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FOUNDATIONS OF S-MATRIX THEORY II. MACROCAUSALITY

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FOUNDATIONS OF S-MATRIX THEORY II. MACROCAUSALITY*

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ABSTRACT

This is the second of a series of reports devoted to a systematic development of S-matrix theory. This report describes the macrocausality property, which is a formulation the condition that the interactions of longest range are those carried by stable particles.

A. MACROCAUSALITY: THE PHYSICAL IDEA

Macrocausality is a mathematical formulation of the physical idea that momentum-energy is transferred over macroscopic distances only by stable particles. More precisely, it expresses the physical idea that any transfer of momentum-energy that cannot be ascribed to stable particles has a probability to occur that falls off exponentially under space-time dilation. This exponential fall-off property is akin to the requirement that potentials have Yukawa-type tails. However, it is formulated without reference to the concept of potentials, and encompasses, for example, transfers of momentum-energy associated with nonlocal interactions, or with unstable particles, or even with a breakdown of the concept of microscopic space-time.

Macrocausality is mathematically equivalent to a set of analytic properties. These analytic properties, called the normal analytic structure, are described in detail in the next chapter. Briefly stated they are this: the physical-region scattering functions are analytic except on positive-α Landau surfaces, where they are specified limits of an analytic function. This normal analytic structure is the primary analytic structure in S-matrix theory. Its equivalence to macrocausality means it can be derived from a space-time causality property. Alternatively, the space-time causality property can be derived from the analytic property. No commitment is made here on the question of which of these two mathematically equivalent properties is more basic.

The general idea of macrocausality in more detail is this: Consider a many-particle scattering process in which the initial and final particles are represented by ensembles of trajectories that lie in well-collimated beams. The scattering transition probability is expected to be small if none of these beams ever pass close to any of the others. If they do not remain far apart then they may all come together in a single space-time "reaction region," as illustrated in Fig. IIA.1.

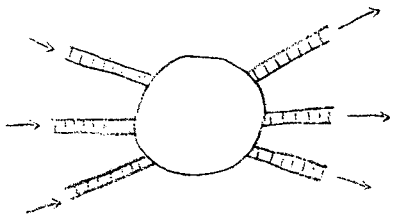

Fig. IIA.1. All the beams may come together in a single space-time reaction region.

Alternatively, the beams may come together in several different space-time reaction regions, as illustrated in Fig. IIA.2.

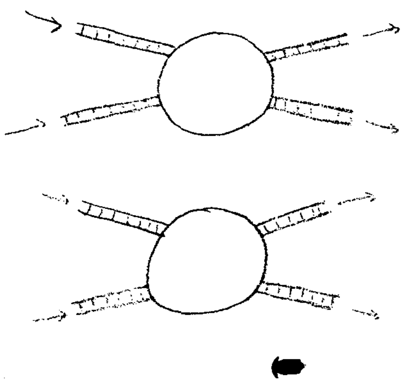

Fig. IIA.2. The beams may come together in several different space-time reaction regions.

If there are several different reaction regions then the scattering transition probability is expected to be small unless momentum-energy can be conserved separately by the particles associated with the separate reaction regions, unless, alternatively, the excess incoming momentum-energy of the earlier reactions can be transferred from the earlier reaction regions to later reaction regions by stable particles, as illustrated in Figs. IIA.3 and IIA.4.

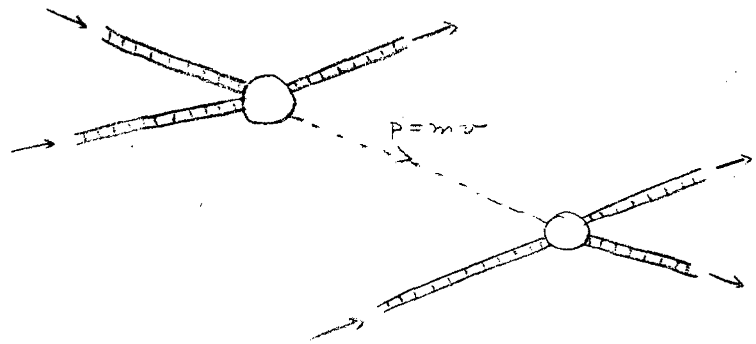

Fig. IIA.3. The space-time locations of the reaction regions are such that the excess momentum-energy of the earlier reaction can be transferred from the earlier reaction region to the later reaction region by a stable particle.

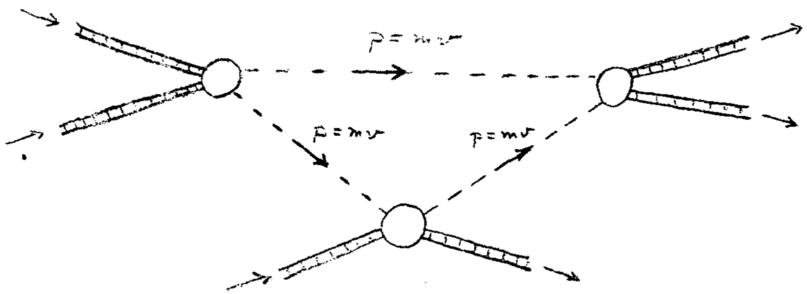

Fig. IIA.4. The space-time locations of the reaction regions are such that the momentum-energy of the initial particles can be transferred to the final particles by a network of stable particles.

Transfers of momentum-energy associated with stable particles are characterized by the relationship p = mv: the momentum-energy carried by a stable particle is proportional to its space-time velocity v, and the proportionality factor m is the mass of the stable particle.

If one considers different space-time positions of the various initial and final beams then there will be certain positions such that the momentum-energy of the initial particles can be transferred to the final particles by a space-time network of stable particles. However, there will be other positions for which no such network exists. Macrocausality asserts that the transition probability is small in these latter cases, and in fact falls off exponentially under certain appropriate space-time dilations. This exponential fall off arises from the necessary occurrence in this case of a transfer of momentum-energy that cannot be ascribed to stable particles, coupled

with the assumed exponential fall off under space-time dilation of the probability for such a transfer to occur.

To formulate this idea carefully certain properties of the functions representing the initial and final particles are needed. These properties, which are generalizations of Ruelle's lemma[1] on the fall-off properties of space-time wave functions, are described in the next section.

B. SPACE-TIME FALL-OFF PROPERTIES

The mathematical formulation of macrocausality is based on certain fall-off properties of the functions that represent the initial and final particles. These fall-off properties are summarized in the propositions stated below. Proofs of the propositions are given in the appendices.

Some definitions are first introduced. The <u>space-time wave function</u> $\tilde{\psi}(x)$ is defined by Fourier transformation:

$$\tilde{\psi}(x) \equiv \int_p \frac{m}{p^0} \frac{d^3p}{(2\pi)^3} \psi(p)\, e^{-ipx} . \qquad (B.1)$$

Here $px = p^0x^0 - \vec{p}\cdot\vec{x}$, and p is a single mass-shell four-vector:

$$p \equiv (p^0,\vec{p}) \equiv \left((m^2 + \vec{p}\cdot\vec{p})^{\frac{1}{2}},\vec{p}\right) . \qquad (B.2)$$

The four-vector x is always real, and the four-vector p is real unless otherwise stated.

The <u>trajectory</u> $\Gamma(p)$ is the space-time straight line that passes through the origin and has direction p:

$$\Gamma(p) \equiv \{x = (x^0,\vec{x}): x = \alpha p;\ \alpha \text{ real}\} . \qquad (B.3a)$$

The <u>displaced trajectory</u> $\Gamma^u(p)$ is the space-time straight line that passes through the point u and has direction p:

$$\Gamma^u(p) \equiv \{x = (x^0,\vec{x}): x = u + \alpha p, \quad \alpha \text{ real}\} . \qquad (B.3b)$$

Clearly

$$\Gamma^u(p) = \Gamma^{u+\lambda p}(p) \qquad (B.3c)$$

for any real λ.

The <u>support of ψ</u> is the set of mass-shell points p such that $\psi(p) \neq 0$, plus the boundary of that set:

$$\text{supp}\ \psi \equiv \text{closure}\{p: \psi(p) \neq 0\} . \qquad (B.4)$$

The <u>velocity cone</u> $V(\psi)$ is the set of points lying on trajectories $\Gamma(p)$ that pass through the support of ψ:

$$V(\psi) \equiv \{x: x\epsilon\Gamma(p),\ p\epsilon \text{ supp } \psi\} . \qquad (B.5)$$

A typical velocity cone is shown in Fig. IIB.1.

