Title

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Publication Date

1999

LBNL-42790

ERNEST ORLANDO LAWRENCE BERKELEY NATIONAL LABORATORY

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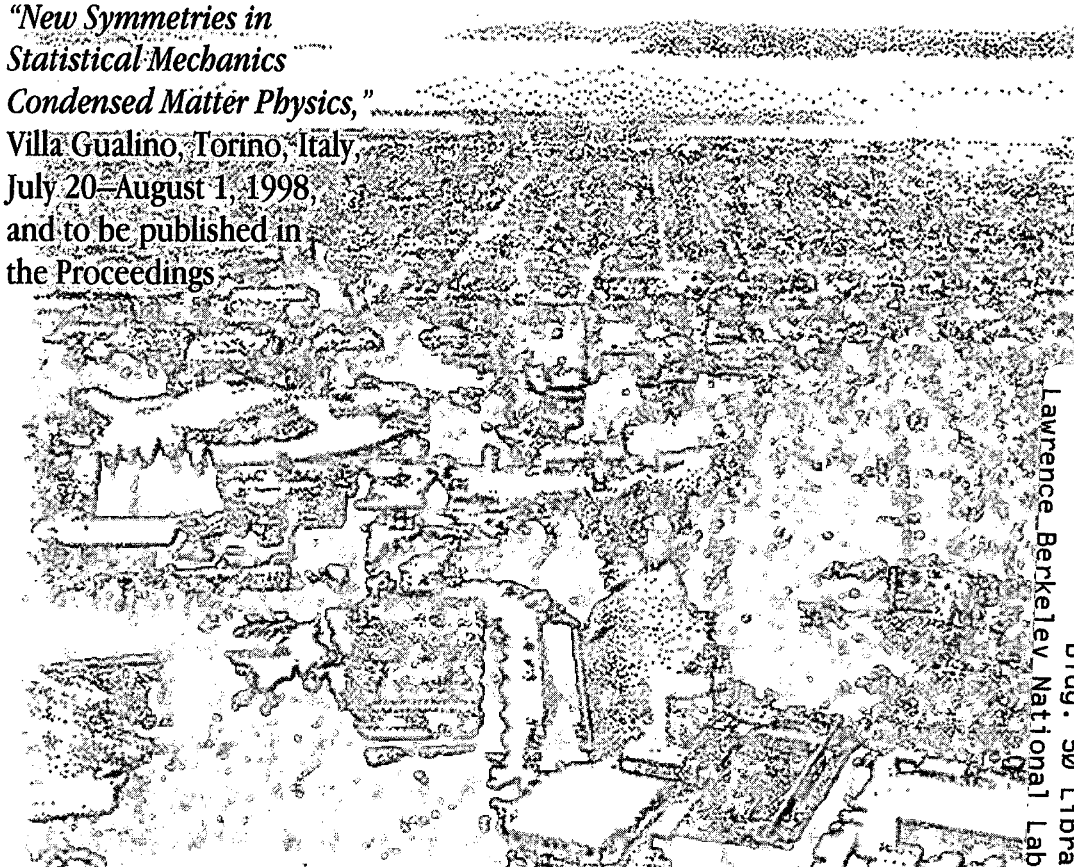
January 1999

Presented at the
Europhysics Conference:
"New Symmetries in
Statistical Mechanics
Condensed Matter Physics,"
Villa Gualino, Torino, Italy,
July 20–August 1, 1998,
and to be published in
the Proceedings



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This work was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098, the National Science Foundation under Grant No. PHY-95-14797, and CNR Grant Bando 203.01.66.

LBNL-42790
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Lie Algebra of the q-Poincaré Group and q-Heisenberg Commutation Relations

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Abstract

We discuss quantum orthogonal groups and their real forms. We review the construction of inhomogeneous orthogonal q-groups and their q-Lie algebras. The geometry of the q-Poincaré group naturally induces a well defined q-deformed Heisenberg algebra of hermitian q-Minkowski coordinates x^a and momenta p_a.

