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UNIVERSITY OF CALIFORNIA SAN DIEGO

Geometry In The Large Of Ricci Flows

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Mathematics

by

Zilu Ma

Committee in charge:

Professor Bennett Chow, Chair
Professor Lei Ni, Co-Chair
Professor Alireza Golsefidy
Professor John Mcgreevy
Professor Luca Spolaor

2022

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The Dissertation of Zilu Ma is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2022

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PUBLICATIONS

Chan, Pak-Yeung; Ma, Zilu; Zhang, Yongjia. *A uniform Sobolev inequality for ancient Ricci flows with bounded Nash entropy.* Int. Math. Res. Not. IMRN, accepted.

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ABSTRACT OF THE DISSERTATION

Geometry In The Large Of Ricci Flows

by

Zilu Ma

Doctor of Philosophy in Mathematics

University of California San Diego, 2022

Professor Bennett Chow, Chair
Professor Lei Ni, Co-Chair

Ricci flow is a powerful and fundamentally innovative tool in the field of geometric analysis introduced by Richard Hamilton [Ha82] in 1982. Many longstanding geometric and topological problems have been solved using Ricci flow. For example, the Poincaré conjecture, Thurston's geometrization conjecture, the 1/4-pinched differentiable sphere theorem, the generalized Smale conjecture, and so on. In the seminal works of Hamilton and Perelman, it is crucial to understand at least the qualitative behavior of the singularities so that Ricci flow can be continued via surgeries. To understand the singularity formation, it is desirable to classify singularity models, which are the blow-up limits along sequences

of space-time points with curvature tending to infinity, or at least to understand them qualitatively well enough for topological applications. However, as compared to dimension 3, the geometry becomes drastically more complicated starting from dimension 4, and so does the singularity analysis of Ricci flow. Recently, Richard Bamler in [Bam20a, Bam20b, Bam20c] established a groundbreaking theory for the weak limits (his $\mathbb{F}$-limits) of Ricci flows on closed manifolds in higher dimensions. This theory will be fundamentally important in the study of higher-dimensional Ricci flow singularities. All the known examples so far suggest that Ricci flow singularity models should be mostly shrinking or steady Ricci solitons. These are the self-similar ancient solutions to Ricci flow, and they can be viewed as generalized Einstein manifolds. Thus, it is vitally important to study shrinking and steady Ricci solitons that arise as singularity models.

In the dissertation, we shall survey some recent results on the geometry in the large of singularity models or more general noncollapsed ancient solutions of Ricci flow, which were jointly investigated by collaborators and the author. We shall streamline some proofs and also present some new unpublished findings. There are two fundamental notions of space-time blow-downs for ancient flows: asymptotic shrinking solitons by Perelman [Per02] and tangent flows at infinity by Bamler [Bam20c]. We will show that the two notions coincide, which is the main theorem of [CMZ21a]. We will present that for ancient flows, various entropy quantities introduced mainly by Perelman converge to those of the tangent flows at infinity, which were proved in [MZ21, CMZ21a, CMZ21b, CMZ21d]. It will also be demonstrated that how tangent flows at infinity determine the geometry in the large of steady solitons in dimension 4, which is the main theorem of [BCDMZ]. Moreover, we will present an optimal volume growth estimate for noncollapsed steady solitons in all dimensions, which is the main result of [BCMZ21].

Chapter 1

Introduction

Richard Hamilton first introduced Ricci flow in his seminal paper [Ha82] in 1982. Ricci flow is a family of metrics $(M, g_t)_{t\in[0,T)}$ on a smooth manifold M satisfying

$$\partial_t g_t = -2\mathrm{Ric}_{g_t},$$

where Ric_{g_t} denotes the Ricci tensor induced by the metric g_t. This geometric equation is weakly parabolic and one could see this by recalling that in a harmonic coordinate,

$$-2R_{ij} = \Delta g_{ij} + Q(g, \partial g),$$

where Q is a rational function quadratic in ∂g. (See, e.g., [Pe16, Lemma 11.2.6].)

During the last forty years, Ricci flow has proved to be a fundamentally innovative and overwhelmingly powerful tool in geometry and topology. Many long-standing problems have been solved by Ricci flow. For example, the Poincaré conjecture, Thurston's geometrization conjecture, the $\frac{1}{4}$-pinched differentiable sphere theorem, the generalized Smale conjecture, and so on.

Ricci flow acts like a heat equation, averaging and smoothing out the metric. Thus, the geometry of the manifold tends to become more uniform under Ricci flow. Under certain curvature positivity conditions, Ricci flow can improve the original metric to a more standard one. In the very first paper of Ricci flow [Ha82], Hamilton proved that any closed three-manifold admitting a metric with positive Ricci curvature must be diffeomorphic to a spherical space form. Later, Hamilton proved in [Ha86] that any closed three-manifold admitting a metric with positive curvature operator must be diffeomorphic to a spherical space form. Böhm and Wilking generalized this result to higher dimensions in [BW08]. In [BS08, BS09], Brendle and Schoen solved the long-standing $\frac{1}{4}$-pinched differentiable sphere theorem using Ricci flow.

However, Ricci flow is also a nonlinear equation, and for both geometric and topological reasons, singularities often arise. Prototypical examples of singularities include the intuitive movies of neck-pinches forming on dumbbells presented by Hamilton [Ha95]. By the singularity formation of a Ricci flow, we mean the space-time points or regions on the manifold where the curvature tends to infinity as one approaches the singularity time. Generally, the goal of Ricci flow is to overcome the singularity formation by controlled topological-geometric surgery so that certain geometric decomposition of the original closed manifold can be clearer. In the seminal works of Hamilton and Perelman, it is crucial to understand at least the qualitative behaviors of the singularities in dimension 3 so that Ricci flow can be continued via surgeries. Perelman solved the Poincaré conjecture and the more general Thurston's geometrization conjecture in his celebrated three papers [Per02, Per03a, Per03b] following Hamilton's framework, and the singularity analysis in dimension 3 is key to his success. In a sequence of their papers [BK17a, BK17b, BK19, BK20], Bamler and Kleiner solved the generalized Samle conjecture using the *singular* Ricci flow whose existence was conjectured by Perelman [Per02, Section 13] and confirmed by Kleiner and Lott [KL17, KL20].

As compared to dimension 3, the geometry becomes drastically more complicated starting from dimension 4, and so does the singularity analysis of Ricci flow. We may encounter more complicated singularity models, and consequently the potential surgeries near such singularities will be more intricate. Recently, Bamler in [Bam20a, Bam20b, Bam20c] established a groundbreaking theory for the weak limits (his $\mathbb{F}$-limits) of Ricci flows on closed manifolds in higher dimensions. This theory can be viewed as a parabolic counterpart of the structure theory of spaces with a lower Ricci bound, which was developed mainly by Cheeger, Colding, Tian, Naber, Jiang and other authors. See, e.g., [CC96, CC97, CC20a, CC20b, CCT02, CN13, CN15, JN21, CJN21] and references therein. Bamler's works will be fundamentally important in the study of higher-dimensional Ricci flow singularities.

Together with my collaborators, we have made some contributions to the geometry in the large of singularity models in Ricci flows. We shall introduce some of them in the following sections and the details will be presented in the following chapters.

1.1 Main Results

We shall briefly summarize the main results we wish to present in the dissertation together with some preliminaries. The details should be referred to the corresponding chapters.

1.1.1 Heat Flow Estimates

Let $(M^n, g_t)_{t\in I}$ be a complete Ricci flow, where $I \subset \mathbb{R}$ is an interval. We say that $u = u(x,t) : M \times I \to \mathbb{R}$ is a **heat flow (coupled with Ricci flow)**, if

$$0 = \Box u := \partial_t u - \Delta_{g_t} u, \qquad \text{on } M \times I,$$

where Δ_{g_t} denotes the Laplacian operator induced by the evolving metric g_t. We say that $v = v(y,s) : M \times I \to \mathbb{R}$ is a **conjugate heat flow (coupled with Ricci flow)**, if

$$0 = \Box^* v := -\partial_s v - \Delta_{g_s} v + R_{g_s} v, \qquad \text{on } M \times I,$$

where R_{g_s} is the scalar curvature of the metric g_s.

As will be seen below, Perelman's monotonicity formulae [Per02] and Bamler's sharp estimates [Bam20a] both rely on heat flows and conjugate heat flows.

Whenever integration by parts is valid, for any smooth functions u, v, we have

$$\frac{d}{dt}\int_M u_t v_t \, dg_t = \int_M (\Box u_t) v_t - u_t(\Box^* v_t)\, dg_t. \tag{1.1.1}$$

Here and throughout the dissertation, we denote by

$$u_t := u(\cdot, t)$$

the restriction of u at time t instead of the time derivative of u.

For any complete Ricci flow $(M^n, g_t)_{t\in I}$, there is always a conjugate heat kernel, which we denote by $K(x, t \,|\, y, s)$. (See, e.g., [CCG+10].) By definition, conjugate heat kernels satisfy the following. For any $x, y \in M$, $s < t$, $s, t \in I$,

$$\begin{aligned} \Box K(\cdot, \cdot \,|\, y, s) = 0, &\quad \lim_{t\to s^+} K(\cdot, t \,|\, y, s) = \delta_y; \\ \Box^* K(x, t \,|\, \cdot, \cdot) = 0, &\quad \lim_{s\to t^-} K(x, t \,|\, \cdot, s) = \delta_x. \end{aligned}$$

We write

$$d\nu_{x,t|s} := K(x, t \,|\, \cdot, s)\, dg_s.$$

When M is closed, $d\nu_{x,t|s}$ clearly integrates to 1 by (1.1.1). When M is noncompact and K is, for example, the minimal kernel, then $d\nu_{x,t|s}$ also integrates to 1, which will be proved in Corollary 3.3.2. So we may always take K as the minimal kernel. For any metric space X, we denote by

$$\mathcal{P}(X)$$

the space of probability measures on X. So for a Ricci flow $(M^n, g_t)_{t\in I}, \nu_{x,t|s} \in \mathcal{P}(M)$ for any $x \in M, s < t, s, t \in I$.

For bounded heat flows coupled with a complete Ricci flow, we shall prove in Chapter 3 some rough gradient estimates and also Bamler's sharp gradient estimates ([Bam20a, Theorem 4.1]) without any curvature conditions. Bamler's gradient estimates improved the previous estimates found by Zhang [Zhq06] and Cao-Hamilton [CH09] and they are foundations of Bamler's sharp Nash entropy estimates.

One of the most important ideas in Bamler's recent theory [Bam20a, Bam20b, Bam20c] is H_n-centers [Bam20a, Definition 3.10]. Roughly speaking, they can be viewed as more natural worldlines of a point in past times and conjugate heat kernels concentrate near H_n-centers. The existence of H_n-centers follows by the following important monotonicity formula [Bam20a, Corollary 3.7] proved by Bamler: Let $(M^n, g_t)_{t\in I}$ be a complete Ricci flow. For any nonnegative conjugate heat flows v_1, v_2 such that $d\mu_{i,t} := v_{i,t}\, dg_t \in \mathcal{P}(M)$,

$$\mathrm{Var}_t(\mu_{1,t}, \mu_{2,t}) + H_n t \tag{1.1.2}$$

is nondecreasing in t, where

$$H_n := \frac{(n-1)\pi^2}{4} + 4,$$

and the variance between $\mu, \nu \in \mathcal{P}(M)$ is defined to be

$$\mathrm{Var}_t(\mu, \nu) := \int_M \int_M |xy|_t^2 \, d\mu(x) d\nu(y).$$

Here and throughout the dissertation, we denote by

$$|xy|_t, \quad |x, y|_t, \quad \text{or} \quad \mathrm{dist}_t(x, y)$$

the distance between two points x, y measured by the distance induced by g_t. By definition,

$(z,s)\in M\times I$ is called an H_n-center of (x,t), if $s<t$ and

$$\mathrm{Var}(\delta_z,\nu_{x,t|s})\le H_n(t-s).$$

The existence of z follows by (1.1.2). We shall prove later that (1.1.2) holds for general noncompact complete Ricci flows without any curvature conditions (but $\mu_{i,t}$ in (1.1.2) should be taken as conjugate heat kernels) and thus we may still consider H_n-centers in this setting.

1.1.2 Perelman's Entropy

One of Perelman's most marvelous contributions is his entropy formulae, which provide noncollapsing estimates while taking limits of flows (in the sense of Cheeger-Gromov-Hamilton) and guarantees that the limit is a smooth flow. In this way, Perelman was able to intensively apply Hamilton's scaling arguments [Ha95] to study singularities of Ricci flow. Recently, Bamler [Bam20a] found some sharp estimates on the Nash entropy, which are foundations of his structure theory of noncollapsed limits [Bam20c]. See also the entropy formula for linear heat equations found by Ni [Ni04].

Let (M^n,g) be a complete manifold. Perelman's $\mathcal{W}$-functional at scale $\tau>0$ is defined to be

$$\mathcal{W}(g,f,\tau):=\int_M\big(\tau\left(|\nabla f|^2+R\right)+f-n\big)\,(4\pi\tau)^{-\frac{n}{2}}e^{-f}dg.$$

Writting $u=(4\pi\tau)^{-n/2}e^{-f}$, we may rewrite the $\mathcal{W}$-functional as

$$\overline{\mathcal{W}}(g,u,\tau):=\int_M\big(\tau\left(|\nabla\log u|^2+R\right)-\log u\big)\,u\,dg-\frac{n}{2}\log(4\pi\tau)-n.$$

For any region $\Omega \subseteq M$, following [Wa18], we may localize Perelman's entropy as following:

$$\mu(\Omega, g, \tau) := \inf \left\{ \overline{\mathcal{W}}(g, u, \tau) : u \geq 0, \sqrt{u} \in C_0^\infty(\Omega), \int_M u \, dg = 1 \right\},$$
$$\nu(\Omega, g, \tau) := \inf_{0 < s \leq \tau} \mu(\Omega, g, s).$$

Indeed, $\nu(\Omega, g, \tau)$ is a local Sobolev constant of the region Ω. Here, $C_0^\infty(\Omega)$ denotes the space of smooth functions defined on Ω with compact support.

We then recall pointed entropy along a Ricci flow. Let $(M^n, g_t)_{t \in [-T, 0]}$ be a complete Ricci flow with bounded curvature. For any $(x_0, t_0) \in M \times [-T, 0], 0 < \tau < t_0 + T$, define

$$\mathcal{W}_{x_0, t_0}(\tau) := \overline{\mathcal{W}}(g_s, K(x_0, t_0 \,|\, \cdot, s), \tau) = \mathcal{W}(g_s, f_s, \tau),$$
$$\mathcal{N}_{x_0, t_0}(\tau) := \int_M f_s \, d\nu_{x_0, t_0 | s} - \frac{n}{2},$$

where

$$s := t_0 - \tau, \quad K(x_0, t_0 \,|\, y, s) := (4\pi\tau)^{-n/2} e^{-f_s(y)}.$$

See Chapter 4 for more detailed properties.

We may now state the main results in Chapter 4 which were proved by Chan, Zhang and the author in [CMZ21d].

We first give the following lower bound on the local ν-entropy in terms of the pointed Nash entropy. In the following, $(M^n, g_t)_{t \in I}$ is a complete Ricci flow with bounded curvature.

Theorem 1.1.1 (= Theorem 4.2.1)**.** *Assume that $[-r^2, 0] \subseteq I$. Then for any point $x_0 \in M$ and any τ, $A > 0$, we have*

$$\nu\left(B_0(x_0, Ar), g_0, \tau r^2\right) \geq \mathcal{N}_{x_0, 0}(r^2) - \sqrt{n} A - \frac{n}{2}\tau - \frac{n}{2}\log(1 + \tau). \tag{1.1.3}$$

As a consequence, in Theorem 4.4.2, we can slightly improve Jian's no local collapsing Theorem [J21], which already improved Perelman's original version [Per02, Theorem 8.2] and Wang's improved version [Wa18, Theorem 1.1].

We also have the following almost monotonicity formula for the local μ-entropy, which is similar to Wang [Wa18, Theorem 5.4] and Tian-Zhang [TZ21], and our proof is inspired by their works. Perelman's original (global) monotonicity formula follows immediately by the following local version.

Theorem 1.1.2 (=Theorem 4.2.2)**.** *Assume that $[-r^2, 0] \subseteq I$. Then for any $x_0 \in M$, any H_n-center $(z, -r^2)$ of $(x_0, 0)$, any $\tau > 0$, and any $A \geq 16$, we have*

$$\begin{aligned}&\mu\left(B_{-r^2}\left(z, 2A\sqrt{H_n}r\right), g_{-r^2}, (1+\tau)r^2\right)\\ \leq\ &\mu\left(B_0\left(x_0, A\sqrt{H_n}r\right), g_0, \tau r^2\right) + \frac{C_n}{A^2}(1+\tau)e^{-\frac{A^2}{20}}.\end{aligned}$$

As a refinement of the local monotonicity formula above, we have the following, which can be viewed as the inverse of Theorem 4.2.1. Note that we should consider the ball centered at H_n-centers for the local μ-entropy.

Theorem 1.1.3 (= Theorem 4.2.3)**.** *Assume that $[-r^2, 0] \subseteq I$. Furthermore, assume that $R_{g_{-r^2}} \geq R_{\min}$. Then, for any $x_0 \in M$, any H_n-center $(z, -r^2)$ of $(x_0, 0)$, and any $A \geq 8$, we have*

$$\mu\left(B_{-r^2}\left(z, 2A\sqrt{H_n}r\right), g_{-r^2}, r^2\right) \leq \mathcal{N}_{x_0,0}(r^2) + C(n, R_{\min}r^2, A), \tag{1.1.4}$$

where

$$C(n, R_{\min}r^2, A) = \tfrac{C_n}{A^2}e^{-\frac{A^2}{20}} + 8\left(e^{-\frac{A^2}{20}} \cdot (n - 2R_{\min}r^2) + e^{-\frac{A^2}{40}} \cdot (n - 2R_{\min}r^2)^{\frac{1}{2}}\right),$$

and C_n is a dimensional constant.

As an application, we can use this theorem to give a simple proof of Bing Wang's improved pseudolocality theorem [Wa20, Theorem 1.2], which generalizes Perelman's original pseudolocality theorem [Per02, 10.1]. As a further application, we can give a simple proof of Peng Lu's local curvature bound [Lu10, Theorem 1.2], which improves Perelman's original local curvature bound [Per02, 10.3].

1.1.3 Geometry at Infinity of Ancient Flows

Following Hamilton, we say a Ricci flow $(M^n, g_t)_{t\in I}$ is **ancient**, if the time-span of the flow $I = (-\infty, C]$, for some constant $C \leq \infty$.

We say that a Ricci flow $(M^n, g_t)_{t\in I}$ arises as a **(finite-time) singularity model**, if there exist a closed Ricci flow $(\overline{M}^n, \bar{g}_t)_{t\in[0,T)}$, a sequence $(x_i, t_i) \in \overline{M} \times [0, T), t_i \to T < \infty$, and scaling factors $\lambda_i \to \infty$, such that

$$(\overline{M}^n, \lambda_i g_{t_i+t/\lambda_i}, (x_i, 0))_{t\in[-\lambda_i t_i, 0]} \to (M^n, g_t)_{t\leq 0},$$

in the sense of Cheeger-Gromov-Hamilton. Note that since $\lambda_i t_i \to \infty$, any (finite-time) singularity model must be an ancient solution, and thus it is vital to study ancient flows.

Recall that a sequence of pointed Ricci flows

$$(M_i^n, g_{i,t}, (p_i, t_i))_{t\in[a_i,b_i]} \to (M^n, g_t, (\bar{p}, \bar{t}))_{t\in[a,b]},$$

in the sense of Cheeger-Gromov-Hamilton, where $a, b \in \mathbb{R}$, if

$$a \geq \limsup a_i, \quad b \leq \liminf b_i, \quad \lim t_i = \bar{t} \in [a, b],$$

and there exist

- an exhaustion $\{U_i\}$ of M by open precompact sets with $p_i \in U_i$;

- a sequence of diffeomorphisms $\Phi_i : U_i \to V_i := \Phi_i(U_i) \subset M_i$ with $\Phi_i(p_i) = \bar{p}$,

such that for any $K \Subset M$, and any $k \in \mathbb{N}$,

$$\sup_{0\le p\le k}\sup_{t\in[a,b]}\sup_{K}\left|\nabla_g^p(\Phi_i^* g_{i,t} - g_t)\right|_g \to 0,$$

as $i \to \infty$, where g is a metric on M comparable to g_t. If $a_i \to -\infty$, then $[a, b]$ should be understood as $(-\infty, b]$, and the convergence should be uniform over any compact subinterval of $(-\infty, b]$.

Let us recall the definition of Ricci solitons, which are self-similar solutions to the Ricci flow. A triple (M^n, g, f) is called a **gradient Ricci soliton** or **GRS** in short, if (M^n, g) is a complete Riemannian manifold and f is a smooth function on M satisfying

$$\mathrm{Ric} + \nabla^2 f = \tfrac{\lambda}{2} g, \tag{1.1.5}$$

for some constant $\lambda \in \mathbb{R}$. There are three types of solitons depending on the sign of λ. (M^n, g, f) is called a **shrinking gradient Ricci soliton** or **shrinker** in short if $\lambda > 0$; it is a **steady GRS** if $\lambda = 0$; and it is an **expanding GRS** if $\lambda < 0$. By normalization on the metric, we may assume that λ can only be $-1, 0, 1$. We remark that we usually assume that $\mathrm{Ric} = \nabla^2 f$ for steady GRS, which will be clear later. Given a GRS satisfying (1.1.5), there is a canonical form of Ricci flow solution induced by the GRS. Let $\tau_t = 1 + \lambda t$, ($\lambda = -1, 0$, or 1,) and Φ_t be a family of diffeomorphisms defined by $\partial_t \Phi_t = \tau_t^{-1} \nabla f|_{\Phi_t}$. Then $g_t := \tau_t \Phi_t^* g$ is a Ricci flow, called the canonical form of the GRS. g_t only moves by scaling and pullback by diffeomorphisms, and thus it is self-similar.

To capture the big pictures of geometry in the large of ancient flows, we would like to understand the geometry at infinity, which are space-time blow-downs. There are two notions of blow-downs for ancient flows. Perelman first introduced his comparison L-geometry for Ricci flows and proved that for κ-solutions ([KL08, Definition 38.1]), we can always get a limit along the ℓ-centers via type-I scalings in the sense of Cheeger-Gromov-Hamilton smooth convergence. Such limit flows are shrinking gradient Ricci solitons by Perelman's monotonicity formula for the reduced volume and they are called the **asymptotic shrinkers**. They are essential in Perelman's singularity analysis. Please see Chapter 5 for detailed definitions.

However, Perelman's machinery cannot be easily generalized to higher dimension in part due to the lack of Hamilton's Harnack inequality [Ha93] in higher dimensions, which holds for Ricci flows with nonnegative curvature operator. By Hamilton-Ivey pinching estimates, it was proved by Chen [Ch09] that any three-dimensional ancient flows must have nonnegative curvautre and Perelman's asymptotic shrinkers are canonical in dimension 3. However, in higher dimensions, Hamilton-Ivey estimate no longer holds and ancient flows may not have nonnegative curvature. (For example, the FIK shrinkers found by Feldman-Ilmanen-Knopf [FIK03].) Thus the existence of asymptotic shrinkers is no longer natural. One may still prove the existence of Perelman's asymptotic shrinkers under weaker curvature conditions. See, e.g., [CZ11, Zhy20, CZ20, MZ21].

In his recent works [Bam20b, Bam20c], Bamler introduced the notion of **tangent flows at infinity**, which can be understood as space-time blow-downs in the sense of his newly introduced $\mathbb{F}$-distance. He developed a completely general compactness theory for super Ricci flows, which is parallel to Gromov's compactness theorem for manifolds with Ricci bounded from below and Brakke's compactness theory for mean curvature flows. The existence of tangent flows at infinity is much more canonical as compared to Perelman's asymptotic shrinkers because of the general $\mathbb{F}$-compactness theory. With the help of his structure theory, parallel to the celebrated Cheeger-Colding-Naber theory, Bamler built up partial regularity results for such tangent flows at infinity and proved that the singular set of any noncollapsed (in the sense of the Nash entropy) limit has codimension no less than 4.

In [Bam20c], Bamler suggested that his (smooth) tangent flows at infinity should coincide with Perelman's asymptotic shrinkers. Chan, Zhang, and the author confirmed this statement in our recent work [CMZ21a]. Roughly speaking, we have the following

Theorem 1.1.4 (= Theorem 5.0.1)**.** *For an ancient Ricci flow $(M^n, g_t)_{t\leq 0}$ with bounded curvature on compact intervals, if a (smooth) asymptotic shrinker exists, then it must be a tangent flow at infinity. Conversely, if a tangent flow at infinity is smooth, then it also must arise as an asymptotic shrinker.*

Tangent flows at infinity resemble asymptotic cones of metric spaces, which are blow-downs centered at some basepoint. It is well-known that such cones do not depend on the basepoint, although they may depend on the sequence of scalings. In [CMZ21c], we showed that tangent flows at infinity of ancient Ricci flows also do not depend on the choice of basepoints. Roughly speaking, we have the following.

Theorem 1.1.5 (=Theorem 5.0.2)**.** *The tangent flow at infinity of an ancient H-concentrated (c.f. [Bam20b, Definition 3.30]) metric flow does not depend on the basepoint.*

1.1.4 On Steady Ricci Solitons

We shall then present several recent results on general steady gradient Ricci solitons with reasonable assumptions that they either arise as singularity models or have bounded curvature. Note that by [CFSZ20, Theorem 1], any four-dimensional steady GRS that arises as a singularity model have bounded curvature. It is important to study steady gradient Ricci solitons as they may arise as singularity models. Moreover, as seen in the recent breakthroughs on the classification of κ-solutions in dimension 3 by Brendle [Br20] and Brendle-Daskalopoulos-Šešum[BDS21], it is important to first classify steady Ricci solitons (by Brendle [Br13]).

As suggested by the known examples and our previous work [CDM22], the geometry of a steady soliton outside of a compact set should be determined by its asymptotic geometry. Tangent flows at infinity reflect the global geometry of a steady soliton since they are space-time blow-downs. Joint with Bamler, Chow, Deng and Zhang, in [BCDMZ], we fully classified the tangent flows at infinity of 4-dimensional steady gradient Ricci solitons that can arise as singularity models. Using a splitting principle for steady solitons and Bamler's structure theory in [Bam20c], we proved that the tangent flow at infinity is *unique*, and we classified them as one of the two types:

$$(\mathbb{S}^3/\Gamma)\times\mathbb{R} \qquad \text{and} \qquad \mathbb{S}^2\times\mathbb{R}^2 \quad \text{or} \quad \big((\mathbb{S}^2\times\mathbb{R})/\mathbb{Z}_2\big)\times\mathbb{R}.$$

Joint with Bamler, Chow, Deng and Zhang, in [BCDMZ], we have a clear qualitative description of steady Ricci solitons whose (unique) tangent flow at infinty is $(\mathbb{S}^3/\Gamma)\times\mathbb{R}$.

Theorem 1.1.6 (= Theorem 6.3.1)**.** *Let (M^4, g, f) be a 4-dimensional complete steady gradient Ricci soliton that is a singularity model. Then the tangent flow at infinity is unique. If the tangent flow at infinity is $(\mathbb{S}^3/\Gamma)\times\mathbb{R}$, then, for any $\epsilon > 0$, outisde a compact set, we have that each point is the center of an ϵ-neck, (M, g) has positive curvature operator, and linear curvature decay.*

Joint with Bamler, Chan, and Zhang, in [BCMZ21], we proved an optimal volume growth estimate for steady gradient Ricci solitons with bounded Nash entropy. First of all, we recall some known results on the volume growth rate for steady solitons. Besides the $r^{\frac{n}{2}}$ volume growth rate lower bound mentioned above (c.f. [CMZ21b]), Munteanu-Šešum [MS13] showed that a steady soliton has at least linear volume growth, Cui [Cui16] proved a volume growth lower bound for steady Kähler Ricci solitons with positive Ricci curvature. The *optimal* volume growth lower bound proved in this paper says that a steady gradient Ricci soliton with *bounded Nash entropy* has volume growth rate no smaller than $r^{\frac{n+1}{2}}$. Since the Bryant soliton (c.f. [Cao09]) as well as Appleton's solitons ([Ap17], which are asymptotic to quotients of the Bryant soliton) have exactly this volume growth rate, our result is indeed optimal.

Theorem 1.1.7 (= Theorem 6.5.1)**.** *Suppose that (M^n, g, f) is a complete steady gradient Ricci soliton such that the canonical form $(M^n, g_t)_{t\in\mathbb{R}}$ is noncollapsed: $\mu_\infty := \inf_{\tau>0}\mathcal{N}_{o,0}(\tau) > -\infty$. Additionally, assume that **either one** of the following conditions is true:*

(1) $(M^n, g_t)_{t\in\mathbb{R}}$ arises as a singularity model; or

(2) (M^n, g) has bounded curvature.

Then

$$c(n,\mu_\infty)r^{\frac{n+1}{2}} \le |B_r(o)| \le C_n r^n \quad \text{for all} \quad r > \bar{r}(n,\mu_\infty),$$

where $c(n,\mu_\infty)$ is a positive constant of the form

$$c(n,\mu_\infty) = \frac{c_n}{\sqrt{1-\mu_\infty}}e^{\mu_\infty}.$$

Furthermore, the upper bound is also true for all $r > 0$ (instead of $r \ge \bar{r}(n,\mu_\infty)$).

1.1.5 Steady Solitons with Nonnegative Ricci Curvature

Under some curvature positivity conditions, we can obtain more results on steady Ricci solitons. As before, for a steady GRS (M^n, g, f), we normalize the metric so that

$$\mathrm{Ric} = \nabla^2 f, \qquad R + |\nabla f|^2 = 1.$$

For shrinking gradient Ricci solitons, by Cao and Zhou [CZ09], the potential function always has a quadratic growth without any curvature conditions. However, it is nontrivial to obtain growth estimates for the potential functions of steady gradient Ricci solitons, partly because Ricci-flat spaces are always steady Ricci solitons and the potential function can be taken as a constant or linear (in Euclidean space, for example). We derive several results on the growth estimates of the potential function assuming certain

Ricci nonnegative conditions. Our proofs are based on existing arguments mainly from [DZ20b, CDM22].

Perelman's compactness theorem for three-dimensional κ-solutions is very important in his resolution of the Poincaré conjecture. To prove his compactness theorem, Perelman used a scaling argument. In [CDM22], we adapted such arguments for four-dimensional steady gradient Ricci solitons with nonnegative Ricci outside a compact set and uniform curvature decay. Similar to [Per02, 11.4], we proved the following in [CDM22].

Theorem 1.1.8 ([CDM22, Theorem 1.10]). *Let (M^4, g, f) be a complete steady gradient Ricci soliton with nonnegative Ricci curvature outside a compact set and uniform scalar curvature decay. If (M, g) is not Ricci-flat, then* $\mathrm{AVR}(g) = 0$.

We remark that for Riemannian manifolds with nonnegative Ricci curvature outside a compact, **asymptotic volume ratio (AVR)** is still well-defined; See Chapter 2 for more details. We shall give a slightly simpler proof for the above Theorem by using a point picking argument for steady Ricci solitons.

1.2 Notations

We collect some of the notations or conventions we use in the dissertation.

- Unless explicitly stated, capital Roman or Greek letters denote large constants (larger than 1), while lowercase Roman or Greek letters denote small constants (less than 1). We denote by $C = C(a, b, \cdots)$ or $C = C_{a,b,\cdots}$ a (large) constant depending on parameters $a, b, \cdots$. Constants may vary from line to line.

- Following [Bam20a, 2.1], we write "if $\epsilon \le \bar{\epsilon}(a, b, \cdots)$" to mean that "there is a (small) constant $\bar{\epsilon}$ depending on $a, b, \cdots$ such that if $\epsilon \le \bar{\epsilon}(a, b, \cdots)$, then $\cdots$." Similarly, we write "if $A \ge \bar{A}(a, b, \cdots)$".

- Sometimes, we use Cheeger's notations as in [Bam20a]. We denote by

$$\Psi(a_1, \cdots, a_k \,|\, b_1, \cdots, b_m)$$

any function that depends on $a_1, \cdots, a_k, b_1, \cdots, b_m$ and tends to 0 if $(a_1, \cdots, a_k) \to 0$ and $b_1, \cdots, b_m$ are fixed. We will write $\Psi = \Psi(a_1, \cdots, a_k \,|\, b_1, \cdots, b_m)$ when there is no ambiguity and we will add lower indices such as $\Psi_1, \Psi_2, \cdots$ to denote some other small quantities to distinguish from Ψ. The exact values of these functions may vary from line to line.

- For a metric space X, we shall denote by

$$|xy|, \quad |x, y|, \quad \text{ or } \mathrm{dist}(x, y)$$

the distance between two points $x, y \in X$. We shall write $|xy|_X$, or $\mathrm{dist}^X(x, y)$, to stress that the underlying space is X. We denote by

$$B_x(r) := B(x, r) := \{y \in X : |xy| < r\}$$

the open ball centered at x with radius r. For $\Omega \subset X$, we write

$$\Omega \Subset X,$$

if the closure $\bar{\Omega}$ is compact. If $X = (M, g)$ is a Riemannian manifold, we denote by dg the volume form induced by the metric g. For any measurable subset $\Omega \subseteq M$, we denote by

$$|\Omega| := |\Omega|_g := \int_\Omega dg$$

the volume of Ω.

- When it is clear from the context, if $(M, g_t)_{t\in I}$ is a family of metrics, we denote by

$$|xy|_t, \quad \mathrm{dist}_t(x,y) \quad \text{ or } \mathrm{dist}_{g_t}(x,y)$$

the distance between $x, y \in M$ measured by the metric g_t. We denote by

$$B_t(x,r) := B(x,r;g_t) := \{y \in M : |xy|_t < r\}$$

the open ball centered at x with radius r with respect to the metric g_t. For any measurable subset $\Omega \subseteq M$, we denote by

$$|\Omega|_t := |\Omega|_{g_t} := \int_\Omega dg_t$$

the volume of Ω with respect to the metric g_t.

- For a function $u = u(x,t) : M \times I \to \mathbb{R}$, we write

$$u_t := u(\cdot, t)$$

as the restriction of u at time t instead of the time derivative of u.

- Let $(M^n, g_t)_{t\in I}$ be a complete Ricci flow. We denote by

$$K(x,t\,|\,y,s)$$

the minimal conjugate heat kernel coupled with the flow. We write

$$d\nu_{x,t|s} := K(x,t\,|\,\cdot,s)\,dg_s.$$

See the previous section for details.

Chapter 2

Manifolds with Nonnegative Ricci near Infinity

In this chapter, we shall study some geometric aspects of Riemannian manifolds with nonnegative Ricci curvature outside a compact set.

Let (M^n, g) be a Riemannian manifold. We write

$$\bar{B}_x(r) = \{y \in M : |xy| \leq r\}, \quad \mathcal{A}_x(r_1, r_2) = B_x(r_2) \setminus \bar{B}_x(r_1).$$

For any measurable set $\Omega \subset M$, we denote by $|\Omega|$ the volume of Ω induced by g. We write

$$V_x(r) := |B_x(r)|.$$

For $\Lambda \geq 0$, we shall denote by

$$v_{-\Lambda}^n(r) := \int_0^r \Lambda^{-(n-1)/2} \sinh^{n-1}(\sqrt{\Lambda} s)\, ds$$

the volume of balls in the n-dimensional space form.

2.1 Preliminaries

Let us recall some concepts related to the Gromov-Hausdorff distance.

First, let us record the standard Bishop-Gromov volume comparison theorem for star-shaped regions. On a manifold (M^n, g), we say that a subset Ω is **star-shaped based at** $x \in M$, if for any $y \in \Omega$, any minimal geodesic from x to y is contained in Ω.

Theorem 2.1.1. *Suppose that (M^n, g) is a manifold and $B_o(2A) \Subset M$ with* Ric $\geq -(n-1)\Lambda$ *on $B_o(2A)$. Then for any $x \in B_o(A)$ and $0 < s < r < A$, if $W_x \subset B_o(2A)$ is a star-shaped region based at x, then*

$$\frac{|W_x \cap B_x(r)|}{v^n_{-\Lambda}(r)} \leq \frac{|W_x \cap B_x(s)|}{v^n_{-\Lambda}(s)}.$$

The proof is a slight variant of the standard Bishop-Gromov comparison theorem. See, for example, [CGT82, Remark 4.1].

Let Z be a metric space. For two subsets $U, V \subset Z$, their **Hausdorff distance**, denoted by $\mathrm{dist}_{\mathrm{H}}(U, V)$, is defined to the smallest number $r > 0$, such that

$$U \subset B_r(V), \quad V \subset B_r(U),$$

where

$$B_r(U) := \{z \in Z : \mathrm{dist}(z, U) < r\}.$$

Let X, Y be two metric spaces. The **Gromov-Hausdorff distance** between X, Y, written as $\mathrm{dist}_{\mathrm{GH}}(X, Y)$, is defined to the smallest number $r > 0$, such that there are embeddings $\phi : X \to Z, \psi : Y \to Z$ satisfying

$$\mathrm{dist}^Z_{\mathrm{H}}(\phi(X), \psi(Y)) < r.$$

Fix $x \in X, y \in Y$. The **pointed Gromov-Hausdorff (pGH) distance** between X, Y, written as $\mathrm{dist}_{\mathrm{pGH}}((X, x), (Y, y))$, is defined to the smallest number $r > 0$, such that there are embeddings $\phi : X \to Z, \psi : Y \to Z$ satisfying

$$\mathrm{dist}_{\mathrm{H}}^{Z}(\phi(X), \psi(Y)) + |\phi(x), \psi(y)| < r.$$

Let $X, X_i, i \in \mathbb{N}$ be a sequence of metric spaces. We say that $(X_i, p_i) \to (X, p)$ in the sense of **pointed Gromov-Hausdorff (pGH) distance**, if for any $A > 0$,

$$\mathrm{dist}_{\mathrm{pGH}}((B_{p_i}(A), p_i), (B_p(A), p)) \to 0.$$

Let X be a metric space. For $\Omega \subset X, \epsilon > 0$, we define the **capacity** of Ω at scale ϵ, denoted by $\mathrm{Cap}_\epsilon(\Omega)$, to be the cardinality of a maximal subset $\{p_\alpha\}_{\alpha=1}^N$ of Ω satisfying

$$|p_\alpha p_\beta| \geq \epsilon, \qquad \text{whenever } \alpha \neq \beta.$$

Clearly, if $\{p_\alpha\}_{\alpha=1}^N$ is such a maximal subset, then the balls $B_{p_\alpha}(\epsilon/2)$ are disjoint and

$$\Omega \subset \bigcup_{\alpha=1}^{N} B_{p_\alpha}(\epsilon).$$

We define the **covering number** of Ω at scale ϵ, written as $\mathrm{Cov}_\epsilon(\Omega)$, to be the cardinality of a minimal subset $\{x_i\}_{i=1}^m$ of Ω satisfying

$$\Omega \subset \bigcup_{i=1}^{m} B_{x_i}(\epsilon).$$

If we relax the requirement that $x_i \in \Omega$, we may define $\mathrm{Cov}_\epsilon^*(\Omega)$ to be the cardinality of a minimal subset $\{y_i\}_{i=1}^k$ of X satisfying

$$\Omega \subset \bigcup_{i=1}^{m} B_{y_i}(\epsilon).$$

We shall collect some elementary observations in the following Lemma.

Lemma 2.1.2. *Let X be a metric space, $\Omega \subset X$, and $\epsilon > 0$.*

- $\mathrm{Cov}_\epsilon(\Omega) \leq \mathrm{Cap}_\epsilon(\Omega)$.
- $\mathrm{Cov}^*_\epsilon(\Omega) \leq \mathrm{Cov}_\epsilon(\Omega) \leq \mathrm{Cov}^*_{\epsilon/2}(\Omega)$.
- *For* $0 < \epsilon_1 \leq \epsilon_2$,

$$\mathrm{Cap}_{\epsilon_2}(\Omega) \leq \mathrm{Cap}_{\epsilon_1}(\Omega), \quad \mathrm{Cov}_{\epsilon_2}(\Omega) \leq \mathrm{Cov}_{\epsilon_1}(\Omega), \quad \mathrm{Cov}^*_{\epsilon_2}(\Omega) \leq \mathrm{Cov}^*_{\epsilon_1}(\Omega).$$

- *For* $\Omega_1 \subset \Omega_2 \subset X$,

$$\mathrm{Cap}_\epsilon(\Omega_1) \leq \mathrm{Cap}_\epsilon(\Omega_2), \quad \mathrm{Cov}^*_\epsilon(\Omega_1) \leq \mathrm{Cov}^*_\epsilon(\Omega_2).$$

- $\mathrm{Cov}_\epsilon, \mathrm{Cov}^*_\epsilon$ *are sub-additive: For any* $U, V \subset X$,

$$\mathrm{Cov}_\epsilon(U \cup V) \leq \mathrm{Cov}_\epsilon(U) + \mathrm{Cov}_\epsilon(V),$$

  and the same holds for Cov^*_ϵ.

Proof. We only prove that $\mathrm{Cov}_\epsilon(\Omega) \leq \mathrm{Cov}^*_{\epsilon/2}(\Omega)$, since the other properties are straightforward. Let $\{y_i\}_{i=1}^m$ be a minimal subset of X satisfying $\Omega \subset \bigcup_{i=1}^m B_{y_i}(\epsilon/2)$. For each i, we may pick $x_i \in B_{y_i}(\epsilon/2) \cap \Omega$. Then $B_{y_i}(\epsilon/2) \subset B_{x_i}(\epsilon)$, and thus $\Omega \subset \bigcup_{i=1}^m B_{x_i}(\epsilon)$. The conclusion follows. $\square$

We have the following standard estimate on the capacity as an application of the standard Bishop-Gromov volume comparison theorem.

Lemma 2.1.3. *Suppose that* $\mathrm{Ric} \geq -(n-1)\Lambda/r^2$ *on* $B_o(3r)$. *Then for any* $\epsilon \in (0, 1)$,

$$\mathrm{Cap}_{\epsilon r}(\bar{B}_o(r)) \leq C(n, \Lambda, \epsilon).$$

Proof. By rescaling, we may assume that $r = 1$. Let $\{p_\alpha\}_{\alpha=1}^N$ be a maximal subset of $\bar{B}_o(1)$ such that $|p_\alpha p_\beta| \geq \epsilon/2$. Then the balls $B_{p_\alpha}(\epsilon/2)$ are disjoint and thus

$$V_o(2) \geq \sum_{\alpha=1}^{N} V_{p_\alpha}(\epsilon/2) \geq c(n, \Lambda, \epsilon) \sum_{\alpha=1}^{N} V_{p_\alpha}(3) \geq c(n, \Lambda, \epsilon) V_o(2) N.$$

The conclusion follows. $\square$

Using the notations introduced above, we can now state Gromov's criterion for precompactness and we refer to, for example, [Pe16, Proposition 11.1.10] for a proof and detailed discussions.