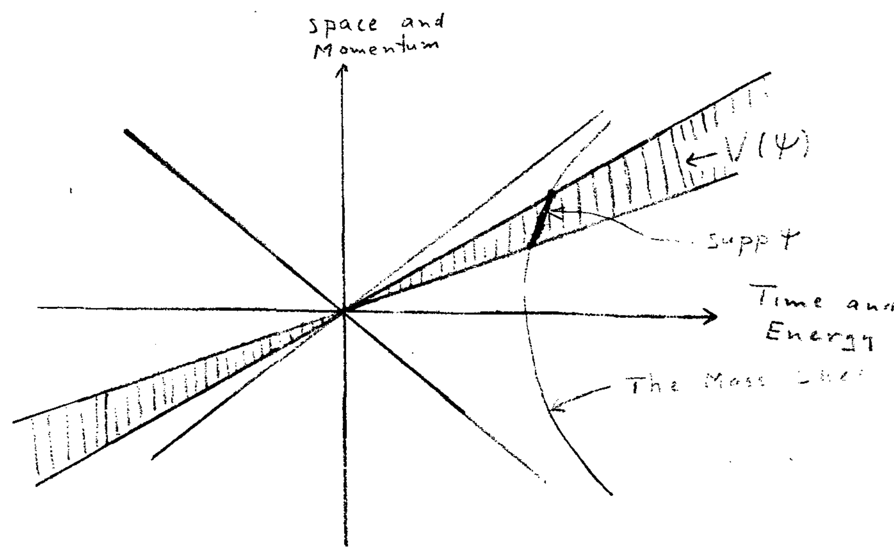

Fig. IIB.1. A typical velocity cone $V(\psi)$.

Let V be any union of space-time trajectories $\Gamma(p)$. Then the set V^u is the set of displaced trajectories $\Gamma^u(p)$:

$$V^u \equiv \{\Gamma^u(p): \Gamma(p) \subset V\} . \qquad (B.6)$$

A typical V^u is shown in Fig. IIB.2.

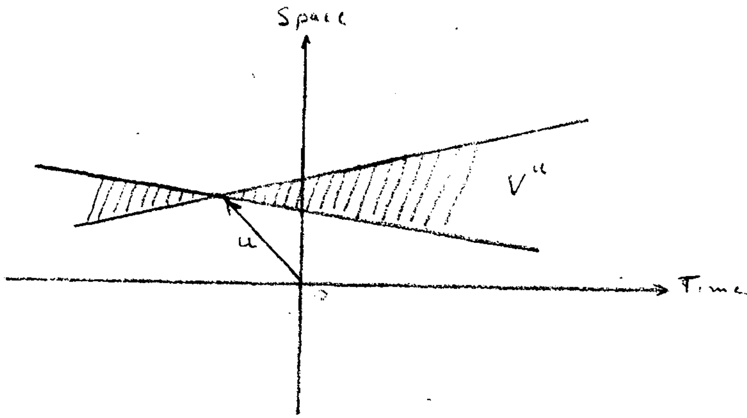

Fig. IIB.2. A typical V^u.

Let V be as above. Then for any $\epsilon > 0$ the set V_ϵ^u consists of all points $\hat{x}$ that lie either in the open ball of Euclidean radius ϵ centered at u, or on some trajectory $\Gamma^u(p')$ that lies at distance $|p - p'| < \epsilon$ from some trajectory $\Gamma^u(p)$ that lies in V^u:

$$V_\epsilon^u \equiv \{\hat{x}: |\hat{x} - u| < \epsilon\} \bigcup \{\hat{x}: \hat{x} \in \Gamma^u(p'), \Gamma^u(p) \subset V^u, \quad |p' - p| < \epsilon\} . \tag{B.6}$$

Here $|a| \equiv \left(\sum (a^u)^2\right)^{\frac{1}{2}}$ represents Euclidean distance. A typical V_ϵ^u is shown in Fig. IIB.3.

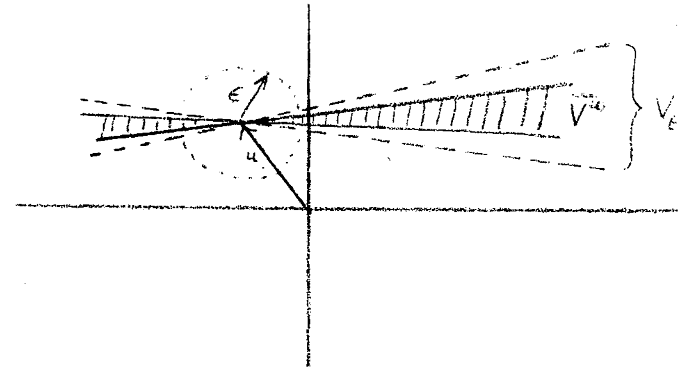

Fig. IIB.3. A typical V_ϵ^u.

The complement of V_ϵ^u is denoted by $\not{C}\, V_\epsilon^u$:

$$\not{C}\, V_\epsilon^u \equiv \{\hat{x}: \hat{x} \notin V_\epsilon^u\} . \tag{B.7}$$

If V is the single trajectory $\Gamma(p)$ then V_ϵ^u is called $\Gamma_\epsilon^u(p)$. The set $\not{C}\, \Gamma_\epsilon^u(p)$ is the complement of $\Gamma_\epsilon^u(p)$.

The wave functions used in the formulation of macrocausality have the following form:[2]

$$\psi(p; P,\gamma\tau) \equiv \chi(\vec{p}) \exp[-(\vec{p} - \vec{P})^2 \gamma\tau] . \tag{B.8a}$$

The function $\chi(\vec{p})$ in (B.8a) is taken to be a C^∞ function of compact support that is analytic at $\vec{p} = \vec{P}$. That is, there is some bounded real neighborhood N in $\vec{p}$ space such that: (1) $\chi(\vec{p})$ vanishes for all $\vec{p}$ not in N, and (2), all partial derivatives of $\chi(\vec{p})$ of all orders are bounded and continuous on N. Moreover, there is an open Euclidean sphere $\mathcal{N}$ in complex $\vec{p}$ space, centered at $\vec{p} = \vec{P}$, such that $\chi(\vec{p})$ is analytic at all points $\vec{p}$ in $\mathcal{N}$. The function $\chi(\vec{p})$ is assumed to be normalized so that $|\chi(\vec{p})| \leq 1$ for all $\vec{p}$ in the union of N with $\mathcal{N}$:

$$|\chi(\vec{p})| \leq 1 \quad \text{for } \vec{p} \text{ in } N \cup \mathcal{N}. \tag{B.8b}$$

The space-time wave function corresponding to (B.8a) is defined by (B.1):

$$\tilde{\psi}(x; P,\gamma\tau) = \int \frac{m}{p^0} \frac{d^3p}{(2\pi)^3} \psi(p; P,\gamma\tau) e^{-ixp} . \tag{B.8c}$$

The corresponding displaced wave function is

$$\tilde{\psi}^{u}(x; P,\gamma\tau) \equiv \tilde{\psi}(x - u; P,\gamma\tau) . \tag{B.8d}$$

In momentum-space this gives

$$\psi^{u\tau}(p; P,\gamma\tau) = \chi(\vec{p}) \exp[-(\vec{p} - \vec{P})^2\gamma + ip\cdot u]\tau . \tag{B.8e}$$

The four-vector P in (B.8) is the mass-shell four-vector

$$P \equiv (P^0,\vec{P}) \equiv \left((m^2 + \vec{P}\cdot\vec{P})^{\frac{1}{2}},\vec{P}\right) , \tag{B.9}$$

where m is the mass of the particle represented by $\psi(p; P,\gamma\tau)$.

One may now state:

<u>Proposition IIB.1</u>. Let $\psi \equiv \psi(p; P,\gamma\tau)$ be as defined in (B.8). Then for any $\epsilon > 0$ there is a set of positive constants C, α, and γ_0 such that

$$|\tilde{\psi}^{u\tau}(\hat{x}\tau; P,\gamma\tau)| \leq C e^{-\alpha\gamma\tau} \tag{B.10a}$$

for all $\tau \geq 0$, all γ satisfying $0 \leq \gamma \leq \gamma_0$, and all $\hat{x}$ in $\not\subset \Gamma_\epsilon^{u}(P)$. Moreover, for each integer $n \geq 0$ there is a constant C_n such that

$$|\tilde{\psi}^{u\tau}(x\tau; P,\gamma\tau)| \leq \frac{C_n e^{-\alpha\gamma\tau}}{|1 + \tau|^n} \tag{B.10b}$$

for all $\tau \geq 0$, all γ satisfying $0 \leq \gamma \leq \gamma_0$, and all $\hat{x}$ in $\not\subset V_\epsilon^{u}(\psi)$.

Equation (B.10a) says that $\tilde{\psi}^{u\tau}(\hat{x}\tau; P,\gamma\tau)$, considered as a function in $\hat{x}$ space, falls exponentially to zero, as $\tau \to \infty$, uniformly on the complement of $\Gamma_\epsilon^{u}(P)$. Equation (B.10b) says that this fall-off is moreover rapid (i.e., faster than any inverse power of τ) on the complement of $V_\epsilon^{u}(\psi)$. [This rapid fall-off is not encompassed by the exponential fall-off because γ is allowed to be any number in the interval $0 \leq \gamma \leq \gamma_0$, including zero.]

Inspection of (B.8a) shows that similar--though stronger--properties hold in p space: for any $\epsilon > 0$ there is a pair of positive constants C and α such that

$$|\psi^{u\tau}(p; P,\gamma\tau)| \leq C e^{-\alpha\gamma\tau} \tag{B.10c}$$

for all p such that $\Gamma^{u}(p)$ enters $\not\subset \Gamma_\epsilon^{u}(P)$. Moreover,

$$|\psi^{u\tau}(p; P,\gamma\tau)| = 0 \tag{B.10d}$$

for all p such that $\Gamma^{u}(p)$ enters $\not\subset V_\epsilon^{u}(\psi)$.