1 Introduction

Quantum groups are deformations of Lie groups that appeared about thirteen years ago in the context of integrable system in 1+1 dimensions. In this talk we study quantum groups structures not as hidden symmetries of a low dimensional dynamical system but as a possible symmetry of space-time itself. We study quantum groups in a kinematic context, as symmetry groups associated to non commutative space-times. Since the coordinates of these spaces do not commute we naturally have uncertainty relations, the position of a particle cannot be measured exactly, a discrete (lattice like) structure emerges, possibly also a minimum lenght. Gedanken experiments of Planck scale gravity and string theory models [1] predict similar qualitative space-time features.

In this perspective it is natural to try to formulate gravity theories on curved non commutative Minkowski space-time with q-Lorentz gauge group and invariant under q-diffeomorphisms. Here q may play the role of regularization parameter preserving the q-symmetries, indeed a non commutative space, contrary to a lattice, provides a space-time structure that is as rich as in the commutative case: q-rotations and q-boosts are symmetries of this space. One can also speculate that since at the level

[1]This work is supported by CNR grant bando 203.01.66. It is in part supported by the Director, Office of Energy Research, Office af High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and by the National Science Foundation under grant PHY-95-14797

of quantized phase-space the parameter q introduces a lattice like structure [2] then the minimal lattice length can play the fundamental role of Planck length in the description of quantum space-time.

A main tool toward the construction of q-gravity theories is the group-geometric approach to gravity [3]. There one considers the softened Poincaré group $\widetilde{ISO}(3,1)$. As a smooth manifold $\widetilde{ISO}(3,1)$ is diffeomorphic to the Poincaré group $ISO(3,1)$, while the deviation from the rigid group structure is encoded in the modified Cartan-Maurer equations: $d\mu^A + \frac{1}{2}C^A_{BC}\mu^B \wedge \mu^C = R^A$, where μ^a are the softened (no more left invariant) one forms and R^A is the curvature two form. On $\widetilde{ISO}(3,1)$ the Lie derivative along a generic vector field $t = \varepsilon^A \tilde{T}_A$, where $\tilde{T}_A$ are the vector fields dual to the one forms μ^A, satisfies: $\ell_t \mu^A = (\nabla t)^A + i_t R^A$ where ∇ is the exterior covariant derivative. After imposing the horizontality conditions along the Lorentz directions (to recover the rigid $SO(3,1)$ group structure) we have a fibre bundle structure with principal fiber $SO(3,1)$ and base space a curved Minkowski space. The Lie derivative along a vector field $\varepsilon = \varepsilon^L T_L$, where now L is a Lorentz index and T_L are the left invariant vector fields associated to $SO(3,1)$, represents an infinitesimal gauge transformation: $\delta^{(gauge)}\mu^A = \ell_\varepsilon \mu^A = (\nabla\varepsilon)^A$. The one-forms μ^A are then identified with the vielbein and the spin connection $[\mu^A = (V^a, \omega^{ab})]$ of the Einstein–Cartan theory (first order formalism).

Following similar steps in the non commutative case one can formulate a geometric definition of curvature, covariant derivative and gauge transformation based on the Cartan-Maurer equation (i.e. the Lie algebra) of the q-Poincaré group.

For an example of this construction in the case of a minimal q-deformation (twist) of the Poincaré group see [4]. There, a generalization of the Einstein–Cartan lagrangian is obtained and found invariant under local q-Lorentz transformations and under q-diffeomorphisms. The results of [4] rely on the bicovariant differential calculus and Lie algebra of the q-Poincaré group.

In the following we briefly define orthogonal quantum groups [5, 6] and study their real forms. We see that there are two different conjugations that give two different quantum $SO(3,1)$ [8].

