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SOME CALCULATIONS ON THE TRIAX PINCH DEVICE



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## ABSTRACT

The hydromagnetic equations of motion for the Triax pinch are linearized by a perturbation expansion about equilibrium. The perturbed equations are then decomposed by the method of normal modes. A numerical calculation is made of the oscillation frequency for two specific modes ($k = 0$, $m = 0$; and $k = 0$, $m = 1$). The $m = 0$, $k = 0$ mode is also analyzed by using a hydromagnetic energy principle.

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## I. INTRODUCTION

The Triax pinch device is a method for containing a heated plasma in a cylindrical sheet configuration. The pinch phenomenon forms the current-carrying plasma into a tubular shape between two concentric cylindrical conductors.[1] This is known as the tubular pinch (Fig. 1).

A macroscopic approach to the pinch stability problem is made by using the method of small perturbations. The basic assumption of an equilibrium plasma configuration during maximum compression enables the magnetohydrodynamical equations of motion to be linearized. It is further assumed that $p_0 / \rho_0$ is a constant. The resultant second-order differential equations are solved by numerical methods, yielding frequency eigenvalues and corresponding modes of oscillation for the perturbed transverse velocity.

## II. EQUATIONS OF MOTION

The macroscopic equations of motion for the plasma are well known:[2]

$$\rho \frac{d\vec{v}}{dt} = \vec{j} \times \vec{B} - \vec{\nabla} p ,$$

$$\vec{\nabla} \times \vec{B} = 4 \pi \vec{j}$$

$$\vec{\nabla} \times \vec{E} = - 4 \frac{\pi}{c} \frac{\partial \vec{B}}{\partial t} , \qquad (1)$$

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = 0 \text{ (setting resistivity } = 0)$$

$$\vec{\nabla} \cdot \vec{B} = 0 ,$$

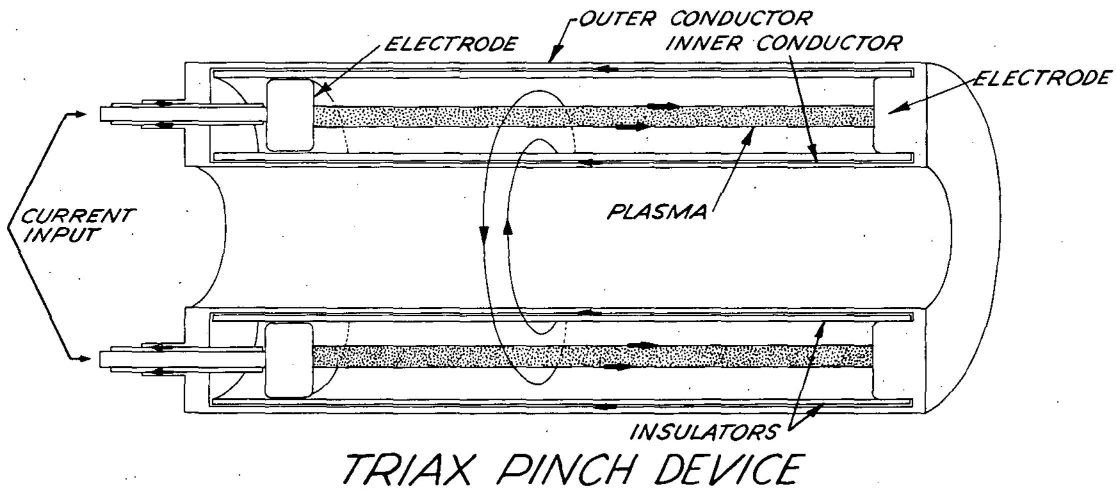


Fig. 1. Triax (schematic).

$$\frac{1}{p} \cdot \frac{dp}{dt} = \frac{\gamma}{\rho} \cdot \frac{d\rho}{dt} \quad (\gamma = cp/cv = 5/3 \text{ for ionized plasma}),$$

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \rho \vec{v} = 0 \quad \text{(continuity equation)} ,$$

$$\left(\frac{\gamma p}{\rho}\right)^{1/2} = v_s \quad \text{(sound speed)}$$

Assume that all variables $q$ behave as $q = q_0 + \hat{q}$ except for $v = \hat{v}$, where perturbation $\hat{q}$ is small compared to equilibrium value $q_0$. Keeping only first-order terms in the perturbed functions, we find the above equations then become

$$\rho_0 \frac{d}{dt} \vec{\hat{v}} = \vec{\hat{j}} \times \vec{B}_0 + \vec{j}_0 \times \vec{\hat{B}} - \vec{\nabla} \hat{p} \qquad \vec{\hat{E}} + \frac{\vec{\hat{v}}}{c} \times \vec{B}_0 = 0 ,$$

$$\vec{\nabla} \times \vec{\hat{B}} = 4\pi \vec{\hat{j}} , \qquad \vec{\nabla} \cdot \vec{\hat{B}} = 0 , \tag{2}$$

$$\vec{\nabla} \times \vec{\hat{E}} = -\frac{1}{4\pi c} \frac{\partial \vec{\hat{B}}}{\partial t} , \qquad \frac{1}{p_0} \frac{d}{dt} \hat{p} = \frac{\gamma}{\rho_0} \frac{d}{dt} \hat{\rho} ,$$

$$\frac{\partial}{\partial t} \hat{\rho} + \vec{\nabla} \cdot \rho_0 \vec{\hat{v}} ,$$

## III. PLASMA EQUATIONS OF MOTION FOR CARTESIAN GEOMETRY

The first plasma model chosen for the calculation of frequency equations for the transverse velocity is a flat ribbon, infinite in the $x$ and $z$ directions. Since the variables $B_0$, $j_0$, $\rho_0$ and $p_0$ are now dependent upon the $y$ coordinate, set $\vec{\hat{q}} = \vec{q}(y) e^{ikz+imx+\omega t}$. Noting that $(B_0)_y = (B_0)_z = (j_0)_y = (j_0)_x = 0$, and using the single subscript 0 for the nonvanishing, unperturbed components, Eq. (2) in component form is

$$\rho_0 \omega v_x = - imp - j_0 B_y ,$$

$$\rho_0 \omega v_y = - p' + j_0 B_x + j_z B_0 ,$$

$$\rho_0 \omega v_z = - ikp - j_y B_0 ,$$

$$4\pi j_x = B_z' - ikB_y \ ,$$

$$4\pi j_y = ikB_x - imB_z \ ,$$

$$4\pi j_z = imB_y - B_x' \ ,$$

$$-\omega \frac{B_x}{c} = E'_z - ikE_y \ ,$$

$$-\omega \frac{B_y}{c} = ikE_x - imE_z \ ,$$

$$-\omega \frac{B_z}{c} = imE_y - E_x'$$

$$E_x = 0 \ , \qquad (3)$$

$$-E_y = \frac{v_z}{c} B_0 \ ,$$

$$-E_z = -\frac{v_y}{c} B_0 \ ,$$

$$-\frac{\omega}{c} B_x = E_z' - ikE_y \ ,$$

$$-\frac{\omega}{c} B_y = ikE_x - imE_z \ ,$$

$$-\frac{\omega}{c} B_z = imE_y \ ,$$

$$B_y' + ikB_z + imB_x = 0 \ ,$$

$$\omega p + v_y p_0' = v_s^2 (\omega\rho + v_y \rho_0') \ ,$$

$$\omega\rho + imv_x \rho_0 + v_y' \rho_0 + v_y \rho_0' + ikv_z \rho_0 = 0 \ ,$$

where the prime symbol indicates differentiation with respect to $y$. Equation (3) reduces to a second-order differential equation in $v_y$ (see Appendix A). In order to render this equation solvable with a desk calculator, $k$ is set equal to zero. With this simplification, the differential equation becomes