The phase-space density functions $w(p,x)$ discussed in Chapter I exhibit similar fall-off properties in both x- and p-space:

<u>Proposition IIB.2</u>. Let $\psi \equiv \psi(p; P,\gamma\tau)$ be as defined in (B.8). Let

$$w(p,x;\, P,\gamma\tau) \;=\; \int \rho(Mv - \tfrac{1}{2}\,q;\; Mv + \tfrac{1}{2}\,q)\; e^{-iq\cdot x}$$

$$\times \left(\frac{M}{m}\right)^{\frac{1}{2}} 2\pi\;\delta(q\cdot v)\;\frac{d^4 q}{(2\pi)^4}\;, \qquad \text{(B.11a)}$$

where $v \equiv p/m$, $M \equiv M(q) \equiv (m^2 - \frac{1}{4}\,q^2)^{\frac{1}{2}}$, and

$$\rho(p'';\, p') \;\equiv\; \psi^*(p'';\, P,\gamma\tau)\;\psi(p';\, P,\gamma\tau)\;. \qquad \text{(B.11b)}$$

Then for any $\epsilon > 0$ there is a set of positive constants C, α, and γ_0 such that

$$|w^{u\tau}(p,\hat{x}\tau;\, P,\gamma\tau)| \;\le\; C\; e^{-\alpha\gamma\tau} \qquad \text{(B.12a)}$$

for all $\tau \ge 0$, all γ satisfying $0 \le \gamma \le \gamma_0$, and for all $(p,\hat{x})$ such that either $\hat{x}$ lies in $\not{c}\,\Gamma_\epsilon^{\,u}(P)$ or $\Gamma^u(p)$ enters $\not{c}\,\Gamma_\epsilon^{\,u}(P)$. Moreover, for each integer $n \ge 0$ there is a constant C_n such that

$$|w^{u\tau}(p,\hat{x}\tau;\, P,\gamma\tau)| \;\le\; \frac{C_n\; e^{-\alpha\gamma\tau}}{(1+\tau)^n} \qquad \text{(B.12b)}$$

for all $\tau \ge 0$, all γ satisfying $0 \le \gamma \le \gamma_0$, and all $(p,\hat{x})$ such that either $\hat{x}$ lies in $\not{c}\,V_\epsilon^{\,u}(\rho)$ or $\Gamma^u(p)$ enters $\not{c}\,V_\epsilon^{\,u}(\rho)$. Here

$$V(\rho) \;\equiv\; \{\Gamma(mv)\colon \Gamma(Mv - \tfrac{1}{2}\,q) \text{ and } \Gamma(Mv + \tfrac{1}{2}\,q) \text{ both lie in } V(\psi) \text{ for some } q \text{ satisfying } q\cdot v = 0\}. \qquad \text{(B.13)}$$

The set $V(\rho)$ includes $V(\psi)$, but is in general larger than $V(\psi)$.

In the formulation of macrocausality one considers scattering processes in which each of the n initial and final particles is represented by a wave function of the form (B.8e). Equation (I.E.4) expresses the scattering transition probability as an integral over the various possible positions and directions of the trajectories $\Gamma^{x_j}(p_j)$ corresponding to the n initial and final particles:

$$\mathcal{P}(P,u\tau,\gamma\tau)$$

$$\equiv \int S(p,x) \prod_{j=1}^{n} \left[w^{u_j\tau}(p_j,x_j;\, P_j,\gamma\tau)\; d^3x_j\;\frac{d^3p_j}{(2\pi)^3} \right]\;. \qquad \text{(B.14)}$$

Here

$$P \;=\; (P_1,\cdots,P_n) \qquad \text{(B.15a)}$$

and

$$u \;=\; (u_1,\cdots,u_n)\;. \qquad \text{(B.15b)}$$

The times $x_j^{\,0}$ in (B.14) can be chosen arbitrarily.

The three-vector variables $\vec{x}_j$ and $\vec{p}_j$ in (B.14) specify the possible positions and directions of the trajectories $\Gamma_j^{\,x_j}(p_j)$. And an integration over $(\vec{p}_j,\vec{x}_j)$ space represents, therefore, an integration over the possible locations of the trajectories $\Gamma^{x_j}(p_j)$. The final, and most important, proposition is an integrated version of Proposition IIB.2:

<u>Proposition IIB.3</u>. Let u be any four-vector and let P be any mass-shell four-vector. Let $R(P,u)$ be the entire six dimensional $(\vec{p},\hat{\vec{x}})$ space minus some arbitrarily small neighborhood of the point $(\vec{P},\vec{u})$.

Let $R(P,u;\tau)$ be the image of $R(P,u)$ in $(p,\ x \equiv \hat{x}\tau)$ space:

$$R(P,u;\ \tau) = \{(p,x):\ (p,x/\tau) \in R(p,u)\}\ . \tag{B.17}$$

Let $w^{u\tau}(p,x;\ P,\gamma\tau)$ be as defined in Proposition IIB.2. Then there is a set of positive constants C, α, and γ_0 such that

$$\int_{R(P,u;\ \tau)} \frac{d^3p}{(2\pi)^3}\, d^3x \left| w^{u\tau}(p,x;\ P,\gamma\tau)\right|_{x^0=u^0\tau} \leq C\, e^{-\alpha\gamma\tau} \tag{B.18a}$$

for all $\tau \geq 0$ and all γ satisfying $0 \leq \gamma \leq \gamma_0$. Moreover, let $R(\rho,u)$ be the entire $(\vec{p},\vec{\hat{x}})$ space minus some open set that contains all points $(\vec{p},\vec{\hat{x}})$ such that $\Gamma^{(u_0,\vec{\hat{x}})}(p)$ lies in $V^u(\rho)$. Let $R(\rho,u;\ \tau)$ be the $x = \hat{x}\tau$ space image of $R(\rho,u)$. Then for any integer $n \geq 0$ there is a constant C_n such that

$$\int_{R(\rho,u;\ \tau)} \frac{d^3p}{(2\pi)^3}\, d^3x \left| w^{u\tau}(p,x;\ P,\gamma\tau)\right|_{x^0=u^0\tau} \leq \frac{C_n\, e^{-\alpha\gamma\tau}}{(1+\tau)^n} \tag{B.18b}$$

for all $\tau \geq 0$ and all γ satisfying $0 \leq \gamma \leq \gamma_0$.

The bound (B.18) plays a central role in the formulation of macrocausality. It says that the probability defined by $w^{u\tau}(p,\hat{x}\tau;\ P,\gamma\tau)$ collapses in $\hat{x}$ space exponentially onto the classical trajectory $\Gamma^u(P)$. [See (E.1) of Chapter I.] This collapse onto a classical trajectory does not conflict with the uncertainty principle. The momentum space width is $\Delta_p = (\gamma\tau)^{-\frac{1}{2}}$, and the coordinate space width is $\Delta_x = (\gamma\tau)^{\frac{1}{2}}$. Thus the width in x space increases like $\tau^{\frac{1}{2}}$ to compensate for the decreasing width in p space. However, in $\hat{x} \equiv x/\tau$ space the width decreases like $\tau^{-\frac{1}{2}}$ just as the width in p space does, and the probability for the particle to lie

on any set of trajectories in $\hat{x}$ space that does not include the classical trajectory $\Gamma^u(P)$ as a limit point drops exponentially to zero as $\tau \to \infty$.

This conclusion that the probability collapses exponentially onto the trajectory $\Gamma^u(P)$ in $\hat{x}$ space is based on the properties of the phase-space density function $w^{u\tau}(p,\hat{x}\tau;\ P,\gamma\tau)$. However, the same conclusion would follow from the properties of the functions $|\psi^{u\tau}(p;\ P,\gamma\tau)|^2$ and $|\tilde{\psi}^{u\tau}(\hat{x}\tau;\ P,\gamma\tau)|^2$, if these are interpreted as p-space and $\hat{x}\tau$-space probability densities. The conclusion is therefore not simply a consequence of some special or particular method of ascribing physical significance to the quantities of the mathematical formalism. It will follow from any reasonable method of interpretation.

If each of the initial and final particles of a scattering process is represented by a wave function of the shrinking gaussian form (B.8), then each of the initial and final particles will be associated in the limit $\tau \to \infty$ with a corresponding $\hat{x}$-space trajectory $\Gamma^{u_j}(P_j)$. Macrocausality is formulated as a fall-off condition on the corresponding transition probability, under certain conditions on the locations of these trajectories. The crucial point is that any transfer of momentum-energy over a <u>finite</u> interval in $\hat{x}$ space becomes for large τ a transfer over a macroscopic space-time distance. Hence the probability for this transfer to occur must, according to the physical idea of macrocausality, fall off exponentially as $\tau \to \infty$, unless it can be ascribed to a stable particle.

C. CAUSAL DIAGRAMS

Certain space-time diagrams called <u>causal diagrams</u> are used in the formulation of macrocausality. These diagrams depict classical multiple-scattering processes in which point particles react at point vertices. Figure IIC.1 is an illustration.