Various deformations of the Poincaré group are known in the literature, we review a canonical method to define inhomogeneous q-groups and in particular explain the construction of a q-Poincaré group and its q-Lie algebra. We thus obtain a differential calculus on a q-Poincaré group [10]. The Lie derivative and contraction operator can be readily defined using this differential calculus. This calculus is a three real parameters deformation of the commutative calculus on $ISO(3,1)$; in particular it contains the q-Poincaré group discussed in [4].

Using a canonical group-geometric procedure we also study a differential calculus on the q-Minkowski plane [11]. We then obtain a deformation of the phase-space with hermitian operators x^a and $p_a \sim \frac{-i\partial}{\partial x^a}$, it is then interesting, in the spirit of [2], the analysis of the representations in Hilbert space of this algebra in order to study the admissible (discrete) values of the momentum and position of particle states.

2 $SO_{q,r}(N)$ multiparametric quantum group

The $SO_{q,r}(N)$ multiparametric quantum group is freely generated by the noncommuting matrix elements $T^a{}_b$ (fundamental representation $a, b = 1, \ldots N$) and the unit element I, modulo the relation $\det_{q,r}T = I$ and the quadratic RTT and CTT (othogonality) relations discussed below. The noncommutativity is controlled by the R matrix:

$$R^{ab}{}_{ef}T^e{}_cT^f{}_d = T^b{}_fT^a{}_eR^{ef}{}_{cd} \tag{2.1}$$

which satisfies the quantum Yang-Baxter equation: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. The R-matrix components $R^{ab}{}_{cd}$ depend continuously on a (in general complex) set of parameters q_{ab}, r. For $q_{ab} = r$ we recover the uniparametric orthogonal group $SO_r(N)$ of ref.[5]. Then $q_{ab} \to 1, r \to 1$ is the classical limit for which $R^{ab}{}_{cd} \to \delta^a_c\delta^b_d$: the matrix entries $T^a{}_b$ commute and become the usual entries of the fundamental representation. The R matrix is upper triangular (i.e. $R^{ab}{}_{cd} = 0$ if $[a = c$ and $b < d]$ or $a < c$), and the parameters q_{ab} appear only in the diagonal components of R: $R^{ab}{}_{ab} = r/q_{ab}, a \neq b, a' \neq b$, where prime indices are defined as $a' \equiv N + 1 - a$. We also define $q_{aa} = q_{aa'} = r$. The following relations reduce the number of independent q_{ab} parameters [6]: $q_{ba} = r^2/q_{ab}$, $q_{ab} = r^2/q_{ab'} = r^2/q_{a'b} = q_{a'b'}$; therefore the q_{ab} with $a < b \leq \frac{N}{2}$ give all the q's.

Orthogonality conditions are imposed on the elements $T^a{}_b$, consistently with the RTT relations (2.1):

$$C^{bc}T^a{}_bT^d{}_c = C^{ad}I \ , \ C_{ac}T^a{}_bT^c{}_d = C_{bd}I \ , \tag{2.2}$$

where the matrix C_{ab} and its inverse C^{ab}, that satisfies $C^{ab}C_{bc} = \delta^a_c = C_{cb}C^{ba}$, are the metric and its inverse. These matrices are antidiagonal; they are equal, and for example for $N = 4$ they read $C_{14} = r^{-1}, C_{23} = 1, C_{32} = 1$ and $C_{41} = r$. The explicit expression of R and C is given in [6], (see the first ref. in [10] for our notational conventions).

The conjugation that from the complex $SO_{q,r}(N)$ leads to the real form $SO_{q,r}(n+1, n-1; \boldsymbol{R})$ and that is in fact needed to obtain the quantum Poincaré group, is defined by $(T^a{}_b)^* = \mathcal{D}^a{}_cT^c{}_d\mathcal{D}^d{}_b$, $\mathcal{D}$ being the matrix that exchanges the index n with the index $n+1$ [7]. This conjugation is well defined if: $|q_{ab}| = |r| = 1$ for a and b both different from n or $n+1$; $q_{ab}/r \in \boldsymbol{R}$ when at least one of the indices a, b is equal to n or $n+1$.