$$v_y\left(\frac{m^2B_0^2}{4\pi\omega}+\rho_0\omega\right)=\frac{dv_y}{dy}\left[\frac{2B_0B_0'}{4\pi\omega}+\frac{\rho_0'\,v_s^2}{\omega\left(1+\frac{m^2v_s^2}{\omega^2}\right)}\right]$$

$$+\frac{d^2v_y}{dy^2}\left[\frac{B_0^2}{4\pi\,\omega}+\frac{\rho_0v_s^2}{\omega\left(1+\frac{m^2v_s^2}{\omega^2}\right)}\right].$$

Let us now limit the treatment in this paper to values of $m$ and $\omega$ such that $\frac{m^2v_s^2}{\omega^2} << 1$.* With this restriction, the above equation simplifies to

$$\left(\omega^2+\frac{m^2B_0^2}{4\pi\rho_0}\right)v_y=\frac{dv_y}{dy}\left(\frac{1}{\rho_0}\,\frac{B_0B_0'}{2\pi}+\frac{v_s^2\rho_0'}{\rho_0}\right)$$

$$+\frac{d^2v_y}{dy^2}\left(\frac{B_0^2}{4\pi\rho_0}+v_s^2\right). \qquad (4)$$

Equation (4) can be put into dimensionless form through a change of variable, and by changing the various coefficients of the $v_y$ derivatives. Let $y=\ell Y$, $v_y=vV$, $B_0^2=\beta^2f(Y)$, $p_0=\frac{\beta^2}{4\pi}\,[1-f(Y)]$, $\rho_0=\rho_m\,g(Y)=k\,\beta^2/8\pi\,(1-f(Y))$, $v_s^2=c^2\,\frac{\beta^2}{4\pi\rho_m}$, $\omega^2=\frac{\beta^2}{4\pi\rho_m}\,w^2$ be the appropriate substitutions, where $\beta=B_0$ at walls, $\rho_m=\rho_{maximum}$; $v, k$ and $c$ are dimensional scaling factors, $\ell$ is the outer radius of the chamber. By normalizing to the mass of the gas present, a relationship between $k$ and $\rho_{bp}$ (before pinching) can be made: for $a \leqslant Y \leqslant b$, we have

*Substitution of the calculated values for $\omega$ into $\frac{m^2v_s^2}{\omega^2}$ leads to maximum value of 0.5.

$$\int_a^b \rho_{bp}\, dY = \int_a^b \rho_m\, g(Y)\, dY = \frac{k\beta^2}{8\pi} \int_a^b [1 - f(Y)]\, dY, \text{ or}$$

$$k = \frac{\rho_{bp} (b-a)\, 8\pi/\beta^2}{\int_a^b [1 - f(Y)]\, dy}$$

In dimensionless form Eq. (4) becomes

$$\left[w^2 + m^2 \frac{f(Y)}{g(Y)}\right] v = \frac{dv}{dY} \left[\frac{f'(Y)}{g(Y)} + c^2 \frac{g'(Y)}{g(Y)}\right] + \frac{d^2v}{dY^2} \left[\frac{f(Y)}{g(Y)} + c^2\right]. \tag{5}$$

This equation is of the form $LV = w^2 V$, where $L$ is a second-order differential operator independent of w. Equation (5) is therefore an eigenvalue equation readily solvable by standard methods, and is the basic equation of this paper. By using data from the 4-inch Triax at Berkeley, stable frequencies are calculated for $m = 0$ and $m = 1$ (see Table I).

## IV. SOLUTION OF EQUATIONS OF MOTION

The solution of Eq. (5) is effected by discretizing the domain of the variable Y into n+1 equally spaced points. The first and second derivatives are approximated by first- and second-order differences. This procedure reduces Eq. (5) to the form

$$\lambda v(Y_i) = - R(Y_i)\, v(Y_i) + p(Y_i) \frac{[v(Y_{i+1}) - v(Y_{i-1})]}{2h}$$

$$= q(Y) \frac{[v(Y_{i+1}) - 2v(Y_i) + v(Y_{i-1})]}{h^2}, \tag{6}$$

where $\lambda = w^2$ and $R(Y_i)$ and $q(Y_i)$ are the coefficients of $v$, $\frac{dv}{dY}$, and $\frac{d^2v}{dY^2}$, at the ith point (see Appendix C). The problem is now to determine the eigenvalues and eigenvectors of a square matrix of order $n - 1$. The method used in this paper for calculations is Sturm's theorem, which is particularly well-suited for use with a desk calculator.[3]

Table I

A. Dimensionless frequency eigenvalues - Cartesian geometry

| Mode | Normalized density distribution $\rho_0 \sim \frac{\beta^2}{8\pi} [1 - f(Y)]$ | | Gaussian distribution (m = 0) $\rho_0 \sim \beta^2 e^{-12(\Delta Y)^2}$ | |
|---|---|---|---|---|
| | m = 0 | m = 1 | 13 subdivisions | 5 subdivisions |
| 1 | 0.47 | 0.49 | 1.33 | 1.69 |
| 2 | 1.04 | 1.05 | 1.84 | 2.71 |
| 3 | 1.58 | 1.58 | 3.13 | 4.39 |

B. Dimensionless frequency eigenvalues - cylindrical geometry

| Mode | Normalized density distribution $\rho_0 \sim \frac{\beta^2}{8\pi} [1 - f(R)]$ | | Gaussian distribution (m = 0) $\rho_0 \sim \beta^2 e^{-12(\Delta R)^2}$ |
|---|---|---|---|
| | m = 0 | m = 1 | 13 subdivisions |
| 1 | 0.56 | | 1.33 |
| 2 | 1.06 | | 1.84 |
| 3 | 1.58 | | 3.13 |

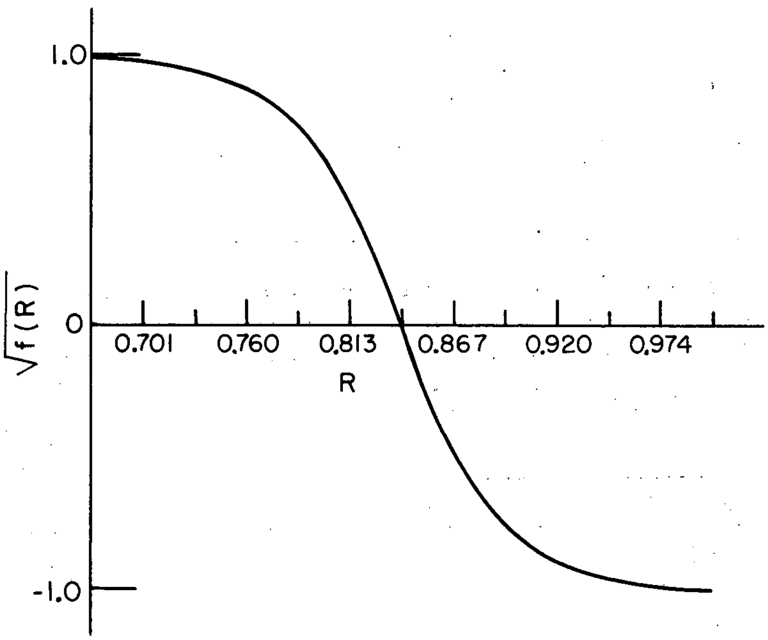


Fig. 2. Plot of the behavior of the magnetic field with transverse distance. The ordinate can be considered as either f(R) or f(Y), depending on whether cylindrical or cartesian geometry is used.