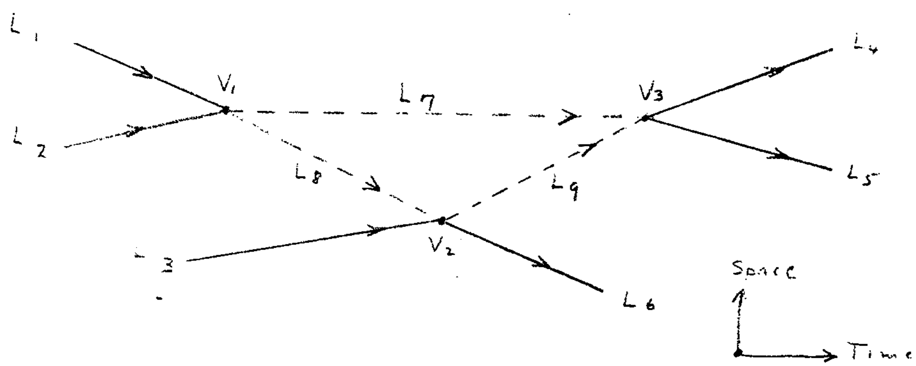

Fig. IIC.1 A causal diagram with initial lines L_1, L_2, and L_3, final lines L_4, L_5, and L_6, and internal lines L_7, L_8, and L_9. The ends of these lines lie in the set of vertices V_1, V_2, and V_3, or at $\pm\infty$.

In general, a causal diagram D consists of a set of external lines $\{L_j,\ j \in \text{Ext}\}$, a set of internal lines $\{L_j,\ j \in \text{Int}\}$, and a set of vertices $\{V_r,\ r \in \text{Ver}\}$. Each line L_j is associated with a type index t_j, hence a mass m_j. Each line L_j lies on a trajectory $\Gamma^{u_j}(p_j)$. It is thereby associated with a momentum-energy vector $p_j = m_j\, v_j$. Each internal line L_i originates at a point L_i^- that coincides with some vertex V_r, and terminates at a point L_i^+ that coincides with some other vertex V_r. The external lines L_j are

either initial lines for final lines. Each initial line L_j originates at $-\infty$, and terminates at a point L_j^+ that coincides with some vertex V_r. Each final line L_j originates at a point L_j^- that coincides with some vertex v_r, and terminates at $+\infty$. Momentum-energy conservation is required at each vertex.

The topological structure of a causal diagram D is defined by the set of t_j and by the set of coefficients ϵ_{jr} defined by the equations

$$\epsilon_{jr} = \begin{cases} +1 & \text{if } L_j^+ = V_r, \\ -1 & \text{if } L_j^- = V_r, \\ 0 & \text{otherwise}. \end{cases} \qquad \text{(C.1)}$$

In terms of these ϵ_{jr} the requirement of momentum-energy conservation is

$$\sum_j p_j\, \epsilon_{jr} = 0 \qquad \text{all } r \in \text{Ver}. \qquad \text{(C.2)}$$

If the vector from some arbitrary origin to the vertex V_r is w_r, then the condition that the internal particle i move in the direction of the momentum-energy it carries [i.e., $v_i = p_i/m_i$] is

$$\sum_r \epsilon_{ir}\, w_r \equiv \Delta_i = \alpha_i p_i \qquad \text{all } i \in \text{Int}, \qquad \text{(C.3)}$$

where α_i is a positive real number:

$$\alpha_i > 0 \qquad \text{all } i \in \text{Int}. \qquad \text{(C.4)}$$

The vector p_i in (C.3) satisfies the mass-shell constraint

$$p_i^0 = (m_i^2 + \vec{p}_i \cdot \vec{p}_i)^{\frac{1}{2}}, \qquad i \in \text{Int} \tag{C.5a}$$

where m_i is the mass of the stable particle corresponding to line i. The external lines also satisfy the mass-shell constraint:

$$p_j^0 = (m_j^2 + \vec{p}_j \cdot \vec{p}_j)^{\frac{1}{2}} \qquad j \in \text{Ext} . \tag{C.5b}$$

Consider now a fixed but arbitrary scattering process with a total of n initial and final particles. Let

$$p \equiv (p_1, \cdots, p_n) \tag{C.6a}$$

represent a set of n mass-shell momentum-energy vectors, one corresponding to each of the n initial and final particles. Let

$$u \equiv (u_1, \cdots, u_n) \tag{C.6b}$$

be a corresponding set of n space-time displacement vectors. The following definitions are then introduced:

<u>Definition IIC.1</u>. The set (p,u) is <u>causal</u> if and only if there is a causal diagram D such that the initial and final lines L_j of D can be placed in one-to-one correspondence with initial and final particles of the scattering process, and hence also with the corresponding pairs (p_j,u_j), and

$$L_j \subset \Gamma^{u_j}(p_j) \qquad \text{for all} \qquad j \in \text{Ext}. \tag{C.7}$$

<u>Definition IIC.2</u>. The <u>causal set $\mathcal{C}$</u> is the set of causal (p,u).

<u>Definition IIC.3</u>. The <u>causal set $\mathcal{C}(p)$</u> is the set of u such that (p,u) is causal:

$$\mathcal{C}(p) \equiv \{u: (p,u) \in \mathcal{C}\} . \tag{C.8}$$

The causal set $\mathcal{C}$ is defined to include the limit points corresponding to diagrams in which various parallel initial trajectories intersect at $-\infty$, or various parallel final trajectories intersect at $+\infty$. The causal set $\mathcal{C}$ is, by virtue of this stipulation, a closed set.

D. FORMULATION OF MACROCAUSALITY

Consider a scattering process in which the initial and final particles are represented by wave functions of the form (B.8):

$$\psi_j = \psi_j(p_j;\, P_j,\gamma\tau)$$
$$= \chi_j(\vec{p}_j)\exp -(\vec{p}_j - \vec{P}_j)^2\,\gamma\tau\,. \qquad \text{(D.1)}$$

The displaced wave functions are

$$\psi_j^{u_j\tau} = \chi_j(\vec{p}_j)\exp[-(\vec{p}_j - \vec{P}_j)^2\,\gamma\tau + i\,p_j\,u_j\,\tau]$$
$$= \psi_j^{u_j\tau}(p_j;\, P_j,\gamma\tau)\,. \qquad \text{(D.2)}$$

The corresponding transition probability is

$$\mathcal{P}[\{\psi_j^{u_j\tau}(p_j;\, P_j,\gamma\tau)\}] = |S[\{\psi_j^{u_j\tau(*)}(p_j;\, P_j,\gamma\tau)\}]|^2$$
$$\equiv \mathcal{P}(P,u\tau,\gamma\tau)\,, \qquad \text{(D.3)}$$

where

$$P = (P_1,\cdots,P_n) = \{P_j\}, \qquad j \in \text{Ext} \qquad \text{(D.4a)}$$

$$u = (u_1,\cdots,u_n) = \{u_j\}, \qquad j \in \text{Ext} \qquad \text{(D.4b)}$$

and $(*)$ is $*$ for final j and no star for initial j.

According to (E.4) of Chapter I, the transition probability $\mathcal{P}(P,u\tau,\gamma\tau)$ is a sum of contributions corresponding to the various possible configurations of the n initial and final trajectories $\Gamma^{\hat{x}_j}(p_j)$. Suppose (P,u) is not causal. Then, by virtue of the definition of causal, it is not possible to transfer the momentum-energy from initial particles lying on the initial trajectories $\Gamma^{u_i}(P_i)$ to

final particles lying on the final trajectories $\Gamma^{u_f}(P_f)$ by a network of stable particles. Moreover, the fact that the causal set $\mathcal{C}$ is closed implies that all points in some neighborhood of (P,u) are not causal. This means that there is a neighborhood of the configuration of n trajectories $\{\Gamma^{u_j}(P_j)\}$ such that for any configuration of n trajectories $\{\Gamma^{\hat{x}_j}(p_j)\}$ in this neighborhood it is not possible to transfer the momentum-energy from initial particles lying on the initial trajectories $\Gamma^{\hat{x}_i}(p_i)$ to final particles lying on the final trajectories $\Gamma^{\hat{x}_f}(p_f)$ by a network of stable particles. That is, for any configuration $\{\Gamma^{\hat{x}_j}(p_j)\}$ in this neighborhood there must be some transfer of momentum-energy that cannot be ascribed to a stable particle. As $\tau \to \infty$ the distance over which this transfer must carry becomes infinite.

The physical idea to be formalized by macrocausality is that any transfer of momentum-energy not ascribable to stable particles has a probability to occur that falls off exponentially under space-time dilation. This physical idea has in the present case a natural meaning: the contributions to $\mathcal{P}(P,u\tau,\gamma\tau)$ coming from configurations $\{\Gamma^{\hat{x}_j}(p_j)\}$ in some sufficiently small neighborhood of the configuration $\{\Gamma^{u_j}(P_j)\}$ should fall off exponentially under space-time dilation. On the other hand, the bound (B.18a) implies that the probability for the configuration $\{\Gamma^{\hat{x}_j}(p_j)\}$ to lie outside this neighborhood has a bound of the form $C\exp -\alpha\gamma\tau$. This latter type of bound is weaker than a simple exponential bound, because γ can be any number satisfying $0 \leq \gamma \leq \gamma_0$. In other words, a simple exponential bound implies a bound of the form $C\exp -\alpha\gamma\tau$ for all γ satisfying $0 \leq \gamma \leq \gamma_0$. Thus a bound of the form $C\exp -\alpha\gamma\tau$ should hold both for the contribution from configurations $\{\Gamma^{\hat{x}_j}(p\ )\}$ that lie inside some neighborhood of the configuration $\{\Gamma^{u_j}_{\hat{x}_j}(P_j)\}$ and also for the contribution from all configurations $\{\Gamma^{u_j}(p_j)\}$ that lie outside this neighborhood. Thus $\mathcal{P}(P,u\tau,\gamma\tau)$ itself should have a bound of the form $C \exp -\alpha\gamma\tau$ if (P,u) is not causal, provided the physical idea to be expressed by macrocausality is indeed valid.

