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UNIVERSITY OF CALIFORNIA SAN DIEGO

Symplectic Stability and New Symplectic Invariants of Integrable Systems

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Xiudi Tang

Committee in charge:

Professor Álvaro Pelayo, Chair
Professor Ioan Bejenaru
Professor Bennett Chow
Professor Daniel Kane
Professor Congjun Wu

2018

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The dissertation of Xiudi Tang is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California San Diego

2018

DEDICATION

To my grandfather Jie Tang (1933–2014).

EPIGRAPH

*Our motherland so rich in beauty
Has made countless heroes vie to pay her their duty.*

—Mao Zedong

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ABSTRACT OF THE DISSERTATION

Symplectic Stability and New Symplectic Invariants of Integrable Systems

by

Xiudi Tang

Doctor of Philosophy in Mathematics

University of California San Diego, 2018

Professor Álvaro Pelayo, Chair

In this dissertation, I prove a number of stability theorems for volume forms and symplectic forms in the noncompact setting, as well as a semiglobal classification result of finite dimensional integrable Hamiltonian systems. Volume forms and symplectic forms are, roughly, structures on smooth manifolds that measure volumes and 2-dimensional areas. A work of Darboux in 1882 ruled out any local invariants of symplectic forms. Moser proved in 1965 that on a compact manifold, we can not get non-diffeomorphic volume forms without changing the volume or non-diffeomorphic symplectic forms with a smooth deformation inside a cohomology class. Moser's result on volume forms was generalized to noncompact manifolds by Greene and Shiohama. I develop these stability results in two directions. For volume forms, I find the extra conditions for

Greene–Shiohama theorem to hold for smooth families of volume forms. The case of smooth families fits into the more general framework of fiber bundles with compact base and noncompact fiber. I define the concept of an exhausted fiber bundle which is exhausted by a smooth function compatible with the fiber bundle structure. On an exhausted fiber bundle, two fiberwise defined volume forms are fiberwise diffeomorphic under similar conditions as the smooth family case. For symplectic forms, the notion of Eliashberg–Gromov convex ends provides a natural restricted setting for the study of analogs of Moser stability theorem in the noncompact case, and this has been significantly developed in work of Cieliebak–Eliashberg. Retaining the end structure on the underlying smooth manifold, but dropping the convexity and completeness assumptions on the symplectic forms at infinity I show that the stability holds for a cohomologous smooth family of symplectic forms subject to a growth condition at the infinity, which I call having bounded log-variation.

Integrable systems are, roughly, dynamical systems with the maximal amount of conserved quantities. The symplectic theory of integrable systems started from the action-angle theorem of Minner in 1937 and Liouville–Arnold in 1963, which was extended to a global version by Duistermaat in 1980. These results clarified the symplectic structures near regular points and compact regular fibers of the momentum map. Eliasson in 1984 (complemented by Vũ Ngọc–Wacheux in 2013) proved that near a nondegenerate singular point the integrable system is symplectomorphic to its linear model, called the Eliasson local normal form. The neighborhoods of a compact connected fiber with only one focus-focus point and without other singular points in a 4-dimensional integrable system is classified by San Vũ Ngọc in 2002, by a formal power series. I prove that a compact connected fiber with multiple focus-focus points and without other singular points in a 4-dimensional integrable system is classified by a tuple formal power series as many as singular points.

Chapter 1

Introduction

Throughout this dissertation *smooth* means C^∞ smooth, and manifolds are always assumed to be smooth manifolds without boundary except where explicitly stated. We use $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}^n = (\mathbb{S}^1)^n$ for $n \in \mathbb{N}$.

A smooth manifold M can carry various geometric structures. A *symplectic form* ω on M is a closed nondegenerate 2-form, which makes (M, ω) a *symplectic manifold*, and a *volume form* α on M is a nondegenerate form of the top degree. Compared to a *Riemannian metric* g on M which is a smooth symmetric 2-form that measures lengths and angles, a symplectic form measures 2-dimensional areas and a volume form measures volumes. These three structures g , ω , and α are different on their levels of details, not only in terms of the dimensions they could measure but also in terms of the rigidity of the structures. If M has dimension n , the Riemannian metric g possesses rich local invariants encoded in the Riemannian curvature which is a tensor of $\frac{1}{2}n^2(n^2 - 1)$ independent components. However, according to a theorem of Darboux [10], the symplectic form ω and the volume form α have no local invariants, and Moser stability theorem [28] implies that when M is compact, the only global invariant of α is the volume $\int_M \alpha$. Greene–Shiohama [19], generalizing Moser stability theorem, showed that on a noncompact M the global invariants of a volume form α are the volume and a boolean datum (finite/infinite) on

each end of M .

The concept of symplectic manifolds is closely connected to and was originally motivated by classical mechanics in the study of systems, such as the simple pendulum, the Kapler system, and a charged particle in an electromagnetic field. Such systems are governed by second-order ordinary differential equations, perhaps Newton's second law. Once we know the initial condition, the position and momentum, we will be able to find the trajectory for all time $(-\infty, \infty)$. It is appropriate to represent the status of such a system by the *phase space* whose coordinates are pairs of position and momentum ones. If we denote by X all possible configurations (positions) of the system, then the phase space is the cotangent bundle T^*X . The energy $H: T^*X \rightarrow \mathbb{R}$ is always preserved along a trajectory, called the *Hamiltonian function*. Let $\omega_{\text{can}} = d\lambda_{\text{can}}$ be the canonical symplectic form on T^*X . By nondegeneracy of ω_{can} there is a unique vector field X_H on T^*X such that

$$\omega_{\text{can}}(X_H, Y) = -\langle dH, Y \rangle, \quad \forall Y \in \mathfrak{X}(T^*X).$$

Here $\mathfrak{X}(T^*X)$ denotes the vector space of vector fields on T^*X and we call X_H the *Hamiltonian vector field* of H . *Hamiltonian mechanics* says that a trajectory γ of the system is a flow line of X_H .

Let $\Sigma \subset T^*X$ be a 2-surface whose boundary $\partial\Sigma$ is a smooth loop γ . The *symplectic area* $\int_{\Sigma} \omega_{\text{can}}$ is unchanged under a deformation of Σ fixing $\partial\Sigma$, by the closedness of ω_{can} . Here since the symplectic form is exact, $\int_{\Sigma} \omega_{\text{can}} = \int_{\gamma} \lambda_{\text{can}}$ only depends on γ , called the *action integral* along γ . A trajectory γ is characterized by, between any two points on it, minimizing¹ the action integral among all paths with the same endpoints. This calculus of variation perspective is summarized in *Lagrangian mechanics*.

An example is the *spherical pendulum* which describes the motion of a particle of unit mass on a frictionless sphere inside a constant force field (for instance, the gravity field on the ground). Here $X = \mathbb{S}^2$ and the phase space is $(M = T^*\mathbb{S}^2, \omega = \omega_{\text{can}})$. The fact that H is

¹More precisely, the trajectory is a critical point of the action integral functional.

invariant under the flow of X_H requires a trajectory (1-dimensional) to lie inside a level set of H (generically 3-dimensional). The energy H is a *conserved quantity*, and flowing along X_H , the time translation, is a *symmetry*. Fortunately, in this case, we have another conserved quantity, the angular momentum J in the z direction (the direction opposite to the force field), and its corresponding symmetry, the rotation around the z -axis (flowing along X_J). These Hamiltonians H and J *Poisson commute* in the sense that the Lie bracket $[X_H, X_J] = 0$. The derivatives of H and J are linearly independent almost everywhere and we can see that an orbit of the joint flow of X_H and X_J has to be contained in a level set of $F \stackrel{\text{def}}{=} (J, H): M \rightarrow \mathbb{R}^2$, both of which are 2-dimensional almost everywhere. Such a system is called *integrable* and F is called the *momentum map* of the integrable system (M, ω, F) .

According to Darboux–Carathéodory theorem near a regular point, or by Liouville–Arnold–Mineur theorem near a compact regular fiber of F , we can change the variables of (J, H) to (a_1, a_2) to make the dynamics as simple as the flow of X_{a_1} translating θ_1 and preserving θ_2 (similarly for X_{a_2}). Here $(a_1, a_2, \theta_1, \theta_2)$ are coordinates of M near the regular point or fiber such that the symplectic form is $d\theta_1 \wedge da_1 + d\theta_2 \wedge da_2$, and $a_i, i = 1, 2$ are action integrals along nontrivial loops on compact regular fibers of F which are 2-tori. These explain the necessity to introduce symplectic manifolds instead of working on coordinates. On the other hand, the nondegenerate (a generic condition) singularities of an integrable system are simple in the sense of Eliasson’s theorem. The only nondegenerate singularities are the Cartesian products of centers and saddles in 2 dimensions, and complex saddles in 4 dimensions. These are called [14] *elliptic*, *hyperbolic*, and *focus-focus* singular points and showed that after a local change of coordinates the integrable systems near the singularities are no different from their linear models. In the spherical pendulum M , resting at the highest point is a focus-focus singular point m . The singularity m is an unstable star node of the flow of X_H , a center of the flow of X_J , and a spiral of the flow of other linear combinations of X_H and X_J .

Focus-focus singularities have so many facets that have nourished a lot of studies. A

compact connected fiber containing only regular and focus-focus points is a torus pinched $k \in \mathbb{N}$ (finite since focus-focus points are isolated) times. In 2003, Vũ Ngọc [35] completely classified the germ at such a fiber with $k = 1$ by a formal power series $\sum_{i,j \geq 0} a_{ij} X^i Y^j$ with $a_{00} = 0$, $a_{10} \in \mathbb{R}/2\pi\mathbb{Z}$, and other $a_{ij} \in \mathbb{R}$.

1.1 The aim of this work

The general goal of this dissertation is to construct invariants of bundles with fiberwise volume forms, noncompact symplectic manifolds, and integrable systems with focus-focus singularities. Questions addressed include the following:

1. The construction in Greene–Shiohama’s proof, of the diffeomorphism intertwining two volume forms with the same global invariants, depends essentially on a geometric operations reading the quantitative properties of the volume forms. It is *a priori* an unstable construction in the sense that if we deform the forms the diffeomorphism may change dramatically. Do we have a method with which we can get a smooth family of diffeomorphisms from two smooth families of volume forms that intertwins them?
2. A smooth family of volume forms can be seen as a smooth form on the product space, which is also a trivial fiber bundle. Suppose instead, the volume forms lie on fibers of a fiber bundle with nontrivial topology. Can we still find a smooth family of diffeomorphisms of the fibers which intertwine the volume forms?
3. Do we have Moser stability for a path of symplectic forms on noncompact manifolds, as Greene–Shiohama did for volume forms?
4. In the spherical pendulum we replace the constant force field by a conservative field whose potential is, for instance, z^2 . Then resting at the two points where z attains its maximum and minimum are two focus-focus singular points of the momentum map $F = (J, H)$ which

happen to lie on the same fiber of F . This fiber is a torus with two pinched points, and Vũ Ngọc's classification does not apply here. Can we classify germs of compact connected fibers containing more than one focus-focus points (and no other singular points)?

In this dissertation I address these questions. The answer to Question 1 is yes in Chapter 3, if we add continuity and smoothness conditions on the families of volume forms. The answer to Question 2 is yes in Chapter 4, if we add continuity and smoothness conditions on the families of volume forms, and topological conditions on the bundle (with which we call an exhausted bundle). The answer to Question 3 is yes in Chapter 5, if we add quantitative conditions on the path of symplectic forms, and topological conditions on the noncompact manifold. The answer to Question 4 is yes in Chapter 6.

Chapter 2

Preliminaries

2.1 Symplectic geometry

2.1.1 Symplectic and volume forms

Definition 2.1.1. Let M be a manifold. A differential 2-form on M is called a *symplectic form* if

- it is closed, that is, $d\omega = 0$;
- it is nondegenerate, that is, for any $p \in M$ and nonzero $v \in T_pM$, there is $w \in T_pM$ such that $\omega_p(v, w) \neq 0$.

Then (M, ω) is called a *symplectic manifold*.

Definition 2.1.2. Let M be a m -dimensional smooth manifold. A differential m -form ω on M is called a *volume form* if it is nowhere vanishing.

Note that, on a symplectic manifold (M, ω) , $\omega^{\wedge n} \stackrel{\text{def}}{=} \omega \wedge \cdots \wedge \omega$ is automatically a volume form.

Definition 2.1.3. Given two symplectic manifolds (M, ω) and (M', ω') , we call a diffeomorphism $\varphi: M \rightarrow M'$ a *symplectomorphism* if $\varphi^*\omega' = \omega$, and in this case (M, ω) and (M', ω') are called *symplectomorphic*.

Given two manifolds M and M' with volume forms ω, ω' respectively, a diffeomorphism $\varphi: M \rightarrow M'$ is a *volume preserving diffeomorphism* if $\varphi^* \omega' = \omega$.

2.1.2 Smooth families

The definition of smooth maps between manifolds or open subsets of manifolds is clear and sound. When the domain or the codomain is a general subset of a manifold we adopt the following definitions for smooth function or sections, throughout the dissertation.

Definition 2.1.4. Let X, Y be manifolds, and $A \subset X, B \subset Y$ be subsets. A map $f: A \rightarrow B$ is *smooth* if any $a \in A$ has an open neighborhood U_a in X and a smooth function $\tilde{f}_a: U_a \rightarrow Y$ which coincides with f in $A \cap U$. A map $f: A \rightarrow B$ is a *diffeomorphism* if f is a homeomorphism and near any $a \in A$, f has a local extension to a diffeomorphism from an open neighborhood of a in X to an open neighborhood of $f(a)$ in Y . If $\pi: Y \rightarrow X$ is a vector bundle and $B = \pi^{-1}(A)$, then a section $f: A \rightarrow B$ is *smooth* if it is smooth as a map.

Next we define smooth families of differential forms and diffeomorphisms.

Definition 2.1.5. Let M be a manifold of dimension m and let B be a compact manifold. Let $q \in \mathbb{N}$ with $0 \leq q \leq m$. Let $\Omega^q(M)$ denote the vector space of q -forms on M . A family of q -forms $\{\omega_p\}_{p \in B} \subset \Omega^q(M)$ is *smooth* if the map $B \times M \rightarrow \wedge^q T^*M, (p, x) \mapsto \omega_p(x)$ is smooth. A family $\{\varphi_p\}_{p \in B}$ of diffeomorphisms of M is *smooth* if the map $B \times M \rightarrow M, (p, x) \mapsto \varphi_p(x)$ is smooth.

2.1.3 Symplectic stability

We define two relations among differential forms on a manifold.

Definition 2.1.6. Let M be a manifold of dimension m and let $q \in \mathbb{N}$ with $0 \leq q \leq m$. Two q -forms $\alpha_0, \alpha_1 \in \Omega^q(M)$ are *diffeomorphic* if there is a diffeomorphism $\varphi: M \rightarrow M$ such that $\varphi^* \alpha_1 = \alpha_0$. A family $\{\omega_p\}_{p \in B} \subset \Omega^q(M)$ is an *isotopy* if it is a smooth path of cohomologous q -forms on M ;

$\{\omega_p\}_{p \in B} \subset \Omega^q(M)$ is a *strong isotopy* if there exists a smooth path φ_t of diffeomorphisms of M such that $\varphi_t^* \omega_t = \omega_0$.

Note that a family of forms being a strong isotopy implies it is an isotopy of diffeomorphic forms. Note also that a convex combination of volume forms (of two forms, that is $(1-t)\alpha_0 + t\alpha_1$ for $t \in [0, 1]$ and α_0, α_1 are volume forms) is a volume form, so for volume forms, having the same volume is equivalent to having an isotopy between them.

Next I introduce several theorems of Darboux, Moser and Greene–Shiohama on the stabilities of volume forms and symplectic forms on manifolds. They either completely or partially solved the question when an isotopy of volume or symplectic forms is a strong isotopy.

Theorem 2.1.1 (Darboux [10]). *Any two symplectic forms on a manifold are locally diffeomorphic, as there is a strong isotopy between them in a neighborhood of any point.*

Darboux’s theorem works for volume forms without changing a word, but the volume form version may have been proved earlier than Darboux’s.

Theorem 2.1.2 (Moser [28]). *Any two volume forms on a compact manifold with equal total volume are diffeomorphic, as there is a strong isotopy between them. Any isotopy of symplectic forms on a compact manifold is a strong isotopy.*

Theorem 2.1.3 (Greene–Shiohama [19]). *Any two volume forms on a noncompact manifold with equal total volume and such that for each end of the manifold they both give finite volumes or both give infinite ones are diffeomorphic.*

The proof of Theorem 2.1.2 deforms the symplectic forms by the time-dependent Hamiltonian field of some 1-form.

Proof of Theorem 2.1.2. Let $\{\omega_t\}_{t \in [0,1]}$ be an isotopy of symplectic forms on a compact manifold M . Equip M with any Riemannian metric g . Use Hodge theory. Let $G: \Omega^2(M) \rightarrow \Omega^2(M)$ be Green’s operator of the Hodge-Laplacian Δ , and $d^*: \Omega^2(M) \rightarrow \Omega^1(M)$ be the codifferential, the

dual of exterior derivative d . Then $d \circ \delta \circ G$ is the identity on exact 2-forms. Note that the time derivative $\dot{\omega}_t$ is exact since $\omega_t, t \in [0, 1]$ are cohomologic. If $\sigma_t = \delta \circ G \dot{\omega}_t$, then $d\sigma_t = \dot{\omega}_t$.

Let $X_t = -\omega_t^{-1} \sigma_t$, the time-dependent Hamiltonian field of σ_t with respect to ω_t . Let φ_t be the flow of $X_t, t \in [0, 1]$. Then by Cartan's formula,

$$\frac{d}{dt}(\varphi_t^* \omega_t) = \varphi_t^*(\dot{\omega}_t + \mathcal{L}_{X_t} \omega_t) = \varphi_t^*(\dot{\omega}_t + d(X_t \lrcorner \omega_t)) = 0.$$

This proves $\varphi_t^* \omega_t = \omega_0$. □

2.2 Integrable systems

2.2.1 Integrable systems

Let (M, ω) be a $2n$ -dimensional symplectic manifold. For a smooth map $f: M \rightarrow \mathbb{R}$ we denote by $X_f = -\omega^{-1}(df) \in \mathfrak{X}(M)$ the *Hamiltonian vector field* of f . For any smooth maps $f, g: M \rightarrow \mathbb{R}$ we define their *Poisson bracket* $\{f, g\} = -\omega(X_f, X_g)$.

Definition 2.2.1. Let $F = (f_1, \dots, f_n): M \rightarrow \mathbb{R}^n$ be a smooth map such that $\{f_i, f_j\} = 0$ for each i, j with $1 \leq i, j \leq n$ and df_1, \dots, df_n are linearly independent almost everywhere. In this case we call F a *momentum map* on M . We say (M, ω, F) is an *integrable system*. Two integrable systems (M, ω, F) and (M', ω', F') are *isomorphic* if there is a symplectomorphism $\varphi: (M, \omega) \rightarrow (M', \omega')$ and a diffeomorphism $G: F(M) \rightarrow F(M')$ such that $F' \circ \varphi = G \circ F$. Let \mathcal{I} be the collection of all integrable systems.

2.2.2 Flow-complete integrable systems

Let (M, ω, F) be an integrable system and $B = F(M)$. For any $b \in B$ and $\beta_b \in T_b^* \mathbb{R}^n$, let $X_{\beta_b} = -\omega^{-1}(F^* \beta_b) \in \mathfrak{X}(F^{-1}(b))$ be the *Hamiltonian vector field* of β_b . If the flow of X_{β_b}

exists until time 1, we denote by $\Psi_{\beta_b} \in \text{Diff}(\mathcal{F})$ the time-1 map of X_{β_b} . As is customary, let $T^*\mathbb{R}^n|_B \times_B M = \{(\beta_b, x) \in T_b^*\mathbb{R}^n \times F^{-1}(b) \mid b \in B\} \subset T^*\mathbb{R}^n|_B \times M$ denote the fiber product of the bundle map of $T^*\mathbb{R}^n|_B$ with the momentum map F . In general, the map $\Psi: \mathcal{U} \rightarrow M, \beta_b \mapsto \Psi_{\beta_b}$ is defined in an open neighborhood \mathcal{U} of the zero section of $T^*\mathbb{R}^n|_B \times_B M \rightarrow B, (\beta_b, x) \mapsto b = F(x)$.

In the dissertation, we focus on integrable systems where Ψ is valid on the entire bundle.

Definition 2.2.2. An integrable system (M, ω, F) is called *flow-complete* if for any $b \in B$ and $\beta_b \in T_b^*\mathbb{R}^n$, X_{β_b} is complete. In this case, $\Psi: T^*\mathbb{R}^n|_B \times_B M \rightarrow M$ defines a fiberwise action of $T^*\mathbb{R}^n|_B$ on M , namely, for any $b \in B$, $T_b^*\mathbb{R}^n$ acts on $F^{-1}(b)$ by $\Psi_{\beta_b} \in \text{Diff}(\mathcal{F})$.

Definition 2.2.3. Let (M, ω, F) be a flow-complete integrable system. Let $U \subset B$ be an open subset. For each $\beta \in \Omega^1(U)$, its *Hamiltonian vector field*¹ $X_\beta = -\omega^{-1}(F^*\beta)$ is a vector field on $F^{-1}(U)$. Let $\Psi_\beta \in \text{Diff}(F^{-1}(U))$ be the time-1 map of X_β , so then $\beta \mapsto \Psi_\beta$ gives the action of $\Gamma(T^*U)$ on $F^{-1}(U)$ by diffeomorphisms, and Ψ_β commutes with F .

Definition 2.2.4. An integrable system (M, ω, F) is called *flow-complete* if for any $b \in B$ and $\beta_b \in T_b^*\mathbb{R}^n$, X_{β_b} is complete. In this case, $\Psi: T^*\mathbb{R}^n|_B \times_B M \rightarrow M$ defines a fiberwise action of $T^*\mathbb{R}^n|_B$ on M , namely, for any $b \in B$, $T_b^*\mathbb{R}^n$ acts on $F^{-1}(b)$ by $\Psi_{\beta_b} \in \text{Diff}(\mathcal{F})$.

Definition 2.2.5. Let (M, ω, F) be a flow-complete integrable system. For any open subset $U \subset B = F(M)$ let $\check{\Lambda}^{(M, \omega, F)}(U) = \{\beta \in \Omega^1(U) \mid \Psi_{2\pi\beta} = \text{id}\}$. We will use $\check{\Lambda}$ omitting the superscripts if there is no ambiguity. We call $\check{\Lambda}$ the *period sheaf* of (M, ω, F) , which is a sheaf of abelian groups over B . The local sections of $\check{\Lambda}$ are *period forms*.

2.2.3 Action-angle coordinates

The goal of this subsection is to give a self-contained proof of the existence of action-angle coordinates. These coordinates identify symplectically a neighborhood of a regular, compact,

¹We define the Hamiltonian vector fields three times and Ψ twice with different meanings. However, the definition should be clear from the context.

and connected fiber in an integrable system to the neighborhood of the zero section of $T^*\mathbb{T}^n$. Let $(M, \omega, F) \in \mathcal{IS}$. Suppose F is proper and has connected fibers. Let B_r be the set of regular values of F in B .

Lemma 2.2.1. *If $U \subset B_r$ is a simply connected open set, then there are $\alpha_1, \dots, \alpha_n \in \Omega^1(U)$ such that $\check{\Lambda}(U) = \bigoplus_{i=1}^n \alpha_i \mathbb{Z}$.*

Proof. Let $b \in B_r$. Consider the action of T_b^*B on $F^{-1}(b)$ by Ψ . Since the $F^{-1}(b)$ consists of regular points, the action is locally free. Thus the orbits are open and closed, and $F^{-1}(b)$ is assumed connected, the action is transitive. The kernel is a discrete subgroup of T_b^*B . Since $F^{-1}(b)$ is compact, the kernel has to be an n -lattice, and $F^{-1}(b)$ is diffeomorphic to an n -torus.

Consider a neighborhood U_0 of 0 in B_r . Take local coordinates $(b_1, \dots, b_n): U_0 \rightarrow U'_0 \subset \mathbb{R}^n$, and a section $P: U_0 \rightarrow F^{-1}(U_0)$. Then the map

$$U'_0 \times \mathbb{R}^n \rightarrow F^{-1}(U_0)$$

$$(b_1, \dots, b_n, \beta^1, \dots, \beta^n) \mapsto \Psi_{\sum_{i=1}^n \beta^i db_i} P(b_1, \dots, b_n)$$

is smooth by the smooth dependence of the solution to an initial value problem of ordinary differential equations on the initial value x and parameters (b_1, \dots, b_n) . Suppose $\beta_0 \in T_0^*B$ is such that $\Psi_{2\pi\beta_0} = \text{id}$. Write $b = (b_1, \dots, b_n)$. By the implicit function theorem, the following equation

$$\Psi_{\sum_{i=1}^n \beta^i(b) db_i} x(b) = x(b)$$

has a smooth solution $(\beta^1, \dots, \beta^n): U_0 \rightarrow \mathbb{R}^n$. This means β_0 can be extended to a smooth 1-form $\beta \in 2\pi\check{\Lambda}(U_0)$, by possibly shrinking U_0 . Hence there are $\alpha_1, \dots, \alpha_n \in \Omega^1(U_0)$ which are a \mathbb{Z} -basis of $\check{\Lambda}(U_0)$.

Now let $U \subset B_r$ be a simply connected open set. Along any path from 0 to some $b \in U$, we can extend $\alpha_i, i \in \{1, \dots, n\}$ to 1-forms near b . Since U is simply connected and the lattice

$\check{\Lambda}(U_0)$ is discrete, the extension is independent of the choice of the path between 0 and b . So $\check{\Lambda}(U_0)$ uniquely extends to $\check{\Lambda}(U)$ as an n -lattice. \square

Theorem 2.2.2 (Action-angle coordinates [2, 27]). *Let $U \subset B^r$ be a simply connected open subset. Let $(\alpha_1, \dots, \alpha_n)$ be a \mathbb{Z} -basis of $\check{\Lambda}(U)$. There are coordinate systems $(A_1, \dots, A_n): U \rightarrow \mathbb{R}^n$, $(\theta_1, \dots, \theta_n, a_1, \dots, a_n): F^{-1}(U) \rightarrow \mathbb{T}^n \times \mathbb{R}^n$ such that*

- $dA_i = \alpha_i$;
- $a_i = F^*A_i$;
- $\omega = \sum_{i=1}^n d\theta_i \wedge da_i$.

We call A_i the action integrals, a_i the action coordinates and θ_i the angle coordinates.

Proof. Since α_i is closed and U is simply connected, we can define smooth functions $A_i: U \rightarrow \mathbb{R}$, uniquely up to a constant, such that $dA_i = \alpha_i$. Let $a_i = F^*A_i$.

Since $\{a_i, a_j\} \in \sum_{i,j=1}^n \mathbb{R}\{f_i, f_j\} = 0$ for any i, j , the vector fields X_{α_i} , $i \in \{1, \dots, n\}$ commute. Choose a Lagrangian section $P: U \rightarrow F^{-1}(U)$. From the choice of α_i we know that the flow of X_{α_i} generates a \mathbb{T}^n -action. By translating along the flow of X_{α_i} we can define θ_i on $F^{-1}(U)$ such that $\theta_i \circ P = 0$ on U and $\frac{\partial}{\partial \theta_i} = X_{\alpha_i}$ on $P(U)$, then we have and $\frac{\partial}{\partial a_i} = P_* \frac{\partial}{\partial A_i}$ on $F^{-1}(U)$. So, $(\frac{\partial}{\partial a_1}, \dots, \frac{\partial}{\partial a_n}, \frac{\partial}{\partial \theta_1}, \dots, \frac{\partial}{\partial \theta_n})$ is a basis of $\mathfrak{X}(F^{-1}(U))$ and $(da_1, \dots, da_n, d\theta_1, \dots, d\theta_n)$ forms the dual basis of $\Omega^1(F^{-1}(U))$.

Since $\frac{\partial}{\partial \theta_i} \lrcorner \omega = da_i$ and level sets of θ are Lagrangian, we have

$$\omega = \sum_{i=1}^n d\theta_i \wedge da_i,$$

that is, the chart (θ, a) is symplectic. \square

2.2.4 Eliasson–Williamson local normal form

Recall that of smooth maps $f, g: M \rightarrow \mathbb{R}$ on a $2n$ -dimensional symplectic manifold (M, ω) the Poisson bracket $\{f, g\} = -\omega(X_f, X_g)$ depend only on their derivatives.

For $x \in M$, let $\mathcal{Q}(T_x M)$ denote the set of quadratic forms on $T_x M$. The Hessian of a smooth map $f: M \rightarrow \mathbb{R}$ at a singular point $x \in M$ ($df_x = 0$) is denoted by $\mathcal{H}_f \in \mathcal{Q}(T_x M)$ and defined by

$$\mathcal{H}_f(x) \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 f}{\partial x_i \partial x_j},$$

where x_1, \dots, x_{2n} are local coordinates of M near x and $1 \leq i, j \leq 2n$. Note that $\mathcal{H}_f(x)$ is independent of the choice of local coordinates (so is well defined in $\mathcal{Q}(T_x M)$) if and only if x is singular.

Define the *Poisson bracket* on $\mathcal{Q}(T_x M)$ as

$$\begin{aligned} \{\cdot, \cdot\}_x: \mathcal{Q}(T_x M) \times \mathcal{Q}(T_x M) &\rightarrow \mathcal{Q}(T_x M), \\ \{\mathcal{H}_f(x), \mathcal{H}_g(x)\}_x &= \mathcal{H}_{\{f, g\}}(x), \end{aligned}$$

where $x \in M$ is a singular point of both f and g . Direct calculations in coordinates check this definition well defined, and then $(\mathcal{Q}(T_x M), \{\cdot, \cdot\}_x)$ is a Lie algebra.