For the solution of Eq. (6) the Y domain is divided into thirteen equal portions. This yields an 11-by-11 matrix for which the three lowest eigenvalues are calculated. These values are given in Table I.

A central problem in calculating the eigenvalues of Eq. (6) is that of determining the coefficients $R(Y_i)$, $p(Y_i)$, and $q(Y_i)$ at the various $Y_i$ points. From the data available for $B_0^2 = \beta_0^2 f(Y)$ (slightly modified from symmetry considerations, as see Fig. 2 ), a plot of $\frac{1}{\sqrt{f(Y)}} \frac{d}{dY} f(Y) \left( \frac{dB_0}{dY} \text{ vs. } Y \right)$ is obtained (see Fig. 3). In order to estimate $\rho_0$ and $v_s^2$ the temperature is assumed constant throughout the plasma at the time of maximum compression. Thus $p_0$ is set proportional to $\rho_0$ and $v_s^2$ is a constant.

Two approaches were used to estimate $\rho_0$. Integrating the expression $\vec{\nabla} p_0 = \vec{j}_0 \times \vec{B}_0$, one obtains $p_0 + B_0^2/8\pi =$ constant. This equation is then used to plot $g(Y) \approx 1 - f(Y)$ versus Y by setting $f(Y) = 1$ at the walls. This graph also gives the density profile, which is normalized to the uncompressed density (see Fig. 4).

Alternative to this method is a Gaussian distribution in which $g(Y) =$ constant $\times e^{-k\Delta Y^2}$, where $\Delta Y = Y - Y_0$, and $Y_0$ is the center of the Y domain. The value of 12 for K best fits the values of $p_0' = \frac{\beta^2}{8\pi} \frac{d}{dY} f(Y)$ calculated from the given data (see Fig. 5).

For the corresponding eigenvectors of v see Fig 6.

To check the validity of the arbitrary division into the thirteen portions, a sample calculation is made with the Y domain divided into five portions. The effect of this is to raise the eigenfrequencies by approximately 25% for each mode of oscillation.

## V. PLASMA EQUATIONS OF MOTION FOR CYLINDRICAL GEOMETRY

To obtain a closer approximation to the Triax geometry, Eq. (2) is solved for a cylindrical sheet infinite in the z direction. The functions $j_0 = (j_0)_z$, $B_0 = (B_0)_\theta$ are now dependent upon the r coordinate alone. For simplicity, set the perturbed quantities $\hat{q} = \vec{q}(r)e^{\omega t}$ with no $\theta$ or z dependence. The plasma equations in cylindrical coordinates then follow:

$$\rho_0 \omega v_\theta = j_0 B_r ,$$

$$\rho_0 \omega v_r = - j_0 B_\theta - j_z B_0 - \frac{dp}{dr} ,$$

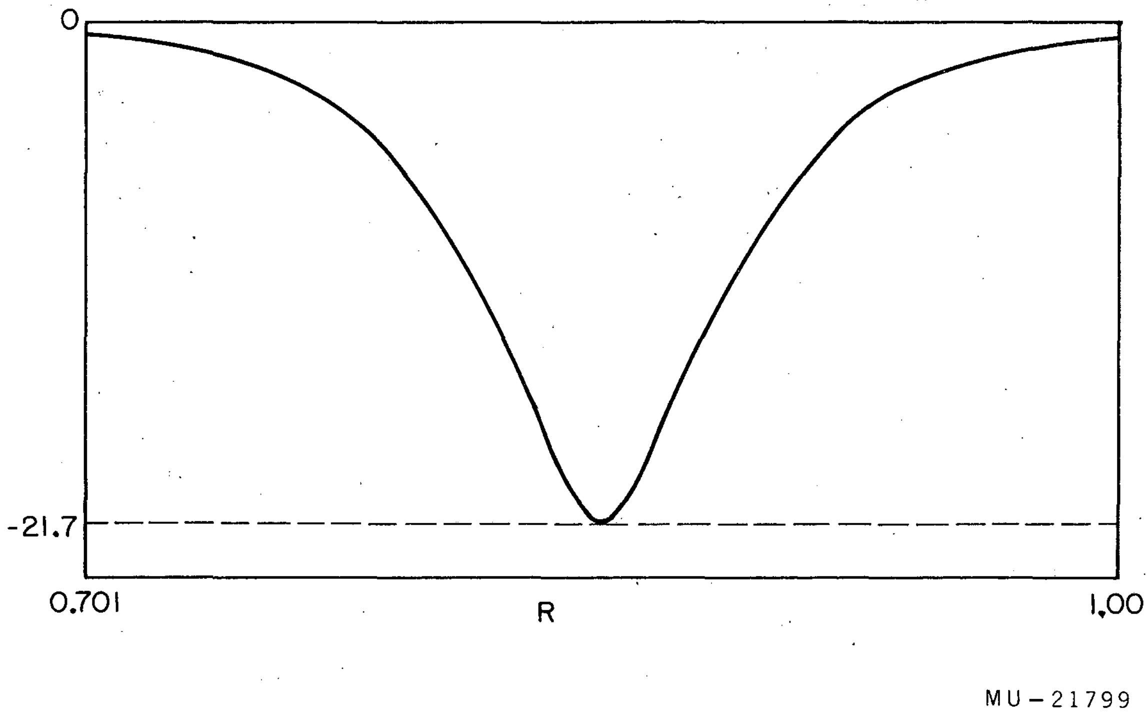


Fig. 3. Plot of the behavior of $\frac{d}{dr} B_0$ (or $\frac{d}{dy} B_0$) with transverse distance.