Theorem 2.1.4. *Let $\mathcal{M}$ be a class of pointed complete metric spaces. Then $\mathcal{M}$ is precompact in the pGH sense if and only if there is a function $N : (0,1) \times (A, \infty) \to (0, \infty)$ such that*

$$\mathrm{Cov}_\epsilon(B_x(r)) \leq N(\epsilon, r),$$

for any $\epsilon \in (0,1), r > A, (X, x) \in \mathcal{M}$.

2.2 A Compactness Theorem

We say that a pointed manifold $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$, for some constants $a > 0$, $\Lambda \geq 0$, if (M^n, g) is a complete noncompact manifold satisfying

$$\mathrm{Ric} \geq 0 \quad \text{ on } M \setminus B_o(a),$$

and

$$\mathrm{Ric} \geq -(n-1)\Lambda/a^2 \quad \text{ on } \bar{B}_o(a).$$

This notation was introduced in [CDM22]. Clearly, for $\lambda > 0$,

$$(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda) \iff (M^n, \lambda^2 g, o) \in \underline{\mathcal{R}}^n(a\lambda, \Lambda).$$

We shall prove the following precompactness theorem in this section.

Theorem 2.2.1. *For any $A > 0, \Lambda \geq 0$,*

$$\bigcup_{a \leq A} \underline{\mathcal{R}}^n(a, \Lambda)$$

is precompact in the sense of pointed Gromov–Hausdorff distance.

Recall that for a metric space X, metric space Z is called an **asymptotic cone** of X, if there exist points $o \in X, z \in Z$, and a sequnce $\lambda_i \to 0$, such that

$$(\lambda_i X, o) \to (Z, z),$$

in the pointed Gromov-Hausdorff sense.

Corollary 2.2.2. *For any $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$ and any sequence $\lambda_i \to 0$, by passing to a subsequence,*

$$(M, \lambda_i^2 g, o) \to (Z, \mathrm{dist}_Z, z),$$

in the pointed Gromov-Hausdorff sense for some length space (Z, dist_Z). Namely, asymptotic cones are well defined and they are independent of the basepoint o.

Proof. Since $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$, $(M, \lambda_i^2 g, o) \in \underline{\mathcal{R}}^n(a\lambda_i, \Lambda)$. The conclusion follows by Theorem 2.2.1. $\square$

To prove Theorem 2.2.1, we shall follow Zhong-dong Liu's arguments in [Liu91] to bound the covering numbers and then apply Gromov's compactness criterion Theorem 2.1.4. Let $U \subset \partial B_o(2a)$ for some $a > 0$. Define

$$\begin{aligned}\mathcal{K}(U) := \mathcal{K}(U; o, 2a) := \{x \in M : &\text{ there is a minimizing geodesic } \gamma : [0, \ell] \to M \\ &\text{passing through } x \text{ with } \gamma(0) = o, \gamma(2a) \in U\}.\end{aligned}$$

We may need the following Lemma from [Liu91, Chapter 3].

Lemma 2.2.3. *Let $U \subset \partial B_o(2a)$ be a subset with* $\operatorname{diam} U \leq 2\eta a$ *for some $\eta \in (0,1]$. Then for any $y_1, y_2 \in \mathcal{K}(U) \setminus B_o(2a)$, and any minimal geodesic $\gamma : [0,\ell] \to M$ from y_1 to y_2, γ never enters $B_o\left((2-\eta)a\right)$.*

Proof. By rescaling, we may assume that $a = 1$. For both $i = 1, 2$, let $\gamma_i : [0, \ell_i] \to M$ be a minimal geodesic from o to y_i such that $\gamma_i(2) \in U$. Then

$$|y_1 y_2| \leq |y_1, \gamma_1(2)| + |\gamma_1(2), \gamma_2(2)| + |\gamma_2(2), y_2| \leq \ell_1 + \ell_2 - 4 + 2\eta.$$

Suppose that there is a point $y_0 = \gamma(s) \in B_o(2-\eta)$ for some $s \in (0,\ell)$. Then

$$|y_1 y_2| = |y_1 y_0| + |y_2 y_0| \geq \ell_1 + \ell_2 - 2|o y_0| > \ell_1 + \ell_2 - 4 + 2\eta,$$

which is a contradiction. $\square$

We use the following notation for simplicity. For any $x \in M, \Omega \subset M$, we denote by

$$\mathcal{S}(x, \Omega)$$

the set of points $y \in M$ such that there is a minimizing geodesic from x to some point in Ω that passes through y. Clearly, $\mathcal{S}(x,\Omega)$ is star-shaped based at x.

We first prove the following volume comparison on $\mathcal{K}(U)$.

Lemma 2.2.4. *Suppose that $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$. Let $U \subset \partial B_o(2a)$ with* $\operatorname{diam} U \leq a$. *If $10a \leq r_1 < r_2$, and $r \leq r_1/2$, then for any $x \in \mathcal{K}(U) \cap \mathcal{A}_o(r_1, r_2)$,*

$$\frac{|\mathcal{K}(U) \cap \mathcal{A}_o(r_1, r_2)|}{|B_x(r)|} \leq (2r_2/r)^n.$$

Proof. Write $\Omega := \mathcal{K}(U) \cap \mathcal{A}_o(r_1, r_2)$. Let

$$W := \mathcal{S}(x, \Omega \cup B_x(r)).$$

For any $y \in W$, any minimal geodesic from x to y stays away from $B_o(a)$ by Lemma 2.2.3. So Ric ≥ 0 on W. For any $y \in W$,

$$|xy| \leq |ox| + |oy| < 2r_2.$$

Thus, $\Omega \subset W \cap B_x(2r_2)$. It follows from Theorem 2.1.1 (the star-shaped version of Bishop-Gromov comparison theorem) that

$$\frac{|\Omega|}{|B_x(r)|} \leq \frac{|W \cap B_x(2r_2)|}{|W \cap B_x(r)|} \leq (2r_2/r)^n.$$

$\square$

Lemma 2.2.5. *Suppose that* $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$ *for some* $a \leq 1$. *Let* $U \subset \partial B_o(2a)$ *with* $\mathrm{diam} U \leq a$. *Then for any* $r \geq 10, \epsilon \in (0, r)$,

$$\mathrm{Cov}_\epsilon \left(\mathcal{K}(U) \cap \mathcal{A}_o(r_1, r)\right) \leq (8r/\epsilon)^n,$$

where $r_1 = \max(10a, \epsilon/2)$.

Proof. Let $r_1 = \max(10a, \epsilon/2)$ and $\Omega := \mathcal{K}(U) \cap \mathcal{A}_o(r_1, r)$. Following the definition of $\mathrm{Cap}_{\epsilon/2}$, let $\{x_i\}_{i=1}^N$ be a maximal subset of Ω such that $|x_i x_j| \geq \epsilon/2,$ whenever $i \neq j$. Then the balls $B_{x_i}(\epsilon/4)$ are pairwise disjoint. Since $\epsilon/4 \leq r_1/2$, by Lemma 2.2.4,

$$|\Omega| \geq \sum_{i=1}^N |B_{x_i}(\epsilon/4)| \geq \left(\frac{\epsilon/4}{2r}\right)^n |\Omega| N.$$

So

$$\mathrm{Cov}_\epsilon(\Omega) \le \mathrm{Cap}_{\epsilon/2}(\Omega) = N \le (8r/\epsilon)^n.$$

□

We can now prove a covering result for balls.

Theorem 2.2.6. *Suppose that* $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$ *with* $a \le 1$. *Then for* $r \ge 100a, \epsilon \in (0, r)$,

$$\mathrm{Cov}_\epsilon(B_o(r)) \le \begin{cases} 1 + C(n,\Lambda)(r/\epsilon)^n, & \textit{if } \epsilon/2 \ge 10a, \\ C(n,\Lambda,\epsilon) + C(n,\Lambda)(r/\epsilon)^n, & \textit{if } \epsilon/2 < 10a. \end{cases}$$

Proof. We follow Liu's construction of a covering of $\partial B_o(2a)$ by subsets with diameter less than $a \le 1$. By Lemma 2.1.3, if $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$, there is a covering

$$\left\{U_\alpha = B_{p_\alpha}(a/2)\right\}_{\alpha=1}^{m}$$

of $\partial B_o(2a)$ and

$$m \le C(n, \Lambda).$$

Clearly, $\mathrm{diam} U_\alpha < a$ for each α, and

$$M \setminus B_o(10a) \subset \bigcup_{\alpha=1}^{m} \mathcal{K}(U_\alpha).$$

Let $r_1 = \max(10a, \epsilon/2)$. By Lemma 2.2.5 and the sub-additivity of Cov_ϵ,

$$\begin{aligned} \mathrm{Cov}_\epsilon\left(\mathcal{A}_o(r_1, r)\right) &\le \sum_{\alpha=1}^{m} \mathrm{Cov}_\epsilon\left(\mathcal{K}(U_\alpha) \cap \mathcal{A}_o(r_1, r)\right) \\ &\le m(8r/\epsilon)^n \le C(n,\Lambda)(r/\epsilon)^n. \end{aligned}$$

It remains to estimate $\mathrm{Cov}_\epsilon(\bar{B}_o(r_1))$. If $\epsilon/2 \ge 10a$, then $r_1 = \epsilon/2$, and $\mathrm{Cov}_\epsilon(\bar{B}_o(r_1)) = 1$.

If $\epsilon/2 < 10a$, then $r_1 = 10a$, and by Lemma 2.1.3,

$$\mathrm{Cov}_\epsilon(\bar{B}_o(r_1)) \leq \mathrm{Cap}_{\epsilon a}(\bar{B}_o(10a)) \leq C(n, \Lambda, \epsilon),$$

where we used the assumption that $a \leq 1$. The conclusion follows by the sub-additivity of Cov_ϵ. □

Now we are ready to prove the compactness theorem 2.2.1.

Proof of Theorem 2.2.1. By rescaling, it suffices to prove the theorem for $A = 1$. Let $\epsilon \in (0, 1), r > 10$ be arbitrarily fixed. Let $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$ for some $a \leq 1, \Lambda \geq 0$. By Theorem 2.2.6,

$$\mathrm{Cov}_\epsilon(B_o(r)) \leq C(n, \Lambda, \epsilon) + C(n, \Lambda)(r/\epsilon)^n.$$

The conclusion follows by Gromov's criterion Theorem 2.1.4. □

Let us also record Liu's covering theorem [Liu91, Theorem 1].

Theorem 2.2.7. *Suppose that $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$. For $r \geq 100a, \alpha > 0$, if $\alpha r \geq 20a$, then*

$$\mathrm{Cov}_{\alpha r}(B_o(r)) \leq C(n, \Lambda, \alpha).$$

Proof. By rescaling, we may assume that $a = 1$ and we may assume that $\alpha \in (0, 1)$. Applying Theorem 2.2.6 with $\epsilon = \alpha r \geq 20$, we have

$$\mathrm{Cov}_{\alpha r}(B_o(r)) \leq 1 + C(n, \Lambda)\alpha^{-n}.$$

□

As an immediate consequence, there are finitely many ends for any $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$ and the number of ends is bounded by $C(n, \Lambda)$. See [Liu91, Cai91].

2.3 Volume Comparison

In this section, we attempt to prove volume comparison theorems for noncompact manifolds with nonnegative Ricci curvature outside a compact set. We remark that all the arguments in this section can be carried out on a single end with nonnegative Ricci curvature.

2.3.1 Asymptotic Volume Ratio

We shall prove that if (M^n, g) has nonnegative Ricci curvature outside a compact set, we can still make sense of the notion of asymptotic volume ratio.

We first prove a variant of the Bishop–Gromov volume comparison theorem.

Proposition 2.3.1. *Let $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$. For any $x \in M$, if $B_o(a) \subset B_x(b)$ for some $b > 0$, then*

$$\frac{V_x(r) - V_x(b)}{(r-b)^n}$$

is non-increasing in r for $r > b$.

Proof. We write $d_x(y) := |xy|$. It is standard to prove that outside of $B_x(b)$,

$$\Delta d_x \leq \frac{n-1}{d_x - b},$$

in the sense of distributions. See, for example, [SY91, Corollary 1.2] or [LT87, Lemma 4.1].

Let $F(r) = |\mathcal{A}_x(b,r)| = V_x(r) - V_x(b)$. For $r > b$,

$$
\begin{aligned}
F'(r) &= \int_{\partial B_x(r)} dA \\
&= \frac{1}{r-b} \int_{\partial B_x(r)} (d_x - b) \frac{\partial}{\partial r}(d_x - b) \\
&= \frac{1}{r-b} \cdot \frac{1}{2} \int_{B_x(r) \setminus B_x(b)} \Delta\big((d_x - b)^2\big) \\
&= \frac{1}{r-b} \int_{B_x(r) \setminus B_x(b)} \big((d_x - b)\Delta(d_x - b) + |\nabla(d_x - b)|^2\big) \\
&\leq \frac{n}{r-b} F(r).
\end{aligned}
$$

Hence

$$
\frac{d}{dr} \frac{V_x(r) - V_x(b)}{(r-b)^n} \leq \frac{nF(r)}{(r-b)^{n+1}} - \frac{nF(r)}{(r-b)^{n+1}} = 0.
$$

$\square$

Lemma 2.3.2. *Let* $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$*. Then*

$$
\mathrm{AVR}(g) := \lim_{r \to \infty} \frac{V_o(r)}{r^n}
$$

is well-defined and does not depend on the basepoint.

Proof. By the monotonicity formula above, $\mathrm{AVR}(g)$ is well-defined.

For any $x, y \in M$, put $\delta = |xy|$. Suppose that $B_o(a) \subset B_y(b)$ for some $b > 0$. For $r > 0$ sufficiently large,

$$
r^{-n} V_x(r) \leq r^{-n} V_y(r + \delta) \leq r^{-n} \big(V_y(r) - V_y(b)\big) \frac{(r + \delta - b)^n}{(r-b)^n} + r^{-n} V_y(b).
$$

Hence

$$
\lim_{r \to \infty} \frac{V_x(r)}{r^n} \leq \lim_{r \to \infty} \frac{V_y(r)}{r^n}.
$$

By the symmetry of the roles of x, y, $\mathrm{AVR}(g)$ does not depend on the basepoint. $\square$

2.3.2 Volume comparison for small radii

Theorem 2.3.3. *Suppose that* $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$. *For any* $x \in M \setminus B_a(o)$, $\frac{V_x(r)}{r^n}$ *is decreasing for* $r < \ell - a$, *where* $\ell := |ox|$. *For* $r \geq \ell$, $0 < s < r$, *we have*

$$\frac{V_x(r)}{V_x(s)} \leq C(\Lambda, n) \left(1 + \tfrac{\Lambda \ell}{a}\right)^{n-1} \left(\frac{r}{s}\right)^n.$$

This result was also previously asserted by Cai in [Cai91]. The constant in this comparison theorem depends on the distance to the origin and it is almost optimal as is seen from the example below.

Example. Let N be a Riemannian manifold with $\mathrm{Rm} > 0, \mathrm{AVR} > 0$. Let $M = N \# (\mathbb{S}^{n-1}_\epsilon \times [0, \infty))$ be the connected sum of N with a thin cylinder of radius ϵ. Pick a point x on the cylinder with $\ell = |ox| \gg 10$. Then

$$\frac{V_x(2\ell)}{V_x(\ell/2)} \sim \frac{\epsilon^{n-1}\ell + \ell^n}{\epsilon^{n-1}\ell} \sim (\ell/\epsilon)^{n-1}.$$

This example has two ends. We may wonder if there is a better volume comparison theorem just assuming in addition that the manifold is connected at infinity. However, we may need the following stronger topological condition, possibly because the topology of a smooth manifold is not as rigid under Ricci curvature restrictions.

We say that a pointed Riemannian manifold (M^n, g, o) has **connected annuli at distances at least** $r_0 > 0$, if for any $r \geq r_0$, there is an open set Ω_r such that

$$\Omega_r \text{ is connected}, \qquad \text{and } \mathcal{A}_o(r/2, 2r) \subset \Omega_r \subset \mathcal{A}_o(r/3, 3r). \tag{CA}$$

Theorem 2.3.4. *Suppose that* $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$ *and satisfies* (CA) *at distances at* *least* $r_0 \geq 10(1+a)$. *Then for any* $r \geq r_0$, $x \in \partial B_o(r)$, $\alpha \in (0, 1/10]$,

$$\frac{|\mathcal{A}_o(r/2, 2r)|}{|B_x(\alpha r)|} \leq C(n, \Lambda, \alpha).$$

Proof. For any $r \geq r_0$, let Ω_r be the connected domain as in (CA). By [Liu91, Theorem 1] or Theorem 2.2.7, there are points $p_1, \ldots, p_N$ in Ω_r satisfying

$$\Omega_r \subset \bigcup_{i=1}^{N} B_{p_i}(\alpha r),$$

where

$$N \leq \bar{N} = \bar{N}(n, \Lambda, \alpha).$$

Claim: For any $1 \leq a, b \leq N$, we can find a subsequence $i_0, i_1, \ldots, i_m$ such that

$$a = i_0, \quad b = i_m, \quad |p_{i_j} p_{i_{j+1}}| \leq 2\alpha r, \quad m \leq \bar{N}.$$

Proof of Claim. This is essentially because Ω_r is connected, c.f. [Liu91, Corollary 2 on p. 21]. Let $W_0 = B_{p_a}(\alpha r)$. For $k \geq 1$, we define W_k to be the union of W_{k-1} with those balls $B_{p_i}(\alpha r)$ satisfying $W_{k-1} \cap B_{p_i}(\alpha r) \neq \emptyset$. This process stops in at most $N \leq \bar{N}$ steps. If there is any ball $B_{p_j}(\alpha r)$ that does not intersect W_N, then we can find two open sets in Ω_r that do not intersect. This is a contradiction to the fact that Ω_r is connected. $\square$

If $y, z \in \Omega_r$, $|yz| \leq 2\alpha r$, then

$$V_y(\alpha r) \leq V_z(3\alpha r) \leq 3^n V_z(\alpha r),$$

since $B_z(3\alpha r) \subset M \setminus B_o(a)$ if $\alpha \leq 1/10$. So for any indices $1 \leq a, b \leq N$,

$$V_{p_a}(\alpha r) \leq 3^{n\bar{N}} V_{p_b}(\alpha r).$$

For any $x \in \partial B_o(r)$, there exists $b \leq N$ such that $x \in B_{p_b}(\alpha r)$. Then $B_{p_b}(\alpha r) \subset B_x(2\alpha r)$, and

$$V_x(\alpha r) \geq 2^{-n} V_x(2\alpha r) \geq 2^{-n} V_{p_b}(\alpha r).$$

It follows that

$$\frac{|\mathcal{A}_o(r/2, 2r)|}{|B_x(\alpha r)|} \leq \frac{|\Omega_r|}{|B_x(\alpha r)|} \leq 2^n \sum_{a=1}^{N} \frac{V_{p_a}(\alpha r)}{V_{p_b}(\alpha r)} \leq 2^n 3^{n\bar{N}} \bar{N}.$$

$\square$

Corollary 2.3.5. *Suppose that* $(M^n, g, o) \in \underline{\mathcal{R}}^n(a, \Lambda)$ *and satisfies* (CA) *at distances at least* $r_0 \geq 10(1+a)$. *If* $\mathrm{AVR}(g) > 0$, *then for any* $x \notin B_o(r_0)$ *and any* $r > 0$,

$$\frac{V_x(r)}{r^n} \geq c(n, \Lambda)\mathrm{AVR}(g).$$

Proof. For any $x \in M \setminus B_o(r_0)$, let $\ell = |ox|$. Then, $x \in \partial B_o(\ell)$ and $\ell \geq r_0$. By Theorem 2.3.4,

$$\frac{V_o(2\ell) - V_o(\ell/2)}{V_x(\ell/10)} \leq C(n, \Lambda). \tag{2.3.1}$$

Note that the Ricci curvature is nonnegative on $B_x(\frac{\ell}{10})$. By (2.3.1), for any $r \in (0, \ell/10]$, we have

$$\frac{V_x(r)}{r^n} \geq \frac{V_x(\ell/10)}{(\ell/10)^n} \geq \frac{V_o(2\ell) - V_o(\ell/2)}{C(n, \Lambda)(2\ell - \ell/2)^n}.$$

For any $s \geq 2\ell$, by the monotonicity formula Proposition 2.3.1, it follows that

$$\frac{V_x(r)}{r^n} \geq \frac{V_o(2\ell) - V_o(\ell/2)}{C(n, \Lambda)(2\ell - \ell/2)^n} \geq \frac{V_o(s) - V_o(\ell/2)}{C(n, \Lambda)(s - \ell/2)^n}.$$

By taking $s \to \infty$, for $r \in (0, \ell/10]$,

$$\frac{V_x(r)}{r^n} \geq c(n, \Lambda)\mathrm{AVR}(g).$$

For any $r \in (\ell/10, 4\ell]$,

$$\frac{V_x(r)}{r^n} \geq \frac{V_x(\ell/10)}{(4\ell)^n} \geq \frac{c(n, \Lambda)}{40^n}\mathrm{AVR}(g).$$

Note that $B_o(a) \subset B_x(2\ell)$. For any $r \geq 4\ell$, by Lemma 2.3.2 and Proposition 2.3.1, we have

$$\frac{V_x(r)}{r^n} \geq \frac{V_x(r) - V_x(2\ell)}{(r-2\ell)^n}\frac{(r-2\ell)^n}{r^n} \geq 2^{-n}\frac{V_x(r) - V_x(2\ell)}{(r-2\ell)^n} \geq 2^{-n}\mathrm{AVR}(g).$$

□

Now we give a criterion for (CA) that will be useful in our setting.

Lemma 2.3.6. *Suppose that (M^n, g) is a complete Riemannian manifold and M is connected at infinity. Suppose that there is a proper positive function β on M and $r_0 > 0$ such that for $r_0 \leq s < r \leq \infty$, $\{s \leq \beta < r\}$ is homeomorphic to $\{\beta = s\} \times [s, r)$. Moreover,*

$$\lim_{x \to \infty} \frac{\beta(x)}{|ox|} = 1,$$

for some $o \in M$. Then (M^n, g, o) satisfies (CA).

Proof. By assumption, $M = \{\beta > r_0\} \cup \{\beta \leq r_0\}$. Note that $\{\beta \leq r_0\}$ is compact and $\{\beta \geq r_0\}$ is homeomorphic to $\{\beta = r_0\} \times [0, \infty)$. Since M is connected at infinity, $\{\beta = r_0\}$ must be connected. For sufficiently large r, we may choose

$$\Omega_r := \left\{x : \tfrac{5r}{12} < \beta(x) < \tfrac{5r}{2}\right\}.$$

Hence, Ω_r is connected. □

We remark that the assumptions in Lemma 2.3.6 are inspired by the notion of finite topological type and an isotopy lemma [C91, Lemma 1.4] due to Grove-Shiohama [GS77]. See, e.g., Cheeger's lecture notes [C91] for a wonderful presentation on these topics and references therein.

Corollary 2.3.7. *Let (M^n, g, f) be a steady Ricci soliton. Suppose that the critical set of f is bounded and*

$$\lim_{x \to \infty} \frac{f(x)}{|ox|} = 1,$$

for some $o \in M$. Then (M^n, g, o) satisfies condition (CA) *at distances at least r_0 for some large $r_0 > 0$. As a consequence of Corollary 2.3.5 , if in addition* $\mathrm{AVR}(g) > 0$, *then for $x \notin B_o(r_0)$, and any $r > 0$,*

$$\frac{V_x(r)}{r^n} \geq c(n, \Lambda)\mathrm{AVR}(g).$$

Proof. If (M, g, f) is a steady soliton with bounded critical set, then the level sets of f are diffeomorphic to each other outside a compact set. In fact, for large $r \geq r_0$,

$$\partial_r F_r = \frac{\nabla f}{|\nabla f|^2}(F_r), \quad F_{r_0} = \mathrm{id}$$

gives such a diffeomorphism. It was proved by Munteanu and Wang in [MW11, Corollary 1.1] that M is connected at infinity. Hence the conditions in Lemma 2.3.6 are satisfied. □

Chapter 2, in part, contains material published on Advances in Mathematics 2022 [CDM22] joint with Chow, Bennett and Deng, Yuxing.

Chapter 3

Heat Flow Estimates

Throughout this chapter, let $(M^n, g_t)_{t\in[0,1]}$ be a complete Ricci flow. For bounded heat flows coupled with (M^n, g_t), we shall prove some rough gradient estimates and also Bamler's sharp gradient estimates ([Bam20a, Theorem 4.1]) without any curvature conditions. The main ingredient is Perelman's cutoff function. With similar ideas, we shall prove that Bamler's H_n-centers are well-defined for (M^n, g_t) without any curvature conditions. Several related Lemmata are included here for future use.

3.1 Rough Gradient Estimates for Heat Flows

We recall Perelman's cutoff function which works well on a large scale and our presentation here mainly follows [Wa18]. See also, e.g., [Ch09, Lu10]. Suppose that for each $t \in (0, 1]$,

$$\mathrm{Ric} \leq (n-1)\Lambda/t, \qquad \text{on } B_t(o, \sqrt{t}),$$

where such a constant Λ always exists by the smoothness of the flow. Without loss of generality, we may assume that $\Lambda \geq 1000n^2$.

By Perelman [Per02, 8.3], for any $t \in (0, 1]$, outside of $B_t(o, \sqrt{t})$,

$$\Box \mathrm{dist}_t(o, \cdot) \geq -2n\left(\frac{\Lambda}{t}\sqrt{t/\Lambda} + \sqrt{\Lambda/t}\right) \geq -\frac{\Lambda}{2\sqrt{t}}.$$

Let $\eta : [0, \infty) \to [0, 1]$ be a smooth non-increasing function such that

$$\eta|_{[0,1]} = 1, \quad \eta|_{[2,\infty)} = 0, \quad -10\sqrt{\eta} \le \eta' \le 0, \quad |\eta''| \le 10\eta.$$

For $(x, t) \in M \times [0, 1]$, define

$$\phi(x,t) := \phi^A(x,t) := \eta\left(\frac{|ox|_t + \Lambda\sqrt{t}}{A}\right). \tag{3.1.1}$$

Then for $A \ge 2\Lambda$,

$$\Box\phi \le \frac{10}{A^2}\phi, \quad |\nabla\phi|^2 \le \frac{100}{A^2}\phi.$$

Note that we only have an upper bound for $\Box\phi$ but this is sufficient for many applications as we will see below.

In the following, we shall use the following temporary notation:

$$\Omega_A := \{(x,t) \in M \times [0,1] : |ox|_t + \Lambda\sqrt{t} < A\}. \tag{3.1.2}$$

Clearly, $\phi = 1$ on Ω_A. The regions Ω_A are comparable to balls: If $A \ge 2\Lambda$, then for each $t \in [0, 1]$,

$$B_t(o, A/2) \times \{t\} \subset \Omega_A \subset B_t(o, A) \times \{t\}.$$

We shall prove a rough gradient estimate of Berstein-Bando-Shi type for bounded heat flows. Roughly speaking, we attempt to obtain C^1-estimates by C^0-estimates on a larger parabolic region. First, we prove the following local version.

Proposition 3.1.1. *Suppose that for each $t \in (0, 1]$,*

$$\mathrm{Ric} \le (n-1)\Lambda/t, \qquad \textit{on } B_t(o, \sqrt{t}),$$

for some constant $\Lambda \geq 1000n^2$. *Suppose that* $\Box u = 0$ *on* $M \times [0,1]$. *Set*

$$\gamma(A) := \sup_{\Omega_A} |u|.$$

Then for $A \geq 2\Lambda$,

$$\sup_{\Omega_A}(t|\nabla u|^2) \leq 10\gamma^2(2A).$$

Proof. Consider $F := F_A := (7\gamma^2(2A)+u^2)|\nabla u|^2$. In the following, we shall write $\gamma = \gamma(2A)$ for simplicity. In Ω_{2A},

$$\begin{aligned}
\Box F &= -2(7\gamma^2 + u^2)|\nabla^2 u|^2 - 2|\nabla u|^4 - 8u\langle \nabla^2 u, \nabla u \otimes \nabla u\rangle \\
&\leq -16u^2|\nabla^2 u|^2 - 2|\nabla u|^4 + 8u|\nabla^2 u||\nabla u|^2 \\
&\leq -|\nabla u|^4 \leq -\frac{1}{64\gamma^4}F^2.
\end{aligned}$$

Let $U = t\phi F$. Then

$$\begin{aligned}
\Box U &= \phi F + t(\phi \Box F + F\Box\phi - 2\langle \phi^{-1}\phi\nabla F, \nabla\phi\rangle) \\
&\leq U/t - \frac{t\phi F^2}{64\gamma^4} + \frac{10}{A^2}U - 2\nabla U \cdot \nabla \ln\phi + 2t|\nabla\phi|^2\phi^{-1}F \\
t\phi\Box U &\leq -\frac{U^2}{64\gamma^4} + \left(1 + \frac{210}{A^2}\right)U - 2t\nabla U \cdot \nabla\phi.
\end{aligned}$$

At any maximum point (p,τ) of U, (clearly $\tau > 0$,)

$$0 \leq -\frac{U^2}{64\gamma^4} + \left(1 + \frac{210}{A^2}\right)U.$$

Thus,

$$\sup_{\Omega_A}(tF) \leq \sup_{\Omega_{2A}} U = U(p,\tau) \leq 64\gamma^4(2A)\left(1 + \tfrac{210}{A^2}\right).$$

Since $7\gamma^2(2A)|\nabla u|^2 \leq F$,

$$\sup_{\Omega_A}(t|\nabla u|^2) \leq 10\gamma^2(2A),$$

where we used the fact that $A \geq 2\Lambda \geq 2000n^2$. $\square$

As a corollary, we have the following gradient estimates for bounded heat flows without any curvature conditions except the completeness of the flow.

Corollary 3.1.2. *Suppose that $(M^n, g_t)_{t\in[0,T]}$ is a complete Ricci flow and $\square u = 0$ on $M \times [0,T]$. If $\sup_{M\times[0,T]}|u| \leq D$, then*

$$\sup_{M\times[0,T]}(t|\nabla u|^2) \leq 10D^2.$$

Proof. By parabolic rescaling, we may assume that $T = 1$. Since the flow is smooth and complete, there is a constant $\Lambda \geq 1000n^2$, such that for any $t \in (0,1]$,

$$\mathrm{Ric} \leq (n-1)\Lambda/t, \qquad \text{on } B_t(o, \sqrt{t}).$$

For any $A \geq 2\Lambda$, by the Lemma above,

$$\sup_{\Omega_A}(t|\nabla u|^2) \leq 10D^2,$$

where Ω_A is defined in (3.1.2). Taking $A \to \infty$, we have the conclusion. $\square$

As another corollary, we have the following Liouville type theorem for ancient heat flows.

Corollary 3.1.3. *Suppose that $(M^n, g_t)_{t\leq 0}$ is a complete ancient Ricci flow and u is an ancient heat flow, i.e., $\square u = 0$, on $M \times (-\infty, 0]$. If u is uniformly bounded, then u is constant.*

Proof. Suppose that u is an ancient heat flow with $\sup_{M\times(-\infty,0]}|u|\le D$. For any $T<\infty$, by Corollary 3.1.2,

$$\sup_M (t+T)|\nabla u|^2 \le 10D^2,$$

for any $t\in[-T,0]$. In particular,

$$\sup_{M\times[-T/2,0]}|\nabla u|^2\le 20D^2/T.$$

By taking $T\to\infty$, $|\nabla u|^2\equiv 0$ and thus u is constant. □

Let us record Bing-Long Chen's estimate using similar techniques. Our proof here is the same as in [Ch09].

Theorem 3.1.4. *Let $(M^n,g_t)_{t\in[a,b]}$ be a complete Ricci flow. Then the scalar curvature satisfies*

$$R_{g_t}\ge \frac{1}{\frac{1}{R_{\min}}-\frac{2}{n}(t-a)}\ge \max\left\{R_{\min},-\tfrac{n}{2(t-a)}\right\},$$

for $t\in(a,b]$, where

$$R_{\min}\le \min\left\{\inf R_{g_a},0\right\}.$$

As a consequence, for any ancient complete Ricci flow, the scalar curvature is nonnegative everywhere.

Proof. By shifting the time, we may assume that $a=0$. We may also assume that $R_{\min}<0$. Recall that

$$\Box R=2|\mathrm{Ric}|^2\ge \tfrac{2}{n}R^2.$$

Suppose that for any $t\in(0,b]$,

$$\mathrm{Ric}\le (n-1)\Lambda/t,\qquad \text{on } B_t(o,\sqrt{t}),$$

for some point $o\in M$ and some constant $\Lambda\ge 1000n^2$. For any $A\ge 2\Lambda$, we consider a cutoff function $\phi = \phi^A$ as in (3.1.1). Let

$$\alpha(t) := \frac{2t}{n} - \frac{1}{R_{\min}}, \quad U := \alpha\phi R.$$

At any point where $R \leq 0$, we have

$$\begin{aligned}
\Box U &= \tfrac{2}{n}\phi R + \alpha\left(\phi\Box R + R\Box\phi - 2\langle\phi^{-1}\phi\nabla R, \nabla\phi\rangle\right) \\
&\geq \frac{2}{n}U/\alpha + \frac{2}{n}\alpha\phi R^2 + \frac{10}{A^2}\alpha\phi R - 2\nabla U\cdot\nabla\ln\phi + 2\alpha R|\nabla\phi|^2/\phi, \\
\tfrac{n}{2}\alpha\phi\Box U &\geq U^2 + U + \frac{210n}{A^2}\alpha(b)U - 2\alpha\nabla U\cdot\nabla\phi.
\end{aligned}$$

Let (p,τ) be a minimum point of U. If $U(p,\tau) \geq 0$, the conclusion follows. So we may assume that $U(p,\tau) < 0$.

If $\tau = 0$, then

$$\inf U = U(p,0) \geq -1.$$

If $\tau > 0$, then at (p,τ),

$$0 \geq U + U^2 + \frac{210n}{A^2}\alpha(b)U,$$

and thus

$$\min_{\Omega_A}(\alpha R) \geq \min U = U(p,\tau) \geq -1 - \frac{210n}{A^2}\alpha(b).$$

By taking $A \to \infty$, the conclusion follows. $\Box$

3.2 Bamler's Gradient estimates

Let

$$\Phi(x) = \int_{-\infty}^{x} (4\pi)^{-1/2} e^{-s^2/4}\, ds, \quad \Phi_t(x) := \Phi(x/\sqrt{t}),$$

for $x \in \mathbb{R}, t > 0$. It is standard that $\Phi_t(x)$ solves the one-variable heat equation, i.e.,

$$\partial_t \Phi_t(x) = \partial_x \partial_x \Phi_t(x),$$

and

$$\lim_{t \to 0} \Phi_t = \chi_{[0,\infty)}.$$

Theorem 3.2.1 (Theorem 4.1 in [Bam20a]). *Let $(M, g_t)_{t \in [t_0, t_1]}$ be a complete Ricci flow. Consider a heat flow u on $M \times [t_0, t_1]$ and assume that u takes values in $(0, 1)$. Let $T \geq 0$ and suppose $|\nabla \Phi_T^{-1}(u_{t_0})|_{g_{t_0}} \leq 1$ if $T > 0$.*

Then $|\nabla \Phi_{T+t-t_0}^{-1}(u_t)|_{g_t} \leq 1$ for any $t \in (t_0, t_1]$.

Proof. We first consider the case where $T > 0$ and $\epsilon \leq u \leq 1 - \epsilon$ for some $\epsilon > 0$. By parabolic rescaling and time shifting, we may assume $0 < t_0 = T < 1$ and $t_1 \geq 1$. It suffices to show $|\nabla \Phi_1^{-1}(u_1)|_{g_1} \leq 1$ under the assumption $|\nabla \Phi_T^{-1}(u_T)|_{g_T} \leq 1$. In the following, we shall omit the subindeces and the reader should keep in mind that the norms of the gradients are computed using the evolving metric.

Define $h_t := \Phi_t^{-1}(u_t)$. Then $|h_t| \leq A_\epsilon \sqrt{t}$, for $t \in [T, 1]$, where $A_\epsilon = \Phi^{-1}(1 - \epsilon)$. Recall that

$$\Box |\nabla h|^2 = \frac{1}{t} |\nabla h|^2 (1 - |\nabla h|^2) - \frac{1}{t} h \langle \nabla |\nabla h|^2, \nabla h \rangle - 2|\nabla^2 h|^2.$$

Since the flow is smooth and complete, there is a constant $\Lambda \geq 1000n^2$, such that for any $t \in (T, 1]$,

$$\mathrm{Ric} \leq (n-1)\Lambda/(t - T), \quad \text{ on } B_t(o, \sqrt{t - T}).$$

Let $\phi = \phi^A$ be the standard cutoff function defined in (3.1.1), where $A \gg A_\epsilon + \Lambda$. Let

$U := \phi|\nabla h|^2$. Then for $t \in (T, 1]$,

$$
\begin{aligned}
t\Box U &= t\phi\Box|\nabla h|^2 + t|\nabla h|^2\Box\phi - 2t\langle\nabla|\nabla h|^2, \nabla\phi\rangle \\
&\le \phi|\nabla h|^2(1-|\nabla h|^2) - h\langle\phi\nabla|\nabla h|^2, \nabla h\rangle \\
&\quad + \frac{10}{A^2}\phi|\nabla h|^2 + 4t|\nabla\phi||\nabla h||\nabla^2 h| - 2t\phi|\nabla^2 h|^2 \\
&\le \phi|\nabla h|^2 - \phi|\nabla h|^4 - h\langle\nabla(\phi|\nabla h|^2), \nabla h\rangle + h|\nabla h|^3|\nabla\phi| \\
&\quad + \frac{10}{A^2}\phi|\nabla h|^2 + 2|\nabla\phi|^2\phi^{-1}|\nabla h|^2,
\end{aligned}
$$

where we obtained the last inequality by applying Cauchy-Schwarz inequality:

$$
4|\nabla\phi||\nabla h||\nabla^2 h| = 4(\phi^{1/2}|\nabla^2 h|)(|\nabla\phi|\phi^{-1/2}|\nabla h|) \le 2\phi|\nabla^2 h|^2 + 2|\nabla\phi|^2\phi^{-1}|\nabla h|^2.
$$

It follows that for any $\delta > 0$,

$$
\begin{aligned}
t\phi\Box U &\le U - U^2 - t\phi h\langle\nabla U, \nabla h\rangle + \frac{10A_\epsilon}{A}U^{3/2} + \frac{210}{A^2}U \\
&\le (1+\delta)U - (1-\delta)U^2 - t\phi h\langle\nabla U, \nabla h\rangle,
\end{aligned}
$$

if $A \ge \bar{A}(\epsilon, \delta, \Lambda)$. Let (p, τ) be a maximum point of U. If $\tau = T$, then

$$
\sup_{t\in(T,1]} \sup_{B_t(o,A/2)} |\nabla h|^2 \le \sup U_T \le \sup|\nabla h_T|^2 \le 1,
$$

because $|\nabla h_T| \le 1$ as the assumption. If $\tau \in (T, 1]$, then at (p, τ),

$$
0 \le (1+\delta)U - (1-\delta)U^2,
$$

and thus

$$
\sup_{t\in(T,1]} \sup_{B_t(o,A/2)} |\nabla h|^2 \le \sup U = U(p,\tau) \le \frac{1+\delta}{1-\delta}.
$$

In summary, for any $\delta > 0$,

$$\sup_{t\in(T,1]} \sup_{B_t(o,A/2)} |\nabla h|^2 \le \sup U \le \frac{1+\delta}{1-\delta}.$$

if $A \ge \bar{A}(\epsilon, \delta, \Lambda)$. By taking $A \to \infty$ and then $\delta \to 0$, we can finish the proof of the case where $T > 0$ and $\epsilon \le u \le 1-\epsilon$.

Now we prove the general case. We shall only prove the case where $T = 0$ and $0 < u < 1$, since it is in fact easier to prove the case where $T > 0$. By parabolic rescaling, we consider a heat flow u on $M \times [0,1]$ taking values in $(0,1)$ and it suffices to show that $|\nabla \Phi_1^{-1}(u_1)| \le 1$. For any $\epsilon \in (0, 1/2)$, let

$$u_t^\epsilon := \epsilon + (1-2\epsilon)u_t.$$

u_t^ϵ take values in $(\epsilon, 1-\epsilon)$. Let $T > 0$ to be determined. For $s \in (0,1/2)$, by Corollary 3.1.2, we have

$$|\nabla u_s|^2_{g_s} \le 10/s.$$

It follows that for any $s \in (0, 1/2)$,

$$\begin{aligned}
|\nabla \Phi_T^{-1}(u_s^\epsilon)|_{g_s} &= \frac{1}{\Phi_T'(\Phi_T^{-1}(u_s^\epsilon))}|\nabla u_s^\epsilon| \\
&= (4\pi T)^{1/2} \exp\left\{\frac{\left(\Phi_T^{-1}(u_s^\epsilon)\right)^2}{4T}\right\}(1-2\epsilon)|\nabla u_s| \\
&\le (40\pi T/s)^{1/2} e^{A_\epsilon^2/4} \le 1,
\end{aligned}$$

if $T \le \bar{T}(\epsilon, s)$. Clearly, $\bar{T}(\epsilon, s) \to 0$ as $s \to 0$ while keeping ϵ fixed. Applying the result of the first case, we have that for any $s \in (0, 1/2)$,

$$|\nabla \Phi_{T+1-s}^{-1}(u_1^\epsilon)|_{g_1} \le 1,$$

if $T \le \bar{T}(\epsilon, s)$. Letting $s \to 0$ and then $\epsilon \to 0$, we obtain $|\nabla \Phi_1^{-1}(u_1)|_{g_1} \le 1$.

$\square$

3.3 Concentration of Heat Kernels

3.3.1 Bamler's Monotonicity

Throughout this subsection, we work on a complete Ricci flow $(M, g_t)_{t \in [0,1]}$. Suppose that

$$\mathrm{Ric}_{g_t} \le (n-1)\Lambda/t, \quad \text{ on } B_t(o, \sqrt{t}), \tag{3.3.1}$$

for some constant Λ independent on t. As seen above and below, the dependence on Λ will disappear for glabal estimates. We may still use Perelman's cutoff function ϕ_t^A defined in (3.1.1).

Lemma 3.3.1. *Suppose that v_t is a smooth solution to $\square^* v_t = 0$. Then for $t \in [0, 1), A \ge 2\Lambda$,*

$$\int \phi_t^A v_t \, dg_t \ge e^{-\frac{10}{A^2}(1-t)} \int \phi_1^A v_1 \, dg_1.$$

By taking $A \to \infty$, we have

$$\int v_t \, dg_t \ge \int v_1 \, dg_1.$$

Corollary 3.3.2. *Suppose that $K(x, t \,|\, y, s)$ is a heat kernel coupled with Ricci flow $(M, g_t)_{t \in [0,1]}$. Then*

$$\int K(x, t \,|\, y, s) \, dg_s(y) \ge 1.$$

In particular, any minimal heat kernel coupled with Ricci flow integrates to 1.