A Cartan subalgebra is a nilpotent subalgebra \mathfrak{h} of a Lie algebra \mathfrak{g} that is self-normalising, in the sense that if $Y \in \mathfrak{g}$ such that $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then $Y \in \mathfrak{h}$.

Definition 2.2.6. Let $(M^{2n}, \omega, F = (f_1, \dots, f_n)) \in \mathcal{I}$. Then a singular point $x \in M$ of F is *nondegenerate* if the Lie subalgebra spanned by $\mathcal{H}_{f_1}(x), \dots, \mathcal{H}_{f_n}(x)$ is a Cartan subalgebra of $(\mathcal{Q}(R_x), \{\cdot, \cdot\}_x)$, where R_x is the quotient of $\ker dF_x$ by $\text{span}(X_{f_1}, \dots, X_{f_n})$.

Based on an algebraic theorem by Williamson [36] on the Lie algebra of quadratic forms, Eliasson showed that the nondegenerate singular points of an integrable system are linearizable.

Theorem 2.2.3 (Eliasson [14]). *Let $(M^{2n}, \omega, F = (f_1, \dots, f_n)) \in \mathcal{I}\mathcal{S}$. Then near any a nondegenerate singular point $x \in M$ of F , local symplectic coordinates $(x, \xi) \stackrel{\text{def}}{=} (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ exist, putting x at the origin, such that $f_i, q_j = 0$, for all indices i, j , where each of the components of $q \stackrel{\text{def}}{=} (q_1, \dots, q_n)$, defined near the origin in \mathbb{R}^{2n} , can be*

(i) *Regular component: $q_j = \xi_j$, where $1 \leq j \leq n$.*

(ii) *Elliptic component: $q_j = \frac{1}{2}(x_j^2 + \xi_j^2)$, where $1 \leq j \leq n$.*

(iii) *Hyperbolic component: $q_j = x_j \xi_j$, where $1 \leq j \leq n$.*

(iv) *Focus-focus component: $q_{j-1} = x_{j-1} \xi_j - x_j \xi_{j-1}$ and $q_j = x_{j-1} \xi_{j-1} + x_j \xi_j$ where $2 \leq j \leq n - 1$.*

Moreover, if x does not have any hyperbolic block, then the condition $f_i, q_j = 0$, for all indices i, j may be replaced by $(F - F(x))\varphi = G \circ q$, where $\varphi = (x, \xi)^{-1}$ and G is a local diffeomorphism of \mathbb{R}^n fixing the origin.

Chapter 3

Moser-Greene-Shiohama stability for families

3.1 Introduction

In Moser [28] and Greene–Shiohama [19] the authors proved that two volume forms on a manifold are diffeomorphic if they have equal total volume and, when the manifold is noncompact, for each end of the manifold they both give finite volumes or both give infinite ones, see Section 2.1.3. The proof of Greene–Shiohama is more complicated than Moser’s because the authors have to deal with the behavior at infinity of the forms. Their proof has three stages: first, they extend Moser’s proof to forms which are compactly supported. Then they chop their noncompact manifold into pieces, and finally, a careful analysis of the behavior at the boundaries and interiors, allows them to construct a global diffeomorphism by pasting together the local diffeomorphisms, bypassing any analytic estimates.

If one mimics the Greene–Shiohama argument in the case of two smooth families of volume forms ω_p, τ_p , indexed by some compact manifold which plays the role of parameter space B , this produces for each p a diffeomorphism φ_p such that $\varphi_p^* \omega_p = \tau_p$, but there is no information

given about how φ_p changes when p changes in B . The goal of this chapter is to give sufficient conditions for the variation of φ_p with respect p to be smooth.

3.2 Main theorem

Definition 3.2.1. Let M be a manifold of dimension m and let B be a compact manifold. Two smooth families of volume forms $\{\omega_p\}_{p \in B}$ and $\{\tau_p\}_{p \in B}$ are *commensurable* on M if for any compact set $K \subset M$, for any connected component C of $M \setminus K$, one of the following holds:

- for any $p \in B$, $\int_C \omega_p = \int_C \tau_p = +\infty$;
- the integrals $\int_C \omega_p$ and $\int_C \tau_p$ are finite and continuous with respect to $p \in B$, and their difference is smooth with respect to $p \in B$.

Our main result is the following parametric version of the Moser and Greene–Shiohama result:

Theorem 3.2.1. *Let M be a noncompact oriented connected manifold. Let B be a compact manifold. Let $\{\omega_p\}_{p \in B}$ and $\{\tau_p\}_{p \in B}$ be commensurable smooth families of volume forms on M such that $\int_M \omega_p = \int_M \tau_p$ for any $p \in B$. Then there is a smooth family of diffeomorphisms $\{\varphi_p: M \rightarrow M\}_{p \in B}$ such that $\varphi_p^* \omega_p = \tau_p$ for each $p \in B$.*

If B is a point, Theorem 3.2.1 was proved by Greene–Shiohama [19]. If $\{\tau_p\}_{p \in B}$ is a constant family, we obtain:

Corollary 3.2.2. *Let M be a noncompact oriented connected manifold and B be a compact manifold. Let $q \in B$. Let $\{\omega_p\}_{p \in B}$ be a smooth family of volume forms on M such that $\int_M \omega_p$ is independent of $p \in B$. Suppose moreover for any connected component C of the complement of a compact subset of M , either $\int_C \omega_p = +\infty$ for all $p \in B$, or $\int_C \omega_p$ is smooth with respect to $p \in B$. Then there is a smooth family of diffeomorphisms $\{\varphi_p: M \rightarrow M\}_{p \in B}$ such that $\varphi_p^* \omega_p = \omega_q$ for each $p \in B$.*

The case M being compact was proved by Moser [28].

The remaining of the paper is devoted to proving this theorem. The proof is inductive and requires the introduction of certain topological-combinatorial constructions (Section 3.3), and geometric-analytic constructions (Section 3.4). This allows us to prove a filtration lemma for noncompact manifolds (Section 3.5), from which Theorem 3.2.1 easily follows (Section 3.6).

3.3 Topological-combinatorial constructions

In this section, we prepare the topological-combinatorial ingredients needed to prove our main theorem. We will first show a result about general topological spaces, which we will then use to give a slicing of a smooth manifold which satisfies certain properties (in terms of an exhaustion function for the manifold). Then we use this slicing to define a tree structure on the manifold itself, which will be an essential ingredient for the proof of the main theorem.

3.3.1 A topological statement about connected components

We state with a general topological statement which we shall need.

Lemma 3.3.1. *Let X be a locally connected locally compact Hausdorff space. Let $\mathcal{K}(X)$ be the collection of compact subsets of X . Let $K \in \mathcal{K}(X)$ and let $A, A' \subset X$ be connected and precompact. If A, A' lie in the same connected component C of X then there is $L \in \mathcal{K}(X)$ such that they lie in the same connected component of $L \cap C$.*

Proof. For any topological space X and a nonempty connected subset A , denote by $\text{conn}(X, A)$ be the unique connected component C of X such that $C \supset A$.

Denote $\mathcal{K}' = \{L \in \mathcal{K}(X) \mid L \supset A \cup A'\}$. For any $A \subset X$ that is connected and precompact, let $C = \text{conn}(X, A)$. Since X is locally connected, C is open in X and locally connected. Let $P(A) = \bigcup_{L \in \mathcal{K}'} \text{conn}(L \cap C, A)$.

For any $L \in \mathcal{K}'$, if $x \in \text{conn}(L \cap C, A)$, then there is a compact connected neighborhood $\bar{U}_x \subset C$ of x . If $L' = L \cup \bar{U}_x \in \mathcal{K}'$, then $\text{conn}(L \cap C, A) \cup \bar{U}_x \subset \text{conn}(L' \cap C, A)$, so $P(A)$ is open.

For any precompact connected open set $U \subset C$, we have

$$\left(\begin{array}{l} U \cap P(A) \neq \emptyset \implies \exists L \in \mathcal{K}', \quad U \cap \text{conn}(L \cap C, A) \neq \emptyset \\ \implies U \subset \text{conn}((L \cup \bar{U}) \cap C, A) \subset P(A) \end{array} \right).$$

Thus $U \setminus P(A) \neq \emptyset$ implies that $U \subset C \setminus P(A)$. Since C has a topology base consisting of connected sets, $P(A)$ is closed in C . Hence $P(A)$ is nonempty and clopen in C and C is connected, so $P(A) = C$.

For any $A' \subset X$ that is connected and precompact, suppose we have $\text{conn}(X, A') = C$ and $\text{conn}(L_1 \cap C, A) \cap \text{conn}(L_2 \cap C, A') \neq \emptyset$, $L_1, L_2 \in \mathcal{K}'$. Let $L' = L_1 \cup L_2 \in \mathcal{K}'$, since $P(A) = P(A') = C$ then

$$\text{conn}(L' \cap C, A) \cap \text{conn}(L' \cap C, A') \supset \text{conn}(L_1 \cap C, A) \cap \text{conn}(L_2 \cap C, A') \neq \emptyset,$$

which means $\text{conn}(L' \cap C, A) = \text{conn}(L' \cap C, A')$. □

3.3.2 Slicing a manifold by an exhaustion function

Let M be a manifold. An *exhaustion function* f for M is a smooth function $f: M \rightarrow \mathbb{R}$ such that for any $\alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha])$ is compact. An exhaustion function for M always exists. Let $\text{Reg}(f)$ be the set of regular values of f (including $\mathbb{R} \setminus f(M)$). Fix as a *basepoint* $x_0 \in M$ a minimum point of f . For any $\alpha \in \text{Reg}(f) \cap f(M)$, let C be the connected component of $f^{-1}((-\infty, \alpha])$ containing x_0 . Define M_α as the union of C and the precompact connected components of $M \setminus C$. Then M_α is compact and connected, see Figure 3.1. For $\alpha \in \mathbb{R} \setminus f(M)$, let $M_\alpha = \emptyset$. We call M_α the *saturated slicing* of M by α . For any set $A \subset M$, let $A_\alpha = A \cap M_\alpha$.

We will need the following technical property of precompact subsets in the proof of

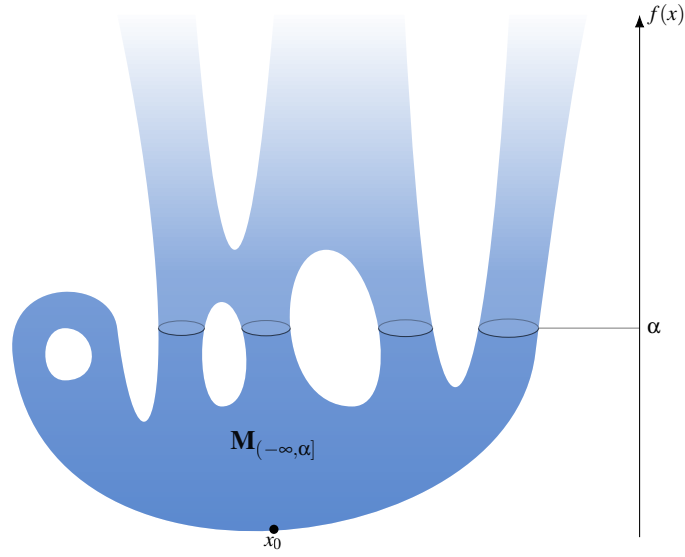


Figure 3.1: The saturated slicing M_α .

Lemma 3.5.1 (which itself is needed to prove the main theorem).

Lemma 3.3.2. *For any connected precompact set $A \subset M$,*

$$\theta_A \stackrel{\text{def}}{=} \inf\{\alpha \in f(A) \mid \forall \beta \in \text{Reg}(f), \beta > \alpha, A_\beta \text{ is connected}\}$$

is finite.

Proof. Fix an $\alpha \in \text{Reg}(f) \cap f(A)$. Since A_α is the interior of a compact manifold with boundary, it can only have finitely many components. By Lemma 3.3.1, there is $K \in \mathcal{K}(M)$ which is connected and contains x_0 and every component of A_α . Suppose $\beta \in \text{Reg}(f)$ and $\beta \geq \max_K f$, then $A_\beta \supset K$ contains every component of A_α . Note that any component of A_β contains a component of A_α , so A_β is connected. Hence $\theta_A \leq \beta < +\infty$. \square

3.3.3 A tree structure on a manifold

Consider the following combinatorial notions of trees which will be very useful for the proof of Theorem 3.2.1.

A *tree* is a strictly partially ordered set (\mathcal{T}, \prec) with the property that for each $x \in \mathcal{T}$, the set $\text{Pre}(x) = \{y \in \mathcal{T} \mid y \prec x\}$ of all *predecessors* of x is well ordered by \prec . We write \mathcal{T} for (\mathcal{T}, \prec) when there is no ambiguity. A *branch* in \mathcal{T} is a maximal linearly ordered subset of \mathcal{T} . Let $\text{Rt}(\mathcal{T}) = \{x \in T \mid \forall y \in T, y \not\prec x\} \neq \emptyset$ be the set of roots of \mathcal{T} . If $\text{Rt}(\mathcal{T})$ is a singleton we call \mathcal{T} *rooted*.

Let $\text{Suc}(x) = \{y \in \mathcal{T} \mid y \succ x\}$ be the set of all *successors* of x , then $(\text{Suc}(x), \prec)$ is a tree. Let

$$\text{Ch}(x) = \text{Rt}(\text{Suc}(x))$$

be the set of *immediate successors* or *children* of x . If for any $x \in \mathcal{T}$, $\text{Ch}(x)$ is finite, we call \mathcal{T} *locally finite*. Let

$$\text{Gch}(x) = \bigcup_{y \in \text{Ch}(x)} \text{Ch}(y)$$

be the set of *grandchildren* of x . Let $\text{Lf}(\mathcal{T}) = \{x \in T \mid \forall y \in T, x \not\prec y\}$ be the set of *pendant vertices* or *leaves* of \mathcal{T} . If $\text{Lf}(\mathcal{T}) = \emptyset$ we call \mathcal{T} *leafless*.

The *depth* of x is the ordinal of $\text{Pre}(x)$, which we denote by $\text{dpt}(x)$. Let

$$\text{hgt}(\mathcal{T}) = \sup\{\text{dpt}(x) + 1 \mid x \in \mathcal{T}\}$$

be the *height* of \mathcal{T} . For any ordinal $\ell < \text{hgt}(\mathcal{T})$, let

$$\text{Lv}(\ell) = \{x \in \mathcal{T} \mid \text{dpt}(x) = \ell\}$$

be the ℓ -th *level* of \mathcal{T} .

Let ω denote the smallest infinite ordinal. If $\text{hgt}(\mathcal{T}) = \omega$, then every node in \mathcal{T} has finite depth, but these depths are unbounded. We have the following essential construction for the combinatorial part of the proof of Theorem 3.2.1.

Lemma 3.3.3. *Let M be a noncompact manifold, $\alpha_0 = -\infty$ and $\{\alpha_\ell\}_{\ell \in \mathbb{N}} \subset \text{Reg}(f) \cap f(M)$ be an unbounded strictly increasing sequence. Let $\mathcal{L}(\ell)$ be the collection of unbounded connected components of $M \setminus M_{\alpha_{\ell-1}}$. Then there is a tree (\mathcal{T}, \supseteq) of open subsets of M such that*

$$\mathcal{T} = \coprod_{\ell \in \mathbb{N} \cup \{0\}} \mathcal{L}(\ell).$$

Moreover, (\mathcal{T}, \supseteq) is a rooted locally finite leafless tree of height ω , and $\mathcal{L}(\ell) = \text{Lv}(\ell)$ for each $\ell \in \mathbb{N} \cup \{0\}$.

Proof. Let $A_i \in \mathcal{L}(\ell_i) \subset \mathcal{T}$ where $\ell_i \in \mathbb{N} \cup \{0\}$, for $i = 1, 2$ and 3. By definition of connected components we have the following: if $A_1 \supseteq A_2$, then $\ell_1 < \ell_2$; if $A_1, A_2 \supseteq A_3$ and $\ell_1 < \ell_2$, then $A_1 \supseteq A_2$. Hence (\mathcal{T}, \supseteq) is a tree.

The only root of \mathcal{T} is $M \in \mathcal{L}(0)$, by induction $\mathcal{L}(\ell)$ is the ℓ -th level of \mathcal{T} , which is finite, so \mathcal{T} is locally finite. For any $A \in \text{Lv}(\ell)$, $A \setminus A_{\alpha_{\ell+1}} \neq \emptyset$, so \mathcal{T} is leafless. Hence $\{\text{dpt}(A) \mid A \in \mathcal{T}\} = \mathbb{N} \cup \{0\}$, and $\text{hgt}(\mathcal{T}) = \omega$. \square

3.4 Geometric-analytic constructions

Throughout this section, M is a noncompact oriented manifold of dimension $m \geq 1$. In this section, we present the analytic statements needed to prove the main theorem. The main tool we use is a version of Hodge theory which applies to certain noncompact manifolds which is sufficient for the purpose of the present paper. We split the content into several subsections for clarity.

3.4.1 Forms with compactly supported difference

In this subsection, we prove (using the work of Bueler–Prokhorenkov on Hodge theory [7]) a parametrized Moser stability theorem for two families $\{\omega_p\}_{p \in B}$, $\{\tau_p\}_{p \in B}$ of volume forms whose differences $\omega_p - \tau_p$, $p \in B$ are supported in some compact submanifold with boundary.

Lemma 3.4.1. *Let f be an exhaustion for M , see Section 3.3.2. Let N be a compact hypersurface of M through regular points of f . Then there exists $\varepsilon > 0$ and a diffeomorphism $\Phi: N \times (-\varepsilon, \varepsilon) \rightarrow V_N$ such that V_N is an open neighborhood of $N \subset M$, $\Phi(y, 0) = y$, $\pi(\Phi(y, s)) = \pi(y)$ and $f(\Phi(y, s)) = f(y) + s$ for any $(y, s) \in N \times (-\varepsilon, \varepsilon)$. If N is connected then V_N is connected too.*

Proof. Pick an arbitrary Riemannian metric g on M . Let \tilde{V}_N be an open neighborhood of $N \subset M$ which consists of regular points of f . Let $X \in \mathfrak{X}(M)$ be such that $X = |\nabla_g f|_g^{-2} \nabla_g f$ in \tilde{V}_N , where $\nabla_g f$ is the gradient of f , then $X(f) = 1$ in \tilde{V}_N . Take the flow of X , $\Phi: N \times (-\varepsilon, \varepsilon) \rightarrow M$, $(y, s) \mapsto x$, that is $\Phi(y, 0) = y$ for all $y \in N$ and $\frac{\partial \Phi}{\partial s}(y, s) = X(\Phi(y, s))$ for all $(y, s) \in N \times (-\varepsilon, \varepsilon)$, for $\varepsilon > 0$ small enough such that the image of Φ is contained in \tilde{V}_N . Then let $V_N = \Phi(N \times (-\varepsilon, \varepsilon))$. Since $X(f) = 1$ in V_N , we have $f(\Phi(y, s)) = f(y) + s$ for any $(y, s) \in N \times (-\varepsilon, \varepsilon)$, and Φ is a diffeomorphism. If N is connected, then V_N is the image of Φ , which is connected. \square

Theorem 3.4.2. *Let W be an open subset of M such that \overline{W} is a submanifold of M with boundary ∂W . Then for any $q \in \mathbb{N}$ with $1 \leq q \leq m$ there is an operator preserving smooth families of q -forms*

$$I_W^q: \{ \xi \in \Omega_c^q(M) \mid \text{supp } \xi \subset W, \xi|_W \in d\Omega_c^{q-1}(W) \} \rightarrow \{ \eta \in \Omega_c^{q-1}(M) \mid \text{supp } \eta \subset \overline{W} \}$$

satisfying $d \circ I_W^q = \text{id}$.

Proof. By [7] there is a weighted Hodge-Laplacian $\Delta_\mu: \Omega_c^q(W) \rightarrow \Omega_c^q(W)$ on W equipped with a specific metric g and measure μ . Its Green operator $G_\mu: \Omega_c^q(W) \rightarrow \Omega_c^q(W)$ and the weighted

codifferential $\delta_\mu: \Omega_c^q(W) \rightarrow \Omega_c^{q-1}(W)$ satisfy the identity $d \circ \delta_\mu \circ G_\mu \circ d = d$. Moreover if a form $\eta \in G_\mu(\Omega_c^q(W))$, then it has an extension $\tilde{\eta} \in \Omega_c^q(M)$, such that $\text{supp } \tilde{\eta} \subset \overline{W}$ and $\tilde{\eta}|_W = \eta$. For $\xi \in \Omega_c^q(M)$ that is supported in W , if $\xi|_W \in d\Omega_c^{q-1}(W)$ we define $I_W^q(\xi)$ as the extension of $(\delta_\mu \circ G_\mu)(\xi|_W)$ to $\Omega_c^{q-1}(M)$, see Figure 3.2. Then we have $d \circ I_W^q = \text{id}$.

The operator I_W^q preserves smooth families. Indeed, the p -derivative of a smooth family $\xi_p, p \in B$ of compactly supported forms is still compactly supported. Since the Green's operator G_μ is an integral operator with a singular kernel, we can pass the p -derivative through the operator G_μ , so $\partial_p G_\mu \xi_p$ exists and is a smooth form, for each $p \in B$. By similar arguments for higher order derivatives, $G_\mu \xi_p, p \in B$ is a smooth family. The map δ_μ preserves smooth families since it is a differential operator. \square

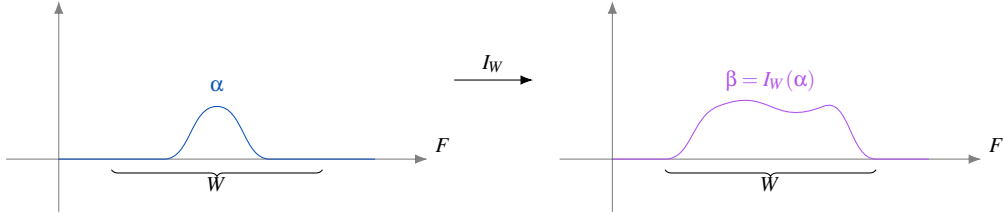


Figure 3.2: From compactly supported forms to forms with zero extensions.

Let B be a compact manifold. We adopt the following notations.

- $\Omega_{F,\text{vol}}M$ is the set of smooth families $\omega = \{\omega_p\}_{p \in B}$ of volume forms on M . Similarly, $\mathcal{F}^\infty(B; \Omega_{\geq 0}^m(M))$ is the set of smooth families of non-negative m -forms on M . Note that $\Omega_{F,\text{vol}}M \subsetneq \mathcal{F}^\infty(B; \Omega_{\geq 0}^m(M))$.
- $\mathcal{F}^\infty(B; \text{Diff}(M))$ is the set of smooth families $\varphi = \{\varphi_p\}_{p \in B}$ of diffeomorphisms of M .
- If $\omega \in \mathcal{F}^\infty(B; \Omega_{\geq 0}^m(M))$, $\int_M \omega$ is the map $B \rightarrow [0, +\infty]$ given by

$$\left(\int_M \omega \right) (p) = \int_M \omega_p.$$

- If $\omega \in \Omega_{F,\text{vol}}M$, $\varphi \in \mathcal{F}^\infty(B; \text{Diff}(M))$, we define

$$\varphi^* \omega = \left\{ \varphi_p^* \omega_p \right\}_{p \in B} \in \Omega_{F,\text{vol}}M.$$

Figure 3.4a illustrates the main point of the following lemma, where the shaded region is the support of $\omega_p - \tau_p$.

Lemma 3.4.3. *Let V be a connected open subset of M such that \bar{V} is a compact submanifold with boundary ∂V . Let $\omega, \tau \in \Omega_{F,\text{vol}}M$ be such that $\text{supp}(\omega_p - \tau_p) \subset V, \forall p \in B$ and $\int_V \omega = \int_V \tau$. Then there is a family $\varphi \in \mathcal{F}^\infty(B; \text{Diff}(M))$ such that $M \setminus V$ has a neighborhood in which φ_p is the identity for $p \in B$ and $\varphi^* \omega = \tau$.*

Proof. Let $N = \partial V$. Applying Lemma 3.4.1 to N there are $\varepsilon > 0$ and V_N a neighborhood of N with the properties stated in the lemma. Since B is compact and $\text{supp}(\omega_p - \tau_p) \subset V, \forall p \in B$, we may decrease ε if necessary so that $\text{supp}(\omega_p - \tau_p) \subset V \setminus \bar{V}_N, \forall p \in B$. Let $W = V \setminus \bar{V}_N$. Since the map $\int_W : H_c^m(W) \rightarrow \mathbb{R}$ is a linear isomorphism, and $\int_V \omega = \int_V \tau$, we have $(\omega_p - \tau_p)|_W \in d\Omega_c^{m-1}(W), \forall p \in B$. Therefore by Theorem 3.4.2 there exists a smooth family $\sigma_p = I_W^m \xi_p \in \Omega_c^{m-1}(M), \forall p \in B$, with $\text{supp} \sigma_p \subset \bar{W}$ such that $d\sigma_p = \omega_p - \tau_p, \forall p \in B$. Let $\omega_t = (1-t)\omega + t\tau \in \Omega_{F,\text{vol}}M$ for any $t \in [0, 1]$.

Since ω_t is nowhere vanishing there exists a unique smooth family of vector fields $\{X_{t,p}\}_{(t,p) \in [0,1] \times B} \subset \mathfrak{X}(M)$ where each $X_{t,p}$ is supported in \bar{W} and such that $\omega_{t,p}(X_{t,p}, \cdot) = \sigma_p$. Since \bar{W} is compact, for each $p \in B$, the flow $\varphi_{t,p}, t \in [0, 1]$ in M generated by $X_{t,p}$ exists and is the identity outside of \bar{W} . For $t \in [0, 1]$, $\varphi_t = \{\varphi_{t,p}\}_{p \in B} \in \mathcal{F}^\infty(B; \text{Diff}(M))$. Then $\varphi_t^* \omega_t = \omega$. If $\varphi = \varphi_1^{-1}$ then we have $\varphi^* \omega = \tau$. Since $X_{t,p} = 0$ in $M \setminus W$ for $(t,p) \in [0, 1] \times B$, $\varphi_{t,p}$ is the identity outside of \bar{W} . \square

3.4.2 The transfer of volumes

In this subsection, we prove a series of lemmas which allow us to transfer volumes of a smooth family of volume forms across the boundaries of compact submanifolds, so as to modify the smooth families $\{\omega_p\}_{p \in B}$, $\{\tau_p\}_{p \in B}$ so that they have the same volume in a certain set of compact submanifolds; then we move the volumes within the compact submanifolds, to pull ω_p back to τ_p for each $p \in B$.

Lemma 3.4.4. *Let $\omega \in \Omega_{F,\text{vol}}M$, and let $V \subset M$ be a precompact open set. Let $w \in C^\infty(B; \mathbb{R})$. Then there exists $\tau \in \Omega_{F,\text{vol}}M$ such that $\text{supp}(\omega_p - \tau_p) \subset V, \forall p \in B$ and $\int_V \tau = w|_V$.*

Proof. We can assume that $V \neq \emptyset$ since otherwise the lemma is trivial. Let $\xi \in \Omega_{F,\text{vol}}M$ be such that $\text{supp}(\xi_p - \omega_p) \subset V, \forall p \in B$ and $\int_V \xi < w$. Let $\eta \in \mathcal{F}^\infty(B; \Omega_{\geq 0}^m(M))$ be such that $\text{supp} \eta_p \subset V, \forall p \in B$ and $\int_V \eta > 0$. Then define $\tau = \xi + \frac{w - \int_V \xi}{\int_V \eta} \eta$. \square

Lemma 3.4.5. *Let N be a compact hypersurface of M and consider $\omega, \tau \in \Omega_{F,\text{vol}}M$. Then there is V_N an open neighborhood of N and $\varphi \in \mathcal{F}^\infty(B; \text{Diff}(M))$ such that, if V_N^+ and V_N^- are the connected components of $V_N \setminus V$, the following hold: $M \setminus V_N$ has a neighborhood in which φ_p is the identity for any $p \in B$; N has a neighborhood in which $\varphi^* \omega = \tau$; $\int_{V_N^+} \varphi^* \omega = \int_{V_N^+} \omega$; and $\int_{V_N^-} \varphi^* \omega = \int_{V_N^-} \omega$.*

Proof. By Lemma 3.4.1, there exists a neighborhood V_N of $N \subset M$, $\varepsilon > 0$ and a diffeomorphism $\Phi: N \times (-\varepsilon, \varepsilon) \rightarrow V_N$ such that $\Phi(y, 0) = y$ and $f(\Phi(y, s)) = f(y) + s$ for any $(y, s) \in N \times (-\varepsilon, \varepsilon)$, see Figure 3.4b. Let $V_N^+ = \Phi(N \times (0, \varepsilon))$ and $V_N^- = \Phi(N \times (-\varepsilon, 0))$.

First we consider $\Phi(N \times [0, \varepsilon))$. Since B is compact, there exists δ with $0 < \delta < \varepsilon/2$ such that

$$\int_{\Phi(N \times (0, \varepsilon - \delta))} \tau > \int_{\Phi(N \times (0, \delta))} \omega, \quad \int_{\Phi(N \times (0, \varepsilon - \delta))} \omega > \int_{\Phi(N \times (0, \delta))} \tau.$$

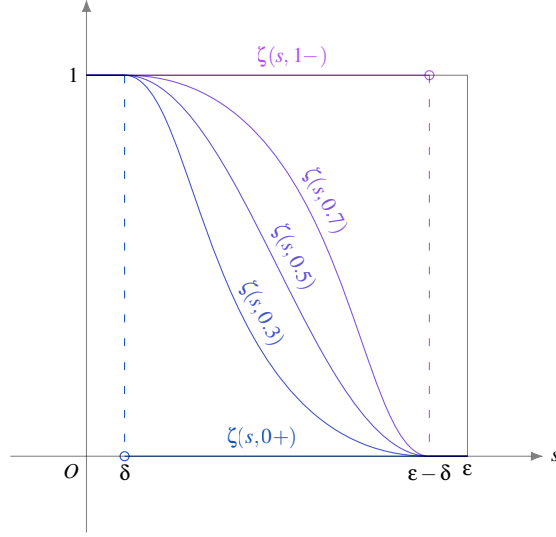


Figure 3.3: The graph of $\zeta(s, t)$.