This argument yields the following conclusion: if (P,u) is not causal then there is a set of positive constants C, α, γ_0 such that

$$\mathcal{P}(P,u\tau,\gamma\tau) \leq C e^{-\alpha\gamma\tau} \qquad (D.5)$$

for all γ satisfying $0 \leq \gamma \leq \gamma_0$. Moreover, if $\mathcal{U}$ is some closed bounded set in a u space that lies outside the causal set $\mathcal{C}(P)$ (see Definition IIC.3) then (D.5) should hold uniformly on $\mathcal{U}$, with C now the maximum of the C's for u in $\mathcal{U}$, and α and γ_0 the minimum of the α's and γ_0's for u in $\mathcal{U}$.

The bound (D.5) is obtained from arguments based on classical ideas. Indeed, it formalizes the requirement that certain classical ideas about the transfer of momentum-energy become valid in the macroscopic domain. This return to macroscopic classical concepts in the formulation of space-time properties stems from the nonoccurence in S-matrix theory, at the fundamental level, of any microscopic space-time description of the flow of momentum-energy, or other conserved quantities.

The limit $\tau \to \infty$ is closely connected to the classical limit $\hbar \to 0$. It has already been emphasized that the initial and final trajectories collapse as τ goes to infinity onto the classical trajectories $\Gamma^{u_j}(P_j)$ in the scaled space $\hat{x} = x/\tau$. On the other

hand, Planck's constant $\hbar$ enters S-matrix theory only as the constant that fixes the scale of physical space-time. The scale change induced by letting $\hbar \to 0$ is equivalent to the scale change induced by letting $\tau \to \infty$. Consequently, macrocausality can be viewed as a formulation of the "correspondence principle," which is the principle that classical ideas become valid in the limit $\hbar \to 0$.

The physical ideas that lead to (D.5) lead also to certain slightly stronger conditions, which express the idea that processes that occur in regions that are far apart are independent, at least within the statistical framework of quantum theory. The stronger version of (D.5) that incorporates this space-time cluster property is now discussed.

If (P,u) is causal then there is a corresponding causal diagram. In fact there may be several. Some or all of these diagrams may be disconnected: they may consist of several distinct subdiagrams that are not connected to each other.

Suppose the n external lines can be partitioned into a set of subgroups g_κ such that no causal diagram corresponding to the fixed (P,u) connects lines of one subgroup to lines of another subgroup. Let $\mathcal{P}_\kappa$ represent the transition probability corresponding to the subgroup g_κ alone. Then the physical ideas that lead to (D.5) lead also to the condition

$$\mathcal{P}(P,u\tau,\gamma\tau) - \prod_\kappa \mathcal{P}_\kappa(P^\kappa,u^\kappa\tau,\gamma\tau) \Longrightarrow 0 , \qquad (D.6)$$

where $\Longrightarrow 0$ represents a fall off of the form (D.5). Property (D.6) arises from the fact that transfers of momentum-energy between the different groups g_κ should fall off exponentially, as $\tau \to \infty$, and hence if dynamical connections are carried by transfers of momentum-energy then the transition probability should exponentially approach the product of the transtion probabilities of the individual subgroups.

The bounds (D.5) and (D.6) can be derived from explicit semi-classical models that incorporate the physical ideas described above. In these models the transition probabilities are assumed to be representable as a sum of contributions corresponding to different ways in which the momentum-energy of the initial particles can be transferred from the initial trajectories to the final trajectories. Elementary transfers can be ascribed either to stable particles or to other mechanisms. But any other mechanism is assumed to have a characteristic exponential damping $\exp -\beta\tau$ under space-time dilations.

The coefficient β of this exponential damping can depend on the momentum-energy p transferred by the mechanism, on the space-time direction v of the transfer, and on any other dilation independent variables. But for any closed region R in all these variables that includes no point satisfying $p = mv$, where m is some stable-particle mass, there is assumed to be a positive constant $\epsilon(R)$ such that $\beta \geq \epsilon(R) > 0$ throughout the region R. If it is assumed that a bound on $\mathcal{P}(P,u\tau,\gamma\tau)$ can be obtained by considering only a finite number of different possible ways of transferring the momentum-energy of the initial particles to the final particles then (D.5) and (D.6) follow from the general arguments given above. Macrocausality can therefore be regarded as the assertion that bounds of the type that hold for all semiclassical models of this kind hold also physically.

In the proof of the equivalence between macrocausality and the normal analytic structure the macrocausality condition is defined to be (D.6), plus two minor variations of (D.6) that express the same physical idea. These variations are now described.

The first variation asserts that the fall off is not only exponential, but also rapid, in the sense of (B.18b), provided u satisfies an extra condition. This extra condition is that (p,u) be noncausal for all p such that each $\Gamma(p_j)$ lies in the corresponding $V(\rho_j)$. Here $V(\rho_j)$ is defined by (B.19) with ψ_j in place of ψ.

To express this variation symbolically let $\mathcal{C}(\rho)$ be the union of the sets $\mathcal{C}(p)$ over those values of $p \equiv (p_1,\cdots,p_n)$ such that each $\Gamma(p_j)$ lies in the corresponding $V(\rho_j)$:

$$\mathcal{C}(\rho) \equiv \{u: u \in \mathcal{C}(p), \Gamma(p_j) \subset V(\rho_j) \text{ all } j\} . \qquad \text{(D.8)}$$

[The set $\mathcal{C}(p)$ is defined at the end of Section C.] If (B.18b) is used in place of (B.18a) then the arguments that give (D.5) give the following condition: for any closed bounded set $\mathcal{U}$ in the complement of $\mathcal{C}(\rho)$, and for any integer $n \geq 0$, there is a set of constants C_n, α, and γ_0 such that

$$\mathcal{P}(P,u\tau,\gamma\tau) \leq \frac{C_n e^{-\alpha\gamma\tau}}{|1+\tau|^n} \qquad \text{(D.9)}$$

for all $\tau \geq 0$, all γ satisfying $0 \leq \gamma \leq \gamma_0$, and all u in $\mathcal{U}$. In other words, if u is such that there is no causal diagram D with external lines $L_j \subset \Gamma^{u_j}(p_j) \subset V^{u_j}(\rho_j)$, then the transition probability $\mathcal{P}(P,u\tau,\gamma\tau)$ falls-off both exponentially and rapidly.

Moreover, this rapid, exponential fall-off is uniform on any closed, bounded set $\mathcal{U}$ of u's that satisfy this condition.

If u lies in $\mathcal{C}(\rho)$ then there must exist one or more causal diagrams D with external lines $L_j \subset L^{u_j}(p_j) \subset V^{u_j}(\rho_j)$. Some of these diagrams may be disconnected. Let κ label partitions of the set of external lines L_j into subgroups g_κ, and let $\mathcal{C}(\rho,g_\kappa)$ be the set of u such that there is some causal diagram D such that: (1) The external lines of D satisfy $L_j \subset \Gamma^{u_j}(p_j) \subset V^{u_j}(\rho_j)$, and (2) D connects lines from different subgroups g_κ. This set $\mathcal{C}(\rho,g_\kappa)$ is such that for all u in the complement of $\mathcal{C}(\rho,g_\kappa)$ there is no causal diagram D with external lines $L_j \subset \Gamma^{u_j}(p_j) \subset V^{u_j}(\rho_j)$ except those that do not link together any lines from different subgroups g_κ. If (B.18b) is used in place of (B.18a) then the argument that gives (D.6) gives the following condition: for any closed bounded set $\mathcal{U}$ in the complement of $\mathcal{C}(\rho,g_\kappa)$ and any integer $n \geq 0$ there is a set of constants C_n, α, and γ_0 such that

$$\mathcal{P}(P,u\tau,\gamma\tau) - \prod_{\kappa} \mathcal{P}_\kappa(P^\kappa,u^\kappa\tau,\gamma\tau) \leq \frac{C_n\, e^{-\alpha\gamma\tau}}{(1+\tau)^n} \tag{D.10}$$

for all $\tau \geq 0$, all γ satisfying $0 \leq \gamma \leq \gamma_0$, and all u in $\mathcal{U}$. This is the first variation of (D.6).

The second variation of (D.6) arises from allowing any one of the n wave functions $\psi_j(P_j;\, u_j\tau,\gamma\tau)$ to be a linear superposition $\lambda\, \psi'(p_j';\, u_j\tau,\gamma\tau) + \mu\, \psi''(P_j'';\, u_j\tau,\gamma\tau)$, where $|\lambda| + |\mu| = 1$. Then the arguments that give (D.6) now give (D.6) with the causal set $\mathcal{C}(P)$ replaced by $\mathcal{C}(P') \cup \mathcal{C}(P'')$, and the set $\mathcal{U}$ on which the bound

holds correspondingly curtailed. This variation of (D.6) is used to fix certain phases in the cluster decomposition of S.