Proof of Lemma 3.3.1. Write $F(t) = \int \phi_t^A v_t \, dg_t$. Then

$$F'(t) = \int \square \phi_t^A v_t \, dg_t \le \frac{10}{A^2} \int \phi_t^A v_t \, dg_t = \frac{10}{A^2} F(t).$$

The conclusion follows by integration. $\square$

In the following, we denote by $K(x,t\,|\,y,s)$ the minimal heat kernel coupled with Ricci flow $(M,g_t)_{t\in[0,1]}$. As shown above,

$$d\nu_{x,t|s} := K(x,t\,|\,\cdot,s)\,dg_s$$

is a probability measure.

Lemma 3.3.3. *Let F be a non-decreasing, and non-negative continuous function, and A as above. Then*

$$\lim_{t\to 1^-}\int_M F(d_t(o,\cdot))\phi_t^A\,d\nu_{o,1|t} = F(0). \tag{3.3.2}$$

Proof. By the smoothness and completeness of the flow, there is a large positive constant $r_1 \gg 3A$ such that for each $t\in[0,1]$,

$$B_t(o,3A)\subset B_0(0,r_1).$$

Since $\overline{B}_0(o,2r_1)$ is compact, one can find a positive constant C_1 such that

$$|\mathrm{Ric}|(x,t)\le C_1 \text{ on } \overline{B}_0(o,2r_1)\times[0,1]. \tag{3.3.3}$$

Hence for any piecewise smooth curve γ in $\overline{B}_0(o,2r_1)$, it holds that

$$e^{-C_1t}L_0(\gamma)\le L_t(\gamma)\le e^{C_1t}L_0(\gamma), \tag{3.3.4}$$

where $L_t(\gamma)$ is the length of γ with respect to g_t. It can be seen that

$$B_0(o,e^{-C_1}A/2)\subset B_t(o,A/2), \tag{3.3.5}$$

and on $B_t(o,2A)$

$$e^{-C_1}d_0(o,\cdot)\le d_t(o,\cdot)\le e^{C_1}d_0(o,\cdot). \tag{3.3.6}$$

Let $\varphi_1(\cdot) = \eta(4e^{C_1} d_0(o,\cdot)/A)$ and $\varphi_2(\cdot) = \eta(d_0(o,\cdot)/r_1)$. Both φ_1, and φ_2 are continuous function with compact support. Using $\operatorname{spt}\phi_t^A \subset B_t(o, 2A)$, and (3.3.5), we have for all $t \in [0,1]$,

$$\varphi_1 \leq \phi_t^A \leq \varphi_2.$$

Hence by (3.3.6) and the monotonicity assumption on F,

$$\begin{aligned} F\left(e^{-C_1} d_0(o,\cdot)\right)\varphi_1 \leq F\left(e^{-C_1} d_0(o,\cdot)\right)\phi_t^A \leq F(d_t(o,\cdot))\phi_t^A \\ \leq F\left(e^{C_1} d_0(o,\cdot)\right)\phi_t^A \leq F\left(e^{C_1} d_0(o,\cdot)\right)\varphi_2. \end{aligned} \tag{3.3.7}$$

Consequently, we have

$$\int_M F\left(e^{C_1} d_0(o,\cdot)\right)\varphi_1 \, d\nu_{o,1|t} \leq \int_M F(d_t(o,\cdot))\phi_t^A \, d\nu_{o,1|t} \leq \int_M F\left(e^{C_1} d_0(o,\cdot)\right)\varphi_2 \, d\nu_{o,1|t}.$$

Result then follows from letting $t \to 1^-$ and the fact that $\lim_{t\to 1^-} \nu_{o,1|t} = \delta_o$ as distribution.

$\square$

Theorem 3.3.4. *Let v_t^1, v_t^2 be two non-negative smooth solutions to $\Box^* v^i = 0$ with $\int v_t^i \, dg_t = 1$, for each $t \in [0,1]$. Write $d\mu_t^i := v_t^i dg_t$. Suppose that*

$$\max_{i=1,2} \int_M \operatorname{dist}_1^p(o,y)\, d\mu_1^i(y) \leq V < \infty, \tag{3.3.8}$$

for some $p > 2$, then

$$\operatorname{Var}_t(\mu_t^1, \mu_t^2) \leq \operatorname{Var}_1(\mu_1^1, \mu_1^2) + H_n(1-t),$$

for $t \in [0,1]$.

Remark. It is not hard to see that condition (3.3.8) implies that $\operatorname{Var}_1(\mu_1^i) < \infty$ for

$i=1,2$. Clearly, conjugate heat kernels $\nu_{x,1|t}$ satisfies (3.3.8) for every $p>2$. As will be proved in the following, we do have a monotonicity formula for conjugate heat kernels, i.e,

$$\mathrm{Var}_t(\nu_{x_1,t_1|t},\nu_{x_2,t_2|t})+H_n t$$

is non-decreasing.

Proof. For a function $U: M\times M\times[0,1]\to\mathbb{R}$, write $U_t=U(\cdot,\cdot,t)$ and

$$\mathcal{P}U_t(x,y):=(\partial_t-\Delta_x-\Delta_y)U_t(x,y),$$

where Δ_x denotes the laplacian operator induced by g_t with respect to variable x. So $\mathcal{P}$ is simply the heat operator for the product Ricci flow $(M\times M, g_t\oplus g_t)$. If U_t has compact support for each t, it is easy to verify that

$$\frac{d}{dt}\int_M\int_M U_t(y_1,y_2)\,d\mu_t^1(y_1)d\mu_t^2(y_2)=\int_{M\times M}\mathcal{P}U_t\,d\mu_t,$$

where

$$\mu_t:=\mu_t^1\times\mu_t^2$$

denotes the product measure.

By [Bam20a, Theorem 3.5],

$$\mathcal{P}\mathrm{dist}_t^2\geq -H_n,$$

in the sense of barriers or distributions. Let

$$\psi_t^A(y_1,y_2):=\phi_t^A(y_1)\phi_t^A(y_2).$$

Then $\operatorname{spt}\psi_A \subset B_t(o,2A) \times B_t(o,2A)$, and

$$\mathcal{P}\psi_t^A(y_1,y_2) = \Box\phi_t^A(y_1)\phi_t^A(y_2) + \phi_t^A(y_1)\Box\phi_t^A(y_2) \le \frac{20}{A^2}\psi_t^A.$$

By Lemma 3.3.1, for $i = 1,2$,

$$\int \phi_t^A \, d\mu_t^i \ge e^{-\frac{10}{A^2}(1-t)} \int \phi_1^A \, d\mu_1^i.$$

On the other hand,

$$\int (1-\phi_1^A)\, d\mu_1^i \le \frac{2^p}{A^p} \int_{M\setminus B_1(o,A/2)} \operatorname{dist}_1^p(o,y)\, d\mu_1^i(y) \le \frac{2^p V}{A^p}. \tag{3.3.9}$$

It follows that

$$\mu_t^i(B_t(o,2A)) \ge \int \phi_t^A \, d\mu_t^i \ge \left(1 - \tfrac{2^p V}{A^p}\right) e^{-\frac{10}{A^2}(1-t)} \ge \left(1 - \tfrac{2^p V}{A^p}\right)\left(1 - \tfrac{10}{A^2}\right).$$

Thus for $i = 1,2$, if $A \ge 10\Lambda$,

$$\mu_t^i(M \setminus B_t(o,A)) \le \frac{C}{A^2},$$

for some constant $C = C(p)(V+1)$. Define

$$U_t(y_1,y_2) := 16A^2 - \operatorname{dist}_t^2(y_1,y_2).$$

Clearly, $U_t \geq 0$ on $\operatorname{spt} \psi_t^A$ and $\mathcal{P} U_t \leq H_n$.

$$
\begin{aligned}
&\frac{d}{dt} \int_{M \times M} U_t \psi_t^A \, d\mu_t = \int_{M \times M} \mathcal{P}(U_t \psi_t^A) \, d\mu_t \\
&\leq H_n \int \psi_t^A \, d\mu_t + \frac{20}{A^2} \int_{\operatorname{spt} \nabla \psi_t^A} U_t \psi_t^A \, d\mu_t \\
&\quad + \frac{C_n A}{A} \left\{ \mu^1(M \setminus B_t(o, A/2)) + \mu^2(M \setminus B_t(o, A/2)) \right\} \\
&\leq H_n + \frac{C(p)(V+1)}{A^2}.
\end{aligned}
$$

It follows that

$$
\int (16A^2 - \operatorname{dist}_1^2) \psi_1^A \, d\mu_1 \leq \int (16A^2 - \operatorname{dist}_t^2) \psi_t^A \, d\mu_t + H_n(1-t) + \frac{C(p)(V+1)}{A^2}(1-t),
$$

By (3.3.9),

$$
\int \psi_1^A \, d\mu_1 \geq 1 - \frac{C(p)V}{A^p}.
$$

Hence

$$
\int \operatorname{dist}_t^2 \psi_t^A \, d\mu_t \leq \int \operatorname{dist}_1^2 \psi_1^A \, d\mu_1 + H_n(1-t) + 16 \frac{C(p)V}{A^{p-2}} + \frac{C(p)(V+1)}{A^2}(1-t).
$$

Taking $A \to \infty$, we finish the proof.

$\square$

Following [Bam20a, Definition 3.1], for a metric space (X, dist) and two probability measures μ, ν, we define p-variance as

$$
\operatorname{Var}^p(\mu, \nu) := \int_X \int_X \operatorname{dist}^p(y_1, y_2) \, d\mu(y_1) d\nu(y_2),
$$

where $p \geq 1$. It is not hard to see that $(\operatorname{Var}^p)^{1/p}$ satisfies triangle inequality, c.f. [Bam20a, Lemma 3.2].

In the following, we shall denote by Var_t^p the p-variance with respect to metric g_t.

We first prove the following lemma locating almost "H-centers".

Lemma 3.3.5. *Suppose that Ricci upper bound condition* (3.3.1) *holds near $o \in M$. Suppose for some $p \geq 2$ and $t \in [0,1], \mathrm{Var}_t^p(\nu_{o,1|t}) \leq V(1-t)^{p/2}$, for some $V \geq 1$. Then there is a point $z_t \in M$ such that*

$$\mathrm{dist}_t(z_t, o) \leq C(V, \Lambda), \qquad \int \mathrm{dist}_t^p(z_t, y)\, d\nu_t(y) \leq V(1-t)^{p/2}.$$

Hence for $A \geq 2C(V, \Lambda)$,

$$\nu_{o,1|t}(M \setminus B_t(o, A)) \leq \frac{2^p V}{A^p}(1-t)^{p/2}.$$

Proof. We may assume that $t = 0$. For any $j \in \mathbb{N}$, there is $z_j \in M$ such that

$$\int \mathrm{dist}_0^p(z_j, y)\, d\nu_0(y) \leq V + 1/j.$$

So for any $A > 0$,

$$\nu_0(M \setminus B_0(z_j, A)) \leq \frac{V+1}{A^p}.$$

For $A = 2\Lambda$, by Lemma 3.3.1,

$$\nu_0(B_0(o, 2A)) \geq \int \phi_0^A\, d\nu_0 \geq e^{-\frac{10}{A^2}}.$$

If $B_0(o, 2A) \cap B_0(z_j, A) = \emptyset$, then

$$1 \geq \nu_0(B_0(o, 2A)) + \nu_0(B_0(z_j, A)) \geq e^{-\frac{10}{A^2}} + 1 - \frac{V+1}{A^p} \geq 2 - \frac{V+11}{A^2},$$

which is impossible if $A = 2\Lambda + 2V$. Hence the two balls intersect and

$$\mathrm{dist}_0(z_j, o) < 3A.$$

Since $\bar{B}_0(o, 3A)$ is compact, there is a subsequence of z_i that converges to a point $z \in \bar{B}_0(o, 3A)$ with the desired properties. $\square$

Corollary 3.3.6. *For any $x \in M$ and any $s \in [0, 1)$, there is $z \in M$ such that (z, s) is an H_n-center of $(x, 1)$, i.e.,*

$$\int \mathrm{dist}_s^2(z, y)\, d\nu_{x,1|s}(y) \le H_n(1 - s).$$

Moreover, if $(z, 0)$ is an H_n-center of $(o, 1)$ with Ricci curvature near o satisfying (3.3.1), then

$$\mathrm{dist}_0(z, o) \le 10\Lambda.$$

Proof. Without loss of generality, we may assume that $x = o, s = 0$. The conclusion follows by the same proof of Lemma 3.3.5. $\square$

Lemma 3.3.7. *For any $p \ge 2, x \in M,$ and $t \in [0, 1)$,*

$$\mathrm{Var}_t^p(\nu_{x,1|t}) \le V_p(1 - t)^{p/2},$$

for some constant constant V_p depending only on p, n. In fact, for $p \in \mathbb{N}$,

$$V_0 = 1, \quad V_{2p} \le (H_n + 8(p - 1))V_{2p-2}.$$

Proof. The proof is similar to above and it suffices to prove the conclusion at time $t = 0$

for $x = o$ by parabolic rescaling. We write

$$\nu_t := \nu_{o,1|t}, \quad \mu_t := \nu_t \times \nu_t.$$

It suffices to prove by induction on $2p$ for $p \in \mathbb{N}$. The case of $p = 1$ has been proved above. Suppose we have proved that there is $D = D(p) < \infty$ such that for $t \in [0, 1)$,

$$\mathrm{Var}_t^{2p-2}(\nu_t) \leq D(1-t)^{p-1}. \tag{3.3.10}$$

Consider

$$U_t(y_1, y_2) := (4A)^{2p} - \mathrm{dist}_t^{2p}(y_1, y_2), \quad F_p(t) := \int U_t \psi_t^A \, d\mu_t.$$

Then

$$\mathcal{P} U_t \leq p \, \mathrm{dist}_t^{2p-2} H_n + 8p(p-1) \mathrm{dist}_t^{2p-2}.$$

It follows that

$$\begin{aligned} F_p'(t) &\leq \int U_t \mathcal{P} \psi_t^A + (H_n p + 8p(p-1)) \int \mathrm{dist}_t^{2p-2} \psi_t^A \, d\mu_t + \frac{C_p}{A} \int_{\mathrm{spt} \nabla(\psi_t^A)} \mathrm{dist}_t^{2p-1} \, d\mu_t \\ &\leq C A^{2p-2} \mu_t(M \times M \setminus B_t(o, A) \times B_t(o, A)) + (H_n p + 8p(p-1)) D(1-t)^{p-1}. \end{aligned} \tag{3.3.11}$$

By the induction hypothesis (3.3.10) and Lemma 3.3.5, if $A \geq \bar{A}(\Lambda)$,

$$\nu_t(M \setminus B_t(o, A)) \leq \frac{C_p D}{A^{2p-2}}.$$

Hence $F_p'(t) \leq C_p D$, and thus

$$(4A)^{2p} = \int ((4A)^{2p} - \mathrm{dist}_1^{2p}) \psi_1^A \, d\mu_1 \leq \int ((4A)^{2p} - \mathrm{dist}_0^{2p}) \psi_0^A \, d\mu_0 + C_p D,$$

if $A \geq \bar{A}(\Lambda)$. By taking $A \to \infty$, we have

$$\int \operatorname{dist}_0^{2p} d\mu_0 \leq C_p D.$$

Now that we have L^{2p} estimates, we have better bounds: $\nu_t(M \setminus B_t(o, A)) \leq C_p A^{-2p}$ for large A. We may plug this back to (3.3.11) and run the argument above again to obtain that we may choose V_{2p} so that

$$V_{2p} \leq (H_n + 8(p-1))V_{2p-2}.$$

$\square$

Theorem 3.3.8. *We have a monotonicity formula for conjugate heat kernels, i.e, for any* $(x_1, t_1), (x_2, t_2) \in M \times (0, 1]$,

$$\operatorname{Var}_t(\nu_{x_1,t_1|t}, \nu_{x_2,t_2|t}) + H_n t$$

is non-decreasing.

Proof. By Lemma 3.3.7, if $t < t_i$,

$$\left(\operatorname{Var}_t^p(\delta_o, \nu_{x_i,t_i|t})\right)^{1/p} \leq (\operatorname{Var}_t^p(\delta_o, \delta_{x_i}))^{1/p} + \left(\operatorname{Var}_t^p(\delta_{x_i}, \nu_{x_i,t_i|t})\right)^{1/p} < \infty.$$

So (3.3.8) holds for $\nu_{x_i,t_i|t}$ at time t and the conclusion follows by parabolic rescaling and Theorem 3.3.4. $\square$

Lemma 3.3.9. *For any $\epsilon \in (0, 1)$, there is a constant C_ϵ depending only on n, ϵ such that*

$$\int_M \int_M \exp\left\{\frac{\operatorname{dist}_s^2(y_1, y_2)}{(8+\epsilon)(t-s)}\right\} d\nu_{x,t|s}(y_1) d\nu_{x,t|s}(y_2) \leq C_\epsilon.$$

Proof. We may assume that $x = o$ with bounds (3.3.1) and $t = 1, s = 0$. Write $\nu_t = \nu_{o,1|t}$

and $\mu_t = \nu_t \times \nu_t$. For any $\epsilon > 0$ and $p \in \mathbb{N}$,

$$V_{2p} \le (H_n + 8(p-1))V_{2p-2} \le \cdots \le C_\epsilon (8 + \epsilon/2)^p (p-1)!.$$

Hence

$$\int \exp\left\{\frac{\mathrm{dist}_0^2}{8+\epsilon}\right\} d\mu_0 = \sum_{p=0}^{\infty} \int \frac{\mathrm{dist}_0^{2p}}{p!(8+\epsilon)^p}\, d\mu_0 \le C_\epsilon \sum_{p=0}^{\infty} \left(\frac{8+\epsilon/2}{8+\epsilon}\right)^p \le C_\epsilon.$$

□

As a straightforward corollary, we can recover Theorem 3.14 in [Bam20a] with a slightly different proof. Note that it is not known yet whether Hein-Naber's log-Sobolev inequality holds for general noncompact Ricci flows and we have avoided using such inequalities.

Theorem 3.3.10 (Theorem 3.14 in [Bam20a])**.** *Let (z, s) be an H_n-center of (x, t). For any $\epsilon > 0$, if $A \ge \bar{A}(n, \epsilon)$,*

$$\nu_{x,t|s}(M \setminus B_s(z, A\sqrt{t-s})) \le C_\epsilon \exp\left(-\frac{A^2}{8+\epsilon}\right).$$

It seems reasonable to make a summary as following.

Theorem 3.3.11. *Any smooth and complete Ricci flow $(M^n, g_t)_{t\in I}$ induces an H_n-concentrated metric flow in the sense of [Bam20b, Definition 3.2].*

Proof. By Theorem 3.2.1, Bamler's sharp gradient estimate holds for bounded heat flows and thus Axiom (6) in [Bam20b, Definition 3.2] holds. The other axioms in [Bam20b, Definition 3.2] are satisfied if we use the minimal conjugate heat kernels. (M^n, g_t) is H_n-concentrated ([Bam20b, Definition 3.30]) because of Corollary 3.3.6. □

3.3.2 Almost Continuity

Lemma 3.3.12. *Let $(M^n, g_t)_{t\in I}$ be a complete Ricci flow. For $s < t, [s,t] \subset I$, and $x \in M$, if (z,s) is an H_n-center of (x,t), then*

$$\nu_{y,t|s}(B_s(z, A\sqrt{t-s})) \geq \Phi\left(\tfrac{A}{3} - \tfrac{|xy|_t}{\sqrt{t-s}}\right),$$

for any $A > \bar{A}(n), y \in M$.

Proof. By parabolic rescaling and shifting the time, we may assume that $s = 0, t = 1$. This is a simple application of the gradient estimates Theorem 3.2.1.

$$u_t(y) := \nu_{y,t|s}(B_s(z, A\sqrt{t-s}))$$

is a heat flow taking values in $(0,1)$. By Theorem 3.2.1, for any $y \in M$,

$$\begin{aligned}\Phi_1^{-1}(u_1(y)) \geq \Phi_1^{-1}(u_1(x)) - |xy|_1 &= \Phi_1^{-1}(\nu_{x,1|0}(B_0(z,A))) - |xy|_1 \\ &=: a - |xy|_1.\end{aligned}$$

By Theorem 3.3.10,

$$\nu_{x,1|0}(B_0(z,A)) \geq 1 - C_n e^{-A^2/9},$$

if $A \geq \bar{A}(n)$. Recall that $\Phi_1 = \Phi$. Then

$$\Phi(a) \geq 1 - C_n e^{-A^2/9},$$

and we may assume that $a \geq 10$, if $A \geq \bar{A}(n)$. By the definition of Φ,

$$\begin{aligned}C_n e^{-A^2/9} &\geq \int_a^\infty (4\pi)^{-1/2} e^{-t^2/4}\, dt \geq \int_a^{a+1} (4\pi)^{-1/2} e^{-t^2/4} \\ &\geq (4\pi)^{-1/2} e^{-(a+1)^2/4}.\end{aligned}$$

Since $a \geq 10$,

$$\tfrac{5}{4}a^2 + 5 \geq (a+1)^2 \geq \tfrac{4}{9}A^2 - C_n, \quad \implies a \geq A/3,$$

if $A \geq \bar{A}(n)$. By the monotonicity of Φ, if $A \geq \bar{A}(n)$,

$$u_1(y) \geq \Phi\left(a - |xy|_1\right) \geq \Phi\left(A/3 - |xy|_1\right).$$

□

We prove the following Lemma which roughly states that the H_n-centers are almost transitive.

Lemma 3.3.13. *Let $(M^n, g_t)_{t\in I}$ be a complete Ricci flow. Fix $(x_0, t_0) \in M \times I$. Suppose that (z_s, s) is an H_n-center of (x_0, t_0) if $s \in I$. Fix $s, t \in I$, such that $s < t < t_0$, and let (z'_s, s) be an H_n-center of (z_t, t). Then*

$$|z_s z'_s|_s \leq C_n \sqrt{t_0 - s}.$$

Proof. By shifting the time, we may assume that $t_0 = 0$. Write $\nu_t := \nu_{x_0,0|t}$ for $t < 0$. By Lemma 3.3.12, for $y \in B_t(z_t, \sqrt{2H_n|t|})$,

$$\begin{aligned}\nu_{y,t|s}(B_s(z'_s, A\sqrt{|s|})) &= \nu_{y,t|s}\left[B_s\left(z'_s, A\tfrac{\sqrt{|s|}}{\sqrt{t-s}}\sqrt{t-s}\right)\right] \\ &\geq \Phi\left(\tfrac{A\sqrt{|s|}}{3\sqrt{t-s}} - \tfrac{\sqrt{2H_n|t|}}{\sqrt{t-s}}\right) \geq \Phi(0) = 1/2,\end{aligned}$$

if $A \geq \bar{A}(n)$. Here we used the fact that $A\frac{\sqrt{|s|}}{\sqrt{t-s}} > A \geq \bar{A}(n)$ to apply Lemma 3.3.12. We now fix $A = \bar{A}(n)$. Then

$$\begin{aligned}\nu_s(B_s(z'_s, A\sqrt{|s|})) &\geq \int_{B_t(z_t,\sqrt{2H_n|t|})} \nu_{y,t|s}(B_s(z'_s, A\sqrt{|s|}))\, d\nu_t(y) \\ &\geq \tfrac{1}{2}\nu_t(B_t(z_t, \sqrt{2H_n|t|})) \geq 1/4.\end{aligned}$$

It follows that

$$|z_s z'_s|_s \leq (A+10)\sqrt{|s|} \leq C_n\sqrt{|s|}.$$

□

Lemma 3.3.14. *Let $(M^n, g_t)_{t\in I}$ be a complete Ricci flow with bounded curvature on compact intervals. For $s < t, [s-(t-s), t] \subset I$, and $x \in M$, let (z, s) be an H_n-center of (x, t) and ℓ be Perelman's ℓ-function based at (x, t). If $p \in M$ such that*

$$\ell(p, s) \leq \Lambda,$$

then

$$|zp|_s^2 \leq C_n(t-s)(\Lambda - \mathcal{N}_{x,t}(t-s)).$$

Proof. By parabolic rescaling and shifting the time, we may assume that $s = -1, t = 0$. By [Per02, 9.5] and [Bam20a, Theorem 7.2],

$$\begin{aligned}(4\pi)^{-n/2}e^{-\Lambda} &\leq (4\pi)^{-n/2}e^{-\ell(p,-1)} \leq K(x, 0\,|\, p, -1) \\ &\leq C_n \exp\left(-\mathcal{N}_{x,0}(1) - |zp|_{-1}^2/9\right).\end{aligned}$$

Thus

$$|zp|_{-1}^2 \leq C_n(\Lambda - \mathcal{N}_{x,0}(1)).$$

□

We present the following result which roughly states that H_n-centers are almost continuous. This result will be useful later.

Proposition 3.3.15. *Let $(M^n, g_t)_{t\in[-2T,0]}$ be a complete Ricci flow with bounded curvature, for some $T > 1$. Fix $x \in M$. Let (z_s, s) be an H_n-center of $(x, 0)$ for each $s \in [-T, 0)$. Suppose that*

$$\int_s^t \sqrt{t-\tau}R(x,\tau)\,d\tau \leq \Lambda\sqrt{t-s},$$

and

$$\mathcal{N}_{x,0}(T) \geq -Y,$$

for some constants $\Lambda, Y > 0$. Then for $-T \leq s < t < -1, t - s < 1$, we have

$$|z_t z_s|_s \leq C(n, \Lambda, Y)\sqrt{|s|}.$$

Proof. Let (z'_s, s) be an H_n-center of (z_t, t). By Lemma 3.3.13,

$$|z_s z'_s|_s \leq C_n\sqrt{|s|}.$$

By Lemma 3.3.14,

$$|z_t z'_s|_s \leq C(n, \Lambda, Y)\sqrt{t-s} \leq C(n, \Lambda, Y)\sqrt{|s|}.$$

So

$$|z_t z_s|_s \leq |z_t z'_s|_s + |z_s z'_s|_s \leq C(n, \Lambda, Y)\sqrt{|s|}.$$

□

We record another useful simple Lemma.

Lemma 3.3.16. *Let $(M^n, g_t)_{t\in[-T,0]}$ be a complete Ricci flow with bounded curvature, for* *some $T > 0$. Let (z_i, s) be any H_n-center of $(x_i, 0), i = 1, 2,$ for some $s \in [-T, 0)$.*

$$|z_1 z_2|_s \le 2\sqrt{H_n|s|} + |x_1 x_2|_0.$$

Proof. By [Bam20a, Lemma 2.7],

$$\begin{aligned}
|z_1 z_2|_s &= \mathrm{dist}^{g_s}_{\mathrm{W}_1}(\delta_{z_1}, \delta_{z_2}) \\
&\le \mathrm{dist}^{g_s}_{\mathrm{W}_1}(\delta_{z_1}, \nu_{x_1,0|s}) + \mathrm{dist}^{g_s}_{\mathrm{W}_1}(\nu_{x_1,0|s}, \nu_{x_2,0|s}) + \mathrm{dist}^{g_s}_{\mathrm{W}_1}(\nu_{x_2,t|0}, \delta_{z_2}) \\
&\le 2\sqrt{H_n|s|} + |x_1 x_2|_0.
\end{aligned}$$

□

Chapter 3, in part, is currently being prepared for submission for publication, which is a joint work with Chan, Pak-Yeung; Cheng, Liang; Zhang, Yongjia [CCMZ]. Chapter 3 also contains material from [CMZ21d] which has been submitted for publication and is a joint work with Chan, Pak-Yeung and Zhang, Yongjia.

Chapter 4

Perelman's Entropy

Throughout this chapter, we assume that $(M^n, g_t)_{t\in I}$ is a complete Ricci flow where $I \subset \mathbb{R}$ is the time-span of the flow. We make a technical assumption in this chapter that

$$\sup_{M\times J} |\mathrm{Rm}| < \infty,$$

for each compact sub-interval $J \subset I$.

4.1 Preliminaries

Let (M^n, g) be a complete manifold. Perelman's $\mathcal{W}$-functional at scale $\tau > 0$ is defined to be

$$\mathcal{W}(g, f, \tau) := \int_M \left(\tau\left(|\nabla f|^2 + R\right) + f - n\right)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dg.$$

Writting $u = (4\pi\tau)^{-n/2}e^{-f}$, we may rewrite the $\mathcal{W}$-functional as

$$\overline{\mathcal{W}}(g, u, \tau) := \int_M \left(\tau\left(|\nabla \log u|^2 + R\right) - \log u\right) u\, dg - \frac{n}{2}\log(4\pi\tau) - n.$$

For any region $\Omega \subseteq M$, following [Wa18], we define

$$\mu(\Omega, g, \tau) := \inf \left\{ \overline{\mathcal{W}}(g, u, \tau) : u \geq 0, \sqrt{u} \in C_0^\infty(\Omega), \int_M u \, dg = 1 \right\},$$
$$\nu(\Omega, g, \tau) := \inf_{0 < s \leq \tau} \mu(\Omega, g, s).$$

Indeed, $\nu(\Omega, g, \tau)$ is a local Sobolev constant of the region Ω.

For $s < t, x \in M$, write

$$\mathcal{N}_s^*(x, t) := \mathcal{N}_{x,t}(t - s).$$

An application of Bamler's sharp gradient estimate ([Bam20a, Theorem 4.1] or Theorem 3.2.1) is the following Harnack inequality for the Nash entropy, which plays an important role in the following discussions.

Theorem 4.1.1 (Theorem 5.11 in [Bam20a]). *If $R_{g_{t^*}} \geq R_{\min}$, and $s < t^* \leq t_1, t_2$, with $s, t^*, t_1, t_2 \in I$, then for $x_1, x_2 \in M$,*

$$\mathcal{N}_s^*(x_1, t_1) - \mathcal{N}_s^*(x_2, t_2) \leq \left(\tfrac{n}{2(t^* - s)} - R_{\min} \right)^{1/2} \operatorname{dist}_{W_1}^{g_{t^*}}(\nu_{x_1, t_1 | t^*}, \nu_{x_2, t_2 | t^*})$$
$$+ \frac{n}{2} \log \frac{t_2 - s}{t^* - s}.$$

We refer to [MZ21] for a proof of the case of noncompact Ricci flows with bounded curvature on compact intervals.

By the standard lower bound on the scalar curvature Theorem 3.1.4, $R_{g_{t^*}} \geq -\frac{n}{2(t^* - s)}$ if $[s, t^*] \subset I$, and thus

$$\mathcal{N}_s^*(x_1, t_1) - \mathcal{N}_s^*(x_2, t_2) \leq \sqrt{n/(t^* - s)} \operatorname{dist}_{W_1}^{g_{t^*}}(\nu_{x_1, t_1 | t^*}, \nu_{x_2, t_2 | t^*})$$
$$+ \frac{n}{2} \log \frac{t_2 - s}{t^* - s}. \tag{4.1.1}$$

It might be more convenient to use this inequality.

4.2 Almost Monotonicity

We first give the following lower bound on the local ν-entropy in terms of the pointed Nash entropy. In the following, $(M^n, g_t)_{t\in I}$ is a complete Ricci flow with bounded curvature.

Theorem 4.2.1. *Assume that $[-r^2, 0] \subseteq I$. Then for any point $x_0 \in M$ and any τ, $A > 0$, we have*

$$\nu\left(B_0(x_0, Ar), g_0, \tau r^2\right) \geq \mathcal{N}_{x_0,0}(r^2) - \sqrt{n}A - \frac{n}{2}\tau - \frac{n}{2}\log(1+\tau). \tag{4.2.1}$$

We also have the following almost monotonicity formula for the local μ-entropy, which is similar to Wang [Wa18, Theorem 5.4] and Tian-Zhang [TZ21], and our proof is inspired by their works. Perelman's original (global) monotonicity formula follows immediately by the following local version.

Theorem 4.2.2. *Assume that $[-r^2, 0] \subseteq I$. Then for any $x_0 \in M$, any H_n-center $(z, -r^2)$ of $(x_0, 0)$, any $\tau > 0$, and any $A \geq 16$, we have*

$$\begin{aligned} &\mu\left(B_{-r^2}\left(z, 2A\sqrt{H_n}r\right), g_{-r^2}, (1+\tau)r^2\right) \\ &\leq \mu\left(B_0\left(x_0, A\sqrt{H_n}r\right), g_0, \tau r^2\right) + \frac{C_n}{A^2}(1+\tau)e^{-\frac{A^2}{20}}, \end{aligned}$$

where C_n is a positive dimensional constant.

As a refinement of the local monotonicity formula above, we have the following, which can be viewed as the inverse of Theorem 4.2.1. Note that we should consider the ball centered at H_n-centers for the local μ-entropy.

Theorem 4.2.3. *Assume that* $[-r^2,0]\subseteq I$*. Furthermore, assume that* $R_{g_{-r^2}}\geq R_{\min}$*. Then, for any* $x_0\in M$*, any* H_n*-center* $(z,-r^2)$ *of* $(x_0,0)$*, and any* $A\geq 8$*, we have*

$$\mu\left(B_{-r^2}\left(z,2A\sqrt{H_n}r\right),g_{-r^2},r^2\right)\leq\mathcal{N}_{x_0,0}(r^2)+C(n,R_{\min}r^2,A),\tag{4.2.2}$$

where

$$C(n,R_{\min}r^2,A)=\tfrac{C_n}{A^2}e^{-\frac{A^2}{20}}+8\left(e^{-\frac{A^2}{20}}\cdot(n-2R_{\min}r^2)+e^{-\frac{A^2}{40}}\cdot(n-2R_{\min}r^2)^{\frac{1}{2}}\right),$$

and C_n *is a dimensional constant.*

4.2.1 Proof of Theorem 4.2.1

In this subsection, we present the proof of Theorem 4.2.1. Our main technique is Bamler's Harnack inequality for the Nash entropy [Bam20a, Corollary 5.11]. The proof is inspired by Bamler's no-collapsing theorem [Bam20a, Theorem 6.1] and Bing Wang's arguments in [Wa18].

By parabolic rescaling, we assume that $r=1$, and we let $\tau>0, A<\infty$ be arbitrarily fixed constants. We shall pick an arbitrary test function and verify that its $\overline{\mathcal{W}}$ functional is bounded from below by the Nash entropy in the way as stated in the theorem. To this end, let u_0 be an arbitrary nonnegative test function such that $\sqrt{u_0}\in C_0^\infty(B_0(x_0,A))$ and $\int_M u_0\,dg_0=1$. We solve the backward conjugate heat equation

$$\Box_t^* u_t:=(-\partial_t-\Delta_{g_t}+R_{g_t})u_t=0\quad\text{ on }\quad M\times[-1,0],$$

with the initial data being u_0 at time $t=0$. Then for all $t\in[-1,0]$, u_t can be written as

$$u_t(x)=\int_M K(y,0\,|\,x,t)u_0(y)dg_0(y),\tag{4.2.3}$$

where K is the fundamental solution to the conjugate heat equation. Then, the evolving probability measure

$$\mu_t(\Omega) := \int_\Omega u_t dg_t, \quad \Omega \subseteq M \text{ is measurable and } t \in [-1, 0]$$

is what Bamler [Bam20b] calls a *conjugate heat flow.* For $t \in [-1, 0)$, we define

$$\begin{aligned}
\tau_t &:= \tau - t, \quad u_t =: (4\pi\tau_t)^{-n/2} e^{-f_t}, \\
\overline{\mathcal{N}}(t) &:= \int_M f_t \, d\mu_t - \frac{n}{2}, \\
\overline{\mathcal{W}}(t) &:= \int_M \big(\tau_t(|\nabla f_t|^2 + R_{g_t}) + f_t - n\big) \, d\mu_t.
\end{aligned}$$

Lemma 4.2.4.

$$\overline{\mathcal{W}}(0) := \lim_{t \to 0^-} \overline{\mathcal{W}}(t) = \overline{\mathcal{W}}(g_0, u_0, \tau), \tag{4.2.4}$$

$$-\frac{d}{dt}\big(\tau_t \overline{\mathcal{N}}(t)\big) = \overline{\mathcal{W}}(t), \tag{4.2.5}$$

$$\frac{d}{dt}\overline{\mathcal{W}}(t) \geq 0. \tag{4.2.6}$$

Proof. When M is closed, the computations are standard and clear. The proof of the lemma when M is noncompact, especially of (4.2.4), is not essentially different from [Wa18, §4], and a detailed treatment can be found in [CMZ21a, §9]. We may apply the heat kernel Gaussian bounds by the assumption that the flow has bounded curvature. (See, e.g., [CCG$^+$10, Theorem 26.25 and Theorem 26.31].)

□

Proof of Theorem 4.2.1. Our goal is to estimate the lower bound of $\overline{\mathcal{W}}(0)$. By (4.2.4)—

(4.2.6), we compute

$$
\begin{aligned}
&\tau_{-1}\overline{\mathcal{N}}(-1) - \tau_0\overline{\mathcal{N}}(0) \\
= & -\int_{-1}^{0}\frac{d}{dt}\left(\tau_t\overline{\mathcal{N}}(t)\right)\,dt = \int_{-1}^{0}\overline{\mathcal{W}}(t)\,dt \\
\leq & \,\overline{\mathcal{W}}(0) = \overline{\mathcal{W}}(g_0, u_0, \tau),
\end{aligned}
$$

where we have defined

$$\overline{\mathcal{N}}(0) := \lim_{t\to 0^-}\overline{\mathcal{N}}(t).$$

Therefore, it remains to estimate $\overline{\mathcal{N}}(-1)$ and $\overline{\mathcal{N}}(0)$.

First, we estimate $\overline{\mathcal{N}}(0)$. By the maximum principle, for $s \in (-1, 0]$, we have

$$R_{g_s} \geq -\frac{n}{2(s+1)} \quad \text{on} \quad M. \tag{4.2.7}$$

For each $s \in (-1, 0)$ close to 0, we have

$$
\begin{aligned}
\overline{\mathcal{N}}(s) &= \overline{\mathcal{W}}(s) + \frac{n}{2} - \int \tau_s\left(|\nabla f_s|^2 + R_{g_s}\right)\,d\mu_s \\
&\leq \overline{\mathcal{W}}(s) + \frac{n}{2} + \frac{n\tau_s}{2(s+1)}.
\end{aligned}
$$

By taking $s \to 0^-$, we have

$$\overline{\mathcal{N}}(0) \leq \overline{\mathcal{W}}(0) + \frac{n}{2}(1+\tau).$$

Then

$$
\begin{aligned}
\overline{\mathcal{W}}(0) &\geq \tau_{-1}\overline{\mathcal{N}}(-1) - \tau_0\mathcal{N}(0) \\
&\geq (1+\tau)\overline{\mathcal{N}}(-1) - \tau\overline{\mathcal{W}}(0) - \frac{n}{2}\tau(1+\tau),
\end{aligned}
$$

and hence

$$\overline{\mathcal{W}}(0) \geq \overline{\mathcal{N}}(-1) - \frac{n}{2}\tau. \tag{4.2.8}$$

It remains to estimate $\overline{\mathcal{N}}(-1)$. To this end, we recall the definition of $\overline{\mathcal{N}}$:

$$\begin{aligned}\overline{\mathcal{N}}(-1) &= \int f_{-1} u_{-1}\, dg_{-1} - \frac{n}{2} \\ &= -\int u_{-1} \log u_{-1}\, dg_{-1} - \frac{n}{2} - \frac{n}{2}\log(4\pi(1+\tau)).\end{aligned} \tag{4.2.9}$$

By (4.2.3), we have

$$u_{-1}(x) = \int K(y,0\,|\,x,-1) u_0(y)\, dg_0(y) = \int K(y,0\,|\,x,-1)\, d\mu_0(y),$$

where μ_0 is a probability measure. Hence, by Jensen's inequality, we have

$$u_{-1} \log u_{-1}(x) \leq \int K(y,0\,|\,x,-1) \log K(y,0\,|\,x,-1)\, d\mu_0(y).$$

It follows from integrating both sides with the measure dg_{-1} that

$$\begin{aligned}&\int u_{-1} \log u_{-1}(x)\, dg_{-1}(x) \\ \leq\ &\int\int K(y,0\,|\,x,-1) \log K(y,0\,|\,x,-1)\, d\mu_0(y) dg_{-1}(x) \\ =\ &\int\int K(y,0\,|\,x,-1) \log K(y,0\,|\,x,-1)\, dg_{-1}(x) d\mu_0(y) \\ =\ &-\int \mathcal{N}_{y,0}(1)\, d\mu_0(y) - \frac{n}{2} - \frac{n}{2}\log(4\pi).\end{aligned} \tag{4.2.10}$$

Here, it is easy to verify that the change of order of the integration is valid, since μ_0 is supported in $B_0(x_0, A)$ and $K(y,0\,|\,\cdot,-1)$ has rapid decay at infinity.

Combining (4.2.9) and (4.2.10), we have

$$\overline{\mathcal{N}}(-1) \geq \int \mathcal{N}_{y,0}(1)\, d\mu_0(y) - \frac{n}{2}\log(1+\tau). \tag{4.2.11}$$

We shall estimate the integral on the right-hand-side. Let us fix an arbitrary $y \in \operatorname{spt}(\mu_0) \subseteq B_0(x_0, A)$. By the definition of W_1-distance, we have

$$\operatorname{dist}_{W_1}^{g_0}(\nu_{y,0\,|\,0}, \nu_{x_0,0\,|\,0}) = \operatorname{dist}_{W_1}^{g_0}(\delta_y, \delta_{x_0}) = \operatorname{dist}_{g_0}(x_0, y) < A.$$

We may then apply (4.1.1) with $x_0 \to x_1$, $y \to x_2$, $0 \to t_1 = t_2 = t^*$, $-1 \to s$, and obtain

$$\begin{aligned} \mathcal{N}_{x_0,0}(1) &\leq \mathcal{N}_{y,0}(1) + \sqrt{n}\operatorname{dist}_{W_1}^{g_0}(\nu_{y,0\,|\,0}, \nu_{x_0,0\,|\,0}) \\ &\leq \mathcal{N}_{y,0}(1) + \sqrt{n}A, \end{aligned} \tag{4.2.12}$$

where we have also applied (4.2.7). It follows from (4.2.8), (4.2.11), and (4.2.12) that

$$\begin{aligned} \overline{\mathcal{W}}(0) &\geq \overline{\mathcal{N}}(-1) - \frac{n}{2}\tau \\ &\geq \int \mathcal{N}_{y,0}(1)\, d\mu_0(y) - \frac{n}{2}\tau - \frac{n}{2}\log(1+\tau) \\ &\geq \mathcal{N}_{x_0,0}(1) - \sqrt{n}A - \frac{n}{2}\tau - \frac{n}{2}\log(1+\tau). \end{aligned}$$

Since the test function u_0 is arbitrarily fixed, we have

$$\mu\left(B_0(x_0, A), g_0, \tau\right) \geq \mathcal{N}_{x_0,0}(1) - \sqrt{n}A - \frac{n}{2}\tau - \frac{n}{2}\log(1+\tau).$$

To see that the local ν-functional is also bounded, one needs only to observe that, for any

$s \in (0, \tau]$, we have

$$\begin{aligned}\mu\left(B_0(x_0, A), g_0, s\right) &\geq \mathcal{N}_{x_0,0}(1) - \sqrt{n}A - \frac{n}{2}s - \frac{n}{2}\log(1+s) \\ &\geq \mathcal{N}_{x_0,0}(1) - \sqrt{n}A - \frac{n}{2}\tau - \frac{n}{2}\log(1+\tau).\end{aligned}$$

This finishes the proof of the theorem. $\square$

4.2.2 Proofs of Theorem 4.2.2 and Theorem 4.2.3

In this subsection, we prove Theorem 4.2.2 and Theorem 4.2.3. As will be clear below, the proof of Theorem 4.2.3 is similar to that of Theorem 4.2.2.