Let $\zeta: (0, \varepsilon) \times (0, 1) \rightarrow [0, 1]$ be a smooth function with the properties (see Figure 3.3):

$$\begin{cases} \zeta(s, \cdot) = 1, & s \in (0, \delta]; \\ \lim_{t \rightarrow 0+} \zeta(s, t) = 0, \frac{\partial \zeta}{\partial t}(s, \cdot) > 0, \lim_{t \rightarrow 1-} \zeta(s, t) = 1, & s \in (\delta, \varepsilon - \delta); \\ \zeta(s, \cdot) = 0, & s \in [\varepsilon - \delta, \varepsilon). \end{cases}$$

Define $\theta: B \times (0, 1) \rightarrow \mathbb{R}$ by

$$\theta(p, t) = \int_{V_N^+} \zeta(s(p), t) \tau_p - \int_{V_N^+} \zeta(s(p), 1-t) \omega_p$$

where $s = \text{pr}_2 \circ \Phi^{-1}: \Phi(N \times (-\varepsilon, \varepsilon)) \rightarrow (-\varepsilon, \varepsilon)$, $\text{pr}_2: N \times (-\varepsilon, \varepsilon) \rightarrow (-\varepsilon, \varepsilon)$ is projection to the second factor. As ζ is smooth and $\omega, \tau \in \Omega_{F, \text{vol}} M$ it follows that θ is smooth. Furthermore

$$\frac{\partial \theta}{\partial t}(p, t) = \int_{V_N^+} \frac{\partial \zeta}{\partial t}(s(p), t) \tau_p + \int_{V_N^+} \frac{\partial \zeta}{\partial t}(s(p), 1-t) \omega_p > 0$$

for any $t \in (0, 1)$ and $\lim_{t \rightarrow 0+} \theta(p, t) < 0 < \lim_{t \rightarrow 1-} \theta(p, t)$ for any $p \in B$. Then for every $p \in B$ there is a unique $t(p)$ solving $\theta(p, t(p)) = 0$. By differentiation rules, the derivatives of t of any

order can be explicitly given in terms of the derivatives of θ , so $t: B \rightarrow \mathbb{R}$ is smooth.

Define $\lambda(p, x) = \zeta(s(p, x), t(p))$ and $\mu(p, x) = \zeta(s(p, x), 1 - t(p))$ in $B \times V_N^+$. The functions λ and μ are smooth in x and satisfy $\int_{V_N^+} \mu \omega = \int_{V_N^+} \lambda \tau$. Analogously we can define λ and μ in $B \times V_N^-$, and let $\lambda = \mu = 1$ on N . Notice that $\lambda = \mu = 1$ in $\Phi(N \times [-\delta, \delta])$, so we obtain smooth extensions of λ, μ which we also denote by $\lambda, \mu: B \times V_N \rightarrow \mathbb{R}$. Hence

$$\begin{aligned} \int_{V_N^+} ((1 - \mu)\omega + \lambda\tau) &= \int_{V_N^+} \omega, \\ \int_{V_N^-} ((1 - \mu)\omega + \lambda\tau) &= \int_{V_N^-} \omega. \end{aligned}$$

By Lemma 3.4.3 applied to $(1 - \mu)\omega + \lambda\tau$ and ω on V_N^+ and V_N^- respectively, combining the results we obtain $\varphi \in \mathcal{F}^\infty(B; \text{Diff}(M))$ such that $\varphi = \text{id}$ in $M \setminus \Phi(N \times (\delta - \varepsilon, \varepsilon - \delta))$ and $\varphi^* \omega = (1 - \mu)\omega + \lambda\tau$. \square

Lemma 3.4.6. *Let $\{L_j\}_{j \in \mathbb{N}}$ be a cover of M by connected compact submanifolds with boundary, which have the same dimension as M , and whose interiors are pairwise disjoint. If $\omega, \tau \in \Omega_{F, \text{vol}} M$ are such that $\int_{L_j} \omega = \int_{L_j} \tau$ for each $j \in \mathbb{N}$ then there is $\varphi \in \mathcal{F}^\infty(B; \text{Diff}(M))$ such that $\varphi^* \omega = \tau$.*

Proof. By the construction of $\{L_j\}_{j \in \mathbb{N}}$, any three different L_j 's for $j \in \mathbb{N}$ do not intersect. Let $\mathcal{C} = \{N \mid N \in \text{Conn}(L_j \cap L_k), j, k \in \mathbb{N}, j \neq k\}$. Then \mathcal{C} is a collection of pairwise disjoint connected hypersurfaces of M . So for each $N \in \mathcal{C}$, let $j, k \in \mathbb{N}$ be such that $N \subset L_j \cap L_k$, by Lemma 3.4.1, we obtain $\varepsilon_N > 0$ and a diffeomorphism $\Phi_N: N \times (-\varepsilon_N, \varepsilon_N) \rightarrow V_N$ where V_N is an open neighborhood of $N \subset M$. We require $V_N \subset L_j \cup L_k$.

We apply Lemma 3.4.5 to V_N to obtain $\varphi_N \in \mathcal{F}^\infty(B; \text{Diff}(M))$ such that $\varphi_N = \text{id}$ in a neighborhood of $M \setminus V_N$, $\varphi_N^* \omega = \tau$ in a neighborhood of N , and

$$\int_{V_N^+} \varphi_N^* \omega = \int_{V_N^+} \omega, \quad \int_{V_N^-} \varphi_N^* \omega = \int_{V_N^-} \omega.$$

Hence $\int_{L_j} \varphi_N^* \omega = \int_{L_j} \omega, \int_{L_k} \varphi_N^* \omega = \int_{L_k} \omega$. See Figure 3.4c.

If necessary, choose ε_N small so that $\overline{V_N}, N \in \mathcal{C}$, are mutually disjoint. Since replacing ω by $\varphi_N^* \omega$ each time does not change the volume of L_j for any $j \in \mathbb{N}$, we compose these φ_N for $N \in \mathcal{C}$, as they are the identity away from disjoint open sets, to obtain $\varphi' \in \mathcal{F}^\infty(B; \text{Diff}(M))$ such that $\omega' = \varphi'^* \omega$ is equal to τ in some neighborhood of $\bigcup_{N \in \mathcal{C}} N$ and $\int_{L_j} \omega' = \int_{L_j} \omega = \int_{L_j} \tau$ for each $j \in \mathbb{N}$. Applying Lemma 3.4.3 to each L_j for $j \in \mathbb{N}$ we get $\psi_j \in \mathcal{F}^\infty(B; \text{Diff}(M))$ such that $\tau = \psi_j^* \omega'$ in L_j and $\psi_j = \text{id}$ in a neighborhood of $M \setminus L_j$. Replacing ω' by $\psi_j^* \omega'$ each time and composing $\{\psi_j\}_{j \in \mathbb{N}}$ we obtain $\psi' \in \mathcal{F}^\infty(B; \text{Diff}(M))$ such that $\tau = \psi'^* \omega'$. Let $\varphi = \varphi' \circ \psi'$. \square

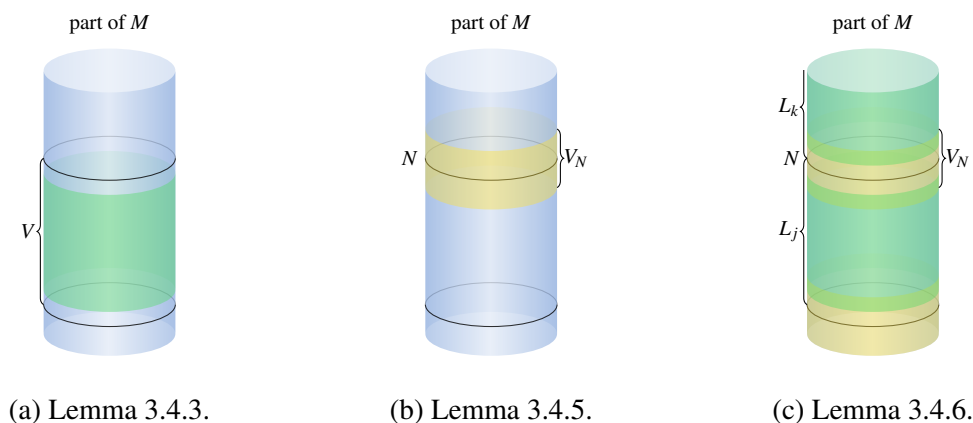


Figure 3.4: Illustrations of the regions affected by the diffeomorphisms.

3.4.3 Approximation lemma for smooth functions

Here we prove a key technical tool for the proof of Lemma 3.5.1, in which proof we often need to express a smooth function as a sum of smooth functions bounded by some continuous functions. The lemma below shows that we can always do that as long as the sum of the bounds is greater than the original smooth function.

In the following lemma, for any $y \in \mathbb{R}$ we let $y^+ = \max(y, 0)$ be its positive part and $y^- = \max(-y, 0)$ be its negative part. For a function $f: B \rightarrow \mathbb{R}$ we denote $f^+(b) = f(b)^+$, $f^-(b) = f(b)^-$ for $b \in B$, so $f^+, f^-: B \rightarrow \mathbb{R}$ are functions.

Lemma 3.4.7. *Let B be a connected compact manifold. Let $k \in \mathbb{N}$. Let $a \in C(B; \mathbb{R})$, $u \in C^\infty(B; \mathbb{R})$ such that $u < a$. Then for any $a_1, \dots, a_k \in C(B; \mathbb{R})$, with $\sum_{j=1}^k a_j = a$, there is $u_1, \dots, u_k \in C^\infty(B; \mathbb{R})$ such that $u_j < a_j$ for $1 \leq j \leq k$ and $\sum_{j=1}^k u_j = u$.*

Proof. Without loss of generality we assume $u = 0$ otherwise we replace a_j by $a_j - u/k$, u_j by $u_j - u/k$ for $1 \leq j \leq k$.

Choose $\varepsilon > 0$ with $k\varepsilon < \min a$. Define $h_j = a_j - \varepsilon$ for $1 \leq j \leq k$, then $\sum_{j=1}^k h_j = a - k\varepsilon > 0$. So $\sum_{j=1}^k h_j^+ > \sum_{j=1}^k h_j^- \geq 0$. Define

$$w_j = \frac{h_j^+}{\sum_{\ell=1}^k h_\ell^+} \sum_{\ell=1}^k h_\ell^- - h_j^-,$$

for $1 \leq j \leq k$. Then $\sum_{j=1}^k w_j = 0$. Moreover, for $1 \leq j \leq k$

$$h_j - w_j = h_j^+ - \frac{\sum_{\ell=1}^k h_\ell^-}{\sum_{\ell=1}^k h_\ell^+} h_j^+ \geq 0.$$

By Whitney Approximation Theorem, for $1 \leq j \leq k$, there is a function $v_j \in C^\infty(B; \mathbb{R})$ such that $|v_j - w_j| < \varepsilon/2$. Then let $u_j = v_j - \frac{1}{k} \sum_{\ell=1}^k v_\ell \in C^\infty(B; \mathbb{R})$. So $|u_j - w_j| < \varepsilon$, and $\sum_{j=1}^k u_j = 0$, hence $a_j - u_j > h_j - w_j \geq 0$ is as required. \square

3.5 Filtration lemma

Now we combine the topological-combinatorial, and geometric-analytic constructions from the previous sections. The objects in the following result are illustrated in Figure 3.5.

For any tree \mathcal{T} of height ω , for any $X \in \text{Lv}(\ell)$, $\ell \in \mathbb{N} \cup \{0\}$, let

$$\text{II}_{\mathcal{T}} X \stackrel{\text{def}}{=} X_{\alpha_{\ell+1}} = X \setminus \coprod_{Y \in \text{Ch}(X)} Y,$$

$$\text{III}_{\mathcal{T}} X \stackrel{\text{def}}{=} X_{\alpha_{\ell+2}} = X \setminus \coprod_{Z \in \text{Gch}(X)} Z.$$

Lemma 3.5.1. *Let M be a noncompact oriented connected manifold. Let B be a compact manifold. Suppose $\omega, \tau \in \Omega_{F, \text{vol}} M$ such that $\int_M \omega = \int_M \tau$, and for any connected component C of the complement of a compact subset of M , either $\int_C \omega = \int_C \tau = +\infty$, or $\int_C \omega$ and $\int_C \tau$ are finite and continuous, and their difference is smooth. Then there is a tree $(\mathcal{T}, \varnothing)$ of connected open subsets of M and $\{\omega_n\}_{n \in \mathbb{N} \cup \{0\}}, \{\tau_n\}_{n \in \mathbb{N} \cup \{0\}} \subset \Omega_{F, \text{vol}} M$ such that $\omega_0 = \omega, \tau_0 = \tau$ and for any $n \in \mathbb{N}$, $p \in B$, we have that*

$$\text{supp}((\omega_n)_p - (\omega_{n-1})_p) \cup \text{supp}((\tau_n)_p - (\tau_{n-1})_p) \subset \bigcup_{C \in \text{Lv}(2n-2)} (\text{III}_{\mathcal{T}} C)^\circ, \quad (3.5.1)$$

as well as that for each $A \in \text{Lv}(2n-3)$ with $n > 1$, $C \in \text{Lv}(2n-2)$, $E \in \text{Lv}(2n-1)$,

$$\int_{\text{II}_{\mathcal{T}} M} \omega_1 = \int_{\text{II}_{\mathcal{T}} M} \tau_1, \quad \int_{\text{III}_{\mathcal{T}} A} \omega_n = \int_{\text{III}_{\mathcal{T}} A} \tau_n \text{ for } n > 1; \quad (3.5.2)$$

$$\int_{\text{III}_{\mathcal{T}} C} \omega_n = \int_{\text{III}_{\mathcal{T}} C} \omega_{n-1}, \quad \int_{\text{III}_{\mathcal{T}} C} \tau_n = \int_{\text{III}_{\mathcal{T}} C} \tau_{n-1}; \quad (3.5.3)$$

$$\int_E \omega_n = \int_E \tau_n. \quad (3.5.4)$$

Proof. The abstract tools we have developed so far in the paper allow us to give an inductive proof of Lemma 3.5.1 with a minimum of technical fuss.

We aim to find $\alpha_0 = -\infty$ and $\{\alpha_\ell\}_{\ell \in \mathbb{N}} \subset \text{Reg}(f) \cap f(M)$ such that \mathcal{T} is constructed by Lemma 3.3.3. Note that, if we know $\{\alpha_\ell\}_{0 \leq \ell \leq m}$ for some $m \in \mathbb{N} \cup \{0\}$ for the sequence $\{\alpha_\ell\}_{\ell \in \mathbb{N} \cup \{0\}}$ defining \mathcal{T} , then we say \mathcal{T} is constructed up to the m -th level, so we know $\text{Lv}(\ell)$

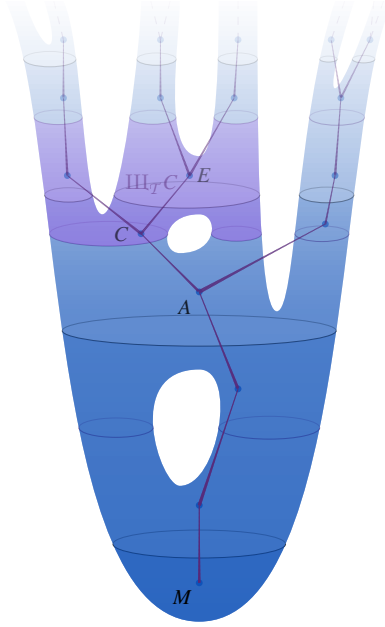


Figure 3.5: The manifold M sliced by f , and the tree \mathcal{T} in Lemma 3.5.1.

of \mathcal{T} for any ℓ with $0 \leq \ell \leq m$.

We proceed by induction on $n \in \mathbb{N} \cup \{0\}$ to find α_{2n-1} , α_{2n} and $\omega_n, \tau_n \in \Omega_{F, \text{vol}} M$ such that $\int_E \omega_n = \int_E \tau_n$ for any $E \in \text{Lv}(2n-1)$ ($E \in \text{Lv}(0)$ if $n = 0$).

Case 0. Set $\alpha_0 = -\infty$, then $M_{\alpha_0} = \emptyset$, and $\text{Lv}(0) = \{M\}$. Since $\omega_0 = \omega$, $\tau_0 = \tau$, we have

$$\int_M \omega_0 = \int_M \tau_0.$$

Case $(n-1)$ for $n \in \mathbb{N}$. Assume by induction

$$\int_A \omega_{n-1} = \int_A \tau_{n-1}. \quad (3.5.5)$$

for any $A \in \text{Lv}(2n-3)$ ($A \in \text{Lv}(0)$ when $n = 1$).

Case n for $n \in \mathbb{N}$. Let $\alpha_{2n-1} \in \text{Reg}(f)$ such that $\alpha_{2n-1} > \max\{\theta_C \mid C \in \text{Lv}(2n-2)\}$, where θ_C is defined by Lemma 3.3.2. Then \mathcal{T} is constructed up to the $(2n-1)$ -th level. Let

$A \in \text{Lv}(2n-3)$ (if $n = 1$ let $A = M$ and replace $\text{Gch}(A)$ by $\text{Ch}(M)$, $\text{III}_{\mathcal{F}}A$ by $\text{II}_{\mathcal{F}}M$ throughout this paragraph). Let $\text{Gch}_0(A)$ (resp. $\text{Gch}_1(A)$) be the subcollection of elements in $\text{Gch}(A)$ with finite (resp. infinite) volume. For any $E \in \text{Gch}(A)$, we define $\delta_E \in C^\infty(B; \mathbb{R})$ as follows: if E has finite volume, let

$$\delta_E = \int_E \tau_{n-1} - \int_E \omega_{n-1};$$

if E has infinite volume, let

$$\delta_E = \frac{1}{\#\text{Gch}_1(A)} \left(\int_{\text{III}_{\mathcal{F}}A} \tau_{n-1} - \int_{\text{III}_{\mathcal{F}}A} \omega_{n-1} - \sum_{E_0 \in \text{Gch}_0(A)} \delta_{E_0} \right).$$

Then by (3.5.5) we have

$$\sum_{E \in \text{Gch}(A)} \delta_E = \int_{\text{III}_{\mathcal{F}}A} \tau_{n-1} - \int_{\text{III}_{\mathcal{F}}A} \omega_{n-1}.$$

For any $C \in \text{Ch}(A)$, let $u_C \in C^\infty(B; \mathbb{R})$ be such that

$$\max \left(- \int_{\text{II}_{\mathcal{F}}C} \omega_{n-1}, - \int_{\text{II}_{\mathcal{F}}C} \tau_{n-1} + \sum_{E \in \text{Ch}(C)} \delta_E \right) < u_C < \int_C \omega_{n-1} - \int_{\text{II}_{\mathcal{F}}C} \omega_{n-1}.$$

Note that if C has finite volume,

$$\begin{aligned} & \left(\int_C \omega_{n-1} - \int_{\text{II}_{\mathcal{F}}C} \omega_{n-1} \right) - \left(- \int_{\text{II}_{\mathcal{F}}C} \tau_{n-1} + \sum_{E \in \text{Ch}(C)} \delta_E \right) \\ &= \int_C \omega_{n-1} + \left(\int_{\text{II}_{\mathcal{F}}C} \tau_{n-1} - \int_{\text{II}_{\mathcal{F}}C} \omega_{n-1} \right) + \sum_{E \in \text{Ch}(C)} \left(\int_E \tau_{n-1} - \int_E \omega_{n-1} \right) \\ &= \int_C \tau_{n-1} > 0, \end{aligned}$$

so such u_C exists. Since

$$u_C < \sum_{E \in \text{Ch}(C)} \int_E \omega_{n-1} = \int_C \omega_{n-1} - \int_{\Pi_{\mathcal{F}} C} \omega_{n-1},$$

by Lemma 3.4.7, we can choose $v_E \in C^\infty(B; \mathbb{R})$ such that $v_E < \int_E \omega_{n-1}$ and $\sum_{E \in \text{Ch}(C)} v_E = u_C$.

For any $E \in \text{Ch}(C)$, if E has infinite volume, take $\beta_E \in \text{Reg}(f)$ that is larger than θ_E .

Otherwise, the function

$$\lambda : B \times \mathbb{R} \rightarrow \mathbb{R},$$

$$(b, \beta) \mapsto \min \left(\left(\int_{E_{(-\infty, \beta]}} \omega_{n-1} \right) (b), \left(\int_{E_{(-\infty, \beta]}} \tau_{n-1} + \delta_E \right) (b) \right) - v_E(b)$$

is continuous in b , is increasing in β . Note that $\lim_{\beta \rightarrow +\infty} \rho(b, \beta) (\int_E \omega_{n-1} - v_E)(b) > 0$ for any $b \in B$. Since B is compact there is $\beta_E > \max\{\alpha_{2n-1}, \theta_E\}$ such that $\lambda(\cdot, \beta_E) > 0$. Let $\alpha_{2n} = \max_{E \in \text{Lv}(2n-1)} \beta_E$, then \mathcal{F} is constructed up to the $2n$ -th level. So $\Pi_{\mathcal{F}} E = E_{\alpha_{2n}}$, then we have $v_E < \int_{\Pi_{\mathcal{F}} E} \omega_{n-1}$, and $v_E - \delta_E < \int_{\Pi_{\mathcal{F}} E} \tau_{n-1}$.

Since all right hand sides are positive smooth functions, by Lemma 3.4.4, there are $\omega_n, \tau_n \in \Omega_{F, \text{vol}} M$ such that

$$\begin{aligned} \int_{\Pi_{\mathcal{F}} C} \omega_n &= \int_{\Pi_{\mathcal{F}} C} \omega_{n-1} + u_C, & \int_{\Pi_{\mathcal{F}} C} \tau_n &= \int_{\Pi_{\mathcal{F}} C} \tau_{n-1} + u_C - \sum_{E \in \text{Ch}(C)} \delta_E, \\ \int_{\Pi_{\mathcal{F}} E} \omega_n &= \int_{\Pi_{\mathcal{F}} E} \omega_{n-1} - v_E, & \int_{\Pi_{\mathcal{F}} E} \tau_n &= \int_{\Pi_{\mathcal{F}} E} \tau_{n-1} - (v_E - \delta_E), \end{aligned}$$

and

$$\text{supp}((\omega_n)_p - (\omega_{n-1})_p) \cup \text{supp}((\tau_n)_p - (\tau_{n-1})_p) \subset (M_{\alpha_{2n}})^\circ \setminus M_{\alpha_{2n-2}}.$$

Then we have

$$\begin{aligned}\int_{\text{III}_{\mathcal{F}A}} \omega_n &= \int_{\text{III}_{\mathcal{F}A}} \omega_{n-1} + \sum_{C \in \text{Ch}(A)} u_C \\ &= \int_{\text{III}_{\mathcal{F}A}} \tau_{n-1} - \sum_{E \in \text{Gch}(A)} (\delta_E - u_C) = \int_{\text{III}_{\mathcal{F}A}} \tau_n,\end{aligned}$$

and

$$\begin{aligned}\int_{\text{III}_{\mathcal{F}C}} \omega_n &= \int_{\text{II}_{\mathcal{F}C}} \omega_n + \sum_{E \in \text{Ch}(C)} \int_{\text{II}_{\mathcal{F}E}} \omega_n = \int_{\text{III}_{\mathcal{F}C}} \omega_{n-1}, \\ \int_{\text{III}_{\mathcal{F}C}} \tau_n &= \int_{\text{II}_{\mathcal{F}C}} \tau_n + \sum_{E \in \text{Ch}(C)} \int_{\text{II}_{\mathcal{F}E}} \tau_n = \int_{\text{III}_{\mathcal{F}C}} \tau_{n-1},\end{aligned}$$

and

$$\begin{aligned}\int_E \omega_n &= \int_{\text{II}_{\mathcal{F}E}} \omega_n + \int_E \omega_{n-1} - \int_{\text{II}_{\mathcal{F}E}} \omega_{n-1} \\ &= \int_E \omega_{n-1} - v_E = \int_E \tau_{n-1} - (v_E - \delta_E) = \int_E \tau_n.\end{aligned}$$

□

3.6 Proof of main theorem

We apply Lemma 3.5.1 to M and ω, τ and obtain the tree \mathcal{F} of connected open subsets of M , such that (3.5.1) to (3.5.4) are satisfied.

For $n \in \mathbb{N}$ and $C \in \text{Lv}(2n-2)$, applying Lemma 3.4.3 to $(\text{III}_{\mathcal{F}C})^\circ$, there are $\varphi_n, \psi_n \in \mathcal{F}^\infty(B; \text{Diff}(M))$ such that we have $\varphi_n^* \omega_{n-1} = \omega_n$, $\psi_n^* \tau_{n-1} = \tau_n$, and $\varphi_n = \psi_n = \text{id}$ outside of $(M\alpha_{2n})^\circ \setminus M\alpha_{2n-2}$. Let

$$\begin{aligned}\omega_\infty &= \lim_{n \rightarrow \infty} \omega_n, & \tau_\infty &= \lim_{n \rightarrow \infty} \tau_n, \\ \varphi_\infty &= \varphi_1 \circ \varphi_2 \circ \cdots, & \psi_\infty &= \psi_1 \circ \psi_2 \circ \cdots.\end{aligned}\tag{3.6.1}$$

Since $\{(\text{III } \mathcal{C})^\circ\}_{C \in \mathcal{T}, 2|\text{dpt}(C)}$ is mutually disjoint, the pointwise limits in (3.6.1) will be stable at a finite n , so $\omega_\infty, \tau_\infty \in \Omega_{F, \text{vol}} M$, $\varphi_\infty, \psi_\infty \in \mathcal{F}^\infty(B; \text{Diff}(M))$,

$$\int_{\text{II } \mathcal{M}} \omega_\infty = \int_{\text{II } \mathcal{M}} \tau_\infty, \quad \int_{\text{III } \mathcal{A}} \omega_\infty = \int_{\text{III } \mathcal{A}} \tau_\infty$$

for each $A \in \mathcal{T}$ with odd depth, $\varphi_\infty^* \omega = \omega_\infty$, and $\psi_\infty^* \tau = \tau_\infty$.

We have left to show that there is $\varphi' \in \mathcal{F}^\infty(B; \text{Diff}(M))$ with $\varphi'^* \omega_\infty = \tau_\infty$. Let $\{L_j\}_{j \in \mathbb{N}}$ be $\{\text{II } \mathcal{M}\} \cup \{\overline{\text{III } \mathcal{A}}\}_{A \in \mathcal{T}, 2 \nmid \text{dpt}(A)}$, then this is the result of Lemma 3.4.6.

Finally,

$$\varphi = \varphi_\infty \circ \varphi' \circ \psi_\infty^{-1} \in \mathcal{F}^\infty(B; \text{Diff}(M))$$

is as required.

3.7 Final remarks

We conclude with a few remarks:

1. We have proved Theorem 3.2.1 using a version of Hodge theory on noncompact manifolds due to Bueler and Prokhorenkov [7]. We believe that there should also be a parametric version of the Greene-Shiohama proof without resorting to Hodge theory. The idea of using Hodge theory is in itself of interest because it can be easily generalized (for instance to symplectic forms Chapter 5).
2. The geometry of volume preserving diffeomorphisms is much simpler than that of their symplectic counterparts (see [17] and [33]).
3. In the way of applications, we would like to mention that the Moser and Greene–Shiohama results are important in classical mechanics, where understanding the geometry of volume forms is relevant [15, 24].

4. There is a version of Theorem 3.2.1 for fiber bundles with nontrivial topology. Theorem 3.2.1 corresponds to the case of trivial bundles over B . The idea and techniques to prove this more general result are similar, but the statement and proof require the introduction of a significant amount of terminology.
5. If $B = [0, 1]$, a version of Theorem 3.2.1 was given for continuous families as [8, Theorem 1] for the case of manifolds M which are the interior of a compact manifold with boundary. The work relies on a version of Moser's theorem for compact manifolds with boundary due to Banyaga [5].

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Chapter 4

Moser-Greene-Shiohama stability for exhausted bundles

4.1 Introduction

In Chapter 3 I prove a parametric form of the Greene-Shiohama result. This may be viewed as a version for trivial fiber bundles $\pi: F \hookrightarrow M \rightarrow B$ with noncompact fiber F , where the total space M is diffeomorphic to $F \times B$.

My goal in this chapter is to generalize this result to a class of nontrivial fiber bundles $\pi: M \rightarrow B$ with noncompact fiber F , whose main property is that they are exhausted by some smooth function $f: M \rightarrow \mathbb{R}$ which is compatible with the fiber bundle structure; I will call these *exhausted* fiber bundles.

The proof developed in the upcoming sections follows the same idea as the proof in Chapter 3 but the implementation of the steps is different and requires me to introduce several new concepts of a topological nature, the key one being the notion of “filled subbundle” which replaces the notion of end. Next I am going to introduce the key notions of the paper. Our main theorem is stated in terms of these notions at the end of the section.

4.1.1 Exhausted and filled (sub)bundles

Let F, M, B be smooth manifolds, where M, B may have boundaries and B is connected. Let $\pi: M \rightarrow B$ be a smooth surjective map. Suppose that for every $p \in B$ there exists an open neighborhood U of p in B and a diffeomorphism $\phi: \pi^{-1}(U) \rightarrow U \times F$ such that $\pi = \phi \circ \text{pr}_1$, where pr_1 is the projection onto the first factor of the product. As usual, (π, M, B, F) is called a *fiber bundle*, with *underlying space* M , *base* B , and *fiber* F . We call U a *trivializing region* of π and (U, ϕ) a *trivialized chart*.

