$$

are of the form

$$\begin{aligned} \mathsf{e} &\mapsto (0, 0, 0) & -\mathsf{e} &\mapsto (s, s, s) \\ \mathsf{i} &\mapsto (s_1, 0, s) & -\mathsf{i} &\mapsto (s_1 + s, s, 0) \\ \mathsf{j} &\mapsto (0, s, s_1) & -\mathsf{j} &\mapsto (s, 0, s_1 + s) \\ \mathsf{k} &\mapsto (s, s_1, 0) & -\mathsf{k} &\mapsto (0, s_1 + s, s) \end{aligned}$$

for some $s, s_1 \in \mathbb{Z}/(2)$. Suppose

$$f_{\mathrm{SL}_{2,3}} : \mathrm{SL}_{2,3} \to N_1$$

is a 1-cocycle such that $f_{\mathrm{SL}_{2,3}}$ is fixed by the action of M_2 (see (3.4.5.1)) and which satisfies $f_{\mathrm{SL}_{2,3}}(\mathsf{g}) = f_{\mathrm{Q}_8}(\mathsf{g})$ for $\mathsf{g} \in \mathrm{Q}_8$. We have

$$\mathsf{M}_2 \cdot f_{\mathrm{SL}_{2,3}}(\mathsf{M}_2^{-1} \cdot \mathsf{g} \cdot \mathsf{M}_2) = f_{\mathrm{SL}_{2,3}}(\mathsf{g})$$

for all $\mathsf{g} \in \mathrm{SL}_{2,3}$; taking $\mathsf{g} = \mathsf{M}_2$ gives $\mathsf{M}_2 \cdot f_{\mathrm{SL}_{2,3}}(\mathsf{M}_2) = f_{\mathrm{SL}_{2,3}}(\mathsf{M}_2)$. Taking $\mathsf{g}_1 = \mathsf{g}_2 = \mathsf{M}_2$ in the 1-cocycle condition (3.4.4.8) then gives $f_{\mathrm{SL}_{2,3}}(\mathsf{M}_2) = 0$. Thus we have

$$f_{\mathrm{SL}_{2,3}}(\mathsf{g} \cdot \mathsf{M}_2) = f_{\mathrm{SL}_{2,3}}(\mathsf{g}) \tag{3.4.7.1}$$

for any $\mathsf{g} \in \mathrm{SL}_{2,3}$, again by (3.4.4.8).

By Shapiro's lemma (3.4.4.5), there is a 1-cocycle

$$f_{\mathrm{GL}_{2,3}} : \mathrm{GL}_{2,3} \to \mathrm{Ind}_{\mathrm{SL}_{2,3}}^{\mathrm{GL}_{2,3}}(N_1)$$

such that precomposing with the inclusion $\mathrm{SL}_{2,3} \subset \mathrm{GL}_{2,3}$ and postcomposing with the projection $\mathrm{Ind}_{\mathrm{SL}_{2,3}}^{\mathrm{GL}_{2,3}}(N_1) \to N_1$ gives $f_{\mathrm{SL}_{2,3}}$. After altering $f_{\mathrm{GL}_{2,3}}$ by a 1-coboundary, we may assume by 3.4.8 that $f_{\mathrm{GL}_{2,3}}$ is given by the formula (3.4.8.1), namely

$$f_{\mathrm{GL}_{2,3}}(\mathsf{g} \cdot \mathsf{M}_1^i)([\mathsf{M}_1^j]) := f_{\mathrm{SL}_{2,3}}(\mathsf{M}_1^j \cdot \mathsf{g} \cdot \mathsf{M}_1^{-j}) \tag{3.4.7.2}$$

for any $i, j \in \{0, 1\}$ and $\mathsf{g} \in \mathrm{SL}_{2,3}$. Any element $\mathsf{g} \in \mathrm{GL}_{2,3}$ may be expressed in the form

$$\mathsf{h} \cdot \mathsf{M}_2^{i_2} \cdot \mathsf{M}_1^{i_1}$$

where $i_1 \in \{0, 1\}$ and $i_2 \in \{0, 1, 2\}$ and $\mathsf{h} \in \mathrm{Q}_8$. We have formulas

$$\begin{aligned} \mathsf{M}_1 \cdot \mathsf{M}_2^{-1} \cdot \mathsf{M}_1 &= \mathsf{M}_2^{-1} \\ \mathsf{M}_1 \cdot \mathsf{i} \cdot \mathsf{M}_1^{-1} &= -\mathsf{i} \\ \mathsf{M}_1 \cdot \mathsf{j} \cdot \mathsf{M}_1^{-1} &= -\mathsf{k} \\ \mathsf{M}_1 \cdot \mathsf{k} \cdot \mathsf{M}_1^{-1} &= -\mathsf{j} \end{aligned} \tag{3.4.7.3}$$

and so

$$\begin{aligned} f_{\mathrm{GL}_{2,3}}(\mathsf{h} \cdot \mathsf{M}_2^{i_2} \cdot \mathsf{M}_1^{i_1})([\mathsf{M}_1^j]) &\overset{1}{=} f_{\mathrm{SL}_{2,3}}(\mathsf{M}_1^j \cdot \mathsf{h} \cdot \mathsf{M}_2^{i_2} \cdot \mathsf{M}_1^{-j}) \\ &= f_{\mathrm{SL}_{2,3}}((\mathsf{M}_1^j \cdot \mathsf{h} \cdot \mathsf{M}_1^{-j}) \cdot (\mathsf{M}_1^j \cdot \mathsf{M}_2^{i_2} \cdot \mathsf{M}_1^{-j})) \\ &\overset{2}{=} f_{\mathrm{SL}_{2,3}}(\mathsf{M}_1^j \cdot \mathsf{h} \cdot \mathsf{M}_1^{-j}) \\ &\overset{3}{=} f_{\mathrm{Q}_8}(\mathsf{M}_1^j \cdot \mathsf{h} \cdot \mathsf{M}_1^{-j}) \end{aligned}$$

where equality 1 is by (3.4.7.2) and equality 2 is by (3.4.7.1) and (3.4.7.3) and equality 3 is since $\mathsf{M}_1^j \cdot \mathsf{h} \cdot \mathsf{M}_1^{-j} \in \mathrm{Q}_8$ (see (3.4.7.3)). This is summarized in (3.4.7.4) below.

$$\begin{aligned} f_{\mathrm{GL}_{2,3}}(\mathsf{e}) &= (f_{\mathrm{Q}_8}(\mathsf{e}), f_{\mathrm{Q}_8}(\mathsf{e})) = ((0,0,0),(0,0,0)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{i}) &= (f_{\mathrm{Q}_8}(\mathsf{i}), f_{\mathrm{Q}_8}(-\mathsf{i})) = ((s_1,0,s),(s_1+s,s,0)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{j}) &= (f_{\mathrm{Q}_8}(\mathsf{j}), f_{\mathrm{Q}_8}(-\mathsf{k})) = ((0,s,s_1),(0,s_1+s,s)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{k}) &= (f_{\mathrm{Q}_8}(\mathsf{k}), f_{\mathrm{Q}_8}(-\mathsf{j})) = ((s,s_1,0),(s,0,s_1+s)) \end{aligned} \tag{3.4.7.4}$$

3.4.8 (The Shapiro isomorphism and inhomogeneous 1-cocycles). [15] Let G be a group, let $H \subseteq G$ be a normal subgroup of finite index such that the projection $G \to G/H$ has a section $G/H \to G$ whose image corresponds to a subgroup Σ of G. Let N be a left H-module and let $\mathrm{Ind}_H^G N := \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], N)$ denote the associated induced left G-module. We recall that the left G-action on $\mathrm{Ind}_H^G N$ sends $\varphi \mapsto \mathsf{g}\varphi$ where $(\mathsf{g}\varphi)(x) = \varphi(x\mathsf{g})$.

We describe the inverse of the Shapiro isomorphism $\mathrm{H}^1(G, \mathrm{Ind}_H^G N) \to \mathrm{H}^1(H, N)$ in terms of inhomogeneous cochains. Suppose given a function

$$f : H \to N$$

which satisfies

$$f(\mathsf{h}_1\mathsf{h}_2) = \mathsf{h}_1 \cdot f(\mathsf{h}_2) + f(\mathsf{h}_1)$$

[15] Ehud Meir's MathOverflow post [68] was helpful in working out the details of this section.

for all $\mathsf{h}_1, \mathsf{h}_2 \in H$. We construct a 1-cocycle

$$s : G \to \mathrm{Ind}_H^G(N)$$

which restricts to f, i.e. satisfies $s(\mathsf{h})(1 \cdot [\mathsf{e}]) = f(\mathsf{h})$ for all $\mathsf{h} \in H$. Note that every element of $\mathsf{g} \in G$ may be written uniquely in the form

$$\mathsf{g} = \mathsf{h}\sigma$$

for $\mathsf{h} \in H$ and $\sigma \in \Sigma$, hence the collection $\{[\sigma]\}_{\sigma \in \Sigma}$ forms a basis for $\mathbb{Z}[G]$ as a left $\mathbb{Z}[H]$-module. We set

$$s(\mathsf{h}\sigma)([\xi]) := f(\xi \mathsf{h} \xi^{-1}) \tag{3.4.8.1}$$

for $\mathsf{h} \in H$ and $\sigma, \xi \in \Sigma$ and extend $\mathbb{Z}[H]$-linearly. Given $\mathsf{g}_1, \mathsf{g}_2 \in G$ where $\mathsf{g}_i = \mathsf{h}_i \sigma_i$ with $\mathsf{h}_i \in H$ and $\sigma_i \in \Sigma$, for any $\xi \in \Sigma$ we have

$$\begin{aligned} s(\mathsf{g}_1\mathsf{g}_2)([\xi]) &= s(\mathsf{h}_1\sigma_1\mathsf{h}_2\sigma_2)([\xi]) \\ &= s(\mathsf{h}_1(\sigma_1\mathsf{h}_2\sigma_1^{-1})\sigma_1\sigma_2)([\xi]) \\ &= f(\xi\mathsf{h}_1(\sigma_1\mathsf{h}_2\sigma_1^{-1})\xi^{-1}) \end{aligned}$$

and

$$\begin{aligned} (\mathsf{g}_1 \cdot s(\mathsf{g}_2))([\xi]) &= s(\mathsf{h}_2\sigma_2)([\xi\mathsf{h}_1\sigma_1]) \\ &= s(\mathsf{h}_2\sigma_2)([(\xi\mathsf{h}_1\xi^{-1})\xi\sigma_1]) \\ &= (\xi\mathsf{h}_1\xi^{-1}) \cdot s(\mathsf{h}_2\sigma_2)([\xi\sigma_1]) \\ &= (\xi\mathsf{h}_1\xi^{-1}) \cdot f((\xi\sigma_1)\mathsf{h}_2(\xi\sigma_1)^{-1}) \end{aligned}$$

and

$$s(\mathsf{g}_1)([\xi]) = s(\mathsf{h}_1\sigma_1)([\xi]) = f(\xi\mathsf{h}_1\xi^{-1})$$

which implies

$$s(\mathsf{g}_1\mathsf{g}_2) = \mathsf{g}_1 \cdot s(\mathsf{g}_2) + s(\mathsf{g}_1)$$

by $\mathbb{Z}[H]$-linearity and since f is a 1-cocycle; hence s is a 1-cocycle. □

3.4.9 (Proof of Theorem 3.1.2). Let k^{sep} be a fixed separable closure of k and let $\mathrm{G}_k := \mathrm{Gal}(k^{\mathrm{sep}}/k) \simeq \widehat{\mathbb{Z}}$ be the absolute Galois group. Set $\mathscr{M} := \mathscr{M}_{1,1,k}$ and $\mathscr{M}^{\mathrm{sep}} := \mathscr{M}_{1,1,k^{\mathrm{sep}}}$. We have $\mathrm{Br}\,\mathscr{M} = \mathrm{Br}'\,\mathscr{M}$ by Lemma 3.2.1. The Leray spectral sequence for the map $\mathscr{M} \to \mathrm{Spec}\,k$ is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{G}_k, \mathrm{H}^q_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m)) \implies \mathrm{H}^{p+q}_{\text{ét}}(\mathscr{M}, \mathbb{G}_m)$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. Here we have $\Gamma(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m) = \Gamma(\mathbb{A}^1_{k^{\mathrm{sep}}}, \mathbb{G}_m) = (k^{\mathrm{sep}})^\times$ since $\mathscr{M}^{\mathrm{sep}} \to \mathbb{A}^1_{k^{\mathrm{sep}}}$ is the coarse moduli space map. Since k is a finite field, we have that $\mathrm{H}^0_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m)$ is a torsion group. Moreover $\mathrm{H}^1_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m) \simeq \mathrm{Pic}(\mathscr{M}^{\mathrm{sep}}) \simeq \mathbb{Z}/(12)$ is a torsion group by [36]. Thus by e.g. [34, 4.3.7] we have $\mathrm{E}_2^{p,q} = 0$ for $(p,q) \in \mathbb{Z}_{\geq 2} \times \{0,1\}$. This means there is an exact sequence

$$0 \to \mathrm{E}_2^{1,1} \to \mathrm{H}^2_{\text{ét}}(\mathscr{M}, \mathbb{G}_m) \to \mathrm{E}_2^{0,2} \to 0 \tag{3.4.9.1}$$

of abelian groups.

By [36], we have that $\mathrm{Pic}(\mathscr{M}^{\mathrm{sep}}) \simeq \mathbb{Z}/(12)$ is generated by the class of the Hodge bundle; since G_k acts trivially on invariant differentials of elliptic curves $E \to S$ where S is a k-scheme, the action of G_k on $\mathrm{Pic}(\mathscr{M}^{\mathrm{sep}})$ is trivial. Hence we have

$$\mathrm{E}_2^{1,1} = \mathrm{H}^1(\mathrm{G}_k, \mathrm{H}^1_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m)) \overset{1}{=} \mathrm{Hom}_{\mathrm{cont}}(\mathrm{G}_k, \mathrm{Pic}(\mathscr{M}^{\mathrm{sep}})) \overset{2}{=} \mathbb{Z}/(12)$$

where equality 1 is by [34, 4.3.7] and equality 2 is since $\mathrm{G}_k \simeq \widehat{\mathbb{Z}}$. We have

$$\mathrm{E}_2^{0,2} = \mathrm{H}^0(\mathrm{G}_k, \mathrm{H}^2_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m)) \overset{1}{=} (\mathbb{Z}/(2))^{\mathrm{G}_k} \overset{2}{=} \mathbb{Z}/(2)$$

where equality 1 is by the computation for an algebraically closed field (Theorem 3.1.1) and also the fact that $\mathrm{H}^2_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m)$ is a torsion group (see [4, Proposition 2.5 (iii)]) and equality 2 is because any group action on the group of order 2 is necessarily trivial. Thus (3.4.9.1) reduces to a natural extension

$$0 \to \mathbb{Z}/(12) \to \mathrm{Br}\,\mathscr{M} \to \mathbb{Z}/(2) \to 0 \tag{3.4.9.2}$$

and it remains to see whether (3.4.9.2) is split. It suffices to compute the size of $(\mathrm{Br}\,\mathscr{M})[2]$, since $(\mathrm{Br}\,\mathscr{M})[2]$ has 4 or 2 elements depending on whether (3.4.9.2) is split or not, respectively.

As in 3.4.2, the fppf Kummer sequence

$$1 \to \mu_2 \to \mathbb{G}_m \overset{\times 2}{\to} \mathbb{G}_m \to 1 \tag{3.4.9.3}$$

gives an exact sequence

$$1 \to \mathbb{Z}/(2) \overset{\partial}{\to} \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}, \mu_2) \to (\mathrm{Br}\,\mathscr{M})[2] \to 1 \tag{3.4.9.4}$$

of abelian groups. We compute $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}, \mu_2)$ using the Leray spectral sequence which is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{G}_k, \mathrm{H}^q_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2)) \implies \mathrm{H}^{p+q}_{\mathrm{fppf}}(\mathscr{M}, \mu_2)$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. We have

$$\mathrm{H}^q_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2) = \begin{cases} 0 & \text{if } q = 0 \\ \mathbb{Z}/(2) & \text{if } q = 1 \\ \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) & \text{if } q = 2 \end{cases}$$

from the fppf Kummer sequence on $\mathscr{M}^{\mathrm{sep}}$, where the $q = 0$ case follows since we are in characteristic 2 and $\Gamma(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m) = \Gamma(\mathbb{A}^1_{k^{\mathrm{sep}}}, \mathbb{G}_m) = (k^{\mathrm{sep}})^\times$, the $q = 1$ case is since the multiplication-by-2 map on $\Gamma(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m) = (k^{\mathrm{sep}})^\times$ is an isomorphism, and the $q = 2$ case is by the computation in the algebraically closed case (combine (3.4.2.5), (3.4.4.5), (3.4.4.6), (3.4.4.14)).

Since k has characteristic 2, the 2-cohomological dimension of k satisfies $\mathrm{cd}_2(k) \leq 1$ by e.g. [38, 6.1.9]; hence $\mathrm{E}_2^{p,q} = 0$ for $p \geq 2$ and any $q \in \{0, 1, 2\}$. Hence there is an exact sequence

$$0 \to \mathrm{H}^1(\mathrm{G}_k, \mathrm{H}^1_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2)) \to \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}, \mu_2) \to \mathrm{H}^0(\mathrm{G}_k, \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2)) \to 0 \tag{3.4.9.5}$$

of abelian groups. As above, the G_k-action on $\mathrm{H}^1_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2)$ is necessarily trivial so we have an isomorphism $\mathrm{H}^1(\mathrm{G}_k, \mathrm{H}^1_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2)) \simeq \mathrm{Hom}_{\mathrm{cont}}(\mathrm{G}_k, \mathbb{Z}/(2)) \simeq \mathbb{Z}/(2)$.

To describe $\mathrm{H}^0(\mathrm{G}_k, \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2))$, we describe the G_k-action on $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2)$. Let

$$\xi \in k^{\mathrm{sep}}$$

be a fixed root of x^2+x+1 (i.e. a primitive 3rd root of unity).

If $\xi \in k$, then G_k acts trivially on $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2)$ by the description in 3.4.3; hence $\mathrm{H}^0(\mathrm{G}_k, \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2))$ has 4 elements, hence $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}, \mu_2)$ has 8 elements by (3.4.9.5), hence $(\mathrm{Br}\,\mathscr{M})[2]$ has 4 elements by (3.4.9.4), hence $\mathrm{Br}\,\mathscr{M} \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(12)$.

Suppose $\xi \notin k$. The k-algebra map

$$k[\mu, \omega, \tfrac{1}{\mu^3-1}]/(\omega^2+\omega+1) \to k^{\mathrm{sep}}[\nu_1, \tfrac{1}{\nu_1^3-1}] \times k^{\mathrm{sep}}[\nu_2, \tfrac{1}{\nu_2^3-1}]$$

sending $\mu \mapsto (\nu_1, \nu_2)$ and $\omega \mapsto (\xi, \xi^2)$ induces an isomorphism

$$k[\mu, \omega, \tfrac{1}{\mu^3-1}]/(\omega^2+\omega+1) \otimes_k k^{\mathrm{sep}} \to k^{\mathrm{sep}}[\nu_1, \tfrac{1}{\nu_1^3-1}] \times k^{\mathrm{sep}}[\nu_2, \tfrac{1}{\nu_2^3-1}] \tag{3.4.9.6}$$

of k^{sep}-algebras. The inverse to (3.4.9.6) sends

$$(f_1(\nu_1), f_2(\nu_2)) \mapsto f_1(\mu)\left(\omega \otimes \frac{1}{\xi-\xi^2} + 1 \otimes \frac{\xi}{\xi-1}\right) + f_2(\mu)\left((-\omega) \otimes \frac{1}{\xi-\xi^2} + (-1) \otimes \frac{1}{\xi-1}\right)$$

for $f_i(\nu_i) \in k[\nu_i, \frac{1}{\nu_i^3-1}]$.

Let

$$\lambda \in \mathrm{G}_k$$

be an automorphism of k^{sep} such that $\lambda(\xi) = \xi^2$. Then the k-algebra automorphism of $k^{\mathrm{sep}}[\nu_1, \frac{1}{\nu_1^3-1}] \times k^{\mathrm{sep}}[\nu_2, \frac{1}{\nu_2^3-1}]$ induced by (3.4.9.6) sends $(\nu_1, 0) \mapsto (0, \nu_2)$ and $(0, \nu_2) \mapsto (\nu_1, 0)$ and $(\xi, 0) \mapsto (0, \xi^2)$ and $(0, \xi) \mapsto (\xi^2, 0)$. We see that the action of λ on M (see (3.4.3.2)) is given by (3.4.9.7).

$$\begin{array}{c|c|c|c|c|c|c} & \nu_1-1 & \nu_1-\xi & \nu_1-\xi^2 & \nu_2-1 & \nu_2-\xi & \nu_2-\xi^2 \\ \lambda & \nu_2-1 & \nu_2-\xi^2 & \nu_2-\xi & \nu_1-1 & \nu_1-\xi^2 & \nu_1-\xi \end{array} \tag{3.4.9.7}$$

A computation with (3.4.9.7) and (3.4.3.6) shows that

$$\lambda \mathsf{g} \lambda^{-1} \cdot m = \mathsf{g} \cdot m \tag{3.4.9.8}$$

for any $m \in M$ and $\mathsf{g} \in \mathrm{GL}_{2,3}$.

Let $f_{\mathrm{GL}_{2,3}} : \mathrm{GL}_{2,3} \to M$ be an inhomogeneous 1-cocycle as in Note 3.4.7. Multiplying the 1-cocycle condition (3.4.4.8) on the left by λ gives

$$\begin{aligned} \lambda \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g}_1 \cdot \mathsf{g}_2) &= \lambda \mathsf{g}_1 \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g}_2) + \lambda \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g}_1) \\ &\overset{1}{=} \mathsf{g}_1 \cdot (\lambda \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g}_2)) + \lambda \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g}_1) \end{aligned}$$

where equality 1 follows from (3.4.9.8). Hence the function $\lambda \cdot f_{\mathrm{GL}_{2,3}} : \mathrm{GL}_{2,3} \to M$ sending $\mathsf{g} \mapsto \lambda \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g})$ is a 1-cocycle as well. Using (3.4.9.7) and (3.4.7.4), we have that

$$\begin{aligned} (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{e}) &= ((0,0,0),(0,0,0)) \\ (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{i}) &= ((s_1+s,0,s),(s_1,s,0)) \\ (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{j}) &= ((0,s,s_1+s),(0,s_1,s)) \\ (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{k}) &= ((s,s_1+s,0),(s,0,s_1)) \end{aligned} \tag{3.4.9.9}$$

and so

$$\begin{aligned} f_{\mathrm{GL}_{2,3}}(\mathsf{e}) - (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{e}) &= ((0,0,0),(0,0,0)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{i}) - (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{i}) &= ((s,0,0),(s,0,0)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{j}) - (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{j}) &= ((0,0,s),(0,s,0)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{k}) - (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{k}) &= ((0,s,0),(0,0,s)) \end{aligned} \tag{3.4.9.10}$$

for the same $s, s_1 \in \mathbb{Z}/(2)$ as in (3.4.7.4).

Suppose $f_{\mathrm{GL}_{2,3}}$ and $\lambda \cdot f_{\mathrm{GL}_{2,3}}$ differ by a 1-coboundary, in other words there exists an element

$$m := ((m_1^1, m_2^1, m_3^1), (m_1^2, m_2^2, m_3^2)) \in M$$

such that

$$f_{\mathrm{GL}_{2,3}}(\mathsf{g}) - (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{g}) = \mathsf{g} \cdot m - m \tag{3.4.9.11}$$

for all $\mathsf{g} \in \mathrm{GL}_{2,3}$. By (3.4.9.10), taking $\mathsf{g} = \mathsf{M}_2$ in (3.4.9.11) gives $m^i := m_1^i = m_2^i = m_3^i$ for $i = 1, 2$; then taking $\mathsf{g} = \mathsf{M}_1$ gives $m^1 = m^2$; then taking $\mathsf{g} = \mathsf{i}$ gives $m = 0$. We see that $f_{\mathrm{GL}_{2,3}}$ and $\lambda \cdot f_{\mathrm{GL}_{2,3}}$ differ by a 1-coboundary if and only if $s = 0$.