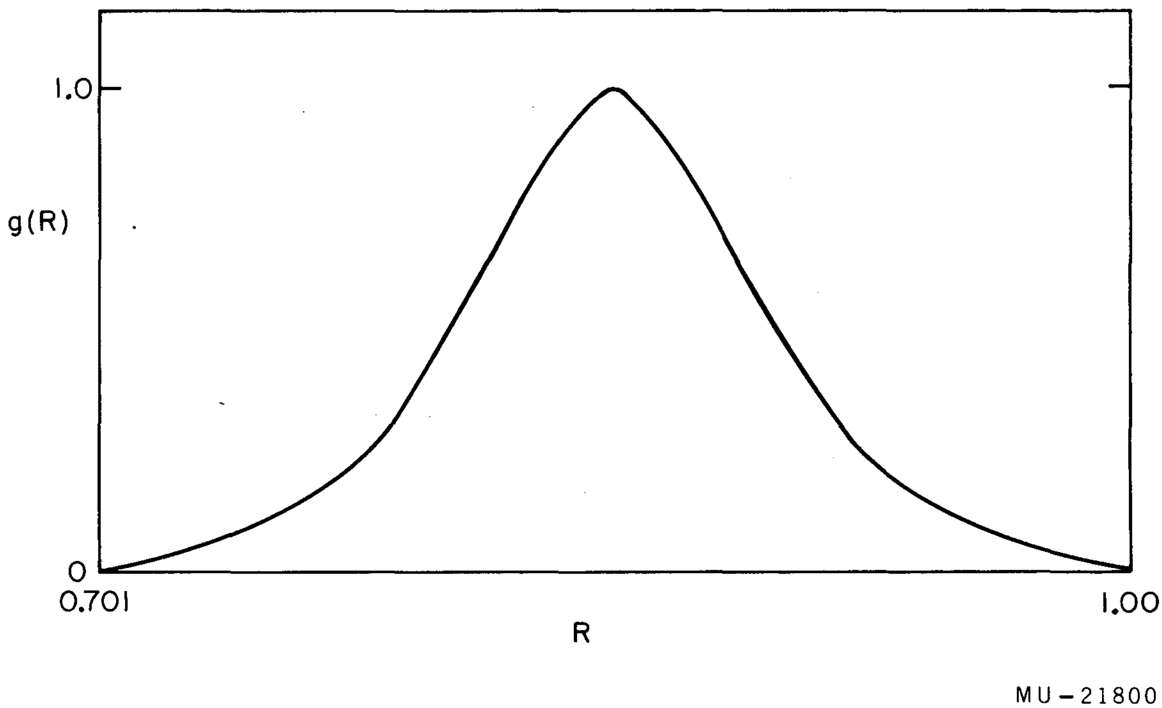


Fig. 4. Plot of the pressure (density) profile with transverse distance.

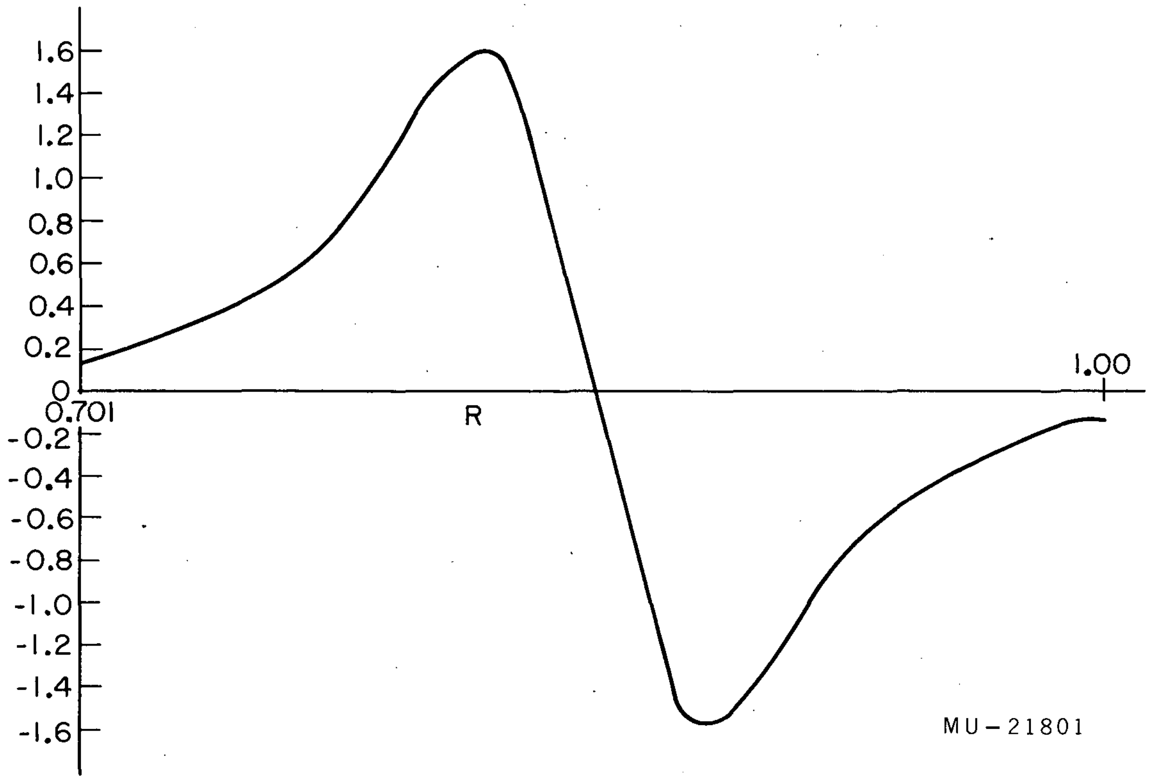


Fig. 5. Plot of $\frac{dp_0}{dr}$ ( or $\frac{dp_0}{dy}$ ) with transverse distance.

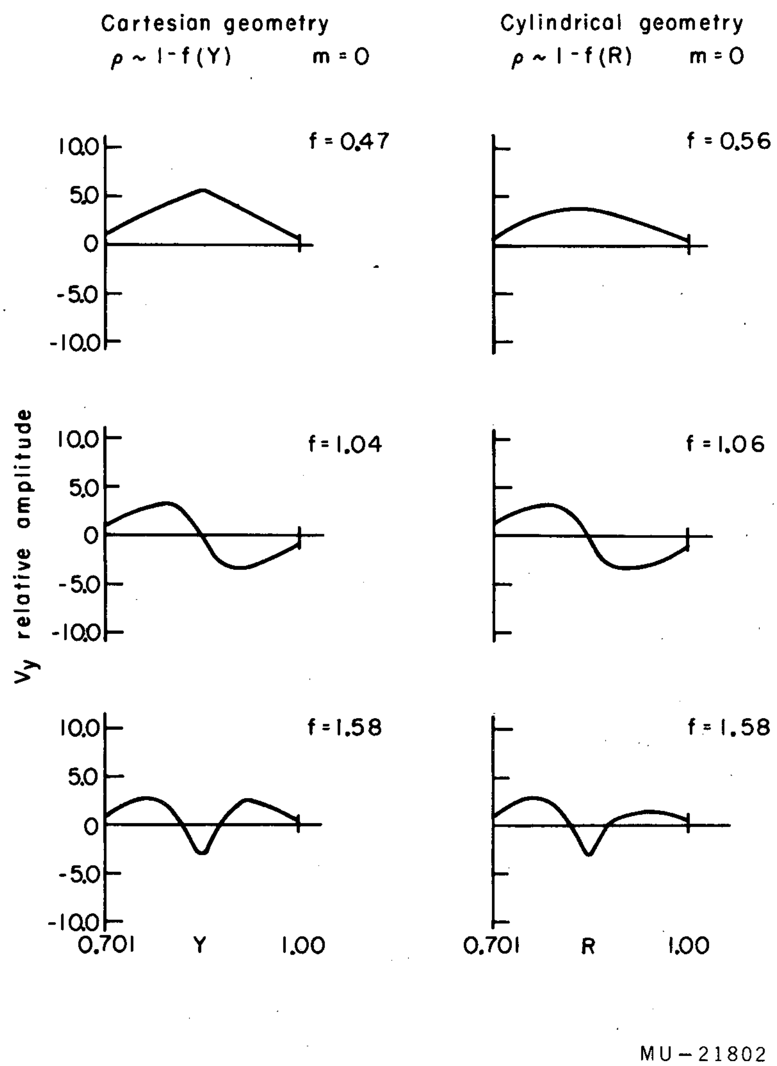


Fig. 6. Plots of transverse velocity relative amplitudes.