Definition 4.1.1. Let (π, M, B, F) be a fiber bundle. Let $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$ be a local trivialization, $f: M \rightarrow \mathbb{R}$ a smooth function, and $\{h_i: F \rightarrow \mathbb{R}\}_{i \in \mathcal{I}}$ a family of smooth functions such that $f \circ \phi_i^{-1} = h_i \circ \text{pr}_2$, where pr_2 is the projection onto the second factor of the product. We call $\mathbf{M} \stackrel{\text{def}}{=} (\pi, M, B, F, f)$ is a *filled bundle*. If f and all the $h_i, i \in \mathcal{I}$, are exhaustion functions, we call \mathbf{M} an *exhausted bundle*.

We call $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$ *compatible* with f . We write $\text{sp}(\mathbf{M}) \stackrel{\text{def}}{=} M$.

Definition 4.1.2. Let $\mathbf{M} = (\pi, M, B, F, f)$ be a filled bundle, A a submanifold of M with or without boundary, and $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$ a local trivialization compatible with f . We call A a *filled subspace* of \mathbf{M} with respect to the trivialization if for any $i \in \mathcal{I}$ there is $P_i \subset F$ with $\phi_i(A \cap \pi^{-1}(U_i)) = U_i \times P_i$. If $U_i \cap U_j \neq \emptyset$ then there is a diffeomorphism of F induced by a change of charts $\phi_j \circ \phi_i^{-1}$ sending P_i to P_j . Since B is connected, the $P_i, i \in \mathcal{I}$, are diffeomorphic; let P be one of them. Then $\mathbf{A} \stackrel{\text{def}}{=} (\pi|_A, A, B, P, f|_A)$ is a filled bundle, which we call a *filled subbundle* of \mathbf{M} . We write $\mathbf{M}|_A \stackrel{\text{def}}{=} \mathbf{A}$.

Here are some examples of filled and exhausted bundles:

1. Let (π, N, B, E) be a compact fiber bundle and let F be a noncompact manifold with a smooth function $h: F \rightarrow \mathbb{R}$. Then $\mathbf{M} = (\pi \circ \text{pr}_1, N \times F, B, E \times F, h \circ \text{pr}_2)$ is a filled bundle which is exhausted if and only if h is an exhaustion function.

2. Let $F = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, y^2 + z^2 > \frac{1}{4}\}$, then F is a noncompact 2-manifold with 4 ends. Let $\phi \in \text{Diff}(F)$ be the diffeomorphism given by $\phi(x, y, z) = (x, -y, -z)$, switching two ends $z \rightarrow +\infty$ and $z \rightarrow -\infty$. Let $h: F \rightarrow \mathbb{R}, h(x, y, z) = z^2 + ((y^2 + z^2) - \frac{1}{4})^{-1}$, then h is an exhaustion function with the property $h \circ \phi = h$. Let $B = \mathbb{S}^1 = (0, 2)/(p \mapsto p + 1)$. Then define $M = (0, 2) \times_{\phi} F$ where $\varphi: (1, 2) \times F \rightarrow (0, 1) \times F$ is given by $\varphi(p, y) = (p - 1, \phi(y))$. Let $\pi: M \rightarrow B$ be the map induced by $\text{pr}_1: (0, 2) \times F \rightarrow (0, 2)$, and $f: M \rightarrow \mathbb{R}$ be the map induced by $h \circ \text{pr}_2: (0, 2) \times F \rightarrow \mathbb{R}$. Then $\mathbf{M} = (\pi, M, B, F, f)$ is an oriented exhausted bundle where M has 3 ends, since ϕ is orientation preserving.

3. Let \mathcal{G} be a subgroup of $\text{SO}(n)$. Let $E \subset \mathbb{R}^k$ be a noncompact complete submanifold, which is invariant under \mathcal{G} . Let $u: \mathbb{R}^k \rightarrow \mathbb{R}$ be a smooth function such that $u \circ \phi = u$ for any $\phi \in \mathcal{G}$. Let $F = E \cap \{u > 0\}$. Let $h: F \rightarrow \mathbb{R}, h(x) = |x|^2 + u(x)^{-1}$, then h is an exhaustion function with the property $h \circ \phi = h$ for any $\phi \in \mathcal{G}$. Let (π, M, B, F) be any fiber bundle with structure group \mathcal{G} such that B is compact. Let f be the unique exhaustion for M such that the transition maps in \mathcal{G} is compatible with f , with same $h = h_i$. Then $\mathbf{M} = (\pi, M, B, F, f)$ is an oriented exhausted bundle with noncompact fiber.

4.1.2 Releasing a filled bundle

A diffeomorphism will be constructed in our main theorem to move volumes within each fiber of a filled bundle \mathbf{M} . To construct it we chop the bundle into subbundles \mathbf{A} , with disconnected fibers. Because of this we cannot transfer volumes between connected components. To resolve the issue we introduce a new filled bundle: RIs \mathbf{A} .

For any topological spaces X, Y let $\text{Conn} X$ be the set of connected components of X and if $\mu: X \rightarrow Y$ is continuous, let $\text{Conn} \mu: \text{Conn} X \rightarrow \text{Conn} Y$ be the map sending C to the connected component of Y containing $\mu(C)$.

Proposition 4.1.1. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a filled bundle with M connected. Let $B_M \stackrel{\text{def}}{=} \text{Conn} \mu$*

$\coprod_{p \in B} \text{Conn} \pi^{-1}(p)$. Let (U, ϕ) be a trivialized chart. Let $\lambda_U: \coprod_{p \in U} \text{Conn} \pi^{-1}(p) \rightarrow U \times \text{Conn} \pi^{-1}(U)$ send a connected component C of $\pi^{-1}(p)$ to p paired with the connected component of $\pi^{-1}(U)$ containing C . Endow B_M with the smooth structure for which λ_U is a diffeomorphism. Then $c_M: B_M \rightarrow B$ given by $c_M(\text{Conn} \pi^{-1}(p)) = \{p\}$, $p \in B$, is a covering. Moreover, $\text{Rls} \pi: M \rightarrow B_M$ given by $(\text{Rls} \pi)|_{\pi^{-1}(p)} \stackrel{\text{def}}{=} \text{Conn}: \pi^{-1}(p) \rightarrow \text{Conn} \pi^{-1}(p)$, $p \in B$, is a fiber bundle with connected fiber, say F_M , and

$$\text{Rls} \mathbf{M} \stackrel{\text{def}}{=} (\text{Rls} \pi, M, B_M, F_M, f) \quad (4.1.1)$$

is a filled bundle.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\pi \times \text{Conn}} & U \times \text{Conn} \pi^{-1}(U) , \\ \downarrow \pi & \searrow \text{Rls} \pi & \uparrow \lambda_U \\ U & \xleftarrow{c_M} & \coprod_{p \in U} \text{Conn} \pi^{-1}(p) \end{array}$$

so c_M is a covering, $\text{Rls} \pi$ is smooth and locally trivial, and $(\text{Rls} \pi)^{-1}(p)$ is connected for each $p \in B_M$. Since M is connected, so is $B_M = (\text{Rls} \pi)(M)$. Hence all fibers of $\text{Rls} \pi$ are diffeomorphic. \square

We call $\text{Rls} \mathbf{M}$ in Proposition 4.1.1 the *releasing* of \mathbf{M} .

4.1.3 Fiber forms

Let $\mathbf{M} = (\pi, M, B, F, f)$ be a filled bundle with oriented fiber. Let $\iota_p: \pi^{-1}(p) \hookrightarrow M$ be the inclusion, $p \in B$. A *fiber k -form* on \mathbf{M} is a family $\{\omega_p\}_{p \in B}$ such that ω_p is a k -form on $\pi^{-1}(p)$ and there exists $\omega \in \Omega^k(M)$ with $\omega_p = \iota_p^* \omega$. We denote $\{\omega_p\}_{p \in B} \stackrel{\text{def}}{=} \omega$. A *fiber top-form* is a fiber $(\dim F)$ -form. A *fiber volume form* is a fiber top-form ω such that ω_p is a volume form on

$\pi^{-1}(p)$. Let $\Omega_F^k(\mathbf{M})$ be the space of fiber k -forms on \mathbf{M} . The space of compactly supported fiber k -forms $(\Omega_F^k)_c(\mathbf{M})$ on \mathbf{M} is defined analogously. Let $\Omega_{F,\text{vol}}(\mathbf{M})$ be the space of fiber volume forms on \mathbf{M} .

4.1.4 Statement of main theorem

Let $\mathbf{M} = (\pi, M, B, F, f)$ be a filled bundle with oriented fibers. If ω is a fiber top-form on \mathbf{M} and for all $p \in B$ the integral $\int_{\pi^{-1}(p)} \omega_p$ exists (it can be $\pm\infty$), we call the map defined by $(\int_{\mathbf{M}} \omega)(p) = \int_{\pi^{-1}(p)} \omega_p$ the *fiber integral of ω on \mathbf{M}* .

Definition 4.1.3. Two forms $\omega, \tau \in \Omega_{F,\text{vol}}(\mathbf{M})$ are *commensurable on a filled subbundle \mathbf{A} of \mathbf{M}* if their *released fiber integrals* on \mathbf{A} :

$$\int_{\mathbf{A}} \omega \stackrel{\text{def}}{=} \int_{\text{Rls}\mathbf{A}} \omega \quad \text{and} \quad \int_{\mathbf{A}} \tau \stackrel{\text{def}}{=} \int_{\text{Rls}\mathbf{A}} \tau: B_{\mathbf{A}} \longrightarrow [-\infty, +\infty]$$

exist and are continuous with smooth difference, or are both infinite. We say that ω, τ are *commensurable* if they are commensurable on the restriction of \mathbf{M} to every unbounded connected component¹ of $f^{-1}(\alpha, +\infty)$, $\alpha \in \text{Reg}(f)$.

Below a diffeomorphism φ of M is a *fiber diffeomorphism* if $\pi \circ \varphi = \varphi \circ \pi$. If ω is a fiber k form we define $\varphi^* \omega = \left\{ \varphi|_{\pi^{-1}(p)}^* \omega_p \right\}_{p \in B}$.

Theorem 4.1.2. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a connected exhausted bundle with compact base B and oriented noncompact connected fiber F . Then for any commensurable fiber volume forms ω, τ on \mathbf{M} with equal fiber integral, there exists a fiber diffeomorphism $\varphi: M \rightarrow M$ such that $\varphi^* \omega = \tau$.*

We conclude with a few remarks:

¹Such restriction is always a filled subbundle, proved in Lemma 4.3.1

1. The following is an interesting problem: give conditions on a fiber bundle so that it admits an exhausted bundle structure.
2. If the fiber bundle in Theorem 4.1.2 is trivial we recover Theorem 3.2.1. If B is a point, this was proved by Greene and Shiohama [19].
3. The proof strategy of Theorem 4.1.2 consists of giving the manifold a tree structure, and then constructing in terms of it a global diffeomorphism intertwining the volume forms by glueing. This strategy is analogous to the one adopted in Chapter 3 but the results we prove do not follow from Chapter 3. On the other hand this is no surprise since the main theorem for smooth families can be stated with essentially no preliminaries but for fiber bundles a lot more preparation was required (Sections 4.1.1 and 4.1.2) to state Theorem 4.1.2. The reason was explained in Section 4.1.2 where the key notion of *releasing a fiber bundle* is given. It is in terms of this notion that we can express the conditions on the integrals over the bundle. The delicate problem has to do with the connectivity of the fibers of fiber bundles not being in general inherited by subbundles (which *cannot* occur for trivial bundles as considered in Chapter 3).
4. Understanding the geometry of volume forms is important in classical mechanics, see for instance [15].

4.2 Category of filled bundles and release functor

Next we define the categories of filled bundles and connected filled bundles for an important reason: later we cut filled subbundles of the exhausted bundle \mathbf{M} into subsubbundles, and we will distribute volumes of the fibers of subbundles into those of subsubbundles. As discussed in Section 4.1.2, the nonconnectivity of the fibers forces us to consider the releasing of subbundles and subsubbundles. This causes another problem, that their bases are different

manifolds. Thanks to the functoriality of the release operation (Lemma 4.2.1), the bases of subsubbundles are covering spaces of those of subbundles, which makes the distribution of volumes feasible.

Let FilBund be the category with objects the filled bundles $\mathbf{M} = (\pi, M, B, F, f)$ and with morphisms from \mathbf{M} to $\mathbf{M}' = (\pi', M', B', F', f')$ given by $\mu = (\mu, \mu_B)$, where μ and μ_B are smooth maps such that

$$\begin{array}{ccc} M & \xrightarrow{\pi} & B \\ f \swarrow & \downarrow \mu & \downarrow \mu_B \\ \mathbb{R} & \xleftarrow{f'} & M' \xrightarrow{\pi'} B' \end{array}$$

commutes. Denote by $\text{Obj}(\text{FilBund})$ the space of objects of FilBund and $\text{Mor}(\mathbf{M}, \mathbf{M}')$ the space of morphisms from \mathbf{M} to \mathbf{M}' .

Lemma 4.2.1. *Let CFilBund be the subcategory of FilBund whose objects have connected underlying spaces. Let $(\text{Rls } \pi, M, B_M, F_M, f)$ be as in Proposition 4.1.1. Then there is a functor $\text{Rls}: \text{CFilBund} \rightarrow \text{CFilBund}$ such that on objects $\text{Rls}(\pi, M, B, F, f) \stackrel{\text{def}}{=} (\text{Rls } \pi, M, B_M, F_M, f)$.*

Proof. Let $\mathbf{M} = (\pi, M, B, F, f)$ and $\mathbf{M}' = (\pi', M', B', F', f')$ be objects of CFilBund . Let $\mu = (\mu, \mu_B) \in \text{Mor}(\mathbf{M}, \mathbf{M}')$. Let $\text{Rls } \mathbf{M} = (\text{Rls } \pi, M, B_M, F_M, f)$ and $\text{Rls } \mathbf{M}' = (\text{Rls } \pi', M', B_{M'}, F_{M'}, f')$ be as in Proposition 4.1.1. We are going to define the functor Rls on morphisms of CFilBund . Let $v: B_M \rightarrow B_{M'}$ be the unique map defined by the commutative diagram (which also clarifies the relationships among v and the maps defined in Proposition 4.1.1)

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\mu} & (\pi')^{-1}(U') \\ \downarrow \pi \times \text{Conn} & & \downarrow \pi' \times \text{Conn} \\ U \times \text{Conn } \pi^{-1}(U) & \xrightarrow{\mu_B \times \text{Conn } \mu} & U' \times \text{Conn } (\pi')^{-1}(U') \\ \downarrow \lambda_U & & \downarrow \lambda_{U'} \\ \coprod_{p \in U} \text{Conn } \pi^{-1}(p) & \xrightarrow{v} & \coprod_{p' \in U'} \text{Conn } (\pi')^{-1}(p') \\ \downarrow c_M & & \downarrow c_{M'} \\ U & \xrightarrow{\mu_B} & U' \end{array}$$

Here U is any open subset of B such that U and $U' \stackrel{\text{def}}{=} \mu_B(U)$ are trivializing regions of π and π' , respectively.² Define

$$\text{Rls}(\mu) \stackrel{\text{def}}{=} (\mu, \nu). \quad (4.2.1)$$

We have defined Rls on objects by formula (4.1.1) and on morphisms by formula (4.2.1). One can verify that Rls assigns the identity map to the identity map and is associative, and therefore Rls is a functor. \square

We call Rls the *release functor* (the relation between \mathbf{M} and $\text{Rls}\mathbf{M}$ is shown in Figure 4.1).

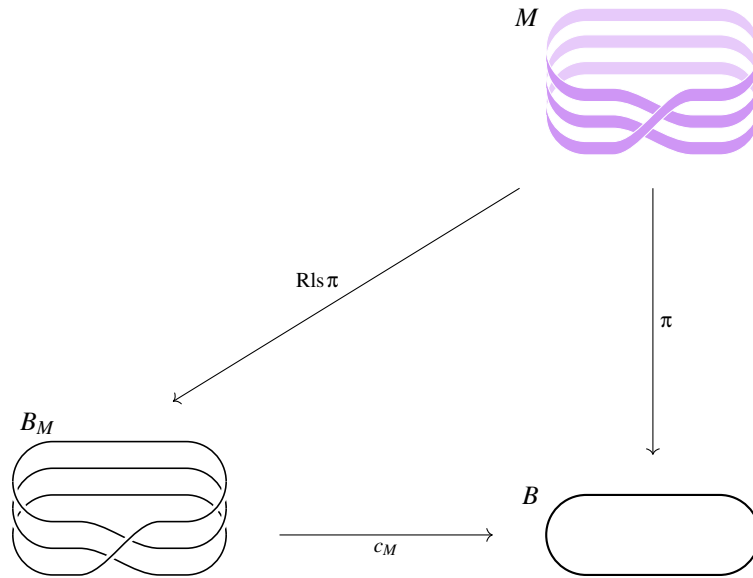


Figure 4.1: $\text{Rls}\mathbf{M} = (\text{Rls}\pi, M, B_M, F_M, f)$.

We apply Lemma 4.2.1 to the inclusion of bundles:

Corollary 4.2.2. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a filled bundle. Let A be a filled subspace of \mathbf{M} and $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{M}|_A$. Let $\iota: A \hookrightarrow M$ be the inclusion and define the morphism $\iota \stackrel{\text{def}}{=}} (\iota, \text{id}_B)$ from \mathbf{A}*

²To define ν we only need the middle rectangle in the diagram but the diagram provides a useful way to keep in mind all maps involved. Also, note that ν is uniquely defined because λ_U is a diffeomorphism; it is well defined because the collection of all such U is a base of the topology of B , and the definitions of ν on the preimages of overlapping regions by c_M coincide.

to \mathbf{M} . Then $\text{Rls } \iota = (\iota, \kappa)$ where $\kappa: B_A \rightarrow B_M$ is the unique map induced by the natural map $\text{Conn } \pi|_A^{-1}(U) \rightarrow \text{Conn } \pi^{-1}(U)$, for any trivializing region U of π .

We call ι in Corollary 4.2.2 the *(inclusion) embedding* of \mathbf{A} into \mathbf{M} .

Lemma 4.2.3. *If the base of \mathbf{M} is compact and the fiber of \mathbf{A} has finitely many connected components then κ in Corollary 4.2.2 is a covering map between compact spaces.*

Proof. For any open $U \subset B$ whose closure is contained in a trivializing region of π , $\bar{U} \times \text{Conn } \pi^{-1}(\bar{U})$ consists of finitely many copies of \bar{U} , so it is compact. Since B has a cover by finitely many such U , B_M is the union of finitely many sets diffeomorphic (by λ_U^{-1}) to $\bar{U} \times \text{Conn } \pi^{-1}(\bar{U})$, so B_M is compact. If the fiber of \mathbf{A} has finitely many connected components, analogous arguments ensure the compactness of B_A .

Since B_A is compact, $\kappa(B_A)$ is compact. But since κ is a local diffeomorphism, $\kappa(B_A)$ is open in B_M , which means κ is surjective, hence a covering map. \square

If $\kappa: B' \rightarrow B$ is a covering space with B connected we denote by $\#\kappa$ the *number of sheets* of κ , that is, the number of $\kappa^{-1}(p)$ for any $p \in B$ (independent of p).

4.3 Ingredients for the proof of the main theorem

In this section we prove most of the intermediate statements needed to prove the main theorem. The results of the section generalize, but do not follow directly from, the results of Sections 3.3 and 3.4, so we had to suitably modify the statements and adapt the proofs. The new difficulty is, as it was explained earlier, that the connectivity of the fibers of a bundle is not inherited by subbundles.

4.3.1 Slicing an exhausted bundle

Lemma 4.3.1. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be an exhausted bundle. Let $A = \bigcup_{k=1}^m A_k$, where $m \in \mathbb{N}$ and $A_k \in \text{Conn}(f^{-1}(I_k))$ for some interval I_k whose endpoints are regular values of f . Then A is a filled subspace of \mathbf{M} .*

Proof. Let $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$ be a local trivialization. By continuity, $\phi_i(A_k \cap \pi^{-1}(U_i)) = U_i \times P_{ik}$ for some P_{ik} , which is the disjoint union of some connected components of $h_i^{-1}(I_k)$. Hence if $P_i = \bigcup_{k=1}^m P_{ik}$, then $\phi_i(A \cap \pi^{-1}(U_i)) = U_i \times P_i$. So A is a filled subspace of \mathbf{M} . \square

From Lemma 4.3.1 we can conclude:

Corollary 4.3.2. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be an exhausted bundle and $\alpha \in \mathbb{R}$. Let $x_0 \in M$ be a minimum³ of f .*

- *If $\alpha \in \mathbb{R} \setminus f(M)$ define $M_{[\alpha]} \stackrel{\text{def}}{=} \emptyset$;*
- *If $\alpha \in \text{Reg}(f) \cap f(M)$ and C_α be a connected component of $f^{-1}(-\infty, \alpha]$ containing x_0 . Define $M_{[\alpha]}$ to be the union of C_α and the precompact connected components of $M \setminus C_\alpha$.*

Then $M_{[\alpha]}$ is a compact and connected filled subspace of \mathbf{M} with respect to any local trivialization.

By Corollary 4.3.2, we can define

$$\mathbf{M}_{[\alpha]} \stackrel{\text{def}}{=} \mathbf{M}|_{M_{[\alpha]}}$$

a filled subbundle of \mathbf{M} . From Lemma 4.3.1 we also have:

Corollary 4.3.3. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be an exhausted bundle and $\alpha \in \mathbb{R}$. Let A be a filled subspace of \mathbf{M} with respect to $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$. Let $P \subset F$ be the fiber of the filled subbundle $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{M}|_A$ of \mathbf{M} . Then $A_{[\alpha]} \stackrel{\text{def}}{=} A \cap M_{[\alpha]}$ is a filled subspace of \mathbf{M} with respect to $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$.*

³The bundles $\mathbf{M}_{[\alpha]}$ and $\mathbf{A}_{[\alpha]}$ will depend on the choice of x_0 . For this reason, we fix the choice of x_0 throughout the chapter.

By Corollary 4.3.2, we can define

$$\mathbf{A}_{[\alpha]} \stackrel{\text{def}}{=} \mathbf{M}|_{A_{[\alpha]}}.$$

a filled subbundle of \mathbf{M} .

The following gives the existence of $\mathbf{A}_{[\alpha]}$ with good properties: *saturated slices*. If $\mathbf{A}_{[\alpha]}$ is a saturated slice of \mathbf{A} then the connected components of the fiber of $\mathbf{A}_{[\alpha]}$ are in one-to-one correspondence with those of \mathbf{A} .

Lemma 4.3.4. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be an exhausted bundle with compact base. Let A be a connected filled subspace of \mathbf{M} and $\mathbf{A} = \mathbf{M}|_A$. For any $\alpha \in \text{Reg}(f) \cap f(A)$, let $\iota_\alpha: \mathbf{A}_{[\alpha]} \hookrightarrow \mathbf{A}$ be the embedding, and let $\text{Rls } \iota_\alpha = (\iota_\alpha, \kappa_\alpha)$. Then map κ_α is a covering, see Lemma 4.2.3. If for any $\alpha' \in \text{Reg}(f)$ with $\alpha' \geq \alpha$, κ_α is a diffeomorphism we call $\mathbf{A}_{[\alpha]}$ a saturated slice of \mathbf{A} by α . Then for any such \mathbf{A} saturated slices of \mathbf{A} exist.*

For the proof of Lemma 4.3.4, see Section 4.5.2.

4.3.2 A tree structure on a connected exhausted bundle

The following generalizes Lemma 3.3.3. See Section 4.5.1 for a review of trees.

Lemma 4.3.5. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a connected exhausted bundle, $\alpha_0 = -\infty$ and $\{\alpha_\ell\}_{\ell \in \mathbb{N}}$ in $\text{Reg}(f) \cap f(M)$ be an unbounded strictly increasing sequence. Let $\mathcal{L}(\ell)$ be the collection of $\mathbf{A} = \mathbf{M}|_A$ where A is any unbounded connected component of $M \setminus M_{[\alpha_{\ell-1}]}$. Then there is a tree (\mathcal{T}, \supseteq) of filled subbundles of \mathbf{M} such that $\mathcal{T} = \coprod_{\ell \in \mathbb{N} \cup \{0\}} \mathcal{L}(\ell)$, where for $\mathbf{A}, \mathbf{C} \in \mathcal{T}$, $\mathbf{A} \supseteq \mathbf{C}$ if \mathbf{C} is a filled subbundle of (not equal to) \mathbf{A} . Moreover, (\mathcal{T}, \supseteq) is a rooted locally finite leafless tree of height ω , and $\mathcal{L}(\ell) = \text{Lv}(\ell)$ for each $\ell \in \mathbb{N} \cup \{0\}$.*

Proof. Let $\mathbf{A}_i \in \mathcal{L}(\ell_i) \subset \mathcal{T}$ where $\ell_i \in \mathbb{N} \cup \{0\}$, for $i = 1, 2$ and 3. By definition of connected components we have the following: if $\mathbf{A}_1 \supseteq \mathbf{A}_2$, then $\ell_1 < \ell_2$; if $\mathbf{A}_1, \mathbf{A}_2 \supseteq \mathbf{A}_3$ and $\ell_1 < \ell_2$, then

$\mathbf{A}_1 \supsetneq \mathbf{A}_2$. Hence $(\mathcal{T}, \supsetneq)$ is a tree. The only root of \mathcal{T} is $\mathbf{M} \in \text{Lv}(0)$, by induction $\mathcal{L}(\ell)$ is the ℓ -th level of \mathcal{T} , which is finite, so \mathcal{T} is locally finite. For any $\mathbf{A} \in \text{Lv}(\ell)$ with $A = \text{sp}(\mathbf{A})$, $A \setminus A_{[\alpha_{\ell+1}]} \neq \emptyset$, so \mathcal{T} is leafless. Hence $\{\text{dpt}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{T}\} = \mathbb{N} \cup \{0\}$, and $\text{hgt}(\mathcal{T}) = \omega$. \square

4.3.3 Transferring volumes within fibers

Next we will use the analytic tool, the work of Bueler–Prokhorov on Hodge theory [7], to prove a series of lemmas that allow us to move the volumes within the fibers of an exhausted bundle in various manners. First we explain how to move volumes within the interiors of compact submanifolds of the fibers.

Lemma 4.3.6. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a filled bundle with compact base. Then following hold:*

- (1) *If $h_i, i \in \mathcal{I}$ in Definition 4.1.1 are exhaustion functions, then \mathbf{M} is an exhausted bundle.*
- (2) *Suppose that \mathbf{M} is also exhausted. Then B, F are compact if and only if M is compact.*

Proof. Since B is compact, let $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$ be a local trivialization of \mathbf{M} such that \mathcal{I} is finite and \overline{U}_i is compact for any $i \in \mathcal{I}$. Now that h_i are exhaustion functions for F , let $\alpha \in \text{Reg}(f)$, then

$$f^{-1}((-\infty, \alpha]) = \bigcup_{i \in \mathcal{I}} f^{-1}((-\infty, \alpha]) \cap \pi^{-1}(\overline{U}_i) = \bigcup_{i \in \mathcal{I}} \phi_i(\overline{U}_i \times h_i^{-1}((-\infty, \alpha]))$$

is compact. Hence f is an exhaustion function for M and then \mathbf{M} is an exhausted bundle. This proves (1).

If \mathbf{M} is exhausted and B, F are compact, then $M = \bigcup_{i \in \mathcal{I}} \pi^{-1}(\overline{U}_i) = \bigcup_{i \in \mathcal{I}} \phi_i(\overline{U}_i \times F)$ is compact. If \mathbf{M} is exhausted and compact, then $B = \pi(M)$ is compact, and f is bounded. So h_i is bounded, which implies the compactness of F . This proves (2). \square

The following generalizes Lemma 3.4.1

Lemma 4.3.7. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a filled bundle with compact base. Suppose that N is a compact connected hypersurface of M through regular points of f , on which π is a submersion.*

Then:

- (i) *there exists $\varepsilon > 0$ and a diffeomorphism $\Phi: N \times (-\varepsilon, \varepsilon) \rightarrow V_N$ such that $V_N \subset M$ is a neighborhood of N , $\Phi(y, 0) = y$, $\pi(\Phi(y, s)) = \pi(y)$ and $f(\Phi(y, s)) = f(y) + s$ for any $(y, s) \in N \times (-\varepsilon, \varepsilon)$.*
- (ii) *The set $V_N \setminus N$ has exactly two connected components, each characterized by the sign of $\text{pr}_2 \circ \Phi^{-1}$.*
- (iii) *If N is a filled subspace of \mathbf{M} , V_N is a filled subspace of \mathbf{M} .*
- (iv) *If $\mathbf{N} = \mathbf{M}|_N$ has connected fiber, $\mathbf{V}_N = \mathbf{M}|_{V_N}$ has connected fiber.*

Proof. Let $\mathbf{VM} = \ker(d\pi: TM \rightarrow TB)$ be the vertical tangent bundle of \mathbf{M} and g be any Riemannian metric on M . Let $Y \in \Gamma(\mathbf{VM})$ be an extension of $\nabla(f|_{\pi^{-1}(p)})$, the gradient of $f|_{\pi^{-1}(p)}$, for any $p \in B$. Then $Y(f)|_{\pi^{-1}(p)} = Y|_{\pi^{-1}(p)}(f|_{\pi^{-1}(p)}) = \left| \nabla(f|_{\pi^{-1}(p)}) \right|_g^2$. Therefore there exists a neighborhood $\tilde{V}_N \supset N$ such that $Y(f) > 0$ in \tilde{V}_N . Let $X \in \Gamma(\mathbf{VM})$ be such that $X(x) = \left| \nabla(f|_{\pi^{-1}(\pi(x))}) \right|_g^{-2} Y(x)$ for $x \in \tilde{V}_N$, then $X(f) = 1$ on \tilde{V}_N . Consider the flow of X , $\Phi: N \times (-\varepsilon, \varepsilon) \rightarrow M, (y, s) \mapsto x$, that is $\Phi(y, 0) = y$ for all $y \in N$ and $\frac{\partial \Phi}{\partial s}(y, s) = X(\Phi(y, s))$ for all $(y, s) \in N \times (-\varepsilon, \varepsilon)$, for $\varepsilon > 0$ small enough such that the image of Φ is contained in \tilde{V}_N . Then we define $V_N = \Phi(N \times (-\varepsilon, \varepsilon))$. Since X is vertical and $X(f) = 1$ in V_N , we have $\pi(\Phi(y, s)) = \pi(y)$ and $f(\Phi(y, s)) = f(y) + s$ for any $(y, s) \in N \times (-\varepsilon, \varepsilon)$, and Φ is a diffeomorphism. The two connected components of $V_N \setminus N$ are $\Phi(N \times (-\varepsilon, 0))$ and $\Phi(N \times (0, \varepsilon))$. If N is a filled subspace of \mathbf{M} then there is $\alpha \in \text{Reg}(f)$ such that N is the union of some connected components of $f^{-1}(\alpha)$, so V_N is the union of some connected components of $f^{-1}((\alpha - \varepsilon, \alpha + \varepsilon))$ and by Lemma 4.3.1 V_N is a filled subspace of \mathbf{M} . If \mathbf{N} has connected fiber, then since the fiber of \mathbf{V}_N is the image of the fiber of \mathbf{N} under the flow of X restricted on the fiber, it is connected. \square

For any filled bundle \mathbf{M} we define the *fiber exterior derivative*

$$\mathbf{d}: \Omega_F^q(\mathbf{M}) \rightarrow \Omega_F^{q+1}(\mathbf{M}), \eta \mapsto \mathbf{d}\eta$$

by $(\mathbf{d}\eta)_p = \mathbf{d}\eta_p$ for $p \in B$.