Hence we have that $\mathrm{H}^0(\mathrm{G}_k, \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2)) \simeq \mathbb{Z}/(2)$, hence $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}, \mu_2)$ has 4 elements by (3.4.9.5), hence $(\mathrm{Br}\,\mathscr{M})[2]$ has 2 elements by (3.4.9.4), hence $\mathrm{Br}\,\mathscr{M} \simeq \mathbb{Z}/(24)$. ☐

Question 3.4.10. In Theorem 3.1.1, "describe" the Azumaya $\mathcal{O}_{\mathscr{M}_{1,1,k}}$-algebra corresponding to the unique nontrivial class in $\mathrm{Br}\,\mathscr{M}_{1,1,k} \simeq \mathbb{Z}/(2)$. Is it a cyclic algebra? Is the cup product map

$$\mathrm{H}^1_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mathbb{Z}/(2^n)) \times \mathrm{H}^1_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mu_{2^n}) \to \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mu_2)$$

surjective for some $n \in \mathbb{N}$?

Remark 3.4.11. In Theorem 3.1.1, the nontrivial Brauer class may be represented by an Azumaya $\mathcal{O}_{\mathscr{M}_{1,1,k}}$-algebra of rank 48^2. Indeed, let $\mathcal{G} \to \mathscr{M}_{1,1,k}$ be the $\mathbb{G}_m$-gerbe corresponding to the nontrivial class in $\mathrm{Br}(\mathscr{M}_{1,1,k})$. Let $\mathcal{G}_{S_{\mathrm{H}}}$ be the fiber product $\mathcal{G} \times_{\mathscr{M}_{1,1,k}} S_{\mathrm{H}}$ (where the second projection is (3.4.1.3)), and let $\pi : \mathcal{G}_{S_{\mathrm{H}}} \to \mathcal{G}$ be the projection. Since $\mathrm{Br}(S_{\mathrm{H}}) = 0$, there exists a 1-twisted invertible sheaf $\mathcal{L}$ on $\mathcal{G}_{S_{\mathrm{H}}}$. The pushforward $\pi_*\mathcal{L}$ is a 1-twisted finite locally free sheaf on $\mathcal{G}$ of rank 48, since π is finite locally free of rank $|\,\mathrm{GL}_{2,3}\,| = 48$. Then the endomorphism algebra $\mathrm{End}_{\mathcal{O}_{\mathcal{G}}}(\pi_*\mathcal{L})$ is a 0-twisted Azumaya $\mathcal{O}_{\mathcal{G}}$-algebra. ☐

4. The cohomological Brauer group of $\mathbb{G}_m$-gerbes

The purpose of this section is to prove Theorem 4.1.2. This material is more or less the same as in [85].

4.1. **Main theorem and introductory remarks.** Gabber in his thesis described the cohomological Brauer group of a Brauer-Severi scheme $X \to S$ as a quotient of the cohomological Brauer group of the base scheme S. More precisely, he proved the following:

Theorem 4.1.1. [37, II, Theorem 2] Let S be a scheme, let $\pi_X : X \to S$ be a Brauer-Severi scheme. Then the sequence

$$\mathrm{H}^0_{\text{ét}}(S, \underline{\mathbb{Z}}) \to \mathrm{Br}' S \overset{\pi_X^*}{\to} \mathrm{Br}' X \to 0 \tag{4.1.1.1}$$

is exact, where the first map sends $1 \mapsto [X]$.

In this section we prove an analogue of the above theorem for torsion $\mathbb{G}_m$-gerbes.

Theorem 4.1.2. Let S be a scheme, let $\pi_{\mathcal{G}} : \mathcal{G} \to S$ be a $\mathbb{G}_{m,S}$-gerbe corresponding to a torsion class $[\mathcal{G}] \in \mathrm{Br}' S$. Then the sequence

$$\mathrm{H}^0_{\text{ét}}(S, \underline{\mathbb{Z}}) \to \mathrm{Br}' S \overset{\pi_{\mathcal{G}}^*}{\to} \mathrm{Br}' \mathcal{G} \to 0 \tag{4.1.2.1}$$

is exact, where the first map sends $1 \mapsto [\mathcal{G}]$.

Remark 4.1.3. Let $\mathcal{A}$ be an Azumaya $\mathcal{O}_S$-algebra and let $\pi_X : X \to S$ and $\pi_{\mathcal{G}} : \mathcal{G} \to S$ be the associated Brauer-Severi scheme and $\mathbb{G}_m$-gerbe of trivializations, respectively. By [78, §8, 4] there exists a finite locally free $\mathcal{O}_X$-module J and an $\mathcal{O}_X$-algebra isomorphism $\pi_X^*\mathcal{A} \simeq \underline{\mathrm{End}}_{\mathcal{O}_X}(J)^{\mathrm{op}}$. There is an $\mathcal{O}_X$-algebra isomorphism $\underline{\mathrm{End}}_{\mathcal{O}_X}(J)^{\mathrm{op}} \simeq \underline{\mathrm{End}}_{\mathcal{O}_X}(J^\vee)$ sending $\varphi \mapsto \varphi^\vee$. Since $\mathcal{G}$ is the gerbe of trivializations of $\mathcal{A}$, we have an S-morphism $f : X \to \mathcal{G}$; this induces a commutative triangle $\pi_X^* = f^*\pi_{\mathcal{G}}^*$ on the cohomological Brauer groups of $S, \mathcal{G}, X$. Since $[\mathcal{G}] \in \ker \pi_{\mathcal{G}}^* \subseteq \ker \pi_X^*$ and $\ker \pi_X^*$ is generated by $[\mathcal{G}]$ by Theorem 4.1.1, we have exactness of (4.1.2.1) at $\mathrm{Br}' S$. The difficulty of Theorem 4.1.2 is in showing that $\pi_{\mathcal{G}}^*$ is surjective.

Remark 4.1.4. Our Theorem 4.1.2 provides a class of algebraic stacks $\mathcal{G}$ for which the Brauer map $\alpha_{\mathcal{G}}$ is surjective. Indeed, if S is a scheme for which α_S is surjective and $\pi_{\mathcal{G}} : \mathcal{G} \to S$ is a torsion $\mathbb{G}_m$-gerbe, then $\alpha_{\mathcal{G}}$ is surjective by Theorem 4.1.2 and functoriality of the Brauer map. On the other hand, if S is a scheme for which α_S is not surjective, then $\alpha_{\mathcal{G}}$ is not surjective for $\mathcal{G} = \mathrm{B}\mathbb{G}_{m,S}$. This follows from the observation that the projection $\pi_{\mathcal{G}} : \mathcal{G} \to S$ has a section $\sigma_{\mathcal{G}} : S \to \mathcal{G}$.

Remark 4.1.5. By Theorem 4.1.2 and Remark 4.1.3, the pullback map

$$f^* : \mathrm{Br}' \mathcal{G} \to \mathrm{Br}' X \tag{4.1.5.1}$$

is an isomorphism, in other words the cohomological Brauer groups of a Brauer-Severi scheme and its associated $\mathbb{G}_m$-gerbe are isomorphic. It would be interesting to give a more direct proof that (4.1.5.1) is an isomorphism. Indeed, smooth-locally on X, the map $f : X \to \mathcal{G}$ is isomorphic to $\mathbb{A}^{n+1} \setminus \{0\}$ and we give a proof using this fact in 4.5.6, under the additional hypotheses that the base S is regular and that its function field has characteristic 0. To remove these hypotheses, it would be necessary to study the differential $\mathrm{E}_3^{0,2}$ on the 3rd page of the Leray spectral sequence associated to f. We note however that showing that (4.1.5.1) is an isomorphism for arbitrary bases S would not be enough to prove Theorem 4.1.2 in case the Brauer map $\alpha_S : \operatorname{Br} S \to \operatorname{Br}' S$ is not surjective, i.e. there exist torsion $\mathbb{G}_{m,S}$-gerbes not corresponding to any Azumaya $\mathcal{O}_S$-algebra.

4.1.6. We outline the proof of Theorem 4.1.2. As in [37], the desired exact sequence (4.1.2.1) comes from the Leray spectral sequence for the map $\pi_{\mathcal{G}}$ and sheaf $\mathbb{G}_{m,\mathcal{G}}$. One step in the proof of Theorem 4.1.2 is to show the vanishing of the higher pushforwards $\mathbf{R}^2\pi_{\mathcal{G},*}\mathbb{G}_{m,\mathcal{G}}$. The stalk of $\mathbf{R}^2\pi_{\mathcal{G},*}\mathbb{G}_{m,\mathcal{G}}$ at a geometric point $\overline{s}$ of S is isomorphic to $\mathrm{H}^2_{\text{ét}}(\mathrm{B}\mathbb{G}_{m,A}, \mathbb{G}_{m,\mathrm{B}\mathbb{G}_{m,A}})$ where $A = \mathcal{O}^{\text{sh}}_{S,\overline{s}}$ is the strict henselization of S at $\overline{s}$. We compute $\mathrm{H}^2_{\text{ét}}(\mathrm{B}\mathbb{G}_{m,A}, \mathbb{G}_{m,\mathrm{B}\mathbb{G}_{m,A}})$ using the descent spectral sequence associated to the covering $\xi : \operatorname{Spec} A \to \mathrm{B}\mathbb{G}_{m,A}$, whose qth row is the Čech complex associated to the cosimplicial abelian group obtained by applying the functor $\mathrm{H}^q_{\text{ét}}(-, \mathbb{G}_m)$ to the simplicial A-scheme $\{\mathbb{G}^{\times p}_{m,A}\}_{p\geq 0}$ obtained by taking fiber products of ξ. In Section 4.3 and Section 4.4, we show that the $\mathrm{E}^{1,1}_2$ and $\mathrm{E}^{2,0}_2$ terms of this spectral sequence vanish, respectively. It is harder to show that $\mathrm{E}^{1,1}_2 = 0$, which comes down to showing that

$$m^* - p_1^* - p_2^* : \operatorname{Pic}(A[t^{\pm}]) \to \operatorname{Pic}(A[t_1^{\pm}, t_2^{\pm}])$$

is injective, where $m, p_1, p_2 : A[t^{\pm}] \to A[t_1^{\pm}, t_2^{\pm}]$ are the A-algebra maps sending $t \mapsto t_1t_2, t_1, t_2$ respectively. If A is a normal domain, then $\operatorname{Pic}(A) \simeq \operatorname{Pic}(A[t^{\pm}]) \simeq \operatorname{Pic}(A[t_1^{\pm}, t_2^{\pm}])$ so the result is trivial. In case A is not normal, we use the Units-Pic sequence associated to the Milnor square of the normalization $A \to \overline{A}$.

4.2. **Gerbes and the transgression map.** The purpose of this section is to prove Lemma 4.2.2, a description of the higher pushforward $\mathbf{R}^1\pi_*\mathbb{G}_{m,\mathcal{G}}$ for a gerbe $\pi : \mathcal{G} \to \mathcal{S}$, and Proposition 4.2.3, a description of the differential $\mathrm{d}^{0,1}_2 : \mathrm{E}^{0,1}_2 \to \mathrm{E}^{2,0}_2$ in the Leray spectral sequence associated to π in terms of torsors and gerbes. This map $\mathrm{d}^{0,1}_2$ is called the transgression map [40, V, §3.2].

In order to describe the higher pushforwards $\mathbf{R}^1\pi_*\mathbb{G}_m$, we will use the following result on the Picard group of $\mathbf{A}$-gerbes.

Remark 4.2.1 (Picard group of $\mathbf{A}$-gerbes). Assume the setup of Lemma 1.3.16. By [15, 5.3.4], for any invertible $\mathcal{O}_{\mathcal{G}}$-module $\mathcal{L}$, there exists a unique character

$$\chi_{\mathcal{L}} \in \widehat{\mathbf{A}}$$

such that the diagram

$$\begin{array}{ccc} \mathbf{A}_{\mathcal{G}} \times \mathcal{L} & \longrightarrow & \mathcal{L} \\ {\scriptstyle \pi^*\chi_{\mathcal{L}} \times \mathrm{id}_{\mathcal{L}}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{\mathcal{L}}} \\ \mathbb{G}_{m,\mathcal{G}} \times \mathcal{L} & \longrightarrow & \mathcal{L} \end{array} \tag{4.2.1.1}$$

commutes, where the top row is the inertial action and the bottom row is the restriction of the $\mathcal{O}_{\mathcal{G}}$-module structure on $\mathcal{L}$. The condition that (4.2.1.1) commutes is equivalent to the condition (1.3.5.1), in other words $\mathcal{L}$ is a $\chi_{\mathcal{L}}$-twisted sheaf. For two invertible $\mathcal{O}_{\mathcal{G}}$-modules $\mathcal{L}_1, \mathcal{L}_2$ we have $\chi_{\mathcal{L}_1\otimes\mathcal{L}_2} = \chi_{\mathcal{L}_1} \cdot \chi_{\mathcal{L}_2}$ by [15, 5.3.6 (2)], hence the assignment $\mathcal{L} \mapsto \chi_{\mathcal{L}}$ defines a group homomorphism

$$\beta_{\mathcal{G}} : \operatorname{Pic}(\mathcal{G}) \to \widehat{\mathbf{A}}$$

of abelian groups. By [15, 5.3.6 (3)], we have that $\chi_{\mathcal{L}} = 0$ if and only if $\mathcal{L}$ is of the form $\pi^* M$ for an invertible $\mathcal{O}_{\mathcal{S}}$-module M; in other words there is an exact sequence

$$0 \to \mathrm{Pic}(\mathcal{S}) \xrightarrow{\pi^*} \mathrm{Pic}(\mathcal{G}) \xrightarrow{\beta_{\mathcal{G}}} \widehat{\mathbf{A}} \tag{4.2.1.2}$$

where injectivity of π^* follows from Lemma 1.3.16. The sequence (4.2.1.2) is functorial on $\mathcal{S}$ in the following sense: if $p : \mathcal{T} \to \mathcal{S}$ is a morphism of locally ringed sites and $\pi_{\mathcal{T}} : \mathcal{G}_{\mathcal{T}} \to \mathcal{T}$ is the $\mathbf{A}_{\mathcal{T}}$-gerbe obtained by pullback, then the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Pic}(\mathcal{S}) & \xrightarrow{\pi^*} & \mathrm{Pic}(\mathcal{G}) & \xrightarrow{\beta_{\mathcal{G}}} & \widehat{\mathbf{A}} \\ & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\ 0 & \longrightarrow & \mathrm{Pic}(\mathcal{T}) & \xrightarrow[\pi_{\mathcal{T}}^*]{} & \mathrm{Pic}(\mathcal{G}_{\mathcal{T}}) & \xrightarrow[\beta_{\mathcal{G}_{\mathcal{T}}}]{} & \widehat{\mathbf{A}}_{\mathcal{T}} \end{array}$$

commutes.

In case $\mathcal{G} := \mathrm{B}\mathbf{A}$, by [15, 5.3.7] the map $\beta_{\mathcal{G}}$ is surjective and the sequence (4.2.1.2) is split; the map $\beta_{\mathcal{G}}$ admits a natural section $\widehat{\mathbf{A}} \to \mathrm{Pic}(\mathcal{G})$ taking a character χ to the trivial $\mathcal{O}_{\mathcal{S}}$-module equipped with the $\mathbf{A}$-action corresponding to χ via the isomorphism $\mathbb{G}_{m,\mathcal{S}} \simeq \underline{\mathrm{Aut}}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}})$.

Lemma 4.2.2. Let $\mathcal{S}$ be a locally ringed site, let $\mathbf{A}$ be an abelian sheaf on $\mathcal{S}$, let $\pi : \mathcal{G} \to \mathcal{S}$ be an $\mathbf{A}$-gerbe. There is a natural isomorphism

$$\mathbf{R}^1 \pi_* \mathbb{G}_{m,\mathcal{G}} \simeq \underline{\mathrm{Hom}}_{\mathrm{Ab}(\mathcal{S})}(\mathbf{A}, \mathbb{G}_{m,\mathcal{S}}) \tag{4.2.2.1}$$

of abelian sheaves on $\mathcal{S}$.

Proof. Let $U \in \mathcal{S}$ be an object. Taking $\mathcal{T} := \mathcal{S}/U$ and $p : \mathcal{S}/U \to \mathcal{S}$ the inclusion of categories, we obtain an exact sequence

$$0 \to \mathrm{Pic}(\mathcal{S}/U) \xrightarrow{\pi^*_{\mathcal{S}/U}} \mathrm{Pic}(\mathcal{G}_{\mathcal{S}/U}) \xrightarrow{\beta_{\mathcal{G}_{\mathcal{S}/U}}} \widehat{\mathbf{A}}_{\mathcal{S}/U} \tag{4.2.2.2}$$

of abelian groups. Letting U range over the objects of $\mathcal{S}$, we obtain an exact sequence of abelian presheaves whose value on U is (4.2.2.2), and sheafifying this sequence gives the desired isomorphism. □

We specialize to the case $\mathbf{A} = \mathbb{G}_{m,\mathcal{S}}$.

Proposition 4.2.3. Let $\mathcal{S}$ be a locally ringed site and let $\pi : \mathcal{G} \to \mathcal{S}$ be a $\mathbb{G}_{m,\mathcal{S}}$-gerbe. Let

$$\mathrm{d}_2^{0,1} : \mathrm{H}^0(\mathcal{S}, \mathbf{R}^1 \pi_* \mathbb{G}_{m,\mathcal{G}}) \to \mathrm{H}^2(\mathcal{S}, \mathbf{R}^0 \pi_* \mathbb{G}_{m,\mathcal{G}}) \tag{4.2.3.1}$$

be the differential in the Leray spectral sequence associated to the map π and sheaf $\mathbb{G}_{m,\mathcal{G}}$. Under the identification (4.2.2.1), the differential $\mathrm{d}_2^{0,1}$ sends the identity $\mathrm{id}_{\mathbb{G}_{m,\mathcal{S}}} \in \mathrm{Hom}_{\mathrm{Ab}(\mathcal{S})}(\mathbb{G}_{m,\mathcal{S}}, \mathbb{G}_{m,\mathcal{S}})$ to the class $-[\mathcal{G}] \in \mathrm{H}^2(\mathcal{S}, \mathbb{G}_{m,\mathcal{S}})$.

Proof. Let $c \in \mathrm{H}^0(\mathcal{S}, \mathbf{R}^1 \pi_* \mathbb{G}_{m,\mathcal{G}})$ be the class corresponding to the identity section $\chi := \mathrm{id}_{\mathbb{G}_{m,\mathcal{S}}} \in \mathrm{Hom}_{\mathrm{Ab}(\mathcal{S})}(\mathbb{G}_{m,\mathcal{S}}, \mathbb{G}_{m,\mathcal{S}})$ via the isomorphism (4.2.2.1). As in [40, V, 3.1.6], let

$$D(c) \to \mathcal{S}$$

denote the category fibered in groupoids whose fiber category $(D(c))(U)$ for an object $U \in \mathcal{S}$ consists of the invertible $\mathcal{O}_{\mathcal{G}_{\mathcal{S}/U}}$-modules whose image under the map

$$\mathrm{H}^1(\mathcal{G}_{\mathcal{S}/U}, \mathbb{G}_{m,\mathcal{S}}) \to \mathrm{H}^0(U, \mathbf{R}^1\pi_*\mathbb{G}_{m,\mathcal{G}})$$

is equal to the image of c under the restriction map

$$\mathrm{H}^0(\mathcal{S}, \mathbf{R}^1\pi_*\mathbb{G}_{m,\mathcal{G}}) \to \mathrm{H}^0(U, \mathbf{R}^1\pi_*\mathbb{G}_{m,\mathcal{G}})$$

of the sheaf $\mathbf{R}^1\pi_*\mathbb{G}_{m,\mathcal{G}}$. By [40, V, 3.2.1], the category $D(c)$ is a $\mathbb{G}_{m,\mathcal{S}}$-gerbe, and the assignment $c \mapsto [D(c)]$ coincides with the differential (4.2.3.1). By the above description of $D(c)$ and by the definition of the isomorphism (4.2.2.1) as the one obtained by sheafifying the maps $\beta_{\mathcal{G}_{\mathcal{S}/U}}$ in (4.2.2.2), we have that an invertible $\mathcal{O}_{\mathcal{G}_{\mathcal{S}/U}}$-module $\mathcal{L}$ is contained in $(D(c))(U)$ if and only if it is $\chi|_{\mathcal{S}/U}$-twisted.

An arbitrary invertible $\mathcal{O}_{\mathcal{G}_{\mathcal{S}/U}}$-module $\mathcal{L}$ arises as the pullback of the tautological bundle under a morphism of stacks $\varphi_{\mathcal{L}} : \mathcal{G}_{\mathcal{S}/U} \to \mathbf{B}\mathbb{G}_{m,\mathcal{S}/U}$. If $u_{\mathcal{L}} : \mathbb{G}_{m,\mathcal{S}/U} \to \mathbb{G}_{m,\mathcal{S}/U}$ is the morphism induced by $\varphi_{\mathcal{L}}$ on bands [40, IV, 2.2.3], then $\mathcal{L}$ is $u_{\mathcal{L}}$-twisted. Hence we have an isomorphism of $\mathbb{G}_{m,\mathcal{S}}$-gerbes $D(c) \simeq \mathrm{HOM}_{\mathrm{id}}(\mathcal{G}, \mathbf{B}\mathbb{G}_{m,\mathcal{S}})$ in the notation of [40, IV, 2.3.1]; the class of the latter in $\mathrm{H}^2_{\text{ét}}(\mathcal{S}, \mathbb{G}_{m,\mathcal{S}})$ is equal to $-[\mathcal{G}]$ by [40, IV, 3.3.2 (iii)]. □

Remark 4.2.4. The category $D(c)$ in the proof of Proposition 4.2.3 may also be described in the following way. Let $\mathscr{P}ic(\mathcal{G}/\mathcal{S})$ denote the relative Picard stack of $\mathcal{G}/\mathcal{S}$, namely the category over $\mathcal{S}$ whose objects lying over some $U \in \mathcal{C}$ is the groupoid of invertible $\mathcal{G}_{\mathcal{S}/U}$-modules. Let $\mathrm{Pic}_{\mathcal{G}/\mathcal{S}} \simeq \mathbf{R}^1\pi_*\mathbb{G}_{m,\mathcal{G}}$ denote the relative Picard functor of $\mathcal{G}/\mathcal{S}$, namely the sheaf associated to the presheaf on $\mathcal{S}$ sending $U \mapsto \mathrm{Pic}(\mathcal{G}_{\mathcal{S}/U})/\mathrm{Pic}(U)$. There is a functor $F : \mathscr{P}ic(\mathcal{G}/\mathcal{S}) \to \mathrm{Pic}_{\mathcal{G}/\mathcal{S}}$ sending an invertible $\mathcal{O}_{\mathcal{G}_{\mathcal{S}/U}}$-module to its isomorphism class. Then $D(c)$ is equivalent to the pullback of F along the morphism $\mathcal{S} \to \mathrm{Pic}_{\mathcal{G}/\mathcal{S}}$ corresponding to the class c. □

4.3. **Picard groups of (Laurent) polynomial rings.** In this section we prove Lemma 4.3.11. For us, the main difficulty is that there are rings A for which the pullback map $\mathrm{Pic}(A) \to \mathrm{Pic}(A[t])$ is not an isomorphism. The ring A is called seminormal [90, p. 210], [94, p. 29] if for every $b, c \in A$ satisfying $b^3 = c^2$ there exists $a \in A$ such that $a^2 = b$ and $a^3 = c$. Seminormal rings are automatically reduced [57, VIII, §7]. By Traverso's theorem [91, Theorem 3.6], [94, Theorem 3.11], the map $\mathrm{Pic}(A) \to \mathrm{Pic}(A[t])$ is an isomorphism if and only if the reduction A_{red} is a seminormal ring. Taking the strict henselization of the cuspidal cubic $k[x,y]/(y^2 = x^3)$ at the cusp gives an example of a reduced strictly henselian local ring A which is not seminormal; by Remark 4.3.9, in this case we also have $\mathrm{Pic}(A[t, t^{-1}]) \neq 0$.

Throughout this section and Section 4.4, we will use Notation 4.3.1 and Notation 4.3.2.

Notation 4.3.1 $(\Delta, \mathrm{C}^\bullet\mathbf{G}, \mathsf{h}^n(\mathrm{C}^\bullet\mathbf{G}))$. Let Δ be the category with objects $[n] := \{0, \dots, n\}$ for each nonnegative integer $n \geq 0$ and whose morphisms $\varphi : [m] \to [n]$ correspond to nondecreasing maps $\varphi : \{0, \dots, m\} \to \{0, \dots, n\}$.