$$\rho_0 \, \omega \, v_z = j_0 B_r \, ,$$

$$4\pi j_\theta = - \frac{d}{dr} B_z \, ,$$

$$4\pi j_r = 0 \, ,$$

$$4\pi j_z = \frac{1}{r} \frac{d}{dr} r B_\theta \, ,$$

$$- \frac{\omega}{c} B_\theta = - \frac{d}{dr} E_z$$

$$- \frac{\omega}{c} B_r = 0 \, , \qquad (7)$$

$$- \frac{\omega}{c} B_z = \frac{1}{r} \frac{d}{dr} r E_\theta \, ,$$

$$E_r = + \frac{v_z B_0}{c} \, ,$$

$$E_\theta = 0 \, ,$$

$$E_z = - \frac{v_r B_0}{c} \, ,$$

$$\omega \rho + \rho_0 \frac{1}{r} \frac{d}{dr} r v_r + v_r \frac{d\rho_0}{dr} = 0 \, ,$$

$$\omega p + v_r \frac{d}{dr} p_0 = v_s^2 \left( \omega \rho + v_r \frac{d\rho_0}{dr} \right) .$$

Through algebraic manipulation similar to that in Sec. III, the equation of motion for $v_r$ is found to be (the prime denoting differentiation with respect to $r$, and $m = 0$).

$$\omega^2 v_r = v_r \left[\frac{v_s^2 \rho_0'}{\rho_0 r} - \frac{v_s^2}{r^2} + \frac{B_0^2}{4\pi\rho_0 r^2}\right]$$

$$+ \frac{dv_r}{dr} \left[\frac{v_s^2 \rho_0'}{\rho_0} + \frac{v_s^2}{r} + \frac{B_0^2}{4\pi\rho_0 r} + \frac{B_0 B_0'}{2\pi\rho_0}\right]$$

$$+ \frac{d^2 v_r}{dr^2} \left[\frac{B_0^2}{4\pi\rho_0} + v_s^2\right] \tag{8}$$

To derive Eq. (8) we have used the relations $4\pi j_0 = \frac{1}{r}\frac{d}{dr} r B_0$, and $\frac{dp_0}{dr} = - j_0 B_0 = - \frac{B_0}{4\pi}\frac{1}{r}\frac{d}{dr} r B_0$.

Note that Eq. (7) reduces to Eq. (4), for $r \rightarrow \infty$. Equation (8) can be put into dimensionless form, as was Eq. (4). The approximation $B_0^2 / 4\pi r << B_0 B_0' / 4\pi$ is made so that the pressure distribution $p_0 = \frac{\beta^2}{8\pi}(1-f(R))$ is employed, as are the relationships $r = \ell R$, $v_r = vV$, $B_0^2 = \beta^2 f(R)$, $\rho_0 = \rho_m g(R) = k \frac{\beta^2}{8\pi}(1-f(R))$, $v_s^2 = c^2 \frac{\beta^2}{4\pi\rho_w^2}$, $\omega^2 = \frac{2}{4\pi\rho_m} w^2$, where $\rho_m$, $\ell$, v, k, and c are defined as in Sec. III.*

Similiarly, for $a \leqslant R \leqslant b$, $k = \frac{\rho_{bp} (b-a) \frac{8\pi}{\beta^2}}{\int_a^b (1-f(R))\, dR}$, Eq. (8) becomes (with $m = 0$):

*Except at the end points, where R = 3.06 and R = 4.50, the above approximation is valid.

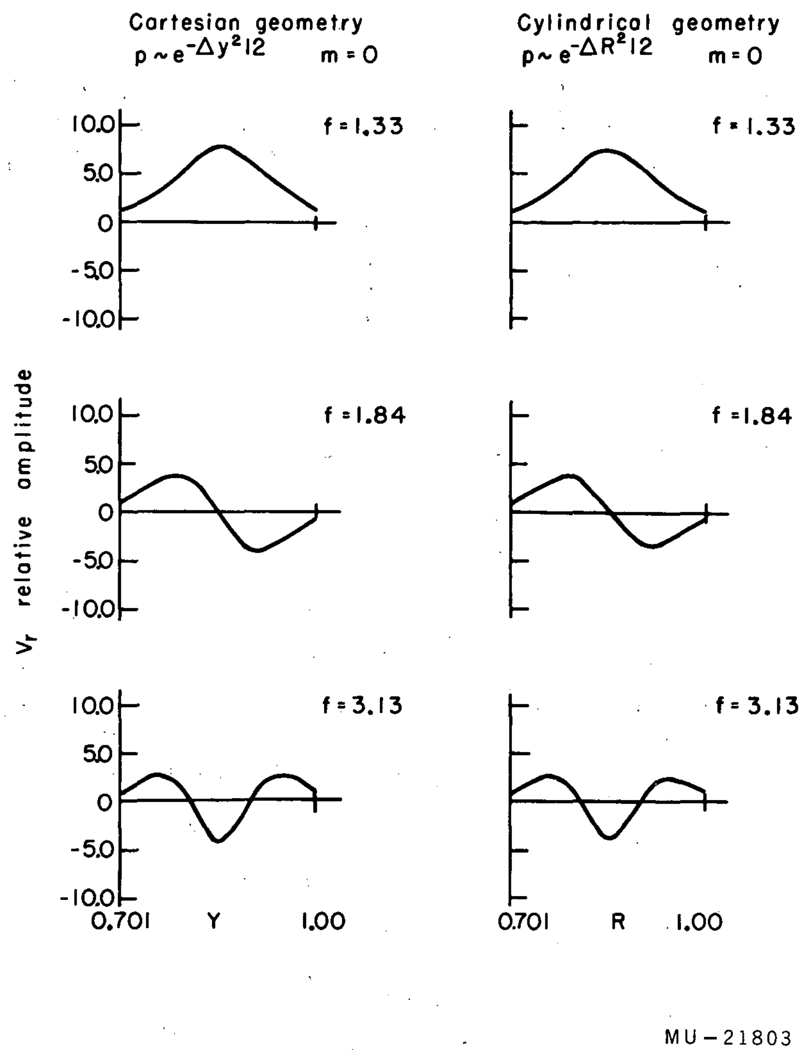


Fig. 7. Transverse velocity eigenvectors (continued).

$$w^2 v = v \left[ \frac{c^2}{R g(R)} \frac{d}{dR} g(R) - \frac{c^2}{R^2} + \frac{1}{R^2} \frac{f(R)}{g(R)} \right]$$

$$+ \frac{dv}{dR} \left[ \frac{c^2}{g(R)} \frac{d}{dR} g(R) + \frac{c^2}{R} + \frac{f(R)}{g(R)} \frac{1}{R} + \frac{1}{g(R)} \frac{d}{dR} f(R) \right]$$

$$+ \frac{d^2 v}{dR^2} \left[ \frac{f(R)}{g(R)} + c^2 \right] \tag{9}$$

Equation (9) can be solved in the same manner as Eq. (5): almost the same frequency eigenvalues are found. This is to be expected, since $3 < R < 5$ implies that the terms in $1/R$ or $1/R^2$ are a good deal smaller than the remaining terms. The corresponding profiles for $v$ are no longer symmetric, but decrease towards the outer circumference (see Figs. 6, 7, and 8).