Lemma 4.3.8. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a filled bundle with compact base. Suppose W is an open subset of M such that \bar{W} is a compact submanifold with boundary ∂W , and $Z \subset F$ makes $\mathbf{W} = (\pi|_W, W, B, Z, f|_W)$ a filled subbundle of \mathbf{M} with connected fiber. Let $\xi \in (\Omega_F^k)_c(\mathbf{M})$. If $\text{supp } \xi \subset W$ and $\xi|_{W \cap \pi^{-1}(p)} \in \mathbf{d}\Omega_c^{k-1}(W \cap \pi^{-1}(p))$ for any $p \in B$, then there is an $\eta \in (\Omega_F^{k-1})_c(\mathbf{M})$ such that $\xi = \mathbf{d}\eta$ and $\text{supp } \eta \subset \bar{W}$.*

Proof. Let $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$ be a local trivialization of \mathbf{M} with respect to which W is a filled subspace of \mathbf{M} , then we can assume $\phi_i(W \cap \pi^{-1}(U_i)) = U_i \times Z$ for any $i \in \mathcal{I}$. By Lemma 4.3.6, \bar{W} is compact. Let $\{\chi_i\}_{i \in \mathcal{I}}$ be a partition of unity subordinated to the open cover $\{U_i\}_{i \in \mathcal{I}}$ of B . We apply Theorem 3.4.2 to Z to get an operator I_Z^q , then define $\eta = \sum_{i \in \mathcal{I}} \phi_i^* I_Z^k (\phi_i^{-1})^* ((\chi_i \circ \pi) \cdot \xi)$. Since $\mathbf{d} \circ I_Z^k = \text{id}$, we have $\xi = \mathbf{d}\eta$ and $\text{supp } \eta \subset \bar{W}$. \square

The following is an extension of Lemma 3.4.3.

Lemma 4.3.9. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a filled bundle with compact base. Let \mathbf{V} be a filled subbundle of \mathbf{M} with connected fiber. Suppose that $V = \text{sp}(\mathbf{V})$ is an open subset of M such that \bar{V} is a compact submanifold with boundary ∂V . Let $\omega, \tau \in \Omega_{F, \text{vol}}(\mathbf{M})$ such that $\text{supp}(\omega - \tau) \subset V$ and*

$$\int_{\mathbf{V}} \omega = \int_{\mathbf{V}} \tau. \tag{4.3.1}$$

Then there is a fiber diffeomorphism $\varphi: M \rightarrow M$ such that φ is the identity in a neighborhood of $M \setminus V$ and $\varphi^ \omega = \tau$.*

Proof. By Lemma 4.3.7 applied to each connected component of $N = \partial V$ there exist $\varepsilon > 0$ and V_N a neighborhood of N satisfying (i)-(iii). Since B is compact and $\text{supp}(\omega - \tau) \subset V$, we may

reduce ε as needed so that $\text{supp}(\omega - \tau) \subset V \setminus \overline{V_N}$. Let $W = V \setminus \overline{V_N}$ and $W_p = W \cap \pi^{-1}(p)$, $p \in B$. It follows from (4.3.1) that $(\omega_p - \tau_p)|_{W_p} \in d\Omega_c^{\dim F - 1}(W_p)$. Therefore by Lemma 4.3.8 there exists $\sigma \in (\Omega_F^{\dim F - 1})_c(\mathbf{M})$ with $\text{supp} \sigma \subset \overline{W}$ such that $d\sigma = \omega - \tau$. Define

$$\omega_t = (1 - t)\omega + t\tau \in \Omega_{F,\text{vol}}(\mathbf{M}) \quad \forall t \in [0, 1].$$

Since ω_t is nowhere vanishing there exists a smooth family of vertical vector fields $\{X_t\}_{t \in [0, 1]} \subset \Gamma(\mathbf{VM})$ where each X_t is supported in \overline{W} and such that

$$\omega_t(X_t, \cdot) = \sigma.$$

Let $\varphi_t : M \rightarrow M$ be a fiber diffeomorphism generated by X_t that is the identity outside of \overline{W} . Then $\varphi = \varphi_1^{-1}$ satisfies the required properties. \square

Now we carry out the transferring of volumes. The following three lemmas correspond to Lemmas 3.4.4 to 3.4.6 in the case of smooth families. The statements and the proofs are analogous but more delicate to implement due to the role that the release functor plays.

Lemma 4.3.10. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a filled bundle with compact base. Let \mathbf{K} be a connected filled subbundle of \mathbf{M} whose underlying space K is a compact manifold with or without boundary which has a nonempty interior. Let B_K be the base of $\mathbf{Rls} \mathbf{K}$, and let $w \in C^\infty(B_K; \mathbb{R})$. If $\omega \in \Omega_{F,\text{vol}}(\mathbf{M})$, then there exists $\tau \in \Omega_{F,\text{vol}}(\mathbf{M})$ such that $\text{supp}(\omega - \tau) \subset K^\circ$ and*

$$\int_{\mathbf{K}} \tau = w.$$

Proof. Let $\xi \in \Omega_{F,\text{vol}}(\mathbf{M})$ be such that $\text{supp}(\xi - \omega) \subset K^\circ$ and $\int_{\mathbf{K}} \xi < w$. Let $\eta \geq 0$ be a fiber

top-form on \mathbf{M} such that $\text{supp } \eta \subset K^\circ$ and $\int_{\mathbf{K}} \eta > 0$. Define

$$\tau = \xi + \text{Rls}(\pi|_K)^* \left(\frac{w - \int_{\mathbf{K}} \xi}{\int_{\mathbf{K}} \eta} \right) \eta,$$

where $\text{Rls}(\pi|_K)$ be the bundle map of $\text{Rls}(\mathbf{K})$. □

Lemma 4.3.11. *Let $\mathbf{W} = (\pi, W, B, Z, f)$ be a filled bundle with compact base and oriented connected fiber. Let \mathbf{N} be a filled subbundle of \mathbf{W} with connected fiber. Suppose that $N = \text{sp}(\mathbf{N})$ is a connected hypersurface of W such that $W \setminus N$ has two components, say W^+ and W^- . Let $\mathbf{W}^+ = \mathbf{W}|_{W^+}$ and $\mathbf{W}^- = \mathbf{W}|_{W^-}$. Let $\omega, \tau \in \Omega_{F, \text{vol}}(\mathbf{W})$. Then there is a neighborhood V_N of N , and a fiber diffeomorphism $\varphi: W \rightarrow W$ with the following properties:*

- (1) φ is the identity in a neighborhood of $W \setminus V_N$;
- (2) $\varphi^* \omega = \tau$ in a neighborhood of N ;
- (3) $\int_{\mathbf{W}^+} \varphi^* \omega = \int_{\mathbf{W}^+} \omega$; and $\int_{\mathbf{W}^-} \varphi^* \omega = \int_{\mathbf{W}^-} \omega$.

Proof. By Lemma 4.3.7, there exists \mathbf{V}_N with underlying space V_N as a filled subbundle of \mathbf{W} , $\varepsilon > 0$ and a diffeomorphism $\Phi: N \times (-\varepsilon, \varepsilon) \rightarrow V_N$ such that $V_N \subset M$ is a neighborhood of N , $\Phi(y, 0) = y$, $\pi(\Phi(y, s)) = \pi(y)$ and $f(\Phi(y, s)) = f(y) + s$ for any $(y, s) \in N \times (-\varepsilon, \varepsilon)$. Let $V_N^+ = \Phi(N \times (0, \varepsilon))$ and $V_N^- = \Phi(N \times (-\varepsilon, 0))$. We consider $\Phi(N \times [0, \varepsilon))$ first. By compactness of B there exists $0 < \delta < \varepsilon/2$ such that

$$\int_{\mathbf{M}|_{\Phi(N \times (0, \varepsilon - \delta))}} \tau > \int_{\mathbf{M}|_{\Phi(N \times (0, \delta))}} \omega \quad \text{and} \quad \int_{\mathbf{M}|_{\Phi(N \times (0, \varepsilon - \delta))}} \omega > \int_{\mathbf{M}|_{\Phi(N \times (0, \delta))}} \tau.$$

Let $s = \text{pr}_2 \circ \Phi^{-1}: \Phi(N \times (-\varepsilon, \varepsilon)) \rightarrow (-\varepsilon, \varepsilon)$ and choose a smooth function $\zeta: (0, \varepsilon) \times (0, 1) \rightarrow [0, 1]$ such that $\zeta(s, \cdot) = 1$ if $s \in (0, \delta]$, $\zeta(s, \cdot) = 0$ if $s \in [\varepsilon - \delta, \varepsilon)$, $\lim_{t \rightarrow 0^+} \zeta(s, t) = 0$,

$\frac{\partial \zeta}{\partial t}(s, \cdot) > 0$, and $\lim_{t \rightarrow 1^-} \zeta(s, t) = 1$ if $s \in (\delta, \varepsilon - \delta)$. Let $\mathbf{V}_N^+ = \mathbf{M}|_{V_N^+}$ and $\mathbf{V}_N^- = \mathbf{M}|_{V_N^-}$. Then

$$\theta(t, \cdot) = \int_{\mathbf{V}_N^+} \zeta(s(\cdot), t) \tau - \int_{\mathbf{V}_N^+} \zeta(s(\cdot), 1-t) \omega$$

is smooth on $(0, 1) \times B$ and

$$\frac{\partial \theta}{\partial t}(t, \cdot) = \int_{\mathbf{V}_N^+} \frac{\partial \zeta}{\partial t}(s(\cdot), t) \tau + \int_{\mathbf{V}_N^+} \frac{\partial \zeta}{\partial t}(s(\cdot), 1-t) \omega > 0$$

for any $t \in (0, 1)$ and $\lim_{t \rightarrow 0^+} \theta(t, p) < 0 < \lim_{t \rightarrow 1^-} \theta(t, p)$ for any $p \in B$. Then for every $p \in B$ there is a unique smooth $t = t(p) : B \rightarrow \mathbb{R}$ solving $\theta(t(p), p) = 0$. The functions $\lambda(x) \stackrel{\text{def}}{=} \zeta(s(x), t(\pi(x)))$ and $\mu(x) \stackrel{\text{def}}{=} \zeta(s(x), 1-t(\pi(x)))$ on V_N^+ are smooth in x . By analogy we define λ and μ in V_N^- , then let $\lambda = \mu = 1$ on N and $\lambda = \mu = 0$ in $W \setminus V_N$. In this way, $\lambda, \mu : W \rightarrow \mathbb{R}$ are defined smoothly, such that $\int_{\mathbf{V}_N^+} \mu \omega = \int_{\mathbf{V}_N^+} \lambda \tau$ and $\int_{\mathbf{V}_N^-} \mu \omega = \int_{\mathbf{V}_N^-} \lambda \tau$.

Hence $\int_{\mathbf{V}_N^+} ((1-\mu)\omega + \lambda\tau) = \int_{\mathbf{V}_N^+} \omega$ and $\int_{\mathbf{V}_N^-} ((1-\mu)\omega + \lambda\tau) = \int_{\mathbf{V}_N^-} \omega$. It follows from Lemma 4.3.9 applied to $(1-\mu)\omega + \lambda\tau$ and ω on \mathbf{V}_N^+ and \mathbf{V}_N^- that there exists a fiber diffeomorphism $\varphi : W \rightarrow W$ such that $\varphi = \text{id}$ in $V \setminus \Phi(N \times (\delta - \varepsilon, \varepsilon - \delta))$ and $\varphi^* \omega = (1-\mu)\omega + \lambda\tau$. Hence φ satisfies the conditions claimed in the statement. \square

Lemma 4.3.12. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a connected exhausted bundle with compact base and oriented noncompact connected fibers. Let $\{L_j\}_{j \in \mathbb{N}}$ be a cover of M by compact connected submanifolds with boundary, which have the same dimension as M , and whose interiors are pairwise disjoint. Suppose for any $j \in \mathbb{N}$, L_j is a filled subspace. Let $\mathbf{L}_j = \mathbf{M}|_{L_j}$. If $\omega, \tau \in \Omega_{F, \text{vol}}(\mathbf{M})$ are such that $\mathfrak{K}_{\mathbf{L}_j} \omega = \mathfrak{K}_{\mathbf{L}_j} \tau$ for each $j \in \mathbb{N}$ then there is a fiber diffeomorphism $\varphi : M \rightarrow M$ such that $\varphi^* \omega = \tau$.*

Proof. By the construction of $\{\mathbf{L}_j\}_{j \in \mathbb{N}}$, any three different L_j 's for $j \in \mathbb{N}$ do not intersect. Let

$$\mathcal{C} = \{\mathbf{M}|_N \mid N \in \text{Conn} L_j \cap L_k, j, k \in \mathbb{N}, j \neq k\}.$$

Then \mathcal{C} is a collection of pairwise disjoint filled subbundles of \mathbf{M} whose underlying spaces are hypersurfaces of M . So for each $\mathbf{N} \in \mathcal{C}$ with underlying space N , let $j, k \in \mathbb{N}$ be such that $N \subset L_j \cap L_k$, by Lemma 4.3.7, we obtain $\varepsilon_N > 0$ and a diffeomorphism $\Phi_N: V_N \times (-\varepsilon_N, \varepsilon_N) \rightarrow V_N$ where V_N is a neighborhood of N and a filled subspace of \mathbf{M} . We require $V_N^- \subset L_j$ and $V_N^+ \subset L_k$. Let $\mathbf{V}_N = \mathbf{M}|_{V_N}$, $\mathbf{V}_N^- = \mathbf{M}|_{V_N^-}$, and $\mathbf{V}_N^+ = \mathbf{M}|_{V_N^+}$. Now apply Lemma 4.3.11 to Rls \mathbf{V}_N in order to obtain a fiber diffeomorphism $\varphi_N: M \rightarrow M$ such that $\varphi_N = \text{id}$ in a neighborhood of $M \setminus V_N$, $\varphi_N^* \omega = \tau$ in a neighborhood of N , and $\mathfrak{F}_{\mathbf{V}_N^+} \varphi_N^* \omega = \mathfrak{F}_{\mathbf{V}_N^+} \omega$ as well as $\mathfrak{F}_{\mathbf{V}_N^-} \varphi_N^* \omega = \mathfrak{F}_{\mathbf{V}_N^-} \omega$ (note that a diffeomorphism preserving the released bundle map also preserves the original bundle map). Therefore $\mathfrak{F}_{\mathbf{L}_j} \varphi_N^* \omega = \mathfrak{F}_{\mathbf{L}_j} \omega$, $\mathfrak{F}_{\mathbf{L}_k} \varphi_N^* \omega = \mathfrak{F}_{\mathbf{L}_k} \omega$. Note that $V_N^- \subset L_j$, so the base of Rls \mathbf{V}_N^- covers that of Rls \mathbf{L}_j .

If necessary, choose ε_N small such that the family $\{\overline{V_N}\}_{N \in \mathcal{C}}$ is mutually disjoint. Since replacing ω by $\varphi_N^* \omega$ each time does not change the released fiber volume of \mathbf{L}_j for any $j \in \mathbb{N}$, we compose these φ_N for $N = \text{sp}(\mathbf{N})$, for $\mathbf{N} \in \mathcal{C}$, as they are the identity away from disjoint open sets, to obtain a fiber diffeomorphism $\varphi': M \rightarrow M$ such that $\omega' = \varphi'^* \omega$ is equal to τ on some neighborhood of $\bigcup_{N \in \mathcal{C}} N$ and $\mathfrak{F}_{\mathbf{L}_j} \omega' = \mathfrak{F}_{\mathbf{L}_j} \omega = \mathfrak{F}_{\mathbf{L}_j} \tau$ for each $j \in \mathbb{N}$. Applying Lemma 4.3.9 to each Rls \mathbf{L}_j for $j \in \mathbb{N}$ we get a fiber diffeomorphism $\psi_j: M \rightarrow M$ such that $\tau = \psi_j^* \omega'$ in L_j and $\psi_j = \text{id}$ in a neighborhood of $M \setminus L_j$. Replacing ω' by $\psi_j^* \omega'$ each time and composing $\{\psi_j\}_{j \in \mathbb{N}}$ we obtain a fiber diffeomorphism $\psi': M \rightarrow M$ such that $\tau = \psi'^* \omega'$. Then $\varphi = \varphi' \circ \psi'$ satisfies the required properties. \square

4.4 Proof of the main result

The analogue of the following lemma appeared in Lemma 3.5.1 for the case of smooth families, the proof strategy is analogous.

For any tree \mathcal{T} of height ω , for any $\mathbf{X} \in \text{Lv}(\ell)$, $\ell \in \mathbb{N} \cup \{0\}$, let

$$\text{II}_{\mathcal{T}}\mathbf{X} = \mathbf{X}_{[\alpha_{\ell+1}]}, \quad \text{III}_{\mathcal{T}}\mathbf{X} = \mathbf{X}_{[\alpha_{\ell+2}]}.$$

Then we have

$$\text{sp}(\text{II}_{\mathcal{T}}\mathbf{X}) = \text{sp}(\mathbf{X}) \setminus \coprod_{\mathbf{Y} \in \text{Ch}(\mathbf{X})} \text{sp}(\mathbf{Y}), \quad \text{sp}(\text{III}_{\mathcal{T}}\mathbf{X}) = \text{sp}(\mathbf{X}) \setminus \coprod_{\mathbf{Z} \in \text{Gch}(\mathbf{X})} \text{sp}(\mathbf{Z}).$$

Lemma 4.4.1. *Let $\mathbf{M} = (\pi, M, B, F, f)$ be a connected exhausted bundle with compact base and oriented noncompact connected fiber. Let $\omega, \tau \in \Omega_{F, \text{vol}}(\mathbf{M})$ be commensurable and suppose that they have equal fiber integral. Then there is a tree (\mathcal{T}, ϱ) of connected filled subbundles of \mathbf{M} and $\{\omega_n\}_{n \in \mathbb{N} \cup \{0\}}, \{\tau_n\}_{n \in \mathbb{N} \cup \{0\}} \subset \Omega_{F, \text{vol}}(\mathbf{M})$ such that $\omega_0 = \omega, \tau_0 = \tau$ and for any $n \in \mathbb{N}$, we have that*

$$\text{supp}(\omega_n - \omega_{n-1}) \cup \text{supp}(\tau_n - \tau_{n-1}) \subset \bigcup_{\mathbf{C} \in \text{Lv}(2n-2)} (\text{sp}(\text{III}_{\mathcal{T}}\mathbf{C}))^\circ, \quad (4.4.1)$$

as well as that for each $\mathbf{A} \in \text{Lv}(2n-3)$ with $n > 1$, $\mathbf{C} \in \text{Lv}(2n-2)$, $\mathbf{E} \in \text{Lv}(2n-1)$,

$$\int_{\text{II}_{\mathcal{T}}\mathbf{M}} \omega_1 = \int_{\text{II}_{\mathcal{T}}\mathbf{M}} \tau_1, \quad \int_{\text{III}_{\mathcal{T}}\mathbf{A}} \omega_n = \int_{\text{III}_{\mathcal{T}}\mathbf{A}} \tau_n \text{ for } n > 1; \quad (4.4.2)$$

$$\int_{\text{III}_{\mathcal{T}}\mathbf{C}} \omega_n = \int_{\text{III}_{\mathcal{T}}\mathbf{C}} \omega_{n-1}, \quad \int_{\text{III}_{\mathcal{T}}\mathbf{C}} \tau_n = \int_{\text{III}_{\mathcal{T}}\mathbf{C}} \tau_{n-1}; \quad (4.4.3)$$

$$\int_{\mathbf{E}} \omega_n = \int_{\mathbf{E}} \tau_n. \quad (4.4.4)$$

Proof. Our goal is to find $\alpha_0 = -\infty$ and $\{\alpha_\ell\}_{\ell \in \mathbb{N}} \subset \text{Reg}(f) \cap f(M)$ such that \mathcal{T} is constructed by Lemma 4.3.5, see Figure 4.2. Note that, if we know $\{\alpha_\ell\}_{0 \leq \ell \leq m}$ for some $m \in \mathbb{N} \cup \{0\}$ for the sequence $\{\alpha_\ell\}_{\ell \in \mathbb{N} \cup \{0\}}$ defining \mathcal{T} , then we say \mathcal{T} is constructed up to the m -th level, so we know $\text{Lv}(\ell)$ of \mathcal{T} for any ℓ with $0 \leq \ell \leq m$. We proceed by induction on $n \in \mathbb{N} \cup \{0\}$ to find $\alpha_{2n-1}, \alpha_{2n}$ and $\omega_n, \tau_n \in \Omega_{F, \text{vol}}(\mathbf{M})$ such that $\int_{\mathbf{E}} \omega_n = \int_{\mathbf{E}} \tau_n$ for any $\mathbf{E} \in \text{Lv}(2n-1)$ ($\mathbf{E} \in \text{Lv}(0)$) if

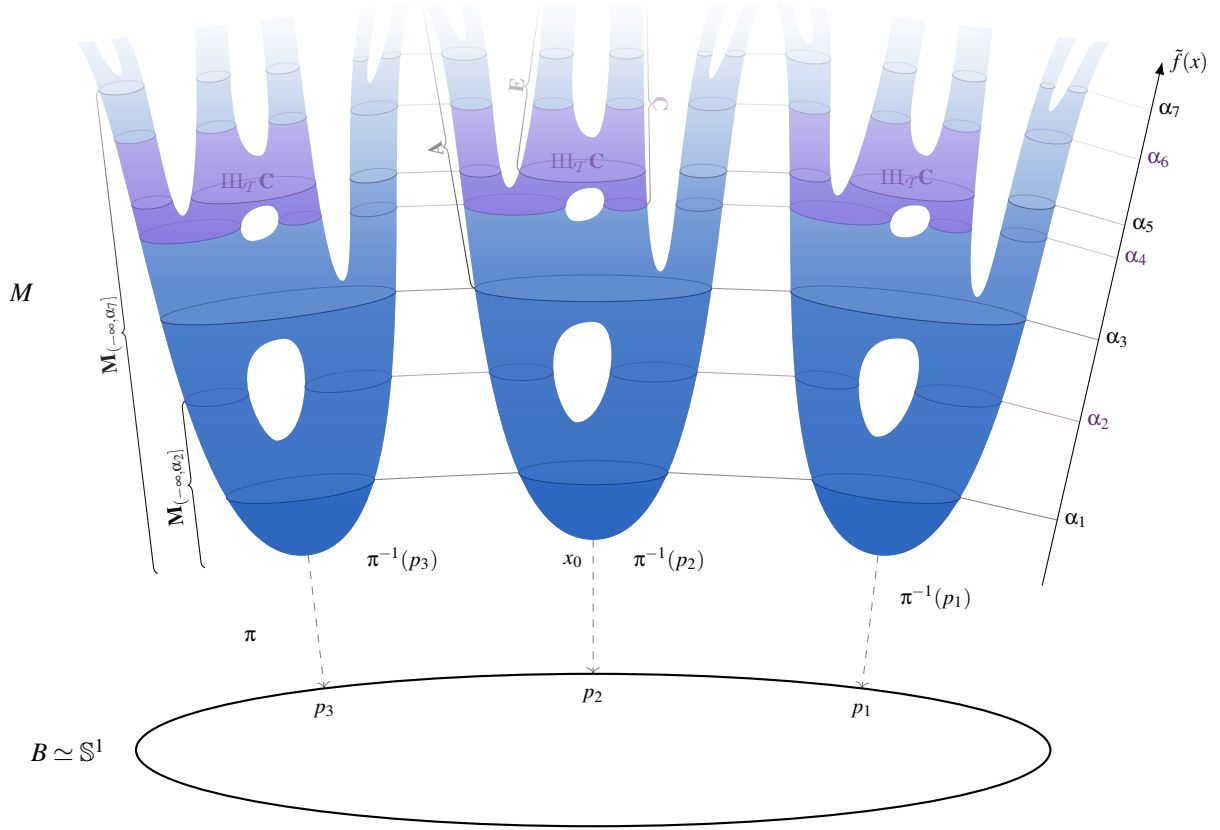


Figure 4.2: $\mathbf{M} = (\pi, M, B, F, f)$ in Lemma 4.4.1.

$n = 0$). For any $\mathbf{X}, \mathbf{Y} \in \mathcal{T}$ with $\mathbf{X} \supseteq \mathbf{Y}$, let $\iota_{\mathbf{X}}^{\mathbf{Y}}: \mathbf{Y} \hookrightarrow \mathbf{X}$ be the embedding and let

$$\text{Rls } \iota_{\mathbf{X}}^{\mathbf{Y}} = (\iota_{\mathbf{X}}^{\mathbf{Y}}, \kappa_{\mathbf{X}}^{\mathbf{Y}}).$$

By Lemma 4.2.3, $\kappa_{\mathbf{X}}^{\mathbf{Y}}$ is a covering map, whose number of sheets is denoted by $\#\kappa_{\mathbf{X}}^{\mathbf{Y}}$.

Case 0. Set $\alpha_0 = -\infty$, then $\mathbf{M}_{[\alpha_0]} = \emptyset$, and $\text{Lv}(0) = \{\mathbf{M}\}$. Since ω and τ has equal fiber integral and \mathbf{M} has connected fiber, we have $\int_{\mathbf{M}} \omega_0 = \int_{\mathbf{M}} \tau_0$.

Case $(n-1)$ for $n \in \mathbb{N}$. By induction we assume that for any $\mathbf{A} \in \text{Lv}(2n-3)$ ($\mathbf{A} \in \text{Lv}(0)$ when $n = 1$) we have:

$$\int_{\mathbf{A}} \omega_{n-1} = \int_{\mathbf{A}} \tau_{n-1}. \quad (4.4.5)$$

Case n for $n \in \mathbb{N}$. Take $\alpha_{2n-1} \in \text{Reg}(f)$ such that $\mathbf{C}_{[\alpha_{2n-1}]}$ is a saturated slice of \mathbf{C} for each

$\mathbf{C} \in \text{Lv}(2n-2)$ (for the concept of saturated slices and the existence of α_{2n-1} , see Lemma 4.3.4). Then \mathcal{T} is constructed up to the $(2n-1)$ -th level. Let $\mathbf{A} \in \text{Lv}(2n-3)$ (if $n=1$ let $\mathbf{A} = \mathbf{M}$ and replace $\text{Gch}(\mathbf{A})$ by $\text{Ch}(\mathbf{M})$, $\text{III}_{\mathcal{T}}\mathbf{A}$ by $\text{II}_{\mathcal{T}}\mathbf{M}$ throughout this paragraph). The base of Rls \mathbf{A} is B_A . Let $\text{Gch}_0(\mathbf{A})$ (resp. $\text{Gch}_1(\mathbf{A})$) be the subcollection of elements in $\text{Gch}(\mathbf{A})$ with finite (resp. infinite) volume. For any $\mathbf{E} \in \text{Gch}(\mathbf{A})$, we define $\delta_{\mathbf{E}} \in C^\infty(B_E; \mathbb{R})$ as follows: if \mathbf{E} has finite volume, let $\delta_{\mathbf{E}} = \int_{\mathbf{E}} \omega_{n-1} - \int_{\mathbf{E}} \tau_{n-1}$; if \mathbf{E} has infinite volume, let

$$\delta_{\mathbf{E}} = \frac{1}{\sum_{\mathbf{G} \in \text{Gch}_1(\mathbf{A})} \#\kappa_{\mathbf{A}}^{\mathbf{G}}} (\kappa_{\mathbf{A}}^{\mathbf{E}})^* \left(\int_{\text{III}_{\mathcal{T}}\mathbf{A}} \tau_{n-1} - \int_{\text{III}_{\mathcal{T}}\mathbf{A}} \omega_{n-1} - \sum_{\mathbf{G} \in \text{Gch}_0(\mathbf{A})} (\kappa_{\mathbf{A}}^{\mathbf{G}})^* \delta_{\mathbf{G}} \right).$$

Then combining equations (4.5.1) and (4.4.5) we obtain

$$\sum_{\mathbf{E} \in \text{Gch}(\mathbf{A})} (\kappa_{\mathbf{A}}^{\mathbf{E}})^* \delta_{\mathbf{E}} = \int_{\text{III}_{\mathcal{T}}\mathbf{A}} \tau_{n-1} - \int_{\text{III}_{\mathcal{T}}\mathbf{A}} \omega_{n-1}.$$

For every $\mathbf{C} \in \text{Ch}(\mathbf{A})$, let B_C be the base of Rls \mathbf{C} , and let $u_C \in C^\infty(B_C; \mathbb{R})$ be such that

$$\max \left(-\int_{\text{II}_{\mathcal{T}}\mathbf{C}} \omega_{n-1}, -\int_{\text{II}_{\mathcal{T}}\mathbf{C}} \tau_{n-1} + \sum_{\mathbf{E} \in \text{Ch}(\mathbf{C})} (\kappa_{\mathbf{C}}^{\mathbf{E}})^* \delta_{\mathbf{E}} \right) < u_C < \int_{\mathbf{C}} \omega_{n-1} - \int_{\text{II}_{\mathcal{T}}\mathbf{C}} \omega_{n-1}.$$

If \mathbf{C} has finite volume then

$$\begin{aligned} & \left(\int_{\mathbf{C}} \omega_{n-1} - \int_{\text{II}_{\mathcal{T}}\mathbf{C}} \omega_{n-1} \right) - \left(-\int_{\text{II}_{\mathcal{T}}\mathbf{C}} \tau_{n-1} + \sum_{\mathbf{E} \in \text{Ch}(\mathbf{C})} (\kappa_{\mathbf{C}}^{\mathbf{E}})^* \delta_{\mathbf{E}} \right) \\ &= \int_{\mathbf{C}} \omega_{n-1} + \left(\int_{\text{II}_{\mathcal{T}}\mathbf{C}} \tau_{n-1} - \int_{\text{II}_{\mathcal{T}}\mathbf{C}} \omega_{n-1} \right) + \sum_{\mathbf{E} \in \text{Ch}(\mathbf{C})} (\kappa_{\mathbf{C}}^{\mathbf{E}})^* \left(\int_{\mathbf{E}} \tau_{n-1} - \int_{\mathbf{E}} \omega_{n-1} \right) \\ &= \int_{\mathbf{C}} \tau_{n-1} > 0, \end{aligned}$$

so such u_C exists. Since

$$u_C < \sum_{\mathbf{E} \in \text{Ch}(\mathbf{C})} (\kappa_{\mathbf{C}}^{\mathbf{E}})_* \int_{\mathbf{E}} \omega_{n-1} = \int_{\mathbf{C}} \omega_{n-1} - \int_{\Pi_{\mathcal{I}} \mathbf{C}} \omega_{n-1},$$

by Lemma 4.5.1 applied to the covering map

$$\coprod_{\mathbf{E} \in \text{Ch}(\mathbf{C})} B_E \rightarrow B_C,$$

we may choose $v_{\mathbf{E}} \in C^\infty(B_E; \mathbb{R})$ (where B_E is the base of $\text{Rls } \mathbf{E}$) such that $v_{\mathbf{E}} < \int_{\mathbf{E}} \omega_{n-1}$ and $\sum_{\mathbf{E} \in \text{Ch}(\mathbf{C})} (\kappa_{\mathbf{C}}^{\mathbf{E}})_* v_{\mathbf{E}} = u_C$.