For $n \geq 0$ and $0 \leq i \leq n+1$, we denote $\delta_i^n : [n] \to [n+1]$ the injective nondecreasing map whose image does not contain i.

For $n \geq 0$ and $0 \leq i \leq n$, we denote $\sigma_i^n : [n+1] \to [n]$ the surjective nondecreasing map satisfying $(\sigma_i^n)^{-1}(i) = \{i, i+1\}$.

A cosimplicial set (resp. abelian group, resp. ring) is a covariant functor from Δ to (Set) (resp. (Ab), resp. (Ring)).

If $\mathbf{G}$ is a cosimplicial abelian group, we denote by

$$\mathrm{C}^\bullet\mathbf{G}$$

the cochain complex where $\mathrm{C}^n\mathbf{G} := \mathbf{G}([n])$ for $n \geq 0$ and where the nth differential $\mathbf{d}^n_{\mathbf{G}} : \mathrm{C}^n\mathbf{G} \to \mathrm{C}^{n+1}\mathbf{G}$ is the alternating sum $\sum_{i=0}^{n+1}(-1)^i\mathbf{G}(\delta^n_i)$. We denote by

$$\mathsf{h}^n(\mathrm{C}^\bullet\mathbf{G}) := \ker(\mathbf{d}^n_{\mathbf{G}})/\operatorname{im}(\mathbf{d}^{n-1}_{\mathbf{G}})$$

the cohomology of $\mathrm{C}^\bullet\mathbf{G}$ at $\mathrm{C}^n\mathbf{G}$.

Notation 4.3.2 ($\mathbf{L}_A, \mathbf{P}_A$). Let A be a ring, let $\pi : \mathrm{B}\mathbb{G}_{m,A} \to \operatorname{Spec} A$ be the trivial $\mathbb{G}_{m,A}$-gerbe, let $\xi : \operatorname{Spec} A \to \mathrm{B}\mathbb{G}_{m,A}$ be the section of π corresponding to the trivial $\mathbb{G}_{m,A}$-torsor. The Čech nerve of the covering ξ corresponds to a cosimplicial A-algebra

$$\mathbf{L}_A : \Delta \to (A\text{-alg})$$

where

$$\mathbf{L}_A([p]) := A[t_1^{\pm}, \dots, t_p^{\pm}]$$

is the Laurent polynomial ring in p indeterminates over A (where by convention $\mathbf{L}_A([0]) := A$).

For $p \geq 0$ and $1 \leq i \leq p$, the ith degeneracy map $\mathbf{L}_A(\delta^p_i) : \mathbf{L}_A([p]) \to \mathbf{L}_A([p+1])$ is the A-algebra map sending $(t_1, \dots, t_p) \mapsto (t_1, \dots, t_i t_{i+1}, \dots, t_{p+1})$; the 0th degeneracy map $\mathbf{L}_A(\delta^p_0)$ sends $(t_1, \dots, t_p) \mapsto (t_2, \dots, t_{p+1})$ and the $(p+1)$th degeneracy map $\mathbf{L}_A(\delta^p_{p+1})$ sends $(t_1, \dots, t_p) \mapsto (t_1, \dots, t_p)$.

For $p \geq 0$ and $0 \leq i \leq p$, the ith face map $\mathbf{L}_A(\sigma^p_i) : \mathbf{L}_A([p+1]) \to \mathbf{L}_A([p])$ is the A-algebra map sending $(t_1, \dots, t_{p+1}) \mapsto (t_1, \dots, t_i, 1, t_{i+1}, \dots, t_p)$.

We also have the cosimplicial A-algebra

$$\mathbf{P}_A : \Delta \to (A\text{-alg})$$

where

$$\mathbf{P}_A([p]) := A[t_1, \dots, t_p]$$

is the polynomial ring in p indeterminates over A, viewed as the subalgebra of $\mathbf{L}_A([p])$, and for which the A-algebra map $\mathbf{P}_A(\varphi) : \mathbf{P}_A([m]) \to \mathbf{P}_A([n])$ is obtained by restricting $\mathbf{L}_A(\varphi) : \mathbf{L}_A([m]) \to \mathbf{L}_A([n])$.

We make explicit the formulas $\mathbf{P}_A(\delta^p_i)$ for $p = 0, 1, 2$ and $\mathbf{P}_A(\sigma^p_i)$ for $p = 0, 1$. For $0 \leq i \leq 1$, the A-algebra map $\mathbf{P}_A(\delta^0_i) : A \to A[t_1]$ is the unique one. For $0 \leq i \leq 2$, the A-algebra map $\mathbf{P}_A(\delta^1_i) : A[t_1] \to A[t_1, t_2]$ sends t_1 to t_2, t_1t_2, t_1 respectively. For $0 \leq i \leq 3$, the A-algebra map $\mathbf{P}_A(\delta^2_i) : A[t_1, t_2] \to A[t_1, t_2, t_3]$ sends (t_1, t_2) to $(t_2, t_3), (t_1t_2, t_3), (t_1, t_2t_3), (t_1, t_2)$ respectively. For $0 \leq i \leq 0$, the A-algebra map $\mathbf{P}_A(\sigma^0_i) : A[t_1] \to A$ sends t_1 to 1. For $0 \leq i \leq 1$, the A-algebra map $\mathbf{P}_A(\sigma^1_i) : A[t_1, t_2] \to A[t_1]$ sends (t_1, t_2) to $(1, t_1), (t_1, 1)$ respectively.

Notation 4.3.3 ($\mathrm{N}_x\mathrm{F}, \mathrm{N}_{x_1,x_2}\mathrm{F}$). Given a functor

$$\mathrm{F} : (\mathrm{Ring}) \to (\mathrm{Ab}) \tag{4.3.3.1}$$

we define new functors

$$\mathrm{N}_x\mathrm{F}, \mathrm{N}_{x_1,x_2}\mathrm{F} : (\mathrm{Ring}) \to (\mathrm{Ab})$$

by

$$\mathrm{N}_x\mathrm{F}(A) := \ker(\mathrm{F}(x = 1) : \mathrm{F}(A[x]) \to \mathrm{F}(A))$$
$$\mathrm{N}_{x_1,x_2}\mathrm{F}(A) := \ker((\mathrm{F}(x_2 = 1), \mathrm{F}(x_1 = 1)) : \mathrm{F}(A[x_1, x_2]) \to \mathrm{F}(A[x_1]) \oplus \mathrm{F}(A[x_2]))$$

for any ring A, where x, x_1, x_2 are indeterminates. The notation "$\mathrm{N}_x\mathrm{F}$" was defined by Weibel in [93, §1].

The operation "N_x" can be iterated, for example if x_1, x_2 are indeterminates, then $\mathrm{N}_{x_1}(\mathrm{N}_{x_2}\mathrm{F})$ is a functor (Ring) $\to$ (Ab).

Lemma 4.3.4. In Notation 4.3.3, we have

$$\mathrm{N}_{x_1,x_2}\mathrm{F}(A) = \mathrm{N}_{x_1}(\mathrm{N}_{x_2}\mathrm{F})(A) = \mathrm{N}_{x_2}(\mathrm{N}_{x_1}\mathrm{F})(A)$$

for any ring A.

Proof. The claim follows from considering the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{N}_{x_1,x_2}\mathrm{F}(A) & \longrightarrow & \mathrm{N}_{x_1}\mathrm{F}(A[x_2]) & \xrightarrow{x_2=1} & \mathrm{N}_{x_1}\mathrm{F}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{N}_{x_2}\mathrm{F}(A[x_1]) & \longrightarrow & \mathrm{F}(A[x_1,x_2]) & \xrightarrow{x_2=1} & \mathrm{F}(A[x_1]) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x_1=1} & & \downarrow{\scriptstyle x_1=1} & & \downarrow{\scriptstyle x_1=1} & & \\
0 & \longrightarrow & \mathrm{N}_{x_2}\mathrm{F}(A) & \longrightarrow & \mathrm{F}(A[x_2]) & \xrightarrow{x_2=1} & \mathrm{F}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.3.4.1}$$

where each row and column is (split) exact. □

Lemma 4.3.5. Assume Notation 4.3.1, Notation 4.3.2, and Notation 4.3.3. We have

$$\mathbf{d}^1_{\mathrm{F}\mathbf{P}_A}(\mathrm{N}_{t_1}\mathrm{F}(A)) \subset \mathrm{N}_{t_1,t_2}\mathrm{F}(A)$$

for any ring A.

Proof. For $0 \leq i \leq 2$, the composition $\mathbf{P}_A(\sigma^1_0)\mathbf{P}_A(\delta^1_i)$ correspond to the A-algebra maps $A[t_1] \to A[t_1]$ sending $t_1 \mapsto 1, t_1, t_1$, respectively; thus $\mathrm{F}(\mathbf{P}_A(\sigma^1_0))(\mathbf{d}^1_{\mathrm{F}\mathbf{P}_A}(\mathrm{N}_{t_1}\mathrm{F}(A))) = 0$. By a similar argument, we have $\mathrm{F}(\mathbf{P}_A(\sigma^1_1))(\mathbf{d}^1_{\mathrm{F}\mathbf{P}_A}(\mathrm{N}_{t_1}\mathrm{F}(A))) = 0$. □

Lemma 4.3.6. Assume Notation 4.3.1, Notation 4.3.2, and Notation 4.3.3. We have

$$\mathsf{h}^1(\mathrm{C}^\bullet(\mathrm{Pic}\mathbf{P}_A)) = 0$$

for any ring A.

Proof. Since $\mathbf{P}_A(\delta^0_0) = \mathbf{P}_A(\delta^0_1)$, the differential $\mathbf{d}^0_{\mathrm{Pic}\mathbf{P}_A} : \mathrm{Pic}(A) \to \mathrm{Pic}(A[t_1])$ is the 0 map. Hence it suffices to show that

$$\mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_A} : \mathrm{Pic}(A[t_1]) \to \mathrm{Pic}(A[t_1,t_2])$$

is injective.

We have that A is the filtered colimit of subrings of A which are finite type $\mathbb{Z}$-algebras, hence by e.g. [88, 0B8W] we may reduce to the case when A is a finite type $\mathbb{Z}$-algebra.

In particular A has finite Krull dimension. We proceed by induction on $\dim A$. Since the Picard group of a ring is invariant under nilpotent thickenings, we may assume that A is reduced. If $\dim A = 0$, then A is a finite product of fields, hence $\ker \mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_A} = 0$ (since in fact $\mathrm{Pic}(A[t]) = 0$ in this case).

Suppose $\dim A > 0$ and let

$$\alpha \in \mathrm{Pic}(A[t_1])$$

be a class such that $\mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_A}(\alpha) = 0$. We have a direct sum decomposition $\mathrm{Pic}(A) \oplus \mathrm{N}_{t_1}\mathrm{Pic}(A) \simeq \mathrm{Pic}(A[t_1])$, and $\mathbf{P}_A(\delta^1_0), \mathbf{P}_A(\delta^1_1), \mathbf{P}_A(\delta^1_2)$ are A-algebra maps, so in fact $\alpha \in \mathrm{N}_{t_1}\mathrm{Pic}(A)$. Let $Q(A)$ denote the total ring of fractions of A, and let

$$A^{\mathrm{sn}} \subset Q(A)$$

denote the seminormalization [90, Lemma 2.2] of A in $Q(A)$. Write

$$A^{\mathrm{sn}} = \varinjlim_{\lambda \in \Lambda} A_\lambda$$

where each $A \subset A_\lambda \subset A^{\mathrm{sn}}$ is a finitely generated subextension of A^{sn}; then $A \subset A_\lambda$ is a finite extension of rings since it is an integral extension. Thus

$$\mathrm{N}_{t_1}\mathrm{Pic}(A^{\mathrm{sn}}) \simeq \varinjlim_{\lambda \in \Lambda} \mathrm{N}_{t_1}\mathrm{Pic}(A_\lambda)$$

by e.g. [88, 0B8W]. By [90, Corollary 3.4], we have that A^{sn} is seminormal, thus $\mathrm{N}_{t_1}\mathrm{Pic}(A^{\mathrm{sn}}) = 0$ by Traverso's theorem [94, Theorem 3.11]. Hence there exists some $\lambda \in \Lambda$ for which α lies in the kernel of $\mathrm{N}_{t_1}\mathrm{Pic}(A) \to \mathrm{N}_{t_1}\mathrm{Pic}(A_\lambda)$.[16] Let

$$I := \{x \in A \ : \ xA_\lambda \subset A\} = \mathrm{Ann}_A(A_\lambda/A)$$

be the conductor ideal of $A \subset A_\lambda$; it is the largest ideal of A_λ contained in A so in particular it is also an ideal of A. We denote

$$\mathrm{U}(A) := A^\times$$

the group of units of A. Let $\mathbf{S}$ denote the commutative cartesian diagram

$$\begin{array}{ccc} A & \hookrightarrow & A_\lambda \\ \downarrow & & \downarrow \\ A/I & \hookrightarrow & A_\lambda/I \end{array} \tag{4.3.6.1}$$

of rings (called a "Milnor square"). By Milnor's theorem [9, IX, (5.3)] there is an exact sequence

$$G(\mathbf{S}) = \left\{ \begin{array}{l} 1 \to \mathrm{U}(A) \xrightarrow{\Delta} \mathrm{U}(A/I) \oplus \mathrm{U}(A_\lambda) \xrightarrow{\pm} \mathrm{U}(A_\lambda/I) \\ \quad \xrightarrow{\partial} \mathrm{Pic}(A) \xrightarrow{\Delta} \mathrm{Pic}(A/I) \oplus \mathrm{Pic}(A_\lambda) \xrightarrow{\pm} \mathrm{Pic}(A_\lambda/I) \end{array} \right\} \tag{4.3.6.2}$$

of abelian groups, called the Units-Pic sequence [94, I, Theorem 3.10]; here we denote by Δ the diagonal map and by $\pm$ the difference map. For any flat A-algebra B we obtain a corresponding Units-Pic sequence $G(\mathbf{S} \otimes_A B)$, and for a morphism $B_1 \to B_2$ of flat A-algebras we obtain a morphism of complexes $G(\mathbf{S} \otimes_A B_1) \to G(\mathbf{S} \otimes_A B_2)$ since the boundary map ∂ of (4.3.6.2) is functorial for morphisms between Milnor squares. We have a morphism of

[16]Here, instead of using the limit argument, we may also use that the extension $A \subset A^{\mathrm{sn}}$ is finite since A is a Nagata ring (it is a finite type $\mathbb{Z}$-algebra) and thus has finite normalization, hence has finite seminormalization.

complexes

$$\mathbf{d}^1_{G(\mathbf{P_S})} : G(\mathbf{S}[t_1]) \to G(\mathbf{S}[t_1, t_2])$$

consisting of the maps $\mathbf{d}^1_{\mathrm{F}\mathbf{P}_B}$ as F ranges over $\mathrm{U}(-), \mathrm{Pic}(-)$ and B ranges over the rings in $\mathbf{S}$. Let $\mathrm{N}_{t_1}G(\mathbf{S})$ be the kernel of the morphism of complexes $G(\mathbf{S}[t_1]) \to G(\mathbf{S})$ sending $t_1 \mapsto 1$; it is an exact sequence since $G(\mathbf{S}[t_1]) \to G(\mathbf{S})$ admits a section. Considering the diagram (4.3.4.1), we obtain a similar exact sequence $\mathrm{N}_{t_1,t_2}G(\mathbf{S})$, and the map $\mathbf{d}^1_{G(\mathbf{P_S})}$ above restricts to a map

$$\mathbf{d}^1_{G(\mathbf{P_S})} : \mathrm{N}_{t_1}G(\mathbf{S}) \to \mathrm{N}_{t_1,t_2}G(\mathbf{S})$$

by Lemma 4.3.5. In particular, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{N}_{t_1}\mathrm{U}(A/I) \oplus \mathrm{N}_{t_1}\mathrm{U}(A_\lambda) & \xrightarrow{\mathbf{d}^1_{\mathrm{U}\mathbf{P}_{A/I}} \oplus \mathbf{d}^1_{\mathrm{U}\mathbf{P}_{A_\lambda}}} & \mathrm{N}_{t_1,t_2}\mathrm{U}(A/I) \oplus \mathrm{N}_{t_1,t_2}\mathrm{U}(A_\lambda) \\
\downarrow \pm_{t_1} & & \downarrow \pm_{t_1,t_2} \\
\mathrm{N}_{t_1}\mathrm{U}(A_\lambda/I) & \xrightarrow{\mathbf{d}^1_{\mathrm{U}\mathbf{P}_{A_\lambda/I}}} & \mathrm{N}_{t_1,t_2}\mathrm{U}(A_\lambda/I) \\
\downarrow \partial_{t_1} & & \downarrow \partial_{t_1,t_2} \\
\mathrm{N}_{t_1}\mathrm{Pic}(A) & \xrightarrow{\mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_A}} & \mathrm{N}_{t_1,t_2}\mathrm{Pic}(A) \\
\downarrow \Delta_{t_1} & & \downarrow \Delta_{t_1,t_2} \\
\mathrm{N}_{t_1}\mathrm{Pic}(A/I) \oplus \mathrm{N}_{t_1}\mathrm{Pic}(A_\lambda) & \xrightarrow{\mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_{A/I}} \oplus \mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_{A_\lambda}}} & \mathrm{N}_{t_1,t_2}\mathrm{Pic}(A/I) \oplus \mathrm{N}_{t_1,t_2}\mathrm{Pic}(A_\lambda)
\end{array} \tag{4.3.6.3}$$

with exact columns, where we denote by $\pm_{t_1}, \partial_{t_1}, \Delta_{t_1}$ and $\pm_{t_1,t_2}, \partial_{t_1,t_2}, \Delta_{t_1,t_2}$ the corresponding maps in $\mathrm{N}_{t_1}G(\mathbf{S})$ and $\mathrm{N}_{t_1,t_2}G(\mathbf{S})$ respectively.

By Lemma 4.3.7 below, we have that I contains a nonzerodivisor of A; hence I is not contained in any minimal prime of A by [3, Lemma (14.10)]; hence A/I has smaller Krull dimension than that of A (c.f. [94, p. 15]); the image of α under $\mathrm{N}_{t_1}\mathrm{Pic}(A) \to \mathrm{N}_{t_1}\mathrm{Pic}(A/I)$ is contained in $\ker \mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_{A/I}}$, which is 0 by the induction hypothesis since $\dim A/I < \dim A$. Hence $\Delta_{t_1}(\alpha) = 0$, so by exactness of the left column of (4.3.6.3), there exists

$$\xi \in \mathrm{N}_{t_1}\mathrm{U}(A_\lambda/I)$$

such that $\alpha = \partial_{t_1}(\xi)$. By e.g. [94, I, Lemma 3.12] we have that ξ is of the form

$$\xi = 1 + \beta(t_1)$$

where

$$\beta \in t_1(\mathrm{nil}(A_\lambda/I)[t_1])$$

is a polynomial with nilpotent coefficients and whose constant coefficient is zero. We have

$$\mathbf{d}^1_{\mathrm{U}\mathbf{P}_{A_\lambda/I}}(\xi) = (1 + \beta(t_1))(1 + \beta(t_1 t_2))^{-1}(1 + \beta(t_2)) \tag{4.3.6.4}$$

in $\mathrm{N}_{t_1,t_2}\mathrm{U}(A_\lambda/I)$. Since $\partial_{t_1,t_2}(\mathbf{d}^1_{\mathrm{U}\mathbf{P}_{A_\lambda/I}}(\xi)) = \mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_A}(\partial_{t_1}(\xi)) = \mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_A}(\alpha) = 0$, by the exactness of the right column of (4.3.6.3) there exists

$$\gamma \in \mathrm{N}_{t_1,t_2}\mathrm{U}(A/I) \oplus \mathrm{N}_{t_1,t_2}\mathrm{U}(A_\lambda)$$

such that $\mathbf{d}^1_{\mathbf{UP}_{A_\lambda/I}}(\xi) = \pm_{t_1,t_2}(\gamma)$. Here by [94, I, Lemma 3.12] the inclusion $\mathrm{U}(A_\lambda) \subset \mathrm{U}(A_\lambda[t_1,t_2])$ is an equality since A_λ is reduced, hence $\mathrm{N}_{t_1,t_2}\mathrm{U}(A_\lambda) = 0$. Moreover $A/I \to A_\lambda/I$ is injective (since I is the largest ideal of A_λ contained in A), hence (4.3.6.4) is in fact contained in $\mathrm{N}_{t_1,t_2}\mathrm{U}(A/I)$. Thus in fact

$$\beta \in (A/I)[t_1]$$

as can be seen for example by setting $t_2 = 0$ in (4.3.6.4). In other words, we have that ξ is in the image of $\pm_{t_1}$; since $\partial_{t_1} \circ \pm_{t_1} = 0$, we conclude $\alpha = 0$. ☐

In the following lemma, we write out the details of a claim in [94, p. 15].

Lemma 4.3.7. Let A be a ring with total ring of fractions $Q(A)$, and let $A \subset B \subset Q(A)$ be a subring.

(i) The inclusion $A \subset B$ preserves nonzerodivisors, and any nonzerodivisor of B is of the form r/u where $r, u \in A$ are nonzerodivisors of A.
(ii) The total ring of fractions of B is $Q(A)$.
(iii) If $A \subset B$ is a finite extension, the conductor ideal $I = \{x \in A \ : \ xB \subset A\} = \mathrm{Ann}_A(B/A)$ contains a nonzerodivisor of A.

Proof. (i) If $x \in A$ is a nonzerodivisor of A, then its image in $Q(A)$ is a nonzerodivisor of $Q(A)$, hence its image in B is a nonzerodivisor of B. An arbitrary element of B is of the form r/u where $r, u \in A$ and u is a nonzerodivisor of A. If $x \in A$ is an element such that $rx = 0$ in A, then $u(r/u)x = 0$ in B and u is a nonzerodivisor of B (by the first part) so $(r/u)x = 0$ in B; then $x = 0$ since r/u is by assumption a nonzerodivisor of B; hence r is a nonzerodivisor of A.

(ii) Let $\varphi : B \to S$ be a ring homomorphism such that φ sends nonzerodivisors of B to units of S. By the first part, φ sends nonzerodivisors of A to units of S, hence there exists a ring map $\xi : Q(A) \to S$ such that $\xi|_A = \varphi|_A$. By definition of ξ, for any $a/u \in B$ we have $\xi(a/u) = \varphi(a) \cdot (\varphi(u))^{-1} = \varphi(a/u)$; hence $\xi|_B = \varphi$.

(iii) Let $x_1/u_1, \ldots, x_n/u_n$ be elements of B which generates B as an A-module; then $u_1 \cdots u_n$ is a nonzerodivisor of A which is contained in I. ☐

Remark 4.3.8. In the proof of Lemma 4.3.6, we may use the normalization instead of the seminormalization. If A is a reduced finite type $\mathbb{Z}$-algebra, then its normalization $\overline{A} \subset Q(A)$ is a finite extension of A; thus $\overline{A}$ is a Noetherian reduced ring which is integrally closed in its total ring of fractions (by e.g. Lemma 4.3.7 (ii)), hence it is finite product of Noetherian normal integral domains [88, 030C]. It is easily checked from the definition of a seminormal ring that normal domains are seminormal.

Remark 4.3.9. For any ring R, by [93, Lemma 1.5.1] and [93, Theorem 5.5] we have an exact sequence

$$0 \to \mathrm{Pic}(R) \xrightarrow{f} \mathrm{Pic}(R[t]) \oplus \mathrm{Pic}(R[t^{-1}]) \xrightarrow{\Sigma} \mathrm{Pic}(R[t^{\pm}]) \to \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\, R, \mathbb{Z}) \to 0 \qquad (4.3.9.1)$$

of abelian groups, where f denotes the map sending $\alpha \mapsto (\alpha, -\alpha)$ and Σ denotes the addition map. For any ring R, by [93, Theorem 2.4] and [93, Theorem 5.5] we have isomorphisms

$$\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\, R, \mathbb{Z}) \simeq \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\, R[t], \mathbb{Z}) \simeq \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\, R[t^{\pm}], \mathbb{Z}) \qquad (4.3.9.2)$$

of abelian groups.

The following is stated in [93]; we write out the details here.