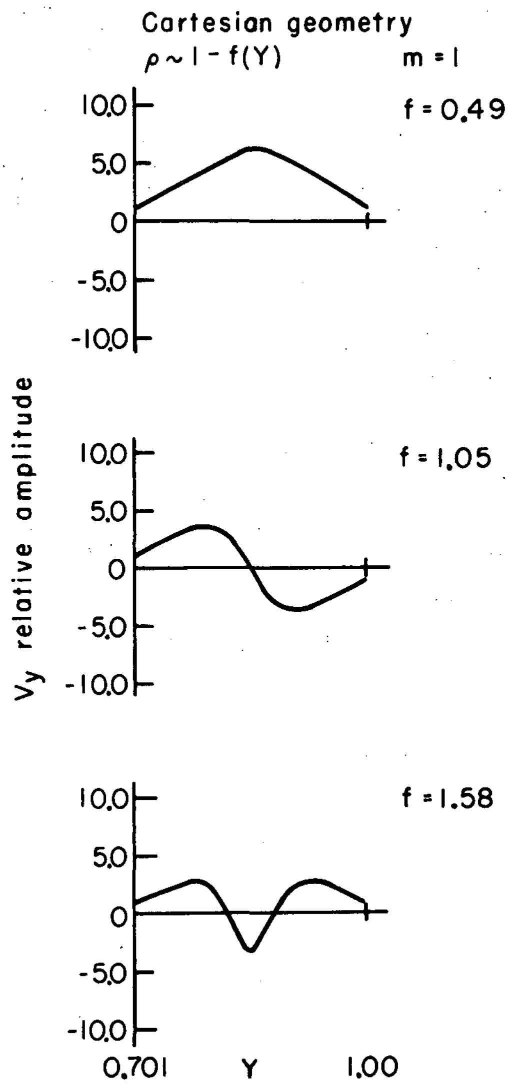


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Fig. 8. Transverse velocity eigenvectors (concluded).

## CONCLUSION

The perturbed transverse displacements of the plasma have the same wave form and oscillation frequency as the perturbed transverse velocities. These displacements furnish a good indication of the behavior of the plasma. They indicate that the tubular pinch is stable with respect to the small perturbations $m = 0, 1$, and $k = 0$, if we use the density profile $\rho \approx \frac{\beta^2}{4\pi}(1-f(Y))$, where $f(Y)$ is the functional behavior of $B_0$, and $\beta = B_0$ maximum. The assumptions made in these calculations are uniform equilibrium temperature, infinite conductivity at the plasma and walls, scalar pressure, zero equilibrium fluid velocity, negligible effects of runaway electrons, no Landau damping, and negligible heat conduction.

The oscillation frequencies calculated by using the data from the Berkeley 4-inch Triax device are 2.2 Mc/sec, for the $m = 0$ oscillation. A short calculation using the energy principle, based upon a highly simplified plasma model, gives substantially the same result (see Appendix D).

Test runs of the Berkeley 4-inch Triax indicate a frequency of 1 Mc/sec. The disparity between these results may be due to inaccurate estimates of the density of material in the pinch.

## ACKNOWLEDGMENTS

The author would like to thank Dr. Wulf Kunkel for suggesting this topic and for his active support and help, Dr. Lloyd Smith for his guidance and assistance, and Dr. Allan N. Kaufman for his criticism and useful suggestions for the improvement of this paper.

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# APPENDICES

## A. Derivation of Differential Equation in $v_y$

Equation (3) can be reduced to three differential equations in $v_y$, $v_z$, and p.

$$
\begin{aligned}
0 &= p' + av_y' + bv_y' - cv_y'' + dv_z - ev_z' \\
0 &= p - fv_y - gv_y' \qquad\qquad + hv_z \qquad\qquad (A1) \\
0 &= kp + \ell v_y + mv_y' \qquad + nv_z
\end{aligned}
$$

Coefficients a, b, c, d, e, f, g, h, k, l, m, n (m and k are not to be confused with the x and z variations of q) are:

$$a = \left[ \frac{j_0 B_0'}{\omega} + \frac{m^2 B_0^2}{4\pi\omega} - \frac{B_0 B_0''}{4\pi\omega} + \rho_0 \omega + \frac{\omega B_0^2}{4\pi c^2} \right]$$

(Where c represents the speed of light)

$$b = \left[ \frac{j_0 B_0}{\omega} - \frac{2B_0 B_0'}{4\pi\omega} \right]$$

$$c = \frac{B_0^2}{4\pi\omega}$$

$$d = \frac{ikj_0 B_0}{c} - \frac{ikB_0 B_0'}{4\pi\omega}$$

$$e = \frac{ikB_0^2}{4\pi\omega}$$

$$f = \frac{BB_0'}{4\pi\omega} \qquad (A2)$$

$$g = \frac{B_0^2}{4\pi\omega}$$

$$h = \frac{m^2+k^2}{ik} \quad \frac{B_0^2}{4\pi\omega} + \frac{B_0^2\omega}{ik4\pi c^2} + \frac{\rho\omega}{ik}$$

$$k = \frac{\omega}{v_s^2} + \frac{m^2}{\omega}$$

$$\ell = \frac{\rho_0'}{v_s^2} + \frac{m^2 j_0 B_0}{\omega^2}$$

$$m = \rho_0$$

$$n = ik\rho_0 \quad .$$

Solving Eq. (A-2) for $v_y$ one obtains

$$0 = v_y \left[ f' + a + \frac{h'+d}{\left(\frac{n}{k} - h\right)} \left(f + \frac{\ell}{k}\right) + \frac{h+e}{\left(\frac{n}{k} - h\right)} \left(f + \frac{\ell}{k}\right)' + \frac{h+e}{\left(\frac{n}{k} - h\right)^2} \left(f + \frac{\ell}{k}\right)\left(h - \frac{n}{k}\right)' \right]$$

$$+ v_y' \left[ f + g' + b + \frac{h'+d}{\frac{n}{k} - h} \left(g + \frac{m}{k}\right) + \frac{h+c}{\frac{n}{k} - h} \left(f + \frac{\ell}{k}\right) + \frac{h+c}{\frac{n}{k} - h} \left(g + \frac{m}{k}\right)' + \frac{h+e}{\left(\frac{n}{k} - h\right)^2} \left(h - \frac{n}{k}\right)' \left(g + \frac{m}{k}\right) \right]$$

$$+ v_y'' \left[ \left(g + \frac{m}{k}\right)\left(\frac{h+e}{\frac{n}{k} - h}\right) + (g - c) \right] .$$