There are certain very exceptional points P at which the derivation of (D.6) from the normal analytic structure breaks down. Hence to obtain a precise equivalence these points are excluded both from the domain referred to by the normal analytic structure, and from the support regions of the wave functions referred to by macrocausality. That is, the normal analytic structure and macrocausality are both restricted so as to exclude any condition involving these exceptional points. These exceptional points are described in Chapter IV.

APPENDIX IIA.

Proof of Proposition IIB.1

Proposition IIB.1. Let $\tilde{\psi}^{u\tau}(\hat{x}\tau;\, P,\gamma\tau)$ be defined by (B.8). Then for any $\epsilon > 0$ there is a set of positive constants C, α, and γ_0 such that

$$|\tilde{\psi}^{u\tau}(\hat{x}\tau;\, P,\gamma\tau)| \leq C\, e^{-\alpha\gamma\tau} \tag{1}$$

for all $\tau \geq 0$, all γ satisfying $0 \leq \gamma \leq \gamma_0$, and all $\hat{x}$ in the complement of $\Gamma_\epsilon{}^u(P)$. Moreover, for each integer $n \geq 0$ there is a constant C_n such that

$$|\tilde{\psi}^{u\tau}(\hat{x}\tau;\, P,\gamma\tau)| \leq \frac{C_n\, e^{-\alpha\gamma\tau}}{(1+\tau)^n} \tag{2}$$

for all $\tau \geq 0$, all γ satisfying $0 \leq \gamma \leq \gamma_0$, and all $\hat{x}$ in the complement of $V_\epsilon{}^u(\psi)$. [The α and γ_0 are independent of n.]

Proof. It is sufficient to consider the case $u = 0$, since the general case is obtained from this case by the substitution $\hat{x} \to \hat{x} - u$.

Equation (B.8) gives

$$\tilde{\psi}(\hat{x}\tau;\, P,\gamma\tau) = \int \frac{m}{p^0}\frac{d^3p}{(2\pi)^3}\,\chi(\vec{p})\,\exp-[(\vec{p}-\vec{P})^2\gamma\tau + ip\hat{x}\tau]. \tag{3}$$

Introducing $q \equiv p - P$ one obtains

$$\tilde{\psi}(\hat{x}\tau;\, P,\gamma\tau) = \exp -iP\hat{x}\tau$$

$$\times \int \frac{m}{(P^0+q^0)}\frac{d^3q}{(2\pi)^3}\,\chi(\vec{P}+\vec{q})\,\exp-[\vec{q}^{\,2}\gamma\tau + iq\hat{x}\tau] \tag{4}$$

where

$$q^0 = \left((\vec{P}+\vec{q})^2 + m^2\right)^{\frac{1}{2}} - \left((\vec{P})^2 + m^2\right)^{\frac{1}{2}} \tag{5}$$

$$= \frac{\vec{q}\cdot\vec{P}}{P^0} + Q(\vec{q}) \,,$$

and $Q(\vec{q})$ is second-order in $\vec{q}$.

Let d be the vector

$$d = d(\hat{x},P) = \hat{x} - \frac{P}{P^0}\,\hat{x}^0 \,. \tag{6}$$

Its time part is identically zero, but its space part is zero if and only if $\hat{x}$ lies on $\Gamma(P)$. Thus $\vec{d}$ is nonzero for all $\hat{x}$ in $\not\subset \Gamma_\epsilon(P)$. In fact, if $\bar{d}$ is defined to be the minimum of $|\vec{d}|$ on the (closed) intersection of $\not\subset \Gamma_\epsilon(P)$ with $\{|\hat{x}| = \epsilon\}$,

$$\bar{d} \equiv \mathrm{Min}\,|\vec{d}(\hat{x},P)| \quad \text{over} \quad \{\hat{x} \in \not\subset \Gamma_\epsilon : |\hat{x}| = \epsilon\} \,, \tag{7}$$

then the linearity of $d(\hat{x},P)$ in $\hat{x}$ gives

$$|\vec{d}(\hat{x},P)| \geq |\hat{x}/\epsilon|\bar{d} \geq \bar{d} > 0 \tag{8}$$

for all $\hat{x}$ in $\not\subset \Gamma_\epsilon(p)$.

Let the 3-axis be taken to lie along $\vec{d}$, and let z be the $\mu = 3$ component of q^{μ}:

$$q^3 = z = x + iy , \tag{9}$$

where $x = \text{Re}\, q^3$, $y = \text{Im}\, q^3$.

All the factors in the integral in (4) are analytic near $\vec{q} = 0$. Thus they are analytic functions of the variable z. The contour of integration, which lies originally on the real x axis, may therefore be distorted into the complex z plane. The distortion to be made is defined as follows: let $r \equiv |\text{Re}\, \vec{q}| \equiv (\text{Re}\, \vec{q}\cdot\text{Re}\, \vec{q})^{\frac{1}{2}}$, let a and b be two small numbers, and let

$$y = y(r) = \begin{cases} 0 & \text{for } r \geq b \\ a\gamma\left(1 - \dfrac{r^2}{b^2}\right) & \text{for } r < b . \end{cases} \tag{10}$$

The constants a, b, and γ_0 are taken small enough so that for all γ satisfying $0 \leq \gamma \leq \gamma_0$ the distorted contour remain inside $\mathcal{N}$, and inside the region where

$$|P^0 + q^0| \geq m/A$$

for some finite number A.

The contribution to (4) from the region $r \geq b$ has a bound

$$B(r \geq b) = V\, e^{-b^2\gamma\tau} \tag{11}$$

where V is the volume of the region N. If V' is the maximum volume of the distorted contour for γ satisfying $0 \leq \gamma \leq \gamma_0$ then the contribution to (7) from the region $r < b$ has a bound

$$B(r < b) = AV'\, e^{-\beta\gamma\tau} , \tag{12}$$

where

$$\beta = \min \text{Re}[\vec{q}^{\,2} + i\, q\, \hat{x}\, \gamma^{-1}] . \tag{13}$$

The minimum in (13) is the minimum over all $(r < b)$-points on the distorted contours (10), for all γ satisfying $0 \leq \gamma \leq \gamma_0$, and all $\hat{x}$ in $\complement\, \Gamma_\epsilon(P)$.

The imaginary part of $\vec{q}$ is

$$\text{Im}\, \vec{q} = y\vec{d}/|\vec{d}| = a\gamma\left(1 - \frac{r^2}{b^2}\right)\frac{\vec{d}}{|\vec{d}|} . \tag{14}$$

Thus (5) gives

$$\text{Im}\, q^0 = a\gamma\left(1 - \frac{r^2}{b^2}\right)\frac{\vec{d}\cdot\vec{P}}{|\vec{d}|P^0} + \text{Im}\, Q . \tag{15}$$

Hence

$$\text{Re}[\vec{q}^{\,2} + i\, q\, \hat{x}\, \gamma^{-1}]$$

$$= (\text{Re}\, \vec{q})^2 - (\text{Im}\, \vec{q})^2 - \text{Im}\, q^0\, \hat{x}^0\, \gamma^{-1} + \text{Im}\, \vec{q}\cdot\vec{\hat{x}}\, \gamma^{-1}$$

$$= r^2 - a^2\gamma^2\left(1 - \frac{r^2}{b^2}\right)^2 - a\left(1 - \frac{r^2}{b^2}\right)\frac{\vec{d}}{|\vec{d}|}\cdot\frac{\vec{P}}{P^0}\, \hat{x}^0$$

$$+ a\left(1 - \frac{r^2}{b^2}\right)\frac{\vec{d}}{|\vec{d}|}\cdot\vec{\hat{x}} - (\text{Im}\, Q)\, \hat{x}^0\, \gamma^{-1}$$

$$= r^2 - a^2\gamma^2\left(1 - \frac{r^2}{b^2}\right)^2 + a\left(1 - \frac{r^2}{b^2}\right)|\vec{d}| - \text{Im}\, Q\, \hat{x}^0\, \gamma^{-1} . \tag{16}$$

If a and γ_0 are chosen small enough so that $a\gamma_0^2 < \bar{d}$ then the sum of the first three terms in the last line of (16) is positive for all r satisfying $0 \leq r \leq b$, all γ that satisfy $0 \leq \gamma \leq \gamma_0$, and all $\hat{x}$ in $\Gamma_\epsilon(P)$.

The function $Q(\vec{q})$ is a real analytic function of $\vec{q}$ near $\vec{q} = 0$. Thus in some neighborhood of $\vec{q} = 0$ one can write

$$\operatorname{Im} Q(\vec{q}) = \operatorname{Im} \vec{q}\cdot\vec{F}(\vec{q}) \tag{17}$$

where $F(\vec{q})$ is first order in $\vec{q}$, due to the quadratic character of $Q(\vec{q})$. Thus the term

$$\operatorname{Im} Q(\vec{q})\, \hat{x}^0\, \gamma^{-1} = a\left(1 - \frac{r^2}{b^2}\right)\frac{\vec{d}}{|\vec{d}|}\cdot\vec{F}(\vec{q})\, \hat{x}^0 \tag{18}$$

can be made arbitrarily small in comparison to the third term in (16) by taking a, b, and γ_0 (and hence $|\vec{q}|$) sufficiently small. Thus for sufficiently small a, b, and γ_0 the β in (13) will be strictly positive:

$$\beta > 0 . \tag{19}$$

The required uniform bound (1) then follows from (11) and (12).