Lemma 4.3.10. Let A be a strictly henselian local ring. Then the canonical map

$$\bigoplus_{(\ell,\diamond)\in\{1,2\}\times\{+,-\}} \mathrm{Pic}(A[t_\ell^\diamond]) \oplus \bigoplus_{(\diamond_1,\diamond_2)\in\{+,-\}^2} \mathrm{N}_{t_1^{\diamond_1},t_2^{\diamond_2}}\mathrm{Pic}(A) \to \mathrm{Pic}(A[t_1^\pm, t_2^\pm])$$

induced by the inclusions $A[t_\ell^\diamond] \to A[t_1^\pm, t_2^\pm]$ and $A[t_1^{\diamond_1}, t_2^{\diamond_2}] \to A[t_1^\pm, t_2^\pm]$ is an isomorphism.

Proof. For notational convenience, we denote $t^+ = t$ and $t^- = t^{-1}$, etc. Since A is strictly henselian local, by (4.3.9.2) the exact sequence (4.3.9.1) reduces to an isomorphism

$$\mathrm{Pic}(A[t^+]) \oplus \mathrm{Pic}(A[t^-]) \xrightarrow{\sim} \mathrm{Pic}(A[t^\pm]) \tag{4.3.10.1}$$

and split exact sequences

$$0 \to \mathrm{Pic}(A[t_2^\diamond]) \to \mathrm{Pic}(A[t_1^+, t_2^\diamond]) \oplus \mathrm{Pic}(A[t_1^-, t_2^\diamond]) \to \mathrm{Pic}(A[t_1^\pm, t_2^\diamond]) \to 0 \tag{4.3.10.2}$$

and

$$0 \to \mathrm{Pic}(A[t_1^\pm]) \to \mathrm{Pic}(A[t_1^\pm, t_2^+]) \oplus \mathrm{Pic}(A[t_1^\pm, t_2^-]) \to \mathrm{Pic}(A[t_1^\pm, t_2^\pm]) \to 0 \tag{4.3.10.3}$$

by taking $R := A, A[t_2^\diamond], A[t_1^\pm]$ respectively for $\diamond \in \{+,-\}$. The sequence (4.3.10.2) induces a natural isomorphism

$$\mathrm{Pic}(A[t_2^\diamond]) \oplus \mathrm{N}_{t_1^+}\mathrm{Pic}(A[t_2^\diamond]) \oplus \mathrm{N}_{t_1^-}\mathrm{Pic}(A[t_2^\diamond]) \xrightarrow{\sim} \mathrm{Pic}(A[t_1^\pm, t_2^\diamond]) \tag{4.3.10.4}$$

of abelian groups. The isomorphism (4.3.10.4) restricts to an isomorphism

$$\mathrm{Pic}(A[t_2^\diamond]) \oplus \mathrm{N}_{t_1^+,t_2^\diamond}\mathrm{Pic}(A) \oplus \mathrm{N}_{t_1^-,t_2^\diamond}\mathrm{Pic}(A) \xrightarrow{\sim} \mathrm{N}_{t_2^\diamond}\,\mathrm{Pic}(A[t_1^\pm]) \tag{4.3.10.5}$$

by taking the subgroups of elements annihilated by setting $t_2^\diamond = 1$. The sequence (4.3.10.3) induces a natural isomorphism

$$\mathrm{Pic}(A[t_1^\pm]) \oplus \mathrm{N}_{t_2^+}\mathrm{Pic}(A[t_1^\pm]) \oplus \mathrm{N}_{t_2^-}\mathrm{Pic}(A[t_1^\pm]) \xrightarrow{\sim} \mathrm{Pic}(A[t_1^\pm, t_2^\pm]) \tag{4.3.10.6}$$

of abelian groups. We combine (4.3.10.6) and (4.3.10.1) and (4.3.10.5) (for $\diamond \in \{+,-\}$) and Lemma 4.3.4 to obtain the desired result. $\square$

Lemma 4.3.11. We have

$$\ker(\mathbf{d}^1_{\mathrm{Pic}\mathbf{L}_A}) = 0$$

for any strictly henselian local ring A.

Proof. The inclusion $\mathrm{N}_{t_1}\mathrm{Pic}(A) \subseteq \mathrm{Pic}(A[t_1])$ is an equality since A is a local ring; recall that $\mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_A}(\mathrm{N}_{t_1}\mathrm{Pic}(A)) \subseteq \mathrm{N}_{t_1,t_2}\mathrm{Pic}(A)$ by Lemma 4.3.5. We have a commutative diagram

$$\begin{array}{ccc}
(\mathrm{Pic}(A[t_1]))^{\oplus 2} & \xrightarrow[f_1]{\simeq} & \mathrm{Pic}(A[t_1^\pm]) \\
\Big\downarrow{\scriptstyle (\mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_A})^{\oplus 2}} & & \Big\downarrow{\scriptstyle \mathbf{d}^1_{\mathrm{Pic}\mathbf{L}_A}} \\
(\mathrm{N}_{t_1,t_2}\mathrm{Pic}(A))^{\oplus 2} & \underset{f_2}{\hookrightarrow} & \mathrm{Pic}(A[t_1^\pm, t_2^\pm])
\end{array}$$

where f_1 and f_2 are the addition maps induced on the Picard groups by the A-algebra maps $A[t_1] \to A[t_1^\pm]$ sending t_1 to t_1, t_1^{-1} and $A[t_1,t_2] \to A[t_1^\pm, t_2^\pm]$ sending $(t_1,t_2) \mapsto (t_1,t_2), (t_1^{-1}, t_2^{-1})$ respectively. Here f_1 is an isomorphism by (4.3.9.1) since A is strictly henselian local, and f_2

is injective by Lemma 4.3.10. Since $\mathbf{d}^1_{\mathrm{Pic}\mathbf{P}_A}$ is injective by Lemma 4.3.6, we have that $\mathbf{d}^1_{\mathrm{Pic}\mathbf{L}_A}$ is injective. □

4.4. **Unit groups of Laurent polynomial rings.** The purpose of this section is to prove Lemma 4.4.2. As in Section 4.3, when it is convenient we will denote $\mathrm{U}(A) := A^\times$ the group of units of a ring A.

Lemma 4.4.1. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a filtered inductive system of rings, and let

$$A := \varinjlim_{\lambda\in\Lambda} A_\lambda$$

be the colimit ring. In the notation of Notation 4.3.1 and Notation 4.3.2, the induced morphism of complexes

$$\varinjlim_{\lambda\in\Lambda} \mathrm{C}^\bullet(\mathrm{U}\mathbf{L}_{A_\lambda}) \to \mathrm{C}^\bullet(\mathrm{U}\mathbf{L}_A)$$

is an isomorphism.

Proof. For any $n \geq 0$, the functor (Ring) $\to$ (Ab) sending $A \mapsto (A[t_1^\pm,\dots,t_n^\pm])^\times$ is locally of finite presentation. □

Lemma 4.4.2. For any ring A, we have $\mathsf{h}^2(\mathrm{C}^\bullet(\mathrm{U}\mathbf{L}_A)) = 0$.

Proof. By writing A as the filtered colimit of subrings which are finite type $\mathbb{Z}$-algebras, by Lemma 4.4.1 we may reduce to the case when A is a finite type $\mathbb{Z}$-algebra. By replacing $\operatorname{Spec} A$ by a connected component, we may assume that $\operatorname{Spec} A$ is connected. Let $\mathfrak{n} \subset A$ be the nilradical of A. By [73, Corollary 6], a unit ξ of $A[t_1^\pm, t_2^\pm]$ is of the form

$$\xi = ut_1^{e_1}t_2^{e_2} + x(t_1,t_2) \tag{4.4.2.1}$$

where $u \in A^\times$ is a unit and $(e_1,e_2) \in \mathbb{Z}^{\oplus 2}$ is an ordered pair of integers and $x(t_1,t_2) \in \mathfrak{n}A[t_1^\pm,t_2^\pm]$ is a Laurent polynomial all of whose coefficients are nilpotent. We have that each unit $u \in A^\times \subset (A[t_1^\pm,t_2^\pm])^\times$ is in the image of $\mathbf{d}^1_{\mathrm{U}\mathbf{L}_A}$, namely the image of the unit $u \in A^\times \subset (A[t_1^\pm])^\times$ since $u\cdot u^{-1}\cdot u = u$. Hence we may assume that the unit u of (4.4.2.1) is equal to 1. We have

$$\begin{aligned}\mathbf{d}^2_{\mathrm{U}\mathbf{L}_A}(ut_1^{e_1}t_2^{e_2}) &= (ut_2^{e_1}t_3^{e_2})\cdot(u(t_1t_2)^{e_1}t_3^{e_2})^{-1}\cdot(ut_1^{e_1}(t_2t_3)^{e_2})\cdot(ut_1^{e_1}t_2^{e_2})^{-1}\\ &= t_1^{-e_1}t_3^{e_2}\end{aligned}$$

by the description of the maps $\mathbf{P}_A(\delta_i^2)$ in Notation 4.3.2. Suppose $\xi \in \ker \mathbf{d}^2_{\mathrm{U}\mathbf{L}_A}$; then $t_1^{-e_1}t_3^{e_2} = 1$, hence $e_1 = e_2 = 0$. This implies that $\mathsf{h}^2(\mathrm{C}^\bullet(\mathrm{U}\mathbf{L}_{A/\mathfrak{n}^1})) = 0$.

We have a sequence

$$A/\mathfrak{n}^s \to A/\mathfrak{n}^{s-1} \to \cdots \to A/\mathfrak{n}^2 \to A/\mathfrak{n}^1$$

where each map is a surjective ring map with square-zero kernel. Hence, since the complex $\mathrm{C}^\bullet(\mathrm{U}\mathbf{L}_A)$ is functorial in A, it suffices to show that, for any ring A and ideal $I \subset A$ satisfying $I^2 = 0$, if $\mathsf{h}^2(\mathrm{C}^\bullet(\mathrm{U}\mathbf{L}_{A/I})) = 0$ then $\mathsf{h}^2(\mathrm{C}^\bullet(\mathrm{U}\mathbf{L}_A)) = 0$. The quotient $A \to A/I$ induces a morphism $\mathrm{C}^\bullet(\mathrm{U}\mathbf{L}_A) \to \mathrm{C}^\bullet(\mathrm{U}\mathbf{L}_{A/I})$ of complexes of abelian groups, part of which is a commutative diagram

$$\begin{array}{ccccc} (A[t_1^{\pm}])^{\times} & \xrightarrow{\mathbf{d}^1_{\mathrm{U}\mathbf{L}_A}} & (A[t_1^{\pm},t_2^{\pm}])^{\times} & \xrightarrow{\mathbf{d}^2_{\mathrm{U}\mathbf{L}_A}} & (A[t_1^{\pm},t_2^{\pm},t_3^{\pm}])^{\times} \\ \downarrow{\scriptstyle \pi^1} & & \downarrow{\scriptstyle \pi^2} & & \downarrow{\scriptstyle \pi^3} \\ ((A/I)[t_1^{\pm}])^{\times} & \xrightarrow{\mathbf{d}^1_{\mathrm{U}\mathbf{L}_{A/I}}} & ((A/I)[t_1^{\pm},t_2^{\pm}])^{\times} & \xrightarrow{\mathbf{d}^2_{\mathrm{U}\mathbf{L}_{A/I}}} & ((A/I)[t_1^{\pm},t_2^{\pm},t_3^{\pm}])^{\times} \end{array} \tag{4.4.2.2}$$

where each vertical arrow π^1, π^2, π^3 is surjective since I is square-zero. By a diagram chase on (4.4.2.2), to show that the top row is exact it suffices to show that every element of $(\ker \mathbf{d}^2_{\mathrm{U}\mathbf{L}_A}) \cap (\ker \pi^2)$ is in the image of $\mathbf{d}^1_{\mathrm{U}\mathbf{L}_A}$. We have $\ker \pi^2 = 1 + IA[t_1^{\pm}, t_2^{\pm}]$; moreover, since I is square-zero, the (multiplicative) condition that $1+x(t_1,t_2) \in \ker \mathbf{d}^2_{\mathrm{U}\mathbf{L}_A}$ is equivalent to the (additive) condition that the element

$$x(t_1,t_2) - x(t_1,t_2t_3) + x(t_1t_2,t_3) - x(t_2,t_3) \tag{4.4.2.3}$$

of $A[t_1^{\pm}, t_2^{\pm}, t_3^{\pm}]$ is equal to zero. Let

$$\begin{aligned} \mathrm{H}_0 &:= \{e_3 = 0\} \\ \mathrm{H}_1 &:= \{e_2 = e_3\} \\ \mathrm{H}_2 &:= \{e_1 = e_2\} \\ \mathrm{H}_3 &:= \{e_1 = 0\} \end{aligned}$$

be hyperplanes of $\mathbb{Z}^{\oplus 3} = \{(e_1,e_2,e_3)\}$ defined by the equations corresponding to the maps $\mathbf{L}_A(\mathrm{p}_0^2), \mathbf{L}_A(\mathrm{p}_1^2), \mathbf{L}_A(\mathrm{p}_2^2), \mathbf{L}_A(\mathrm{p}_3^2)$ in the sense that the image of $\mathbb{Z}^{\oplus 2}$ under $\mathbf{L}_A(\mathrm{p}_i^2)$ is $\mathrm{H}_i \subset \mathbb{Z}^{\oplus 3}$. Then the pairwise intersections

$$\begin{aligned} \mathrm{H}_0 \cap \mathrm{H}_1 &= \mathbb{Z}(1,0,0) & \mathrm{H}_1 \cap \mathrm{H}_2 &= \mathbb{Z}(1,1,1) \\ \mathrm{H}_0 \cap \mathrm{H}_2 &= \mathbb{Z}(1,1,0) & \mathrm{H}_1 \cap \mathrm{H}_3 &= \mathbb{Z}(0,1,1) \\ \mathrm{H}_0 \cap \mathrm{H}_3 &= \mathbb{Z}(0,1,0) & \mathrm{H}_2 \cap \mathrm{H}_3 &= \mathbb{Z}(0,0,1) \end{aligned}$$

are all distinct. Let

$$x_{e_1,e_2} \in I$$

be the coefficient of $t_1^{e_1} t_2^{e_2}$ in $x(t_1,t_2)$. Then if $(e_1,e_2) \in \mathbb{Z}^{\oplus 2}$ is an ordered pair for which $x_{e_1,e_2} \neq 0$, then we must have

$$(e_1,e_2) \in \mathbb{Z}(1,0) \cup \mathbb{Z}(1,1) \cup \mathbb{Z}(0,1)$$

in $\mathbb{Z}^{\oplus 2}$. Moreover, saying that (4.4.2.3) is equal to zero translates to the collection of equations

$$\begin{aligned} x_{e,0} - x_{e,0} &= 0 & x_{e,e} - x_{e,e} &= 0 \\ x_{e,e} + x_{e,0} &= 0 & x_{0,e} + x_{e,e} &= 0 \\ x_{0,e} - x_{e,0} &= 0 & x_{0,e} - x_{0,e} &= 0 \end{aligned}$$

for all $e \in \mathbb{Z}$, which simplifies to

$$x_{e,0} = x_{0,e} = -x_{e,e}$$

for all $e \in \mathbb{Z}$. Then

$$1 + x(t_1,t_2) = \mathbf{d}^1_{\mathrm{U}\mathbf{L}_A}(1 - \textstyle\sum_{e \in \mathbb{Z}} x_{e,e} t_1^e)$$

so we have the desired result. $\square$

4.5. **Proof of the main theorem.** In this section we prove Theorem 4.1.2.

Our argument, in outline, is that of the proof of [37, II, Lemma 1′]. Namely, we compute $\mathrm{H}^2_{\text{ét}}(\mathcal{G}, \mathbb{G}_{m,\mathcal{G}})$ using the Leray spectral sequence associated to the map π and sheaf $\mathbb{G}_{m,\mathcal{G}}$, which is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}(S, \mathbf{R}^q\pi_*\mathbb{G}_{m,\mathcal{G}}) \implies \mathrm{H}^{p+q}_{\text{ét}}(\mathcal{G}, \mathbb{G}_{m,\mathcal{G}}) \tag{4.5.0.1}$$

with differentials $\mathrm{d}_2^{p,q} : \mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$.

The stalks of $\mathbf{R}^2\pi_*\mathbb{G}_{m,\mathcal{G}}$ are described by Lemma 4.5.2.

Setup 4.5.1 (Descent spectral sequence for $\mathrm{B}\mathbb{G}_m$). Let A be a ring and let

$$\xi : \operatorname{Spec} A \to \mathrm{B}\mathbb{G}_{m,A}$$

be the smooth cover associated to the trivial $\mathbb{G}_{m,A}$-torsor. The cohomological descent spectral sequence associated to ξ gives a spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q_{\text{ét}}(\mathbb{G}_{m,A}^{\times p}, \mathbb{G}_m) \implies \mathrm{H}^{p+q}_{\text{ét}}(\mathrm{B}\mathbb{G}_{m,A}, \mathbb{G}_m) \tag{4.5.1.1}$$

where the qth row $\mathrm{E}_1^{\bullet,q} = \mathrm{H}^q_{\text{ét}}(\mathbb{G}_{m,A}^{\times\bullet}, \mathbb{G}_m)$ can be realized as the complex $\mathrm{C}^\bullet(\mathrm{F}\mathbf{L}_A)$ (see Notation 4.3.1 and Notation 4.3.2) where the functor $\mathrm{F} : (\mathrm{Ring}) \to (\mathrm{Ab})$ is defined by $\mathrm{F}(R) := \mathrm{H}^q_{\text{ét}}(\operatorname{Spec} R, \mathbb{G}_m)$. The lower-left part of the E_1-page of the spectral sequence (4.5.1.1) is

$$\begin{array}{c}
\mathrm{H}^3_{\text{ét}}(\mathbb{G}_{m,A}^{\times 0}, \mathbb{G}_m) \longrightarrow \mathrm{H}^3_{\text{ét}}(\mathbb{G}_{m,A}^{\times 1}, \mathbb{G}_m) \longrightarrow \mathrm{H}^3_{\text{ét}}(\mathbb{G}_{m,A}^{\times 2}, \mathbb{G}_m) \longrightarrow \mathrm{H}^3_{\text{ét}}(\mathbb{G}_{m,A}^{\times 3}, \mathbb{G}_m) \\
\mathrm{H}^2_{\text{ét}}(\mathbb{G}_{m,A}^{\times 0}, \mathbb{G}_m) \longrightarrow \mathrm{H}^2_{\text{ét}}(\mathbb{G}_{m,A}^{\times 1}, \mathbb{G}_m) \longrightarrow \mathrm{H}^2_{\text{ét}}(\mathbb{G}_{m,A}^{\times 2}, \mathbb{G}_m) \longrightarrow \mathrm{H}^2_{\text{ét}}(\mathbb{G}_{m,A}^{\times 3}, \mathbb{G}_m) \\
\mathrm{H}^1_{\text{ét}}(\mathbb{G}_{m,A}^{\times 0}, \mathbb{G}_m) \longrightarrow \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{m,A}^{\times 1}, \mathbb{G}_m) \longrightarrow \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{m,A}^{\times 2}, \mathbb{G}_m) \longrightarrow \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{m,A}^{\times 3}, \mathbb{G}_m) \\
\mathrm{H}^0_{\text{ét}}(\mathbb{G}_{m,A}^{\times 0}, \mathbb{G}_m) \longrightarrow \mathrm{H}^0_{\text{ét}}(\mathbb{G}_{m,A}^{\times 1}, \mathbb{G}_m) \longrightarrow \mathrm{H}^0_{\text{ét}}(\mathbb{G}_{m,A}^{\times 2}, \mathbb{G}_m) \longrightarrow \mathrm{H}^0_{\text{ét}}(\mathbb{G}_{m,A}^{\times 3}, \mathbb{G}_m)
\end{array}$$

where $\mathrm{d}_1^{0,q}$ is the zero map for all $q \geq 0$ since $\mathrm{B}\mathbb{G}_{m,A}$ is the quotient of $\operatorname{Spec} A$ by the trivial action of $\mathbb{G}_{m,A}$.

Lemma 4.5.2. Assume the setup of Setup 4.5.1. For any strictly henselian local ring A, we have $\mathrm{H}^2_{\text{ét}}(\mathrm{B}\mathbb{G}_{m,A}, \mathbb{G}_m) = 0$.

Proof. We have $\mathrm{E}_1^{0,q} = \mathrm{H}^q_{\text{ét}}(\operatorname{Spec} A, \mathbb{G}_m) = 0$ for any $q \geq 1$ since A is strictly henselian. We have $\mathrm{E}_2^{1,1} = 0$ by Lemma 4.3.11 and $\mathrm{E}_2^{2,0} = 0$ by Lemma 4.4.2. □

Remark 4.5.3. We show that, in the proof of Lemma 4.5.2, it is possible to reduce to the case when A is a reduced ring; the reducedness assumption simplifies the proof of Lemma 4.4.2. By standard limit arguments, we may assume that A is a finite type $\mathbb{Z}$-algebra. Then the reduction $A \to A_{\text{red}}$ can be factored as a finite sequence of square-zero thickenings. Thus we reduce to showing that if $A \to A_0$ is a surjection of rings whose kernel I is square-zero, then the reduction map

$$\mathrm{H}^2_{\text{ét}}(\mathrm{B}\mathbb{G}_{m,A}, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(\mathrm{B}\mathbb{G}_{m,A_0}, \mathbb{G}_m) \tag{4.5.3.1}$$

is an isomorphism. Set $\mathscr{X} := \mathrm{B}\mathbb{G}_{m,A}$ and $\mathscr{X}_0 := \mathrm{B}\mathbb{G}_{m,A_0}$ and let $i : \mathscr{X}_0 \to \mathscr{X}$ be the closed immersion. We may use either the big étale site $(\mathrm{Sch}/\mathscr{X})_{\text{ét}}$ or the lisse-étale site $\text{Lis-Et}(\mathscr{X})$

to compute cohomology on $\mathscr{X}$, since the inclusion functor of sites

$$u : \text{Lis-Et}(\mathscr{X}) \to (\text{Sch}/\mathscr{X})_{\text{ét}}$$

induces a restriction functor on abelian sheaves

$$u^{-1} : \text{Ab}((\text{Sch}/\mathscr{X})_{\text{ét}}) \to \text{Ab}(\text{Lis-Et}(\mathscr{X}))$$

which is exact and admits an exact left adjoint $u_!$ (see [88, 0788 (1)]). There is an exact sequence

$$1 \to 1 + I \to \mathbb{G}_{m,\mathscr{X}} \to i_*\mathbb{G}_{m,\mathscr{X}_0} \to 1$$

of abelian sheaves on Lis-Et($\mathscr{X}$); here left exactness follows from the fact that for any scheme X and smooth morphism $X \to \mathscr{X}$ the composition $X \to \mathscr{X} \to \operatorname{Spec} A$ is flat. We have an induced long exact sequence

$$\cdots \to \mathrm{H}^p_{\text{ét}}(\mathscr{X}, I) \to \mathrm{H}^p_{\text{ét}}(\mathscr{X}, \mathbb{G}_{m,\mathscr{X}}) \to \mathrm{H}^p_{\text{ét}}(\mathscr{X}, i_*\mathbb{G}_{m,\mathscr{X}_0}) \to \mathrm{H}^{p+1}_{\text{ét}}(\mathscr{X}, I) \to \cdots$$

in cohomology. We have $\mathrm{H}^p_{\text{ét}}(\mathscr{X}, i_*\mathbb{G}_{m,\mathscr{X}_0}) \simeq \mathrm{H}^p_{\text{ét}}(\mathscr{X}_0, \mathbb{G}_{m,\mathscr{X}_0})$ for $p \geq 0$ since pushforward along a closed immersion in the étale topology is exact (using e.g. [88, 04E3]); see also [15, A.3.5]. It suffices now to show that if $\mathscr{F}$ is any quasi-coherent $\mathcal{O}_{\mathscr{X}}$-module then $\mathrm{H}^p_{\text{ét}}(\mathscr{X}, \mathscr{F}) = 0$ for all $p > 0$. The category of quasi-coherent $\mathcal{O}_{\mathscr{X}}$-modules corresponds to the category $\mathcal{C}$ of $\mathbb{Z}$-graded A-modules. Denoting by $\pi : \mathscr{X} \to \operatorname{Spec} A$ the structure map, the pushforward functor $\pi_* : \text{QCoh}(\mathscr{X}) \to \text{QCoh}(A)$ corresponds to sending a $\mathbb{Z}$-graded module $M_\bullet = \bigoplus_{n\in\mathbb{Z}} M_n$ to the degree zero component M_0. Since this is an exact functor, we have that π is cohomologically affine [2, Definition 3.1]. Since π has affine diagonal, we have the desired result by [2, Remark 3.5].