The proof of Theorem 4.2.2 is inspired by [Wa18] and [TZ21]. The idea is to evolve the minimizer of the local μ-functional at $t = 0$ using conjugate heat flow, and we may then compare the local μ-functional at different time-slices using the differential Harnack inequality in [Wa18]. The error terms come from the concentration estimate of the heat kernel measure near an H_n-center [Bam20a, Proposition 3.14]. The proof of Theorem 4.2.3 is similar to that of Theorem 4.2.2.

Proof of Theorem 4.2.2. By parabolic rescaling, we may assume that $r = 1$. For simplicity, we write

$$B = B_{-1}\left(z, \tfrac{7}{4}A\sqrt{H_n}\right), \quad B' = B_{-1}\left(z, 2A\sqrt{H_n}\right),$$

where $(z, -1)$ is an H_n-center of $(x_0, 0)$. Let u_0 be the minimizer of

$$\mu := \mu\left(B_0\left(x_0, A\sqrt{H_n}\right), g_0, \tau\right).$$

Following [Wa18], we let u_t be the solution to the conjugate heat equation $\Box_t^* u_t = 0$ on $M \times [-1, 0]$ with initial data being u_0 at $t = 0$. Defining $\tau_t := \tau - t$, by [Wa18, Theorem

4.2], we have

$$\Big\{\tau_t\big(-2\tfrac{\Delta u_t}{u_t}+|\nabla \log u_t|^2+R\big)-\log u \quad -n-\mu-\tfrac{n}{2}\log(4\pi\tau_t)\Big\}u_t\le 0,$$

on $M\times[-1,0)$. As before, $d\mu_t := u_t\, dg_t$ is a probability measure for $t\in[-1,0]$.

Claim: If $A\ge 16$, then we have

$$\mu_{-1}(M\setminus B)\le 2e^{-\frac{A^2}{20}}<\tfrac{1}{2}.$$

Proof of the Claim. For any $x\in B_0\left(x_0, A\sqrt{H_n}\right)$, let $(z_x,-1)$ be an H_n-center of $(x,0)$. Then

$$\begin{aligned}
&\operatorname{dist}_{-1}(z_x,z)=\operatorname{dist}_{W_1}^{g_{-1}}(\delta_{z_x},\delta_z)\\
&\le \operatorname{dist}_{W_1}^{g_{-1}}(\delta_{z_x},\nu_{x,0\,|\,-1})+\operatorname{dist}_{W_1}^{g_{-1}}(\nu_{x,0\,|\,-1},\nu_{x_0,0\,|\,-1})+\operatorname{dist}_{W_1}^{g_{-1}}(\delta_z,\nu_{x_0,0\,|\,-1})\\
&\le 2\sqrt{H_n}+\operatorname{dist}_{W_1}^{g_0}(\nu_{x,0\,|\,0},\nu_{x_0,0\,|\,0})\\
&\le 2\sqrt{H_n}+A\sqrt{H_n}\\
&\le \tfrac{5}{4}A\sqrt{H_n},
\end{aligned}$$

where we used the monotonicity of the W_1-Wassernstein distance (c.f. [Bam20a, Lemma 2.7]). Then $B_{-1}\left(z_x,\frac{1}{2}A\sqrt{H_n}\right)\subseteq B$ for any $x\in B_0\left(x_0,A\sqrt{H_n}\right)$. By [Bam20a, Proposition 3.14], for any $x\in B_0\left(x_0,A\sqrt{H_n}\right)$, if $A\ge 16$, then we have

$$\nu_{x,0\,|\,-1}(M\setminus B)\le \nu_{x,0\,|\,-1}\left((M\setminus B_{-1}\left(z_x,\tfrac{1}{2}A\sqrt{H_n}\right)\right)\le 2e^{-\frac{A^2}{20}}.$$

By the standard semi-group property, we have

$$\mu_{-1}(M\setminus B)=\int_{B_0\left(x_0,A\sqrt{H_n}\right)}\nu_{x,0\,|\,-1}(M\setminus B)\cdot u_0(x)\,dg_0(x)\le 2e^{-\frac{A^2}{20}}.$$

$\square$

Let η be a smooth cutoff function supported in B' such that $\eta|_B = 1$, $0 \le \eta \le 1$, and $|\nabla \eta|^2 \le \frac{160}{A^2 H_n}\eta$. Define

$$\alpha := \int_M \eta^2 u_{-1}\, dg_{-1} = \int_M \eta^2\, d\mu_{-1}.$$

By the Claim, we have

$$\tfrac{1}{2} \le 1 - 2e^{-\frac{A^2}{20}} \le \alpha \le 1.$$

Let $\tilde{u} = \eta^2 u_{-1}/\alpha$. Then $\tilde{u} \in C_0^\infty(B')$ and $\int \tilde{u}\, dg_{-1} = 1$. In the following, we omit the measure dg_{-1} and write $u = u_{-1}$ when there is no ambiguity. Integrating by parts, we have

$$\begin{aligned}\int |\nabla \log \tilde{u}|^2 \tilde{u} &= \alpha^{-1} \int |\nabla \log u|^2 \eta^2 u + 2\langle \nabla \eta^2, \nabla u\rangle + 4|\nabla \eta|^2 u \\ &= \alpha^{-1} \int \big(|\nabla \log u|^2 u - 2\Delta u\big)\eta^2 + 4|\nabla \eta|^2 u.\end{aligned}$$

Then

$$\mu(B', g_{-1}, 1+\tau) \le \int \big(\tau_{-1}(|\nabla \log \tilde{u}|^2 + R)\tilde{u} - \log \tilde{u} \cdot \tilde{u}\big)\, dg_{-1} - n - \frac{n}{2}\log(4\pi\tau_{-1}) \tag{4.2.13}$$

$$\begin{aligned}&= \alpha^{-1} \int \Big\{\tau_{-1}\big(|\nabla \log u|^2 - 2\tfrac{\Delta u}{u} + R\big) - \log u - n - \tfrac{n}{2}\log(4\pi\tau_{-1})\Big\}\eta^2 u \\ &\quad + \alpha^{-1}\tau_{-1} \int 4|\nabla \eta|^2 u - \int \log \tfrac{\eta^2}{\alpha} \cdot \tfrac{\eta^2}{\alpha} u \\ &\le \mu + \frac{C_n}{\alpha A^2}(1+\tau)\mu_{-1}(M \setminus B) \le \mu + \frac{C_n}{A^2}(1+\tau)e^{-\frac{A^2}{20}},\end{aligned}$$

where we have implemented the following consequence of Jensen's inequality applied to the convex function $t \mapsto t\log t$ and the probability measure $u_{-1}\, dg_{-1}$

$$-\int \log \tfrac{\eta^2}{\alpha} \cdot \tfrac{\eta^2}{\alpha} u \le -\int \tfrac{\eta^2}{\alpha} u \cdot \log \int \tfrac{\eta^2}{\alpha} u = 0.$$

$\square$

Proof of Theorem 4.2.3. The proof of Theorem 4.2.3 is similar to the above proof. We shall again assume $r = 1$ by parabolic rescaling. We define

$$B = B_{-1}\left(z, A\sqrt{H_n}\right), \quad B' = B_{-1}\left(z, 2A\sqrt{H_n}\right),$$
$$u_t = (-4\pi t)^{-\frac{n}{2}} e^{-f_t} =: K(x_0, 0 \,|\, \cdot, t) \quad \text{for} \quad t \in [-1, 0),$$

where $(z, -1)$ is an H_n-center of $(x_0, 0)$. As before, let η be the cut-off function defined on (M, g_{-1}), satisfying $\eta|_B = 1$, $\eta|_{M\setminus B'} = 0$, $0 \le \eta \le 1$, and $|\nabla\eta|^2 \le \frac{10}{A^2 H_n}\eta$, and let

$$\alpha := \int_M u_{-1}\eta^2 dg_{-1}.$$

We shall then use

$$\tilde{u} := \alpha^{-1} u_{-1}\eta^2$$

as a test function to estimate $\mu(g_{-1}, B', 1)$.

First of all, [Bam20a, Proposition 3.14] implies that

$$\tfrac{1}{2} \le 1 - 2e^{-\frac{A^2}{20}} \le \nu_{x_0,0\,|\,-1}(B) \le \alpha \le \nu_{x_0,0\,|\,-1}(M) = 1. \tag{4.2.14}$$

Then, we may follow the same computation as in (4.2.13). Since this computation is on the fixed time-slice at $t = -1$, we shall omit the subindex -1 and the measure notation

dg_{-1} when there is no ambiguity.

$$\begin{aligned}
\mu(B', g_{-1}, 1) \le & \int \big((|\nabla \log \tilde{u}|^2 + R)\tilde{u} - \log \tilde{u} \cdot \tilde{u}\big)\, dg_{-1} - n - \frac{n}{2}\log(4\pi) \\
= \alpha^{-1} & \int \Big\{ \big(|\nabla \log u|^2 - 2\tfrac{\Delta u}{u} + R\big) - \log u - n - \tfrac{n}{2}\log(4\pi) \Big\} \eta^2 u \\
& + \alpha^{-1} \int 4|\nabla \eta|^2 u - \int \log \tfrac{\eta^2}{\alpha} \cdot \tfrac{\eta^2}{\alpha} u \\
= \alpha^{-1} & \int \big(2\Delta f - |\nabla f|^2 + R + f - n\big)\, \eta^2 u + \alpha^{-1} \int 4|\nabla \eta|^2 u - \int \log \tfrac{\eta^2}{\alpha} \cdot \tfrac{\eta^2}{\alpha} u \\
= \alpha^{-1} & \int (|\nabla f|^2 + R + f - n)\eta^2 u - \alpha^{-1} \int 4\langle \nabla f, \nabla \eta \rangle \eta u \\
& + \alpha^{-1} \int 4|\nabla \eta|^2 u - \int \log \tfrac{\eta^2}{\alpha} \cdot \tfrac{\eta^2}{\alpha} u.
\end{aligned} \tag{4.2.15}$$

The last two terms are easily dealt with using the same argument as in the proof of the previous theorem, we shall consider the first two terms.

First of all, we observe that the first term can be split into two terms.

$$\begin{aligned}
& \alpha^{-1} \int (|\nabla f|^2 + R + f - n)\eta^2 u \\
= {} & \alpha^{-1} \int (|\nabla f|^2 + R)\eta^2 u + \alpha^{-1} \int \big(f - \tfrac{n}{2}\big)\, \eta^2 u - \frac{n}{2} \\
=: {} & \mathrm{I} + \mathrm{II} - \frac{n}{2}.
\end{aligned} \tag{4.2.16}$$

Applying [Bam20a, Proposition 5.13] (see also [CMZ21a, Proposition 3.3] for the proof under our current condition), we may estimate the term I as follows.

$$\begin{aligned}
\mathrm{I} &= \alpha^{-1}\int(|\nabla f|^2+R)\eta^2 u \qquad (4.2.17)\\
&= \alpha^{-1}\int(|\nabla f|^2+R-R_{\min})\eta^2 u + R_{\min}\\
&\le \alpha^{-1}\int(|\nabla f|^2+R-R_{\min})u + R_{\min}\\
&= \alpha^{-1}\int(|\nabla f|^2+R)u - (\alpha^{-1}-1)R_{\min}\\
&\le \alpha^{-1}\cdot\frac{n}{2} - (\alpha^{-1}-1)R_{\min}.
\end{aligned}$$

We may apply [Bam20a, Proposition 5.13] again as well as the definition of the Nash entropy to estimate the term II as follows. For the sake of notational simplicity, we have defined $\mathcal{N} := \mathcal{N}_{x_0,0}(1)$.

$$\begin{aligned}
\mathrm{II} &= \alpha^{-1}\int\left(f-\tfrac{n}{2}-\mathcal{N}\right)\eta^2 u + \mathcal{N} \qquad (4.2.18)\\
&= \alpha^{-1}\int\left(f-\tfrac{n}{2}-\mathcal{N}\right)(\eta^2-1)u + \mathcal{N}\\
&\le \alpha^{-1}\left(\int\left(f-\tfrac{n}{2}-\mathcal{N}\right)^2 u\right)^{\frac12}\left(\int\left(\eta^2-1\right)^2 u\right)^{\frac12} + \mathcal{N}\\
&\le \alpha^{-1}(\nu_{-1}(M\setminus B))^{\frac12}(n-2R_{\min})^{\frac12} + \mathcal{N}\\
&\le \sqrt{2}\alpha^{-1}\cdot(n-2R_{\min})^{\frac12}\cdot e^{-\frac{A^2}{40}} + \mathcal{N},
\end{aligned}$$

where we have applied (4.2.14). Combining (4.2.16), (4.2.17), and (4.2.18), we have

$$\begin{aligned}
&\alpha^{-1}\int(|\nabla f|^2+R+f-n)\eta^2 u \qquad (4.2.19)\\
&\le \mathcal{N} + \tfrac12(\alpha^{-1}-1)\cdot(n-2R_{\min}) + \sqrt{2}\alpha^{-1}e^{-\frac{A^2}{40}}\cdot(n-2R_{\min})^{\frac12}\\
&\le \mathcal{N} + 4\left(e^{-\frac{A^2}{20}}\cdot(n-2R_{\min}) + e^{-\frac{A^2}{40}}\cdot(n-2R_{\min})^{\frac12}\right),
\end{aligned}$$

where we have applied (4.2.14).

Next, we apply the Cauchy-Schwarz inequality to estimate the second term on the right-hand-side of (4.2.15).

$$
\begin{aligned}
-\alpha^{-1}\int 4\langle\nabla f,\nabla\eta\rangle\eta u &\le 4\alpha^{-1}\left(\int|\nabla f|^2u\right)^{\frac{1}{2}}\left(\int|\nabla\eta|^2\eta^2u\right)^{\frac{1}{2}} \\
&\le 4\alpha^{-1}\left(\int(|\nabla f|^2+R)u-R_{\min}\right)^{\frac{1}{2}}(\nu_{-1}(M\setminus B))^{\frac{1}{2}} \\
&\le 4\alpha^{-1}e^{-\frac{A^2}{40}}\cdot(n-2R_{\min})^{\frac{1}{2}}.
\end{aligned}
\tag{4.2.20}
$$

Finally, arguing as in the proof of Theorem 4.2.2, we have

$$
\alpha^{-1}\int 4|\nabla\eta|^2u\le\alpha^{-1}\frac{C_n}{A^2}e^{-\frac{A^2}{20}},\quad -\int\log\tfrac{\eta^2}{\alpha}\cdot\tfrac{\eta^2}{\alpha}u\le 0. \tag{4.2.21}
$$

Combining (4.2.15), (4.2.19), (4.2.20), and (4.2.21), the conclusion follows. □

4.3 Entropy for Ancient Flows

In this subsection, we shall show that for a *noncollapsed* complete ancient Ricci flow $(M^n,g_t)_{t\le 0}$ with bounded curvature on compact intervals, roughly speaking, all known entropy quantities converge to the same quantity μ_∞, and the limits of pointed entropy quantities do not depend on the basepoints. Here, following Bamler, we call an ancient Ricci flow $(M^n,g_t)_{t\le 0}$ **noncollasped**, if there is a point $(p_0,t_0)\in M\times(-\infty,0]$, such that

$$
\inf_{\tau>0}\mathcal{N}_{p_0,t_0}(\tau)>-\infty.
$$

Theorem 4.3.1. *Let $(M^n,g_t)_{t\le 0}$ be a complete ancient Ricci flow. Suppose that for some $p_0\in M, t_0\le 0$,*

$$
\mu_\infty:=\inf_{\tau>0}\mathcal{N}_{p_0,t_0}(\tau)>-\infty.
$$

Then for any $(p_1, t_1) \in M \times (-\infty, 0]$,

$$\mu_\infty = \inf_{\tau>0} \mathcal{N}_{p_1,t_1}(\tau) = \inf_{\tau>0} \mathcal{W}_{p_1,t_1}(\tau) = \inf_{t\leq 0} \nu(g_t).$$

We remark that by [CMZ21a, CMZ21b], μ_∞ is just the shrinker entropy of any asymptotic shrinker or tangent flow at infinity of $(M^n, g_t)_{t\leq 0}$.

Proof. We first show that $\mu_\infty = \inf_{\tau>0} \mathcal{N}_{p_1,t_1}(\tau)$. If suffices to show that $\mu_\infty \leq \mathcal{N}_{p_1,t_1}(\tau)$ by the symmetry of the roles of the basepoints. By the Harnack inequality for the Nash entropy (4.1.1), for any $s \ll -1, \epsilon \in (0,1)$, letting $t^* = (1-\epsilon)s, \bar{t} = \min\{t_0, t_1\} \geq t^*$,

$$\begin{aligned}
\mathcal{N}_s^*(p_0,t_0) - \mathcal{N}_s^*(p_1,t_1) &\leq \sqrt{\tfrac{n}{\epsilon|s|}}\, \mathrm{dist}_{\mathrm{W}_1}^{g_{t^*}}(\nu_{p_0,t_0|t^*}, \nu_{p_1,t_1|t^*}) + \frac{n}{2}\log\frac{t_1-s}{\epsilon|s|} \\
&\leq \sqrt{\tfrac{n}{\epsilon|s|}}\, \mathrm{dist}_{\mathrm{W}_1}^{g_{\bar{t}}}(\nu_{p_0,t_0|\bar{t}}, \nu_{p_1,t_1|\bar{t}}) + \frac{n}{2}\log\frac{t_1-s}{\epsilon|s|}.
\end{aligned}$$

Letting $s \to -\infty$ and then $\epsilon \to 1$, we have

$$\mu_\infty \leq \inf_{\tau>0} \mathcal{N}_{p_1,t_1}(\tau).$$

Next, we prove that for any $p \in M$,

$$\inf_{\tau>0} \mathcal{W}_{p,0}(\tau) = \inf_{\tau>0} \mathcal{N}_{p,0}(\tau) = \mu_\infty.$$

For any $\epsilon \in (0,1)$,

$$\begin{aligned}
\mathcal{W}_{p,0}(\tau) \leq \mathcal{N}_{p,0}(\tau) &= \frac{1}{\tau}\int_0^\tau \mathcal{W}_{p,0}(s)\, ds \\
&\leq \frac{1}{\tau}\int_{\epsilon\tau}^\tau \mathcal{W}_{p,0}(s)\, ds \leq (1-\epsilon)\mathcal{W}_{p,0}(\epsilon\tau).
\end{aligned}$$

Letting $\tau \to \infty$ and then $\epsilon \to 0$, we have proved the claim.

We then prove that $\mu_\infty \leq \inf_{t\leq 0} \nu(g_t)$ by using the local version Theorem 4.2.1. By shifting the time, it suffices to prove that $\mu_\infty \leq \nu(g_0)$. For any $\epsilon \in (0,1), r > 0$, by Theorem 4.2.1,

$$\nu(B_0(p, \epsilon r), g_0, \epsilon r^2) \geq \mathcal{N}_{p,0}(r^2) - \sqrt{n}\epsilon - \frac{n}{2}\epsilon - \frac{n}{2}\log(1+\epsilon).$$

By taking $r \to \infty$ and then $\epsilon \to 0$, we have proved that $\mu_\infty \leq \nu(g_0)$.

Finally, we prove that $\mu_\infty \geq \inf_{t\leq 0} \nu(g_t)$. For any $\epsilon > 0$, $\mathcal{N}_{p,0}(\tau) \leq \mu_\infty + \epsilon$, if $\tau \geq \bar{\tau}(\epsilon)$. For large τ,

$$\mu_\infty + \epsilon \geq \mathcal{N}_{p,0}(\tau) \geq \mathcal{W}_{p,0}(\tau) \geq \nu(g_{-\tau}) \geq \inf_{t\leq 0} \nu(g_t).$$

The conclusion follows by taking $\epsilon \to 0$. □

4.4 Perelman's No Local Collapsing

We first show that a volume lower bound near (almost) ℓ-centers gives a lower bound on the Nash entropy. This leads to an improved version of Perelman's no local collapsing theorem and it will be also useful later. In fact, the following theorem simiplies [CMZ21a, Theorem 1.7] and does not assume any local curvature bounds.

Theorem 4.4.1. *Assume that $[-2r^2, 0] \subseteq I$. Let $x \in M$ and $\ell = \ell_{x,0}$ be Perelman's ℓ-function based at $(x, 0)$. Suppose that*

$$\ell(p, -r^2) \leq \Lambda, \quad |B_{-r^2}(p, r)|_{-r^2} \geq \alpha r^n,$$

for some $p \in M, \alpha > 0, \Lambda \geq n/2$. Then

$$\mathcal{N}_{x,0}(2r^2) \geq -C(n, \Lambda, \alpha).$$

As a corollary, we obtain the following Perelman's no local collapsing, which was later generalized by Bing Wang [Wa18], Wangjian Jian [J21], and [CMZ21d].

Theorem 4.4.2 (Noncollapsing improving)**.** *Assume that* $[-2r^2, 0] \subseteq I$. *Let* $x_0 \in M$ *be a fixed point. If*

$$\int_{-r^2}^{0} \sqrt{|t|} R(x_0, t)\, dt \le Ar,$$
$$|B_{-r^2}(x_0, r))|_{-r^2} \ge A^{-1} r^n,$$

then

$$\mathcal{N}_{x_0,0}(r^2) \ge -C(n, A),$$

and by Theorem 4.2.1,

$$\nu(B_0(x_0, Ar), g_0, A^2 r^2) \ge -C(n, A).$$

As a consequence, there is a constant $\kappa = \kappa(n, A) > 0$*, such that for any ball* $B := B_0(y, \rho)$ *satisfying* $y \in B_0(x_0, Ar)$ *and* $\rho \in (0, Ar]$*, we have*

$$\sup_B R_{g_0} \le \rho^{-2} \implies |B|_0 \ge \kappa \rho^n.$$

Proof of Theorem 4.4.2 assuming Theorem 4.4.1. By parabolic rescaling, we may assume that $r = 1$. Let $\ell = \ell_{x_0,0}$ be Perelman's ℓ-function based at $(x_0, 0)$. By the assumption on the scalar curvature, if we consider the constant curve at $(x_0, 0)$,

$$\ell(x_0, 0) \le \frac{1}{2} \int_{-1}^{0} \sqrt{|t|} R(x_0, t)\, dt \le A/2.$$

Applying Theorem 4.4.1 with $x \leftarrow x_0, p \leftarrow x_0$, we have

$$\mathcal{N}_{x_0,0}(1) \geq \mathcal{N}_{x_0,0}(2) \geq -C(n, A),$$

and the other conclusions follow. $\square$

We now prove Theorem 4.4.1. The proof is almost identical to Jian [J21].

Proof of Theorem 4.4.1. By parabolic rescaling, we may assume that $r = 1$. By Theorem 8.1 in [Bam20a],

$$\alpha \leq |B_{-1}(p,1)|_{-1} \leq C_n e^{\mathcal{N}_{p,-1}(1)},$$

and thus

$$\mathcal{N}_{p,-1}(1) \geq \log \alpha - C_n. \tag{4.4.1}$$

Let $(z, -1)$ be an H_n-center of $(x, 0)$. By Lemma 3.3.14,

$$|zp|_{-1} \leq C_n \left(\Lambda - \mathcal{N}_{x,0}(1)\right)^{1/2} \leq C_n \left(\Lambda - \mathcal{N}_{x,0}(2)\right)^{1/2}.$$

On the other hand, by (4.1.1),

$$\begin{aligned}
\mathcal{N}_{p,-1}(1) &\leq \mathcal{N}_{x,0}(2) + \sqrt{n}\,\mathrm{dist}_{W_1}^{g_{-1}}(\delta_p, \nu_{x_0,0|-1}) + \tfrac{n}{2}\log 2 \\
&\leq \mathcal{N}_{x,0}(2) + \sqrt{n}\left(|pz|_{-1} + \mathrm{dist}_{W_1}^{g_{-1}}(\delta_z, \nu_{x_0,0|-1})\right) + \tfrac{n}{2}\log 2 \\
&\leq \mathcal{N}_{x,0}(2) + C_n + C_n\left(\Lambda - \mathcal{N}_{x,0}(2)\right)^{1/2} \\
&\leq \mathcal{N}_{x,0}(2) + C_n + \tfrac{1}{2}\left(\Lambda - \mathcal{N}_{x,0}(2)\right) \\
&\leq \tfrac{1}{2}\mathcal{N}_{x,0}(2) + C_n + \Lambda/2,
\end{aligned}$$

where we used Cauchy-Schwarz inequality: $C\sqrt{t} \leq t/2 + C^2/2$. Combining with (4.4.1),

$$\mathcal{N}_{x,0}(2) \geq 2\mathcal{N}_{p,-1}(1) - C_n - \Lambda \geq 2\log\alpha - C_n - \Lambda.$$

□

4.5 Perelman's Pseudolocality Theorems

As an application of the local monotonicity inequalities, we give a simple proof of Bing Wang's improved pseudolocality theorem [Wa20, Theorem 1.2], which generalizes Perelman's original pseudolocality theorem [Per02, 10.1]. As a further application, we give a simple proof of Peng Lu's local curvature bound [Lu10, Theorem 1.2], which improves Perelman's original local curvature bound [Per02, 10.3].

Theorem 4.5.1 (Pseudolocality theorem [Per02, Wa20]). *For any $\alpha \in (0, \frac{1}{100n})$, there is a $\delta = \delta(n, \alpha) > 0$, such that the following holds. Assume that $[0, T] \subset I$. Let $x_0 \in M$ be a point satisfying*

$$\inf_{0<t\leq T} \mu\left(B_0\left(x_0, \sqrt{t}/\delta\right), g_0, t\right) \geq -\delta^2. \tag{4.5.1}$$

Then, for any $t \in (0, T]$ and any $x \in B_t\left(x_0, \sqrt{t}/\alpha\right)$, we have

$$|\mathrm{Rm}_{g_t}|(x) \leq \alpha/t, \tag{4.5.2}$$

$$|B_t(x,\rho)|_t \geq (1-\alpha)\omega_n \rho^n, \qquad \forall\, \rho \in (0, \sqrt{t}/\alpha), \tag{4.5.3}$$

$$\mathrm{inj}_{g_t}(x) \geq \sqrt{t}/\alpha, \tag{4.5.4}$$

where ω_n is the volume of the n-dimensional unit ball in the Euclidean space.

Theorem 4.5.2 (Local curvature bounds [Per02, Lu10]). *For any $n \geq 3, v > 0$, there is a constant $\epsilon_0 = \epsilon_0(n, v) > 0$ with the following properties. Suppose that $(M^n, g_t)_{t\in[0,(\epsilon r_0)^2]}$ is a complete Ricci flow for some $\epsilon \in [0, \epsilon_0]$ and $r_0 > 0$, and the initial metric g_0 satisfies*

$$\sup_{B_0(x_0,r_0)} |\mathrm{Rm}_{g_0}| \leq r_0^{-2}, \quad |B_0(x_0, r_0)| \geq v r_0^n.$$

Then for any $t \in [0, (\epsilon r_0)^2]$,

$$\sup_{B_t(x_0, \epsilon_0 r_0)} |\mathrm{Rm}_{g_t}| \leq (\epsilon_0 r_0)^{-2}.$$

We record Bamler's ϵ-regularity Theorem here, since it will be very important in our new proofs of Theorem 4.5.1 and Theorem 4.5.2.

Theorem 4.5.3 ([Bam20a, Theorem 10.3])**.** *For any* $\varepsilon > 0$ *there is* $\delta = \delta(n, \varepsilon) > 0$, *such that the following holds. Let* $(x,t) \in M \times I$ *and* $r > 0$ *satisfy* $[t - r^2, t] \subset I$ *and* $\mathcal{N}_{x,t}(r^2) \geq -\delta$. *Then we have*

$$|\mathrm{Rm}| \leq \varepsilon r^{-2} \quad \text{on} \quad B_t(x, \varepsilon^{-1} r) \times ([t - (1-\varepsilon) r^2, \varepsilon^{-1} r^2] \cap I), \tag{4.5.5}$$

$$\inf_{\rho \in (0, \varepsilon^{-1} r)} \rho^{-n} |B_t(x, \rho)|_t \geq (1-\varepsilon)\omega_n, \tag{4.5.6}$$

where ω_n *is the volume of the* n*-dimensional unit ball in the Euclidean space.*

The proof of (4.5.6) follows verbatim as in [Bam20a, Theorem 10.3] and we omit the proof here.

Before proving Theorem 4.5.1, we shall give the following proposition.

Proposition 4.5.4. *Assume that* $[-r^2, 0] \subseteq I$. *Then, for any* $x_0 \in M$, *any* H_n*-center* $(z, -r^2)$ *of* $(x_0, 0)$, *and any* $A \geq \bar{A}(n)$, *we have*

$$\mu\left(B_{-r^2}\left(z, Ar\right), g_{-r^2}, r^2\right) \leq \mathcal{N}_{x_0,0}\left(r^2/2\right) + C_n e^{-cA^2}.$$

Proof. By parabolic rescaling, we may assume that $r = 1$. Let $(w, -1/2)$ be an H_n-center of $(x_0, 0)$ and $(z', -1)$ be an H_n-center of $(w, -1/2)$. By Lemma 3.3.13,

$$|zz'|_{-1} \leq C_n.$$

By Theorem 4.2.3,

$$\mathcal{N}_{x_0,0}(1/2) \geq \mu(B_{-1/2}(w, A/10), 1/2) - C_n e^{-cA^2},$$

if $A \geq \bar{A}(n)$. By Theorem 4.2.2,

$$\mu(B_{-1/2}(w, A/10), 1/2) \geq \mu(B_{-1}(z', A/2), 1) - C_n e^{-cA^2},$$

if $A \geq \bar{A}(n)$. It follows that

$$\mathcal{N}_{x_0,0}(1/2) \geq \mu(B_{-1}(z', A/2), 1) - C_n e^{-cA^2} \geq \mu(B_{-1}(z', A), 1) - C_n e^{-cA^2},$$

if $A \geq \bar{A}(n)$, where we used the fact that $|zz'|_{-1} \leq C_n$. $\square$

We are now ready to prove Theorem 4.5.1.

Proof of Theorem 4.5.1. We prove the curvature bound (4.5.2) by contradiction. Let $\bar{t} < T$ be the first time that (4.5.2) fails. That is, for $t \in (0, \bar{t})$,

$$|\mathrm{Rm}|(x,t) < \alpha/t, \qquad \text{for any } x \in B_t(x_0, \sqrt{t}/\alpha); \tag{4.5.7}$$

and there is $\bar{x} \in B_{\bar{t}}(x_0, \sqrt{\bar{t}}/\alpha)$ such that

$$\bar{t}|\mathrm{Rm}|(\bar{x}, \bar{t}) = \alpha. \tag{4.5.8}$$

Clearly, $\bar{t} > 0$. By parabolic rescaling, we may assume that $\bar{t} = 1$.

Let $(z', 0)$ be an H_n-center of $(x_0, 1)$ and $(z, 0)$ be an H_n-center of $(\bar{x}, 1)$. Now that we have a local curvature bound (4.5.7) until time $t = 1$, by Corollary 3.3.6,

$$|z'x_0|_0 \leq 10^4 n^2.$$

(Recall that $\Lambda \geq 1000n^2$ in Corollary 3.3.6.) By Lemma 3.3.16,

$$|zz'|_0 \leq 2\sqrt{H_n} + |\bar{x}x_0|_1 \leq 2/\alpha.$$

So

$$|zx_0|_0 \leq 10^4 n^2 + 2/\alpha.$$

By Proposition 4.5.4 and the assumption,

$$\begin{aligned}
\mathcal{N}_{\bar{x},1}(1/2) &\geq \mu(B_0(z, \tfrac{1}{2\delta}), g_0, 1) - C_n e^{-c/\delta^2} \\
&\geq \mu(B_0(x_0, \tfrac{1}{\delta}), g_0, 1) - C_n e^{-c/\delta^2} \\
&\geq -\delta^2 - C_n e^{-c/\delta^2} > -2\delta^2,
\end{aligned}$$

if $\delta < \bar{\delta}(n, \alpha)$. By picking $\delta < \bar{\delta}(n, \alpha)$ in Bamler's ϵ-regularity Theorem 4.5.3, we have

$$|\mathrm{Rm}|(\bar{x}, 1) < \alpha/2,$$

which is a contradiction to (4.5.8). So no such $\bar{t}$ exists and (4.5.2) holds.

The proof of (4.5.4) is similar and (4.5.3) follows by a well-known result proved by Cheeger-Gromov-Taylor [CGT82].

□

Now we start to prove the local curvature bounds: Theorem 4.5.2. Our proof follows the same lines as in [Lu10]. We will make use of Wang's improved version of pseudolocality theorem ([Wa20], Theorem 4.5.1), while [Lu10] applied Perelman's original pseudolocality theorem. However, the original proof in [Lu10] was divided into three cases and used a point-picking argument. We can avoid such technicalities and our proof should be slightly cleaner and hopefully more accessible. New ingredients in our proof include Bamler's Bamler's ϵ-regularity Theorem 4.5.3 and the local entropy monotonicity (a variant of Theorem 4.2.3).

Following [Wa18, (3.18)], we denote by

$$\mathbf{I}(\Omega) := \inf \frac{|\partial D|}{|D|^{\frac{n-1}{n}}},$$

the (best) isoperimetric constant of a region Ω, where the infimum is taken over all regular compact domains $D \subset \Omega$. Let $\mathbf{I}_n := n\omega_n^{1/n}$ be the best isoperimetric constant of the Euclidean space.

Lemma 4.5.5. *For any $n \geq 3, \epsilon, v > 0$, there is $\rho = \rho(n, \epsilon, v) > 0$ with the following property. Suppose that (M^n, g) is a manifold and $B(x_0, r_0) \Subset M$ with*

$$\sup_{B(x_0,r_0)} |\mathrm{Rm}| \leq r_0^{-2}, \quad |B(x_0, r_0)| \geq v r_0^n.$$

Then

$$\nu(B(x_0, \rho r_0), g, (\rho r_0)^2) \geq -\epsilon^2.$$

Proof. By parabolic rescaling, we may assume that $r_0 = 1$. By [Wa18, (3.25) in Lemma 3.5],

$$\nu(B(x_0, \rho), g, \tau) \geq n \log \frac{\mathbf{I}(B(x_0, \rho))}{\mathbf{I}_n} - n(n-1)\tau,$$

for any $\rho < 1, \tau > 0$. By [Lu10, Lemma 3.1], for any $\delta > 0$, there is $\rho = \rho(n, \delta, v) > 0$, such that

$$\frac{\mathbf{I}(B(x_0, \rho))}{\mathbf{I}_n} \geq 1 - \delta.$$

The Lemma follows by choosing $\delta = \delta(n, \epsilon, v) > 0$ such that

$$n \log(1 - \delta) - n(n-1)\rho^2 > -\epsilon^2.$$

$\square$

Proof of Theorem 4.5.2. Let $\alpha \in (0, \frac{1}{(10n)^{10}}]$ to be determined later and let $\delta = \delta(n, \alpha) \ll 1$ be given by the pseudolocality Theorem 4.5.1. By parabolic rescaling, we may assume that r_0 is a large number to be determined and r_0 depends only on n, α, v.

By Lemma 4.5.5, there is $\rho = \rho(n, \delta, v) = \rho(n, \alpha, v) > 0$, such that

$$\nu(B_0(x_0, \rho r_0), g_0, (\rho r_0)^2) \geq -\delta^2.$$

Write

$$\Lambda := 1000n^2, \quad A = 100\Lambda.$$

We then choose $\epsilon_0 = \epsilon_0(n, \alpha, v) > 0$ such that

$$\epsilon_0 = \delta\rho/A,$$

We now fix $r_0 = r_0(n, \alpha, v)$ so that

$$\epsilon_0 r_0 = A.$$

Then

$$\rho r_0 = A\epsilon_0 r_0/\delta = A^2/\delta. \tag{4.5.9}$$

Let $\epsilon \leq \epsilon_0$ and $(M^n, g_t)_{t\in[0,(\epsilon r_0)^2]}$ be a complete Ricci flow with initial metric g_0. Write

$$T := (\epsilon r_0)^2 \leq (\epsilon_0 r_0)^2 = A^2.$$

For any $t \in (0, T]$,

$$\sqrt{t}/\delta \leq \epsilon_0 r_0/\delta = \rho r_0/A,$$

and thus

$$\mu(B_0(x_0, \sqrt{t}/\delta), g_0, t) \geq \nu(B_0(x_0, \rho r_0), g_0, (\rho r_0)^2) \geq -\delta^2.$$

By Theorem 4.5.1, for any $t \in (0, T]$,

$$\sup_{B_t(x_0,\sqrt{t})} |\mathrm{Rm}_{g_t}| \le \alpha/t. \tag{4.5.10}$$

We next obtain curvature bounds on a larger ball.

Claim: For any $t \in (0, T]$ and $y \in B_t(x_0, A)$,

$$|\mathrm{Rm}|(y,t) \le \alpha/t.$$

Proof of Claim. Fix $t \in (0, T]$. Let $(z', 0), (z, 0)$ be any H_n-centers of $(x_0, t), (y, t)$, respectively. Now that we have the local curvature bound (4.5.10), by Corollary 3.3.6,

$$|x_0 z'|_0 \le 10\Lambda\sqrt{t} \le A^2/10.$$

Then by Lemma 3.3.16,

$$\begin{aligned}
|x_0 z|_0 &\le |x_0 z'|_0 + |z' z|_0 \\
&\le |x_0 z'|_0 + 2\sqrt{H_n t} + |x_0 y|_t \\
&\le A^2/10 + 2\sqrt{H_n t} + A \le A^2/2.
\end{aligned}$$

It follows by (4.5.9) that

$$B_0(z, \tfrac{1}{2}\delta^{-1}\sqrt{t}) \subset B_0(x_0, \rho r_0),$$

and thus

$$\begin{aligned}
\mathcal{N}_{y,t}(t/2) &\ge \nu(B_0(z, \tfrac{1}{2}\delta^{-1}\sqrt{t}), g_0, t) - C_n e^{-c_n\delta^{-2}} \\
&\ge \nu(B_0(x_0, \rho), g_0, \rho^2) - C_n e^{-c_n\delta^{-2}} \\
&\ge -\delta^2 - C_n e^{-c_n\delta^{-2}} > -2\delta^2.
\end{aligned}$$

The claim then follows by Bamler's ϵ-regularity theorem. $\square$

Let $\bar{t} \in (0, T]$ be the first time violating the property that

$$\sup_x |\mathrm{Rm}|(x,t)\,(A - |x_0 x|_t)_+^2 < 1. \tag{4.5.11}$$

Because of the initial curvature bound and the smoothness of the metrics, $\bar{t} > 0$. Let $(\bar{x}, \bar{t})$ be a point that achieves the maximum. Suppose that

$$|x_0 \bar{x}|_{\bar{t}} = (1 - 3\theta)A,$$

where $\theta > 0$ may depend on the geometry of (M, g_t). We may assume that $\theta \leq 1/100$ because it is much easier to deal with the case where $\theta \geq 1/100$ as will be clear below. Write

$$\epsilon_1 := \theta A.$$

Since $(\bar{x}, \bar{t})$ achieves the maximum, by the Claim above,

$$(3\epsilon_1)^{-2} = |\mathrm{Rm}|(\bar{x}, \bar{t}) \leq \alpha/\bar{t},$$

and thus

$$\bar{t} \leq (3\epsilon_1)^2 \alpha.$$

Meanwhile, for any $t \in (0, \bar{t})$ and $x \in \bar{B}_t(x_0, (1-\theta)A)$,

$$|\mathrm{Rm}|(x,t) \leq \epsilon_1^{-2}.$$

Let $\eta : [0, \infty) \to [0, 1]$ be a smooth cutoff function such that

$$\eta|_{[0,1-2\theta]} = 1, \quad \eta|_{[1-\theta,\infty)} = 0, \quad -10\sqrt{\eta}/\theta \leq \eta' \leq 0, \quad |\eta''| \leq 10\eta/\theta^2.$$

Let

$$\psi(x,t) = \eta\left(\frac{|x_0 x|_t + \Lambda\sqrt{t}}{A}\right).$$

Since

$$|x_0\bar{x}|_{\bar{t}} + \Lambda\sqrt{\bar{t}} = (1-3\theta)A + 3\Lambda\theta A\sqrt{\alpha} < (1-2\theta)A,$$

if $\alpha < \bar{\alpha}(n)$, we have

$$\psi(\bar{x},\bar{t}) = 1, \quad \operatorname{spt}\psi_t \subset B_t(x_0,(1-\theta)A).$$

Then

$$\Box\psi \leq= 10\epsilon_1^{-2}\psi, \quad |\nabla\psi|^2 \leq 100\epsilon_1^{-2}\psi.$$

Recall that under Ricci flow,

$$\Box|\mathrm{Rm}|^2 \leq -2|\nabla\mathrm{Rm}|^2 + 16|\mathrm{Rm}|^3.$$

Thus

$$\begin{aligned}
\Box(\psi|\mathrm{Rm}|^2) &\leq -2\psi|\nabla\mathrm{Rm}|^2 + 16\psi|\mathrm{Rm}|^3 + |\mathrm{Rm}|^2\Box\psi + 4|\nabla\psi||\mathrm{Rm}||\nabla\mathrm{Rm}| \\
&\leq 2|\nabla\psi|^2\psi^{-1}|\mathrm{Rm}|^2 + 300\epsilon_1^{-2}\psi|\mathrm{Rm}|^2 \\
&\leq 1000\epsilon_1^{-2}|\mathrm{Rm}|^2.
\end{aligned}$$

As in [Lu10], consider

$$U(t) := \int_M \psi_t|\mathrm{Rm}_{g_t}|^2\, d\nu_{\bar{x},\bar{t}|t}.$$

Then for $t \in (0,\bar{t})$,

$$U'(t) = \int_M \Box(\psi|\mathrm{Rm}|^2)\, d\nu_{\bar{x},\bar{t}|t} \leq 10^3\epsilon_1^{-6}.$$

Recall that $\bar{t} \leq (3\epsilon_1)^2\alpha$. So

$$
\begin{aligned}
(3\epsilon_1)^{-4} &\leq |\mathrm{Rm}|^2(\bar{x}, \bar{t}) = U(\bar{t}) \leq U(0) + 10^3\epsilon_1^{-6}\bar{t} \\
&\leq r_0^{-2} + 10^4\epsilon_1^{-4}\alpha, \\
1 &\leq (\delta\rho/A)^2 + 10^6\alpha.
\end{aligned}
$$

We may choose $\alpha < \bar{\alpha}(n)$ such that the inequality above does not hold. Therefore, no such $\bar{t}$ exists, which means

$$
\sup_{t\in[0,T]} \sup_x |\mathrm{Rm}|(x,t)\,(A - |x_0x|_t)_+^2 < 1.
$$

Recall that $T = (\epsilon r_0)^2$, $A = \epsilon_0 r_0$. So we have proved the theorem by replacing ϵ_0 with $\epsilon_0/2$.