Terms in $1/c^2$ are neglected, and it is noted that $4\pi\vec{j}_0 = \vec{\nabla}\times\vec{B}_0 = \vec{e}_z \frac{\partial B_0}{\partial z}$ and $\nabla p_0 = p_0' = j_0 \times B_0 = \frac{\vec{\nabla}(B_0^2)}{8\pi}$ . Upon substitution of the values for the symbols a through n, and after simplification, the general differential equation in $v_y$ is obtained.

$$0 = v_y \left(\omega' \rho_0 + \frac{m^2 B_0^2}{4\pi\omega}\right)$$

$$+ v_y' \frac{\left\{(m^2+2k^2)\frac{B_0^2}{4\pi\omega} + \omega\rho_0\right\}\left(\frac{B_0^2}{4\pi\omega} + \frac{\rho_0\omega}{\omega^2/v_s^2 + m^2}\right)}{-(k^2+m^2+\frac{\omega^2}{v_s^2})\frac{(\rho_0\omega)}{\omega^2/v_s^2+m^2} - (m^2+k^2)\frac{B_0^2}{4\pi\omega}}$$

$$+\left(\frac{m^2B_0^2}{4\pi\omega} + \rho_0\omega\right)\frac{d}{dy}\left\{\frac{v_y'\left(\frac{B_0^2}{4\pi\omega} + \frac{\rho_0\omega}{\omega^2/v_s^2+m^2}\right)}{-(k^2+m^2+\frac{\omega^2}{v_s^2})\frac{\rho_0\omega}{\omega^2/v_s^2+m^2} - (m^2+k^2)\frac{B_0^2}{4\pi\omega}}\right\}$$

(A3)

The final form for $v_y$ is:

$$v_y\left(\frac{m^2B_0^2}{4\pi\omega} + \rho_0\omega\right) = \left(\frac{m^2B_0^2}{4\pi\omega} + \rho_0\omega\right)$$

$$\frac{d}{dy}\left\{\frac{v_y'\left(\frac{B_0^2}{4\pi\omega} + \frac{\rho_0\omega}{\omega^2/v_s^2+m^2}\right)}{\left(k^2+m^2+\frac{\omega^2}{v_s^2}\right)\frac{\rho_0\omega}{\omega^2/v_s^2+m^2} + (m^2+k^2)\frac{B_0^2}{4\pi\omega}}\right\}$$

$$+\left(\frac{B_0^2}{4\pi\omega} + \frac{\rho_0\omega}{\frac{\omega^2}{v_s^2}+m^2}\right)v_y'\frac{\left[(m^2+2k^2)\frac{B_0^2}{4\pi\omega} + \omega\rho_0\right]'}{\left(k^2+m^2+\frac{\omega^2}{v_s^2}\right)\frac{\rho_0\omega}{\frac{\omega^2}{v_s^2}+m^2} + (m^2+k^2)\frac{B_0^2}{4\pi\omega}}$$

(A4)

## B. Relevent Numerical Data

| Quantity | Value |
| --- | --- |
| $B_0$ | $2.6 \times 10^4$ gauss to $-2.6 \times 10^4$ gauss |
| $p_0$ | 300 microns of Hg. (uncompressed) |
| $\rho_0$ | $6 \times 10^{-8}$ gm/cm$^3$ (uncompressed) |
| $v_s$ (sound speed) | $1.4 \times 10^7$ cm/sec. |
| n | $1.8 \times 10^{16}$ atoms/cm$^3$ (uncompressed) |

## C. Method of Solving Equation (5)

Given the differential equation

$$\left[w^2 + m^2 \frac{f(Y)}{g(Y)}\right] v(Y) = \frac{d\,v(Y)}{d(Y)} \left[\frac{f'(Y)}{g(Y)} + c^2 \frac{g'(Y)}{g(Y)}\right] + \frac{d^2 v(Y)}{dY^2} \left[\frac{f(Y)}{g(Y)} + c^2\right], \quad (5)$$

which is of the form

$$\lambda\, v(Y) = -R(Y)\, v(Y) + p(Y)\, v'(Y) + q(Y)\, v''(Y),$$

replace

$$v''(Y) \text{ by } \frac{v(Y_{i+1}) - 2v(Y_i) + v(Y_{i-1})}{h^2}$$

and

$$v'(Y) \text{ by } \frac{v(Y_{i+1}) - v(Y_{i-1})}{2h}$$

The equation of motion thus becomes:

$$\lambda v(Y_i) = - R(Y_i)\, v(Y_i) + p(Y_i)\, \frac{v(Y_{i+1}) - v(Y_{i-1})}{2h}$$

$$+ q\,(Y_i)\left( v\, \frac{(Y_{i+1} - 2\, v(Y_i) - v(Y_{i-1})}{h^2} \right) \qquad \text{(C1)}$$

In matrix form, Eq. (C1) is:

$$\lambda v(Y_i) = \Big(0 \cdots \frac{-p(Y_i)}{2h} + \frac{q(Y_i)}{h^2}$$

$$- \frac{R(Y_i) - 2q(Y_i)}{h^2} \qquad \frac{p(Y_i)}{2h} + \frac{q(Y_i)}{h^2} \cdots 0\Big) \begin{pmatrix} v(Y_{i-1}) \\ v(Y_i) \\ v(Y_i+1) \end{pmatrix}$$

The problem now consists of solving the matrix equation $\lambda \vec{v} = A\vec{v}$. Write

$$A = \frac{1}{h^2} \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & \\ 0 & c_2 & a_3 & b_3 & \\ c & 0 & c_3 & a_4 & \\ & & & & \\ 0 & & & c & c_{n-1} \quad a_n \end{pmatrix}$$

with the coefficients

$$a_1 = -2q(Y_1) - h^2 R(Y_1) \qquad b_1 = \frac{h}{2} p(Y_1) + q(Y_1) \qquad c_1 = -\frac{h}{2} p(Y_2) + q(Y_2)$$

$$a_2 = -2q(Y_2) - h^2 R(Y_2) \qquad b_2 = \frac{h}{2} p(Y_2) + q(Y_2) \qquad c_2 = -\frac{h}{2} p(Y_3) + q(Y_3)$$

$$a_n = -2q(Y_n) - h^2 R(Y_n) \qquad b_n = \frac{h}{2} p(Y_n) + q(Y_n) \qquad c_{n-1} = -\frac{h}{2} p(Y_n) + q(Y_n)$$

The determinantal equation $|A - \lambda| = 0$ was solved by Sturm's Theorem.[3]

## D. Energy Principle Application

The energy principle can be used to obtain an approximation to the frequency of the plasma oscillations. We employ the expression:

$$\omega^2 = \frac{-\delta W}{\frac{1}{2}\int \rho \xi^2 \tau}$$ (see Ref. 4, p. 207, formula 4.19), where δw

is the change in potential energy under an infinitesimal displacement $\vec{\xi}$ of the plasma from its equilibrium position.