4.5.4 (Proof of Theorem 4.1.2). For any strictly henselian local ring A, we have

$$\mathrm{H}^2_{\text{ét}}(\mathrm{B}\mathbb{G}_{m,A}, \mathbb{G}_{m,\mathrm{B}\mathbb{G}_{m,A}}) = 0$$

by Lemma 4.5.2, hence

$$\mathbf{R}^2\pi_{\mathcal{G},*}\mathbb{G}_{m,\mathcal{G}} = 0$$

since its stalks vanish. By Lemma 4.2.2, we have

$$\mathbf{R}^1\pi_{\mathcal{G},*}\mathbb{G}_{m,\mathcal{G}} \simeq \underline{\mathrm{Hom}}_{\mathrm{Ab}(S)}(\mathbb{G}_{m,S}, \mathbb{G}_{m,S}) = \underline{\mathbb{Z}}$$

so the Leray spectral sequence (4.5.0.1) gives an exact sequence

$$\mathrm{H}^0_{\text{ét}}(S, \underline{\mathbb{Z}}) \xrightarrow{\dagger} \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S}) \xrightarrow{\pi^*_{\mathcal{G}}} \mathrm{H}^2_{\text{ét}}(\mathcal{G}, \mathbb{G}_{m,\mathcal{G}}) \to \mathrm{H}^1_{\text{ét}}(S, \underline{\mathbb{Z}}) \tag{4.5.4.1}$$

where by Proposition 4.2.3 the first map $\dagger$ sends $1 \mapsto [\mathcal{G}]$.

Suppose $\beta \in \mathrm{H}^2_{\text{ét}}(\mathcal{G}, \mathbb{G}_{m,\mathcal{G}})$ is a class annihilated by some locally nonzero $n \in \Gamma(\mathcal{G}, \underline{\mathbb{Z}})$. Since $\pi : \mathcal{G} \to S$ is a gerbe morphism, the pullback $\pi^* : \Gamma(S, \underline{\mathbb{Z}}) \to \Gamma(\mathcal{G}, \underline{\mathbb{Z}})$ is an isomorphism and a section $n \in \Gamma(S, \underline{\mathbb{Z}})$ is locally nonzero if and only if $\pi^* n \in \Gamma(\mathcal{G}, \underline{\mathbb{Z}})$ is locally nonzero. By Lemma A.0.3 the last term $\mathrm{H}^1_{\text{ét}}(S, \underline{\mathbb{Z}})$ is n-torsion free, hence there exists $\alpha \in \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S})$ such that $\beta = \pi^*_{\mathcal{G}}(\alpha)$. Then $\pi^*_{\mathcal{G}}(n\alpha) = n\pi^*_{\mathcal{G}}(\alpha) = n\beta = 0$, hence $n\alpha = m[\mathcal{G}]$ for some $m \in \Gamma(S, \underline{\mathbb{Z}})$ (not necessarily locally nonzero); by assumption there exists locally nonzero $n' \in \Gamma(S, \underline{\mathbb{Z}})$ such that $n'[\mathcal{G}] = 0$, hence $n'n\alpha = 0$, hence $\alpha \in \mathrm{Br}' S$; in other words the restriction $\pi^*_{\mathcal{G}} : \mathrm{Br}' S \to \mathrm{Br}' \mathcal{G}$ is surjective. Hence we have the desired result. □

Remark 4.5.5. As pointed out to me by Siddharth Mathur, in Theorem 4.1.2, the restriction map

$$\pi^*_{\mathcal{G}} : \mathrm{Br}'(S) \to \mathrm{Br}'(\mathcal{G})$$

is not necessarily surjective if $[\mathcal{G}] \in \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S})$ is a nontorsion class. Let S be a scheme for which $\mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S})$ is not a torsion group; let $\alpha \in \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S})$ be a nontorsion element, and let $\pi_{\mathcal{G}} : \mathcal{G} \to S$ be the $\mathbb{G}_{m,S}$-gerbe corresponding to the class $2\alpha \in \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S})$. Then $\pi^*_{\mathcal{G}}(\alpha)$ is a 2-torsion class of $\mathrm{H}^2_{\text{ét}}(\mathcal{G}, \mathbb{G}_{m,\mathcal{G}})$. We show that there does not exist any torsion element $\beta \in \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S})$ such that $\pi^*_{\mathcal{G}}(\alpha) = \pi^*_{\mathcal{G}}(\beta)$. If so, then $\alpha - \beta = n[\mathcal{G}] = 2n\alpha$ for some n, which means $(2n-1)\alpha$ is torsion, which contradicts our assumption that α is nontorsion. Taking as our S above the normal surface of Mumford [42, Remarques 1.11, b] for which $\mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S})$ is not a torsion group, we obtain an example of a $\mathbb{G}_{m,S}$-gerbe $\pi_{\mathcal{G}} : \mathcal{G} \to S$ for which the restriction

$$\pi^*_{\mathcal{G}} : \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S}) \to \mathrm{H}^2_{\text{ét}}(\mathcal{G}, \mathbb{G}_{m,\mathcal{G}})$$

is surjective (by (4.5.4.1), using that $\mathrm{H}^1_{\text{ét}}(S, \mathbb{Z}) = 0$ by [45, VIII, Prop. 5.1] since S is geometrically unibranch) but the restriction to the torsion subgroups is not surjective. □

4.5.6 (Alternate proof of Theorem 4.1.2 under additional hypotheses). We give an alternate proof of Theorem 4.1.2 in case the base scheme S and the $\mathbb{G}_{m,S}$-gerbe $\mathcal{G}$ satisfy the following conditions:

(i) the scheme S is regular Noetherian,
(ii) the function field of S has characteristic 0, and
(iii) the class $[\mathcal{G}]$ lies in the image of Brauer map $\alpha_S : \mathrm{Br}\, S \to \mathrm{Br}'\, S$.

By (iii), there exists an Azumaya $\mathcal{O}_S$-algebra $\mathcal{A}$ (say of rank r^2, where we may assume $r \geq 2$) such that $\alpha_S([\mathcal{A}]) = [\mathcal{G}]$. Let $\pi_X : X \to S$ be a Brauer-Severi scheme corresponding to $\mathcal{A}$. By Lemma 4.5.7, there exists an S-morphism

$$f : X \to \mathcal{G}$$

and the induced pullback morphism

$$f^* : \mathrm{Br}'\, \mathcal{G} \to \mathrm{Br}'\, X$$

is an isomorphism by Lemma 4.5.8, whose hypotheses are satisfied by Lemma 4.5.7 and conditions (i), (ii) above. □

Lemma 4.5.7 (Comparing Brauer-Severi scheme and $\mathbb{G}_m$-gerbe). Let S be a scheme and let $\mathcal{A}$ be an Azumaya $\mathcal{O}_S$-algebra of rank n^2. Let $X \to S$ be the Brauer-Severi scheme associated to $\mathcal{A}$, and let $\mathscr{X} \to S$ be the $\mathbb{G}_{m,S}$-gerbe of trivializations of $\mathcal{A}$. There is a natural S-morphism

$$\pi : X \to \mathscr{X}$$

which is smooth-locally on $\mathscr{X}$ isomorphic to the projection $\mathbb{A}^n_S \setminus \{0\} \to S$ (in particular, it is flat and surjective).

Proof. The first claim follows from [78, §8, 4], as explained in Remark 4.1.3. We give a description here using the functor of points of X. For an S-scheme T, the set $X(T)$ consists of the isomorphism classes of pairs $(\mathcal{P}, \eta)$ where $\mathcal{P}$ is a left $\mathcal{A}_T$-module which is of rank n as an $\mathcal{O}_T$-module, and $\eta : \mathcal{A}_T \to \mathcal{P}$ is a surjective $\mathcal{A}_T$-linear map; two pairs $(\mathcal{P}_1, \eta_1)$ and $(\mathcal{P}_2, \eta_2)$ are isomorphic if there exists an $\mathcal{A}_T$-linear isomorphism $\zeta : \mathcal{P}_1 \to \mathcal{P}_2$ such that $\eta_2 = \zeta\eta_1$.

The map $\pi(T) : X(T) \to \mathscr{X}(T)$ sends the pair $(\mathcal{P}, \eta)$ to the pair $(\mathcal{P}, c_\eta)$ where $c_\eta : \mathcal{A}_T \to \mathrm{End}_{\mathcal{O}_T\text{-mod}}(\mathcal{P})$ is the $\mathcal{O}_T$-algebra isomorphism sending $a \mapsto m_a$ where the latter is left multiplication by a.

We prove that there exists a scheme S' and a smooth surjection $S' \to \mathscr{X}$ such that $S' \times_{\mathscr{X}} X$ is S'-isomorphic to $\mathbb{A}^n_{S'} \setminus \{0\}$. After an étale base change on S, we may assume that $\mathcal{A} = \mathrm{Mat}_{n\times n}(\mathcal{O}_S)$. In this case we have a 2-commutative diagram

$$\begin{array}{ccc} \mathbb{P}^{n-1}_S & \xrightarrow{\omega_1} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathrm{B}\mathbb{G}_{m,S} & \xrightarrow[\omega_2]{} & \mathscr{X} \end{array}$$

where, if we view $\mathbb{P}^{n-1}_S$ as the functor parametrizing pairs $(\mathcal{L}, \mathbf{s})$ where $\mathcal{L}$ is a line bundle and $\mathbf{s} = (s_1, \dots, s_n)$ is an ordered tuple of globally generating sections and $\mathrm{B}\mathbb{G}_{m,S}$ as the stack of line bundles, the map ω_1 sends $(\mathcal{L}, \mathbf{s}) \mapsto (\mathcal{L}^{\oplus n}, \eta_{\mathbf{s}})$ where $\eta_{\mathbf{s}} : \mathrm{Mat}_{n\times n}(\mathcal{O}_S) \to \mathcal{L}^{\oplus n}$ is the map sending the matrix $E_{i,i}$ to the section s_i in the ith component of $\mathcal{L}^{\oplus n}$, the map ω_2 sends a line bundle $\mathcal{L}$ to the pair $(\mathcal{L}^{\oplus n}, \sigma^{\mathrm{can}}_{\mathcal{L}})$ where $\sigma^{\mathrm{can}}_{\mathcal{L}} : \mathrm{Mat}_{n\times n}(\mathcal{O}_S) \to \mathrm{End}_{\mathcal{O}_S}(\mathcal{L}^{\oplus n})$ is the canonical isomorphism, and π' is the functor forgetting $\mathbf{s}$. The horizontal maps ω_1 and ω_2 are isomorphisms.

If $\rho : S \to \mathrm{B}\mathbb{G}_{m,S}$ denotes the section corresponding to the trivial line bundle, then the 2-fiber product $S \times_{\rho, \mathrm{B}\mathbb{G}_{m,S}, \pi'} \mathbb{P}^n_S$ is the scheme representing ordered n-tuples $\mathbf{s}$ of sections of $\mathcal{O}_S$ that are globally generating; this is representable by $\mathbb{A}^n_S \setminus \{0\}$. $\square$

Lemma 4.5.8. Let $\pi : X \to S$ be a morphism of algebraic stacks such that there exists a smooth surjection $S' \to S$, where S' is a scheme and $X' := S' \times_S X$ is isomorphic to $\mathbb{A}^m_{S'} \setminus \{0\}$. Assume that $m \geq 2$, that S' is regular Noetherian, and that the function field of S' has characteristic 0. Then the pullback

$$\mathrm{Br}'(S) \to \mathrm{Br}'(X)$$

is an isomorphism.[17]

Proof. The Leray spectral sequence for the map π and sheaf $\mathbb{G}_{m,X}$ is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}(S, \mathbf{R}^q\pi_*(\mathbb{G}_{m,X})) \implies \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_{m,X})$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. The induced map $\mathbb{G}_{m,S} \to \pi_*\mathbb{G}_{m,X}$ is an isomorphism. Thus it suffices to show that $\mathbf{R}^1\pi_*(\mathbb{G}_{m,X}) = 0$ and $\mathbf{R}^2\pi_*(\mathbb{G}_{m,X}) = 0$. For this we may replace S by S' and assume that S is a regular Noetherian scheme whose function field has characteristic 0 and that $X \simeq \mathbb{A}^m_S \setminus \{0\}$. Let $\overline{s}$ be a geometric point of S, and let $A := \mathcal{O}^{\mathrm{sh}}_{S,\overline{s}}$ be the strict henselization of S at $\overline{s}$. We have

$$(\mathbf{R}^1\pi_*(\mathbb{G}_{m,X}))_{\overline{s}} \simeq \mathrm{Pic}(\mathbb{A}^m_A \setminus \{0\}) \overset{1}{\simeq} \mathrm{Pic}(\mathbb{A}^m_A) \overset{2}{\simeq} \mathrm{Pic}(A) = 0$$

where we have isomorphism 1 since the codimension of the origin is at least 2 and isomorphism 2 follows by e.g. [48, Proposition II.6.6] since A is regular. We have

$$(\mathbf{R}^2\pi_*(\mathbb{G}_{m,X}))_{\overline{s}} \simeq \mathrm{H}^2_{\text{ét}}(\mathbb{A}^m_A \setminus \{0\}, \mathbb{G}_m) \overset{1}{\simeq} \mathrm{H}^2_{\text{ét}}(\mathbb{A}^m_A, \mathbb{G}_m) \overset{2}{\simeq} \mathrm{Br}(\mathbb{A}^m_A) \overset{3}{\simeq} \mathrm{Br}(A) = 0$$

where isomorphism 1 follows from purity for the cohomological Brauer group on regular Noetherian schemes (see Gabber [35] and Česnavičius [18]), isomorphism 2 holds since $\mathbb{A}^m_A$

[17]A similar result is proved in [33, Proposition 1.3].

is regular Noetherian and affine, the isomorphism 3 holds by [8, 7.7] since the function field of S has characteristic 0. $\square$

5. Variants

5.1. **The Azumaya Brauer group of a Brauer-Severi scheme.** In this section we investigate the Azumaya Brauer group of a Brauer-Severi scheme. This question was asked by Pieter Belmans in his blog post [10].

5.1.1. Let S be a scheme, and let $\pi : X \to S$ be a Brauer-Severi scheme of relative dimension d. We are interested in the Azumaya Brauer group $\mathrm{Br}(X)$. In general, we have a commutative diagram

$$\begin{array}{ccccccc} \Gamma(S,\underline{\mathbb{Z}}) & \xrightarrow{\xi} & \mathrm{Br}(S) & \xrightarrow{\pi^*} & \mathrm{Br}(X) & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle \alpha_S} & & \downarrow{\scriptstyle \alpha_X} & & \\ \Gamma(S,\underline{\mathbb{Z}}) & \xrightarrow[\xi']{} & \mathrm{Br}'(S) & \xrightarrow[(\pi^*)']{} & \mathrm{Br}'(X) & \longrightarrow & 0 \end{array} \tag{5.1.1.1}$$

where the map ξ sends $1 \mapsto [X]$ and the map ξ' is the composite $\alpha_S \circ \xi$. The bottom row is exact by Gabber [37, I, Theorem 2]. This implies that the top row of (5.1.1.1) is exact at $\mathrm{Br}(S)$. Indeed, given a class $\alpha \in \ker \pi^*$, we have $(\pi^*)'(\alpha_S(\alpha)) = 0$ so there exists $\beta \in \Gamma(S,\underline{\mathbb{Z}})$ such that $\xi'(\beta) = \alpha_S(\alpha)$; then $\alpha_S(\xi(\beta)) = \alpha_S(\alpha)$, but α_S is injective so $\xi(\beta) = \alpha$.

We may investigate $\mathrm{Br}(X)$ by asking whether α_X is an isomorphism and whether π^* is surjective. We have only partial answers to these questions:

(1) If X is isomorphic to projective space, then π^* is an isomorphism.[18] Indeed, there is a section $s : S \to X$ of π, which induces a commutative diagram

$$\begin{array}{ccccc} \mathrm{Br}(S) & \xrightarrow{\pi^*} & \mathrm{Br}(X) & \xrightarrow{s^*} & \mathrm{Br}(S) \\ \downarrow{\scriptstyle \alpha_S} & & \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_S} \\ \mathrm{Br}'(S) & \xrightarrow[(\pi^*)']{} & \mathrm{Br}'(X) & \xrightarrow[(s^*)']{} & \mathrm{Br}'(S) \end{array} \tag{5.1.1.2}$$

where the vertical arrows are injective and the composites of the horizontal arrows are the identity. We know that $(\pi^*)'$ is an isomorphism, hence $(s^*)'$ is also an isomorphism; given $\alpha \in \ker s^*$, we have $\alpha_X(\alpha) \in \ker(s^*)' = 0$, hence $\alpha = 0$. Thus π^*, s^* are isomorphisms.

(2) If α_S is an isomorphism, then α_X is an isomorphism and π^* is surjective. Indeed, we have that $(\pi^*)'$ is surjective by Gabber's result.

(3) (The following was explained to me by Siddharth Mathur.) If S is quasi-compact quasi-separated, then π^* is surjective.[19] Write S as a filtered inverse limit $S \simeq \varprojlim_{\lambda\in\Lambda} S_\lambda$ where each S_λ is of finite type over $\mathbb{Z}$ and the transition maps $S_\lambda \to S_{\lambda'}$ are affine morphisms, then descend the Brauer-Severi scheme $\pi : X \to S$ to Brauer-Severi schemes $\pi_\lambda : X_\lambda \to S_\lambda$. In this case $\mathrm{Br}(X) \simeq \varinjlim_{\lambda\in\Lambda} \mathrm{Br}(X_\lambda)$ and $\mathrm{Br}(S) \simeq$

[18] It may be true that π^* is an isomorphism if more generally X is a "trivial Brauer-Severi scheme", more precisely, the projectivization of a vector bundle on S. However, the argument given here does not apply in that situation since $\pi : X \to S$ would not necessarily have a section (take a scheme S and a vector bundle $\mathcal{E}$ for which there does not exist any line bundle $\mathcal{L}$ and surjection $\mathcal{E} \to \mathcal{L}$, then apply [44, II, (4.2.3)]).

[19] This is proved in [79, Proposition 1.1], which cites [28, Theorem 3.6], who assume that the base S is Noetherian.

$\varinjlim_{\lambda\in\Lambda} \mathrm{Br}(S_\lambda)$, so after replacing S by S_λ we may assume that S is a Noetherian scheme.

Let $\mathcal{A}$ be an Azumaya $\mathcal{O}_X$-algebra; let $\mathcal{G}_X \to X$ be the $\mathbb{G}_{m,X}$-gerbe corresponding to the gerbe of trivializations of $\mathcal{A}$. The surjectivity of $(\pi^*)'$ implies that $\mathcal{G}_X$ is the pullback of a $\mathbb{G}_m$-gerbe over S, namely there exists a $\mathbb{G}_{m,S}$-gerbe $\mathcal{G}_S \to S$ such that $\mathcal{G}_S \times_S X \simeq \mathcal{G}_X$; let $\pi_{\mathcal{G}} : \mathcal{G}_X \to \mathcal{G}_S$ be the projection. It suffices to show that $\mathcal{G}_S$ admits a locally free 1-twisted sheaf. We know that on $\mathcal{G}_X$ there is a 1-twisted finite locally free sheaf $\mathcal{E}_X$. Choose a π-relatively ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ and let $\mathcal{L}|_{\mathcal{G}_X}$ be its pullback to $\mathcal{G}_X$. Set $\mathcal{E}_X(n) := \mathcal{E}_X \otimes_{\mathcal{O}_{\mathcal{G}_X}} (\mathcal{L}|_{\mathcal{G}_X})^{\otimes n}$ for $n \in \mathbb{Z}$; we will show that $(\pi_{\mathcal{G}})_*(\mathcal{E}_X(n))$ is finite locally free for sufficiently large $n \in \mathbb{Z}$. This is local for the étale topology on S; let $S' \to S$ be a quasi-compact étale surjection such that $\mathcal{G}_S \times_S S'$ is the trivial $\mathbb{G}_{m,S'}$-gerbe; after replacing S by S', we may assume that $\mathcal{G}_S$ is trivial. Let $\xi_S : S \to \mathcal{G}_S$ and $\xi_X : X \to \mathcal{G}_X$ be the sections corresponding to the trivial $\mathbb{G}_m$-torsor.

$$\begin{array}{ccccc}
X & \xrightarrow{\xi_X} & \mathcal{G}_X & \longrightarrow & X \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{\mathcal{G}}} & & \downarrow{\scriptstyle \pi} \\
S & \xrightarrow[\xi_S]{} & \mathcal{G}_S & \longrightarrow & S
\end{array}$$

There is a natural map

$$(\xi_S)^*(\pi_{\mathcal{G}})_*(\mathcal{E}_X(n)) \to (\pi)_*(\xi_X)^*(\mathcal{E}_X(n))$$

which is an isomorphism for $n \gg 0$ by [30, Lemma 5.4], and $(\pi)_*(\xi_X)^*(\mathcal{E}_X(n))$ is finite locally free for $n \gg 0$ by cohomology and base change [48, III, Theorem 12.11] and Serre vanishing [48, III, Theorem 5.2]

□

5.2. **The Azumaya Brauer group of a $\mathbb{G}_m$-gerbe.**

5.2.1. Let S be a scheme, and let $\pi : \mathcal{X} \to S$ be a $\mathbb{G}_{m,S}$-gerbe corresponding to a torsion class $[\mathcal{X}] \in \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{m,S})$. We are interested in the Azumaya Brauer group $\mathrm{Br}(\mathcal{X})$. In general, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Br}(S) & \xrightarrow{\pi^*} & \mathrm{Br}(\mathcal{X}) \\
\downarrow{\scriptstyle \alpha_S} & & \downarrow{\scriptstyle \alpha_{\mathcal{X}}} \\
\mathrm{Br}'(S) & \xrightarrow[(\pi^*)']{} & \mathrm{Br}'(\mathcal{X})
\end{array} \qquad (5.2.1.1)$$

where α_S and $\alpha_{\mathcal{X}}$ are the Brauer maps. We may investigate $\mathrm{Br}(\mathcal{X})$ by asking whether $\alpha_{\mathcal{X}}$ is an isomorphism and whether π^* is surjective. (We observe that there is no natural candidate for the kernel of π^*, as opposed to the case of Brauer-Severi schemes 5.1.1.) We have only partial answers to these questions:

(1) If $\mathcal{X}$ is trivial, then π^* is an isomorphism, as in 5.1.1.

(2) If α_S is an isomorphism, then $\alpha_{\mathcal{X}}$ is an isomorphism and π^* is surjective. Indeed, we have that $(\pi^*)'$ is surjective by Theorem 4.1.2. Hence $\alpha_{\mathcal{X}}$ must be surjective, hence an isomorphism.
(3) On the other hand, it may be the case that $\alpha_{\mathcal{X}}$ is an isomorphism even though α_S is not. Let S be the example of [28, Corollary 3.11]. By Hoobler's comment to the Corollary (or Bertuccioni's proof [12, §3]), we have $\mathrm{Br}(S) = 0$ and $\mathrm{Br}'(S) = \mathbb{Z}/(2)$. Let $\mathcal{X}$ be the $\mathbb{G}_{m,S}$-gerbe corresponding to the nontrivial class in $\mathrm{Br}'(S)$. Then $\mathrm{Br}'(\mathcal{X}) = 0$ since the kernel of $(\pi^*)'$ is generated by $[\mathcal{X}]$. Thus $\alpha_{\mathcal{X}}$ is an isomorphism (see (5.2.1.1)).