For any $\mathbf{E} \in \text{Ch}(\mathbf{C})$, if \mathbf{E} has infinite volume, take $\beta_{\mathbf{E}} \in \text{Reg}(f)$ such that $\mathbf{E}_{[\beta_{\mathbf{E}}]}$ is a saturated slice (for the concept of saturated slices and the existence of α_{2n-1} , see Lemma 4.3.4).

Otherwise, the function

$$(\cdot, \beta) \mapsto \min \left(\int_{(\text{Rls } \mathbf{E})|_{E[\beta]}} \omega_{n-1}, \int_{(\text{Rls } \mathbf{E})|_{E[\beta]}} \tau_{n-1} + \delta_{\mathbf{E}} \right) - v_{\mathbf{E}} \quad (4.4.6)$$

$B_E \times \mathbb{R}$ is continuous in the first variable, is increasing in β , and converges to $\int_{\mathbf{E}} \omega_{n-1} - v_{\mathbf{E}} > 0$ as $\beta \rightarrow +\infty$ pointwise. Note $\text{Rls } \mathbf{E}$ is not a subbundle of \mathbf{M} , so we cannot slice it by β . Since B_E is compact there is $\beta_{\mathbf{E}} > \alpha_{2n-1}$ such that (4.4.6) is positive when $\beta = \beta_{\mathbf{E}}$. Let $\alpha_{2n} = \max_{\mathbf{E} \in \text{Lv}(2n-1)} \beta_{\mathbf{E}}$, then \mathcal{I} is constructed up to the $2n$ -th level. So $\Pi_{\mathcal{I}} \mathbf{E} = \mathbf{E}_{[\alpha_{2n}]}$, then we have $v_{\mathbf{E}} < \int_{\Pi_{\mathcal{I}} \mathbf{E}} \omega_{n-1}$, and $v_{\mathbf{E}} - \delta_{\mathbf{E}} < \int_{\Pi_{\mathcal{I}} \mathbf{E}} \tau_{n-1}$. Since all the right hand sides of these expressions are positive, by Lemma 4.3.10, there are $\omega_n, \tau_n \in \Omega_{F, \text{vol}}(\mathbf{M})$ such that

$$\begin{aligned} \int_{\Pi_{\mathcal{I}} \mathbf{C}} \omega_n &= \int_{\Pi_{\mathcal{I}} \mathbf{C}} \omega_{n-1} + u_C, & \int_{\Pi_{\mathcal{I}} \mathbf{C}} \tau_n &= \int_{\Pi_{\mathcal{I}} \mathbf{C}} \tau_{n-1} + u_C - \sum_{\mathbf{E} \in \text{Ch}(\mathbf{C})} (\kappa_{\mathbf{C}}^{\mathbf{E}})_* \delta_{\mathbf{E}}, \\ \int_{\Pi_{\mathcal{I}} \mathbf{E}} \omega_n &= \int_{\Pi_{\mathcal{I}} \mathbf{E}} \omega_{n-1} - v_{\mathbf{E}}, & \int_{\Pi_{\mathcal{I}} \mathbf{E}} \tau_n &= \int_{\Pi_{\mathcal{I}} \mathbf{E}} \tau_{n-1} - (v_{\mathbf{E}} - \delta_{\mathbf{E}}), \end{aligned}$$

and

$$\begin{aligned} \text{supp}(\omega_n - \omega_{n-1}) \cup \text{supp}(\tau_n - \tau_{n-1}) \\ \subset (M_{[\alpha_{2n}]}^\circ \setminus M_{[\alpha_{2n-2}]} = \bigcup_{\mathbf{C} \in \text{Lv}(2n-2)} (\text{sp}(\text{III}_{\mathcal{T}}\mathbf{C}))^\circ. \end{aligned}$$

Then we have

$$\begin{aligned} \int_{\text{III}_{\mathcal{T}}\mathbf{A}} \omega_n &= \int_{\text{III}_{\mathcal{T}}\mathbf{A}} \omega_{n-1} + \sum_{\mathbf{C} \in \text{Ch}(\mathbf{A})} (\kappa_{\mathbf{A}}^{\mathbf{C}})_* u_{\mathbf{C}} \\ &= \int_{\text{III}_{\mathcal{T}}\mathbf{A}} \tau_{n-1} - \sum_{\mathbf{E} \in \text{Gch}(\mathbf{A})} (\kappa_{\mathbf{A}}^{\mathbf{C}})_* \left((\kappa_{\mathbf{C}}^{\mathbf{E}})_* \delta_{\mathbf{E}} - u_{\mathbf{C}} \right) = \int_{\text{III}_{\mathcal{T}}\mathbf{A}} \tau_n, \end{aligned}$$

and

$$\begin{aligned} \int_{\text{III}_{\mathcal{T}}\mathbf{C}} \omega_n &= \int_{\text{II}_{\mathcal{T}}\mathbf{C}} \omega_n + \sum_{\mathbf{E} \in \text{Ch}(\mathbf{C})} (\kappa_{\mathbf{C}}^{\mathbf{E}})_* \int_{\text{II}_{\mathcal{T}}\mathbf{E}} \omega_n = \int_{\text{III}_{\mathcal{T}}\mathbf{C}} \omega_{n-1}, \\ \int_{\text{III}_{\mathcal{T}}\mathbf{C}} \tau_n &= \int_{\text{II}_{\mathcal{T}}\mathbf{C}} \tau_n + \sum_{\mathbf{E} \in \text{Ch}(\mathbf{C})} (\kappa_{\mathbf{C}}^{\mathbf{E}})_* \int_{\text{II}_{\mathcal{T}}\mathbf{E}} \tau_n = \int_{\text{III}_{\mathcal{T}}\mathbf{C}} \tau_{n-1}, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbf{E}} \omega_n &= \int_{\text{II}_{\mathcal{T}}\mathbf{E}} \omega_n + \int_{\mathbf{E}} \omega_{n-1} - \int_{\text{II}_{\mathcal{T}}\mathbf{E}} \omega_{n-1} \\ &= \int_{\mathbf{E}} \omega_{n-1} - \nu_{\mathbf{E}} = \int_{\mathbf{E}} \tau_{n-1} - (\nu_{\mathbf{E}} - \delta_{\mathbf{E}}) = \int_{\mathbf{E}} \tau_n. \end{aligned}$$

□

Now we can apply Lemma 4.4.1 to \mathbf{M} and ω, τ , in which way we obtain the tree \mathcal{T} of filled subbundles of \mathbf{M} such that (4.4.1) to (4.4.4) hold.

For $n \in \mathbb{N}$ and $\mathbf{C} \in \text{Lv}(2n-2)$, applying Lemma 4.3.9 to $(\text{III}_{\mathcal{T}}\mathbf{C})^\circ$, there are fiber diffeomorphisms $\varphi_n, \psi_n: M \rightarrow M$ such that $\varphi_n^* \omega_{n-1} = \omega_n$, $\psi_n^* \tau_{n-1} = \tau_n$, and $\varphi_n = \psi_n = \text{id}$

outside of $(M_{[\alpha_{2n}]})^\circ \setminus M_{[\alpha_{2n-2}]}$. Let

$$\begin{aligned}\omega_\infty &= \lim_{n \rightarrow \infty} \omega_n, & \tau_\infty &= \lim_{n \rightarrow \infty} \tau_n, \\ \varphi_\infty &= \varphi_1 \circ \varphi_2 \circ \cdots, & \psi_\infty &= \psi_1 \circ \psi_2 \circ \cdots.\end{aligned}\tag{4.4.7}$$

Since the interiors of $\text{sp}(\text{III}_{\mathcal{T}} \mathbf{C})$, $\mathbf{C} \in \mathcal{T}$ with even depths are mutually disjoint, the pointwise limits in (4.4.7) will be stable at a finite n , so $\omega_\infty, \tau_\infty \in \Omega_{F, \text{vol}}(\mathbf{M})$, $\varphi_\infty, \psi_\infty: M \rightarrow M$ must be fiber diffeomorphisms,

$$\int_{\text{II}_{\mathcal{T}} \mathbf{M}} \omega_\infty = \int_{\text{II}_{\mathcal{T}} \mathbf{M}} \tau_\infty, \quad \int_{\text{III}_{\mathcal{T}} \mathbf{A}} \omega_\infty = \int_{\text{III}_{\mathcal{T}} \mathbf{A}} \tau_\infty$$

for each $\mathbf{A} \in \mathcal{T}$ with odd depth, $\varphi_\infty^* \omega = \omega_\infty$, and $\psi_\infty^* \tau = \tau_\infty$.

Let $\{\mathbf{L}_j\}_{j \in \mathbb{N}}$ be the set of $\text{II}_{\mathcal{T}} \mathbf{M}$ and the closures of $\text{III}_{\mathcal{T}} \mathbf{A}$ for $\mathbf{A} \in \mathcal{T}$ with even depths.

By Lemma 4.3.12, there is a fiber diffeomorphism $\varphi': M \rightarrow M$ such that $\varphi'^* \omega_\infty = \tau_\infty$.

Finally, $\varphi = \varphi_\infty \circ \varphi' \circ \psi_\infty^{-1}: M \rightarrow M$, which concludes the proof.

4.5 Technical tools

4.5.1 Trees

A *tree* is a strictly partially ordered set (\mathcal{T}, \prec) with the property that for each $x \in \mathcal{T}$, the set $\text{Pre}(x) = \{y \in \mathcal{T} \mid y \prec x\}$ of all predecessors of x is well ordered by \prec . We write \mathcal{T} for (\mathcal{T}, \prec) when there is no ambiguity. Let $\text{Rt}(\mathcal{T}) = \{x \in T \mid \forall y \in T, y \not\prec x\} \neq \emptyset$ be the set of roots of \mathcal{T} . If $\text{Rt}(\mathcal{T})$ is a singleton we call \mathcal{T} *rooted*.

Let $\text{Suc}(x) = \{y \in \mathcal{T} \mid y \succ x\}$ be the set of all successors of x , then $(\text{Suc}(x), \prec)$ is a tree. Let $\text{Ch}(x) = \text{Rt}(\text{Suc}(x))$ be the set of *children* of x . If for any $x \in \mathcal{T}$, $\text{Ch}(x)$ is finite, we call \mathcal{T} *locally finite*. Let $\text{Gch}(x) = \bigcup_{y \in \text{Ch}(x)} \text{Ch}(y)$ be the set of *grandchildren* of x . Let $\text{Lf}(\mathcal{T}) = \{x \in T \mid \forall y \in T, x \not\prec y\}$ be the set of leaves of \mathcal{T} . If $\text{Lf}(\mathcal{T}) = \emptyset$ we call \mathcal{T} *leafless*.

The *depth* of x is the ordinal of $\text{Pre}(x)$, which we denote by $\text{dpt}(x)$. Let $\text{hgt}(\mathcal{T}) =$

$\sup\{\text{dpt}(x) + 1 \mid x \in \mathcal{T}\}$ be the *height* of \mathcal{T} . For any ordinal $\ell < \text{hgt}(\mathcal{T})$, let $\text{Lv}(\ell) = \{x \in \mathcal{T} \mid \text{dpt}(x) = \ell\}$ be the ℓ -th *level* of \mathcal{T} .

4.5.2 Three auxiliary technical lemmas

Now we prove an auxiliary lemma which will ensure the smooth dependence of volumes on parameters after distributed into many connected components (for the proof of the main technical tool below Lemma 4.4.1). Let $\kappa: B' \rightarrow B$ be a covering map. We define the *pullback* κ^* and the *pushforward* κ_* of functions as follows

$$\begin{aligned} \kappa^*: C(B; \mathbb{R}) &\rightarrow C(B'; \mathbb{R}), & \kappa_*: C(B'; \mathbb{R}) &\rightarrow C(B; \mathbb{R}); \\ (\kappa^*u)(p') &= u(\kappa(p')), & (\kappa_*u)(p) &= \sum_{p' \in \kappa^{-1}(p)} u(p'). \end{aligned}$$

If B is connected, recall that $\#\kappa \in \mathbb{N}$ is the number of sheets of κ . Then for any $u \in C(B; \mathbb{R})$,

$$(\kappa_*\kappa^*u)(p) = \sum_{p' \in \kappa^{-1}(p)} u(\kappa(p')) = \#\kappa \cdot u(p). \quad (4.5.1)$$

The following generalizes Lemma 3.4.7.

Lemma 4.5.1. *Let $\kappa: B' \rightarrow B$ be a covering map with B' compact (so B is compact). Let $a \in C(B; \mathbb{R})$, $u \in C^\infty(B; \mathbb{R})$ such that $u < a$. Then for any $a' \in C(B'; \mathbb{R})$ with $\kappa_*a' = a$, there is $u' \in C^\infty(B'; \mathbb{R})$ such that $u' < a'$ and $\kappa_*u' = u$.*

Proof. Without loss of generality we assume B is connected and $u = 0$ otherwise we can deal with each connected component of B one by one and replace a' by $a' - u/\#\kappa$, u' by $u' - u/\#\kappa$.

Choose $\varepsilon > 0$ such that $\#\kappa \cdot \varepsilon < \min a$. Define $h' = a' - \varepsilon$, then $\kappa_*h' = a - \#\kappa \cdot \varepsilon > 0$. So $\kappa_*(h')^+ > \kappa_*(h')^- \geq 0$. Here $(h')^+(p) = \min\{h'(p), 0\}$ and $(h')^-(p) = \min\{-h'(p), 0\}$ denote the positive and negative parts of h' , respectively. Since h' is bounded from below we set

$c = \max \kappa_*(h')^- > 0$. Define $w' = \frac{(h')^+}{\kappa^* \kappa_*(h')^+} \kappa^* \kappa_*(h')^- - (h')^-$, then $\kappa_*(w') = 0$. Moreover,

$$h' - w' = (h')^+ - \frac{\kappa^* \kappa_*(h')^-}{\kappa^* \kappa_*(h')^+} (h')^+ \geq 0.$$

By Whitney Approximation Theorem there is $v' \in C^\infty(B'; \mathbb{R})$ such that $|v' - w'| < \varepsilon/2$. Then let $u' = v' - \frac{1}{\#\kappa} \kappa^* \kappa_*(v') \in C^\infty(B'; \mathbb{R})$. So $|u' - w'| < \varepsilon$, and $\kappa_* u' = 0$ by (4.5.1), hence $a' - u' > h' - w' \geq 0$ is as required. \square

Finally, Lemma 4.3.4 follows from Lemma 3.3.1.

Proof of Lemma 4.3.4. Let P be the fiber of \mathbf{A} , so $\mathbf{A} = (\pi|_A, A, B, P, f|_A)$. Let B_A, P_A be manifolds such that $\text{Rls } \mathbf{A} = (\text{Rls}(\pi|_A), A, B_A, P_A, f)$. Fix $\alpha \in \text{Reg}(f) \cap f(A)$ and then we consider $\mathbf{A}_{[\alpha]}$. Since the fiber of $\mathbf{A}_{[\alpha]}$ is a precompact submanifold of F with boundary, it can only have finitely many components. Let P_α denote the fiber of $(\text{Rls } \mathbf{A})|_{A_{[\alpha]}}$, so then P_α has finitely many components. By Lemma 3.3.1, there is $K \in \mathcal{K}(\overline{P_A})$ which is connected and contains both x_0 and P_α , where the closure of P_A is taken in F . Suppose $\beta \in \text{Reg}(f)$ and $\beta \geq \max_{i \in \mathcal{I}} \max_K h_i$, then $P_\beta \supset K$. Let $\iota_{\alpha, \beta}: (\text{Rls } \mathbf{A})|_{A_{[\alpha]}} \hookrightarrow (\text{Rls } \mathbf{A})|_{A_{[\beta]}}$ be the embedding, and let $\text{Rls } \iota_{\alpha, \beta} = (\iota_{\alpha, \beta}, \kappa_{\alpha, \beta})$. Let $B_{A, \alpha}$ and $B_{A, \beta}$ be the bases of $(\text{Rls } \mathbf{A})|_{A_{[\alpha]}}$ and $(\text{Rls } \mathbf{A})|_{A_{[\beta]}}$, respectively. Then $\kappa_{\alpha, \beta}: B_{A, \alpha} \rightarrow B_{A, \beta}$ is a covering map. Note that a component of P_β contains of P_α , this means the image of $\kappa_{\alpha, \beta}$ is a one-fold covering of B_A . Let $\iota_\beta: (\text{Rls } \mathbf{A})|_{A_{[\beta]}} \hookrightarrow \text{Rls } \mathbf{A}$ be the embedding, and let $\text{Rls } \iota_\beta = (\iota_\beta, \kappa_\beta)$, then $\kappa_\beta: B_{A, \beta} \rightarrow B_A$ is a diffeomorphism. For the same reason, $\#\kappa_{\beta'} = 1$ for any $\beta' \in \text{Reg}(f)$ no less than β . \square

4.6 Examples of exhausted bundles

Example 4.6.1. Let $(\pi, M, B, \mathbb{R}^k)$ be a vector bundle with rank k and let $g: M \times_B M \rightarrow B \times \mathbb{R}$ be a metric on the bundle, that is, for any $b \in B$, $g|_{\pi^{-1}(b)}: \pi^{-1}(b) \times \pi^{-1}(b) \rightarrow \mathbb{R}$ is an inner product.

Let $f: M \rightarrow \mathbb{R}, x \mapsto g(x, x)$. Then $(\pi, M, B, \mathbb{R}^k, f)$ is a filled bundle, which is exhausted if and only if B is compact.

Example 4.6.2. Let (π, N, B, E) be a compact fiber bundle and let F be a noncompact manifold with a smooth function $h: F \rightarrow \mathbb{R}$. Then $(\pi \circ \text{pr}_1, N \times F, B, E \times F, h \circ \text{pr}_2)$ is a filled bundle which is exhausted if and only if h is an exhaustion function.

Example 4.6.3. Let $F = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, y^2 + z^2 > \frac{1}{4}\}$, then F is a noncompact 2-manifold with 4 ends. Let $\phi \in \text{Diff}(F)$ be the diffeomorphism given by $\phi(x, y, z) = (x, -y, -z)$, switching two ends $z \rightarrow +\infty$ and $z \rightarrow -\infty$. Let $h: F \rightarrow \mathbb{R}, h(x, y, z) = z^2 + ((y^2 + z^2) - \frac{1}{4})^{-1}$, then h is an exhaustion function with the property $h \circ \phi = h$. Let $B = \mathbb{S}^1 = (0, 2)/(p \mapsto p + 1)$. Then define $M = (0, 2) \times_{\phi} F$ where $\phi: (1, 2) \times F \rightarrow (0, 1) \times F$ is given by $\phi(p, y) = (p - 1, \phi(y))$. Let $\pi: M \rightarrow B$ be the map induced by the $\text{pr}_1: (0, 2) \times F \rightarrow (0, 2)$, and $f: M \rightarrow \mathbb{R}$ be the map induced by the $h \circ \text{pr}_2: (0, 2) \times F \rightarrow \mathbb{R}$. Then $\mathbf{M} = (\pi, M, B, F, f)$ is an oriented exhausted bundle where M has 3 ends, since ϕ is orientation preserving.

Example 4.6.4. Let $k \in \mathbb{N}$ and \mathcal{G} be a subgroup of $\text{SO}(n)$. Let $E \subset \mathbb{R}^k$ be a noncompact complete submanifold, which is invariant under \mathcal{G} . Let $u: \mathbb{R}^k \rightarrow \mathbb{R}$ be a smooth function such that $u \circ \phi = u$ for any $\phi \in \mathcal{G}$. Let $F = E \cap \{u > 0\}$. Let $h: F \rightarrow \mathbb{R}, h(x) = |x|^2 + u(x)^{-1}$, then h is an exhaustion function with the property $h \circ \phi = h$ for any $\phi \in \mathcal{G}$. Let (π, M, B, F) be any fiber bundle with structure group \mathcal{G} such that B is compact. Let f be the unique exhaustion for M such that the transition maps in \mathcal{G} is compatible with f , with the uniform h in place of h_i , in Definition 4.1.1. Then $\mathbf{M} = (\pi, M, B, F, f)$ is an oriented exhausted bundle with noncompact fiber.

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Chapter 5

Moser stability on noncompact manifolds

5.1 Introduction

A fundamental problem in symplectic topology is that of determining when two symplectic forms are equivalent. Recall in Section 2.1.3 the symplectic stability result of Moser [28] (1965) that an isotopy of symplectic forms on a compact manifold is always a strong isotopy. In his early work on the h-principle Gromov showed that two cohomologous symplectic forms ω_0 and ω_1 on a noncompact manifold may be joined by an isotopy if and only if they are connected by a path of nondegenerate forms [20] (1969). On the other hand, in his paper on pseudoholomorphic curves [21] (1985) Gromov proved the existence of exotic symplectic structures on \mathbb{R}^{2n} , $n \geq 2$ (not symplectomorphic to the standard structure). See also [6, 29]. So there can be no straightforward generalization of Moser's result to the noncompact case; indeed Moser's argument depends strongly on the assumption of compactness. In order to give a natural setting within which one may attempt to generalize stability and other results from compact to noncompact symplectic manifolds Eliashberg and Gromov [13] (1991) formalized the notion of symplectic manifolds with convex ends, which has become a fundamental concept in symplectic topology. In particular it led to important work of Cieliebak and Eliashberg, e.g., in their book on Stein and Weinstein

manifolds [9] where stability results are established for special classes of symplectic manifolds with convex ends, namely for Liouville manifolds and Weinstein manifolds.

My goal is to drop the assumption that the symplectic forms be convex on the ends, keeping only the assumption that the underlying manifold has an end structure, i.e. can be viewed as the interior of a manifold with boundary. In order to do so, one must impose a growth condition on the path of symplectic forms, for which a metric is required. In this chapter I give a natural condition for an isotopy of symplectic forms on a manifold with cylindrical ends to be a strong isotopy. I am going to recall the definition of manifolds with cylindrical ends and state the main theorem in the next section.

5.2 Main theorem

Topologically, having cylindrical ends corresponds to the assumption, standard in symplectic topology, that the noncompact manifold may be viewed as the interior of a compact manifold with boundary. Recall that a Riemannian manifold (M, g) has *cylindrical ends* if there exists a compact codimension 0 submanifold K whose boundary ∂K is a smooth hypersurface, and an isometry $M \setminus K \rightarrow \partial K \times (1, \infty)$ where ∂K has the induced metric. The second component of the isometry may be smoothly extended to a function $M \rightarrow \mathbb{R}_+$ with values less than 1 on K° , referred to as the *radial coordinate function* of (M, g) . The reciprocal of the radial coordinate is a defining function for the *boundary at infinity* ∂M , diffeomorphic to ∂K . Let $\|\cdot\|_r$ denote the uniform norm with respect to the metric over the points with radial coordinate r . Let $S_a(M)$ be the set of symplectic forms on M with cohomology class $a \in H^2(M, \mathbb{R})$. We define the *log-variation* $\text{LV}: S_a(M) \times d\Omega^1(M) \rightarrow [0, \infty]$ by

$$\text{LV}(\omega, \beta) = \sup_{r \geq 1} r^{-1} \|\omega^{-1}\|_r \|\beta\|_r.$$

Our main result gives a sufficient condition for symplectic stability on these manifolds.

Theorem 5.2.1. *Let M be a manifold with cylindrical ends and $H^1(\partial M, \mathbb{R}) = 0$. If $\omega_t, t \in [0, 1]$, is a symplectic isotopy with total log-variation*

$$\int_0^1 \text{LV}(\omega_t, \dot{\omega}_t) dt < \infty$$

then it is a strong isotopy.

The condition in the theorem is not necessary, see Example 5.5.3, however it is a natural and practical sufficient condition.

Corollary 5.2.2. *Let M be a manifold with cylindrical ends and $H^1(\partial M, \mathbb{R}) = 0$. Then a symplectic isotopy $\omega_t, t \in [0, 1]$, is a strong isotopy if there exists $C > 0$ such that $\|\omega_t^{-1}\|_r \|\dot{\omega}_t\|_r \leq Cr$ for $r \gg 0, t \in [0, 1]$.*

Corollary 5.2.3. *Let M be a manifold with cylindrical ends and $H^1(\partial M, \mathbb{R}) = 0$, and fix $a \in H^2(M, \mathbb{R})$. Then $S_a(M) \times S_a(M) \rightarrow [0, \infty]$ given by $(\alpha, \beta) \mapsto \inf(\int_0^1 \text{LV}(\omega_t, \dot{\omega}_t) dt)$, where the infimum is taken over all isotopies from α to β , is a pseudometric. Moreover, forms at finite distance are strongly isotopic.*

Corollaries 5.2.2 and 5.2.3 follow immediately from the main theorem.

Corollary 5.2.4. *Let M be a manifold with cylindrical ends and radial coordinate function r , with $H^1(\partial M, \mathbb{R}) = 0$. Let ω be a symplectic form and σ a 1-form on M . Suppose that $\sup_{r \in \mathbf{r}(M)} \|\omega^{-1}\|_r \|\text{d}\sigma\|_r < 1$. Then $\omega + t \text{d}\sigma, t \in [0, 1]$, is a strong isotopy of symplectic forms.*

Corollary 5.2.5. *Let M be an even dimensional compact manifold, $\dim M \geq 4$, and let F be a finite set of points on M . If $\omega_t, t \in [0, 1]$, is a symplectic isotopy on $M \setminus F$ for which ω_t^{-1} and $\dot{\omega}_t$ are bounded uniformly in t with respect to any fixed metric on M , then ω_t is a strong isotopy on $M \setminus F$.*

Corollary 5.2.6. *A symplectic isotopy ω_t , $t \in [0, 1]$, on \mathbb{R}^{2n} , $2n \geq 4$, is a strong isotopy if there exists $C > 0$ such that $\|\omega_t^{-1}\|_r \|\dot{\omega}_t\|_r \leq C \log r$ for $r \gg 1, t \in [0, 1]$, where $\|\cdot\|_r$ is the uniform Euclidean norm over the sphere of radius r .*

Theorem 5.2.1 and Corollaries 5.2.4 and 5.2.5 are proved in Section 5.4. Corollary 5.2.6 follows by noting that, away from the origin, Euclidean space is conformal to a cylinder.

In fact, we may divide $\|\omega_t^{-1}\|_r \|\dot{\omega}_t\|_r$ in Corollary 5.2.6 by any positive function asymptotic to $\log r$ as r goes to infinity.