□

Remark. (4.5.11) is inspired by Bing Wang's work (e.g. [Wa20, Proposition 4.1]). We define such a quantity in hope that the maximum point of $|\mathrm{Rm}|$ is not too close to the boundary of the ball $B_t(x_0, A)$. The term $(A - |xx_0|_t)_+^2$ plays a role as a penalty, which is prevalent and fundamental in optimization.

Chapter 4, in part, has been submitted for publication joint with Chan, Pak-Yeung and Zhang, Yongjia [CMZ21d]. Chapter 4 also contains material from [MZ21] which is published on the Journal of Functional Analysis 2021 joint with Zhang, Yongjia.

Chapter 5

Geometry at Infinity of Ancient Flows

In this chapter, we study the geometry at infinity of complete *noncollapsed* ancient Ricci flow $(M^n, g_t)_{t\leq 0}$. Throughout this chapter, we make a technical assumption that

$$\sup_{M\times J} |\mathrm{Rm}| < \infty,$$

for any compact interval $J \subset (-\infty, 0]$.

There are currently two notions of blow-downs of an ancient Ricci flow, which characterize the geometry at infinity when time goes to $-\infty$. We shall recall such two notions introduced by Perelman [Per02, Section 11] and Bamler [Bam20b, Section 6.8]. Roughly speaking, one of our main theorems in this chapter is that the two notions of blow-downs coincide given that the blow-down under consideration is *smooth*. We shall also include the result that Bamler's notion of blow-downs, tangent flows at infinity for general H-concentrated ancient metric flows, do not depend on the choice of base points.

Let $p_0 \in M$ be a fixed point. Then, for a sequence $\{\tau_i\}_{i=1}^{\infty}$ with $\tau_i \nearrow \infty$, we may find $\{p_i\}_{i=1}^{\infty}$ such that $(p_i, -\tau_i)$ are ℓ-centers of $(p_0, 0)$, namely, $\ell_{p_0,0}(p_i, \tau_i) \leq \frac{n}{2}$; see Definition 5.1.4 for more details of the definitions. If Perelman's asymptotic shrinkers ever exist for an ancient Ricci flow, the following assumption must hold. (See the following subsection for the detailed definitions of asymptotic shrinkers.)

Assumption B: For the fixed point $(p_0, 0)$ and the sequences $\{\tau_i\}_{i=1}^{\infty}$ and $\{p_i\}_{i=1}^{\infty}$ as described above, there exists a smooth and complete Ricci flow $\big(M_\infty, g_\infty(t), p_\infty\big)_{t\in[-2,-1]}$, such that

$$\big(M, g_i(t), p_i\big)_{t\in[-2,-1]} \longrightarrow \big(M_\infty, g_\infty(t), p_\infty\big)_{t\in[-2,-1]} \tag{5.0.1}$$

in the smooth Cheeger-Gromov-Hamilton sense, where the Ricci flow $g_i(t)$ is obtained by the Type I scaling

$$g_i(t) := \tau_i^{-1} g(\tau_i t). \tag{5.0.2}$$

The statement of (5.0.1) is involved with a base point $(p_0, 0)$, a sequence of positive scales $\{\tau_i\}_{i=1}^{\infty}$, and the choices of ℓ-centers $(p_i, -\tau_i)$. Hence, if necessary, we shall refer to an ancient solution as "satisfying Assumption B with respect to $\big(p_0, 0, \{\tau_i\}_{i=1}^{\infty}, \{p_i\}_{i=1}^{\infty}\big)$". For the limit Ricci flow $(M_\infty, g_\infty(t))$ in (5.0.1), we do not assume any shrinker structure, neither do we make any geometric assumption except for its completeness. However, it will soon be clear that, because of [CZ20, Theorem 6.1], this limit is Perelman's asymptotic shrinker; see the statement of Theorem 5.0.1(1) below.

We are now ready to state the main theorem in this chapter.

Theorem 5.0.1. *Under Assumption B, the following hold (see section 2 for all the definitions involved).*

(1) $\big(M_\infty, g_\infty(t), p_\infty\big)_{t\in[-2,-1]}$ *admits a shrinker structure, which makes it an asymptotic shrinker in the sense of Perelman ([Per02, Proposition 11.2]).*

(2) We have

$$\lim_{\tau\to\infty} \mathcal{N}_{p_0,0}(\tau) = \mu_\infty,$$

where $\mathcal{N}_{p_0,0}$ is the Nash entropy based at $(p_0, 0)$ and μ_∞ is the entropy of the asymptotic shrinker. In particular, μ_∞ is the infimum of $\mathcal{N}_{p_0,0}(\tau)$, $\tau > 0$.

(3) Any $\mathbb{F}$-limit of the sequence $\left\{\left((M, g_i(t))_{t\in[-2,-1]}, (\nu_t^i)_{t\in[-2,-1]}\right)\right\}_{i=1}^{\infty}$, where $\nu_t^i := \nu_{p_0,0\,|\,\tau_i t}$, given by Bamler's compactness theorem [Bam20b, Theorem 7.6] is of the form

$$\left((M_\infty, g_\infty(t))_{t\in[-2,-1]}, (\nu_t^\infty)_{t\in[-2,-1)}\right),$$

up to isometry, where $(\nu_t^\infty)_{t\in[-2,-1)}$ is a conjugate heat flow made of a shrinker potential function.

(4) Conversely, if a tangent flow at infinity of $(M, g_t)_{t\le 0}$ is smooth, then it also arises as an asymptotic shrinker.

Tangent flows at infinity resemble asymptotic cones of metric spaces, which are blow-downs centered at some basepoint. It is well-known that such cones do not depend on the basepoint, (see, e.g., [BBI01, Proposition 8.2.8],) although they may depend on the sequence of scalings. In [CMZ21c], Chan, Zhang, and the author showed that tangent flows at infinity of ancient Ricci flows also do not depend on the choice of basepoints.

Theorem 5.0.2. *The tangent flow at infinity of an ancient H-concentrated (c.f. [Bam20b, Definition 3.30]) metric flow does not depend on the basepoint.*

5.1 Preliminaries

5.1.1 Perelman's Comparison Geometry

We briefly review Perelman's reduced distance and reduced volume introduced by Perelman in [Per02, Section 11]. Let $(M, g(t))_{t\in[-T,0]}$ be a Ricci flow with bounded curvature. Let $(p_0, t_0) \in M \times (-T, 0]$ be a fixed point in space-time. Then, Perelman's *reduced distance* is defined as

$$\ell_{p_0,t_0}(x,\tau) := \frac{1}{2\sqrt{\tau}} \inf_{\gamma} \int_0^\tau \sqrt{s}\left(|\dot{\gamma}(s)|^2_{g(t_0-s)} + R(\gamma(s), t_0 - s)\right) ds, \tag{5.1.1}$$

where $x \in M$, $\tau \in (0, T - |t_0|]$, and the infimum is taken over all piecewise smooth curves $\gamma : [0,\tau] \to M$ satisfying $\gamma(0) = p_0$ and $\gamma(\tau) = x$. The minimizer of (5.1.1) is usually called a minimal $\mathcal{L}$-geodesic from (p_0, t_0) to $(x, t_0 - \tau)$. (p_0, t_0) is called the base point of ℓ, and whenever the base point is understood, we shall suppress the subindex in the notaion $\ell_{p_0,t_0}(\cdot,\cdot)$. The *reduced volume based at* (p_0, t_0) is defined as

$$\mathcal{V}_{p_0,t_0}(\tau) := \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-\ell_{p_0,t_0}(\cdot,\tau)} dg_{t_0-\tau}. \tag{5.1.2}$$

Perelman's reduced distance and reduced volume satisfy many nice equations and inequalities. Among them the most important one is the monotonicity of the reduced volume.

Proposition 5.1.1. *Perelman's reduced volume $\mathcal{V}(\tau)$ is an increasing function in time (and hence a decreasing function in τ).*

The underlying reason for the monotonicity of the reduced volume is the fact that its integrand is a "sub"-conjugate heat kernel.

Proposition 5.1.2. *Let $\ell = \ell_{p,0}$ be the reduced distance based at $(p, 0)$. Then, $u(x,t) := (4\pi|t|)^{-\frac{n}{2}} e^{-\ell_{p,0}(x,|t|)}$ is a subsolution to the conjugate heat equation, and it also converges to the Dirac delta measure based at p as $\tau \to 0+$. Precisely,*

$$\frac{\partial \ell}{\partial \tau} - \Delta_{g_{-\tau}} \ell + \left| \nabla_{g_{-\tau}} \ell \right|^2_{g_{-\tau}} - R_{g_{-\tau}} + \frac{n}{2\tau} \geq 0,$$
$$\lim_{\tau \to 0+} (4\pi\tau)^{-\frac{n}{2}} e^{-\ell_{p,0}(\cdot,\tau)} = \delta_p.$$

Both of the formulae above are understood in the sense of distribution. As a consequence, *we have*

$$(4\pi|t|)^{-\frac{n}{2}} e^{-\ell_{p,0}(x,|t|)} \leq K(p,0\,|\,x,t) \quad \textit{for all } (x,t) \in M \times [-T,0), \tag{5.1.3}$$

where $K(p,0\,|\,\cdot,\cdot)$ *is the conjugate heat kernel based at* $(p,0)$.

As an elementary application of the maximum principle, Perelman [Per02] proved that $\ell(\cdot,\tau)$ always attains its minimum. This minimum point should be viewed as the "center" of the reduced distance.

Proposition 5.1.3. *Let* $\ell_{p,0}$ *be the reduced distance based at* $(p,0)$. *Then we have*

$$\min_M \ell(\cdot,\tau) \leq \frac{n}{2} \quad \textit{for all} \quad t \in (0,T]. \tag{5.1.4}$$

The point(s) where the minimum in formula (5.1.4) is attained plays an important role in our arguments. In most of the cases, it turns out that such a minimum point is not far from Bamler's H_n-centers. Hence, we would like to assign a special term to these points.

Definition 5.1.4. Let $(M^n, g(t))_{t\in I}$ be a Ricci flow, and let $(x,t) \in M \times I$ be a point in space time. Let $s \in I \cap (-\infty, t)$. Then, (z,s) is called an **ℓ-center** of (x,t) if

$$\ell_{x,t}(z, t-s) \leq \frac{n}{2}.$$

Remark: Similar to the case of the H_n-center, the ℓ-center is not necessarily unique at a fixed time s for a fixed base point (x,t). Furthermore, in practice (especially when considering the base points for the blow-down sequence from which we obtain an asymptotic shrinker), a sequence of space-time points along which ℓ is uniformly bounded serves equally well as a sequence of ℓ-centers.

Let us recall Perelman's asymptotic shrinkers.

Definition 5.1.5. A Ricci flow $(M^n_\infty, g_\infty(t))_{t\in[-2,-1]}$ is called an **asymptotic shrinker** of an ancient Ricci flow $(M^n, g(t))_{t\le 0}$, if there are $\tau_i \to \infty, p_i \in M$ satisfying

$$\sup_i \ell_{p_0,0}(p_i, \tau_i) < \infty,$$

for some fixed point $p_0 \in M$; and

$$(M^n, g_i(t), (p_i, -1))_{t\in[-2,-1]} \to (M^n_\infty, g_\infty(t), (p_\infty, -1))_{t\in[-2,-1]},$$

in the sense of smooth Cheeger-Gromov-Hamilton convergence, where

$$g_i(t) := \tau_i g(t/\tau_i).$$

Cheng and Zhang in [CZ20, Theorem 6.1] proved that if $(M^n_\infty, g_\infty(t))_{t\in[-2,-1]}$ satisfies the assumption above in the definition of asymptotic shrinkers, then it indeed admits a shrinking gradient Ricci soliton structure. In fact, they proved the existence of the shrinker structure only assuming that the underlying flow $(M^n, g(t))$ is *locally uniformly type-I*; See [CZ20, Definition 4.1] for the details. The interval $[-2,-1]$ can be clearly replaced by, e.g., intervals of the form $[-A, -1/A]$ for some $A > 1$, and we shall use $[-2,-1]$ for simplicity.

5.1.2 Bamler's Metric Flows and $\mathbb{F}$-convergence

In this subsection we briefly recall the notions of the metric flows introduced by Bamler [Bam20b]. The readers are encouraged to refer to [Bam20b] for more details. We shall denote by $\mathcal{P}(X)$ the space of probability measures on a metric space X. The metric flow is introduced as a natural generalization of the Ricci flow space-time. A **metric flow over** $I \subset \mathbb{R}$ is a tuple

$$\big(\mathcal{X}, \mathfrak{t}, (\mathrm{dist}_t)_{t\in I}, (\nu_{x\,|\,s})_{x\in\mathcal{X}, s\in I, s\le \mathfrak{t}(x)}\big),$$

where $\mathfrak{t} : \mathcal{X} \to I$ is the time function and $\mathcal{X}_t := \mathfrak{t}^{-1}(t)$ is called the time slice at t, dist_t is a metric on $\mathcal{X}_t$, $\nu_{x\,|\,s} \in \mathcal{P}(\mathcal{X}_s)$ is a family of probability measures called the *conjugate heat kernel* based at $x \in \mathcal{X}$, and it satisfies the usual reproduction formula: for any $t_1 \le t_2 \le t_3$ in I and for any $x \in \mathcal{X}_{t_3}$, we have

$$\nu_{x\,|\,t_1} = \int_{\mathcal{X}_{t_2}} \nu_{\cdot\,|\,t_1}\, d\nu_{x\,|\,t_2}.$$

The sharp gradient estimate of Bamler [Bam20b, Theorem 4.1] is also axiomized into the definition of the metric flow. More details could be found in [Bam20b, Definition 3.2]. We have generalized [Bam20b, Theorem 4.1] to general complete Ricci flows (without any curvature conditions) in Theorem 3.2.1. Hence, the following observation is straightforward.

Theorem 5.1.6 (Bamler). *Let $(M^n, g(t))_{t\in I}$ be a complete Ricci flow. Then it induces a canonical metric flow in the sense of [Bam20b, Definition 3.2].*

We shall then introduce some definitions and results for the metric flow. In particular, they can be applied to smooth Ricci flows satisfying the condition of the above theorem. Before we proceed to define the notion of H-concentration, let us recall the *variance* of two probability measures. Let $\mu, \nu \in \mathcal{P}(X)$, where X is a metric space, then their variance is defined as

$$\mathrm{Var}(\mu, \nu) := \int_{X\times X} \mathrm{dist}^2(y_1, y_2)\, d\mu(y_1) d\nu(y_2),$$

where dist is the metric on X. If μ is the same as ν, then we also denote $\mathrm{Var}(\mu) := \mathrm{Var}(\mu, \mu)$. H-concentration is defined as follows.

Definition 5.1.7 (Definition 3.30 in [Bam20b]). A metric flow $\mathcal{X}$ over $I \subset \mathbb{R}$ is said to be H-concentrated, where H is a positive number, if for any $s, t \in I$, $s \le t$, and $x_1, x_2 \in \mathcal{X}_t$, we have,

$$\mathrm{Var}(\nu_{x_1\,|\,s}, \nu_{x_2\,|\,s}) \le \mathrm{dist}_t^2(x_1, x_2) + H(t-s). \tag{5.1.5}$$

Combining (5.1.5) with the reproduction formula, we have that, on an H-concentrated metric flow $\mathcal{X}$, for any $x_1, x_2 \in \mathcal{X}_t$,

$$\mathrm{Var}(\nu_{x_1\,|\,s}, \nu_{x_2\,|\,s}) + Hs$$

is non-decreasing in $s \in I \cap (-\infty, t]$, and is bounded from above by $\mathrm{dist}_t^2(x_1, x_2) + Ht$ (c.f. [Bam20b, Proposition 3.34]). This fact guarantees the existence of H-centers (c.f. [Bam20b, Proposition 3.36]), which are defined as follows.

Definition 5.1.8 (Definition 3.35 in [Bam20b])**.** Let $s, t \in I$ and $s \le t$. A point $z \in \mathcal{X}_s$ is called an H-center of $x \in \mathcal{X}_t$ if

$$\mathrm{Var}(\delta_z, \nu_{x\,|\,s}) \le H(t-s).$$

The H-center adopted its name partially because the conjugate heat kernel accumulates its measure around it. Precisely, we have ([Bam20a, Proposition 3.13] and [Bam20b, Lemma 3.37]):

Proposition 5.1.9 (Bamler)**.** *Let $\mathcal{X}$ be an H-concentrated metric flow over $I \subset \mathbb{R}$. Let $x \in \mathcal{X}_t$ be a fixed point, and let $z \in \mathcal{X}_s$ be an H-center of x, where $s < t$ and $s, t \in I$. Then, for any $A > 1$, we have*

$$\nu_{x\,|\,s}\big(B_s(z, \sqrt{AH(t-s)})\big) \ge 1 - \frac{1}{A}.$$

Remark: Bamler [Bam20a, Proposition 3.2] proved that if $\mathcal{X} = M^n \times I$ is a Ricci flow space-time on a closed manifold M^n over an interval I, then $\mathcal{X}$ must be H_n-concentrated, where

$$H_n := \frac{(n-1)\pi^2}{2} + 4.$$

The same argument also works when the Ricci flow is complete and has bounded curvature within compact time intervals. In general, given $x \in \mathcal{X}_t$, and $s \le t$, H-centers of x may not be unique in $\mathcal{X}_s$.

Suppose $\mathcal{X}$ is a metric flow over $I \subset \mathbb{R}$. A family of probability measures $\mu_s \in \mathcal{P}(\mathcal{X}_s)$, where $s \in I' \subset I$, is called a *conjugate heat flow* if it satisfies the reproduction formula: for any $s, t \in I', s \le t$, we have

$$\mu_s = \int_{\mathcal{X}_t} \nu_{x|s} \, d\mu_t(x).$$

A metric flow pair is then defined to be a metric flow coupled with a conjugate heat flow.

Definition 5.1.10 (Definition 5.1 in [Bam20b]). A pair $(\mathcal{X}, (\mu_t)_{t \in I'})$ is called a *metric flow pair* over $I \subset \mathbb{R}$ if the following conditions are satisfied:

- $I' \subset I$, and $|I \setminus I'| = 0$, where $|\cdot|$ is the Lebesgue measure;
- $\mathcal{X}$ is a metric flow over I';
- $(\mu_t)_{t \in I'}$ is a conjugate heat flow on $\mathcal{X}$ with $\mathrm{spt}\mu_t = \mathcal{X}_t$ for all $t \in I'$.

The definition of $\mathbb{F}$-convergence requires the notions of coupling and 1-Wasserstein distance between probability measures. Let X, Y be metric spaces. For any $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, we denote by $\Pi(\mu, \nu)$ the space of *couplings* between μ and ν, namely, the set of all the probability measures $q \in \mathcal{P}(X \times Y)$ satisfying

$$q(A \times Y) = \mu(A), \quad q(X \times B) = \nu(B),$$

for any measurable subsets $A \subset X$ and $B \subset Y$. The 1-Wasserstein distance between $\mu, \nu \in \mathcal{P}(X)$ is defined to be

$$\mathrm{dist}_{W_1}(\mu, \nu) := \inf_{q \in \Pi(\mu, \nu)} \int_{X \times X} \mathrm{dist}(x, y) \, dq(x, y).$$

By the Kantorovich-Rubinstein Theorem, this definition is equivalent to

$$\mathrm{dist}_{W_1}(\mu,\nu)=\sup_f\left(\int f\,d\mu-\int f\,d\nu\right),$$

where the supremum is taken over all bounded 1-Lipschitz functions f on X.

It is to be noted that for any metric flow $\mathcal{X}$, the 1-Wassernstein distance between two conjugate heat flows satisfies a monotonicity property ([Bam20b, Proposition 3.24(2)]), namely, for any conjugate heat flows $(\mu^1_s)_{s\in I'}$ and $(\mu^2_s)_{s\in I''}$, we have

$$\mathrm{dist}^{\mathcal{X}_s}_{W_1}(\mu^1_2,\mu^2_s)\quad\text{is non-decreasing in}\quad s\in I'\cap I''. \tag{5.1.6}$$

Consequently, for any x_1 and $x_2\in\mathcal{X}_t$, we have

$$\mathrm{dist}^{\mathcal{X}_s}_{W_1}(\nu_{x_1\,|\,s},\nu_{x_2\,|\,s})\le\mathrm{dist}_t(x_1,x_2)\quad\text{for all}\quad s<t. \tag{5.1.7}$$

In fact, this monotonicity property is a consequence of the prescribed gradient estimate in the definition of the metric flow ([Bam20b, Definition 3.2(6)]).

We now introduce the definition of $\mathbb{F}$-convergence within a correspondence, because this is the only version we shall use, and because it is essentially equivalent to the definition of $\mathbb{F}$-convergence itself. See more details in [Bam20b, Section 6]. Let $\left\{(\mathcal{X}^i,(\mu^i_t)_{t\in I',i})\right\}_{i\in\mathbb{N}\cup\{\infty\}}$ be a sequence of metric flow pairs over a finite interval $I\subset\mathbb{R}$. A correspondence $\mathfrak{C}$ is a collection of complete and separable metric spaces $(Z_t,\mathrm{dist}^{Z_t})_{t\in I}$ together with isometric embeddings $\phi^i_t:(\mathcal{X}^i_t,\mathrm{dist}^i_t)\to(Z_t,\mathrm{dist}^{Z_t})$ for $t\in I'^{,i}$. Then, $(\mathcal{X}^i,(\mu^i_t)_{t\in I',i})$ $\mathbb{F}$-converges to $(\mathcal{X}^\infty,(\mu^\infty_t)_{t\in I',\infty})$ within the correspondence $\mathfrak{C}$ uniformly on $J\subset I$, denoted as

$$\left(\mathcal{X}^i,(\mu^i_t)_{t\in I',i}\right)\xrightarrow{\mathbb{F},\mathfrak{C},J}\left(\mathcal{X}^\infty,(\mu^\infty_t)_{t\in I',\infty}\right),$$

if for any $\varepsilon>0$, there is an $\bar{i}\in\mathbb{N}$, such that if $i\ge\bar{i}$, there is a measurable subset $E_i\subset I$

with

$$J \subset I \setminus E_i \subset I'^{,i} \cap I'^{,\infty},$$

and there are couplings $q_t^i \in \Pi(\mu_t^i, \mu_t^\infty)$ for $t \in I \setminus E_i$ with the following properties:

- $|E_i| \le \varepsilon^2$.
- For any $s, t \in I \setminus E_i, s \le t$, it holds that

$$\int_{\mathcal{X}_t^i \times \mathcal{X}_t^\infty} \mathrm{dist}_{W_1}^{Z_s} \left(\phi_{s*}^i \nu_{x_1 \,|\, s}^i, \phi_{s*}^\infty \nu_{x_2 \,|\, s}^\infty \right) \, dq_t^i(x_1, x_2) \le \varepsilon.$$

If J above can be taken as any compact sub-interval of I, we say that the convergence is uniform over any compact sub-intervals.

When introducing the notion of tangent flow, we implement the same notations as in [Bam20b], to which the reader is encouraged to refer for more details. For a metric flow $\mathcal{X}$ over $I \subset \mathbb{R}$, we denote by $\mathcal{X}^{-t_0,\lambda}$ the metric flow obtained by first applying a $-t_0$ time shift to $\mathcal{X}$ and then a parabolic rescaling by factor λ. Let $\mathcal{X}$ be a metric flow over I and $|(-\infty, 0] \setminus I| = 0$. For any $x_0 \in \mathcal{X}_{t_0}$, we call a metric flow pair (that is, a metric flow coupled with a conjugate heat flow; see [Bam20b, Section 5]) $(\mathcal{X}^\infty, (\nu_{x_{\max} \,|\, t}^\infty)_{t \in I'^{,\infty}})$ a **tangent flow at infinity based at** x_0 if there is a sequence $\lambda_j \searrow 0$, such that,

$$\left(\mathcal{X}_{[-T,0]}^{-t_0,\lambda_j}, \left(\nu_{x_0 \,|\, t}^{-t_0,\lambda_j} \right)_{t \in \lambda_j^2 (I - t_0) \cap [-T,0]} \right) \xrightarrow{\mathbb{F}} \left(\mathcal{X}_{[-T,0]}^\infty, \left(\nu_{x_{\max} \,|\, t}^\infty \right)_{t \in I'^{,\infty} \cap [-T,0]} \right).$$

for any $T < \infty$. We can then define

$$\mathcal{T}_{x_0}^\infty := \{ \text{ tangent flows at infinity based at } x_0 \},$$

which is nonempty by Bamler's compactness theory in [Bam20b, Section 7]. Now we may restate Theorem 5.0.2 more precisely as follows. Note that in the statement of Theorem

5.1.11, the base points x_0 and y_0 need not lie in the same time-slice.

Theorem 5.1.11. *Suppose that $\mathcal{X}$ is an H-concentrated ([Bam20b, Definition 3.30]) metric flow over $(-\infty, 0]$. Then for any x_0, $y_0 \in \mathcal{X}$, we have, up to isometry,*

$$\mathcal{T}_{x_0}^{\infty} = \mathcal{T}_{y_0}^{\infty}.$$

5.1.3 Convergence of Heat Kernels.

Let $\{(M_i, g_i(t), o_i)_{t\in(-T_i,0]}\}_{i=1}^{\infty}$ be a sequence of complete Ricci flows with base point, and assume that each Ricci flow therein has bounded curvature within every compact time interval. Assume moreover that this sequence converges to $(M_\infty, g_\infty(t), o)_{t\in(-T_\infty,0]}$ in the pointed smooth Cheeger-Gromov-Hamilton sense, where $T_\infty = \limsup_{i\to\infty} T_i \in (0, \infty]$. Note that we do not make any assumption on the limit $(M_\infty, g_\infty(t), o)_{t\in(-T_\infty,0]}$.

By the definition of smooth convergence, we may find an increasing sequence of pre-compact open sets $U_i \subset M_\infty$ with $\cup_{i=1}^{\infty} U_i = M_\infty$, a sequence of diffeomorphisms $\Psi_i : U_i \to V_i \subset M_i$, and a sequence of positive numbers $\varepsilon_i \searrow 0$, such that

$$\Psi_i(o) = o_i,$$
$$\|\Psi_i^* g_i - g_\infty\|_{C^{[\varepsilon_i^{-1}]}(U_i\times[-T_\infty+\varepsilon_i,0])} < \varepsilon_i,$$

where we let $-T_\infty + \varepsilon_i = -\varepsilon_i^{-1}$ in the case $T_\infty = \infty$.

Let $K_i(x, t\,|\,y, s)$ be the heat kernel coupled with $(M_i, g_i(t))$ as introduces in section 2.1. For any $x, y \in U_i$ and for any $-T_\infty + \varepsilon_i \le s < t \le 0$, we shall define

$$\bar{K}_i(x, t\,|\,y, s) := (\Psi_i^* K_i)(x, t\,|\,y, s) = K_i(\Psi_i(x), t\,|\,\Psi_i(y), s).$$

As a slight generalization of [Lu12], we shall prove the following.

Theorem 5.1.12 ([CMZ21a, Theorem B.1])**.** *There is a positive heat kernel K_∞ coupled* *with $(M_\infty, g_\infty(t))$, such that, after passing to a subsequence, we have*

$$\bar{K}_i \to K_\infty,$$

in the C_c^∞-topology, and the convergence is uniform on any compact subset of

$$\mathcal{M} := \left\{ (x,t,y,s) \,\middle|\, x,y \in M_\infty, s,t \in (-T_\infty, 0], s < t \right\}.$$

See [CMZ21a, Appendix B] for a proof.

5.2 $\mathbb{F}$-convergence to Asymptotic Shrinkers

In this subsection, we shall prove Theorem 5.0.1.

Proof of Theorem 5.0.1(1). By assumption B, we have smooth convergence in the sense of Cheeger-Gromov-Hamilton, and thus the ancient Ricci flow in question $(M^n, g_t)_{t\le 0}$ satisfies *locally uniformly type-I* introduced by Cheng and Zhang [CZ20, Definition 4.1] and hence the limit flow admits a shrinker structure by [CZ20, Theorem 6.1]. See [CMZ21a, Section 5] for more details. □

Proof of Theorem 5.0.1 (2). We first show that

$$\inf_{\tau>0} \mathcal{N}_{p_0,0}(\tau) > -\infty.$$

We shall apply Theorem 4.4.1. By assumption B, for large i,

$$\tau_i^{-n/2} |B_{-\tau_i}(p_i, \sqrt{\tau_i})|_{-\tau_i} = |B_{g_{i,-1}}(p_i, 1)|_{g_{i,-1}} \ge \alpha,$$

where $\alpha = |B_{g_{\infty,-1}}(p_\infty, 1)|_{g_{\infty,-1}}/2$. Since $(p_i, -\tau_i)$ are ℓ-centers of $(p_0, 0)$, by Theorem 4.4.1, for large i,

$$\mathcal{N}_{p_0,0}(\tau_i) \geq -C(n,\alpha).$$

Therefore, $\inf_{\tau>0} \mathcal{N}_{p_0,0}(\tau) > -\infty$.

We refer to [CMZ21a, Section 7] for the proof that

$$\inf_{\tau>0} \mathcal{N}_{p_0,0}(\tau) = \mu_\infty,$$

where μ_∞ is the shrinker entropy of (M_∞, g_∞). □

The rest of this section is devoted to prove Theorem 5.0.1 (3). In fact, we shall prove a slightly stronger version. Let $\{(M_i, g_i(t))_{t\in(-T_i,0]}\}_{i=1}^\infty$ be a sequence of complete Ricci flows, each one with bounded curvature within each compact time interval, where $\infty \geq T_i > c > 0$ for some constant c. For each i, let $p_i \in M_i$ be a fixed point and $(\nu_t^i)_{t\in(-T_i,0]}$ the conjugate heat kernel on (M_i, g_i) based at $(p_i, 0)$. According to [Bam20b, Theorem 7.4, Corollary 7.5], $\left\{\left(M_i, g_i(t))_{t\in(-T_i,0]}, (\nu_t^i)_{t\in(-T_i,0]}\right)\right\}_{i=1}^\infty$ has an $\mathbb{F}$-convergent subsequence.

Theorem 5.2.1. *Assume that there exist a compact interval $I = [a,b] \subset (-T_\infty, 0)$, where $T_\infty := \limsup_{i\to\infty} T_i$, and a smooth Ricci flow $(M_\infty, g_\infty(t), z_\infty)_{t\in I}$, such that*

$$(M_i, g_i(t), z_i)_{t\in I} \longrightarrow (M_\infty, g_\infty(t), z_\infty)_{t\in I} \tag{5.2.1}$$

in the smooth Cheeger-Gromov-Hamilton sense, where (z_i, b) is an H_n-center of $(p_i, 0)$ for each $i \in \mathbb{N}$. Then $(M_\infty, g_\infty(t))_{t\in[a,b]}$ induces an H_n-concentrated continuous metric flow $\mathcal{X}^\infty$ and there is a conjugate heat flow $(\nu_t^\infty)_{t\in[a,b)}$ on $\mathcal{X}^\infty$, such that, by passing to a subsequence, we have

$$\left((M_i, g_i(t))_{t\in[a,b]}, (\nu_t^i)_{t\in[a,b]}\right) \xrightarrow{\mathbb{F}} \left(\mathcal{X}^\infty, (\nu_t^\infty)_{t\in[a,b)}\right), \tag{5.2.2}$$

where the convergence is uniform over any compact sub-interval of $[a,b)$.

Remarks:

1. This theorem implies that, for any subsequence of $\left\{\left((M_i, g_i(t))_{t\in[a,b]}, (\nu_t^i)_{t\in[a,b]}\right)\right\}_{i=1}^{\infty}$, there is a further subsequence that $\mathbb{F}$-converges to $\left(\mathcal{X}^\infty, (\nu_t^\infty)_{t\in[a,b)}\right)$, where ν_t^∞ is a conjugate heat flow on $\mathcal{X}^\infty$. So any continuous metric flow pair $(\mathcal{Y}^\infty, (\mu_t)_{t\in[a,b)})$ that arises as an $\mathbb{F}$-limit given by the compactness theorem [Bam20b, Theorem 7.6] should be of the form $\left(\mathcal{X}^\infty, (\nu_t^\infty)_{t\in[a,b)}\right)$. However, this does not imply that the $\mathbb{F}$-convergence in (5.2.2) holds without passing to a subsequence, since we are not able to show that ν_t^∞ is independent of the subsequence.

2. We may replace the H_n-centers (z_i, b) in the assumptions with ℓ-centers and the conclusions still hold. This is because H_n-centers are not far away from ℓ-centers by Lemma 3.3.14. This point will become clear from the proof.

Throughout this section, we assume that the conditions of Theorem 5.2.1 hold. By the definition of smooth convergence, we may find an increasing sequence of precompact open sets $U_i \subset M_\infty$ with $\cup_{i=1}^\infty U_i = M_\infty$ and a sequence of diffeomorphisms $\Psi_i : U_i \to V_i \subset M_i$, such that $\Psi_i(z_\infty) = z_i$ and

$$\left\|\Psi_i^* g_i - g_\infty\right\|_{C^{[\varepsilon_i^{-1}]}(U_i\times[a,b])} < \varepsilon_i, \tag{5.2.3}$$

for some $\varepsilon_i \searrow 0$.

The proof of Theorem 5.2.1 is divided into several components. We shall first of all show that the smooth limit flow is indeed a metric flow. This is not as obvious as it appears to be, since we do not make any geometric assumption for $(M_\infty, g_\infty(t))_{t\in[a,b]}$ except for its smoothness and completeness. Fortunately, by Theorem 3.3.11, e.g. Corollary 3.3.6, it is sufficient to have the following only assuming smoothness and completeness.

Lemma 5.2.2. *$(M_\infty, g_\infty(t))_{t\in[a,b]}$ induces an H_n-concentrated continuous metric flow $\mathcal{X}^\infty$.*

See [CMZ21a] for a proof using the converging sequence.

Lemma 5.2.3. *There is a positive solution to the conjugate heat equation* $v : M_\infty \times [a,b) \to \mathbb{R}$ *coupled with* $(M_\infty, g_\infty(t))$*, satisfying* $d\nu_s^\infty := v_s\, dg_{\infty,s} \in \mathcal{P}(M_\infty)$ *and*

$$\Psi_i^* K_i(p_i, 0 \,|\, \cdot, \cdot) \to v$$

locally smoothly on $M_\infty \times [a,b)$.

Proof. Arguing in the same way as the proof of [Lu12, Theorem 2.1], we can find a nonnegative solution $v : M_\infty \times [a,b) \to \mathbb{R}$ to the conjugate heat equation, such that

$$\Psi_i^* K_i(p_i, 0 \,|\, \cdot, \cdot) \to v \tag{5.2.4}$$

locally smoothly on $M_\infty \times [a,b)$.

In this case, Theorem 5.1.12 does not imply that $\int v_s\, dg_{\infty,s} = 1$. This is because the base point $(p_i, 0)$ of the conjugate heat kernel $K_i(p_i, 0 \,|\, \cdot, \cdot)$ is not in the region of the Cheeger-Gromov-Hamilton convergence. We first of all observe from (5.2.3) that, there is a positive function $C : (0,\infty) \to (0,\infty)$ with the following property: for any $r > 0$, we have

$$|\mathrm{Rm}_{g_i}| \le C(r) \quad \text{on} \quad B_{g_{i,b}}(z_i, r) \times [a,b] \tag{5.2.5}$$

whenever i is large enough. Since we also have

$$\liminf_{i\to\infty} \mathrm{Vol}_{g_{i,b}}\big(B_{g_{i,b}}(z_i, 1)\big) = \mathrm{Vol}_{g_{\infty,b}}\big(B_{g_{\infty,b}}(z_\infty, 1)\big) > 0,$$

By Theorem 4.4.1, there is a positive number Y independent of i, such that

$$\mathcal{N}^i_{p_i,0}(|b|) \ge -Y \quad \text{for all} \quad i \in \mathbb{N},$$

where $\mathcal{N}^i$ is the Nash entropy of the Ricci flow $(M_i, g_i(t))$. By Lemma 3.3.14,

$$\mathrm{dist}_{g_{i,b}}(z_i, p_i') \leq C \quad \text{for all} \quad i \in \mathbb{N}, \tag{5.2.6}$$

where (p_i', b) is an ℓ-center of $(p_i, 0)$, and C is a constant depending only on Y.

Now we will use (5.1.3) to obtain a uniform lower bound for $K_i(p_i, 0 \,|\, p_i', b - \varepsilon)$, where ε is an arbitrarily fixed number in $(0, b - a]$. Let us fix such an ε. (5.2.5) and (5.2.6) imply

$$\sup_{t\in[a,b]} |R_{g_i}(p_i', t)| \leq C \quad \text{for all } i \in \mathbb{N},$$

where C is a constant independent of i. We may concatenate a minimal $\mathcal{L}$-geodesic from $(p_i, 0)$ to (p_i', b) and the static curve from (p_i', b) to $(p_i', b - \varepsilon)$ as a test curve in (5.1.1). This yields

$$\begin{aligned}
\ell_{p_i,0}(p_i', |b| + \varepsilon) &\leq \frac{1}{2\sqrt{|b| + \varepsilon}} \left(2\sqrt{|b|}\ell_{p_i,0}(p_i', |b|) + \int_{|b|}^{|b|+\varepsilon} \sqrt{\tau} R_{g_i}(p_i', -\tau)\, d\tau \right) \\
&\leq \frac{1}{2\sqrt{|b| + \varepsilon}} \left(2\sqrt{|b|} \cdot \frac{n}{2} + C(|b| + \varepsilon)^{3/2} \right) \\
&\leq C \quad \text{for all} \quad i \in \mathbb{N}.
\end{aligned}$$

Hence, by (5.1.3), we have

$$K_i(p_i, 0 \,|\, p_i', b - \varepsilon) \geq \frac{1}{4\pi(|b| + \varepsilon)^{\frac{n}{2}}} e^{-\ell_{p_i,0}(p_i', |b|+\varepsilon)} \geq c(\varepsilon) > 0 \quad \text{for all} \quad i \in \mathbb{N}.$$

By (5.2.6) again, we may find a point $p_\infty' \in M_\infty$, such that $\Psi_i^{-1}(p_i') \to p_\infty'$ after possibly passing to a subsequence. Consequently

$$v(p_\infty', b - \varepsilon) = \lim_{i\to\infty} K_i(p_i, 0 \,|\, p_i', b - \varepsilon) \geq c(\varepsilon) > 0.$$

Since $\varepsilon \in (0, b - a]$ is arbitrary, it then follows from the strong maximum principle that

$v > 0$ everywhere on $M_\infty \times [a, b)$.

Once we know that v is positive, by exactly the same argument as in the proof of [CMZ21a, Theorem B.2], we have

$$\int_{M_\infty} v_s \, dg_{\infty,s} = 1,$$

for any $s \in [a, b)$. $\square$

Lemma 5.2.4. *Let $(M^n, g(t))_{t\in I}$ be a complete Ricci flow over a compact interval I and $o \in M$. Then for any $A > 0, \Omega := \cup_{t\in I} \bar{B}_t(o, A)$ is compact.*

Proof. Let $x_j \in \bar{B}_{s_j}(o, A) \subset \Omega$ be an arbitrary sequence. Assume that $s_j \to \bar{s} \in I$. We shall prove that $\{x_j\}$ has a convergent subsequence. Suppose that

$$\sup_{B_{\bar{s}}(o,10A)\times I} |\mathrm{Ric}| \le \Lambda.$$

We claim that for any $\epsilon \in (0, 1)$, there is $\bar{j}$, such that if $j \ge \bar{j}$, $x_j \in B_{\bar{s}}(o, (1+\epsilon)A)$. Suppose not. Then by passing to a subsequence, we may assume that there are $g(s_j)$-minimal geodesics $\gamma_j : [0, \sigma_j] \to M$ with $\gamma_j(0) = o, \gamma_j(\sigma_j) = x_j, \sigma_j < A$ but there is a first time $\lambda_j < \sigma_j$ such that $\gamma_j(\lambda_j) \in \partial B_{\bar{s}}(o, (1 + \epsilon/2)A)$. Pick $\delta > 0$ such that $e^{-\Lambda\delta} > 1 - \epsilon/4$. When $|s_j - \bar{s}| < \delta$, we have

$$\begin{aligned} A \ge L_{s_j}(\gamma_j|_{[0,\lambda_j]}) &\ge e^{-\Lambda|s_j-\bar{s}|} L_{\bar{s}}(\gamma_j|_{[0,\lambda_j]}) \ge (1 - \epsilon/4)\mathrm{dist}_{\bar{s}}(o, \gamma_j(\lambda_j)) \\ &= (1 - \epsilon/4)(1 + \epsilon/2)A > (1 + \epsilon/8)A, \end{aligned}$$

which is a contradiction. Hence, after passing to a subsequence, we have $x_j \to \bar{x}$ for some $\bar{x} \in \bar{B}_{\bar{s}}(o, A)$. $\square$

We are now ready to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. We divide the proof into several steps.

Step 1: Construction of the correspondence. For $t \in I$, set $Z_t^i := M_i \sqcup M_\infty$ and we shall extend the metrics on $(M_i, g_i(t))$ and $(M_\infty, g_\infty(t))$ to Z_t^i. For any $y_i \in M_i, y \in M_\infty$, define

$$\mathrm{dist}^{Z_t^i}(y, y_i) = \mathrm{dist}^{Z_t^i}(y_i, y) := \inf_{w \in U_i} \mathrm{dist}_{g_{\infty,t}}(y, w) + \mathrm{dist}_{g_{i,t}}(\Psi_i(w), y_i) + \varepsilon_i.$$

It is routine to verify that this is indeed a metric, and $M_i, M_\infty \to M_i \sqcup M_\infty$ are isometric embeddings. By [Bam20b, Lemma 2.13], we may assume that Z_t^i are isometrically embedded into a common metric space Z_t that is complete and separable. Let $\phi_t^i : (M_i, g_i(t)) \to Z_t$ be the isometric embedding for $i \in \mathbb{N} \cup \{\infty\}$. Note that for any $x \in U_i$,

$$\mathrm{dist}^{Z_t}(\phi_t^\infty(x), \phi_t^i(\Psi_i(x))) = \varepsilon_i.$$

Step 2: Construction of the couplings. Henceforth until the end of the proof of the theorem, we shall fix an arbitrarily small $\varepsilon > 0$ and denote $E = (b - \varepsilon^2, b]$. By Lemma 5.2.3, there is a conjugate heat flow

$$d\nu_s^\infty := v_s \, dg_{\infty,s},$$

where $\nu_s^\infty \in \mathcal{P}(M_\infty)$ for $s \in [a, b)$, and $\Psi_i^* \nu_s^i \to \nu_s^\infty$ on $M_\infty \times [a, b)$ in the C_c^∞-topology as smooth n-forms.

Claim 1: For $i = \infty$ or for all i sufficiently large, if $s \in I \setminus E$ and $r \geq 10\sqrt{|a|}$, then

$$\nu_s^i(M_i \setminus B_{g_{i,s}}(z_i, r)) \leq C e^{-cr^2}, \tag{5.2.7}$$

where c and C are constants depending only on the geometry of $(M_\infty, g_\infty(t))_{t \in I \setminus E}$.