To get the bound (2) let $\vec{\nabla}$ be the gradient in $\vec{p}$ space, let

$$e \equiv d(\hat{x},p) \equiv \hat{x} - \frac{\hat{x}^0 p}{p^0} , \tag{20}$$

and write

$$\tau^n \tilde{\psi}(\hat{x}\tau;\, P,\gamma\tau) = \int \frac{m}{p^0}\frac{d^3p}{(2\pi)^3}\,\chi(\vec{p})$$

$$\times \left[\frac{1}{\vec{e}\cdot\vec{\nabla}(-ip\hat{x} - \vec{q}^{\,2}\gamma)}\,\vec{e}\cdot\vec{\nabla}\right]^n \exp[(-ip\hat{x} - \vec{q}^{\,2}\gamma)\tau] . \tag{21}$$

The denominator contains

$$\vec{e}\cdot\vec{\nabla}(-ip\hat{x} - \vec{q}^{\,2}\gamma) = i\,\vec{e}\cdot\vec{\epsilon} - 2\vec{e}\cdot\vec{q}\gamma . \tag{22}$$

Note that $|\vec{e}| \equiv |\vec{e}(\hat{x},p)|$ is bounded away from zero for all (real) $(\hat{x},p)$ such that $\hat{x}$ is in $\not\subset V_\epsilon(\psi)$ and $\Gamma(p)$ is in $V(\psi)$. [The argument is essentially the same as the one leading to (8).] Thus if a, b, and γ_0 are taken sufficiently small--so that $\operatorname{Im} q$ is sufficiently small--then the function (22) will remain bounded away from zero for all $(\hat{x},\vec{p})$ such that $\hat{x}$ is in $\not\subset V_\epsilon(\psi)$ and $\vec{p}$ is in the range of integration. An n-fold integration by parts followed by a distortion of contours (10) then gives

$$|\tau^n \tilde{\psi}(\hat{x}\tau;\, P,\gamma\tau)| \leq C_n' \exp -\alpha\gamma\tau \tag{23}$$

for all $\tau \geq 0$, all γ satisfying $0 \leq \gamma \leq \gamma_0$, and all $\hat{x}$ in the complement of $V_\epsilon(\psi)$. [The contour can be smoothed out at $r = b$ so that $\chi(\vec{p} + iy(\vec{p})\vec{d})$ becomes an infintely differentiable function of $\vec{p}$.]

Given a set of C_n' satisfying (23) it is easy to find a set of C_n satisfying (2). One choice is

$$C_n = (C_n'^{1/n} + C_0'^{1/n})^n . \tag{24}$$

With this choice the right-hand side of (2) is larger than $C_0' \exp -\alpha\gamma\tau$ for $|\tau|^n < C_n'/C_0'$ and is larger than $C_n' \exp -\alpha\gamma\tau/|\tau|^n$ for $|\tau|^n > C_n'/C_0'$. This completes the proof.

APPENDIX IIB.

Proof of Proposition IIB.2

Proposition IIB.2. Let $\psi \equiv \psi(p,P;\gamma\tau)$ be defined by (B.8). Let

$$w(p,x;P,\gamma\tau) \equiv \int \rho(Mv - \tfrac{1}{2}q;\ Mv + \tfrac{1}{2}q)\, e^{-iq\cdot x} \times \left(\frac{M}{m}\right)^{\frac{1}{2}} 2\pi\, \delta(q\cdot v) \frac{d^4q}{(2\pi)^4} \tag{1.a}$$

where $v \equiv p/m$, $M \equiv M(q) \equiv (m^2 - \frac{1}{4}q^2)^{\frac{1}{2}}$, and

$$\begin{aligned}\rho(p'';p') &= \psi^*(p'';P,\gamma\tau)\,\psi(p';P,\gamma\tau) \\ &= \chi^*(\vec{p}'')\,\chi(\vec{p}')\exp-[(\vec{p}'' - \vec{P})^2 + (\vec{p}' - \vec{P})^2]\gamma\tau .\end{aligned} \tag{1.b}$$

Then for any $\epsilon > 0$ there is a set of positive constants C, α, and γ_0 such that

$$|w^{u\tau}(p,\hat{x}\tau;P,\gamma\tau)| \leq C\, e^{-\alpha\gamma\tau} \tag{2}$$

for all $\tau \geq 0$, all γ satisfying $0 \leq \gamma \leq \gamma_0$, and all $(p,\hat{x})$ such that either $\hat{x}$ lies in $\not\subset \Gamma_\epsilon^u(P)$, or $\Gamma^u(p)$ enters $\not\subset \Gamma_\epsilon^u(P)$.

Moreover, for each integer $n \geq 0$ there is a constant C_n such that

$$|w^{u\tau}(p,\hat{x}\tau;P,\gamma\tau)| \leq \frac{C_n\, e^{-\alpha\gamma\tau}}{(1+\tau)^n} \tag{3.a}$$

for all $\tau \geqslant 0$, all γ satisfying $0 \leq \gamma \leq \gamma_0$, and all $(p,\hat{x})$ such that either $\hat{x}$ lies in $\not\subset V_\epsilon^u(\rho)$ or $\Gamma^u(p)$ enters $\not\subset V_\epsilon^u(\rho)$. Here

$$V(\rho) \equiv \{\Gamma(mv): \Gamma(Mv - \tfrac{1}{2} q) \text{ and } \Gamma(Mv + \tfrac{1}{2} q) \text{ both lie in } V(\psi) \text{ for some } q \text{ satisfying } q\cdot v = 0\}. \quad (3.b)$$

Proof. It is sufficient to consider the case $u = 0$, since the general result is obtained from this case by the substitution $\hat{x} \to \hat{x} - u$.

The definition (1), gives

$$w(p,x,P,\gamma\tau) = \int \chi^*(M\vec{v} - \tfrac{1}{2}\vec{q})\, \chi(M\vec{v} + \tfrac{1}{2}\vec{q}) \left(\frac{M}{m}\right)^{\frac{1}{2}}$$

$$\times \exp[-(M\vec{v} - \tfrac{1}{2}\vec{q} - \vec{P})^2 \gamma\tau - (M\vec{v} + \tfrac{1}{2}\vec{q} - \vec{P})^2 \gamma\tau]$$

$$\times \exp\left[i\vec{q}\cdot\left(\vec{x} - \frac{\vec{v}}{v^0} x^0\right)\right] \frac{1}{v^0} \frac{d^3q}{(2\pi)^3}$$

$$= \exp[-2(M\vec{v} - \vec{P})^2 \gamma\tau]$$

$$\times \int \chi^*(M\vec{v} - \tfrac{1}{2}\vec{q})\, \chi(M\vec{v} + \tfrac{1}{2} q) \left(\frac{M}{m}\right)^{\frac{1}{2}}$$

$$\times \exp[-\tfrac{1}{2}\vec{q}^2 \gamma\tau] \exp\left[i\vec{q}\cdot\left(\vec{x} - \frac{\vec{v}}{v^0} x^0\right)\right] \frac{1}{v^0} \frac{d^3q}{(2\pi)^3} . \quad (4)$$

The region of integration is divided into the two parts $|\vec{q}| \geq b$ and $|\vec{q}| < b$. Because the full region of integration is bounded, the contribution from $|\vec{q}| \equiv r \geq b$ has a bound

$$B(r \geq b) = C' e^{-\frac{1}{2}b^2\gamma\tau} \quad (5)$$

coming from the gaussian factor under the integral.

The definitions of $\Gamma_\epsilon(P)$ and $\Gamma(P)$ imply that there is a $\delta > 0$ such that $\Gamma(p)$ enters $\notin \Gamma_\epsilon(P)$ only if $|\vec{p} - \vec{P}| \geq \delta$. Now divide p space into two parts $|\vec{p} - \vec{P}| \geq b'$ and $|\vec{p} - \vec{P}| < b' \leq \delta$.

For any $b'' > 0$ one can find numbers b and b' satisfying $b > 0$ and $0 < b' \leq \delta$, such that

$$|M\vec{v} \pm \tfrac{1}{2}\vec{q} - \vec{P}| \equiv |(M\vec{v} - m\vec{v}) \pm \tfrac{1}{2}\vec{q} - (\vec{P} - \vec{p})| < b''_\pm \leq b'' \quad (6)$$

for all $(\vec{q},\vec{p})$ in $(|\vec{q}| \leq b,\ |\vec{p} - \vec{P}| \leq b')$, and

$$|M\vec{v} - \vec{P}| = \left|\frac{M}{m}\vec{p} - \vec{P}\right| \geq b''' > 0 \quad (7)$$

for all $(\vec{q},\vec{p})$ in $(|\vec{q}| \leq b, |\vec{p} - \vec{P}| \geq b')$. The conditions $b'' \gg b' \gg b > 0$ imply (6) and (7).

For all p in $\{p: |\vec{p} - \vec{P}| \geq b'\}$ the contribution to (4) from the region $|\vec{q}| < b$ has a bound $C'' \exp -2(b''')^2\gamma\tau$ coming from the gaussian factor in front of the integral in (4). [See Eq. (7).] This leaves only the region $(|\vec{q}| < b,\ |\vec{p} - \vec{P}| < b')$ for consideration.