□

Remark 5.2.2. We give a ring-theoretic argument to show that $\mathrm{Br}(\mathrm{B}\mathbb{G}_{m,k}) \simeq \mathrm{Br}(k)$ for any field k. Let $\pi : \mathrm{B}\mathbb{G}_{m,k} \to \operatorname{Spec} k$ be the projection and let $s : \operatorname{Spec} k \to \mathrm{B}\mathbb{G}_{m,k}$ be the section corresponding to the trivial $\mathbb{G}_m$-torsor. It suffices to show that $\ker(s^* : \mathrm{Br}(\mathrm{B}\mathbb{G}_{m,k}) \to \mathrm{Br}(k))$ is trivial. Let $\mathcal{A}$ be an Azumaya $\mathcal{O}_{\mathrm{B}\mathbb{G}_{m,k}}$-algebra of rank r^2 such that $s^*\mathcal{A}$ is a trivial Azumaya algebra over k; this corresponds to $\mathbb{Z}$-grading on the matrix algebra $\mathrm{Mat}_{r\times r}(k)$, and it is trivial if and only if it is isomorphic as $\mathbb{Z}$-graded algebras to the endomorphism algebra $\mathrm{End}_k(V)$ of a $\mathbb{Z}$-graded k-vector space V (which is necessarily of dimension r). This follows from Proposition 1.2 and Corollary 1.5 of [21]. □

5.3. **$\mathbb{A}^1$-homotopy invariance of the Brauer group.** To investigate questions regarding the (cohomological) Brauer group of an algebraic stack $\mathcal{X}$, one approach is to choose a scheme U and a smooth surjection $U \to \mathcal{X}$, then study the properties of the pullback $\mathrm{Br}'(\mathcal{X}) \to \mathrm{Br}'(U)$. Locally for the smooth topology on both $\mathcal{X}$ and U, the map $U \to \mathcal{X}$ is isomorphic to affine space. Hence we are fundamentally interested in the Brauer groups of polynomial rings:

Question 5.3.1. For which rings A is the pullback
$$\mathrm{Br}(A) \to \mathrm{Br}(A[t]) \tag{5.3.1.1}$$
an isomorphism?

Remark 5.3.2. We have that $A^\times = (A[t])^\times$ if and only if A is reduced, and $\mathrm{Pic}(A) = \mathrm{Pic}(A[t])$ if and only if A_{red} is seminormal (by Traverso's theorem). It would be interesting to find a similar ring-theoretic condition on A which is equivalent to saying $\mathrm{Br}(A) = \mathrm{Br}(A[t])$, or $\mathrm{H}^2_{\text{ét}}(\operatorname{Spec} A, \mathbb{G}_m) = \mathrm{H}^2_{\text{ét}}(\operatorname{Spec} A[t], \mathbb{G}_m)$. □

Theorem 5.3.3. [39, 4.5] [20] Let k be a field of positive characteristic $p > 0$ and let $d \geq 1$ be an integer. Then we have
$$\mathrm{Br}(k[t_1, \dots, t_d]) = \mathrm{Br}(k) \oplus (\mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z})^{\oplus I}$$
where the set I is nonempty if and only if k is not perfect or $d \geq 2$. □

Remark 5.3.4. Here we list some known cases of Question 5.3.1.

(1) If A is a field, then (5.3.1.1) is an isomorphism if and only if A is perfect.

[20] This is a generalization of [54, Theorem 5.7]. Negron [72, 6.4] proves that if $\operatorname{char} k \geq 3$ and $d \geq 2$ then $\mathrm{Br}(k[t_1, \dots, t_d])$ contains the additive group of k.

(2) For every prime p which is invertible in A, the p-primary subgroups of $\mathrm{Br}(A)$ and $\mathrm{Br}(A[t])$ coincide (see Lemma 5.3.5); thus if A contains $\mathbb{Q}$, then (5.3.1.1) is an isomorphism.
(3) If A is a regular Noetherian domain whose fraction field has characteristic 0, then (5.3.1.1) is an isomorphism.
(4) For any field k of positive characteristic and $A = k[t]$, the map (5.3.1.1) is not surjective (see Theorem 5.3.3).

Lemma 5.3.5. [21] [32, 13.6.4] Let A be a ring. If p is invertible on A, then the inclusion

$$\mathrm{Br}(A)[p^\infty] \to \mathrm{Br}(A[t])[p^\infty]$$

is an isomorphism.

Proof. For any ℓ we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^2_{\text{ét}}(A[t], \mu_{p^\ell}) & \twoheadrightarrow & \mathrm{Br}(A[t])[p^\ell] \\
\uparrow \beta & & \uparrow \gamma \\
\mathrm{H}^2_{\text{ét}}(A, \mu_{p^\ell}) & \twoheadrightarrow & \mathrm{Br}(A)[p^\ell]
\end{array}$$

where the horizontal arrows are surjective and β is an isomorphism by acyclicity [69, VI, 4.20] since p is invertible on A. □

We also have the following positive result by Knus and Ojanguren:

Theorem 5.3.6. [53, Theorem 3.6] Let A be a Dedekind domain of characteristic 0 and such that its residue fields are characteristic 0 or perfect and C_1. Let R be a finite, faithfully flat A-algebra. Then the inclusion $R \to R[t_1, \dots, t_n]$ induces an isomorphism $\alpha : \mathrm{Br}(R) \to \mathrm{Br}(R[t_1, \dots, t_n])$. □

Using the results of [53], we can describe more rings A (on which no prime is invertible) for which (5.3.1.1) is an isomorphism:

Theorem 5.3.7. [22]Let A be a Noetherian integral domain with normalization $\overline{A}$ and conductor $\mathfrak{a}$. Suppose

(i) $\mathrm{char}(\mathrm{Frac}(A)) = 0$,
(ii) the inclusion $A \to \overline{A}$ is finite,
(iii) the rings $\overline{A}, (A/\mathfrak{a})_{\mathrm{red}}, (\overline{A}/\mathfrak{a})_{\mathrm{red}}$ are regular rings whose fraction fields have characteristic 0,
(iv) the induced map $(A/\mathfrak{a})_{\mathrm{red}} \to (\overline{A}/\mathfrak{a})_{\mathrm{red}}$ is an isomorphism, and
(v) vector bundles on polynomial rings over $\overline{A}, (A/\mathfrak{a})_{\mathrm{red}}, (\overline{A}/\mathfrak{a})_{\mathrm{red}}$ are trivial.

Then $\mathrm{Br}(A) \simeq \mathrm{Br}(A[t_1, \dots, t_n])$.

[21]This is proved in [79, 1.5] under the hypotheses that A is normal, integral Noetherian and all the strict henselizations of local rings of polynomial rings over A are UFDs.
[22]This is a variant of Theorem 5.3.6. See Proposition 5.3.8 for an example of a ring A satisfying these conditions.

Proof. We denote $A[\mathbf{t}] := A[t_1, \ldots, t_n]$, etc. Note that if A satisfies the above five conditions, then $A[\mathbf{t}]$ also satisfies the same five conditions since $\overline{A}[\mathbf{t}]$ is the normalization of $A[\mathbf{t}]$ and $\mathfrak{a} \otimes_A A[\mathbf{t}]$ is the conductor of $\overline{A}[\mathbf{t}]/A[\mathbf{t}]$. We have the Milnor square

$$\begin{array}{ccc} A & \longrightarrow & \overline{A} \\ \downarrow & & \downarrow \\ A/\mathfrak{a} & \longrightarrow & \overline{A}/\mathfrak{a} \end{array}$$

which we call $\mathbf{S}$. We apply [53, Theorem 2.2] to the morphism of Milnor squares $\mathbf{S} \to \mathbf{S}[\mathbf{t}]$, which will show that there is a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Pic}(\overline{A}/\mathfrak{a}) & \longrightarrow & \mathrm{Br}(A) & \longrightarrow & \mathrm{Br}(A/\mathfrak{a}) \oplus \mathrm{Br}(\overline{A}) & \longrightarrow & \mathrm{Br}(\overline{A}/\mathfrak{a}) \\ \downarrow{\scriptstyle \xi_1} & & \downarrow{\scriptstyle \xi_2} & & \downarrow{\scriptstyle \xi_3} & & \downarrow{\scriptstyle \xi_4} \\ \mathrm{Pic}((\overline{A}/\mathfrak{a})[\mathbf{t}]) & \longrightarrow & \mathrm{Br}(A[\mathbf{t}]) & \longrightarrow & \mathrm{Br}((A/\mathfrak{a})[\mathbf{t}]) \oplus \mathrm{Br}(\overline{A}[\mathbf{t}]) & \longrightarrow & \mathrm{Br}((\overline{A}/\mathfrak{a})[\mathbf{t}]) \end{array}$$

with exact rows. Here $\mathrm{Pic}(\overline{A}/\mathfrak{a}) = 0$ and $\mathrm{Pic}((\overline{A}/\mathfrak{a})[\mathbf{t}]) = 0$ by (v), and ξ_3 and ξ_4 are isomorphisms by (iii) and [8, Proposition 7.7].

In the notation of [53], we have $K_0\mathbf{FP}(A/\mathfrak{a}) \simeq \mathbb{Z}$ and $K_0\mathbf{FP}(\overline{A}/\mathfrak{a}) \simeq \mathbb{Z}$ and $K_0\mathbf{CRP}(\overline{A}/\mathfrak{a}) \simeq K_0\mathbf{FP}(\overline{A}/\mathfrak{a}) \simeq \mathbb{Z}$ and $\mathrm{Pic}(\overline{A}/\mathfrak{a}) = 0$ by (v). Here vector bundles on polynomial rings over $A/\mathfrak{a}, \overline{A}/\mathfrak{a}$ are trivial since there are no nontrivial deformations of the trivial vector bundle over a square-zero thickening of affine schemes. This verifies the conditions of [53, Theorem 2.2]. □

Proposition 5.3.8. Let R be a regular Noetherian domain with $\mathrm{Pic}(R) = 0$ and $\mathrm{char}(\mathrm{Frac}(R)) = 0$ and such that vector bundles on polynomial rings over R are trivial. Let $\mathrm{M} \subset \mathbb{Z}^d$ be a submonoid $\mathrm{M} \subset \mathbb{N}^d$ such that the complement $\mathbb{N}^d \setminus \mathrm{M}$ is a finite set (e.g. the submonoid generated by two coprime natural numbers $\langle m, n \rangle \subseteq \mathbb{N}$). Set $A := R[\mathrm{M}]$, the monoid algebra associated to M over R. Then (5.3.1.1) is an isomorphism.

Proof. We verify the conditions of Theorem 5.3.7. The normalization of A is $\overline{A} = R[t_1, \ldots, t_d]$ and the conductor is $\mathfrak{a} = \langle \{\chi_{\mathrm{m}}\}_{\mathrm{m} \in \mathrm{M} \setminus \{0\}} \rangle A$, thus $A/\mathfrak{a} \simeq R$ and $\overline{A}/\mathfrak{a} \simeq R[\{\chi_{\mathrm{m}}\}_{\mathrm{m} \in \mathbb{N}^d}]/\langle \{\chi_{\mathrm{m}}\}_{\mathrm{m} \in \mathrm{M} \setminus \{0\}} \rangle$ so $(\overline{A}/\mathfrak{a})_{\mathrm{red}} = R$. □

Appendix A. Torsors under torsion-free abelian groups

In [93, Corollary 7.9.1], it is proved that $\mathrm{H}^1_{\text{ét}}(S, \underline{\mathbb{Z}})$ is a torsion-free abelian group if S is a quasi-compact quasi-separated scheme. In this section, we record a different proof which works over an arbitrary site. This argument is from [88, 093J].

Let $\mathcal{S}$ be a site. For any set S, let $\underline{\mathrm{S}}$ denote the constant sheaf on $\mathcal{S}$ associated to S.

Lemma A.0.1. Let $f : \mathrm{S} \to \mathrm{T}$ be a surjective function between sets. Then the induced map

$$\Gamma(\mathcal{S}, f) : \Gamma(\mathcal{S}, \underline{\mathrm{S}}) \to \Gamma(\mathcal{S}, \underline{\mathrm{T}})$$

is surjective.

Proof. Choose a function $g : \mathrm{T} \to \mathrm{S}$ satisfying $fg = \mathrm{id}_{\mathrm{T}}$. By functoriality of the "constant sheaf" functor, we have $\Gamma(\mathcal{S}, f) \circ \Gamma(\mathcal{S}, g) = \mathrm{id}_{\Gamma(\mathcal{S}, \underline{\mathrm{T}})}$. □

Lemma A.0.2. Let $0 \to \mathrm{A} \to \mathrm{B} \to \mathrm{C} \to 0$ be an exact sequence of abelian groups. Then the induced map

$$\mathrm{H}^1(\mathcal{S}, \underline{\mathrm{A}}) \to \mathrm{H}^1(\mathcal{S}, \underline{\mathrm{B}})$$

is injective.

Proof. As part of the long exact sequence in cohomology, we obtain an exact sequence

$$\Gamma(\mathcal{S}, \underline{\mathrm{B}}) \to \Gamma(\mathcal{S}, \underline{\mathrm{C}}) \to \mathrm{H}^1(\mathcal{S}, \underline{\mathrm{A}}) \to \mathrm{H}^1(\mathcal{S}, \underline{\mathrm{B}})$$

where the first arrow is surjective by Lemma A.0.1, hence the third arrow is injective. □

Lemma A.0.3. Let A be a torsion-free abelian group. For any locally nonzero (see Definition 1.2.10) section $n \in \Gamma(\mathcal{S}, \underline{\mathrm{A}})$, the $\Gamma(\mathcal{S}, \underline{\mathrm{A}})$-module $\mathrm{H}^1(\mathcal{S}, \underline{\mathrm{A}})$ is n-torsion free.

Proof. We first consider the case when n is in the image of $\mathrm{A} \to \Gamma(S, \underline{\mathrm{A}})$. Let n be a positive integer. Applying Lemma A.0.2 to the exact sequence

$$0 \to \mathrm{A} \stackrel{\times n}{\to} \mathrm{A} \to \mathrm{A}/n\mathrm{A} \to 0$$

implies that the multiplication-by-n map on $\mathrm{H}^1(\mathcal{S}, \underline{\mathrm{A}})$ is injective.

In general, for $\ell \in \mathrm{A}$, let $\mathcal{S}_\ell \subseteq \mathcal{S}$ be the full subcategory consisting of objects $U \in \mathcal{S}$ for which the restriction $n|_U \in \Gamma(U, \underline{\mathrm{A}})$ equals the image of ℓ under $\mathrm{A} \to \Gamma(U, \underline{\mathrm{A}})$. We have a functor

$$\textstyle\coprod_{\ell \in \mathrm{A}} \mathcal{S}_\ell \to \mathcal{S}$$

and the morphism of topoi

$$\mathrm{Sh}(\mathcal{S}) \to \textstyle\prod_{\ell \in \mathrm{A}} \mathrm{Sh}(\mathcal{S}_\ell)$$

induced by restriction is an equivalence; the point is that any object $U \in \mathcal{S}$ admits a covering by objects lying in at least one of the $\mathcal{S}_\ell$, and if an object $U \in \mathcal{S}$ lies in two subsites $\mathcal{S}_{\ell_1}$ and $\mathcal{S}_{\ell_2}$, then the sheafification map $\mathrm{A} \to \Gamma(U, \underline{\mathrm{A}})$ is not injective so the empty family is a covering of U in $\mathcal{S}$, hence $\Gamma(U, \mathcal{F})$ is a singleton for every sheaf $\mathcal{F}$ [88, 04B6] (in this case U is said to be "sheaf theoretically empty"). In particular, an $\underline{\mathrm{A}}$-torsor $\mathcal{P}$ is trivial if and only if each $\underline{\mathrm{A}}|_{\mathcal{S}_\ell}$-torsor $\mathcal{P}|_{\mathcal{S}_\ell}$ is trivial. □

Appendix B. Cohomology and spectral sequences

In this section, we recall some tools we require for explicit computations. The material in this section is standard and we claim no originality.

For a category $\mathcal{C}$, we denote by $\mathrm{PSh}(\mathcal{C})$ (resp. $\mathrm{PAb}(\mathcal{C})$) the category of presheaves (resp. abelian presheaves) on $\mathcal{C}$. If $\mathcal{C}$ is a site, we denote by $\mathrm{Sh}(\mathcal{C})$ (resp. $\mathrm{Ab}(\mathcal{C})$) the category of sheaves (resp. abelian sheaves) on $\mathcal{C}$.

B.1. **Cohomological descent spectral sequence.**

B.1.1. [76, 2.4.25, 2.4.26], [69, III, Proposition 2.7] [23] Let $\mathcal{C}$ be a site, let $\mathscr{X}$ be a stack over $\mathcal{C}$, let $\pi : X \to \mathscr{X}$ by a covering be a sheaf of sets X over $\mathcal{C}$. Let

$$X_n := X \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} X$$

be the $(n+1)$-fold 2-fiber product of X over $\mathscr{X}$, and let

$$\{X_\bullet\}_{n\in\mathbb{Z}_{\geq 0}}$$

be the simplicial sheaf of sets thus obtained. For an abelian sheaf $\mathbf{A}$ on $\mathscr{X}$, we have a spectral sequence of the form

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q(X_p, \mathbf{A}|_{X_p}) \implies \mathrm{H}^{p+q}(\mathscr{X}, \mathbf{A}) \tag{B.1.1.1}$$

with differentials $\mathrm{E}_1^{p,q} \to \mathrm{E}_1^{p+1,q}$.

Suppose that $\mathscr{X} = [X/G]$ for a sheaf of groups G acting on X. In this case, the E_2 page of (B.1.1.1) is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(G, \mathrm{H}^p(X, \mathbf{A})) \implies \mathrm{H}^{p+q}([X/G], \mathbf{A}) \tag{B.1.1.2}$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. □

B.2. **Higher direct images of sheaves on classifying stacks of discrete groups.**

Setup B.2.1. Let $\mathcal{C}$ be a site, let G be a finite (discrete) group, let $\mathrm{B}G_{\mathcal{C}}$ be the classifying stack associated to G over $\mathcal{C}$. Let

$$\pi : \mathrm{B}G_{\mathcal{C}} \to \mathcal{C}$$

be the projection and let

$$\varphi : \mathcal{C} \to \mathrm{B}G_{\mathcal{C}}$$

be the canonical section of π. We view any fibered category $p : \mathcal{F} \to \mathcal{C}$ as a site via the Grothendieck topology inherited from $\mathcal{C}$ via p.

Lemma B.2.2. Assume Setup B.2.1. For any abelian sheaf $\mathscr{F} \in \mathrm{Ab}(\mathrm{B}G_{\mathcal{C}})$ the higher pushforward $\mathbf{R}^i\pi_*\mathscr{F}$ is naturally isomorphic to the sheaf associated to the presheaf whose value on an object $U \in \mathcal{C}$ is $\mathrm{H}^i(G, \Gamma(U, \varphi^*\mathscr{F}))$.

Proof. Let $\mathrm{P}G_{\mathcal{C}}$ denote the category whose objects are the objects of $\mathcal{C}$ and where a morphism $X_1 \to X_2$ in $\mathrm{P}G_{\mathcal{C}}$ is a pair (φ, g) where $\varphi \in \mathrm{Mor}_{\mathcal{C}}(X_1, X_2)$ and $g \in G$. (In other words, there is an equivalence of categories $\mathrm{P}G_{\mathcal{C}} \simeq \mathcal{C} \times [*/G]$ where $[*/G]$ is the category with one object $*$ and where $\mathrm{Hom}_{[*/G]}(*, *)$ is isomorphic to G.) The fibered category $\mathrm{P}G_{\mathcal{C}}$ is a (separated) prestack whose associated stack is $\mathrm{B}G_{\mathcal{C}}$, and the inclusion $\mathrm{P}G_{\mathcal{C}} \to \mathrm{B}G_{\mathcal{C}}$ induces an equivalence of topoi $\mathrm{Sh}(\mathrm{P}G_{\mathcal{C}}) \simeq \mathrm{Sh}(\mathrm{B}G_{\mathcal{C}})$. Hence in the statement of the lemma we may replace $\mathrm{B}G_{\mathcal{C}}$ by $\mathrm{P}G_{\mathcal{C}}$ where by abuse of notation we also denote

$$\pi : \mathrm{P}G_{\mathcal{C}} \to \mathcal{C}$$

the projection morphism. Since sheafification is an exact functor, the diagram

[23] This is also called the "Cech-to-global spectral sequence" [1] and the "spectral sequence relative to a covering" [15, A.2.1].

$$\begin{array}{ccc} \mathrm{PAb}(\mathrm{P}G_{\mathcal{C}}) & \xrightarrow{\pi_*^{\mathrm{pre}}} & \mathrm{PAb}(\mathcal{C}) \\ {\scriptstyle \mathrm{sh}}\downarrow & & \downarrow{\scriptstyle \mathrm{sh}} \\ \mathrm{Ab}(\mathrm{P}G_{\mathcal{C}}) & \xrightarrow[\pi_*]{} & \mathrm{Ab}(\mathcal{C}) \end{array}$$

is (2-)commutative. For the same reason, we have a natural isomorphism

$$(\mathbf{R}\pi_*^{\mathrm{pre}}(\mathscr{F}))^{\mathrm{sh}} \simeq \mathbf{R}\pi_*(\mathscr{F}^{\mathrm{sh}}) \tag{B.2.2.1}$$

in $\mathrm{D}^+(\mathrm{Ab}(\mathcal{C}))$ for any abelian presheaf $\mathscr{F} \in \mathrm{PAb}(\mathrm{P}G_{\mathcal{C}})$. Presheaves on $\mathrm{P}G_{\mathcal{C}}$ correspond to presheaves $\mathscr{F}$ on $\mathcal{C}$ equipped with a G-action, and under this identification $\pi_*^{\mathrm{pre}}(\mathscr{F}) = \mathscr{F}^G$ where $\Gamma(U, \mathscr{F}^G) := (\Gamma(U, \mathscr{F}))^G$ for all $U \in \mathcal{C}$. Let $\mathscr{F} \in \mathrm{Ab}(\mathrm{P}G_{\mathcal{C}})$ be an abelian sheaf, and let

$$\mathscr{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

be a resolution of $\mathscr{F}$ by injective abelian presheaves $\mathcal{I}^i \in \mathrm{PAb}(\mathrm{P}G_{\mathcal{C}})$. Then $\mathbf{R}\pi_*^{\mathrm{pre}}(\mathscr{F})$ is isomorphic to

$$(\mathcal{I}^\bullet)^G = \{(\mathcal{I}^0)^G \to (\mathcal{I}^1)^G \to (\mathcal{I}^2)^G \to \cdots\} \tag{B.2.2.2}$$

in $\mathrm{D}^+(\mathrm{PAb}(\mathcal{C}))$, and $\Gamma(U, \mathbf{R}\pi_*^{\mathrm{pre}}(\mathscr{F}))$ is isomorphic to

$$\Gamma(U, (\mathcal{I}^\bullet)^G) = \{(\Gamma(U, \mathcal{I}^0))^G \to (\Gamma(U, \mathcal{I}^1))^G \to (\Gamma(U, \mathcal{I}^2))^G \to \cdots\} \tag{B.2.2.3}$$

in $\mathrm{D}^+(\mathrm{PAb}(\mathcal{C}))$. Furthermore $\Gamma(U, \mathcal{I}^i) \simeq (i_U)^*\mathcal{I}^i$ is an injective G-module for all i by Lemma B.2.3, thus we have an isomorphism

$$h^i(\Gamma(U, (\mathcal{I}^\bullet)^G)) \simeq \mathrm{H}^i(G, \Gamma(U, \mathscr{F}))$$

of abelian groups. □

Lemma B.2.3. Let $\mathcal{C}$ be a category, let $U \in \mathcal{C}$ be an object, let $\mathcal{A}_{\mathcal{C},U}$ denote the full subcategory of $\mathcal{C}$ containing exactly U, and let $i_U : \mathcal{A}_{\mathcal{C},U} \to \mathcal{C}$ denote the inclusion. The inverse image functor $(i_U)^* : \mathrm{PAb}(\mathcal{C}) \to \mathrm{PAb}(\mathcal{A}_{\mathcal{C},U})$ preserves injectives.