Proof of Claim 1. By the smooth convergence of $\Psi_i^* K_i(p_i, 0 \,|\, \cdot, \cdot)$ and the fact that $v > 0$, there is $c_0 > 0$, such that for any $s \in I \setminus E$, we have

$$\nu_s^i \left(B_{g_{i,s}}(z_i, \sqrt{|s|}) \right) \geq \tfrac{1}{2} \nu_s^\infty \left(B_{g_{\infty,s}}(z_\infty, \sqrt{|s|}) \right) \geq c_0, \tag{5.2.8}$$

if $i \geq \bar{i}$ is sufficiently large. For $i \geq \bar{i}$ and $r \geq 10\sqrt{|a|}$, by the Gaussian concentration [Bam20a, Theorem 3.14] and (5.2.8), we have

$$\begin{aligned} c_0 \nu_s^i \left(M_i \setminus B_{g_{i,s}}(z_i, r) \right) &\leq \nu_s^i \left(B_{g_{i,s}}(z_i, \sqrt{|s|}) \right) \nu_s^i \left(M_i \setminus B_{g_{i,s}}(z_i, r) \right) \\ &\leq \exp\left\{ -\frac{(r - \sqrt{|s|})^2}{8|s|} \right\} \leq e^{-cr^2}, \end{aligned}$$

for some c depending on a and b. Note that the Gaussian concentration is also true for ν_s^∞ by Fatou's lemma. Thus, (5.2.7) also holds for $i = \infty$. $\square$

For all $s \in I \setminus E$, set $\Omega_s = \bar{B}_{g_{\infty,s}}(z_\infty, A)$, where A is some large number to be determined. Let us also denote $\Omega := \cup_{s \in I \setminus E} \Omega_s$, which is compact by Lemma 5.2.4. For all $s \in I \setminus E$ and for i large enough or $i = \infty$, define

$$\mu_s^\infty := \nu_s^\infty|_{\Omega_s} + \eta_s \delta_{z_\infty}, \quad \mu_s^i := \Psi_{i*}(\mu_s^\infty),$$

where the push-forward by Ψ_i makes sense because $\mathrm{spt}\mu_s^\infty \subset U_i$ when i is large enough.

Claim 2: We can fix A large enough, such that for $i = \infty$ or for $i \in \mathbb{N}$ sufficiently large, we have

$$\sup_{s \in I \setminus E} \mathrm{dist}_{W_1}^{(M_i, g_i(s))}(\nu_s^i, \mu_s^i) < \varepsilon.$$

Proof of Claim 2. For any $s \in I \setminus E$ and any 1-Lipschitz function ϕ on $(M_\infty, g_\infty(s))$, by

(5.2.7), we have

$$\int \phi \, d(\nu_s^\infty - \mu_s^\infty) = \int (\phi - \phi(z_\infty)) \, d(\nu_s^\infty - \mu_s^\infty) \leq \int_{M_\infty \setminus \Omega_s} \mathrm{dist}_{g_{\infty,s}}(z_\infty, x) \, d\nu_s^\infty(x)$$
$$\leq ACe^{-cA^2} + C \int_A^\infty re^{-cr^2} dr < \varepsilon,$$

if A is large enough. Here we have used a standard real analysis result (c.f. [MZ21, Lemma 3.3]). Let $s \in I \setminus E$ and ϕ be any 1-Lipschitz function on $(M_i, g_i(s))$. By (5.2.7), if A is fixed large enough, then whenever i is sufficiently large (depending on A), we have

$$\begin{aligned}
&\int \phi \, d(\nu_s^i - \mu_s^i) = \int [\phi - \phi(z_i)] \, d(\nu_s^i - \mu_s^i) \\
\leq & A \int_{\Omega_s} d(\Psi_i^* \nu_s^i - \nu_s^\infty) + \int_{M_i \setminus \Psi_i(\Omega_s)} \mathrm{dist}_{g_{i,s}}(z_i, x) \, d\nu_s^i(x) \\
\leq & 2A|\Omega|_{g_{\infty,s}} \left\| K_i(p_i, 0 \,|\, \Psi_i(\cdot), s) - v_s^\infty \right\|_{C^0(\Omega)} + ACe^{-cA^2} + C \int_A^\infty re^{-cr^2} dr < \varepsilon.
\end{aligned}$$

Here we have used the smooth convergence (5.2.4) and the fact $\Psi_i(z_\infty) = z_i$. This finishes the proof of the claim.

□

Next, we define a sequence of coupling by

$$\tilde{q}_s^i := (\mathrm{id}, \Psi_i)_*(\mu_s^\infty) \in \Pi(\mu_s^\infty, \mu_s^i).$$

Then, for any $s, t \in I$ with $s < t$, we have

$$\begin{aligned}
&\int_{M_\infty \times M_i} d_{W_1}^{Z_s} \left(\phi_{s*}^\infty \nu_{x,t\,|\,s}^\infty, \phi_{s*}^i \nu_{y,t\,|\,s}^i \right) \, d\tilde{q}_t^i(x,y) \\
= & \int_{\Omega_t} d_{W_1}^{Z_s} \left(\phi_{s*}^\infty \nu_{x,t\,|\,s}^\infty, \phi_{s*}^i \nu_{\Psi_i(x),t\,|\,s}^i \right) \, d\nu_t^\infty(x) + \eta_t \cdot d_{W_1}^{Z_s} \left(\phi_{s*}^\infty \nu_{z_\infty,t\,|\,s}^\infty, \phi_{s*}^i \nu_{z_i,t\,|\,s}^i \right).
\end{aligned}$$

Claim 3: There is a large $\bar{i} \in \mathbb{N}$, such that if $i \geq \bar{i}$, then, for any $s, t \in I \setminus E = [a, b - \varepsilon^2]$

with $s < t$ and for any $x \in \Omega_t$, we have

$$\mathrm{dist}_{W_1}^{Z_s}\left(\phi^{\infty}_{s*}\nu^{\infty}_{x,t\,|\,s}, \phi^{i}_{s*}\nu^{i}_{\Psi_i(x),t\,|\,s}\right) < \varepsilon.$$

Proof of Claim 3. Suppose not. By passing to a subsequence, we may assume that there are $s_i, t_i \in I \setminus E, s_i < t_i, x_i \in \Omega_{t_i}$, such that

$$\mathrm{dist}_{W_1}^{Z_{s_i}}\left(\phi^{\infty}_{s_i*}\nu^{\infty}_{x_i,t_i\,|\,s_i}, \phi^{i}_{s_i*}\nu^{i}_{\Psi_i(x_i),t_i\,|\,s_i}\right) \geq \varepsilon. \tag{5.2.9}$$

By passing to a further subsequence, we may assume that $t_i \to \bar{t}, s_i \to \bar{s} \leq \bar{t}, x_i \to \bar{x} \in \Omega_{\bar{t}}$.
Case A: $\bar{s} = \bar{t}$. Write $\bar{x}_i = \Psi_i(\bar{x})$.

$$\begin{aligned}
&\mathrm{dist}_{W_1}^{Z_{s_i}}\left(\phi^{\infty}_{s_i*}\nu^{\infty}_{x_i,t_i\,|\,s_i}, \phi^{i}_{s_i*}\nu^{i}_{\Psi_i(x_i),t_i\,|\,s_i}\right)\\
\leq\ &\mathrm{dist}_{W_1}^{Z_{s_i}}\left(\phi^{\infty}_{s_i*}\nu^{\infty}_{x_i,t_i\,|\,s_i}, \phi^{\infty}_{s_i*}\delta_{\bar{x}}\right) + \mathrm{dist}_{W_1}^{Z_{s_i}}\left(\phi^{\infty}_{s_i*}\delta_{\bar{x}}, \phi^{i}_{s_i*}\delta_{\Psi_i(\bar{x})}\right)\\
&\quad + \mathrm{dist}_{W_1}^{Z_{s_i}}\left(\phi^{i}_{s_i*}\delta_{\Psi_i(\bar{x})}, \phi^{i}_{s_i*}\nu^{i}_{\Psi_i(x_i),t_i\,|\,s_i}\right)\\
\leq\ &\mathrm{dist}_{W_1}^{g_{\infty,s_i}}\left(\nu^{\infty}_{x_i,t_i\,|\,s_i}, \delta_{\bar{x}}\right) + \varepsilon_i + \mathrm{dist}_{W_1}^{g_{i,s_i}}\left(\delta_{\bar{x}_i}, \nu^{i}_{\Psi_i(x_i),t_i\,|\,s_i}\right)\\
\leq\ &\mathrm{dist}_{W_1}^{g_{\infty,s_i}}\left(\nu^{\infty}_{x_i,t_i\,|\,s_i}, \nu^{\infty}_{\bar{x},t_i\,|\,s_i}\right) + \mathrm{dist}_{W_1}^{g_{\infty,s_i}}\left(\nu^{\infty}_{\bar{x},t_i\,|\,s_i}, \delta_{\bar{x}}\right) + \varepsilon_i\\
&\qquad + \mathrm{dist}_{W_1}^{g_{i,s_i}}\left(\nu^{i}_{\bar{x}_i,t_i\,|\,s_i}, \nu^{i}_{\Psi_i(x_i),t_i\,|\,s_i}\right) + \mathrm{dist}_{W_1}^{g_{i,s_i}}\left(\nu^{\infty}_{\bar{x}_i,t_i\,|\,s_i}, \delta_{\bar{x}_i}\right)\\
\leq\ &2\mathrm{dist}_{g_{\infty,t_i}}(x_i, \bar{x}) + 2\varepsilon_i + \mathrm{dist}_{W_1}^{g_{\infty,s_i}}\left(\nu^{\infty}_{\bar{x},t_i\,|\,s_i}, \delta_{\bar{x}}\right) + \mathrm{dist}_{W_1}^{g_{i,s_i}}\left(\nu^{\infty}_{\bar{x}_i,t_i\,|\,s_i}, \delta_{\bar{x}_i}\right) \to 0,
\end{aligned}$$

as $i \to \infty$, which is a contradiction to (5.2.9). Here we have also applied (5.1.6). The last convergence above is due to [Bam20a, Proposition 9.5] which is but a consequence of Proposition 5.1.9.

Case B: $\bar{s} < \bar{t}$. By Theorem 5.1.12, after possibly passing to a subsequence, we have

$$\Psi_i^* K_i(\Psi_i(x_i), t_i\,|\,\cdot,\cdot) \to K_\infty(\bar{x}, \bar{t}\,|\,\cdot,\cdot)$$

in the C_c^∞-topology and the convergence is uniform on compact subsets of $M_\infty \times [a, \bar{t})$. In particular,

$$\left\| \Psi_i^* \nu_{\Psi_i(x_i), t_i \,|\, s_i} - \nu^\infty_{\bar{x}, \bar{t} \,|\, s_i} \right\|_{C^0(\mathcal{K})} \to 0,$$

as n-forms for any compact subset $\mathcal{K} \subset M_\infty$. Let $(z, \bar{s})$ be an H_n-center of $(\bar{x}, \bar{t})$ and let $B := B_{g_{\infty, \bar{s}}}(z, 10D)$ for some large constant D to be determined. First choose D large enough so that $\nu^\infty_{\bar{x}, \bar{t} \,|\, \bar{s}}(M_\infty \setminus B) < \frac{\varepsilon}{10D}$. This is possible because of Proposition 5.1.9. Then we have

$$\begin{aligned} &\operatorname{dist}_{W_1}^{Z_{s_i}} \left(\phi^\infty_{s_i *} \nu^\infty_{x_i, t_i | s_i}, \phi^i_{s *} \nu^i_{\Psi_i(x_i), t_i | s_i} \right) \\ \le\ &\operatorname{dist}_{W_1}^{g_{\infty, s_i}} \left(\nu^\infty_{x_i, t_i | s_i}, \nu^\infty_{\bar{x}, \bar{t} | s_i} \right) + \operatorname{dist}_{W_1}^{Z_{s_i}} \left(\phi^\infty_{s_i *} \nu^\infty_{\bar{x}, \bar{t} | s_i}, \phi^i_{s *} \nu^i_{\Psi_i(x_i), t_i | s_i} \right). \end{aligned}$$

The first term above clearly converges to 0. For the second term, we argue in the same way as Claim 2 above. By the local distance distortion estimates, we may assume that $B_i = B_{g_{\infty, s_i}}(z, D) \subset B$. Consider any bounded 1-Lipschitz function ϕ defined on Z_{s_i}. We may assume that $\phi(\phi^\infty_{s_i}(z)) = 0$, for otherwise we may replace it with $\phi - \phi(\phi^\infty_{s_i}(z))$. Then, we compute

$$\begin{aligned} &\int_{M_\infty} \phi \circ \phi^\infty_{s_i} \, d\nu^\infty_{\bar{x}, \bar{t} \,|\, s_i} - \int_{M_i} \phi \circ \phi^i_{s_i} \, d\nu^i_{\Psi_i(x_i), t_i \,|\, s_i} \\ \le\ &\int_{B_i} \phi \circ \phi^\infty_{s_i} \, d\nu^\infty_{\bar{x}, \bar{t} \,|\, s_i} - \int_{\Psi_i(B_i)} \phi \circ \phi^i_{s_i} \, d\nu^i_{\Psi_i(x_i), t_i \,|\, s_i} \\ &+ CDe^{-cD^2} + C \int_D^\infty s e^{-cs^2} \, ds, \end{aligned}$$

where we used the Gaussian concentration as in Claim 1. We can fix D so that the last line above is less than $\varepsilon/4$ for all i large. Then, we have

$$\begin{aligned}
&\int_{B_i} \phi \circ \phi^{\infty}_{s_i}\, d\nu^{\infty}_{\bar{x},\bar{t}\,|\,s_i} - \int_{\Psi_i(B_i)} \phi \circ \phi^{i}_{s_i}\, d\nu^{i}_{\Psi_i(x_i),t_i\,|\,s_i} \\
&\leq \int_{B_i} \left\{ \phi \circ \phi^{\infty}_{s_i} - \phi \circ \phi^{i}_{s_i} \circ \Psi_i \right\} d\nu^{\infty}_{\bar{x},\bar{t}\,|\,s_i} + \int_{B_i} \phi \circ \phi^{i}_{s_i} \circ \Psi_i \left\{ d\nu^{\infty}_{\bar{x},\bar{t}\,|\,s_i} - \Psi_i^* d\nu^{i}_{\Psi_i(x_i),t_i\,|\,s_i} \right\} \\
&\leq \varepsilon_i + 2D \left\| \Psi_i^* \nu_{\Psi_i(x_i),t_i\,|\,s_i} - \nu^{\infty}_{\bar{x},\bar{t}\,|\,s_i} \right\|_{C^0(B)}.
\end{aligned}$$

Note that the last line does not depend on ϕ and converges to 0 since B is compact. Hence, we have

$$\operatorname{dist}_{W_1}^{Z_{s_i}} \left(\phi^{\infty}_{s_i *} \nu^{\infty}_{x_i,t_i\,|\,s_i}, \phi^{i}_{s_i *} \nu^{i}_{\Psi_i(x_i),t_i\,|\,s_i} \right) < \varepsilon/2,$$

when i is large enough; this is a contradiction agains (5.2.9). $\square$

By the definition of the 1-Wassernstein distance, there are couplings $\theta^i_s \in \Pi(\mu^i_s, \nu^i_s)$ such that

$$\int_{M_i \times M_i} \operatorname{dist}_{g_{i,s}}(x,y)\, d\theta^i_s(x,y) < \operatorname{dist}_{W_1}^{(M_i, g_{i,s})}(\nu^i_s, \mu^i_s) + \varepsilon < 2\varepsilon,$$

if $i \geq \bar{i}$. Applying [Bam20b, Lemma 2.2] for three times, there is $Q^i_s \in \mathcal{P}(M_i \times M_i \times M_\infty \times M_\infty)$ such that the marginal into the first and second factors equals θ^i_s, the marginal into the third and first factors equals $\tilde{q}^i_s$, and the marginal into the third and fourth factors equals θ^∞_s. Define q^i_s to be the marginal of Q^i_s into the second and fourth factors. Then $q^i_s \in \Pi(\nu^i_s, \nu^\infty_s)$.

Step 4: Final verification. For any $s, t \in I \setminus E = [a, b - \varepsilon^2]$, $s \le t$, we have

$$\begin{aligned}
&\int_{M_\infty \times M_i} \mathrm{dist}_{W_1}^{Z_s} \left(\phi_{s*}^\infty \nu_{x,t\,|\,s}^\infty, \phi_{s*}^i \nu_{y,t\,|\,s}^i\right) \, dq_t^i(x,y) \\
&= \int_{M_i \times M_i \times M_\infty \times M_\infty} \mathrm{dist}_{W_1}^{Z_s} \left(\phi_{s*}^\infty \nu_{x,t\,|\,s}^\infty, \phi_{s*}^i \nu_{y,t\,|\,s}^i\right) \, dQ_t^i(y, y_1, x, x_1) \\
&\le \int \left\{\mathrm{dist}_{g_{\infty,t}}(x, x_1) + \mathrm{dist}_{g_{i,t}}(y, y_1) + \mathrm{dist}_{W_1}^{Z_s} \left(\phi_{s*}^\infty \nu_{x_1,t\,|\,s}^\infty, \phi_{s*}^i \nu_{y_1,t\,|\,s}^i\right)\right\} \, dQ_t^i(y, y_1, x, x_1) \\
&= \int_{M_\infty \times M_\infty} \mathrm{dist}_{g_{\infty,t}} \, d\theta_t^\infty + \int_{M_i \times M_i} \mathrm{dist}_{g_{i,t}} \, d\theta_t^i \\
&\qquad + \int_{M_\infty \times M_i} \mathrm{dist}_{W_1}^{Z_s} \left(\phi_{s*}^\infty \nu_{x,t\,|\,s}^\infty, \phi_{s*}^i \nu_{y,t\,|\,s}^i\right) \, d\tilde{q}_t^i(x,y) \\
&< 10\varepsilon,
\end{aligned}$$

if $i \ge \bar{i}$, where we have also used the monotonicity formula (5.1.6).

□

As a special case of this Theorem, under the assumptions of Theorem 5.0.1, we have, after passing to a subsequence,

$$\Big((M, g_i(t))_{t\in[-2,-1]}, (\nu_s^i)_{s\in[-2,-1]}\Big) \xrightarrow{\ \mathbb{F}\ } \Big((M_\infty, g_\infty(t))_{t\in[-2,-1]}, (\nu_s^\infty)_{s\in[-2,-1)}\Big).$$

To finish the proof of Theorem 5.0.1, we only need to show that any ν_s^∞ in the proof of Theorem 5.2.1 is induced by the shrinker potential function.

5.3 Independence of Base Points

In this section, we prove that for an H-concentrated metric flow, the tangent flow at infinity do not depend on the base point. For all the basic definitions involved, such as the variance, the Weissernstein distance, etc., the author may refer to [Bam20b]. We will first prove that time shifting and parabolic scaling are continuous with respect to the $\mathbb{F}$-distance, which are natural but fundamentally important. The main theorem then follows as a consequence in the same spirit as [BBI01, Proposition 8.2.8].

Let $\mathcal{X}$ be a metric flow over some $I \subset \mathbb{R}$. When necessary, we will put $\mathcal{X}$ as an upper index for geometric quantities to stress that they are quantities on $\mathcal{X}$. For example, $\nu^{\mathcal{X}}_{x\,|\,s}$ represents the conjugate heat kernel on $\mathcal{X}$ at time s based at $x \in \mathcal{X}$.

For any metric flow $\mathcal{X}$ over some $I \subset \mathbb{R}$, $t_0 \in \mathbb{R}$, and $\lambda > 0$, we denote by

$$\mathcal{X}^{-t_0,\lambda}$$

the metric flow obtained by first applying a $-t_0$ time shift to $\mathcal{X}$ and then a parabolic rescaling by factor λ. To be more specific, if we write $\mathcal{Y} = \mathcal{X}^{-t_0,\lambda}$, then $\mathcal{Y}$ is a metric flow defined over $J := \lambda^2(I - t_0)$, such that for each $t \in J$, we have

$$\mathcal{Y}_t := \mathcal{X}_{\lambda^{-2}t+t_0}, \quad \mathrm{dist}^{\mathcal{Y}_t} := \lambda \cdot \mathrm{dist}^{\mathcal{X}_{\lambda^{-2}t+t_0}}.$$

For any $y \in \mathcal{Y}_t = \mathcal{X}_{\lambda^{-2}t+t_0}$ and $s, t \in J$ with $s \le t$, we define the conjugate heat kernels by

$$\nu^{\mathcal{Y}}_{y|s} := \nu^{\mathcal{X}}_{y|\lambda^{-2}s+t_0}.$$

For any conjugate heat flow $(\mu_t)_{t\in I'}$ on $\mathcal{X}$ over $I' \subset I$, we define

$$\mu_t^{-t_0,\lambda} := \mu_{\lambda^{-2}t+t_0}, \quad t \in \lambda^2(I' - t_0).$$

For simplicity, we write

$$\mathcal{X}^{-t_0} := \mathcal{X}^{-t_0,1}, \quad \mu_t^{-t_0} := \mu_t^{-t_0,1}$$

for any metric flow $\mathcal{X}$ and conjugate heat flow μ_t. We first prove that time shifting is continuous with respect to the $\mathbb{F}$-distance for an H-concentrated metric flow.

Proposition 5.3.1. *For any H, V, $T < \infty$, and $\epsilon > 0$, there is a $\delta = \delta(H, V, T, \epsilon) > 0$*

such that the following holds. Let $\left(\mathcal{X}, (\mu_t)_{t\in[-T-1,0)}\right)$ *be an* H*-concentrated metric flow pair over* $[-T-1,0]$*. Suppose that*

$$\sup_t \operatorname{Var}(\mu_t) \leq V.$$

If $0 \leq \sigma \leq \delta$*, then*

$$\operatorname{dist}_{\mathbb{F}}\left(\left(\mathcal{X}_{[-T,0]}, (\mu_t)_{t\in[-T,0)}\right), \left(\mathcal{X}^{\sigma}_{[-T,0]}, (\mu^{\sigma}_t)_{t\in[-T,0)}\right)\right) < \epsilon.$$

Remark. According to the definitions of metric flow pair and $\mathbb{F}$-distance, the exact form of the existence interval I' of μ_t does not matter. It will be clear in the proof that we only need to assume that $|[-T,0]\setminus I'| = 0$, and the future completion (see [Bam20b, Definition 4.42]) of I' is $[-T,0]$ because of the assumptions of [Bam20b, Proposition 4.1]. For simplicity, we assumed $I' = [-T-1,0)$ in the proposition above. In applications, we will use conjugate heat kernels which exist over, e.g., $[-T-1,0]$.

Proof. Let $I = [-T,0)$ and let $\beta,\delta \in (0,1/2)$ be constants to be determined later. Then the function $v(t) := \operatorname{Var}(\mu_t) + Ht$ is non-decreasing in t by [Bam20b, Proposition 3.34]. Let

$$E := E_{\delta,\beta} := \{t \in [-T,0) : v(t) - v(t-\delta) \geq \beta\}.$$

Let us find a maximal finite sequence $t_1, \cdots, t_N \in E$ such that $\{[t_k-\delta, t_k]\}_{k=1}^N$ are disjoint, then we have

$$A := V + H(T+1) \geq v(0) - v(-T-1) \geq \sum_{k=1}^{N} v(t_k) - v(t_k-\delta) \geq \beta N.$$

By the maximality of $\{t_k\}_{k=1}^N$, we have that, for any $t \in E$, there is $1 \leq k \leq N$ such that $[t-\delta, t]$ intersects $[t_k-\delta, t_k]$. Hence $t \in [t_k-\delta, t_k+\delta]$ and we have

$$E \subset \bigcup_{1\leq k\leq N} [t_k-\delta, t_k+\delta], \quad |E| \leq 2N\delta \leq \frac{2A\delta}{\beta}.$$

Let $\tau_j = 2^{-3(j+1)}/H$ for each $j \geq 0$ and we define $b : (0,1] \to (0,1)$ by

$$b(s) := \tfrac{1}{2}\Phi\left(-\sqrt{\tfrac{8V}{s\tau_j}}\right) \quad \text{for} \quad s \in (2(\tau_j H)^{1/3}, 2(\tau_{j-1}H)^{1/3}],$$

where $\Phi : \mathbb{R} \to (0,1)$ is the same function as defined in [Bam20b, (3.1)], satisfying

$$\Phi'(x) = (4\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{4}}, \quad \lim_{x\to-\infty} \Phi(x) = 0, \quad \lim_{x\to\infty} \Phi(x) = 1.$$

So b is a positive increasing function defined on $(0,1]$. Applying [Bam20b, Proposition 4.1] with $r = 1$, we have that, given $\sigma \in [0,\delta]$, for each $t \in I \setminus \big(E \cup (-H^{-1}\sigma^3, 0]\big)$, it holds that

$$b_1^{(\mathcal{X}_t, \mathrm{dist}_t, \mu_t)}(\varepsilon) \geq b(\varepsilon) \quad \text{for all} \quad \varepsilon \in [\sigma, 1], \tag{5.3.1}$$

where $b_r^{(X,d,\mu)} : (0,1] \to (0,1]$ is the mass distribution function at scale $r > 0$; see [Bam20b, Definition 2.17].

Next, we shall apply [Bam20b, Proposition 4.14]. To this end, we verify that the assumptions therein are satisfied by $\mathcal{X}$ with $t \in I \setminus E$ and $t' = t - \sigma$, where $\sigma \in [0,\delta]$. Indeed, applying [Bam20b, Lemma 4.7] and the definition of E, we have

$$\begin{aligned} \int_{\mathcal{X}_t \times \mathcal{X}_t} \mathrm{dist}_t \, d\mu_t d\mu_t - \int_{\mathcal{X}_{t'} \times \mathcal{X}_{t'}} \mathrm{dist}_{t'} \, d\mu_{t'} d\mu_{t'} &\leq \sqrt{v(t) - v(t')} + 2\sqrt{H(t-t')} \\ &\leq \sqrt{v(t) - v(t-\delta)} + 2\sqrt{H(t-t')} \\ &\leq \sqrt{\beta} + 2\sqrt{H\delta} < \Psi(\delta, \beta \,|\, H). \end{aligned} \tag{5.3.2}$$

Given (5.3.1) and (5.3.2), we can now apply [Bam20b, Proposition 4.14] with $r = 1$, and conclude that for each $t \in I \setminus \big(E \cup (-H^{-1}\delta^3, 0]\big)$ and $\sigma \in [0,\delta]$, writing $t' = t - \sigma r^2 = t - \sigma \in [t-\delta, t]$, there is a closed subset $W_t \subset \mathcal{X}_t$ such that:

(1) $\mu_t(\mathcal{X}_t \setminus W_t) < \Psi(\delta, \beta \,|\, H, V)$.

(2) For any $y_1, y_2 \in W_t$, we have

$$0 \leq \mathrm{dist}_t(y_1, y_2) - \mathrm{dist}_{W_1}^{\mathcal{X}_{t'}}(\nu_{y_1\,|\,t'}, \nu_{y_2\,|\,t'}) < \Psi(\delta, \beta\,|\,H, V).$$

Furthermore, there exist a metric space $(Z_t, \mathrm{dist}^{Z_t})$ and isometric embeddings $\psi_t : \mathcal{X}_t \to Z_t$, $\phi_t : \mathcal{X}_{t'} \to Z_t$, such that:

(3) For any $x \in \mathcal{X}_{t'}, y \in W_t$, we have

$$\mathrm{dist}^{Z_t}(\phi_t(x), \psi_t(y)) \leq \mathrm{dist}_{W_1}^{\mathcal{X}_{t'}}(\delta_x, \nu_{y\,|\,t'}) + \Psi(\delta, \beta\,|\,H, V).$$

(4) We can construct the following coupling between μ_t and $\mu_{t'}$

$$q_t := \int_{\mathcal{X}_t} (\nu_{y\,|\,t'} \otimes \delta_y)\, d\mu_t(y) \in \Pi(\mu_{t'}, \mu_t)$$

satisfying the estimate

$$\mathrm{dist}_{W_1}^{Z_t}(\phi_{t*}\mu_{t'}, \psi_{t*}\mu_t) \leq \int_{\mathcal{X}_{t'} \times \mathcal{X}_t} \mathrm{dist}^{Z_t}(\phi_t(x), \psi_t(y)) dq_t(x, y) < \Psi(\delta, \beta\,|\,H, V).$$

For $t \in I \cap \big(E \cup (-H^{-1}\delta^3, 0]\big)$, we let $(Z_t, \mathrm{dist}^{Z_t})$ be an arbitrary separable metric space into which $(\mathcal{X}_t, \mathrm{dist}_t)$ and $(\mathcal{X}_{t-\sigma}, \mathrm{dist}_{t-\sigma})$ can be embedded. Now $\mathfrak{C} = \big((Z_t, \mathrm{dist}^{Z_t}), (\psi_t, \phi_t)\big)_{t \in I}$ serves as a correspondence between $\mathcal{X}_I$ and $\mathcal{X}_I^\sigma$.

The goal is to show that for each $s \leq t, s, t \in I \setminus \big(E \cup (-H^{-1}\delta^3, 0]\big)$, it holds that

$$\text{if } \delta \leq \bar{\delta}(\epsilon, H, V, T), \quad \text{ then } \int_{\mathcal{X}_t^\sigma \times \mathcal{X}_t} \mathrm{dist}_{W_1}^{Z_s}\left(\phi_{s*}\nu_{x^1\,|\,s-\sigma}, \psi_{s*}\nu_{x^2\,|\,s}\right) dq_t(x^1, x^2) < \epsilon,$$

where q_t is the coupling defined in item (4) above. We need only to verify the case where

$s < t$, because the equality case is equivalent to item (4) above. We write

$$s' := s - \sigma, \quad t' := t - \sigma.$$

Note that

$$\begin{aligned}
&\int_{\mathcal{X}_t^\sigma \times \mathcal{X}_t} \mathrm{dist}_{W_1}^{Z_s}\left(\phi_{s*}\nu_{x\,|\,s'}, \psi_{s*}\nu_{y\,|\,s}\right) dq_t(x,y) \\
&= \int_{\mathcal{X}_t} \int_{\mathcal{X}_t^\sigma} \mathrm{dist}_{W_1}^{Z_s}\left(\phi_{s*}\nu_{x\,|\,s'}, \psi_{s*}\nu_{y\,|\,s}\right)\, d\nu_{y\,|\,t'}(x)\, d\mu_t(y) \\
&\leq \int_{\mathcal{X}_t} \int_{\mathcal{X}_t^\sigma} \mathrm{dist}_{W_1}^{\mathcal{X}_{s'}}(\nu_{x\,|\,s'}, \nu_{y\,|\,s'})\, d\nu_{y\,|\,t'}(x)\, d\mu_t(y) \\
&\quad + \int_{\mathcal{X}_t} \int_{\mathcal{X}_t^\sigma} \mathrm{dist}_{W_1}^{Z_s}\left(\phi_{s*}\nu_{y\,|\,s'}, \psi_{s*}\nu_{y\,|\,s}\right)\, d\nu_{y\,|\,t'}(x)\, d\mu_t(y) \\
&=: I_1 + I_2.
\end{aligned}$$

For I_1, by the monotonicity formula [Bam20b, Propsition 3.24(b)], the definition of the W_1-Weissernstein distance, and the definition of variance, we have

$$\begin{aligned}
I_1 &\leq \int_{\mathcal{X}_t} \int_{\mathcal{X}_t^\sigma} \mathrm{dist}_{W_1}^{\mathcal{X}_{t'}}(\delta_x, \nu_{y\,|\,t'})\, d\nu_{y\,|\,t'}(x)\, d\mu_t(y) \\
&= \int_{\mathcal{X}_t} d\mu_t(y) \int_{\mathcal{X}_{t'} \times \mathcal{X}_{t'}} \mathrm{dist}_{t'}\, d\nu_{y\,|\,t'} d\nu_{y\,|\,t'} \\
&\leq \int_{\mathcal{X}_t} \mathrm{Var}(\nu_{y\,|\,t'})^{1/2}\, d\mu_t(y) \leq \sqrt{H\delta},
\end{aligned}$$

where we used [Bam20b, Proposition 3.34] in the last inequality.

I_2 can be simplified as

$$I_2 = \int_{\mathcal{X}_t} \mathrm{dist}_{W_1}^{Z_s}\left(\phi_{s*}\nu_{y\,|\,s'}, \psi_{s*}\nu_{y\,|\,s}\right)\, d\mu_t(y).$$

We shall use the same argument as in the proof of [Bam20b, Lemma 4.18] to obtain an estimate of I_2. Fix any $x \in \mathcal{X}_{s'}$ and $w \in W_s^\delta \subset \mathcal{X}_s$, where we let

$$W^\delta = B(W, \delta),$$

following the notations in [Bam20b, Lemma 4.18]. Let $w' \in W_s$ such that $\mathrm{dist}_s(w, w') < \delta$, then, by item (3) above, we have

$$\begin{aligned}
\mathrm{dist}^{Z_s}(\phi_s(x), \psi_s(w)) &\le \mathrm{dist}^{Z_s}(\phi_s(x), \psi_s(w')) + \delta \le \mathrm{dist}_{W_1}^{\mathcal{X}_{s'}}(\delta_x, \nu_{w'\,|\,s'}) + \Psi(\delta, \beta \,|\, H, V) \\
&\le \mathrm{dist}_{W_1}^{\mathcal{X}_{s'}}(\delta_x, \nu_{w\,|\,s'}) + \mathrm{dist}_s(w, w') + \Psi(\delta, \beta \,|\, H, V) \qquad (5.3.3) \\
&\le \mathrm{dist}_{W_1}^{\mathcal{X}_{s'}}(\delta_x, \nu_{w\,|\,s'}) + \Psi(\delta, \beta \,|\, H, V),
\end{aligned}$$

where we have applied [Bam20b, Propsition 3.24(b)] again. Define

$$q := q_{y,s} := \int_{\mathcal{X}_s} \nu_{z\,|\,s'} \otimes \delta_z \, d\nu_{y\,|\,s}(z) \in \Pi(\nu_{y\,|\,s'}, \nu_{y\,|\,s}) \quad \text{for each} \quad y \in \mathcal{X}_t,$$

we may use the definition of the W_1-Weissernstein distance to estimate the integrand of I_2 as follows.

$$\begin{aligned}
\mathrm{dist}_{W_1}^{Z_s}\left(\phi_{s*}\nu_{y\,|\,s'}, \psi_{s*}\nu_{y\,|\,s}\right) &\le \int_{\mathcal{X}_{s'} \times \mathcal{X}_s} \mathrm{dist}^{Z_s}(\phi_s(x_1), \psi_s(x_2)) \, dq(x_1, x_2) \\
&= \int_{\mathcal{X}_s} \int_{\mathcal{X}_{s'}} \mathrm{dist}^{Z_s}(\phi_s(x), \psi_s(z)) \, d\nu_{z\,|\,s'}(x) \, d\nu_{y\,|\,s}(z) \\
&= \int_{W_s^\delta} \int_{\mathcal{X}_{s'}} + \int_{\mathcal{X}_s \setminus W_s^\delta} \int_{\mathcal{X}_{s'}} =: A_1(y) + A_2(y).
\end{aligned}$$

On one hand, by (5.3.3) and applying [Bam20b, Proposition 3.34] again, we have

$$
\begin{aligned}
A_1(y) &= \int_{W_s^\delta} \int_{\mathcal{X}_{s'}} \operatorname{dist}^{Z_s}(\phi_s(x), \psi_s(w))\, d\nu_{w\,|\,s'}(x)\, d\nu_{y\,|\,s}(w) \\
&\leq \int_{W_s^\delta} \int_{\mathcal{X}_{s'}} \operatorname{dist}_{W_1}^{\mathcal{X}_{s'}}(\delta_x, \nu_{w\,|\,s'})\, d\nu_{w\,|\,s'}(x)\, d\nu_{y\,|\,s}(w) + \Psi(\delta, \beta\,|\,H, V) \\
&= \int_{W_s^\delta} \left(\int_{\mathcal{X}_{s'} \times \mathcal{X}_{s'}} \operatorname{dist}_s(x, x')\, d\nu_{w\,|\,s'}(x)\, d\nu_{w\,|\,s'}(x') \right) d\nu_{y\,|\,s}(w) + \Psi(\delta, \beta\,|\,H, V) \\
&\leq \int_{W_s^\delta} \operatorname{Var}(\nu_{w\,|\,s'})^{1/2}\, d\nu_{y\,|\,s}(w) + \Psi(\delta, \beta\,|\,H, V) \\
&\leq \sqrt{H\delta} + \Psi(\delta, \beta\,|\,H, V).
\end{aligned}
$$

On the other hand, if δ, β are sufficiently small, then $\mu_s(W_s^\delta) \geq 1/2$, and we may compute as follows using (5.3.3) and [Bam20b, Proposition 3.34].

$$
\begin{aligned}
A_2(y)/2 &= \frac{1}{2}\int_{\mathcal{X}_s\setminus W_s^\delta}\int_{\mathcal{X}_{s'}} \mathrm{dist}^{Z_s}(\phi_s(x),\psi_s(z))\,d\nu_{z\,|\,s'}(x)\,d\nu_{y\,|\,s}(z) && (5.3.4)\\
&\le \int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}\int_{\mathcal{X}_{s'}} \mathrm{dist}^{Z_s}(\phi_s(x),\psi_s(z))\,d\nu_{z\,|\,s'}(x)\,d\nu_{y\,|\,s}(z)\,d\mu_s(w)\\
&\le \int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}\int_{\mathcal{X}_{s'}} \left\{\mathrm{dist}^{Z_s}(\phi_s(x),\psi_s(w))+\mathrm{dist}_s(w,z)\right\}d\nu_{z\,|\,s'}(x)\,d\nu_{y\,|\,s}(z)\,d\mu_s(w)\\
&\le \int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}\int_{\mathcal{X}_{s'}} \mathrm{dist}_{W_1}^{\mathcal{X}_{s'}}(\delta_x,\nu_{w\,|\,s'})\,d\nu_{z\,|\,s'}(x)\,d\nu_{y\,|\,s}(z)\,d\mu_s(w)+\Psi(\delta,\beta\,|\,H,V)\\
&\quad+\int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}\mathrm{dist}_s(w,z)\,d\nu_{y\,|\,s}(z)\,d\mu_s(w)\\
&= \int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}\left(\int_{\mathcal{X}_{s'}\times\mathcal{X}_{s'}}\mathrm{dist}^{\mathcal{X}_{s'}}(x,x')\,d\nu_{z\,|\,s'}(x)\,d\nu_{w\,|\,s'}(x')\right)d\nu_{y\,|\,s}(z)\,d\mu_s(w)\\
&\quad+\int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}\mathrm{dist}_s(w,z)\,d\nu_{y\,|\,s}(z)\,d\mu_s(w)+\Psi(\delta,\beta\,|\,H,V)\\
&\le \int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}\mathrm{Var}(\nu_{z\,|\,s'},\nu_{w\,|\,s'})^{1/2}\,d\nu_{y\,|\,s}(z)\,d\mu_s(w)\\
&\quad+\int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}\mathrm{dist}_s(w,z)\,d\nu_{y\,|\,s}(z)\,d\mu_s(w)+\Psi(\delta,\beta\,|\,H,V)\\
&\le \int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}\left(\sqrt{\mathrm{dist}_s^2(w,z)+H\delta}+\mathrm{dist}_s(w,z)\right)d\nu_{y\,|\,s}(z)\,d\mu_s(w)\\
&\quad\;+\Psi(\delta,\beta\,|\,H,V) && (5.3.5)\\
&\le \left(2\int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}\left(2\mathrm{dist}_s^2(w,z)+H\delta\right)\,d\nu_{y\,|\,s}(z)\,d\mu_s(w)\right)^{\frac{1}{2}}\\
&\quad\;\times\left(\int_{W_s^\delta}\int_{\mathcal{X}_s\setminus W_s^\delta}d\nu_{y\,|\,s}(z)\,d\mu_s(w)\right)^{\frac{1}{2}}+\Psi(\delta,\beta\,|\,H,V) && (5.3.6)\\
&\le 2\left\{\mu_s(W_s^\delta)\cdot\nu_{y\,|\,s}(\mathcal{X}_s\setminus W_s^\delta)\right\}^{1/2}\left(\mathrm{Var}(\nu_{y\,|\,s},\mu_s)+\tfrac{1}{2}H\delta\right)^{1/2}+\Psi(\delta,\beta\,|\,H,V),
\end{aligned}
$$

where we have also applied the definition of H-concentration. Since

$$\int_{\mathcal{X}_t}\nu_{z\,|\,s}(\mathcal{X}_s\setminus W_s^\delta)\,d\mu_t(z)=\mu_s(\mathcal{X}_s\setminus W_s^\delta)<\Psi(\delta,\beta\,|\,H,V),$$

taking $\Psi_1 = \sqrt{\Psi}$ and $\Omega_t := \{\nu_{\cdot\,|\,s}(\mathcal{X}_s \setminus W_s^\delta) < \Psi_1\}$, we have

$$\mu_t(\Omega_t) \geq 1 - \Psi_1(\delta, \beta \,|\, H, V), \quad \nu_{z\,|\,s}(\mathcal{X}_s \setminus W_s^\delta) < \Psi_1(\delta, \beta \,|\, H, V) \;\text{ for each }\; z \in \Omega_t.$$

It follows from (5.3.4) that

$$\begin{aligned}
&\int_{\mathcal{X}_t} A_2(y)\, d\mu_t(y) = \int_{\Omega_t} + \int_{\mathcal{X}_t \setminus \Omega_t} \\
&\leq 4\Psi_1(\delta, \beta \,|\, H, V)^{1/2} \int_{\mathcal{X}_t} \left(\mathrm{Var}(\nu_{y\,|\,s}, \mu_s) + \tfrac{1}{2} H\delta\right)^{1/2} d\mu_t(y) \\
&\quad + \Psi(\delta, \beta \,|\, H, V) + 4 \int_{\mathcal{X}_t \setminus \Omega_t} \left(\mathrm{Var}(\nu_{y\,|\,s}, \mu_s) + \tfrac{1}{2} H\delta\right)^{\frac{1}{2}} d\mu_t(y) \\
&\leq 4\Psi_1(\delta, \beta \,|\, H, V)^{1/2} \left(\int_{\mathcal{X}_t} \mathrm{Var}(\delta_y, \mu_t)\, d\mu_t(y) + H(t-s) + \tfrac{1}{2} H\delta\right)^{1/2} + \Psi(\delta, \beta \,|\, H, V) \\
&\quad + 4\mu_t(\mathcal{X}_t \setminus \Omega_t)^{1/2} \left(\int_{\mathcal{X}_t} \mathrm{Var}(\delta_y, \mu_t)\, d\mu_t(y) + H(t-s) + \tfrac{1}{2} H\delta\right)^{1/2} \\
&\leq 10\Psi_1(\delta, \beta \,|\, H, V)^{1/2} (V + HT + H\delta)^{1/2} + \Psi(\delta, \beta \,|\, H, V) < \Psi_2(\delta, \beta \,|\, H, V, T).
\end{aligned}$$

Combining the estimates above, we have

$$\int_{\mathcal{X}_t^\sigma \times \mathcal{X}_t} \mathrm{dist}_{W_1}^{Z_s} \left(\phi_{s*}\nu_{x^1\,|\,s'}, \psi_{s*}\nu_{x^2\,|\,s}\right) dq_t(x^1, x^2) < \Psi_2(\delta, \beta \,|\, H, V, T).$$

For a given $\epsilon > 0$, we may first choose $\beta \leq \bar{\beta}(\epsilon, H, V, T)$ and $\delta \leq \bar{\delta}(\epsilon, H, V, T)$ so that Ψ_2 above satisfies $\Psi_2 < \epsilon$. Then choose $\delta \leq \bar{\delta}'(\beta, \epsilon, H, V, T)$, such that

$$|E \cup (-H^{-1}\delta^3, 0]| \leq \frac{2A\delta}{\beta} + H^{-1}\delta^3 < \epsilon^2.$$

(Recall that $A = V + H(T+1)$.) Then, by the definition of the $\mathbb{F}$-distance, we have

$$\mathrm{dist}_{\mathbb{F}} \left(\left(\mathcal{X}_{[-T,0)}, (\mu_t)_{t \in [-T,0)}\right), \left(\mathcal{X}^\sigma_{[-T,0)}, (\mu^\sigma_t)_{t \in [-T,0)}\right)\right) < \epsilon.$$

$\square$

Using essentially the same arguments as above, we can show that the operation of parabolic scaling is continuous at scale 1 (and hence at any scale). We do not need the following proposition in this article, and the detailed proof is left to the reader.