Another conjugation [8] that also gives the Lorentz group and more in general the real form $SO_{q,r}(2n-1, 1; \boldsymbol{R})$ is defined by $(T^a{}_b)^* = \mathcal{D}^a{}_cT^{\star c}{}_d\mathcal{D}^d{}_b$, where $T^\star \equiv C^tTC^t$ and ^{t} means matrix transposition. The definition of $T^\star$ can also be given using the antipode κ: $T^\star = [\kappa(T)]^t$. This conjugation requires $r \in \boldsymbol{R}$ and: $q_{ab}\bar{q}_{ab} = r^2$ for a and b both different from n or $n+1$; $q_{ab} = \bar{q}_{ab}$ when at least one of the indices a, b is equal to n or $n+1$.

3 $ISO_{q,r}(N)$ as a projection from $SO_{q,r}(N+2)$

In the commutative case $ISO(N)$ is a subgroup of $SO(N+2)$, similarly the Lie algebra and the universal enveloping algebra of $ISO(N)$ are subalgebras of $SO(N+2)$. Dually, if we consider $Fun(ISO(N))$, the algebra of smooth functions on $ISO(N)$, we have that $Fun(ISO(N))$ is a quotient of $Fun(SO(N+2))$: to obtain $Fun(ISO(N))$ we identify different functions on $SO(N+2)$ if they have the same value on $ISO(N)$. We proceed similarly in the quantum case. $ISO_{q,r}(N) \equiv Fun_{q,r}(ISO(N))$ [the non commutative deformation of $Fun(ISO(N))$] is defined requiring its universal enveloping algebra to be a subalgebra of the universal enveloping algebra of $SO_{q,r}(N+2)$; this means that $ISO_{q,r}(N)$ is a quotient of the Hopf algebra $SO_{q,r}(N+2)$. Let $T^A{}_B$ be the $SO_{q,r}(N+2)$ generators, and split the index A of $SO_{q,r}(N+2)$ as A=$(\circ, a, \bullet)$, with $a = 1, ...N$. With this notation:

$$ISO_{q,r}(N) = \frac{SO_{q,r}(N+2)}{H} , \tag{3.3}$$

where H is the left and right ideal in $SO_{q,r}(N+2)$ generated by the relations:

$$T^a{}_\circ = T^\bullet{}_b = T^\bullet{}_\circ = 0 . \tag{3.4}$$

Following [10] the projection $P : SO_{q,r}(N+2) \to SO_{q,r}(N+2)/H$ is an epimorphism between Hopf algebras, and defining the projected matrix elements $t^A{}_B = P(T^A{}_B)$, we can give an R-matrix formulation of $ISO_{q,r}(N)$. We set $u \equiv P(T^\circ{}_\circ)$, $y_b \equiv P(T^\circ{}_b)$, $z \equiv P(T^\circ{}_\bullet)$, $x^a \equiv P(T^a{}_\bullet)$ and (with abuse of notation) $T^a{}_b \equiv P(T^a{}_b)$, then we have

Theorem The quantum group $ISO_{q,r}(N)$ is generated by the matrix entries

$$t \equiv \begin{pmatrix} u & y_b & z \\ 0 & T^a{}_b & x^a \\ 0 & 0 & v \end{pmatrix} \quad \text{and the unity } I \tag{3.5}$$

modulo the Rtt and Ctt relations

$$R^{AB}{}_{EF} t^E{}_C t^F{}_D = t^B{}_F t^A{}_E R^{EF}{}_{CD} , \tag{3.6}$$

$$C^{BC} t^A{}_B t^D{}_C = C^{AD}, \ C_{AC} t^A{}_B t^C{}_D = C_{BD} , \tag{3.7}$$

where R and C are the multiparametric R-matrix and metric of $SO_{q,r}(N+2)$□□□

Using the explicit expression of the R matrix one can check that relations (3.6), (3.7) contain in particular the $SO_{q,r}(N)$ relations (2.1), (2.2) and the quantum orthogonal plane commutation relations:

$$P_A^{ab}{}_{cd} x^c x^d = 0 \tag{3.8}$$

where the q-antisymmetrizer P_A is given by $P_A = \frac{1}{r+r^{-1}}[-\hat{R} + rI - (r - r^{1-N})P_0]$ and $P_0{}^{ab}{}_{cd} \equiv (C_{ef}C^{ef})^{-1} C^{ab} C_{cd}$. Moreover, due to the Ctt relations, the y and z

elements are polynomials in u, x and T, and we also have $uv = vu = I$. A set of independent generators of ISO is then

$$T^a{}_b, x^a, u, v \equiv u^{-1} \text{ and the identity } I \,. \tag{3.9}$$

Their commutations are (2.1), (3.8) and

$$T^b{}_d x^a = \frac{r}{q_{d\bullet}} R^{ab}{}_{ef} x^e T^f{}_d \,, \qquad u T^b{}_d = \frac{q_{b\bullet}}{q_{d\bullet}} T^b{}_d u \,, \qquad u x^b = q_{b\bullet} x^b u \,.$$

The deformation parameters of $ISO_{q,r}(N)$ are the same as those of $SO_{q,r}(N+2)$; they are r and q_{AB} i.e. r, q_{ab} and $q_{a\bullet}$ ($q_{a\bullet} = r^2/q_{a\circ} = q_{\bullet a'} = q_{\circ a}$). In the limit $q_{a\bullet} \to 1 \; \forall a$, which implies $r \to 1$, the dilatation u commutes with x and T, and can be set equal to the identity I; then, when also $q_{ab} \to 1$ we recover the classical algebra $Fun(ISO(N))$.

Only the first of the two $SO_{q,r}(N+2)$ real forms mentioned in the previous section is inherited by $ISO_{q,r}(N)$. In particular the q-Poincaré group $ISO_{q,r}(3,1;\boldsymbol{R})$ is obtained by setting $|q_{1\bullet}| = |r| = 1$, $q_{2\bullet}/r \in \boldsymbol{R}$, $q_{12} \in \boldsymbol{R}$. A dilatation-free q-Poincaré group is found after the further restriction $q_{1\bullet} = q_{2\bullet} = r = 1$. The only free parameter remaining is then $q_{12} \in \boldsymbol{R}$.

4 Differential Calculus on $SO_{q,r}(N+2)$, $ISO_{q,r}(N)$ and q-Heisenberg algebra

In the classical case the differential calculus on a Lie group is determined by the Lie algebra of left invariant vector fields. These vector fields are uniquely defined by their value at the origin 1_G of the group. Let $\{\chi_i\}$ be a basis of the $SO(N+2)$ Lie algebra, the χ_i's are linear functions on $Fun(SO(N+2))$, their action is: $\forall \; f \in Fun(ISO(N))$, $\chi_i(f) = \partial_i f|_{1_G}$. They satisfy the Leibniz rule $\forall f, h \in SO_{q,r}(N+2)$, $\chi_i(fh) = \chi_i(f)\, h|_{1_G} + f|_{1_G}\, \chi_i(b)$. Similarly, in the quantum case a quantum (orthogonal) Lie algebra [9] is given by a set of linear functions $\{\chi_i\}$ on $SO_{q,r}(N+2) \equiv Fun_{q,r}(SO(N+2))$. The functionals χ_i satisfy a deformed Leibniz rule: $\forall a, b \in SO_{q,r}(N+2)$, $\chi_i(ab) = \varepsilon(a)\chi_i(b) + \chi_j(a) O^j{}_i(b)$ where $O^j{}_i$ are linear functions that in the commutative limit approach $\varepsilon\delta^j{}_i$ and ε is the counit [in the commutative limit $\varepsilon(f) = f|_{1_G}$]. Moreover the functionals χ_i close on the quantum Lie algebra:

$$[\chi_i, \chi_j]_q \equiv \chi_i \chi_j - \Lambda^{kl}{}_{ij} \chi_k \chi_l = \mathbf{C}_{ij}{}^k \chi_k \tag{4.1}$$

where $\chi_i\chi_j - \Lambda^{kl}{}_{ij}\chi_k\chi_l \xrightarrow{q,r\to 1} \chi_i\chi_j - \chi_j\chi_i$, and the q-structure constants are given by

$$\mathbf{C}^{A_1}{}_{A_2}{}^{B_1}{}_{B_2}|^{C_2}_{C_1} = \frac{1}{r - r^{-1}} [-\delta^{B_1}_{B_2} \delta^{A_1}_{C_1} \delta^{C_2}_{A_2} + \Lambda_B{}^B{}_{C_1}^{C_2}|^{A_1}{}_{A_2}{}^{B_1}{}_{B_2}] \,, \tag{4.2}$$

here the index pairs ${}^{A_1}{}_{A_2}$ or ${}_{A_1}{}^{A_2}$ replace the indices ${}_i$ or ^{i}. The Λ matrix is given by

$$\Lambda^{A_2}_{A_1}{}^{D_2}_{D_1}|^{C_1}_{C_2}{}^{B_1}_{B_2} = d^{F_2} d^{-1}_{C_2} R^{F_2 B_1}{}_{C_2 G_1} (R^{-1})^{C_1 G_1}{}_{E_1 A_1} (R^{-1})^{A_2 E_1}{}_{G_2 D_1} R^{G_2 D_2}{}_{B_2 F_2} \tag{4.3}$$

with $d^A = C^{AB}C_{AB}$ where the sum is only on B. Notice that in (4.1) the number of linearly independent generators χ_i is $(N+2)^2$ and not $(N+2)(N+1)/2$ as in the commutative case. The q-Jacoby identities read: $[\chi_i, [\chi_j, \chi_k]] = [[\chi_i, \chi_j], \chi_k] - \Lambda^{lm}_{jk}[[\chi_i, \chi_l], \chi_m]$

We now study the differential calculus and q-Lie algebra of $ISO_{q,r}(N)$. In the commutative case the $ISO(N)$ Lie algebra is a subalgebra of the $SO(N+2)$ Lie algebra. In the quantum case it turns out [10] that for an arbitrary value of the deformation parameter r, inside the $SO_{q,r}(N+2)$ q-Lie algebra there is not a q-Lie algebra that includes the $SO_{q,r}(N)$ q-Lie algebra and that becomes the $ISO(N)$ Lie algebra in the commutative limit. However when $r=1$, i.e. for minimal deformations (twists), we have strong simplifications and an $ISO_{q,r}(N)$ q-Lie algebra:
1) some q-Lie algebra generators become linearly dependent: $C_{AC}\chi^C{}_B = -q_{AB}C_{BD}\chi^D{}_A$ so that as in the commutative case we have $(N+2)(N+1)/2$ generators.
2) the $SO_{q,r=1}(N+2)$ q-Lie algebra elements $\chi^a{}_b$, $a > N+1-b,\ a,b = 1,...N$ and $\chi^\bullet{}_b$ generate the $ISO_q(N)$ q-Lie algebra, while the $\chi^a{}_b$ alone generate the $SO_q(N)$ q-Lie algebra.

In the case $N=4$, when $r=1$, we are left with 3 deformation parameters: the phase $q_{1\bullet}$ and the real numbers $q_{2\bullet}$ and q_{12}. The calculus on the q-Poincaré group used in [4] is obtained fixing $q_{12}=1$. A calculus on a q-Poincaré group without the dilatation u is obtained fixing $q_{1\bullet} = q_{2\bullet} = 1$.