Remark 5.2.1. Symplectic stability is known to hold for symplectic manifolds with convex ends in the sense of Eliashberg-Gromov [13], provided that the isotopy is given in terms of a smooth path of Liouville forms (i.e. 1-forms whose exterior derivatives give the smooth path of symplectic forms) with respect to which the convex end structure also varies suitably [9, Proposition 11.8]. Our result, by contrast, imposes no restriction on the asymptotic behavior of the symplectic forms. Rather, our assumption is on the growth of $\dot{\omega}_t$ relative to ω_t . Our results can therefore be applied to study isotopies starting from any symplectic form on \mathbb{R}^{2n} , for example. The main theorem does not completely cover the result [9, Proposition 11.8] for the case of convex ends, because the product $\|\omega_t^{-1}\|_r \|\dot{\omega}_t\|_r$ can grow faster than $\mathcal{O}(r)$. See Example 5.5.3. Our proof combines a slicing strategy, similar to that used in [13, 9], with the original method of Moser. In order to make the argument work without any convexity assumption on the symplectic forms at the ends one needs to impose the growth condition that we give. The role of the condition $\|\omega_t^{-1}\|_r \|\dot{\omega}_t\|_r \leq Cr$ in Corollary 5.2.2 is intuitive and natural: it prevents finite time blow up for the ordinary differential inequality of the form $\dot{r}(t) \leq \|\omega_t^{-1}\|_r \|\dot{\omega}_t\|_r$. Heuristically, this inequality controls the escape to infinity of the integral curves for the time dependent vector field X_t , constructed by generalizing Moser's path method (Section 5.3), whose flow gives the strong isotopy. Actually, a differential inequality of the form $\dot{r}(t) \leq \varepsilon + C \|\omega_t^{-1}\|_r \|\dot{\omega}_t\|_r$ can only be made to hold for r in a set of intervals with arbitrarily small gaps between them. In our approach we therefore control the length of the integral curves more directly, leading to the result obtained

in the main theorem.

Remark 5.2.2. The assumption $H^1(\partial M, \mathbb{R}) = 0$ is equivalent to the natural map $H_c^2(M, \mathbb{R}) \rightarrow H^2(M, \mathbb{R})$ being injective. This allows one to handle the compact part of M separately in constructing the generator X_t of the strong isotopy via the path method (Section 5.3). More importantly, this assumption implies injectivity of the map $H_c^2(V, \mathbb{R}) \rightarrow H^2(V, \mathbb{R})$ for sets V of the form $\mathbf{r}^{-1}((r - \varepsilon, r + \varepsilon))$, where \mathbf{r} is the radial coordinate function of M . Without this assumption it is impossible to construct the time dependent vector field X_t with bounds on X_t which are localized in the radial coordinate on the ends. This makes the assumption natural, and apparently necessary for our kind of results. The assumption $H^1(\partial M, \mathbb{R}) = 0$ also implies $\dim M > 2$. If $\dim M = 2$ a symplectic isotopy is a strong isotopy if $\int_M \omega_0 = \int_M \omega_1$ and the set of ends where ω_0 and ω_1 give infinite volume coincide up to permutation by a diffeomorphism, see [19] and Chapter 4.

Remark 5.2.3. In stating our main theorem and some of its corollaries we have made use of a Riemannian metric with cylindrical ends. This metric plays only an auxiliary role, allowing us to give the simplest formulation of our result. Metrics with different asymptotics can be used. This is demonstrated for the most basic case of the Euclidean metric in Corollary 5.2.6. For concrete examples our conditions are also very easy to check. The following is a simple application of Corollary 5.2.6: If f_1, f_2 are smooth functions bounded away from zero and with bounded time derivative and c is any constant, then the isotopy of symplectic forms $\omega_t = f_1(t, x_1, y_1) dx_1 \wedge dy_1 + f_2(t, x_2, y_2) dx_2 \wedge dy_2 + c dx_1 \wedge dx_2$, $t \in [0, 1]$, on \mathbb{R}^4 is a strong isotopy. More generally, the time derivatives of f_1 and f_2 may have logarithmic growth in r , the radial coordinate on \mathbb{R}^4 .

5.3 Path method on noncompact manifolds

For M compact, Moser proved his symplectic stability result by differentiating $\varphi_t^* \omega_t = \omega_0$ to get $0 = \frac{d}{dt}(\varphi_t^* \omega_t) = \varphi_t^*(\dot{\omega}_t + \mathcal{L}_{X_t} \omega_t)$, where $\dot{\omega}_t$ is the time derivative of ω_t and X_t is the time-dependent vector field generating the family φ_t , and then solving for φ_t in terms of X_t . Since $[\omega_t]$ is constant, $\dot{\omega}_t$ is exact for all $t \in [0, 1]$. By Hodge theory on compact manifolds there exists a smooth family σ_t of 1-forms such that $\dot{\omega}_t = d\sigma_t$ for all $t \in [0, 1]$. By Cartan's formula $\mathcal{L}_{X_t} \omega_t = d(X_t \lrcorner \omega_t)$ since ω_t is closed for each $t \in [0, 1]$. So $\dot{\omega}_t + \mathcal{L}_{X_t} \omega_t = d(\sigma_t + X_t \lrcorner \omega_t)$. If one chooses X_t to be the vector field determined by $\sigma_t + X_t \lrcorner \omega_t = 0$ then, since M is compact, we may integrate X_t to determine a family φ_t such that $\varphi_t^* \omega_t = \omega_0$ for all $t \in [0, 1]$. This technique is usually called the *path method*. In the noncompact case, the argument above does not work, and the conclusion is false. The problem lies in being able to solve $\dot{\omega}_t = d\sigma_t$ for a smooth family of 1-forms σ_t in such a way that $X_t, t \in [0, 1]$, is complete.

The following is the outline of the steps we carry out to construct the vector field X_t and provide the L^∞ estimates needed to determine the existence of the flow when M is not compact: In the first step we consider a compact Riemannian manifold (N, g_N) of dimension m and an open interval J . Combining Hodge theory on (N, g_N) with the Poincaré Lemma one has, for any k with $1 \leq k \leq m$, an operator $I_{N \times J}^k: \Omega^k(N \times J) \rightarrow \Omega^{k-1}(N \times J)$ satisfying $dI_{N \times J}^k \omega = \omega$ for all $\omega \in d\Omega^{k-1}(N \times J)$. We bound the L^∞ norm of $I_{N \times J}^k$ by proving (for $m \geq 3$) that $I_N^k = d^* \circ G: \Omega^k(N) \rightarrow \Omega^{k-1}(N)$ has finite L^∞ norm, where G is the Green's operator for the Hodge Laplacian on k -forms and d^* is the codifferential.

In the second step we solve the d -equation for compactly supported forms. Let M be a smooth manifold and let V be an open submanifold of M with compact closure and smooth boundary. We use the weighted Hodge theory of Bueler-Prohorenkov [7] on noncompact manifolds to construct an operator $I_{M,V}^k: \Omega_c^k(M, V) \rightarrow \Omega^{k-1}(M, V)$ on forms compactly supported in V satisfying $d \circ I_{M,V}^k \omega = \omega$ for all $\omega \in d\Omega_c^{k-1}(M, V)$.

In the final step, given an isotopy of symplectic forms ω_t , $t \in [0, 1]$, we put the previous steps together to construct a time-dependent vector field X_t satisfying $d(X_t \lrcorner \omega_t) = -\dot{\omega}_t$ with explicit L^∞ estimates in terms of the L^∞ norms of $\dot{\omega}_t$, ω_t^{-1} , and the operators $I_{N \times J}^k$ and $I_{M, V}^k$ for a collection of precompact pieces $U \cong N \times J$ and V of the underlying manifold M . To define these pieces we pick a proper smooth function f and a covering of $f(M)$ by intervals whose preimages give the sets U and V . For intervals J not containing any critical values of f we identify $U = f^{-1}(J)$ with $N \times J$, where $N = f^{-1}(r_0)$ for some $r_0 \in J$, and define $\sigma_t = I_{N \times J}^2 \dot{\omega}_t$, for which we have explicit L^∞ estimates from the first step. We then smoothly extend σ_t across the remaining gluing regions, corresponding to the remaining intervals J' , to solve $d\sigma_t = \dot{\omega}_t$. This requires using the operator $I_{M, V}^2$ from the second step with $V = f^{-1}(J')$. This gluing step is topologically obstructed, and we must assume that $H_c^2(V, \mathbb{R}) \rightarrow H^2(V, \mathbb{R})$ is injective (this is the reason for the condition $H^1(\partial M, \mathbb{R}) = 0$ in our main theorem). We then let $X_t = -\omega_t^{-1} \sigma_t$. Since $d(X_t \lrcorner \omega_t) = -\dot{\omega}_t$, the local flow of φ_t of X_t starting from $t_0 = 0$ satisfies $\frac{d}{dt}(\varphi_t^* \omega_t) = 0$, where this makes sense. So the problem reduces to studying the global existence of the flow φ_t for $t \in [0, 1]$. This is done in Section 5.4 using the precise estimates on X_t which appear in Lemma 5.3.4.

Step 1: L^∞ estimates for solving the d-equation

Let N be a manifold and J an open interval. The Poincaré Lemma for de Rham cohomology states that $H^k(N \times J, \mathbb{R}) = H^k(N, \mathbb{R})$ for any k . This is proved by fixing any $r_0 \in J$ and constructing a de Rham homotopy operator for the pair of maps $\pi: N \times J \rightarrow N$, the projection, and $\iota: N \hookrightarrow N \times J$, the inclusion $y \mapsto (y, r_0)$. An example of such a homotopy operator is the map $I_0^k: \Omega^k(N \times J) \rightarrow \Omega^{k-1}(N \times J)$ given by $(I_0^k \omega)(y, r) = \int_{r_0}^r \partial_s \lrcorner \omega(y, s) ds$ for each $(y, r) \in N \times J$, where ∂_s is the coordinate vector field along J . A straightforward calculation shows that $dI_0^k \omega + I_0^k d\omega = \omega - \pi^* \iota^* \omega$ for any $\omega \in \Omega^k(N \times J)$, with $0 \leq k \leq \dim N + 1$. We will make use of the following trivial consequence.

Lemma 5.3.1. *Let N be a manifold and J an open interval. Let $k \in \{1, \dots, \dim N\}$ and let*

$I_N^k: \Omega^k(N) \rightarrow \Omega^{k-1}(N)$ be a smooth operator such that $dI_N^k = \text{id}$ on $d\Omega^{k-1}(N)$. Fix $r_0 \in J$ and let $\iota: N \hookrightarrow N \times J$ be the map $y \mapsto (y, r_0)$. Then the operator $I_{N \times J}^k: \Omega^k(N \times J) \rightarrow \Omega^{k-1}(N \times J)$ given by

$$(I_{N \times J}^k \omega)(y, r) = \int_{r_0}^r \partial_s \lrcorner \omega(y, s) ds + (I_N^k \iota^* \omega)(y)$$

satisfies $dI_{N \times J}^k \omega = \omega$ for all $\omega \in d\Omega^{k-1}(N \times J)$.

We will be applying Lemma 5.3.1 in the case of a compact Riemannian manifold (N, g_N) . In order to bound the L^∞ norm of $I_{N \times J}^k$ it suffices to prove that the natural Hodge theoretic operator I_N^k has finite L^∞ norm.

Theorem 5.3.2. *Let (N, g_N) be a compact Riemannian manifold of dimension $m \geq 3$. Let $k \in \{1, \dots, m\}$ and let $I_N^k = d^* \circ G: \Omega^k(N) \rightarrow \Omega^{k-1}(N)$ where G is the Green's operator for the Hodge Laplacian on k -forms, and d^* is the codifferential. Then $d \circ I_N^k$ is the identity on $d\Omega^{k-1}(N)$ and*

$$\|I_N^k\|_{L^\infty} = \sup_{\omega \in \Omega^k(N)} \frac{\|I_N^k \omega\|_{L^\infty(N, g_N)}}{\|\omega\|_{L^\infty(N, g_N)}} < \infty.$$

Here $\|\cdot\|_{L^\infty(N, g_N)}$ is the uniform norm with respect to g_N over N .

Proof. The Green's operator $G: \Omega^k(N) \rightarrow \Omega^k(N)$ is characterized by $\Delta G\omega = \omega$ for $\omega \in (\ker \Delta)^\perp$ and $G\omega = 0$ for $\omega \in \ker \Delta$, where $\Delta: \Omega^k(N) \rightarrow \Omega^k(N)$ is the Hodge Laplacian. It is possible to construct an integral kernel for G ; the only difficulty is that the Green's kernel must be thought of as a distributional section of the bundle $\pi_1^* \Lambda^2 N \otimes \pi_2^* (\Lambda^2 N)^* \rightarrow N \times N$ where $\pi_1, \pi_2: N \times N \rightarrow N$ are the projections onto the first and second factor respectively. We will show that the Green's kernel has the same asymptotic behavior at leading order near the diagonal as the Euclidean Green's function (cf. [4] for the case of functions). To construct the Green's kernel we solve $\Delta_{q, \text{distr.}} G(p, q) = \delta_p(q) - V^{-1}$ where $\Delta_{q, \text{distr.}}$ is the distributional Laplacian, $\delta_p(q)$ is the Dirac delta function at p , and V is the volume of (N, g_N) . We start by formally approximating $G(p, q)$ near the diagonal. Let $f \in C_0^\infty(\mathbb{R})$ be the standard bump

function equal to 1 on $(-\frac{\delta}{2}, \frac{\delta}{2})$ and supported in $(-\delta, \delta)$ where δ is the injectivity radius of (N, g_N) . Let $H(p, q) = \frac{\text{dist}(p, q)^{2-m}}{(m-2)\sigma_{m-1}} f(\text{dist}(p, q))$ where σ_{m-1} is the volume of the $(m-1)$ -sphere. Let n be an integer larger than $\frac{m}{2}$. Let $\Gamma_1(p, q) = -\Delta_q H(p, q)$ and for $1 \leq i \leq n$ let $\Gamma_{i+1}(p, q) = -\int_N \Gamma_i(p, r) \Delta_q H(r, q) d\text{Vol}_q$. We write

$$G(p, q) = H(p, q) + \sum_{i=1}^n \int_N \Gamma_i(p, r) H(r, q) d\text{Vol}_q + F(p, q)$$

where $F(p, q)$ is a distributional section of $\pi_1^* \Lambda^2 N \otimes \pi_2^* (\Lambda^2 N)^* \rightarrow N \times N$, and seek to solve for $F(p, q)$. Taking the Laplacian of $G(p, q)$, using that $\Delta_{q, \text{distr.}} H(p, q) = \Delta_q H(p, q) + \delta_p(q)$ by Green's third identity (see for instance Page 107 in [4]), and canceling,

$$V^{-1} = \Gamma_{n+1}(p, q) + \Delta_{q, \text{distr.}} F(p, q). \quad (5.3.1)$$

By a standard Lemma of Giraud [16, p. 150] $\Gamma_n(p, q)$ is bounded, and consequently $\Gamma_{n+1}(p, q)$ is C^1 . By elliptic theory, for each fixed p there is a weak solution $F(p, q)$ of (5.3.1). Then by elliptic regularity for elliptic operators between vector bundles whose principal part has scalar coefficients the solution $F(p, q)$ is C^2 . It follows from the definition of $H(p, q)$ and the ansatz for $G(p, q)$ above that $G(p, q) = \frac{\text{dist}(p, q)^{2-m}}{(m-2)\sigma_{m-1}} (1 + \mathcal{O}(\text{dist}(p, q)))$ near the diagonal. Thus $\left| \int_{B_\delta(p)} d_p^* G(p, q) \omega(q) d\text{Vol}_q \right|$ is at most

$$\left| \int_{B_\delta(p)} \frac{r^{1-m}}{(m-2)\sigma_{m-1}} (1 + \mathcal{O}(r)) r^{m-1} dr d\text{Vol}_{\mathbb{S}^{m-1}} \right| \|\omega\|_{L^\infty(N, g)}$$

where $r = \text{dist}(p, q)$. Since the derivative of $G(p, q)$ is bounded outside of the ball $B_\delta(p)$ and N

is compact there exists $C > 0$ for which

$$\begin{aligned} \left| (I_N^k \omega)(p) \right|_g &= \left| \int_N d_p^* G(p, q) \omega(q) d\text{Vol}_q \right|_g \\ &\leq \left| \int_{B_\delta(p)} d_p^* G(p, q) \omega(q) d\text{Vol}_q \right|_g + \left| \int_{N \setminus B_\delta(p)} d_p^* G(p, q) \omega(q) d\text{Vol}_q \right|_g \\ &\leq C \|\omega\|_{L^\infty(N, g)} \end{aligned}$$

for all $p \in N$. □

Step 2: Solving the d-equation for compactly supported forms

Lemma 5.3.3. *Let M be a smooth manifold and let V be an open submanifold of M with smooth compact boundary. Let $\Omega^k(M, V)$ be the space of k -forms which vanish outside of V . For $k \in \{1, \dots, \dim M\}$ there exists a smooth operator $I_{M, V}^k: \Omega_c^k(V) \rightarrow \Omega^{k-1}(M, V)$ such that $(d \circ I_{M, V}^k \omega)|_V = \omega$ for all $\omega \in d\Omega_c^{k-1}(V)$.*

Proof. Let g_N be a metric on $N = \partial V$. Let U be a tubular neighborhood of N and let ρ be a defining function for N such that $U = \rho^{-1}(-1, 1)$ and $\rho > 0$ on V . Fix a diffeomorphism $U \rightarrow N \times (-1, 1)$ with the second component being ρ . Let $f = \rho^{-1}$ on V and use this diffeomorphism to identify $U \cap V$ with $N \times (1, \infty)$. The metric $g_N \oplus dr^2$ on $N \times (1, \infty)$ may be extended to a complete metric g_V on V . Let $\mathcal{S}(\Lambda^k V)$ be the space of smooth k -forms ω on V with rapid decay in the sense that $\lim_{r \rightarrow \infty} |f^\ell \partial^\alpha \omega|(y, r) = 0$ for any multiindex α , $\ell \in \mathbb{N}$, and choice of local coordinates on N (here the coordinate derivatives are with respect to (y, r) and act only on the coefficients of the differential form). A k -form ω in $e^{-2f^2} \mathcal{S}(\Lambda^k V)$ vanishes to infinite order on $N = \partial V$, and thus extends smoothly by zero to all of M . Let $\mu = e^{2f^2} d\text{vol}_{g_V}$ where $d\text{vol}_{g_V}$ is the Riemann-Lebesgue measure. Then $d_\mu^* = e^{-2f^2} d^* e^{2f^2}$ is the formal adjoint of d with respect to μ . Let $\Delta_\mu = dd_\mu^* + d_\mu^* d$. By the Hodge decomposition of [7] there exists a Green's operator G_μ for Δ_μ with domain and codomain equal to $e^{-2f^2} \mathcal{S}(\Lambda^k V)$, which properly contains $\Omega_c^k(V)$.

By definition we then have $\text{d}d_\mu^* G_\mu \omega = \omega$ for all $\omega \in \text{d}\Omega_c^{k-1}(V)$, and we define $I_{M,V}^k$ to be $d_\mu^* G_\mu$ composed with extension by zero. \square

Step 3: Piecewise construction of X_t with estimates

Given a compact Riemannian manifold (N, g_N) we denote, as in Theorem 5.3.2, the Hodge theoretic right inverse to the exterior derivative $\text{d} : \Omega^1(N) \rightarrow \Omega^2(N)$ by I_N^2 .

Lemma 5.3.4. *Let M be a manifold, $\dim M \geq 4$, and $f : M \rightarrow \mathbb{R}$ an exhaustion. Let $[a_i, b_i]$, $i \in \mathbb{N}$, be intervals containing no critical values of f such that $a_i < b_i < a_{i+1}$ and $b_i \rightarrow \infty$. Let $X = \cup_{i \in \mathbb{N}} [a_i, b_i]$ and suppose that $H_c^2(M \setminus f^{-1}(X), \mathbb{R}) \rightarrow H^2(M \setminus f^{-1}(X), \mathbb{R})$ is injective. Let g be a Riemannian metric on M such that ∇f is a unit Killing vector field on $f^{-1}(X)$. If ω_t , $t \in [0, 1]$, is an isotopy of symplectic forms on M , then there exists a time-dependent vector field X_t on M , $t \in [0, 1]$, satisfying $\text{d}(X_t \lrcorner \omega_t) = -\dot{\omega}_t$ and on each $U_i = f^{-1}(a_i, b_i)$*

$$\|X_t\|_{L^\infty(U_i, g)} \leq \left(\frac{b_i - a_i}{2} + \left\| I_{f^{-1}(\frac{a_i+b_i}{2})}^2 \right\|_{L^\infty} \right) \|\omega_t^{-1}\|_{L^\infty(U_i, g)} \|\dot{\omega}_t\|_{L^\infty(U_i, g)}$$

for each $i \in \mathbb{N}$, $t \in [0, 1]$.

Proof. For each $i \in \mathbb{N}$ let $J_i = (a_i, b_i)$ and choose enlarged intervals $\tilde{J}_i = (\tilde{a}_i, \tilde{b}_i)$ such that the closures $[\tilde{a}_i, \tilde{b}_i]$ do not contain critical points of f , and $\tilde{a}_i < a_i < b_i < \tilde{b}_i < \tilde{a}_{i+1}$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ let $\hat{J}_i = (\frac{a_i+2\tilde{a}_i}{3}, \frac{b_i+2\tilde{b}_i}{3})$, so that $J_i \subsetneq \hat{J}_i \subsetneq \tilde{J}_i$, and let $U_i = f^{-1}(J_i)$, $\hat{U}_i = f^{-1}(\hat{J}_i)$, and $\tilde{U}_i = f^{-1}(\tilde{J}_i)$. Let $r_i = \frac{a_i+b_i}{2}$, and let $\iota_{r_i} : f^{-1}(r_i) \rightarrow M$ be the inclusion. Using the flow ψ of ∇f we may identify \tilde{U}_i with $f^{-1}(r_i) \times \tilde{J}_i$. We thus define the 1-form σ_t^i on \tilde{U}_i by

$$\sigma_t^i(y, r) = \int_{r_i}^r \nabla f \lrcorner \psi_{s-r}^* (\dot{\omega}_t(y, s)) \, ds + (I_{f^{-1}(r_i)}^2 \iota_{r_i}^* \dot{\omega}_t)(y), \quad (5.3.2)$$

for $(y, r) \in f^{-1}(r_i) \times \tilde{J}_i$. By Lemma 5.3.1 we have $\text{d}\sigma_t^i = \dot{\omega}_t$ on \tilde{U}_i . Let $\lambda_i : M \rightarrow [0, 1]$ be a smooth function supported in \tilde{U}_i and equal to 1 in a neighborhood of \hat{U}_i . Let $\alpha_t = \dot{\omega}_t -$

$\sum_{i=1}^{\infty} d(\lambda_i \sigma_t^i) = -\sum_{i=1}^{\infty} d\lambda_i \wedge \sigma_t^i + (1 - \sum_{i=1}^{\infty} \lambda_i) \dot{\omega}_t$. Let $J_{0,1} = (-\infty, \frac{2a_1 + \tilde{a}_1}{3})$ and for each $i \in \mathbb{N}$ let $J_{i,i+1} = (\frac{2b_i + \tilde{b}_i}{3}, \frac{2a_{i+1} + \tilde{a}_{i+1}}{3})$. For all $i \in \mathbb{N} \cup \{0\}$ let $V_{i,i+1} = f^{-1}(J_{i,i+1})$. Note that α_t is supported in the union of the gluing regions $V_{i,i+1}$, moreover $\alpha_t|_{V_{i,i+1}}$ is compactly supported in $V_{i,i+1}$. Since α_t is exact on M and, by assumption, $H_c^2(V_{i,i+1}) \rightarrow H^2(V_{i,i+1})$ is injective we have $[\alpha_t|_{V_{i,i+1}}]_{H_c^2(V_{i,i+1})} = 0$. By Lemma 5.3.3, there is $\beta_t^{i,i+1} \in \Omega^1(M)$ which vanishes outside $V_{i,i+1}$, and satisfies $d\beta_t^{i,i+1} = \alpha_t$ on $V_{i,i+1}$. Let $\beta_t = \sum_{i=0}^{\infty} \beta_t^{i,i+1}$ and let $\sigma_t = \sum_{i=1}^{\infty} (\lambda_i \sigma_t^i) + \beta_t$. Then $\dot{\omega}_t = d\sigma_t$. Hence the time-dependent vector field given by $X_t = -\omega_t^{-1} \sigma_t$ for $t \in [0, 1]$ satisfies $d(X_t \lrcorner \omega_t) = -\dot{\omega}_t$. The estimate follows from (5.3.2). \square

5.4 Symplectic stability on manifolds with cylindrical ends

Lemma 5.4.1. *Let M be a smooth manifold and X_t , $t \in [0, 1]$, a smooth time-dependent vector field on M . Let $\gamma: J \rightarrow M$ be the maximal flow line of X_t with $\gamma(0) = x_0$. If $\gamma(J)$ is contained in a compact set then $J = [0, 1]$.*

Proof. Suppose that $J \neq [0, 1]$, then there is $T \in (0, 1]$ such that $J = [0, T)$. Define \tilde{X} on $M \times [0, 1]$ by $\tilde{X} = X_t + \partial_t$, and let $\tilde{\gamma}$ be the maximal integral curve of \tilde{X} with $\tilde{\gamma}(0) = (x_0, 0)$. Then $\tilde{\gamma}$ has maximal domain J and is given by $\tilde{\gamma}(t) = (\gamma(t), t)$. By the standard Escape Lemma [26, Lemma 9.19] $\tilde{\gamma}(J)$ is not contained in any compact subset of $M \times [0, 1]$. But this implies that $\gamma(J)$ is not contained in any compact subset of M . \square

Lemma 5.4.2. *Suppose that M is a noncompact manifold and $f: M \rightarrow \mathbb{R}$ is an exhaustion such that $H^1(f^{-1}(r), \mathbb{R}) = 0$ for $r > R$. Let $\{r_i\}_{i \in \mathbb{N}} \subset \mathbb{R}_{>R}$, $\{\delta_i\}_{i \in \mathbb{N}} \subset \mathbb{R}_{>0}$, and $\{\alpha_i\}_{i \in \mathbb{N}} \subset \mathbb{R}_{\geq 0}$ be sequences such that the intervals $[r_i - \delta_i, r_i + \delta_i]$ are disjoint and contain no critical values of f , and $\sum_{i=1}^{\infty} \alpha_i \delta_i = \infty$. Let g be a metric on M which is a product on each $U_i = f^{-1}((r_i - \delta_i, r_i + \delta_i)) \cong f^{-1}(r_i) \times (r_i - \delta_i, r_i + \delta_i)$. Then an isotopy of symplectic forms ω_t , $t \in [0, 1]$, such that*

$$\int_0^1 \sup_{i \in \mathbb{N}} \alpha_i \left(\delta_i + \left\| I_{f^{-1}(r_i)}^2 \right\|_{L^\infty} \right) \left\| \omega_t^{-1} \right\|_{L^\infty(U_i, g)} \left\| \dot{\omega}_t \right\|_{L^\infty(U_i, g)} dt < \infty.$$

is a strong isotopy.

Proof. Let $J_i = (a_i, b_i) = (r_i - \delta_i, r_i + \delta_i)$ for each $i \in \mathbb{N}$. The assumption that $H^1(f^{-1}(r), \mathbb{R}) = 0$ for all $r > R$ implies that $\dim M > 2$ and $H_c^2(M \setminus f^{-1}(X), \mathbb{R}) \rightarrow H^2(M \setminus f^{-1}(X), \mathbb{R})$ is injective for $X = \cup_{i \in \mathbb{N}} [a_i, b_i]$. Hence $\dim M \geq 4$ and we can apply Lemma 5.3.4. Let X_t be the time-dependent vector field of Lemma 5.3.4. It suffices to show that the flow of X_t starting at $t_0 = 0$ exists globally for all $t \in [0, 1]$. Let $x \in M$. Fix i_0 such that $f(x) < a_{i_0}$. Let γ be the maximal flow line of X_t with $\gamma(0) = x$. Suppose that the maximal domain of γ is $[0, T)$ with $0 < T \leq 1$. Then by Lemma 5.4.1 the image $\gamma([0, T))$ must not be contained in any compact set. So $\lim_{t \rightarrow T} f(\gamma(t)) = \infty$. It follows that γ must pass through each set U_i with $i \geq i_0$. For each i let $\ell_i = \int_{\gamma^{-1}(U_i)} |X(\gamma(t))|_g dt$. Then $\ell_i \geq \delta_i$ for each $i \geq i_0$, so $\sum_{i=i_0}^{\infty} \alpha_i \ell_i \geq \sum_{i=i_0}^{\infty} \alpha_i \delta_i = \infty$.