Proposition 5.3.2. *For any H, V, $T < \infty$ and $\epsilon > 0$, there is $\delta = \delta(H,V,T,\epsilon) > 0$ suth that the following holds. Let $(\mathcal{X},(\mu_t))$ be an H-concentrated metric flow pair over $[-T-1,0]$. Suppose*

$$\sup_t \mathrm{Var}(\mu_t) \le V.$$

If $|\lambda - 1| \le \delta$, then

$$\mathrm{dist}_{\mathbb{F}}\left(\left(\mathcal{X}_{[-T,0]},(\mu_t)_{t\in[-T,0)}\right),\left(\mathcal{X}^{0,\lambda}_{[-T,0]},(\mu^{0,\lambda}_t)_{t\in[-T,0)}\right)\right) < \epsilon.$$

Proposition 5.3.3. *For any H, $T < \infty$ and $\epsilon > 0$, there is $\delta = \delta(\epsilon,H,T) > 0$ such that the following holds. Let $\mathcal{X}$ be an H-concentrated metric flow over $(-\infty,1)$ and $x_0 \in \mathcal{X}_0$. If $\sigma \in (0,\delta)$ and $y_0 \in \mathcal{P}^*(x_0;\delta) \cap \mathcal{X}_{-\sigma}$, then*

$$\mathrm{dist}_{\mathbb{F}}\left(\left(\mathcal{X}_{[-T,0]},(\nu_{x_0\,|\,t})_{t\in[-T,0]}\right),\left(\mathcal{X}^{\sigma}_{[-T,0]},(\nu^{\sigma}_{y_0\,|\,t})_{t\in[-T,0]}\right)\right) < \epsilon.$$

Here $\mathcal{P}^*$ (as well as $\mathcal{P}^{*+}$ and $\mathcal{P}^{*-}$ in the proof below) is the W_1-parabolic neighborhood defined in [Bam20b, Definition 3.38, Definition 3.39].

Proof. We may assume that $y_0 \in \mathcal{P}^{*-}(x_0;\delta)$ because if $y_0 \in \mathcal{P}^{*+}(x_0;\delta)$, then $x_0 \in \mathcal{P}^{*-}(y_0;\delta)$ and we switch the role of x and y.

Let $\delta = \delta_{5.3.1}(H, H(T+1), T, \epsilon/2) > 0$ be given by Proposition 5.3.1 such that

$$\mathrm{dist}_{\mathbb{F}}\left(\left(\mathcal{X}_{[-T,0]},(\nu_{x_0\,|\,t})_{t\in[-T,0]}\right),\left(\mathcal{X}^{\sigma}_{[-T,0]},(\nu^{\sigma}_{x_0\,|\,t})_{t\in[-T,0]}\right)\right) < \epsilon/2.$$

By the monotonicity of the W_1-Weinssernstein distance [Bam20b, Proposition 3.24(b)] and the definition of $\mathcal{P}^{*-}(x_0,\delta)$, for each $t \in [-T, -\delta^2]$, we have

$$\mathrm{dist}_{W_1}^{\mathcal{X}_t^\sigma}(\nu^\sigma_{x_0\,|\,t}, \nu^\sigma_{y_0\,|\,t}) \le \mathrm{dist}_{W_1}^{\mathcal{X}_{-\delta^2}}(\nu_{x_0\,|\,-\delta^2}, \nu_{y_0\,|\,-\delta^2}) < \delta.$$

By [Bam20b, Lemma 5.19], if $\delta < \epsilon/2$, then we have

$$\mathrm{dist}_{\mathbb{F}}\left(\left(\mathcal{X}^\sigma_{[-T,0]}, (\nu^\sigma_{x_0\,|\,t})_{t\in[-T,0]}\right), \left(\mathcal{X}^\sigma_{[-T,0]}, (\nu^\sigma_{y_0\,|\,t})_{t\in[-T,0]}\right)\right) < \epsilon/2,$$

The conclusion follows from the triangle inequality of $\mathrm{dist}_{\mathbb{F}}$.

□

We are now ready to prove Theorem 5.0.2 and Theorem 5.1.11.

Proof of Theorem 5.0.2 and Theorem 5.1.11. Suppose $x_0 \in \mathcal{X}_{t_0}$ and $y_0 \in \mathcal{X}_{s_0}$ with $s_0 \le t_0$. Suppose that $\lambda_j \to 0$ is a sequence such that

$$\left(\mathcal{X}^{-t_0,\lambda_j}_{[-T,0]}, (\nu^{-t_0,\lambda_j}_{x_0\,|\,t})_{t\in[-T,0]}\right) \xrightarrow[j\to\infty]{\mathbb{F}} \left(\mathcal{X}^\infty_{[-T,0]}, (\nu^\infty_{x_{\max}\,|\,t})_{t\in[-T,0]}\right) \in \mathcal{T}_{x_0},$$

for each $T < \infty$.

Suppose that $y_0 \in \mathcal{P}^*(x_0;\rho)$, for some $\rho < \infty$. In fact, such a number ρ exists because we may take z_0 to be an H-center of x_0 at time s_0 and take ρ to be any large number so that

$$\rho > \mathrm{dist}_{s_0}(y_0, z_0) + \sqrt{H(t_0-s_0)} + \sqrt{t_0-s_0}.$$

Then

$$\begin{aligned}\mathrm{dist}_{W_1}^{\mathcal{X}_{t_0-\rho^2}}(\nu_{y_0\,|\,t_0-\rho^2}, \nu_{x_0\,|\,t_0-\rho^2}) &\le \mathrm{dist}_{W_1}^{\mathcal{X}_{s_0}}(\delta_{y_0}, \nu_{x_0\,|\,s_0})\\ \le \mathrm{dist}_{s_0}(y_0,z_0) + \mathrm{dist}_{W_1}^{\mathcal{X}_{s_0}}(\delta_{z_0}, \nu_{x_0\,|\,s_0}) &\le \mathrm{dist}_{s_0}(y_0,z_0) + \sqrt{H(t_0-s_0)} < \rho.\end{aligned}$$

If we write

$$\tilde{\mathcal{X}}^j := \mathcal{X}^{-t_0,\lambda_j}, \quad \tilde{\nu}^j_{x\,|\,t} := \nu^{-t_0,\lambda_j}_{x\,|\,t} = \nu_{x\,|\,t_0+\lambda_j^{-2}t},$$

then

$$x_0 \in \tilde{\mathcal{X}}^j_0, \quad \mathcal{X}^{-s_0,\lambda_j} = (\tilde{\mathcal{X}}^j)^{\lambda_j^2(t_0-s_0)}, \quad y_0 \in \mathcal{P}^*_{\tilde{\mathcal{X}}^j}(x_0;\lambda_j\rho)\cap\tilde{\mathcal{X}}^j_{-\lambda_j^2(t_0-s_0)}.$$

Note that H-concentration is invariant under parabolic rescaling. Hence for any $\epsilon>0$, by Proposition 5.3.3, there is $\delta>0$ depending on ϵ, T, H such that if j is sufficiently large so that $\lambda_j\rho<\delta$, then

$$\begin{aligned}
&\operatorname{dist}_{\mathbb{F}}\left(\left(\mathcal{X}^{-t_0,\lambda_j}_{[-T,0]}, (\nu^{-t_0,\lambda_j}_{x_0\,|\,t})_{t\in[-T,0]}\right), \left(\mathcal{X}^{-s_0,\lambda_j}_{[-T,0]}, (\nu^{-s_0,\lambda_j}_{y_0\,|\,t})_{t\in[-T,0]}\right)\right)\\
=&\operatorname{dist}_{\mathbb{F}}\left(\left(\tilde{\mathcal{X}}^j_{[-T,0]}, (\tilde{\nu}^j_{x_0\,|\,t})_{t\in[-T,0]}\right), \left(\left(\tilde{\mathcal{X}}^j_{[-T,0]}\right)^{\lambda_j^2(t_0-s_0)}, \left(\left(\tilde{\nu}^j_{y_0\,|\,t}\right)^{\lambda_j^2(t_0-s_0)}\right)_{t\in[-T,0]}\right)\right) < \epsilon.
\end{aligned}$$

It follows from taking $\epsilon\to 0$ that $\mathcal{T}^\infty_{x_0}\subset\mathcal{T}^\infty_{y_0}$. The proof of the other direction is the same. $\square$

Chapter 5, in part, has been submitted for publication joint with Chan, Pak-Yeung and Zhang, Yongjia [CMZ21a]. Chapter 5 also contains material from [CMZ21c] which has been submitted for publication and is a joint work with Chan, Pak-Yeung and Zhang, Yongjia.

Chapter 6

On Steady Ricci Solitons

By definition, a triple (M^n, g, f) is called a **steady gradient Ricci soliton** or **steady GRS** in short, if f is a smooth function on M such that

$$\mathrm{Ric} = \nabla^2 f.$$

f is usually called the **potential function**. We shall assume that (M^n, g) is an n-dimensional *complete* Riemannian manifold.

Let $\{\Phi_t\}_{t\in\mathbb{R}}$ be the 1-parameter group of diffeomorphisms generated by $-\nabla f$, i.e.,

$$\partial_t \Phi_t = -\nabla f|_{\Phi_t}, \quad \Phi_0 = \mathrm{id}.$$

Then $g_t := \Phi_t^* g$ is a Ricci flow, called the **canonical form** of (M^n, g, f). It follows that

$$B_t(x, r) = \{y : |\Phi_t(x)\Phi_t(y)| < r\} = \Phi_{-t}(B(\Phi_t(x), r)),$$

for $x \in M, r > 0, t \in \mathbb{R}$.

As a consequence of the second Bianchi identity,

$$\nabla R = -2\mathrm{Ric}(\nabla f, \cdot).$$

As observed by Hamilton,

$$\nabla R = -2\mathrm{Ric}(\nabla f, \cdot) = -2\nabla^2 f(\nabla f, \cdot) = -\nabla|\nabla f|^2.$$

Thus $|\nabla f|^2 + R$ is constant on M. We may normalize the metric so that

$$|\nabla f|^2 + R = 1. \tag{6.0.1}$$

Throughout this chapter, we shall assume that the *smooth* steady GRS (M^n, g, f) under consideration either

1. arises as a finite-time singularity model; or
2. has bounded curvature, i.e., $\sup_M |\mathrm{Rm}| < \infty$.

These requirements are natural.

6.1 Preliminaries

Let us recall the full classification of three-dimensional shrinking gradient Ricci solitons.

Theorem 6.1.1. *Three-dimensional shrinking Ricci solitons can only be one of the following.*

$$\mathbb{S}^3/\Gamma, \qquad \mathbb{S}^2 \times \mathbb{R}, \qquad (\mathbb{S}^2 \times \mathbb{R})/\mathbb{Z}_2,$$

where Γ *is a discrete group.*

Proof. See Hamilton [Ha95, §26], Perelman [Per03a, Lemma 1.2], Cao, Chen, and Zhu [CCZ08], Ni and Wallach [NW08], and Petersen and Wylie [PW10]. □

6.2 Tangent Flows at Infinity in Dimension 4

Let us summarize Bamler's recent structure theory for noncollapsed limits of Ricci flows in dimension 4.

Theorem 6.2.1. *If $(M^4, g(t))$, $t \in (-\infty, 0]$, is a 4-dimensional singularity model on an orbifold with isolated singularities, then any tangent flow at infinity $(M^4_\infty, g_\infty(t))$, $t \in (-\infty, 0)$, of $(M, g(t))$ is a 4-dimensional, smooth, complete, shrinking gradient Ricci soliton on a Riemannian orbifold with (isolated) conical singularities. Moreover, either (M_∞, g_∞) is isometric to $\mathbb{R}^4/\Gamma$ for some nontrivial finite subgroup $\Gamma \subset O(4)$ or $R_{g_\infty(t)} > 0$ on all of M_∞. For each $t < 0$, the convergence to $(M_\infty, g_\infty(t))$ is in the smooth Cheeger–Gromov sense outisde of the discrete set of conical singularities.*

Proposition 6.2.2. *Any tangent flow at infinity of a nontrivial $(M^4, g_t)_{t\le 0}$ can only be one of the following*

$$\mathbb{R}^4/\Gamma' \text{ (but not } \mathbb{R}^4\text{)}, \quad (\mathbb{S}^3/\Gamma) \times \mathbb{R}, \quad \mathbb{S}^2 \times \mathbb{R}^2, \quad ((\mathbb{S}^2 \times \mathbb{R})/\mathbb{Z}_2) \times \mathbb{R}.$$

If a tangent flow at infinity is $\mathbb{R}^4/\Gamma$, then (M^4, g) is a Ricci-flat ALE space.

Proof. Firstly, we remark that the definition of a tangent flow at infinity, which uses a space-time basepoint $(x_0, t_0) \in M \times (-\infty, 0]$ and a sequence $\lambda_i \to 0$, may depend on λ_i but is independent of the choice of (x_0, t_0); see [Bam20b, Definition 6.55] and [CMZ21b, Theorem 1.6](= Theorem 5.0.2). By [Bam20b, Theorem 6.58], any tangent flow at infinity of a finite time singularity model can be realized as an $\mathbb{F}$-limit of a sequence of compact Ricci flows (rescalings of the original Ricci flow). By [Bam20c, §2.7], the Nash entropy of the sequence is uniformly bounded away from $-\infty$ and thus the tangent flows at infinity of singularity models always exist (even if they do not have bounded curvature).

We claim that each tangent flow at infinity is either $\mathbb{R}^4/\Gamma$ ($\Gamma \neq 1$ by [Bam20c, Theorem 2.40]) or splits off a line. In the latter case, since it is a smooth orbifold with conical singularities, by Theorem 6.2.1 it must be the product of $\mathbb{R}$ with a complete shrinking gradient Ricci soliton (not necessarily with bounded curvature) on a 3-dimensional smooth manifold with $R > 0$. The proposition now follows since these have been classified as $\mathbb{S}^3/\Gamma$, $\mathbb{S}^2 \times \mathbb{R}$ and $(\mathbb{S}^2 \times \mathbb{R})/\mathbb{Z}_2$; see Theorem 6.1.1.

Now $\mathbb{F}$-convergence (see [Bam20b, Definition 6.2], when the limit is an orbifold with conical singularities, can be upgraded to pointed Cheeger–Gromov convergence with respect to H_n-centers smoothly on compact subsets of the limit minus the conical singularities; see [Bam20b, §9.4].

To prove the claim in the first paragraph of this proof, we consider two cases: (1) ∇f remains locally bounded and (2) ∇f goes to infinity. Suppose that the rescalings $(M, \lambda_i g(-\lambda_i^{-1}), z_i)$ of a steady soliton model, where $(z_i, -\lambda_i^{-1})$ is an H_n-center of $(x_0, 0)$ and $\lambda_i \to 0$, limit to a complete shrinking gradient Ricci soliton $(M_\infty, g_\infty, z_\infty)$ on a 4-orbifold with conical singularities, after pulling back by diffeomorphisms ϕ_i. Let $\mathcal{S}_\infty$ denote the set of conical singularities of M_∞, which is a discrete set of points, and let $\mathcal{R}_\infty = M_\infty - \mathcal{S}_\infty$.

We may assume that the steady soliton solution to the Ricci flow $g(t)$ is equal to $\Phi_t^* g$, where Φ_t is the 1-parameter group of diffeomorphisms generated by $-\nabla_g f$. We define $f(x,t) = f(\Phi_t(x))$, so that $\mathrm{Ric}_{g(t)} = \nabla^2_{g(t)} f(t)$. Let $g_i = \lambda_i g(-\lambda_i^{-1})$ and $f_i := f(\cdot, -\lambda_i^{-1})$. We have $z_\infty \in \mathcal{R}_\infty$ and we have smooth pointed Cheeger–Gromov convergence of (M, g_i, z_i) to the limit on compact subsets of $\mathcal{R}_\infty$ (see [Bam20b, §9]).

Case 1: Suppose that, for a subsequence, $|df_i|_{g_i}(z_i)$ is uniformly bounded. Pass to this subsequence. Let $\bar{f}_i = f_i - f_i(z_i)$. From the smooth convergence, we have that $|\mathrm{Rm}_{g_i}|$ is uniformly bounded away from the conical singularities of the limit (after pulling back by the diffeomorphisms ϕ_i). In particular, the consequent Ricci curvature bound and the steady soliton equation imply that $|\nabla^2_{g_i} \bar{f}_i|_{g_i} \leq C$ on compact subsets of $\mathcal{R}_\infty$. Since $\mathcal{R}_\infty$ is connected and $|d\bar{f}_i|_{g_i}(z_i) \leq C$, this implies that $|d\bar{f}_i|_{g_i} \leq C$ on compact subsets of $\mathcal{R}_\infty$.

Thus, by $\bar{f}_i(z_i) = 0$, $|\nabla \bar{f}_i|_{g_i} \leq C_1(d(\cdot, z_i))$, and Shi's local derivative of curvature estimates, we have that $|\nabla^k \bar{f}_i|_{g_i} \leq C_k(d(\cdot, z_i))$ for all $k \geq 0$ on compact subsets of $\mathcal{R}_\infty$. Hence the $\bar{f}_i$ subconverge to a smooth function f_∞ on $\mathcal{R}_\infty$. By taking the limit of the steady soliton equation $\mathrm{Ric}_{g_i} = \nabla^2_{g_i} \bar{f}_i$, we obtain $\mathrm{Ric}_{g_\infty} = \nabla^2_{g_\infty} f_\infty$ on M_∞ minus the conical singularities. On the other hand, since (M_∞, g_∞) has a shrinking gradient Ricci soliton structure, there exists a function f_0 such that $\mathrm{Ric}_{g_\infty} = \nabla^2_{g_\infty} f_0 + \frac{1}{2} g_\infty$, so that $h := f_\infty - f_0$ satisfies $\mathcal{L}_{\nabla h} g_\infty = 2\nabla^2_{g_\infty} h = g_\infty$ on M_∞ minus the conical singularities. By adjusting h by an additive constant if necessary, this implies that $|\nabla h|^2_{g_\infty} = \frac{1}{2} h$. Hence $\rho := 2\sqrt{h}$ satisfies $|\nabla \rho|_{g_\infty} \equiv 1$ and $\nabla_{\nabla \rho} \nabla \rho \equiv 0$ on $\mathcal{R}_\infty$, so that the integral curves of $\nabla \rho$ are unit speed geodesics. This implies that (M_∞, g_∞) is a flat cone whose cross sections are the level sets of h.

Since the conical singularities are orbifold points, this implies that $(M_\infty, g_\infty) = \mathbb{R}^4/\Gamma$, where Γ is a finite subgroup of $O(4)$. Therefore, on (M, g), we have $R_g(w_i) = \lambda_i R_{g_i}(z_i) \to 0$, where $w_i = \Phi_{-1/\lambda_i}(z_i)$. We also have that $|df|^2_g(w_i) = \lambda_i |df_i|^2_{g_i}(z_i) \to 0$. So on (M, g), $R + |df|^2 = C = 0$, which implies $\mathrm{Ric}_g = 0$. Since the steady soliton singularity model has $R_g \equiv 0$, by the first author's generalization of Perelman's no local collapsing theorem [Bam20a, Theorem 6.1], there exists $\kappa > 0$ such that $\mathrm{Vol}_g(B^g_r(x_0)) \geq \kappa r^4$ for $r > 0$; hence, by definition, g has Euclidean volume growth. It now follows from Cheeger and Naber [CN15, Corollary 8.85] that (M, g) is an ALE space. Note that $\Gamma \neq 1$ also follows from the equality case of the Bishop–Gromov volume comparison theorem.

Case 2: Suppose that, for a subsequence, $|df_i|_{g_i}(z_i) := \beta_i^{-1} \to \infty$. Pass to this subsequence. Let $\bar{f}_i := \beta_i(f_i - f_i(z_i))$. Then $\bar{f}_i(z_i) = 0$, $|d\bar{f}_i|_{g_i}(z_i) = 1$, and $\nabla^2_{g_i} \bar{f}_i \to 0$ on compact subsets of $\mathcal{R}_\infty$. Again, we have higher derivative estimates for $\bar{f}_i$. Thus, the $\bar{f}_i$ subconverge to a smooth function f_∞ on $\mathcal{R}_\infty$ satisfying $\nabla^2_{g_\infty} f_\infty = 0$ on $\mathcal{R}_\infty$ and $|df_\infty|_{g_\infty}(z_\infty) = 1$. This implies the splitting of $(\mathcal{R}_\infty, g_\infty)$. Since the singularities are conical, there are no singularities and hence (M_∞, g_∞) splits. $\square$

The discreteness of the space of 3-dimensional shrinking solitons occurring in

Proposition 6.2.2 implies the following.

Proposition 6.2.3. *Any 4-dimensional steady gradient Ricci soliton singularity model $(M^4, g(t))$, with potential function $f(t)$, has a unique tangent flow at infinity.*

Proof. If one tangent flow at infinity is $\mathbb{R}^4/\Gamma$, then $(M, g(t))$ is a Ricci flat ALE space as we have seen in the proof of Proposition 6.2.2, and thus in this case any tangent flow at infinity is $\mathbb{R}^4/\Gamma$. So we may assume that no tangent flow at infinity is $\mathbb{R}^4/\Gamma$.

Let $\mathcal{X}$ be the metric flow induced by the Ricci flow $(M^4, g(t))$; see [Bam20b, Definition 3.2]. Let $I = [-2, -1/2]$ and let

$$\mathcal{T} := \big\{\text{metric solitons } (\mathcal{Y}, (\mu_t)) \text{ that arise as} \\ \text{tangent flows at infinity of } \mathcal{X}, \text{ restricted to } I\big\};$$

see [Bam20b, Definition 3.57] for the definition of metric soliton, and see [Bam20b, Definition 3.10] for the definition of the restriction of a metric flow. By Proposition 6.2.2, the elements of $\mathcal{T}$ are the metric solitons associated to $N \times \mathbb{R}$, where N is a 3-dimensional complete shrinking gradient Ricci soliton structure that is isometric to $\mathbb{S}^3/\Gamma$, $\mathbb{S}^2 \times \mathbb{R}$, or $(\mathbb{S}^2 \times \mathbb{R})/\mathbb{Z}_2$. Note that these are the splitting quotients of $\mathbb{S}^k \times \mathbb{R}^{4-k}$, with the metrics $2(k-1)g_{\mathbb{S}^k} + g_{\mathbb{R}^{4-k}}$, $k = 2, 3$. Hence the metric space $(\mathcal{T}, d_{\mathbb{F}}^J)$ is discrete, where $d_{\mathbb{F}}^J$ denotes the $\mathbb{F}$-distance introduced in [Bam20b, §5.1] and where J is taken to be $\{-1\}$ for convenience. By [Bam20b, Theorem 7.4], $\mathcal{T}$ is compact and thus finite.

Let 10ϵ be the smallest distance between elements of $(\mathcal{T}, d_{\mathbb{F}}^J)$ and suppose that this distance is attained by $(\mathcal{Y}_I^k, \mu_t^k) \in \mathcal{T}, k = 0, 1$, i.e.,

$$10\epsilon = d_{\mathbb{F}}^J\left((\mathcal{Y}_I^0, (\mu_t^0)), (\mathcal{Y}_I^1, (\mu_t^1))\right).$$

Then there are sequences of scales $\lambda_{k,j} \to 0$ as $j \to \infty$ such that

$$\lim_{j\to\infty} d_{\mathbb{F}}^J \left((\mathcal{Y}_I^k, (\mu_t^k)), \left(\mathcal{X}_I^{0,\lambda_{k,j}}, (\nu_{x_0;t}^{0,\lambda_{k,j}}) \right) \right) \to 0,$$

for $k = 0, 1$ and where $\mathcal{X}^{-\Delta T,\lambda}$ denotes the time-shift by $-\Delta T$ and then parabolic rescaling by λ of $\mathcal{X}$ as in [Bam20b, §6.8].

By discarding some scales, we may assume that $\lambda_{0,j} < \lambda_{1,j}$. There is a $\bar{j}$ such that if $j \geq \bar{j}$,

$$d_{\mathbb{F}}^J \left((\mathcal{Y}_I^k, (\mu_t^k)), \left(\mathcal{X}_I^{0,\lambda_{k,j}}, (\nu_{x_0;t}^{0,\lambda_{k,j}}) \right) \right) < \epsilon.$$

It follows that

$$d_{\mathbb{F}}^J \left(\big(\mathcal{X}_I^{0,\lambda_{0,j}}, (\nu_{x_0;t}^{0,\lambda_{0,j}})\big), \big(\mathcal{X}_I^{0,\lambda_{1,j}}, (\nu_{x_0;t}^{0,\lambda_{1,j}})\big) \right) > 8\epsilon.$$

Note that there is a continuous curve connecting the two rescaled flows:

$$\gamma_j(\eta) = \big(\mathcal{X}_I^{0,\eta}, (\nu_{x_0;t}^{0,\eta})\big)$$

for $\eta \in [\lambda_{0,j}, \lambda_{1,j}]$. So there is some $\eta_j \in (\lambda_{0,j}, \lambda_{1,j})$ such that

$$d_{\mathbb{F}}^J \left(\gamma_j(\eta_j), \big(\mathcal{X}_I^{0,\lambda_{0,j}}, (\nu_{x_0;t}^{0,\lambda_{0,j}})\big) \right) \in [2\epsilon, 4\epsilon]\,;$$

meanwhile,

$$d_{\mathbb{F}}^J \left(\gamma_j(\eta_j), \big(\mathcal{X}_I^{0,\lambda_{1,j}}, (\nu_{x_0;t}^{0,\lambda_{1,j}})\big) \right) > 2\epsilon.$$

By the existence of tangent flows at infinity, a subsequence of $\gamma_j(\eta_j)$ converges to a splitting metric soliton $(\mathcal{Z}, (\mu_t))$. Hence

$$d_{\mathbb{F}}^J \big((\mathcal{Z}_I, (\mu_t)), (\mathcal{Y}_I^0, (\mu_t^0)) \big) \in [2\epsilon, 4\epsilon], \quad d_{\mathbb{F}}^J \big((\mathcal{Z}_I, (\mu_t)), (\mathcal{Y}_I^1, (\mu_t^1)) \big) \geq 2\epsilon,$$

which is a contradiction to the definition of ϵ.

□

6.3 Structure of 3-Cylindrical Steady Solitons

In this section, we aim to prove the following main theorem in [BCDMZ].

Theorem 6.3.1. *Let (M^4, g, f) be a 4-dimensional complete steady gradient Ricci soliton that is a singularity model. Then the tangent flow at infinity is unique. If the tangent flow at infinity is $(\mathbb{S}^3/\Gamma)\times\mathbb{R}$, then, for any $\epsilon > 0$, outisde a compact set we have that each point is the center of an ϵ-neck, has positive curvature operator, and linear curvature decay.*

We shall mainly focus on proving that outside a compact set, each point is the center of an ϵ-neck, which is similar to Perelman's canonical neighborhood theorem [Per02, §12]. Our proof here is slightly simpler than [BCDMZ, Proposition] and we will use the rough continuity of H_n-centers proved in Proposition 3.3.15.

Proposition 6.3.2. *Let $(M^n, g_t)_{t\in\mathbb{R}}$ be the canonical form of a steady GRS. Fix $o \in M$. Suppose that (M^n, g_t) is noncollapsed, i.e.,*

$$\mu_\infty := \inf_{\tau>0} \mathcal{N}_{o,0}(\tau) > -\infty.$$

Let (z_s, s) be an H_n-center of $(o, 0)$ for each $s < 0$. Then for $s < t < -1, t - s < 1$, we have

$$|z_t z_s|_s \leq C(n, \mu_\infty)\sqrt{|s|}.$$

Proof. Note that $R \leq 1$ on a (normalized) steady GRS. So this is a direct application of Proposition 3.3.15. □

When a tangent flow at infinity is $(\mathbb{S}^3/\Gamma)\times\mathbb{R}$, we obtain a canonical neighborhood-type result. The idea of the proof is that in lieu of proving continuity of H_n-centers (which are not unique) in the variable λ, we show an overlapping property for ϵ-necks centered at H_n-centers.

Proposition 6.3.3. *Suppose that a 4-dimensional steady gradient Ricci soliton singularity model $(M^4, g(t), f(t))$ has a tangent flow at infinity isometric to $(\mathbb{S}^3/\Gamma) \times \mathbb{R}$. Then, for any $\epsilon > 0$, there exists a compact set $K_\epsilon \subset M$ such that any $x \in M - K_\epsilon$ is the center of an ϵ-neck with respect to $g = g(0)$.*

Proof. By Proposition 6.2.3, there exists a finite subgroup Γ of $O(4)$ such that each tangent flow at infinity of $(M, g(t))$ is $((\mathbb{S}^3/\Gamma) \times \mathbb{R}, g_{\mathrm{cyl}})$, where

$$g_{\mathrm{cyl}} = 4g_{\mathbb{S}^3/\Gamma} + g_{\mathbb{R}}.$$

Let μ_∞ be the shrinker entropy of g_{cyl}. Let $A \geq \bar{A}(n, \mu_\infty)$ be a constant such that

$$A \geq 10C(n, \mu_\infty),$$

where $C(n, \mu_\infty)$ is given by Proposition 6.3.2.

Let $\lambda > 0$, let $(z_\lambda, -1/\lambda)$ be an H_4-center of $(x_0, 0)$, and define $g_\lambda(t) = \lambda g(t/\lambda)$. By the above, there exist $\epsilon = \epsilon(\lambda) > 0$ and a diffeomorphism $\Psi_\lambda : B^{\mathrm{cyl}}_{1/\epsilon} \to B(z_\lambda, 1/\epsilon; g_\lambda(-1))$ such that $\lim_{\lambda \to 0} \epsilon(\lambda) = 0$ and

$$\|\Psi_\lambda^* g_\lambda(-1) - g_{\mathrm{cyl}}\|_{C^{[1/\epsilon]}(B^{\mathrm{cyl}}_{1/\epsilon})} \leq \epsilon,$$

where $B^{\mathrm{cyl}}_{1/\epsilon}$ denotes a ball of radius $1/\epsilon$ in $((\mathbb{S}^3/\Gamma) \times \mathbb{R}, g_{\mathrm{cyl}})$. That is, z_λ is the center of an ϵ-neck in $(M, g_\lambda(-1))$. Note that $g_\lambda(-1) = \lambda \Phi^*_{-1/\lambda} g$, where $g := g(0)$ and $\Phi_t : M \to M$ is the 1-parameter group of diffeomorphisms generated by $-\nabla_g f$. We have the composition of diffeomorphisms

$$B^{\mathrm{cyl}}_{1/\epsilon} \xrightarrow{\Psi_\lambda} B(z_\lambda, 1/\epsilon\,; g_\lambda(-1)) \xrightarrow{\Phi_{-1/\lambda}} B\big(w_\lambda, 1/(\sqrt{\lambda}\epsilon); g\big) =: \mathfrak{N}_\lambda,$$

where $w_\lambda := \Phi_{-1/\lambda}(z_\lambda)$. So

$$\|\lambda(\Phi_{-1/\lambda} \circ \Psi_\lambda)^* g - g_{\mathrm{cyl}}\|_{C^{[1/\epsilon]}(B^{\mathrm{cyl}}_{1/\epsilon})} \leq \epsilon.$$

In particular,

$$|\mathrm{Rm}_g|(x) \sim c\lambda \quad \text{for all } x \in \mathfrak{N}_\lambda.$$

Choose $\bar{\lambda} > 0$ to be small enough so that if $\lambda \leq \bar{\lambda}$, then $\epsilon(\lambda) < 10^{-6}$ and

$$V_\lambda := B\big(z_\lambda, A\,; g_\lambda(-1)\big) = B_{-1/\lambda}\big(z_\lambda, A/\sqrt{\lambda}\big)$$

is diffeomorphic to the corresponding ball in $(\mathbb{S}^3/\Gamma) \times \mathbb{R}$. Write

$$U_\lambda := B\big(w_\lambda, A/\sqrt{\lambda}\big) = \Phi_{-1/\lambda}(V_\lambda). \tag{6.3.1}$$

We will next show that

$$M - K_0 \subset \bigcup_{\lambda > 0} 10 U_\lambda$$

for some compact set K_0, where we denote by

$$\alpha B(x, r; g) := B(x, \alpha r; g)$$

for any $\alpha > 0$. This suffices to show that every point outside of K_0 is the center of an ϵ-neck if we enlarge ϵ slightly.

Claim: For any $\lambda_0 > 0$, there is a $\delta(\lambda_0) > 0$ such that if $|\lambda - \lambda_0| < \delta$, then

$$U_\lambda \cap U_{\lambda_0} \neq \emptyset.$$

Proof of the claim. Let $|\lambda^{-1} - \lambda_0^{-1}| < 1$ and we shall consider the case where $-\lambda^{-1} \leq -\lambda_0^{-1}$.

The proof of the case where $-\lambda_0^{-1} \leq -\lambda^{-1}$ is similar. We write

$$s = -1/\lambda, \quad t = -1/\lambda_0.$$

By Proposition 6.3.2,

$$|\Phi_s(z_s)\Phi_s(z_t)| = |z_s z_t|_s \leq C(n, \mu_\infty)\sqrt{|s|} \leq A\sqrt{|s|}/10.$$

Since $|\nabla f| \leq 1$, for any $x \in M$,

$$|\Phi_s(x)\Phi_t(x)| \leq \int_s^t |\partial_\tau \Phi_\tau(x)| \, d\tau \leq (t-s) < 1.$$

So

$$|w_s w_t| = |\Phi_s(z_s)\Phi_t(z_t)| \leq A\sqrt{|s|}/5,$$

and thus $U_\lambda \cap U_{\lambda_0} \neq \emptyset$.

$\square$

By Munteanu and Wang [MW11], M is connected at infinity if it does not split for *smooth* steady solitons. Thus $M - U_\lambda$ has two components when $\lambda < \bar{\lambda}$. Let W_λ^∞ be the unbounded component of $M - U_\lambda$ and let $W_\lambda^0 = M - W_\lambda^\infty$, which is clearly bounded.

Now let $K_0 = \overline{W_{\bar{\lambda}}^0}$. Then K_0 is compact. Fix $x \notin K_0$. Consider

$$\Lambda := \left\{\lambda \in (0,1) : x \in W_\lambda^\infty\right\}.$$

Let $\lambda_0 = \inf \Lambda$. We claim that $\lambda_0 \in (0, \bar{\lambda}]$. In fact, $\lambda_0 \leq \bar{\lambda}$ directly follows from the definition. If $\lambda_0 = 0$, then there is a sequence $\lambda_j \to 0$ such that $x \in W_{\lambda_j}^\infty$ and thus there is a sequence $y_j \in \partial W_{\lambda_j}^\infty \subset \partial U_{\lambda_j}$ that stays bounded. By passing to a subsequence, we may assume that $y_j \to y$ for some point $y \in M$. Then $|\mathrm{Rm}|(y) = \lim_{j\to\infty} |\mathrm{Rm}|(y_j) \leq$

$\lim_{j\to\infty} C_n\lambda_j = 0$, which is a contradiction to the assumption that $R > 0$ on M.

By definition, there exists $\lambda_1 \geq \lambda_0$ such that $\lambda_1 \in \Lambda$ and $\lambda_1 - \lambda_0 < \delta(\lambda_0)/2$. Pick $\lambda_2 \in (0, \lambda_0)$ such that $\lambda_0 - \lambda_2 < \delta/2$. We proved above that

$$U_{\lambda_1} \cap U_{\lambda_2} \neq \emptyset.$$

Since $x \in W^0_{\lambda_2}$, we have $x \in 10U_{\lambda_1}$. Thus

$$M - K_0 \subset \bigcup_{\lambda>0} 10U_\lambda.$$

As $10 \ll \frac{1}{10\epsilon(\lambda)}$ and $10U_\lambda$ lies in the middle of the neck region $\mathfrak{N}_\lambda := \frac{1}{10\epsilon(\lambda)}U_\lambda$, we have that every point outside of K_0 is the center of an ϵ-neck. This completes the proof of the proposition. $\square$

As a result, we can see that if $(M^4, g(t), f(t))$ is a steady gradient Ricci soliton singularity model whose tangent flow at infinity is $(\mathbb{S}^3/\Gamma) \times \mathbb{R}$, then it is **asymptotically (quotient) cylindrical** in the following sense: for any sequence $x_j \to \infty$,

$$(M, R(x_j)g, x_j) \to ((\mathbb{S}^3/\Gamma) \times \mathbb{R}, \bar{g}, x_\infty)$$

(without passing to a subsequence), where $\bar{g}$ is the rescaling of the standard cylindrical metric with scalar curvature $R(\bar{g}) = 1$. In fact, for any $x_j \to \infty$, by the last proposition, $x_j \in 10U_{\lambda_j}$ for some $\lambda_j > 0$. Since $R(x_j) = 1.5\lambda_j + o(1)$ and $10U_{\lambda_j} \subset \mathfrak{N}_{\lambda_j}$ is an ϵ-neck, we have the convergence.

We refer to [BCDMZ] for the rest of the proof of Theorem 6.3.1. We remark that Deng and Zhu [DZ20b] proved that if a 4-dimensional steady GRS is asymptotically cylindrical, then it has positive curvature operator outside a compact set and the curvature decays linearly. In [BCDMZ], we proved this result for general dimensions.

6.4 Location of ℓ-centers

Since the canonical form of the steady soliton (M^n, g, f) moves only by diffeomorphism, we may work with Perelman's $\mathcal{L}$-geometry [Per02, §7] on the background of the static manifold (M^n, g).

As mentioned before, we will use g_t to represent the canonical form of the steady soliton (M^n, g, f) satsifying the conditions of Theorem 6.5.1. Recall that Perelman defined the $\mathcal{L}$-length in [Per02, §7]. For any $\tau > 0$, and any piecewisely smooth curve $\Gamma : [0, \tau] \to M$ with $\Gamma(0) = o$,

$$\mathcal{L}(\Gamma) := \int_0^\tau \sqrt{s}(R_{g_{-s}} + |\dot{\Gamma}|^2_{g_{-s}})(\Gamma(s))\, ds.$$

To reinterpret the $\mathcal{L}$-geometry on the static background (M, g), let

$$\gamma(s) = \Phi_{-s}(\Gamma(s)) \quad \text{for} \quad s \in [0, \tau].$$

Then

$$\dot{\gamma} = \nabla f|_\Gamma + \Phi_{-s*}(\dot{\Gamma}),$$

and

$$\mathcal{L}(\Gamma) = \int_0^\tau \sqrt{s}\big(R_g + |\dot{\gamma} - \nabla f|^2_g\big)(\gamma(s))\, ds,$$

and this expression only uses the static metric g. If we perform a change of variables: $u = \sqrt{s}$, and write $\tilde{\gamma}(u) = \gamma(u^2)$, then

$$\mathcal{L}(\Gamma) = \int_0^{\sqrt{\tau}} \Big(\tfrac{1}{2}\left|\dot{\tilde{\gamma}} - 2u\nabla f\right|^2 + 2u^2 R(\tilde{\gamma}(u))\Big)\, du.$$

For any $x \in M$ and $\tau > 0$, we define

$$L(\Phi_\tau(x), \tau) := \inf_\Gamma \mathcal{L}(\Gamma),$$

where the infimum is taken over all $\Gamma : [0, \tau] \to M$ with $\Gamma(0) = o$ and $\Gamma(\tau) = \Phi_\tau(x)$. On the static metric background, we may define an equivalent function:

$$\Lambda(x, \tau) := L(\Phi_\tau(x), \tau) = \inf \int_0^\tau \sqrt{s}(R_g + |\dot{\gamma} - \nabla f|_g^2)(\gamma(s))\, ds, \tag{6.4.1}$$

where the infimum is taken over all $\gamma : [0, \tau] \to M$ with $\gamma(0) = o$ and $\gamma(\tau) = x$, and a curve at which the above infimum is attained shall be called a **Λ-geodesic**. Accordingly, define

$$\lambda(x, \tau) := \ell(\Phi_\tau(x), \tau) := \frac{1}{2\sqrt{\tau}} \Lambda(x, \tau).$$

Arguing as Perelman in [Per02, Section 7.1], we have that, for any $\tau > 0$, there is a point $p_\tau \in M$ such that $\lambda(p_\tau, \tau) = \ell(\Phi_\tau(p_\tau), \tau) \leq n/2$. Any such point p_τ is called an **ℓ-center** at time $-\tau$. Note that in our current case we are considering the ℓ-center on a static metric background, hence it differs from the ℓ-center defined in [CMZ21a] by a diffeomorphism.

Lemma 6.4.1. $\lambda(o, \tau) \geq \tau/12$, *for any* $\tau > 0$.