To calculate $\omega^2$ several assumptions are made:

1. during the compression the plasma is compressed into a region of thickness $2\delta$, where $\delta << r_0$, and $r_0$ is the mean radius of the plasma;
2. The density and pressure are uniform in the plasma, with vacuum outside the plasma;
3. The current is confined to the inner and outer sufaces of the plasma;
4. Walls and plasma have infinite conductivity;
5. Half the return current flows on the inner conductor and half on the outer conductor;
6. The plasma is assumed to be incompressible during perturbation, which implies that $\vec{\nabla} \cdot \vec{\xi} = 0$; and
7. For purposes of simplicity m and k are chosen to be zero. Furthermore, the infinitesimal displacement $\vec{\xi}$ is set equal to $\frac{\xi_0}{r}\hat{r}$ ($\xi_0$ constant), with no $\theta$ or z components.

The energy change $\delta W$[4] is given by $\delta W = \delta W_f + \delta W_s + \delta W_r$ where:

$$\delta W_f = \frac{1}{2}\int \left[\frac{Q^2}{4\pi} - \vec{J} \cdot (\vec{Q} \times \vec{\xi})\right] d\tau_f$$

$$\delta W_s = \frac{1}{2}\int ds\, (\vec{\xi} \cdot \vec{n})^2 \vec{n} \cdot \left[\nabla\left(p + \frac{B^2}{8\pi}\right)\right] =$$

$$\frac{1}{2}\int ds\, (\vec{\xi} \cdot \vec{n})^2 \vec{n} \cdot \left[\vec{B} \cdot \vec{\nabla}\, \vec{B}\right]$$

$\delta W_v = \frac{1}{8\pi} \int d\tau_v (\vec{\nabla} \times \vec{A})^2$, where $\vec{Q} \equiv \vec{\nabla} \times (\vec{\xi} \times \vec{B})$, $\vec{A}$ = perturbed vector potential $d\tau_f$ = volume element in the plasma, $d\tau_v$ = volume element in the vacuum, and $ds$ = vacuum-to-plasma surface element. Here we write $\vec{B}$, $\rho$, $\vec{J}$, as unperturbed quantities. Bracketed terms in $d_s$ indicate the jump of $\vec{B} \cdot \vec{\nabla} \vec{B}$ and $\vec{\nabla}\ (p + \frac{B^2}{8\pi})$ across the surfaces of the plasma.

For the Triax geometry, the expression $\vec{n} \cdot [\vec{B} \cdot \vec{\nabla} \vec{B}]$ becomes $\vec{n} \cdot [-\vec{n}\frac{B_\theta^2}{r}]$. In the plasma region $B_\theta$ varies as $1/r$ and can be considered constant because of assumption (1). In the vacuum region $B_\theta$ has the same magnitude at both plasma-to-vacuum surfaces. Hence the jump in $B_\theta^2$ is the same across the two surfaces, and we can apply assumption (1) to conclude that $\delta W_s$ vanishes.

The displacement $\vec{\xi}$ can be considered as constant in the plasma region, because of assumptions (1) and (7), while $B_\theta$ can be considered as constant in the plasma, as noted above. Therefore the integrand in $\delta W_s$ can be considered as constant, while the region of integration is of order $\delta$ in size. Consequently, $\delta W_f$ can be neglected.

To evaluate $\delta W_v$, use the condition that the fluxes internal and external to the plasma region must be conserved. Writing $B_e = b_e/r$ for the field in the region $r > r_0$, the external flux

$\Phi_e$ is $\Phi_e = \int_{r_0}^{r_e} B_\theta \, drdz = Lb_e \log \frac{r_e}{r_0}$ (+) where $r_e$ is the radius of the outer conductor. The change in flux equals zero, and is given by

$$\delta \Phi_e = L \left\{ \delta b_e \log \frac{r_e}{r_0} - \frac{b_e}{r_0} \frac{\xi_\theta}{r} \right. = 0.$$

Now we substitute $\vec{\nabla} \times \vec{A} = \delta \vec{B} = \hat{e}_\theta \frac{\delta b_e}{r}$ into $\delta W_v \big|_{ext}$ which is the energy change in the region $r > r_0$. The result is:

$$\delta W_v \big|_{ext} = \int_{r_0}^{r_e} d\tau_v \frac{(\nabla \times A)^2}{8\pi} = \frac{1}{8\pi} \int_{r_0}^{r_e} dz\ 2\pi r dr \left(\frac{\delta b_e}{r}\right)^2$$

$$= \frac{L}{4} \left(\frac{b_e}{r_0}\right)^2 \frac{\xi_0^2}{r_0^2} \left[\log \frac{r_e}{r_0}\right]^{-1}$$

where $\xi_0/r = \xi_0/r_0$ because of assumption (1).

By means of identical arguments; setting the variation in internal flux equal to zero, we obtain an expression for the energy change $\delta W_v \Big|_{int}$ in the region $r < r_0$. This result is:

$$\delta W_v \Big|_{int} = \int_{r_i}^{r_0} \frac{(\nabla \times A)^2}{8\pi} = \frac{L}{4} \frac{\xi_0^2}{r_0^2} \left(\frac{b_i}{r_0}\right)^2 \left[\log \frac{r_0}{r_i}\right]^{-1} \quad \text{where}$$

$\xi_0/r = \xi_0/r_0$, $B_\theta$ for $r < r_0$ is $b_i/r$, and $r_i$ is the radius of the inner conductor. Thus the total energy change $\delta W = \delta W_v \Big|_{int} + \delta W_v \Big|_{ext}$ becomes:

$$\frac{L}{4} \frac{\xi_0^2}{r_0^2} \beta^2 \left[\frac{1}{\log \frac{r_e}{r_0}} + \frac{1}{\log \frac{r_0}{r_i}}\right] \quad \text{(D2)}$$

Here we have written $\beta$ for $b_i/r_0$ and $b_e/r_0$.

Substituting Eq. (D2) into Eq. (D1), the angular frequency of oscillation squared becomes:

$$-\omega^2 = \frac{L}{4} \beta^2 \frac{\xi_0^2}{r_0^2} \left[\frac{1}{\log \frac{r_e}{r_0}} + \frac{1}{\log \frac{r_0}{r_i}}\right] \Big/ \frac{1}{2} \int \rho\, \xi^2\, d\tau =$$

$$\frac{\beta^2 \left[\frac{1}{\log \frac{r_e}{r_0}} + \frac{1}{\log \frac{r_0}{r_i}}\right]}{2\pi\rho \left[r_e^2 - r_i^2\right]}.$$

This angular frequency of oscillation is imaginary, indicating stable oscillations. Substituting the appropriate Triax data for $\beta$, $\rho$, $r_0$, $r_e$, and $r_i$, we obtain $\omega^2 = -1.47 \times 10^{14}$ (rad/sec)$^2$

The frequency $f = \frac{|\omega|}{2\pi}$ is then $1.93 \times 10^6$ cycles/second.

If the dimensionless frequency of 0.47 listed on Table I of this report is multiplied by the value of $\frac{\beta^2}{4\pi\rho_m}$ as applicable to the Triax, f is found to be $2.2 \times 10^6$ cycles/second.

Using the above simplified pinch model, the energy principle method and the perturbation approach thus give substantially the same plasma oscillation frequency for the $m = k = 0$ mode.

## References

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