Proof. The functor $(i_U)^* : \mathrm{PAb}(\mathrm{P}G_{\mathcal{C}}) \to \mathrm{PAb}(\mathcal{A}_{\mathcal{C},U})$ has an exact left adjoint, namely the "extension by zero" functor $i_{U,!} : \mathrm{PAb}(\mathcal{A}_{\mathcal{C},U}) \to \mathrm{PAb}(\mathrm{P}G_{\mathcal{C}})$ which sends $M \in \mathrm{PAb}(\mathcal{A}_{\mathcal{C},U})$ to the abelian presheaf $i_{U,\dagger}(M)$ where $\Gamma(V, i_{U,\dagger}(M)) = M$ if $V = U$ and 0 otherwise (with the only nontrivial restriction morphisms being those corresponding to the endomorphisms of U). □

Appendix C. Inverse image of gerbes

C.0.1. Let $f : X \to Y$ be a morphism of sites, let $\mathbf{A}$ (resp. $\mathbf{B}$) be an abelian sheaf on X (resp. Y), and let $\varphi : \mathbf{B} \to f_*\mathbf{A}$ be a morphism of abelian sheaves on Y. For a $\mathbf{B}$-gerbe $\mathcal{G}$, let $\varphi^*\mathcal{G}$ be denote the $\mathbf{A}$-gerbe corresponding to the image of $\mathcal{G}$ under the pullback morphism $\varphi^* : \mathrm{H}^2(Y, \mathbf{B}) \to \mathrm{H}^2(X, \mathbf{A})$. The purpose of this section is to describe $\varphi^*\mathcal{G}$ and the accompanying morphism of sites $\varphi^*\mathcal{G} \to \mathcal{G}$, in order to verify that the pushforward of an n-twisted sheaf is n-twisted Lemma 1.3.7 and that the Brauer map is functorial for morphisms of sites 1.4.3. For a more general notion of inverse image of a fibered category via a morphism of sites, see [40, II, 3.1.5] and [88, 04WA].

C.0.2. Let
$$f = (f^{-1}, f_*) : X \to Y$$
be a morphism of sites, i.e. induced by a continuous functor
$$u : Y \to X$$
which commutes with finite limits [88, 00X6]; in this case $f^{-1} = u_s$ and $f_* = u^s$ in the notation of [88, 00WU]. In particular $\Gamma(V, f_*\mathscr{F}) = \Gamma(u(V), \mathscr{F})$ for any sheaf $\mathscr{F}$ on X and any object $V \in Y$.

C.0.3. Let $\mathbf{A}$ (resp. $\mathbf{B}$) be an abelian sheaf on X (resp. Y), and let $\varphi : \mathbf{B} \to f_*\mathbf{A}$ be a morphism of abelian sheaves. There are induced pullback morphisms
$$\varphi^* : \mathrm{H}^i(Y, \mathbf{B}) \to \mathrm{H}^i(X, \mathbf{A})$$
for all $i \geq 0$ on cohomology. There is a bijective correspondence between classes in $\mathrm{H}^2(X, \mathbf{A})$ (resp. $\mathrm{H}^2(Y, \mathbf{B})$) and isomorphism classes of $\mathbf{A}$-gerbes (resp. $\mathbf{B}$-gerbes) [40], [76, 12.2.8]. Let $\mathcal{G}$ be a $\mathbf{B}$-gerbe with corresponding class $[\mathcal{G}] \in \mathrm{H}^2(Y, \mathbf{B})$; we describe the $\mathbf{A}$-gerbe $\varphi^*\mathcal{G}$ corresponding to the class $\varphi^*[\mathcal{G}] \in \mathrm{H}^2(X, \mathbf{A})$.

Let $\mathbf{A} \to \mathcal{I}^\bullet$ (resp. $\mathbf{B} \to \mathcal{J}^\bullet$) be a resolution of $\mathbf{A}$ (resp. $\mathbf{B}$) by injective abelian sheaves on X (resp. Y); then φ lifts to a chain map $\varphi^\bullet : \mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$ since each $f_*\mathcal{I}^i$ is an injective sheaf on Y.

$$\begin{array}{ccccccccc} f_*\mathbf{A} & \xrightarrow{f_*d_{\mathcal{I}}^{-1}} & f_*\mathcal{I}^0 & \xrightarrow{f_*d_{\mathcal{I}}^{0}} & f_*\mathcal{I}^1 & \xrightarrow{f_*d_{\mathcal{I}}^{1}} & f_*\mathcal{I}^2 & \longrightarrow & \cdots \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi^0} & & \uparrow{\scriptstyle\varphi^1} & & \uparrow{\scriptstyle\varphi^2} & & \\ \mathbf{B} & \xrightarrow[d_{\mathcal{J}}^{-1}]{} & \mathcal{J}^0 & \xrightarrow[d_{\mathcal{J}}^{0}]{} & \mathcal{J}^1 & \xrightarrow[d_{\mathcal{J}}^{1}]{} & \mathcal{J}^2 & \longrightarrow & \cdots \end{array} \tag{C.0.3.1}$$

Set $Z^i_{\mathcal{I}} = \ker d^i_{\mathcal{I}}$ and $Z^i_{\mathcal{J}} = \ker d^i_{\mathcal{J}}$; let $\beta \in \Gamma(Y, Z^2_{\mathcal{J}})$ be a lift of $[\mathcal{G}] \in \mathrm{H}^2(Y, \mathbf{B})$. The desired $\mathbf{A}$-gerbe $\varphi^*\mathcal{G}$ corresponds to $(\Gamma(Y, \varphi^2))(\beta) \in \Gamma(Y, f_*Z^2_{\mathcal{I}}) = \Gamma(X, Z^2_{\mathcal{I}})$ and the continuous functor
$$u_{\mathcal{G}} : \mathcal{G} \to \varphi^*\mathcal{G} \tag{C.0.3.2}$$
has the following description. We use the explicit construction of a gerbe given a second cohomology class [22, 2.5]. An object of $\mathcal{G}$ corresponds to a local lift of β to $\mathcal{J}^1$, namely a pair
$$(V, \xi)$$
where $V \in Y$ is an object and $\xi \in \Gamma(V, \mathcal{J}^1)$ is a section such that
$$(\Gamma(V, d^1_{\mathcal{J}}))(\xi) = \beta|_V$$
as elements of $\Gamma(V, \mathcal{J}^2)$. A morphism
$$(a, \rho) : (V_1, \xi_1) \to (V_2, \xi_2)$$
consists a morphism $a \in \mathrm{Mor}_Y(V_1, V_2)$ and a section $\rho \in \Gamma(V_1, \mathcal{J}^0)$ such that
$$(\Gamma(V_1, d^0_{\mathcal{J}}))(\rho) = \xi_1 - a^*\xi_2$$
as elements of $\Gamma(V_1, \mathcal{J}^1)$. The isomorphisms
$$\iota_{(V,\xi)} : \Gamma(V, \mathbf{B}) \to \mathrm{Aut}_{\mathcal{G}(V)}((V, \xi))$$

of (1.3.1.1) are induced by $d_{\mathcal{J}}^{-1}$. It may be verified that $(\mathcal{G}, \{\iota_{(V,\xi)}\})$ is a $\mathbb{G}_{m,Y}$-gerbe. The analogous construction applied to $(\Gamma(Y,\varphi^2))(\beta)$ in place of β gives our desired $\mathbb{G}_{m,X}$-gerbe $(\varphi^*\mathcal{G}, \{\iota_{(U,\omega)}\})$. The functor $u_{\mathcal{G}}$ sends

$$(V,\xi) \mapsto (u(V), (\Gamma(V,\varphi^1))(\xi))$$

on objects and

$$(a,\rho) \mapsto (u(a), (\Gamma(V_1,\varphi^0))(\rho))$$

on morphisms. This makes sense as

$$\begin{aligned}(\Gamma(u(V_1), d_{\mathcal{I}}^0))((\Gamma(V_1,\varphi^0))(\rho)) &= \Gamma(V_1,\varphi^1)\Gamma(V_1,d_{\mathcal{J}}^0)(\rho)\\ &= \Gamma(V_1,\varphi^1)(\xi_1 - a^*\xi_2)\\ &= \Gamma(V_1,\varphi^1)(\xi_1) - a^*\Gamma(V_2,\varphi^1)(\xi_2)\end{aligned}$$

as elements of $\Gamma(u(V_1),\mathcal{I}^1) = \Gamma(V_1, f_*\mathcal{I}^1)$.

For all objects (V,ξ) with image $(U,\omega) := (u(V), (\Gamma(V,\varphi^1))(\xi))$, the diagram

$$\begin{array}{ccc} \Gamma(V,\mathbf{B}) & \xrightarrow{\iota_{(V,\xi)}} & \mathrm{Aut}_{\mathcal{G}(V)}((V,\xi)) \\ \downarrow{\scriptstyle \Gamma(V,\varphi)} & & \downarrow{\scriptstyle u_{\mathcal{G}}} \\ \Gamma(U,\mathbf{A}) & \xrightarrow[\iota_{(U,\omega)}]{} & \mathrm{Aut}_{\varphi^*\mathcal{G}(U)}((U,\omega)) \end{array} \tag{C.0.3.3}$$

commutes.

C.0.4. If $V \in Y$ is an object such that $\mathcal{G}|_{Y/V}$ is trivial (i.e. isomorphic to $\mathrm{B}\mathbf{B}|_{Y/V}$), then $(\varphi^*\mathcal{G})|_{X/u(V)}$ is trivial, and the functor (C.0.3.2) is isomorphic to the canonical morphism of gerbes $\mathrm{B}\mathbf{B}|_{Y/V} \to \mathrm{B}\mathbf{A}|_{X/u(V)}$; locally on $\mathrm{B}\mathbf{B}|_{Y/V}$, this isomorphic to the localization $Y/V \to X/u(V)$. A trivialization cover for $\mathcal{G}$ pulls back by u to a trivialization cover for $\varphi^*\mathcal{G}$.

C.0.5. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{u_{\mathcal{G}}} & \varphi^*\mathcal{G} \\ \downarrow{\scriptstyle p_{\mathcal{G}}} & & \downarrow{\scriptstyle p_{\varphi^*\mathcal{G}}} \\ Y & \xrightarrow[u]{} & X \end{array} \tag{C.0.5.1}$$

of functors on the underlying categories, where $p_{\mathcal{G}}$ and $p_{\varphi^*\mathcal{G}}$ are the projections. The functors u and $u_{\mathcal{G}}$ induce morphisms of topoi $f : X \to Y$ and $F : \varphi^*\mathcal{G} \to \mathcal{G}$ [88, 00XC] and the functors $p_{\mathcal{G}}$ and $p_{\varphi^*\mathcal{G}}$ induce morphisms of topoi $P_{\mathcal{G}} : \mathcal{G} \to Y$ and $P_{\varphi^*\mathcal{G}} : \varphi^*\mathcal{G} \to X$ [88, 06NW].

C.0.6 (Proof of Lemma 1.3.7). Assume the notation of C.0.2 and C.0.3 and set $\mathscr{X} := \mathcal{G}$ and $\mathscr{Y} := \varphi^*\mathcal{G}$ and $F := u_{\mathcal{G}}$. Let $\rho_{\mathscr{F}} : \mathcal{O}_{\mathscr{X}} \times \mathscr{F} \to \mathscr{F}$ be the action map of $\mathscr{F}$ as an $\mathcal{O}_{\mathscr{X}}$-module. The $\mathcal{O}_{\mathscr{Y}}$-module structure on $F_*\mathscr{F}$ is given by the composition $\rho_{F_*\mathscr{F}} := (F_*\rho_{\mathscr{F}}) \circ (f^\flat \times \mathrm{id})$ as in the following diagram:

$$\begin{array}{ccc} \mathcal{O}_{\mathscr{Y}} \times F_*\mathscr{F} & \xrightarrow{\rho_{F_*\mathscr{F}}} & F_*\mathscr{F} \\ \downarrow{\scriptstyle f^\flat \times \mathrm{id}} & & \| \\ F_*\mathcal{O}_{\mathscr{X}} \times F_*\mathscr{F} & \xrightarrow[F_*\rho_{\mathscr{F}}]{} & F_*\mathscr{F} \end{array} \tag{C.0.6.1}$$

We show that $F_*\mathscr{F}$ is $\chi_{\mathbf{B}}$-twisted. Let $(V,\xi) \in \mathscr{Y}$ be an object, let $t \in \Gamma((V,\xi), F_*\mathscr{F})$ be a section, let $(V',\xi') \to (V,\xi)$ be a morphism in $\mathscr{Y}$, and let $b \in \Gamma((V',\xi'), \mathbf{B}_{\mathscr{Y}})$ be a section. Let $F((V',\xi') \to (V,\xi)) = (U',\omega') \to (U,\omega)$ and let $s \in \Gamma((U,\omega),\mathscr{F})$ be the section corresponding to t under the identification $\Gamma((V,\xi),F_*\mathscr{F}) \simeq \Gamma((U,\omega),\mathscr{F})$. We have

$$\begin{aligned}(\iota_{\mathscr{Y}}(b)^*)(t|_{(V',\xi')}) &\overset{1}{=} (\iota_{\mathscr{X}}(\varphi(b))^*)(s|_{(U',\omega')}) \\ &\overset{2}{=} \rho_{\mathscr{F}}(\chi_{\mathbf{A}}(\varphi(b)), s|_{(U',\omega')}) \\ &\overset{3}{=} \rho_{\mathscr{F}}(f^\flat(\chi_{\mathbf{B}}(b)), s|_{(U',\omega')}) \\ &\overset{4}{=} \rho_{F_*\mathscr{F}}(\chi_{\mathbf{B}}(b), t|_{(V',\xi')})\end{aligned}$$

where equality 1 is by definition of $\iota_{\mathscr{Y}}$, equality 2 is since $\mathscr{F}$ is $\chi_{\mathbf{A}}$-twisted, equality 3 is by hypothesis on $\chi_{\mathbf{B}}$, equality 4 is by definition of $\rho_{F_*\mathscr{F}}$. □

C.0.7. We verify that the diagram (1.4.3.1) commutes. Assume the situation of C.0.3 with $\mathbf{A} := \mathbb{G}_{m,X}$ and $\mathbf{B} := \mathbb{G}_{m,Y}$ and $\varphi := f^\flat : \mathbb{G}_{m,Y} \to f_*\mathbb{G}_{m,X}$. Let $\mathcal{B}$ be an Azumaya $\mathcal{O}_Y$-algebra, let $\mathcal{A} := f^*\mathcal{B}$ be the Azumaya $\mathcal{O}_X$-algebra obtained by pullback, let $\mathscr{Y}_{\mathrm{AZ}}$ (resp. $\mathscr{X}_{\mathrm{AZ}}$) be the $\mathbb{G}_{m,Y}$-gerbe (resp. $\mathbb{G}_{m,X}$-gerbe) of trivializations of $\mathcal{B}$ (resp. $\mathcal{A}$), let $\beta \in \Gamma(Y, Z^2_{\mathcal{J}})$ be a cohomology class corresponding to $\mathscr{Y}_{\mathrm{AZ}}$, and set $\mathscr{Y}_{\mathrm{CO}} := \mathcal{G}$ and $\mathscr{X}_{\mathrm{CO}} := \varphi^*\mathcal{G}$ as in C.0.3.

By construction, there is an isomorphism of $\mathbb{G}_{m,Y}$-gerbes $\mathscr{Y}_{\mathrm{AZ}} \simeq \mathscr{Y}_{\mathrm{CO}}$. The desired claim reduces to showing that there is an isomorphism of $\mathbb{G}_{m,X}$-gerbes $\mathscr{X}_{\mathrm{AZ}} \simeq \mathscr{X}_{\mathrm{CO}}$.

We will construct the isomorphism locally on $\mathscr{X}_{\mathrm{AZ}}$ then glue. For convenience, we will assume $\mathcal{B}$ has constant rank r^2. Let

$$\mathfrak{V} = \{Y_i \to Y\}_{i\in I}$$

be a covering such that there exist isomorphisms

$$\alpha_i : \mathrm{Mat}_{r\times r}(\mathcal{O}_{Y_i}) \to \mathcal{B}|_{Y_i}$$

of $\mathcal{O}_{Y_i}$-algebras. On $Y_{i_1,i_2} := Y_{i_1} \times_Y Y_{i_2}$, there are $\mathcal{O}_{Y_{i_1,i_2}}$-algebra automorphisms

$$\alpha_{i_1,i_2} := (\alpha_{i_2}|_{Y_{i_1,i_2}})^{-1} \circ (\alpha_{i_1}|_{Y_{i_1,i_2}}) \in \mathrm{Aut}_{\mathcal{O}_{Y_{i_1,i_2}}}(\mathrm{Mat}_{r\times r}(\mathcal{O}_{Y_{i_1,i_2}}))$$

which, by the proof of Lemma 1.1.10, corresponds to a $\mathbb{G}_{m,Y_{i_1,i_2}}$-torsor $\mathcal{P}_{i_1,i_2}$ naturally embedded as a subsheaf of $\mathrm{GL}_r(\mathcal{O}_{Y_{i_1,i_2}})$. Moreover, on triple intersections $Y_{i_1,i_2,i_3} := Y_{i_1} \times_Y Y_{i_2} \times_Y Y_{i_3}$, the equality

$$\alpha_{i_1,i_3}|_{Y_{i_1,i_2,i_3}} = \alpha_{i_2,i_3}|_{Y_{i_1,i_2,i_3}} \circ \alpha_{i_1,i_2}|_{Y_{i_1,i_2,i_3}}$$

of algebra automorphisms corresponds to the equality

$$\mathcal{P}_{i_1,i_3}|_{Y_{i_1,i_2,i_3}} = \mathcal{P}_{i_2,i_3}|_{Y_{i_1,i_2,i_3}} \cdot \mathcal{P}_{i_1,i_2}|_{Y_{i_1,i_2,i_3}} \tag{C.0.7.1}$$

of $\mathbb{G}_{m,Y_{i_1,i_2,i_3}}$-torsors, with respect to the group law on $\mathrm{GL}_r(\mathcal{O}_{Y_{i_1,i_2,i_3}})$. Set $X_i := u(Y_i)$ and $X_{i_1,i_2} := u(Y_{i_1,i_2})$ and $X_{i_1,i_2,i_3} := u(Y_{i_1,i_2,i_3})$; since the functor u is continuous, the collection

$$f^{-1}\mathfrak{U} = \{X_i \to X\}_{i\in I}$$

is a covering of X and satisfies $X_{i_1,i_2} \simeq X_{i_1} \times_X X_{i_2}$ and $X_{i_1,i_2,i_3} := X_{i_1} \times_X X_{i_2} \times_X X_{i_3}$. The triple

$$(Y_i, \mathcal{O}_{Y_i}^{\oplus r}, \alpha_i)$$

is an object of $\mathscr{Y}_{\mathrm{AZ}}$. Let

$$(Y_i, \xi_i)$$

be the corresponding object of $\mathscr{Y}_{\mathrm{CO}}$ and let

$$(U_i, \omega_i) := (u(Y_i), (\Gamma(Y_i, \varphi^1))(\xi_i))$$

be its image in $\mathscr{X}_{\mathrm{CO}}$. For each i, we may define an isomorphism of $\mathbb{G}_{m,X_i}$-gerbes

$$f^{-1}\alpha_i : \mathrm{B}\mathbb{G}_{m,X_i} \to \mathscr{X}_{\mathrm{CO}}|_{X_i}$$

sending the trivial $\mathbb{G}_{m,X_i}$-torsor to (U_i, ω_i). The transition map

$$\mathrm{B}\mathbb{G}_{m,X_{i_1}}|_{X_{i_1,i_2}} \to \mathrm{B}\mathbb{G}_{m,X_{i_2}}|_{X_{i_1,i_2}}$$

is given by the $\mathbb{G}_{m,X_{i_1,i_2}}$-torsor $f^\flat(\mathcal{P}_{i_1,i_2})$, which is identified with a subsheaf of $\mathrm{GL}_r(\mathcal{O}_{X_{i_1,i_2}})$; these transition maps satisfy the cocycle condition analogous to (C.0.7.1), and hence, via stackification, the $f^{-1}\alpha_i$ glue to give the desired morphism $\mathscr{X}_{\mathrm{AZ}} \to \mathscr{X}_{\mathrm{CO}}$ of $\mathbb{G}_{m,x}$-gerbes.

Appendix D. The Weierstrass and Hesse presentations of $\mathscr{M}_{1,1}$

The purpose of this section is to prove Proposition D.2.1 below, which we could not find proved in the literature. For completeness of exposition, we first recall the definition of a full level N structure on an elliptic curve E/S.

D.1. Full level N structures.

D.1.1. [51, Ch. 3] Let N be a positive integer. We define $[\Gamma(N)]$ to be the category of pairs

$$(E/S, \xi)$$

where

$$E/S = (f : E \to S, e : S \to E)$$

is an elliptic curve and

$$\xi : (\mathbb{Z}/(N))^2_S \to E$$

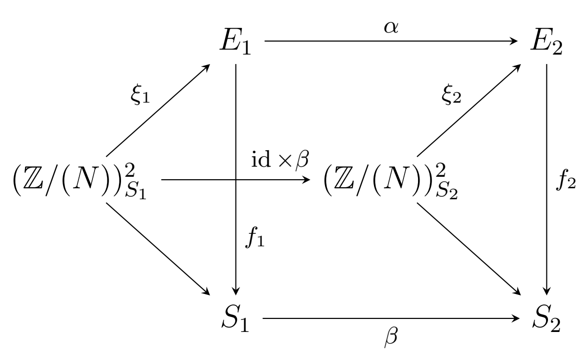
is a morphism of S-group schemes inducing an isomorphism $(\mathbb{Z}/(N))^2_S \simeq E[N]$. A morphism

$$(E_1/S_1, \xi_1) \to (E_2/S_2, \xi_2)$$

is a pair

$$(\alpha : E_1 \to E_2\ ,\ \beta : S_1 \to S_2)$$

of morphisms of schemes such that the diagram



(D.1.1.1)

commutes, where the morphism $\mathrm{id} \times \beta$ is the one induced by the identity on $(\mathbb{Z}/(3))^2_{\mathbb{Z}}$ and β, and such that α induces an isomorphism of S_1-group schemes $E_1 \simeq S_1 \times_{\beta, S_2} E_2$.

There is a functor

$$[\Gamma(N)] \to \mathscr{M}_{1,1,\mathbb{Z}}$$

sending $(E/S,\xi) \mapsto E/S$ on objects and $(\alpha,\beta) \mapsto (\alpha,\beta)$ on morphisms. If E/S admits a full level N structure, then N is invertible on S by [51, 2.3.2], hence the above functor factors through $\mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{N}]}$. If $N \geq 3$, then for any scheme S the fiber category $[\Gamma(N)](S)$ is equivalent to a set by [51, 2.7.2], so $[\Gamma(N)]$ is fibered in sets over the category of schemes.

D.1.2 (The $\mathrm{GL}_2(\mathbb{Z}/(N))$-action on $[\Gamma(N)]$). Fix a scheme S. For any element

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

in $\mathrm{GL}_2(\mathbb{Z}/(N))$, let

$$\varphi_\sigma : (\mathbb{Z}/(N))_S^2 \to (\mathbb{Z}/(N))_S^2$$

be the S-group scheme automorphism of $(\mathbb{Z}/(N))_S^2$ corresponding to the abelian group homomorphism $(\mathbb{Z}/(N))^2 \to (\mathbb{Z}/(N))^2$ defined by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11}x_1 + \sigma_{12}x_2 \\ \sigma_{21}x_1 + \sigma_{22}x_2 \end{bmatrix}$$

for $x_1, x_2 \in \mathbb{Z}/(N)$, i.e. acting by multiplication on the left on $(\mathbb{Z}/(N))^2$ viewed as vertical vectors. We have

$$\varphi_{\sigma_1}\varphi_{\sigma_2} = \varphi_{\sigma_1\sigma_2}$$

for $\sigma_1, \sigma_2 \in \mathrm{GL}_2(\mathbb{Z}/(N))$.

Fix an object $(E/S,\xi) \in [\Gamma(N)](E/S)$; then $(E/S, \xi\circ\varphi_\sigma)$ is another object of $[\Gamma(N)](E/S)$, i.e. corresponds to another full level N structure on E/S. This implies that there is a natural action of $\mathrm{GL}_2(\mathbb{Z}/(N))$ on each fiber category $[\Gamma(N)](E/S)$; the action is a right action since it is defined by precomposition.

Theorem D.1.3. [51, 4.7.2] If $N \geq 3$, the category $[\Gamma(N)]$ is representable by a smooth affine curve $Y(N)$ over $\mathbb{Z}[\frac{1}{N}]$.