Proof. Let $\gamma : [0, \tau] \to M$ be a loop at o and let $\tilde{\gamma} : [0, \sqrt{\tau}] \to M$ be the reparametrization: $\tilde{\gamma}(u) = \gamma(u^2)$. Then

$$\begin{aligned}
\int_0^\tau \sqrt{s}(R + |\dot{\gamma} - \nabla f|^2) &= \int_0^{\sqrt{\tau}} \left(\tfrac{1}{2} \left|\dot{\tilde{\gamma}} - 2u \nabla f\right|^2 + 2u^2 R(\tilde{\gamma}(u)) \right) du. \\
&= \int_0^{\sqrt{\tau}} \left(\tfrac{1}{2} |\dot{\tilde{\gamma}}|^2 - 2u(f \circ \tilde{\gamma} - f(o))' + 2u^2 \right) du \\
&= \frac{2}{3}\tau^{3/2} + \int_0^{\sqrt{\tau}} \left(\tfrac{1}{2} |\dot{\tilde{\gamma}}|^2 + 2(f \circ \tilde{\gamma}(u) - f(o)) \right) du,
\end{aligned} \tag{6.4.2}$$

where in the second equality we have applied (6.0.1). Let $F(u) = f \circ \tilde{\gamma}(u) - f(o)$ and define

$$L := \sup_{u \in [0, \sqrt{\tau}]} \operatorname{dist}(o, \tilde{\gamma}(u)) =: \operatorname{dist}(o, \tilde{\gamma}(u_1)),$$

for some $u_1 \in [0, \sqrt{\tau}]$. Then we have

$$\begin{aligned}\frac{1}{2}\int_0^{\sqrt{\tau}} |\dot{\tilde{\gamma}}|^2 &\geq \frac{1}{2}\int_0^{u_1} |\dot{\tilde{\gamma}}|^2 + \frac{1}{2}\int_{u_1}^{\sqrt{\tau}} |\dot{\tilde{\gamma}}|^2 \\ &\geq \frac{L^2}{2}\left(\frac{1}{u_1} + \frac{1}{\sqrt{\tau} - u_1}\right) \geq \frac{2L^2}{\sqrt{\tau}},\end{aligned}$$

where we have applied the Cauchy-Schwarz inequality (e.g. $L^2 \leq (\int_0^{u_1} |\dot{\tilde{\gamma}}|)^2 \leq \int_0^{u_1} |\dot{\tilde{\gamma}}|^2 \cdot \int_0^{u_1} 1^2$). Since $|\nabla f| \leq 1$ by (6.0.1), we have

$$|F(u)| \leq \operatorname{dist}(\tilde{\gamma}(u), o) \leq L, \quad \forall\, u \in [0, \sqrt{\tau}],$$

and thus

$$\int_0^{\sqrt{\tau}} 2(f \circ \tilde{\gamma}(u) - f(o))du \geq -2L\sqrt{\tau}.$$

In summary, we have

$$\begin{aligned}\int_0^{\tau} \sqrt{s}(R + |\dot{\gamma} - \nabla f|^2) &\geq \frac{2}{3}\tau^{3/2} + \frac{2L^2}{\sqrt{\tau}} - 2L\sqrt{\tau} \\ &= \frac{2}{3}\tau^{3/2} + \frac{2}{\sqrt{\tau}}(L^2 - L\tau) \\ &= \frac{1}{6}\tau^{3/2} + \frac{2}{\sqrt{\tau}}(L - \tau/2)^2 \geq \frac{1}{6}\tau^{3/2},\end{aligned}$$

and the conclusions follow by taking the infimum on the left hand side.

□

The following Lemma is straightforward and is similar to the standard triangle inequality; c.f. [CMZ21a, §4, Claim 3].

Lemma 6.4.2. *For any $x, y \in M, \tau > 0$ and any $\delta \in (0, 1)$,*

$$\lambda(x, (1+\delta)^2\tau) \leq \lambda(y, \tau) + \frac{\operatorname{dist}^2(x, y)}{\delta\tau} + 5\delta\tau.$$

Proof. Let $\gamma_1 : [0, \tau] \to M$ be a minimizing Λ-geodesic from o to y, namely, a curve at which the infimum in (6.4.1) is attained. Let $\tilde{\gamma}_2 : [\sqrt{\tau}, (1+\delta)\sqrt{\tau}] \to M$ be a minimizing g-geodesic from y to x with constant speed. Define $\gamma_2 : [\tau, (1+\delta)^2\tau] \to M$ by $\gamma_2(s) = \tilde{\gamma}_2(\sqrt{s})$.

$$\begin{aligned}
\Lambda(x, (1+\delta)^2\tau) \le & \int_0^\tau \sqrt{s}(R + |\dot{\gamma}_1 - \nabla f|^2)(\gamma_1(s))\, ds \\
& + \int_\tau^{(1+\delta)^2\tau} \sqrt{s}(R + |\dot{\gamma}_2 - \nabla f|^2)(\gamma_2(s))\, ds \\
& \le \Lambda(y, \tau) + \int_{\sqrt{\tau}}^{(1+\delta)\sqrt{\tau}} \left(\tfrac{1}{2}|\dot{\tilde{\gamma}}_2|^2 + 2u|\dot{\tilde{\gamma}}_2||\nabla f| + 2u^2(R + |\nabla f|^2)\right) du \\
& \le \Lambda(y, \tau) + \int_{\sqrt{\tau}}^{(1+\delta)\sqrt{\tau}} \left(|\dot{\tilde{\gamma}}_2|^2 + 4u^2\right) du \\
& \le \Lambda(y, \tau) + \frac{\mathrm{dist}^2(x, y)}{\delta\sqrt{\tau}} + 4\frac{(1+\delta)^3 - 1}{3}\tau^{3/2} \\
& \le \Lambda(y, \tau) + \frac{\mathrm{dist}^2(x, y)}{\delta\sqrt{\tau}} + 10\delta\tau^{3/2}.
\end{aligned}$$

The conclusion follows by dividing $2(1+\delta)\sqrt{\tau}$ on both sides. $\square$

Lemma 6.4.3. *There is a universal constant $\alpha \in (0, 1)$, such that for any $\tau \ge \bar{\tau}(n)$ and any ℓ-center p_τ, we have*

$$\mathrm{dist}(p_\tau, o) \ge \alpha\tau.$$

Proof. By Lemma 6.4.1 and Lemma 6.4.2, for any $\delta \in (0, 1)$, if $\tau \ge \bar{\tau}(n, \delta)$, then we have

$$\begin{aligned}
\frac{(1+\delta)^2\tau}{12} \le \lambda(o, (1+\delta)^2\tau) \le \lambda(p_\tau, \tau) + \frac{\mathrm{dist}^2(p_\tau, o)}{\delta\tau} + 5\delta\tau \\
\le \frac{\mathrm{dist}^2(p_\tau, o)}{\delta\tau} + 10\delta\tau,
\end{aligned}$$

where we have used the fact that $\lambda(p_\tau, \tau) \le \frac{n}{2}$. We may take, e.g., $\delta = 10^{-3}$ to obtain the inequality.

$\square$

Lemma 6.4.4. *For any $\tau \ge \bar{\tau}(n)$, there is $x_\tau \in M$ such that $\mathrm{dist}(x_\tau, o) = \tau$ and*

$\lambda(x_\tau, \tau_0) \leq C$ for some $\tau_0 \in [c\tau, \tau/\alpha]$, where $c > 0$ and $C < \infty$ are dimensional constants and α is given by Lemma 6.4.3.

Proof. Let $\gamma : [0, \tau/\alpha] \to M$ be a minimizing Λ-geodesic from o to $p := p_{\tau/\alpha}$. By Lemma 6.4.3, $\operatorname{dist}(p, o) \geq \tau$. So we can define

$$\tau_0 := \sup\{s \in [0, \tau/\alpha] : \operatorname{dist}(\gamma(s), o) \leq \tau\}, \quad x_\tau := \gamma(\tau_0).$$

We first show that $\tau_0 \geq c\tau$ for some universal constant $c > 0$. Define $\tilde{\gamma} : \left[0, \sqrt{\tau/\alpha}\right] \to M$ by $\tilde{\gamma}(u) = \gamma(u^2)$. Note that, arguing in the same way as (6.4.2), we have

$$\begin{aligned}
\frac{1}{2}\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2 &\leq \Lambda(p, \tau/\alpha) + \int_0^{\sqrt{\tau_0}} 2u\langle \dot{\tilde{\gamma}}, \nabla f\rangle \\
&\leq n\sqrt{\tau/\alpha} + \frac{1}{4}\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2 + 4\int_0^{\sqrt{\tau_0}} u^2, \\
\frac{1}{4}\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2 &\leq n\sqrt{\tau/\alpha} + \frac{4}{3}\tau_0^{3/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{4}\tau^2 = \frac{1}{4}\operatorname{dist}(o, x_\tau)^2 &\leq \frac{1}{4}\left(\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|\right)^2 \leq \frac{1}{4}\sqrt{\tau_0}\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2 \\
&\leq n\sqrt{\tau_0\tau/\alpha} + \frac{4}{3}\tau_0^2 \leq \frac{1}{8}\tau^2 + \frac{4}{3}\tau_0^2,
\end{aligned}$$

if $\tau \geq \bar{\tau}(n)$. Hence $\tau_0 \geq c\tau$ for some dimensional constant $c > 0$. Then

$$\lambda(x_\tau, \tau_0) \leq \frac{\sqrt{\tau/\alpha}}{\sqrt{\tau_0}}\lambda(p, \tau/\alpha) \leq \frac{n}{2\sqrt{c\alpha}}.$$

$\square$

6.5 Optimal Volume Growth

In this section, the main result is the following.

Theorem 6.5.1. *Suppose that (M^n, g, f) is a complete steady gradient Ricci soliton normalized as in* (6.0.1) *and the canonical form $(M^n, g_t)_{t\in\mathbb{R}}$ has bounded Nash entropy:*

$$\mu_\infty := \inf_{\tau>0} \mathcal{N}_{o,0}(\tau) > -\infty.$$

Additionally, assume that ***either one*** *of the following conditions is true:*

(1) $(M^n, g_t)_{t\in\mathbb{R}}$ arises as a singularity model; or

(2) (M^n, g) has bounded curvature.

Then

$$c(n, \mu_\infty) r^{\frac{n+1}{2}} \leq |B_r(o)| \leq C(n) r^n \quad \text{for all} \quad r > \bar{r}(n, \mu_\infty),$$

where $c(n, \mu_\infty)$ is a positive constant of the form

$$c(n, \mu_\infty) = \frac{c(n)}{\sqrt{1-\mu_\infty}} e^{\mu_\infty}.$$

Furthermore, the upper bound is also true for all $r > 0$ (instead of $r \geq \bar{r}(n, \mu_\infty)$).

Lemma 6.5.2. *Suppose that (M^n, g, f) satisfies the assumptions in Theorem 6.5.1. Then for any $\tau \geq \bar{\tau}(n)$, there is $x_\tau \in M$ such that $\operatorname{dist}(x_\tau, o) = \tau$ and*

$$|B(x_\tau, \sqrt{A\tau})| \geq c e^{\mu_\infty} \tau^{n/2},$$

where $A = C_n(1-\mu_\infty), c = c(n) > 0$.

Proof. Let $\nu_t = \nu_{o,0|t}$ be the conjugate heat kernel (c.f. [Bam20a, Definition 2.4]) based at $(o, 0)$ coupled with the canonical form (M, g_t). Let x_τ, τ_0 be given by Lemma 6.4.4. Recall that $c\tau \le \tau_0 \le \tau/\alpha$ and $\lambda(x_\tau, \tau_0) \le C$, for some dimensional constants c, C and α is given by Lemma 6.4.3. By the proof of [Bam20a, Theorem 6.2], it suffices to show that

$$\nu_{-\tau_0}\left(B_{g_{-\tau_0}}\left(y_\tau, \sqrt{\alpha A \tau_0}\right)\right) \ge 1/2, \tag{6.5.1}$$

where $y_\tau = \Phi_{\tau_0}(x_\tau)$. Because once we can show (6.5.1), by the proof of [Bam20a, Theorem 6.2], we have

$$\begin{aligned}\left|B_g\left(x_\tau, \sqrt{A\tau}\right)\right|_g \ge \left|B_g\left(x_\tau, \sqrt{\alpha A \tau_0}\right)\right|_g = \left|B_{g_{-\tau_0}}\left(y_\tau, \sqrt{\alpha A \tau_0}\right)\right|_{g_{-\tau_0}} \\ \ge c_n e^{\mu_\infty} \tau_0^{n/2} \ge c_n e^{\mu_\infty} \tau^{n/2},\end{aligned} \tag{6.5.2}$$

where we used the fact that $\tau/\alpha \ge \tau_0 \ge c\tau$ for some dimensional constant $c > 0$. We leave the details of the proof of (6.5.2) to the reader. Note that, by [CMZ21a, Proposition 3.3], [Bam20a, Theorem 6.2] also holds for Ricci flows with bounded curvature on compact intervals.

Now we prove (6.5.1). Let $(z, -\tau_0)$ be an H_n-center of $(o, 0)$. By [Per02, 9.5] and [Bam20a, Theorem 7.2] (or [CMZ21a, Theorem 3.2]), we have

$$\begin{aligned}(4\pi\tau_0)^{-n/2} e^{-C} \le (4\pi\tau_0)^{-n/2} e^{-\ell(y_\tau, \tau_0)} \le K(o, 0 \,|\, y_\tau, -\tau_0) \\ \le C_n e^{-\mu_\infty} \tau_0^{-n/2} \exp\left(-\frac{\mathrm{dist}^2_{-\tau_0}(y_\tau, z)}{9\tau_0}\right),\end{aligned}$$

where K is the fundamental solution to the conjugate heat equation, and we also used Lemma 6.4.4 and the fact that $\ell(y_\tau, \tau_0) = \lambda(x_\tau, \tau_0) \le C$. Hence

$$\mathrm{dist}^2_{-\tau_0}(y_\tau, z) \le 9(-\mu_\infty + C_n)\tau_0.$$

We choose A so that

$$\alpha A = 18(-\mu_\infty + C_n) + 10H_n.$$

By [Bam20a, Proposition 3.13], we have

$$\nu_{-\tau_0}\left(B_{g_{-\tau_0}}\left(y_\tau, \sqrt{\alpha A \tau_0}\right)\right) \geq \nu_{-\tau_0}\left(B_{g_{-\tau_0}}\left(z, \sqrt{\alpha A \tau_0/2}\right)\right) \geq 1 - \frac{2H_n}{\alpha A} \geq \frac{1}{2}.$$

So we finished the proof of (6.5.1).

□

Now we are ready to prove Theorem 6.5.1.

Proof of Theorem 6.5.1. Let $\bar{\tau}(n) < \infty$ be given by Lemma 6.5.2. For each $r > 10A+\bar{\tau}(n)$, we construct a decreasing sequence $r = \tau_1 > \tau_2 > \cdots > \tau_N > 0$, such that $\tau_N < r/10$ and for $1 \leq j \leq N-1$,

$$\tau_j - \tau_{j+1} = \sqrt{A\tau_j} + \sqrt{A\tau_{j+1}}.$$

As long as $\tau_j \geq r/10$, the above equation is solvable for positive τ_{j+1} since the discriminant $A + 4(\tau_j - \sqrt{A\tau_j}) = 4(\sqrt{\tau_j} - \sqrt{A}/2)^2 \geq 0$. Since $\tau_j \geq r/10$ and $r > 10A$, there is a unique positive solution for τ_{j+1}. Moreover, $\tau_j - \tau_{j+1} \geq \sqrt{A\tau_j} \geq \sqrt{Ar/10}$, hence we can find a finite positive integer N such that $0 < \tau_N < r/10$. For each j, by Lemma 6.5.2, there is $x_j \in M$ such that $\mathrm{dist}(x_j, o) = \tau_j$, and

$$\left|B\left(x_j, \sqrt{A\tau_j}\right)\right| \geq c(n)e^{\mu_\infty}\tau_j^{n/2}.$$

By the construction of $\{\tau_j\}$, the balls $\left\{B\left(x_j, \sqrt{A\tau_j}\right)\right\}_{j=1}^N$ are pairwise disjoint. It follows that

$$
\begin{aligned}
|B_{2r}(o)| &\geq \sum_{j=1}^{N} \left| B\left(x_j, \sqrt{A\tau_j}\right)\right| \geq \sum_{j=1}^{N} c(n) e^{\mu_\infty} \tau_j^{n/2} \\
&\geq \tfrac{c(n)}{2\sqrt{A}} e^{\mu_\infty} \sum_{j=1}^{N-1} \tau_j^{\frac{n-1}{2}} (\tau_j - \tau_{j+1}) \\
&\geq \tfrac{c(n)}{\sqrt{A}} e^{\mu_\infty} \sum_{j=1}^{N-1} \int_{\tau_{j+1}}^{\tau_j} \tau^{\frac{n-1}{2}}\, d\tau \\
&= \tfrac{c(n)}{\sqrt{A}} e^{\mu_\infty} \int_{\tau_N}^{r} \tau^{\frac{n-1}{2}}\, d\tau \\
&\geq \tfrac{c(n)}{\sqrt{A}} e^{\mu_\infty} r^{\frac{n+1}{2}}.
\end{aligned}
$$

□

Chapter 6, in part, contains material published on Advances in Mathematics 2022 [BCDMZ] joint with Bamler, Richard H; Chow, Bennett; Deng, Yuxing; Zhang, Yongjia. Chapter 6 also contains material from [BCMZ21] which has been submitted for publication and is a joint work with Bamler, Richard H; Chan, Pak-Yeung; Zhang, Yongjia.

Chapter 7

Steady Solitons with Nonnegative Ricci Curvature

In this chapter, we shall study some aspects of steady gradient Ricci solitons with certain reasonable curvature conditions. As expected, we should have better results under such conditions.

We shall follow the same notations as in the last chapter. By definition, (M^n, g, f) is called a **steady gradient Ricci soliton** if f is a smooth function on M such that

$$\mathrm{Ric} = \nabla^2 f.$$

f is usually called the **potential function**. We shall assume that (M^n, g) is an n-dimensional *complete* Riemannian manifold.

Let $\{\Phi_t\}_{t\in\mathbb{R}}$ be the 1-parameter group of diffeomorphisms generated by $-\nabla f$, i.e.,

$$\partial_t \Phi_t = -\nabla f|_{\Phi_t}, \quad \Phi_0 = \mathrm{id}.$$

Then $g_t := \Phi_t^* g$ is a Ricci flow, called the **canonical form** of (M^n, g, f).

By Hamilton [Ha95, Section 20], $|\nabla f|^2 + R$ is constant on M. We may normalize the metric so that

$$|\nabla f|^2 + R = 1.$$

7.1 Preliminaries

We state a useful Lemma of the similar spirit to Lemma 2.2.3, although we remark that the following Lemma was found before the authors were aware of Lemma 2.2.3 from [Liu91].

Lemma 7.1.1 ([CDM22, Lemma 9.7])**.** *Let (M^n, g) be a complete Riemannian manifold and $K \subset M$ be a compact subset. Then for any sequence $p_k \to \infty$, there is a subsequence $\{p_{k_j}\}_{j\geq 0}$ such that every minimizing geodesic from p_{k_i} to p_{k_j} does not intersect K, for any $i, j \geq 0$.*

Proof. Fix $p = p_0$. Suppose that $K \subset B_p(r-1)$ for some $r > 1$. Let $\gamma_j(t) = \exp_p(tv_j)$ be a minimizing geodesic from p to p_j with $v_j \in T_pM, |v_j| = 1$. By passing to a subsequence, we may assume that $v_j \to v$ for some $v \in T_pM$. Let $\gamma(t) = \exp_p(tv)$.

We claim that there is $j_0 > 1$, such that any minimizing geodesic from p_i to p_j stays away from $B_p(r)$, if $i, j \geq j_0$. The Lemma follows from this claim immediately.

We prove the claim by contradiction. Suppose that the claim fails, which means that for any $j > 1$, there exist indecies $a_j, b_j \geq j$, a minimizing geodesic $\sigma_j : [0, \ell_j] \to M$ from $x_j := p_{a_j}$ to $y_j := p_{b_j}$ and there is $s_j \in [0, \ell_j]$ such that

$$z_j := \sigma_j(s_j) \in B_p(r).$$

Fix $\ell > 2r$. Since $z_j \in B_p(r)$,

$$\begin{aligned}|x_jy_j| &= |x_jz_j| + |z_jy_j| \geq |x_jp| - |z_jp| + |y_jp| - |z_jp| \\ &\geq |x_jp| + |y_jp| - 2r.\end{aligned}$$

On the other hand,

$$\begin{aligned}|x_jy_j| &\le |x_j\gamma_{a_j}(\ell)| + |\gamma_{a_j}(\ell)\gamma_{b_j}(\ell)| + |\gamma_{b_j}(\ell)y_j| \\ &= |\gamma_{a_j}(\ell)\gamma_{b_j}(\ell)| + |x_jp| + |y_jp| - 2\ell.\end{aligned}$$

It follows that

$$|\gamma_{a_j}(\ell)\gamma_{b_j}(\ell)| \ge 2(\ell - r) \ge 2r > 0.$$

However, as $j \to \infty$, by the convergence $v_j \to v$,

$$|\gamma_{a_j}(\ell)\gamma_{b_j}(\ell)| \le |\gamma_{a_j}(\ell)\gamma(\ell)| + |\gamma_{b_j}(\ell)\gamma(\ell)| \to 0,$$

which is a contradiction.

□

7.2 Growth of the Potential Function

Throughout this section, (M^n, g, f) is a complete steady gradient Ricci soliton satisfying

$$\mathrm{Ric} = \nabla^2 f, \quad R + |\nabla f|^2 = 1.$$

We shall investigate the conditions for f to grow linearly.

For simplicity, we say that a function u on (M^n, g) has **linear growth**, if there is a point $o \in M$ and constants $c, C > 0$, such that

$$c|ox| - C \le u(x) \le C|ox| + C, \tag{7.2.1}$$

for all $x \in M$.

For the potential function f, by our normalization $R + |\nabla f|^2 = 1$ and the fact that

$R \geq 0$, (by [Ch09] or Theorem 3.1.4,) we have $|\nabla f| \leq 1$ and thus

$$f(x) \leq f(o) + |ox|,$$

for all $x \in M$. However, it does not seem to be simple to find a lower bound for f.

Let us record the following result in [CDM22].

Theorem 7.2.1 ([CDM22, Theorem 2.1])**.** *Suppose that* $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$ *outside a compact set and the scalar curvature decays uniformly. Then* f *has a linear growth.*

For a steady gradient Ricci soliton (M^n, g, f), it is obvious that if R has uniform decay, then the critical set of f

$$\mathcal{C} := \{x \,|\, \nabla f(x) = 0\}$$

is nonempty and compact. The converse is not correct. For example, consider the product of two Bryant solitons. Thus, it is strictly weaker than the uniform decay of R to assume that $\mathcal{C}$ is nonempty and compact.

The following arguments are indeed contained in [CDM22, Section 9]. We shall record them for future use.

Proposition 7.2.2. *Suppose that* $\mathrm{Ric} \geq 0$ *everywhere on* M*. If* $\mathcal{C}$ *is nonempty and compact, then* f *has linear growth.*

Proof. Pick $o \in \mathcal{C}$. Suppose that $\mathcal{C} \subset B_o(A)$ for some $A > 0$.

Claim: For any $r \geq 2A$ and any $x \notin \bar{B}_o(r)$, there is $t_x > 0$, such that

$$\Phi_{t_x}(x) \in \partial B_o(r); \qquad \Phi_t(x) \notin \bar{B}_o(r), \quad t \in [0, t_x).$$

Proof. Since $\mathrm{Ric} \geq 0$ on M, for $t \geq 0$ and any $x \in M$,

$$\mathrm{dist}(o, \Phi_t(x)) = |ox|_t \leq |ox|,$$

by the standard distance distorsion estimate. (See, e.g., [Ha95, Lemma 17.13].) So the curve $\Phi_t(x)$ stays bounded for $t \geq 0$.

We show that $\Phi_t(x)$ accumulates on $\mathcal{C}$. Suppose to the contrary that there is a subsequence $t_j \to \infty, \Phi_{t_j}(x) \to y$ for some y but $y \notin \mathcal{C}$. Let $2c = |\nabla f|^2(y) > 0$. For $t \geq 0$,

$$\partial_t |\nabla f|^2(\Phi_t(x)) = -2\mathrm{Ric}(\nabla f, \nabla f)(\Phi_t(x)) \leq 0.$$

Thus for large j,

$$f(\Phi_{t_j}(x)) - f(x) = -\int_0^{t_j} |\nabla f|^2(\Phi_s(x))\, ds \leq -|\nabla f|^2(\Phi_{t_j}(x)) t_j \leq -ct_j,$$

which is impossible since $f(\Phi_{t_j}(x)) \to f(y)$ while $t_j \to \infty$. Hence $\Phi_t(x)$ accumulates on $\mathcal{C}$. Since $\mathcal{C} \subset B_o(A)$, there must be such a number $t_x > 0$ for each $x \notin \bar{B}_o(r)$.

□

Now we fix $r_0 \geq 2A$. Let $C, \theta > 0$ be constants such that

$$\min_{\partial B_o(r_0)} f \geq -C, \quad \min_{\partial B_o(r_0)} |\nabla f| \geq \theta.$$

Then

$$\begin{aligned} f(x) &= f(\Phi_{t_x}(x)) + \int_0^{t_x} |\nabla f|^2(\Phi_s(x)) ds \\ &\geq -C + |\nabla f|(\Phi_{t_x}(x)) \int_0^{t_x} |\partial_s \Phi_s(x)| ds \\ &\geq -C + \theta \cdot d(x, \partial B_o(r_0)) \geq \theta |ox| - (C + \theta r_0). \end{aligned}$$

Thus, f has linear growth. $\square$

Lemma 7.2.3. *Suppose that* $\mathrm{Ric} \geq 0$ *outside a compact set and* f *is proper. Then* $\mathcal{C}$ *is nonempty and compact.*

Proof. As f is proper, it has a minimum and thus $\mathcal{C}$ is nonempty.

Suppose that $\mathrm{Ric} \geq 0$ outside $B_o(A)$ for some $A > 0, o \in M$.

Suppose that $\mathcal{C}$ is not compact. Then there is a sequence $x_j \in \mathcal{C}$ and $x_j \to \infty$. By Lemma 7.1.1, we may assume that there is $x_0 \in \mathcal{C}$ such that for each $j \geq 1, x_j \in \mathcal{C}$ and any minimal geodesic from x_0 to x_j does not intersect $B_o(A)$. Let $\gamma_j : [0, \ell_j] \to M$ be a normal minimizing geodesic joining x_0 to x_j. Note that $\Phi_t(x_j) \equiv x_j$ for all t since $x_j \in \mathcal{C}$. Thus, for any $t \in \mathbb{R}$,

$$|x_0 x_j|_t = \mathrm{dist}(\Phi_t(x_0), \Phi_t(x_j)) = |x_0 x_j| = \ell_j.$$

We denote by $L_t(\gamma)$ the length of a curve γ with respect to metric g_t. For $t \in (-1, 1)$,

$$L_t(\gamma_j) \geq |x_0 x_j|_t = \ell_j, \quad L_0(\gamma_j) = \ell_j.$$

So

$$\left.\frac{d}{dt}\right|_{t=0} L_t(\gamma_j) = 0.$$

On the other hand, by the standard distance distorsion formula under Ricci flow [Ha95, Section 17],

$$0 = \left.\frac{d}{dt}\right|_{t=0} L_t(\gamma_j) = -\int_0^{\ell_j} \mathrm{Ric}(\dot\gamma_j, \dot\gamma_j)|_{\gamma(s)}\, ds.$$

Since γ_j does not intersect $B_o(A)$, $\mathrm{Ric}(\dot\gamma_j, \dot\gamma_j)|_{\gamma(s)} \geq 0$ for $s \in [0, \ell_j]$. It follows that

$$\mathrm{Ric}(\dot\gamma_j, \dot\gamma_j)|_{\gamma_j(s)} = 0,$$

for all $s \in [0, \ell_j]$. So $\dot{\gamma}_j(s)$ is a null eigenvector of $\mathrm{Ric}|_{\gamma_j(s)}$ and

$$\partial_s R(\gamma_j(s)) = -2\mathrm{Ric}(\dot{\gamma}_j, \nabla f)(\gamma_j(s)) = 0, \quad \forall s \in [0, \ell_j].$$

Then

$$|\nabla f|(\gamma_j(s)) \equiv |\nabla f|(x_0) = 0, \quad \forall s \in [0, \ell_j].$$

Thus, for any $j \geq 1$,

$$f(x_j) = f(x_0),$$

which is a contradiction to the assumption that f is proper.

□

In summary, we have the following.

Proposition 7.2.4. *Suppose* $\mathrm{Ric} \geq 0$ *on a complete steady GRS* (M^n, g, f)*. Then the following are equivalent:*

- *f is proper;*
- *f has linear growth;*
- *$\mathcal{C}$ is nonempty and compact.*

We shall refine the constants in (7.2.1) given that f is bounded below. We record the following argument from [DZ20b, Lemma 2.2].

Proposition 7.2.5. *Suppose that* $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$ *outside a compact set and f is bounded below. Suppose that*

$$\limsup_{x\to\infty} R(x) \leq 1 - \theta^2,$$

for some $\theta \in (0, 1]$. Then f has a linear growth. In fact, for some $o \in M$, (and thus for all $o \in M$,)

$$\liminf_{x\to\infty} \frac{f(x)}{|ox|} \geq \theta.$$

Proof. Suppose that $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$ outside $B_o(A)$ for some $o \in M, A > 0$. For simplicity, we write

$$K_r := \inf_{M \setminus B_o(r)} |\nabla f|.$$

Then the assumption implies that

$$\lim_{r \to \infty} K_r = \sup_{r > 0} K_r \geq \theta.$$

Hence, for any $\epsilon \in (0, \theta)$, there is $A_\epsilon \geq 2A$ such that

$$K_r \geq \theta - \epsilon, \qquad \text{for all } r \geq A_\epsilon.$$

Since f is bounded below, by adding a constant, we may assume that

$$f \geq 0, \qquad \text{on } M.$$

Claim: for any $\epsilon > 0$, $r > A_\epsilon$, and any $x \notin \bar{B}_o(r)$, there is $t_{x,r} > 0$, such that

$$\Phi_{t_{x,r}}(x) \in \partial B_o(r); \qquad \Phi_t(x) \notin \bar{B}_o(r), \quad t \in [0, t_{x,r}).$$

Proof of Claim. The proof is similar to Proposition 7.2.2. Suppose to the contrary that $\Phi_t(x) \notin \bar{B}_o(r)$ for any $t \geq 0$. Thus for any $t \geq 0$,

$$-f(x) \leq f(\Phi_t(x)) - f(x) = -\int_0^t |\nabla f|^2(\Phi_s(x))\, ds \leq -K_r^2 t \leq -(\theta - \epsilon)^2 t,$$

which is impossible since t can be arbitrarily large. Hence such a number $t_{x,r}$ exists. $\square$

For any $\epsilon > 0$, $r > A_\epsilon$, and any $x \notin \bar{B}_o(r)$, let $y_r := \Phi_{t_{x,r}}(x)$. Since $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$

outside $\bar{B}_o(A)$, for $t \in [0, t_{x,r})$,

$$\partial_t |\nabla f|^2(\Phi_t(x)) = -2\mathrm{Ric}(\nabla f, \nabla f) \leq 0.$$

Then

$$\begin{aligned} f(x) &= f(y_r) + \int_0^{t_{x,r}} |\nabla f|^2(\Phi_t(x))\, dt \\ &\geq |\nabla f|(y_r) \int_0^{t_{x,r}} |\partial_t \Phi_t(x)|\, dt \\ &\geq K_r |xy_r| \geq (\theta - \epsilon)(|ox| - r). \end{aligned}$$

In particular, the inequality above holds for $r = \sqrt{|ox|}$, if $|ox| > A_\epsilon^2$. So

$$\frac{f(x)}{|ox|} \geq (\theta - \epsilon)(1 - 1/r).$$

The conclusion follows since ϵ is arbitrary. $\square$

7.3 Asymptotic Volume Ratio

We first prove a point picking lemma similar to Appendix H of [KL08] for a relatively general setting.

Let (M^n, g) be a complete manifold and $f : M \to [0, \infty)$ is a proper continuous function. For applications, f can be taken as the potential function of a steady GRS.

We define some annuli around $p \in M$ of relative width $\alpha > 0$ as

$$\mathcal{A}_\alpha(p) := \{x : |f(x) - f(p)| < \alpha\sqrt{f(p)}\}.$$

Note that Deng and Zhu first considered such annuli which are written as $M_{p,\alpha}$ in their works [DZ19, DZ20a, DZ20b].

Lemma 7.3.1. *Let $Q : M \to [0,\infty)$ be a continuous function and $(M^n, g), f$ be as above. For any p with $f(p) > 100\alpha^2$, there is $x \in \mathcal{A}_{3\alpha}(p)$ and $\bar{\alpha} \in (0, \alpha]$, such that $\bar{\alpha}^2 Q(x) \geq \alpha^2 Q(p)$ and*

$$Q \leq 4Q(x), \qquad \text{on } \mathcal{A}_{\bar{\alpha}}(x).$$

Proof. Suppose that no such x exists. Let $y_0 = p, \alpha_0 = \alpha, Q_0 = Q(p)$. We shall construct a sequence (y_i, α_i) and prove inductively that

$$\alpha_i = \alpha_{i-1}/2, \quad Q_i > 4Q_{i-1}, \quad \alpha_i^2 f(y_i) < \tfrac{1}{2}\alpha_{i-1}^2 f(y_{i-1}),$$

where $Q_i = Q(y_i)$.

Suppose that we have chosen $y_0, \cdots, y_j$ with the estimates above. If $Q \leq 4Q_j$ on $\mathcal{A}_{\alpha_j}(y_j)$, then we may choose $x = y_j$, which is a contradiction to our assumption. Thus, there is $y_{j+1} \in A_{\alpha_j}(y_j)$ such that $Q_{j+1} = Q(y_{j+1}) > 4Q_j$. Set $\alpha_{j+1} = \alpha_j/2$. Then $\alpha_{j+1}^2 Q_{j+1} > \alpha_j^2 Q_j$ and

$$\alpha_{j+1}^2 f(y_{j+1}) < \tfrac{1}{4}\alpha_j^2 \left(f(y_j) + \alpha_j \sqrt{f(y_j)} \right) < \tfrac{1}{4}\alpha_j^2 [f(y_j) + \tfrac{1}{2} f(y_j)] < \tfrac{1}{2}\alpha_j^2 f(y_j),$$

where we may use a continuity argument to show that $f(y_j) > 4\alpha^2$ and thus we have $\alpha_j \sqrt{f(y_j)} < \frac{1}{2} f(y_j)$. It follows that for $k = 0, \cdots, j$,

$$|f(y_{k+1}) - f(y_k)| < \alpha_k \sqrt{f(y_k)} < \alpha_{k-1}\sqrt{f(y_{k-1})/2} < \cdots < \alpha\sqrt{f(p)} 2^{-k/2},$$

and thus

$$|f(y_{j+1}) - f(p)| \leq \sum_{k=0}^{j} |f(y_{k+1}) - f(y_k)| < \alpha\sqrt{f(p)} \sum_{k=0}^{\infty} 2^{-k/2} < 3\alpha\sqrt{f(p)}.$$

So we can find an infinite sequence $\{y_j\}_{j=1}^{\infty}$ living in a compact region, but $Q(y_j) \to \infty$, which is a contradiction. $\square$

Proposition 7.3.2. *If* $|\mathrm{Rm}| \le C/r$ *and* $\mathrm{AVR}(g) > 0$*, then* $\mathrm{Ric} \equiv 0$*.*

Proof. Steady solitons with linear curvature decay and nonnegative Ricci have been studied extensively in a series of work by Deng and Zhu [DZ19, DZ20a, DZ20b]. It's not hard to adapt their proofs to our settings, i.e., $\mathrm{Ric} \ge 0$ outside K.

For any $x_i \to \infty$,

$$(M, |\mathrm{Rm}|(x_i)g, x_i) \to (M_\infty, g_\infty, x_\infty),$$

in the sense of Cheeger-Gromov, such that

$$X^{(k)} = |\mathrm{Rm}|(x_i)^{-1/2}\nabla^g f \to X^\infty, \quad \nabla X^\infty = 0.$$

Thus $M_\infty = \mathbb{R} \times N$ and N has maximal volume growth. But N is diffeomorphic to $\{f = r_0\}$ which is compact. This is a contradiction. See [DZ20b, Section 6] for a complete argument.

$\square$

We would like to rule out the case where Rm has no linear decay under the assumption that M has maximal volume growth.

Proposition 7.3.3. *Suppose* Rm *has no linear decay and* $\mathrm{AVR}(g) > 0$*. Then there is a sequence* $x_j \to \infty, r_j \to \infty$*, such that* $|\mathrm{Rm}|(x_j)f(x_j) \to \infty$*,* $B_j = B(x_j, r_j)$ *are pairwise disjoint, and*

$$(B_j, |\mathrm{Rm}|(x_j)g, x_j) \to (\mathbb{R} \times N, ds^2 + g_N(t), (0, x_\infty))$$

as an ancient Ricci flow for $t \le 0$*.* $g_N(0)$ *satisfies* $\mathrm{Ric} \ge 0, |\mathrm{Rm}|(x_\infty) = 1, \mathrm{AVR}(g_N) > 0$*. Curvature of* $g_N(t)$ *is bounded on bounded time intervals.*

Proof. There is a sequence $y_j \to \infty$, such that $|\mathrm{Rm}|(y_j)f(y_j) \to \infty$. Fix $\alpha > 1$. By point picking, there is $x_j \in A_{3\alpha}(y_j)$ and $\alpha_j \in (0, \alpha/2]$ such that

$$4\alpha_j^2|\mathrm{Rm}|(x_j)f(x_j) \geq \alpha^2|\mathrm{Rm}|(y_j)f(x_j) > \tfrac{\alpha^2}{2}|\mathrm{Rm}|(y_j)f(y_j) \to \infty,$$

and

$$|\mathrm{Rm}| \leq 4|\mathrm{Rm}|(x_j), \quad \text{ on } A_{2\alpha_j}(x_j).$$

Let $r_j = \alpha_j\sqrt{f(x_j)}$. Since $|f(x) - f(y)| \leq d(x, y)$,

$$B_j := B(x_j, r_j) \subseteq A_{2\alpha_j}(x_j).$$

Clearly $\{B_j\}$ can be taken to be pairwise disjoint. Set

$$Q_j = |\mathrm{Rm}|(x_j), \quad g_j(t) = Q_j\Phi^*_{t/Q_j}[g],$$

where Φ_t is the flow of $-\nabla f$. We denote by

$$B(x, r; h)$$

the ball with respect to metric h. As $\mathrm{Ric} \geq 0$,

$$B(x_j, \bar{r}_j; g_j(t)) \subseteq B(x_j, r_j), \quad \forall t \leq 0,$$

where

$$\bar{r}_j = \alpha_j(|\mathrm{Rm}|(x_j)f(x_j))^{1/2} = Q_j^{1/2}r_j \to \infty.$$

In particular, $r_j \to \infty$.

Fix any $T > 0$, we show that there is a constant C_T, such that $|\mathrm{Rm}|_{g_j} \leq C_T$ on

$B(x_j, \bar{r}_j; g_j) \times [-T, 0]$. For $x \in B(x_j, \bar{r}_j; g_j) = B(x_j, r_j), t \in [-T, 0]$,

$$|\Phi_t(x), x| \le \int_t^0 |\partial_s \Phi_s(x)| ds \le T < r_j/2,$$

when j is sufficiently large. It follows that

$$\Phi_t(x) \in B(x_j, 2r_j) \subseteq A_{2\alpha_j}(x_j),$$

and thus

$$|\mathrm{Rm}|_{g_j}(x, t) = Q_j^{-1} |\mathrm{Rm}|(\Phi_t(x)) \le 4, \quad \forall (x, t) \in B(x_j, \bar{r}_j; g_j(0)) \times [-T, 0].$$

By E.2 of [KL08], after passing to a subsequence if necessary,

$$(B_j, g_j(t), x_j) \to (M_\infty, g_\infty(t), x_\infty),$$

for all $t \le 0$. $g_\infty(t)$ is an ancient solution to the Ricci flow with complete time slices. $\mathrm{Ric}[g_\infty] \ge 0$ and g_∞ has maximal volume growth. $|\mathrm{Rm}|(x_\infty, 0) = 1$, and $|\mathrm{Rm}|_{g_\infty} \le 4$ on M_∞.

Derivative estimates on B_j. For any $x \in B_j$, let

$$B = B(x, 2; g_j(-1)) \subseteq B(x_j, 1.1\bar{r}_j; g_j) = B(x_j, 1.1r_j),$$

when j is large. Similar to above, for any $(y, t) \in B \times [-2, 0]$, $\Phi_t(y) \in B(x_j, 2r_j) \subseteq A_{2\alpha_j}(x_j)$ and thus

$$|\mathrm{Rm}|_{g_j}(y, t) \le 4.$$

By Shi's local derivative estimates, there are constants $C_k > 0$ such that for any $(y, t) \in$

$B(x, 1; g_j(-1)) \times [-1, 0]$,

$$|\nabla^k \mathrm{Rm}|_g(y,t) = Q_j^{\frac{k+2}{2}} |\nabla^k \mathrm{Rm}|_{g_j}(y,t) \le C_k Q_j^{\frac{k+2}{2}}.$$

In particular,

$$\sup_{B_j} |\nabla^k \mathrm{Rm}| \le C_k Q_j^{\frac{k+2}{2}}.$$

Splitting at infinity. Define for $x \in B_j$,

$$f_j(x) \doteqdot Q_j^{1/2}[f(x) - f(x_j)].$$

Then

$$\nabla^{g_j} f_j = Q_j^{-1/2} \nabla^g f, \quad (\nabla^{g_j})^2 f_j = Q_j^{1/2} (\nabla^g)^2 f = Q_j^{1/2} \mathrm{Ric}[g_j].$$

It follows that on B_j,

$$|\nabla^{g_j} f_j|_{g_j} = |\nabla f|_g, \quad \left|(\nabla^{g_j})^{k+2} f_j\right|_{g_j} = Q_j^{1/2} |(\nabla^{g_j})^k \mathrm{Ric}_{g_j}|_{g_j} \le C_k Q_j^{1/2} \to 0,$$

for any $k \ge 0$. For any $\rho > 0$ and $x \in B(x_j, Q_j^{-1/2}\rho) = B(x_j, \rho; g_j) \sim B(x_\infty, \rho)$,

$$|f_j(x)| \le Q_j^{1/2} d(x, x_j) < \rho.$$

Combined with the estimates above, by passing to a subsequence if necessary, f_j converges to some function f_∞ in C^∞_{loc} such that

$$|\nabla f_\infty| \equiv 1, \quad \nabla^2 f_\infty \equiv 0.$$

So g_∞ splits off an $\mathbb{R}$-factor and f_∞ serves as a coordinate for that $\mathbb{R}$-factor with $f_\infty(x_\infty) = 0$. $\square$

Theorem 7.3.4 ([CDM22, Theorem 1.10]). *Suppose that* (M^4, g, f) *is a complete steady GRS with nonnegative Ricci outside a compact set and* $\lim_{x\to\infty} R(x) = 0$. *If* $\mathrm{AVR}(g) > 0$, *then* $\mathrm{Ric} \equiv 0$.

Proof. Suppose that g is not Ricci-flat. By [Ch19, Lemma 5], there is constant C such that

$$|\mathrm{Rm}| \leq CR.$$

R cannot decay linearly by Proposition 7.3.3. Then by the previous Proposition, we have dimension reduction at infinity. By [Ch09, Corollary 2.4], $(N, g_N(t))$ is an ancient κ-solution. (See the definition of κ-solutions in [KL08, Section 38] which comes from [Per02, 11.1].) By Theorem 7.2.1, we may apply Corollary 2.3.7 to obtain that $\mathrm{AVR}(g_N) > 0$. However, $\mathrm{AVR}(g_N) = 0$ by Perelman's work in [Per02, 11.4], which is a contradiction. □

Chapter 7, in part, contains material published on Advances in Mathematics 2022 [CDM22] joint with Chow, Bennett and Deng, Yuxing.

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