In the general case $r \neq 1$ only the functionals $\chi^\bullet{}_a$ (and $\chi^\bullet{}_\bullet$, $\chi^\bullet{}_\circ$) are well defined on the quotient $ISO_{q,r}(N) = SO_{q,r}(N+2)/H$. We miss the functionals $\chi^a{}_b$ relative to the homogeneous q-group $SO_{q,r}(N)$. However the functionals $\chi^\bullet{}_a$ define a differential calculus on the orthogonal q-plane. Their q-Lie algebra is a subset of the $SO_{q,r}(N+2)$ q-Lie algebra (4.1). Similarly to (3.8) it reads

$$q_{\bullet a} P_A^{ab}{}_{cd} \chi^\bullet{}_b \chi^\bullet{}_a = 0 \ . \tag{4.4}$$

If we denote by $\chi^\bullet{}_b*$ the left invariant vector field associated to the q-tangent vector $\chi^\bullet{}_b$ and by ω^b the dual left invariant one form, then the exterior differential reads $da = \chi^\bullet{}_b * a\ \omega^b$. Comparing this expression with the equivalent one $da = \partial_b a\, dx^b$ ($dx^b \equiv \chi^\bullet{}_c * x^b\ \omega^c$) we determine the relation between partial derivatives ∂_b and left invariant vector fields $\chi^\bullet{}_b*$. From the q-plane Lie algebra (4.4) we can then derive the q-commutation relations between the partial derivatives. Similarly the q-commutations between the differentials dx^a are deduced from the calculus on the $SO_{q,r}(N+2)$ q-group. We thus obtain the following calculus on the q-orthogonal plane:

$$P_A^{ab}{}_{cd} x^c x^d = 0 \ , \quad P_A^{ab}{}_{cd} \partial_b \partial_a = 0 \ , \quad \partial_c x^b = r R^{eb}{}_{cd} x^d \partial_e + \delta^b_c I \tag{4.5}$$

$$x^a dx^b = r R^{ba}_{ef}(dx^e x^f) \ , \quad dx^a \wedge dx^b = -r R^{ba}_{ef}(dx^e \wedge dx^f) \tag{4.6}$$

This calculus generalizes to the multiparametric case the results of [12]. The derivation of (4.5) and (4.6) sketched here (for the details see [11]) is a canonical group-geometric procedure to restrict the calculus on the quantum group $SO_{q,r}(N+2)$ to the calculus on the N dimensional q-orthogonal plane. As a bonus the covariance under $SO_{q,r}(N)$ and $ISO_{q,r}(N)$ is easily studied. Moreover the $*$-structure present on $ISO_{q,r}(N+2)$ canonically induces a $*$-structure on the q-plane calculus. For $N=4$ this gives a q-Minkowski calculus. Explicitly

$$(x^a)^* = \mathcal{D}^a{}_b x^b,\ (dx^a)^* = \mathcal{D}^a{}_b dx^b,\ (\partial_a)^* = -r^N d_a^{-1} \mathcal{D}^b{}_a \partial_b\ . \tag{4.7}$$

We can now finally construct hermitian operators and the q-Heisenberg algebra. Let $a' = N+1-a$, then the transformation

$$X^a = \frac{1}{\sqrt{2}}(x^a + x^{a'})\ ,\ a \le n\ ;\quad X^{n+1} = \frac{i}{\sqrt{2}}(x^n - x^{n+1})\ ;\quad X^a = \frac{1}{\sqrt{2}}(x^a - x^{a'})\ ,\ a > n+1$$

defines real coordinates X^a. A similar linear combination of the partial derivatives gives hermitian momenta P_a. In terms of these hermitian operators we have the q-Heisenberg algebra

$$P_a X^b - r S^{bc}{}_{ad} X^d P_c = -i\hbar E^b_a I \tag{4.8}$$

here $S^{bc}{}_{ad}$ and $E^b_a \equiv \frac{i}{\hbar} P_a(X^b)$ are C-number matrices [11] that reduce to unity for $q_{12}, q_{1\bullet}, q_{2\bullet}, r \to 1$.

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Prepared for the U.S. Department of Energy under Contract No. DE-AC03-76SF00098