Since q is real in the domain of integration one can replace $\chi^*(M\vec{v} - \frac{1}{2}\vec{q})$ by $\chi^*(M\vec{v} - \frac{1}{2}\vec{q}^*)$, which is an analytic function of $\vec{q}$ for $M\vec{v} - \frac{1}{2}\vec{q} - \vec{P}$ near zero. Thus (6) ensures that for some sufficiently small $b'' > 0$ the region $|\vec{q}| \leq b$ lies in a region over which the integrand is analytic. Condition (6), evaluated at $\vec{q} = 0$, also ensures that by taking b'' sufficiently small the $\Gamma(p)$ corresponding to $|\vec{p} - \vec{P}| < b'$ will lie in an arbitrarily small open cone about the axis $\Gamma(P)$.

For any $\epsilon > 0$ one can take $b'' > 0$ small enough so that the trajectories $\Gamma(p)$ corresponding to $|\vec{p} - \vec{P}| \leq b'$ lie in a closed cone about $\Gamma(P)$ that does not intersect $\not\subset \Gamma_\epsilon(P)$. Thus the arguments of Appendix IIA give the bound

$$\left|d(\hat{x},p)\right| \equiv \left|\hat{x} - \frac{v}{v^0}\hat{x}^0\right| \geq |\hat{x}/\epsilon|\bar{d} \geq \bar{d} > 0 \qquad (8)$$

for all $(\hat{x},p)$ such that $\hat{x}$ lies in $\not\subset \Gamma_\epsilon(P)$ and $\vec{p}$ satisfies $|\vec{p} - \vec{P}| < b'$.

The situation is now essentially the same as the one encountered in Appendix IIA, and the distortion of contours introduced there, with now $d(\hat{x},p)$ in place of $d(\hat{x},P)$, yields a bound $C''' \exp -\alpha'\gamma\tau$ on the contribution from $|\vec{q}| < b$. And this bound is uniform over $\hat{x}$ in $\not\subset \Gamma_\epsilon(P)$, and over p in $\{p: (\vec{p} - \vec{P}) \leq b'\}$. Thus the identifications

$$C = \text{Max}(C', C'', C''')$$

$$\alpha = \text{Min}\left(\tfrac{1}{2} b^2, 2(b'')^2, \alpha'\right)$$

yield the required uniform bound (3).

The uniform bound (4) follows from essentially the same argument that gives the bound (2) of Appendix IIA. If $\Gamma(p)$ enters $\not\subset V(\rho)$ then $w(p,x; P,\gamma\tau)$ is in fact zero. Thus $\Gamma(p)$ can be assumed to lie in $V(\rho)$. But then the denominator in the equation

$$\tau^n w(p,\hat{x}\tau; P,\gamma\tau) = \exp[-2(M\vec{v} - \vec{P})^2 \gamma\tau]$$

$$= \int \chi^*(M\vec{v} - \tfrac{1}{2}\vec{q}^*)\, \chi(M\vec{v} + \tfrac{1}{2}\vec{q}) \left(\frac{M}{m}\right)^{\frac{1}{2}} \frac{1}{v^0}$$

$$\times \left[\frac{1}{(i\vec{e}\cdot\vec{e} - \vec{e}\cdot\vec{q}\gamma)}\, e\cdot\nabla\right]^n \exp\left[\left(i\vec{q}\cdot\left(\vec{\hat{x}} - \frac{\vec{p}}{p^0}\hat{x}^0\right) - \frac{1}{2}\vec{q}^2\gamma\right)\tau\right]$$

$$d^3q/(2\pi)^3 \qquad (9)$$

is bounded away from zero for all $\hat{x}$ in $V_\epsilon(\rho)$, just as is the denominator in (21) of Appendix IIA. The bound (4) then follows from the argument given there.

APPENDIX IIC.

Proof of Proposition IIB.3

Proposition IIB.3. Let $w(p,x;\ P,\gamma\tau)$ be as defined in Proposition IIB.2. Let u be any four-vector, and let P be any mass-shell four-vector. Let $R(P,u)$ be the entire $(\vec{p},\hat{\vec{x}})$ space minus some arbitrarily small neighborhood of the point $(\vec{P},\vec{u})$. Let $R(P,u;\ \tau)$ be the image of $R(P,u)$ in $x = \hat{x}\tau$ space:

$$R(P,u;\ \tau) \;=\; \{(p,x):\ (p,x/\tau) \in R(p,u)\}\ . \tag{1}$$

Then there is a set of positive constants C, α, and γ_0 such that

$$\int_{R(P,u;\ \tau)} \frac{d^3p}{(2\pi)^3}\, d^3x \left| w^{u\tau}(p,x;\ P,\gamma\tau)\right|_{x^0=u^0\tau} \;\leq\; C\, e^{-\alpha\gamma\tau} \tag{2}$$

for all $\tau \geq 0$ and all γ satisfying $0 \leq \gamma \leq \gamma_0$.

Moreover, let $R(\rho,u)$ be the entire $(\vec{p},\hat{\vec{x}})$ space minus some open set that contains all points $(\vec{p},\hat{\vec{x}})$ such that $\Gamma^{(u_0\vec{x})}\big(p(\vec{p})\big)$ lies in $V^u(\rho)$. Let $R(\rho,u;\ \tau)$ be the $x = \hat{x}\tau$ space image of $R(\rho,u)$. Then for any integer $n \geq 0$ there is a constant C_n such that

$$\int_{R(\rho,u;\ \tau)} \frac{d^3p}{(2\pi)^3}\, d^3x \left| w^{u\tau}(p,x;\ P,\gamma\tau)\right|_{x^0=u^0\tau} \;\leq\; \frac{C_n\, e^{-\alpha\gamma\tau}}{(1+\tau)^n} \tag{3}$$

for all $\tau \geq 0$ and all γ satisfying $0 \leq \gamma \leq \gamma_0$.

Proof. It is sufficient to consider the case $u = 0$. The definitions give, for $x^0 = u^0 = 0$,

$$\begin{aligned} w(p,x;\ P,\gamma\tau) \;=\;& \exp[-2(M\vec{v} - \vec{p})^2\,\gamma\tau] \\ &\times \int \chi^*(M\vec{v} - \tfrac{1}{2}\vec{q})\ \chi(M\vec{v} + \tfrac{1}{2}\vec{q}) \left(\frac{M}{m}\right)^{\frac{1}{2}} \\ &\quad \times \exp[-\tfrac{1}{2}\vec{q}^{\,2}\,\gamma\tau]\ \exp[i\vec{q}\cdot\vec{x}]\ \frac{1}{v^0}\ \frac{d^3q}{(2\pi)^3}\ . \end{aligned} \tag{4}$$

A four-fold integration by parts gives

$$\begin{aligned} |\vec{x}|^4\, w(p,x;\ P,\gamma\tau) \;=\;& \exp[-2(M\vec{v} - \vec{P})^2\,\gamma\tau] \int \frac{d^3q}{(2\pi)^3}\ \exp(\vec{q}\cdot\vec{x}) \\ &\times (\nabla^2)^2\left[\chi^*(M\vec{v} - \tfrac{1}{2}\vec{q})\ \chi(M\vec{v} + \tfrac{1}{2}\vec{q}) \left(\frac{M}{m}\right)^{\frac{1}{2}} \frac{1}{v^0}\ \exp - \tfrac{1}{2}\vec{q}^{\,2}\ \gamma\tau\right]. \end{aligned} \tag{5}$$

Note that the factors $\gamma\tau$ that arise from the derivatives can be bounded by factors $C_\eta \exp \eta\gamma\tau$ for arbitrarily small $\eta > 0$, by making C_η sufficiently large. Hence they can be absorbed into factors $C \exp -\alpha\gamma\tau$. Thus the method of Appendix IIB gives a uniform bound over $R(P,u;\ \tau)$ of the form

$$|w(p,x;\ P,\gamma\tau)| \;\leq\; \frac{C' \exp -\alpha\gamma\tau}{(1 + |\vec{x}|)^4}\ . \tag{6}$$

[The "one" in $(1 + |x|)$ is obtained by using also the bound (2) asserted by Proposition IIB.2, and applying Eq. (24) of Appendix IIA.]

Let R_p represent the bounded domain in p space where $w(p,x;\ P,\gamma\tau)$ is nonvanishing. Then the bound (6) gives

$$\int_{R(\rho,u;\ \tau)} \frac{d^3p}{(2\pi)^3} \int d^3x\ |w(p,x;\ P,\gamma\tau)|$$

$$\leq\ C'\ \exp -\alpha\gamma\tau \int_{R_p} \frac{d^3p}{(2\pi)^3} \int_0^\infty \frac{(4\pi)\ r^2\ dr}{(r+1)^4}$$

$$=\ C\ \exp -\alpha\gamma\tau \qquad (7)\ .$$

This is the required uniform bound (2). The uniform bound (3) follows from an application of essentially the same argument that gives (2) of Appendix IIA, and (3) of Appendix IIB.

FOOTNOTES AND REFERENCES

1. D. Ruelle, Helv. Phys. Acta 35, 147 (1962).

2. These wave functions of the "shrinking-gaussian" form were first used in this connection by R. Omnes, Phys. Rev. 146, 1123 (1966). See also D. Iagolnitzer and H. P. Stapp, Commun. Math. Phys. 14, 15 (1969), and J. Bros and D. Iagolnitzer, Saclay Report DPh-T/71-33, July 1971; Presented at the Marseille Meeting on Renormalization Theory.

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