D.2. **Comparing the Weierstrass and Hesse presentations.** We are primarily interested in the case $N = 3$. The 3-torsion points of an elliptic curve correspond to its inflection points (also "flex points"). In [51, (2.2.11)] it is shown that $Y(3) \simeq \operatorname{Spec} A_{\mathrm{W}}$ where

$$A_{\mathrm{W}} := \mathbb{Z}[\tfrac{1}{3}, B, C, \tfrac{1}{C}, \tfrac{1}{a_3}, \tfrac{1}{a_1^3-27a_3}]/(B^3 - (B+C)^3)$$

and the universal elliptic curve over A_{W} with full level 3 structure is the pair

$$\begin{cases} E_{\mathrm{W}} := \operatorname{Proj} A_{\mathrm{W}}[X,Y,Z]/(Y^2Z + a_1XYZ + a_3YZ^2 = X^3) \\ [0:0:1], [C : B+C : 1] \end{cases} \tag{D.2.0.1}$$

where

$$a_1 = 3C - 1 \tag{D.2.0.2}$$

$$a_3 = -3C^2 - B - 3BC\ . \tag{D.2.0.3}$$

The formulas (D.2.0.2) and (D.2.0.3) are obtained by imposing the condition that the line $Y = X+BZ$ is a flex tangent to E_{W} at $[C : B+C : 1]$. The ring A_{W} is isomorphic to $TMF(3)_0$ (3.4.6.1), with mutually inverse ring isomorphisms $TMF(3)_0 \to A_{\mathrm{W}}$ and $A_{\mathrm{W}} \to TMF(3)_0$ given by $(\zeta, t) \mapsto (\frac{B+C}{C}, \frac{1}{3C})$ and $(B,C) \mapsto (\frac{1}{3(\zeta-1)t}, \frac{1}{3t})$ respectively.

In this paper, however, we use the "Hesse presentation" of $Y(3)$ as in [36, 5.1]. The following is claimed without proof in the Introduction to [25] and [47, 5.2.30].

Proposition D.2.1. There is an isomorphism $Y(3) \simeq \operatorname{Spec} A_{\mathrm{H}}$ where

$$A_{\mathrm{H}} := \mathbb{Z}[\tfrac{1}{3}, \mu, \omega, \tfrac{1}{\mu^3-1}]/(\omega^2+\omega+1)$$

and the universal elliptic curve over A_{H} with full level 3 structure is the pair

$$\begin{cases} E_{\mathrm{H}} := \operatorname{Proj} A_{\mathrm{H}}[X,Y,Z]/(X^3+Y^3+Z^3 = 3\mu XYZ) \\ [-1:0:1], [1:-\omega:0] \end{cases} \tag{D.2.1.1}$$

with identity section $[1:-1:0]$.

The explicit $\mathbb{Z}[\frac{1}{3}]$-algebra isomorphisms $A_{\mathrm{H}} \to A_{\mathrm{W}}$ and $A_{\mathrm{W}} \to A_{\mathrm{H}}$ are given in (D.2.5.7) and (D.2.5.8) respectively.

D.2.2. By [87, §4], the group law of an elliptic curve $E = \operatorname{Proj} A[X,Y,Z]/(X^3+Y^3+Z^3 = 3\mu XYZ)$ in Hessian form over a ring A is as follows. If $P = [x:y:z]$, then $2P = [x':y':z']$ where

$$\begin{aligned} x' &= y(z^3-x^3) \\ y' &= x(y^3-z^3) \\ z' &= z(x^3-y^3) \end{aligned}$$

and if $P_i = [x_i : y_i : z_i]$ are points of E_{H} for $i = 1,2,3$ satisfying $P_1 + P_2 = P_3$, then

$$\begin{aligned} x_3 &= x_2y_1^2z_2 - x_1y_2^2z_1 \\ y_3 &= x_1^2y_2z_2 - x_2^2y_1z_1 \\ z_3 &= x_2y_2z_1^2 - x_1y_1z_2^2 \end{aligned}$$

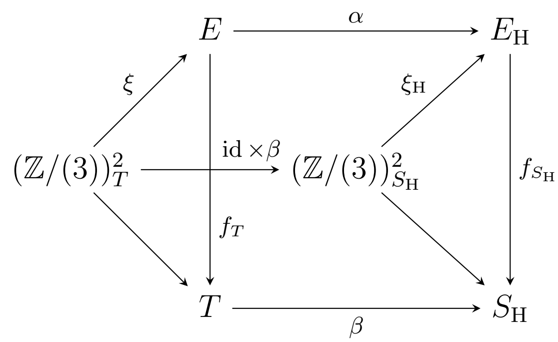
which only makes sense if $P_1 \neq P_2$.

Using the above formulas, we may check that the full level 3 structure $\xi_{\mathrm{H}} : (\mathbb{Z}/(3))^2_{A_{\mathrm{H}}} \to E_{\mathrm{H}}$ is given by the table (D.2.2.1).

$$\xi_{\mathrm{H}}\left(\begin{bmatrix} (0,0) & (1,0) & (2,0) \\ (0,1) & (1,1) & (2,1) \\ (0,2) & (1,2) & (2,2) \end{bmatrix}\right) = \begin{bmatrix} [1:-1:0] & [-1:0:1] & [0:1:-1] \\ [1:-\omega:0] & [-\omega:0:1] & [0:1:-\omega] \\ [1:-\omega^2:0] & [-\omega^2:0:1] & [0:1:-\omega^2] \end{bmatrix} \tag{D.2.2.1}$$

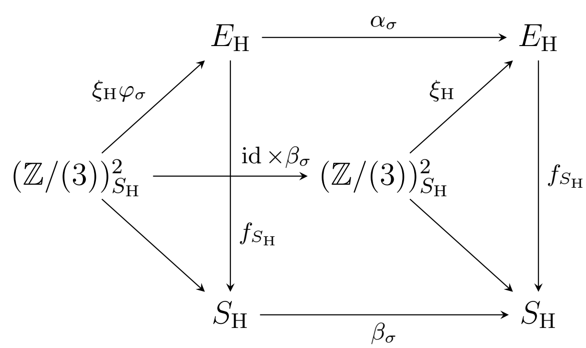
The Hesse presentation (D.2.1.1) is sometimes easier to work with than the Weierstrass presentation (D.2.0.1) since the equation of the universal elliptic curve is symmetric in X, Y, Z, which means that there is also considerable symmetry in the 3-torsion points (D.2.2.1).

D.2.3. We describe the $\mathrm{GL}_2(\mathbb{Z}/(3))$-action on $E_{\mathrm{H}}/A_{\mathrm{H}}$. Set $S_{\mathrm{H}} := \operatorname{Spec} A_{\mathrm{H}}$. The functor $[\Gamma(3)]$ being representable by S_{H} means explicitly that for any $\mathbb{Z}[\frac{1}{3}]$-scheme T and object $(E/T, \xi) \in ([\Gamma(3)])(T)$, there exists a unique pair (α, β) of morphisms of schemes $\alpha : E \to E_{\mathrm{H}}$ and $\beta : T \to S_{\mathrm{H}}$ such that the diagram

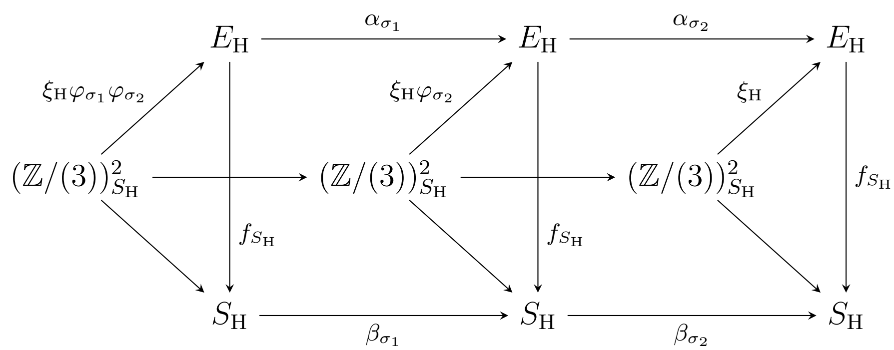


commutes and induces an isomorphism of T-group schemes $E \simeq T \times_{\beta, S_{\mathrm{H}}} E_{\mathrm{H}}$ as in (D.1.1.1).

As in D.1.2, for every $\sigma \in \mathrm{GL}_2(\mathbb{Z}/(3))$, let φ_σ be the S_{H}-automorphism of $(\mathbb{Z}/(3))^2_{S_{\mathrm{H}}}$ induced by σ; then precomposition $\xi_{\mathrm{H}}\varphi_\sigma$ defines another full level 3 structure on $E_{\mathrm{H}}/S_{\mathrm{H}}$. Taking $T = S_{\mathrm{H}}$ and $\xi = \xi_{\mathrm{H}}\varphi_\sigma$ above, there is a unique pair $(\alpha_\sigma, \beta_\sigma)$ of morphisms of schemes $\alpha_\sigma : E_{\mathrm{H}} \to E_{\mathrm{H}}$ and $\beta_\sigma : S_{\mathrm{H}} \to S_{\mathrm{H}}$ such that the diagram



commutes and induces an isomorphism of S_{H}-group schemes $E_{\mathrm{H}} \simeq S_{\mathrm{H}} \times_{\beta_\sigma, S_{\mathrm{H}}} E_{\mathrm{H}}$. Given two elements $\sigma_1, \sigma_2 \in \mathrm{GL}_2(\mathbb{Z}/(3))$, we have a commutative diagram



which implies

$$\beta_{\sigma_2}\beta_{\sigma_1} = \beta_{\sigma_1\sigma_2}$$

since $\varphi_{\sigma_1\sigma_2} = \varphi_{\sigma_1}\varphi_{\sigma_2}$ (see D.1.2). Thus the assignment

$$\sigma \mapsto \beta_\sigma \tag{D.2.3.1}$$

defines a right action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on the scheme S_{H}.

In terms of the generators

$$\mathsf{M}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \mathsf{M}_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \ \mathsf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

of $\mathrm{GL}_2(\mathbb{Z}/(3))$, the action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on $E_{\mathrm{H}}/A_{\mathrm{H}}$ is as follows. (We refer to (D.2.2.1) for the additive structure on $E_{\mathrm{H}}[3]$.)

(1) For $\sigma = \mathsf{M}_1$, the new level 3 structure $\xi_{\mathrm{H}}\varphi_{\mathsf{M}_1}$ is

$$\big[[-1:0:1] \quad [1:-\omega:0]\big] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \big[[-1:0:1] \quad [1:-\omega^2:0]\big]$$

and the scheme morphisms $\alpha_{\mathsf{M}_1} : E_{\mathrm{H}} \to E_{\mathrm{H}}$ and $\beta_{\mathsf{M}_1} : S_{\mathrm{H}} \to S_{\mathrm{H}}$ correspond to the ring homomorphisms sending

$$\begin{cases} (X, Y, Z) \hookleftarrow (X, Y, Z) \\ (\mu, \omega^2) \hookleftarrow (\mu, \omega) \end{cases}$$

respectively.

(2) For $\sigma = \mathsf{M}_2$, the new level 3 structure $\xi_{\mathrm{H}}\varphi_{\mathsf{M}_2}$ is

$$\big[[-1:0:1] \quad [1:-\omega:0]\big] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \big[[-\omega^2:0:1] \quad [1:-\omega:0]\big]$$

and the scheme morphisms $\alpha_{\mathsf{M}_2} : E_{\mathrm{H}} \to E_{\mathrm{H}}$ and $\beta_{\mathsf{M}_2} : S_{\mathrm{H}} \to S_{\mathrm{H}}$ correspond to the ring homomorphisms sending

$$\begin{cases} (X, Y, \omega^2 Z) \hookleftarrow (X, Y, Z) \\ (\omega\mu, \omega) \hookleftarrow (\mu, \omega) \end{cases}$$

respectively.

(3) For $\sigma = \mathsf{i}$, the new level 3 structure $\xi_{\mathrm{H}}\varphi_{\mathsf{i}}$ is

$$\big[[-1:0:1] \quad [1:-\omega:0]\big] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \big[[1:-\omega:0] \quad [0:1:-1]\big]$$

and the scheme morphisms $\alpha_{\mathsf{i}} : E_{\mathrm{H}} \to E_{\mathrm{H}}$ and $\beta_{\mathsf{i}} : S_{\mathrm{H}} \to S_{\mathrm{H}}$ correspond to the ring homomorphisms sending

$$\begin{cases} (\omega X + \omega^2 Y + Z, \omega^2 X + \omega Y + Z, X + Y + Z) \hookleftarrow (X, Y, Z) \\ (\frac{\mu+2}{\mu-1}, \omega) \hookleftarrow (\mu, \omega) \end{cases}$$

respectively.

Remark D.2.4. According to our convention, the action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on the fiber category $[\Gamma(3)](E_{\mathrm{H}}/\operatorname{Spec} A_{\mathrm{H}})$ is by precomposition, hence the action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on pairs of points on the right hand side of (D.2.2.1) is a *right* action; thus the induced action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on the scheme $\operatorname{Spec} A_{\mathrm{H}}$ is a *right* action (as described in (D.2.3.1)) and the corresponding action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on the coordinate ring A_{H} is a *left* action.

D.2.5 (Proof of Proposition D.2.1). In fact, it turns out that the identities

$$a_1^3 - 27a_3 = (3C + 9B - 1)^3 \tag{D.2.5.1}$$

$$a_3 = B(6C + 9B - 1) \tag{D.2.5.2}$$

hold in A_{W} which yields a simpler description

$$A_{\mathrm{W}} \simeq \mathbb{Z}[\tfrac{1}{3}, B, C, \tfrac{1}{C}, \tfrac{1}{3C+9B-1}, \tfrac{1}{6C+9B-1}]/(C^2+3CB+3B^2)$$

of A_{W}. (For (D.2.5.1), write out $a_1^3 - 27a_3$ in terms of B, C and notice that it is of the form $9C + 27B - 1$ plus higher order terms; then check that the naive guess works. To see (D.2.5.2), substitute $C^2 = -3CB - 3B^2$ into (D.2.0.3).)

We follow the argument of [6, 2.1]; see also [20, §1.4.1, §1.4.2]. Working "generically", we will assume that a_1 is a unit to obtain the coordinate change formula (D.2.5.9), then observe that it applies also to the case when a_1 is not a unit. Starting with

$$Y_1Z_1(Y_1 + a_1X_1 + a_3Z_1) = X_1^3 \tag{D.2.5.3}$$

we define X_2, Y_2, Z_2 by the system

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} u^2 & & \\ & u^3 & \\ & & 1 \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

where $u = a_1/3$ and substitute into (D.2.5.3) to get

$$Y_2Z_2(Y_2 + 3X_2 + \tfrac{27a_3}{a_1^3}Z_2) = X_2^3\ . \tag{D.2.5.4}$$

We define X_3, Y_3, Z_3 by the system

$$\begin{bmatrix} 1 & 1 & \\ 1 & & \frac{27a_3}{a_1^3} \\ 1 & & \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \omega & \omega^2 & \\ \omega^2 & \omega & \\ & & 1 \end{bmatrix} \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix}$$

where $\omega = \frac{C+B}{B}$ [24] and substitute into (D.2.5.4) to get

$$(\omega X_3 + \omega^2 Y_3 - Z_3)(\omega^2 X_3 + \omega Y_3 - Z_3)(-X_3 - Y_3 + Z_3) = \tfrac{27a_3}{a_1^3}Z_3^3$$

or equivalently

$$X_3^3 + Y_3^3 + \tfrac{27a_3 - a_1^3}{a_1^3}Z_3^3 = -3X_3Y_3Z_3\ . \tag{D.2.5.5}$$

We know that the coefficient of Z_3^3 in (D.2.5.5) is a cube (D.2.5.1) so we normalize by defining X_4, Y_4, Z_4 by the system

$$\begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{-a_1}{3C+9B-1} \end{bmatrix} \begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix}$$

and substitute into (D.2.5.5) to get

$$X_4^3 + Y_4^3 + Z_4^3 = 3\tfrac{a_1}{3C+9B-1}X_4Y_4Z_4\ . \tag{D.2.5.6}$$

To summarize the above, there is a ring homomorphism $\varphi_{21} : A_{\mathrm{H}} \to A_{\mathrm{W}}$ sending

$$\begin{aligned} \mu &\mapsto \frac{3C-1}{3C+9B-1} \\ \omega &\mapsto \frac{C+B}{B} \end{aligned} \tag{D.2.5.7}$$

[24] Since 3 is invertible, if x is a root of the polynomial $T^2 + 3T + 3$ then $x + 1$ is a root of the polynomial $T^2 + T + 1$, thus it is natural to take $\frac{C+B}{B}$ as our ω.

and solving for B, C in terms of μ, ω implies that the inverse $\varphi_{12} : A_{\mathrm{W}} \to A_{\mathrm{H}}$ sends

$$\begin{aligned} B &\mapsto \frac{\mu - 1}{3(\omega+2)(\mu-\omega)} \\ C &\mapsto \frac{(\omega-1)(\mu-1)}{3(\omega+2)(\mu-\omega)} \end{aligned} \tag{D.2.5.8}$$

where $\omega + 2$ is a unit of A_{H} since $(\omega+2)(\omega-1) = -3$ and $\mu - \omega$ is a unit of A_{H} since $\mu^3 - 1 = (\mu-1)(\mu-\omega)(\mu-\omega^2)$. We may check that the product

$$\begin{bmatrix} u^2 & & \\ & u^3 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \\ 1 & & \frac{27a_3}{a_1^3} \\ 1 & & \end{bmatrix}^{-1} \begin{bmatrix} \omega & \omega^2 & \\ \omega^2 & \omega & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{-a_1}{3C+9B-1} \end{bmatrix}$$

is "projectively equivalent" to the matrix

$$X := \begin{bmatrix} 0 & 0 & \frac{-3}{3C+9B-1} \\ \omega & \omega^2 & \frac{3u}{3C+9B-1} \\ \frac{\omega^2}{a_3} & \frac{\omega}{a_3} & \frac{3u}{a_3(3C+9B-1)} \end{bmatrix} \tag{D.2.5.9}$$

whose determinant is a unit of A_{W}. Given a section $[s_X : s_Y : s_Z]$ of (D.2.5.3), the corresponding section of (D.2.5.6) is $X^{-1} \cdot [s_X : s_Y : s_Z]^{\mathrm{T}}$ where

$$X^{-1} = \begin{bmatrix} \frac{-a_1}{3} & \frac{B}{C} & \frac{-9CB-18B^2-C}{3} \\ \frac{-a_1}{3} & \frac{-B}{C+3B} & \frac{-9CB-9B^2+C+3B}{3} \\ \frac{-3C-9B+1}{3} & 0 & 0 \end{bmatrix}.$$

The above implies that the sections

$$[0:1:0]\ ,\ [0:0:1]\ ,\ [C : B+C : 1]$$

of (D.2.5.3) (i.e. the identity section and ordered basis for the 3-torsion) correspond to the sections

$$[1 : -\omega : 0]\ ,\ [1 : -\omega^2 : 0]\ ,\ [-1 : 0 : 1] \tag{D.2.5.10}$$

of (D.2.5.6). We may apply an automorphism of the pair $(A_{\mathrm{H}}, E_{\mathrm{H}}/A_{\mathrm{H}}) \in \mathscr{M}_{1,1,\mathbb{Z}}$ of the form D.2.3(2) (for Y instead of Z) to (D.2.5.10) to get

$$[1 : -1 : 0]\ ,\ [1 : -\omega : 0]\ ,\ [-1 : 0 : 1] \tag{D.2.5.11}$$

and using the fact that there is a simply transitive action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on the set of ordered bases of the 3-torsion in $E_{\mathrm{H}}/A_{\mathrm{H}}$, we may switch the second and third sections of (D.2.5.11) to obtain

$$[1 : -1 : 0]\ ,\ [-1 : 0 : 1]\ ,\ [1 : -\omega : 0] \tag{D.2.5.12}$$

as desired. □

Remark D.2.6. For (D.2.5.1), see also Stojanoska's derivation [89, §4.1].

Remark D.2.7. There are coordinate change formulas in [87, §3] transforming a Weierstrass equation into Hesse normal form, but there it is assumed that the base ring is a finite field $\mathbb{F}_q$ where $q \equiv 2 \pmod 3$, in order to take cube roots of $a_1^3 - 27a_3$, but from this description it is not clear that the cube root is an algebraic function. As shown in (D.2.5.1), it turns out that in fact $a_1^3 - 27a_3$ is a cube in the ring A_{W}. One suspects that this is the case after tracing through the proof of [6, 2.1] and arriving at the equation $x^3+y^3+\frac{27a_3-a_1^3}{a_1^3}z^3=3xyz$, in which case we know that $\frac{27a_3-a_1^3}{a_1^3}$ is a cube by Lemma D.2.8.

Lemma D.2.8. Let k be a field of characteristic not 3, and let

$$x^3+y^3+\beta=3xy \tag{D.2.8.1}$$

be a curve in $\mathbb{A}^2_k$. Suppose that

$$ax+by+c=0 \tag{D.2.8.2}$$

is the tangent line to a flex point of E and suppose that $a^3\neq b^3$. Then β is a cube in k.

Proof. If $a=0$, then $b\neq 0$ and substituting $y=-\frac{c}{b}$ into (D.2.8.1) and rearranging gives $x^3+\frac{3c}{b}x-(\frac{c}{b})^3+\beta=0$ which by assumption is of the form $(x+\ell)^3$ for some $\ell\in k$. Comparing coefficients, we have $\ell=0$ and so $\beta=(\frac{c}{b})^3$.

By symmetry we may assume that $a,b\neq 0$. By scaling (D.2.8.2), we may assume that $b=-1$. Substituting $y=ax+c$ into E gives

$$(a^3+1)x^3+3(a)(ac-1)x^2+3(c)(ac-1)x+(c^3+\beta)$$

and dividing by the leading coefficient gives

$$x^3+3\left(\frac{a(ac-1)}{a^3+1}\right)x^2+3\left(\frac{c(ac-1)}{a^3+1}\right)x+\left(\frac{c^3+\beta}{a^3+1}\right)$$

and comparing this to

$$x^3+3\ell x^2+3\ell^2x+\ell^3$$

gives either $ac-1=0$ in which case $c^3+\beta=0$ as well (so that $\beta=(-1/a)^3=(-c)^3$), otherwise if $ac-1\neq 0$ then

$$\frac{c}{a}=a\left(\frac{ac-1}{a^3+1}\right)$$

which implies $c=-a^2$ so that the original equation of the tangent line is $y=ax-a^2$. Substituting this back into E gives $\beta=(-a)^3$. $\square$

Appendix E. Computation using Magma

We compute $\mathrm{H}^1(\mathrm{GL}_2(\mathbb{Z}/(3)),M)$ in 3.4.4 using Magma [14]. Here `G` is defined as the subgroup of $\mathrm{GL}_2(\mathbb{Z}/(3))$ generated by the matrices in (3.4.3.6), but the specified matrices constitute a generating set so in fact `G` $=\mathrm{GL}_2(\mathbb{Z}/(3))$. The group `G` acts on the abelian group $M=(\mathbb{Z}/(2))^{\oplus 6}$ by the three specified elements of $\mathrm{Mat}_{6\times 6}(\mathbb{Z})$, where each $\mathbf{x}\in M$ is viewed as a horizontal vector and each 6×6 matrix $\mathbf{A}$ acts on M by right multiplication $\mathbf{x}\mapsto\mathbf{x}\cdot\mathbf{A}$. The last line computes $\mathrm{H}^1(G,(\mathbb{Z}/(2))^{\oplus 6})$.

```
G := MatrixGroup< 2 , FiniteField(3) |
  [ 1,0 , -1,1 ] , [ 0,-1 , 1,0] , [ 1,0 , 0,-1 ]
>;
mats := [
  Matrix(Integers() , 6 , 6 , [
    0, 0, 1, 0, 0, 0 ,
    1, 0, 0, 0, 0, 0 ,
    0, 1, 0, 0, 0, 0 ,
```

```
    0, 0, 0, 0, 1, 0 ,
    0, 0, 0, 0, 0, 1 ,
    0, 0, 0, 1, 0, 0 ]) ,
  Matrix(Integers() , 6 , 6 , [
    1, 0, 0, 0, 0, 0 ,
    1, 0, 1, 0, 0, 0 ,
    1, 1, 0, 0, 0, 0 ,
    0, 0, 0, 1, 0, 0 ,
    0, 0, 0, 1, 0, 1 ,
    0, 0, 0, 1, 1, 0 ]) ,
  Matrix(Integers() , 6 , 6 , [
    0, 0, 0, 1, 0, 0 ,
    0, 0, 0, 0, 1, 0 ,
    0, 0, 0, 0, 0, 1 ,
    1, 0, 0, 0, 0, 0 ,
    0, 1, 0, 0, 0, 0 ,
    0, 0, 1, 0, 0, 0 ])
];
CM := CohomologyModule(G,[2,2,2,2,2,2],mats);
CohomologyGroup(CM,1);
```

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