On the other hand, by the bound on $\|X_t(x)\|_{L^\infty(U_i, g)}$ from Lemma 5.3.4

$$\ell_i \leq \int_{\gamma^{-1}(U_i)} \left(\delta_i + \left\| I_{f^{-1}(r_i)}^2 \right\|_{L^\infty} \right) \left\| \omega_t^{-1} \right\|_{L^\infty(U_i, g)} \left\| \dot{\omega}_t \right\|_{L^\infty(U_i, g)} dt$$

and thus $\alpha_i \ell_i \leq \int_{\gamma^{-1}(U_i)} \alpha_i \left(\delta_i + \left\| I_{f^{-1}(r_i)}^2 \right\|_{L^\infty} \right) \left\| \omega_t^{-1} \right\|_{L^\infty(U_i, g)} \left\| \dot{\omega}_t \right\|_{L^\infty(U_i, g)} dt$. Since $\gamma^{-1}(U_i)$ as subsets of $[0, 1]$ are disjoint we have that

$$\sum_{i=i_0}^{\infty} \alpha_i \ell_i \leq \int_0^1 \sup_{i \in \mathbb{N}} \alpha_i \left(\delta_i + \left\| I_{f^{-1}(r_i)}^2 \right\|_{L^\infty} \right) \left\| \omega_t^{-1} \right\|_{L^\infty(U_i, g)} \left\| \dot{\omega}_t \right\|_{L^\infty(U_i, g)} dt < \infty,$$

a contradiction. We conclude that γ has domain $[0, 1]$. □

Proof of Theorem 5.2.1. Let (M, g) be a Riemannian manifold with cylindrical ends and let $f: M \rightarrow \mathbb{R}_+$ be its radial coordinate function. Let $\omega_t, t \in [0, 1]$, be an isotopy of symplectic forms with total log-variation $\int_0^1 LV(\omega_t, \dot{\omega}_t) dt < \infty$. Since M is symplectic and $H^1(\partial M, \mathbb{R}) = 0$, $\dim M \geq 4$. Let Δ denote the diagonal in $(1, \infty) \times (1, \infty)$. The finiteness of the total log-variation is equivalent to $\int_0^1 \sup_{(f(x), f(x')) \in \Delta} f(x)^{-1} \left| \omega_t^{-1}(x) \right|_g \left| \dot{\omega}_t(x') \right|_g dt < \infty$. By continuity, any point $(c, c, t) \in \Delta \times [0, 1]$ has an open neighborhood $W_{c,t}$ such that $f(z)^{-1} \left| \omega_s^{-1}(z) \right|_g \left| \dot{\omega}_s(z') \right|_g$ is less than

$\sup_{(f(x), f(x')) \in \Delta} f(x)^{-1} |\omega_s^{-1}(x)|_g |\dot{\omega}_s(x')|_g + 1$ for all z, z' and s with $(f(z), f(z'), s) \in W_{c,t}$. So by the compactness of $[0, 1]$ there exists a neighborhood W of $\Delta \subset (1, \infty) \times (1, \infty)$ such that

$$\int_0^1 \sup_{(f(x), f(x')) \in W} f(x)^{-1} |\omega_t^{-1}(x)|_g |\dot{\omega}_t(x')|_g dt < \infty. \quad (5.4.1)$$

Let $\mu: (1, \infty) \rightarrow \mathbb{R}_+$ be a continuous function such that $r \mapsto r + \mu(r)$ is strictly increasing and $\{(r, s) \in \mathbb{R}_+^2 \mid -\mu(r) < r - s < \mu(s)\} \subset W$. We find disjoint subintervals $\{(r_i - \delta_i, r_i + \delta_i)\}_{i \in \mathbb{N}}$ in $(1, \infty)$ as follows. Let $\delta_1 = \min\{1, \mu(2)/2\}$ and $r_1 = 2 + \delta_1$. Then inductively let $\delta_{i+1} = \min\{1, \mu(r_i + \delta_i)/2\}$ and $r_{i+1} = r_i + \delta_i + \delta_{i+1}$ for all $i \in \mathbb{N}$. We have $r_i \rightarrow \infty$ as $i \rightarrow \infty$, since otherwise the sequence r_i would converge to some point r_∞ with $\mu(r_\infty) = 0$.

For each $i \in \mathbb{N}$, let $\alpha_i = 1/(r_i + \delta_i)$. Then

$$\sum_{i=1}^{\infty} 2\alpha_i \delta_i = \sum_{i=1}^{\infty} \left(1 - \frac{r_i - \delta_i}{r_{i+1} - \delta_{i+1}}\right) \geq \sum_{i=1}^{\infty} \min\left\{\frac{1}{2}, \frac{1}{2} \log\left(\frac{r_{i+1} - \delta_{i+1}}{r_i - \delta_i}\right)\right\}.$$

In the last sum there are either infinitely many i for which the i^{th} summand is $\frac{1}{2}$, or there is some fixed $i_0 \in \mathbb{N}$ such that i^{th} summand is $\frac{1}{2} \log\left(\frac{r_{i+1} - \delta_{i+1}}{r_i - \delta_i}\right)$ for all $i \geq i_0$. In either case the sum diverges. So $\sum_{i=1}^{\infty} \alpha_i \delta_i = \infty$. By reducing each δ_i a little bit we can ensure that $r_i + \delta_i < r_{i+1} - \delta_{i+1}$ with $\sum_{i=1}^{\infty} \alpha_i \delta_i$ still being ∞ . For each $i \in \mathbb{N}$ let $J_i = (r_i - \delta_i, r_i + \delta_i)$ and $U_i = f^{-1}(J_i)$. Note that (5.4.1) holds with W replaced by the subset $\cup_{i \in \mathbb{N}} J_i \times J_i$, which implies $\int_0^1 \sup_{i \in \mathbb{N}} \alpha_i \|\omega_t^{-1}\|_{L^\infty(U_i, g)} \|\dot{\omega}_t\|_{L^\infty(U_i, g)} dt < \infty$, since $\alpha_i \leq f(x)^{-1}$ for $x \in U_i$, $i \in \mathbb{N}$. Now since the hypersurfaces $f^{-1}(r_i)$ are all isometric to $f^{-1}(r_1)$, the quantity $\|I_{f^{-1}(r_i)}^2\|_{L^\infty}$ is independent of i , and since $\delta_i \leq 1$ for all i we have

$$\begin{aligned} & \int_0^1 \sup_{i \in \mathbb{N}} \alpha_i \left(\delta_i + \|I_{f^{-1}(r_i)}^2\|_{L^\infty} \right) \|\omega_t^{-1}\|_{L^\infty(U_i, g)} \|\dot{\omega}_t\|_{L^\infty(U_i, g)} dt \\ & \leq \left(1 + \|I_{f^{-1}(r_1)}^2\|_{L^\infty}\right) \int_0^1 \sup_{i \in \mathbb{N}} \alpha_i \|\omega_t^{-1}\|_{L^\infty(U_i, g)} \|\dot{\omega}_t\|_{L^\infty(U_i, g)} dt. \end{aligned}$$

The result then follows from Lemma 5.4.2. □

Proof of Corollary 5.2.4. Suppose $A = \sup_{r \in f(M)} \|\omega^{-1}\|_r \|\mathbf{d}\sigma\|_r < 1$. For any $x \in M$, the map $\omega^{-1}(x) \mathbf{d}\sigma(x)$ is an endomorphism of $(T_x M, |\cdot|_g)$ with operator norm at most A . If $t < A^{-1}$, then $|t\omega^{-1}(x) \mathbf{d}\sigma(x)|_g < 1$ for any $x \in M$, which means $1 + t\omega^{-1}(x) \mathbf{d}\sigma(x)$ is invertible. So $\omega_t = \omega + t \mathbf{d}\sigma$ is symplectic for all $t \in [0, 1]$. Moreover, for any $x \in M$

$$|\omega_t^{-1}(x)|_g \leq |(1 + t\omega^{-1} \mathbf{d}\sigma)^{-1}(x)|_g |\omega^{-1}(x)|_g \leq (1 - tA)^{-1} |\omega^{-1}(x)|_g.$$

Thus by assumption we have

$$\int_0^1 \text{LV}(\omega_t, \dot{\omega}_t) dt \leq \int_0^1 \sup_{r \geq 1} (1 - tA)^{-1} \|\omega^{-1}\|_r \|\mathbf{d}\sigma\|_r dt \leq \frac{A}{1 - A} < \infty.$$

□

Proof of Corollary 5.2.5. Note that Corollary 5.2.6 generalizes trivially to manifolds equipped with a metric which is Euclidean on the end(s). We will make use of this generalization, rather than arguing directly from Theorem 5.2.1, because it makes the coordinate computations easier. It suffices to treat the case where F contains just one point p . Let g be a metric on M , and let U be geodesic ball about p . Scaling g if necessary we may take U to be a unit geodesic ball, and we may use normal (exponential) coordinates to identify (U, p) with $(B^{2n}, 0)$ where B^{2n} is the unit ball in \mathbb{R}^{2n} . Let $\phi : U \setminus \{p\} \rightarrow \mathbb{R}^{2n} \setminus \overline{B^{2n}}$ be the diffeomorphism which in normal coordinates sends $x \in B^{2n} \setminus \{0\}$ to $\frac{x}{|x|^2}$. Under ϕ the radial coordinate r on $\mathbb{R}^{2n} \setminus \overline{B^{2n}}$ pulls back to the reciprocal of the geodesic distance from p on $U \setminus \{p\}$. Let (x_i) denote the standard coordinates on B^{2n} and (\bar{x}_i) those on $\mathbb{R}^{2n} \setminus \overline{B^{2n}}$. Then $\phi_* dx_i = \sum_{j=1}^{2n} (\frac{\delta_{ij}}{|x|^2} + 2\frac{\bar{x}_i \bar{x}_j}{|x|^4}) d\bar{x}_j$ and $\phi_* \partial_{x_i} = \sum_{j=1}^{2n} (|\bar{x}|^2 \delta_{ij} + 2\bar{x}_i \bar{x}_j) \partial_{\bar{x}_j}$. Since $\dot{\omega}_t$ is bounded with respect to g , uniformly in t , the corresponding forms $\tilde{\omega}_t = \phi_* \dot{\omega}_t$ on $\mathbb{R}^{2n} \setminus \overline{B^{2n}}$ are $O(r^{-4})$, uniformly in t . Similarly, from the differential of ϕ one has that $\tilde{\omega}_t^{-1} = \phi_* \omega_t^{-1}$ is $O(r^4)$ uniformly in t . Pulling the Euclidean metric on $\mathbb{R}^{2n} \setminus \overline{B^{2n}}$ back to $U \setminus \{p\}$ and extending this to a metric g' on $M \setminus F$ we may apply (a trivial

generalization of) Corollary 5.2.6 to conclude that ω_t is a strong isotopy on $M \setminus F$. □

5.5 Applications of Moser stability theorem

Example 5.5.1. Consider \mathbb{R}^{2n} , $2n \geq 4$, with coordinates $(x_1, y_1, \dots, x_n, y_n)$. Let U be an open subset of \mathbb{R}^{2n} . Let $f_i \in C^\infty(U)$, for $i = 1, \dots, n$. Then $\omega = \sum_{i=1}^n f_i dx_i \wedge dy_i$ is a symplectic form if and only if each of the f_i is nowhere vanishing and depends only on the coordinates x_i and y_i . The isotopy of symplectic forms $\omega_t = \sum_{i=1}^n f_i(t, x_i, y_i) dx_i \wedge dy_i$, $t \in [0, 1]$, satisfies the assumption of Corollary 5.2.6 if the functions f_i are bounded away from zero and have bounded time derivative. Suppose $a_i \in \mathbb{R} \setminus \{0\}$. Consider the symplectic forms $\omega_t = a_1 \sqrt{x_1^2 + y_1^2 + 1 + t^2} dx_1 \wedge dy_1 + \sum_{i=2}^n a_i dx_i \wedge dy_i$, $t \in [0, 1]$. By Corollary 5.2.6 there is a smooth path of diffeomorphisms φ_t of \mathbb{R}^{2n} , $t \in [0, 1]$, such that $\varphi_t^* \omega_t = \omega_0$.

Example 5.5.2. Here we apply our result to an isotopy ω_t , $t \in [0, 1]$, for which the norm of the derivative grows with r , while the norm of the inverse decays. Let $\phi: [0, +\infty) \rightarrow [0, +\infty)$ be a diffeomorphism such that $\phi|_{[0,1]} = \text{id}$, and $\phi(r)/r$ is increasing. Then $\hat{\phi}: \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $\hat{\phi}(x) = \frac{\phi(|x|)}{|x|}x$ is a diffeomorphism. If $\omega = \hat{\phi}^* \omega_0$, then with $r = |x|$ we have

$$\begin{aligned} \omega(x_1, \dots, x_4) &= (A + B(x_1^2 + x_2^2)) dx_1 \wedge dx_2 + (A + B(x_3^2 + x_4^2)) dx_3 \wedge dx_4 \\ &\quad - B(x_1 x_4 - x_2 x_3)(dx_1 \wedge dx_3 + dx_2 \wedge dx_4) \\ &\quad + B(x_1 x_3 + x_2 x_4)(dx_1 \wedge dx_4 - dx_2 \wedge dx_3), \end{aligned}$$

where $A = (\frac{\phi(r)}{r})^2$ and $B = \frac{\phi(r)}{r^2} (\frac{\phi(r)}{r})' \geq 0$. Let us fix $p > 1$, $c \in (0, 1)$ and define ϕ by $\phi(r) = r^p$ for $r \geq 1$. Since we want ϕ to be smooth, we should perturb it in a neighborhood of $r = 1$. None of our estimates are affected if this perturbation is sufficiently small, so we proceed as if ϕ were

given by the exact formula. Then for $r \geq 1$ we have $A = r^{2p-2}$, $B = (p-1)r^{2p-4}$, and

$$\begin{aligned} & r^{4-2p} \omega(x_1, \dots, x_4) \\ &= (px_1^2 + px_2^2 + x_3^2 + x_4^2) dx_1 \wedge dx_2 + (x_1^2 + x_2^2 + px_3^2 + px_4^2) dx_3 \wedge dx_4 \\ &\quad - (p-1)(x_1x_4 - x_2x_3)(dx_1 \wedge dx_3 + dx_2 \wedge dx_4) \\ &\quad + (p-1)(x_1x_3 + x_2x_4)(dx_1 \wedge dx_4 - dx_2 \wedge dx_3). \end{aligned}$$

Let $\lambda: [0, +\infty) \rightarrow [0, +\infty)$ be an increasing smooth function which vanishes on $[0, \frac{1}{2}]$, equals 1 in $[1, +\infty)$, and satisfies $\lambda' \leq 3$. Let

$$\sigma = \frac{cp}{6(2p-1)^2} \lambda(r) r^{2p-1} (dx_1 + dx_2 + dx_3 + dx_4).$$

Then $d\sigma = \frac{cp}{6(2p-1)^2} ((2p-1)\lambda + \lambda' r) r^{2p-3} \sum_{i < j} (x_i - x_j) dx_i \wedge dx_j$. For an $m \times m$ -matrix $Q = (q_{ij})$, the ℓ^1 operator norm is $|Q|_{\ell^1} = \max_{1 \leq i \leq m} \sum_{j=1}^m |q_{ij}|$. For convenience we define $\|\cdot\|_r$ as the supremum over the sphere of radius r of this pointwise norm (rather than of the equivalent ℓ^2 norm). We then have $\|\omega^{-1}\|_r \leq (2-p^{-1})r^{2-2p}$ if $r \geq 1$, and $\|\omega^{-1}\|_r = 1$ if $r < 1$. Similarly $\|d\sigma\|_r \leq \frac{cp}{2p-1} r^{2p-2}$ if $r \geq 1$, and $|d\sigma(x)| \leq c$ if $r < 1$. Since $|\omega^{-1}(x)| \|d\sigma(x)\| \leq c < 1$ the 2-form $\omega_t = \omega + t d\sigma$ is nondegenerate for every $t \in [0, 1]$ (cf. the proof of Corollary 5.2.4). Moreover, $\int_0^1 \sup_{r \geq 1} \|\omega_t^{-1}\|_r \|\dot{\omega}_t\|_r dt < \infty$. So $\omega_t, t \in [0, 1]$, is a strong isotopy by Corollary 5.2.6.

Example 5.5.3. Here we give an example of a strong isotopy with infinite log variation. Consider the unit sphere S^3 contained in \mathbb{R}^4 with coordinates (x_1, y_1, x_2, y_2) , and let $\alpha_0 = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)$ be the standard contact form on S^3 . Consider the rescaled contact form $\alpha = (2x_1^2 + y_1^2)\alpha_0$ on S^3 . The structure (S^3, α) can be realized as the boundary of a Liouville domain (Ω, ω, V) in the sense of [9] (in fact this may be taken to be the boundary of a star convex domain in \mathbb{R}^4 with the standard symplectic form). The Liouville completion of (Ω, ω, V) is constructed by attaching $S^3 \times [0, \infty)$ to Ω , where the symplectic form on $S^3 \times [0, \infty)$ is $d(e^r \alpha)$

with r the coordinate on $[0, \infty)$. The resulting symplectic manifold (M, ω) is symplectomorphic to the standard \mathbb{R}^4 , but this construction allows us to more easily write down the required family of diffeomorphisms of M . For $t \in \mathbb{R}$ define $\phi_t : S^3 \times [0, \infty) \rightarrow S^3 \times [0, \infty)$ by $(x, r) \mapsto (e^{it r^p} \cdot x, r)$, where $e^{i\theta}$ acts on S^3 by a rotation through angle θ in the (x_1, y_1) -plane. The family ϕ_t may be extended to a smooth 1-parameter family of diffeomorphism of M , which we still denote ϕ_t . Let $\omega_t = \phi_t^* \omega$, $t \in [0, 1]$. Then on $M \setminus \Omega = S^3 \times [0, \infty)$ we have $\omega_t = e^r [(1 + \cos^2(tr^p))x_1^2 - \sin(2tr^p)x_1y_1 + (1 + \sin^2(tr^p))y_1^2](d\alpha_0 + dr \wedge \alpha) + e^r [2(1 + \cos^2(tr^p))x_1 dx_1 - \sin(2tr^p)(x_1 dy_1 + y_1 dx_1) + 2(1 + \sin^2(tr^p))y_1 dy_1] \wedge \alpha_0$. From this it is easy to see that $\|\omega_t^{-1}\|_r \sim e^{-r}$ whereas $\|\dot{\omega}_t\|_r \sim r^p e^r$, so that $\|\omega_t^{-1}\|_r \|\dot{\omega}_t\|_r \sim r^p$ and hence the Main Theorem does not apply. Although this is a path of Liouville structures by construction, it is not obvious from the formula for ω_t .

5.6 Other stability results

5.6.1 Naïve symplectic stability on \mathbb{R}^{2n}

For \mathbb{R}^{2n} it is possible to get a naive symplectic stability result with a completely elementary proof as follows. Let ω_t , $t \in [0, 1]$, be an isotopy of symplectic forms on \mathbb{R}^{2n} with $\int_0^1 \sup_{x \in \mathbb{R}^n, s \in [0, 1]} s|x| |\omega_t^{-1}(x)|_{g_{\mathbb{E}}} |\dot{\omega}_t(sx)|_{g_{\mathbb{E}}} dt$ finite. Then ω_t is a strong isotopy. To verify this let E be the Euler vector field on \mathbb{R}^{2n} and $I : \Omega^2(\mathbb{R}^{2n}) \rightarrow \Omega^1(\mathbb{R}^{2n})$ be given by $I\omega(x) = \int_0^1 E(sx) \lrcorner \omega(sx) ds$. Then $dI\omega = \omega$ for any exact 2-form ω . Let $\sigma_t = I\dot{\omega}_t$ and let $X_t = -\omega_t^{-1}\sigma_t$. Let $x \in \mathbb{R}^{2n}$ and let γ be the maximal flow line of X_t with $\gamma(0) = x$. If the maximal domain of γ is $[0, T)$ with $0 < T \leq 1$, then by Lemma 5.4.1 the image of γ must not be contained in any compact set. But the length of γ is bounded by $\int_0^T \sup_{x \in \mathbb{R}^n, s \in [0, 1]} s|x| |\omega_t^{-1}(x)|_{g_{\mathbb{E}}} |\dot{\omega}_t(sx)|_{g_{\mathbb{E}}} dt < \infty$ so $\gamma([0, T))$ is precompact. So the flow ϕ_t of X_t starting from $t_0 = 0$ exists for all $t \in [0, 1]$.

5.6.2 Symplectic stability for compactly supported isotopies

Using Lemma 5.3.3 one can generalize Moser’s stability theorem to apply to compactly supported isotopies: Let $\omega_t, t \in [0, 1]$, be an isotopy of symplectic forms on a manifold M such that $\text{supp}(\omega_t - \omega_0) \subset W$ for all t , where $W \subset M$ is an open submanifold with compact closure and smooth boundary, and the cohomology class of $(\omega_t - \omega_0)|_W$ in $H_c^2(W, \mathbb{R})$ is trivial for all t . Then for any smoothly bounded precompact open submanifold V of M with $\overline{W} \subset V$ there exists a smooth path of diffeomorphisms of M fixing $M \setminus V$ such that $\varphi_t^* \omega_t = \omega_0$ for all t . Indeed, let $I_{M,V}^2$ be as in Lemma 5.3.3. Let $\sigma_t = I_{M,V}^2 \dot{\omega}_t$, then $d\sigma_t = \dot{\omega}_t$. Then $X_t = -\omega_t^{-1} \sigma_t$ is compactly supported in W and therefore complete; the flow of X_t fixes points in $M \setminus V$. By the path method the flow φ_t of X_t satisfies $\varphi_t^* \omega_t = \omega_0$ for all t . In fact, the result holds for W any precompact open set, cf. [9, Theorem 6.8] or [17, Lemma, page 617] for alternative approaches (we chose to keep with the Hodge theoretic approach in establishing Lemma 5.3.3).

This result was used in the proof of the stability result [9, Proposition 11.8] for “Liouville homotopies” of Liouville manifolds, where it plays a role analogous to our use of Lemma 5.3.3 on the gluing regions: by assuming the existence of smoothly varying families of compact hypersurfaces transverse to the (radial) Liouville vector field Cieliebak and Eliashberg are able to construct the required 1-parameter family of diffeomorphisms on certain primary regions by applying Gray’s theorem [18] to these hypersurfaces and then using the local product structure coming from the Liouville vector field; the resulting 1-parameter family of diffeomorphisms can be fixed up on the remaining gluing regions by using the above generalization of Moser’s theorem. Without the convexity assumptions on the symplectic forms, however, and the compatible “Liouville homotopy” giving the smooth families of contact hypersurfaces on which one can apply Gray’s theorem, the generator X_t for the strong symplectic isotopy one is trying to construct needs to be estimated to determine its integrability.

5.6.3 Punctured compact manifolds

Considering punctured compact manifolds allows for a comparison of sorts between our result and the original result of Moser. Corollary 5.2.5 states that a symplectic isotopy on a punctured compact manifold $M \setminus F$ such that ω_t^{-1} and $\dot{\omega}_t$ are uniformly bounded with respect to a metric defined on M is a strong isotopy, provided $\dim M \geq 4$. Slightly modifying Moser's proof in the compact case one has a direct elementary proof of the weaker result: Let M be a compact manifold and let F be a finite set of points on M . If $\omega_t, t \in [0, 1]$, is a symplectic isotopy on $M \setminus F$ which is the restriction of a symplectic isotopy on M , then ω_t is a strong isotopy on $M \setminus F$. To demonstrate this let ω_t also denote the symplectic isotopy on M whose restriction is the isotopy ω_t on $M \setminus F$. Construct X_t on M as in the usual proof of Moser's theorem. Since F is finite, for each t one can choose a Hamiltonian vector field Y_t (Hamiltonian with respect to ω_t) for which $Y_t|_F = -X_t|_F$. Since X_t is smooth in t , Y_t can be chosen smooth in t . By the usual argument the flow $\varphi_t, t \in [0, 1]$, generated by $X_t + Y_t$ satisfies $\varphi_0 = \text{id}$ and $\varphi_t^* \omega_t = \omega_0$. Moreover, by construction φ_t preserves F . So $\varphi_t|_{M \setminus F}$ is the required strong isotopy.

5.6.4 Contact stability

The previous ideas apply trivially to contact manifolds. Let (M, g) be a complete oriented odd dimensional Riemannian manifold. Let $\theta_t, t \in [0, 1]$, be a smooth path of contact forms on M with $\int_0^1 \sup_M |(d\theta_t|_{H_t})^{-1} \dot{\theta}_t|_{H_t}|_g dt < \infty$ where $H_t = \ker \theta_t$. Then there exists a smooth path φ_t of diffeomorphisms of M and f_t of positive smooth functions on M such that $\varphi_0 = \text{id}$ and $\varphi_t^* \theta_t = f_t \theta_0$ for $t \in [0, 1]$. Indeed, this case is easy because one does not need to invert the exterior derivative to construct the time-dependent vector field (using the 'path method' of Gray [18]). Let $H_t = \ker \theta_t$, and let $H = H_0$. Let X_t be the time dependent vector field $-(d\theta_t|_{H_t})^{-1}(\dot{\theta}_t|_{H_t})$. Let $x \in M$ and let γ be the maximal flow line of X_t with $\gamma(0) = x$. If the maximal domain of γ is $[0, T)$ with $0 < T \leq 1$, then by Lemma 5.4.1 the image of γ must not be contained in any compact

set. However, the length of γ is bounded by $\int_0^1 \sup_M |(d\theta_t|_{H_t})^{-1} \dot{\theta}_t|_{H_t}|_g dt$ and therefore $\gamma([0, T))$ is precompact. So the flow φ_t of X_t starting from $t_0 = 0$ exists for all $t \in [0, 1]$. Let R_t denote the Reeb vector field of θ_t and let $h_t = \dot{\theta}_t(R_t)$. We compute, using Cartan's formula and $\theta_t(X_t) = 0$,

$$\frac{d}{dt}(\varphi_t^* \theta_t) = \varphi_t^*(\mathcal{L}_{X_t} \theta_t + \dot{\theta}_t) = \varphi_t^*(-\dot{\theta}_t|_{H_t} + \dot{\theta}_t) = \varphi_t^*(\dot{\theta}_t(R_t)\theta_t) = h_t \varphi_t^* \theta_t.$$

Since $\varphi_0^* \theta_0 = \theta_0$ there exists f_t such that $\varphi_t^* \theta_t = f_t \theta_0$ for all $t \in [0, 1]$.

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Chapter 6

Symplectic invariants of integrable systems with multiply pinched singular fibers

6.1 Introduction

The construction of computable invariants – topological, smooth, or symplectic – is fundamental in the study of integrable systems. Ideally one would like to classify important classes of integrable systems up to isomorphisms in terms of a collection of such invariants. This was achieved in the 1980s in the seminal work of Atiyah [3], Guillemin–Sternberg [22], and Delzant [11], who gave a classification of toric integrable systems on compact manifolds on any dimension. Here *toric* means that the flows of the Hamiltonian vector fields of the components f_1, \dots, f_n of the momentum map generate a Hamiltonian n -dimensional torus action. In the past ten years, there has been intense activity trying to extend this classification beyond the toric case. In [30, 31], this goal was achieved for semitoric systems in dimension 4, where *semitoric* means that f_1 generates an \mathbb{S}^1 -action, f_1 is a proper map, the singularities of F are non-degenerate without hyperbolic blocks, and singular fibers of F cannot be wedge sums of 2 or more spheres.

Throughout this chapter, with very few exceptions that I point out explicitly, I assume

that F is a proper map and has connected fibers. According to Theorem 2.2.3, in dimension 4 a nondegenerate singular point containing focus-focus blocks has the same local structure. Suppose our integrable system only has focus-focus singular points. Then a singular fiber \mathcal{F} can contain at most finitely many singularities whose number $k \in \mathbb{N}$ is the only topological invariant of a compact focus-focus fiber is the of focus-focus points in the fiber, and the fiber is homeomorphic to a torus pinched k times (Zung [38, 39]). If \mathcal{F} contains multiple singular points, I label the points with $\mathbb{Z}_k \stackrel{\text{def}}{=} \mathbb{Z}/k\mathbb{Z}$. In the following, we classify the germs at singular fibers (the invariants are called *semiglobal*) in the sense that two germs are *isomorphic* if there is a symplectomorphism between two neighborhoods of the fibers preserving the momentum maps, the orientation of images of momentum maps, and the labels of singular points.

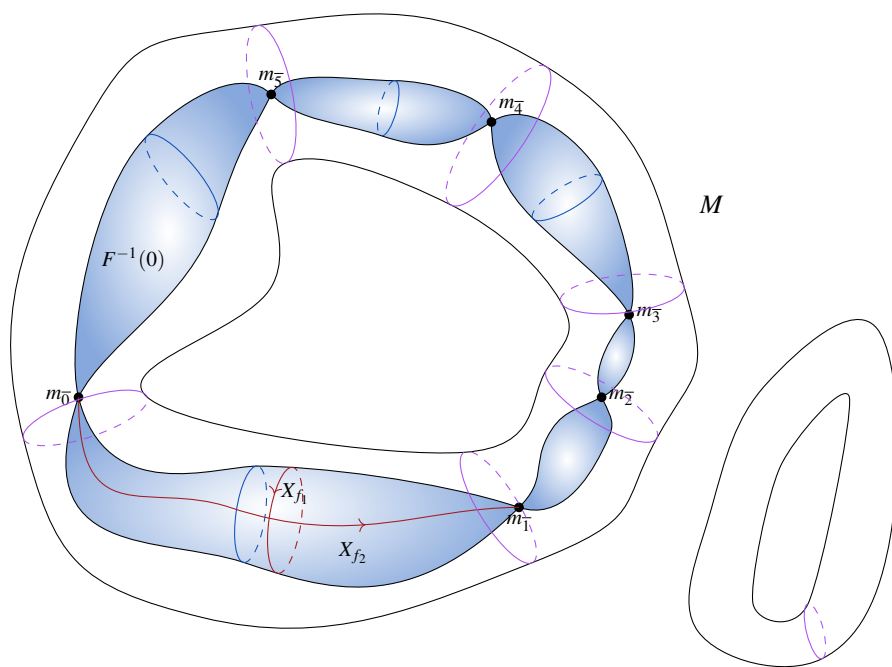


Figure 6.1: A singular fiber of focus-focus type and its neighborhood.

In 2003, Vũ Ngọc proved [35] that the semiglobal germ at a compact connected focus-focus fiber with one non-degenerate singularity is classified by a formal power series which is the Taylor series of the action integral in a neighborhood of the critical fiber, vanishing at the origin, desingularized at each singular point.

In his paper, he also stated a conjecture for what he thought should hold in the case when $k > 1$, which already appeared in the arXiv version of the paper in 2002, and in 2003 [35, Section 7]. I prove his conjecture that a neighborhood of a compact connected fiber with precisely $k \in \mathbb{N}$ focus-focus points (Figure 6.1) is classified up to isomorphisms by k formal power series as their invariants, and give a step by step explicit construction of the invariants; for the case of $k = 1$, the only invariant is the Taylor series of the desingularized action integral, and this has been computed (at least some of its terms have been computed) by several authors for important cases such as the coupled spin-oscillator system [32, 1], the spherical pendulum [12], and the coupled angular-momenta [25]. Roughly speaking, the first Taylor series measure the global singular behavior of the Hamiltonian vector fields X_{f_1} and X_{f_2} , where $F = (f_1, f_2)$, near the singular fiber containing the focus-focus point. The travel times of the flows of these vector fields exhibit a singular behavior, of logarithmic type, as they approach the singularities. The remaining $(k - 1)$ series account for the difference between the Eliasson normal forms at the subsequent pairs of singularities. The theorem says that the entire collection of these Taylor series is a complete symplectic invariant of a tubular neighborhood of a compact focus-focus fiber up to isomorphisms, where an isomorphism preserves the leaves of the foliation induced by F near the singular fiber. See Theorem 6.5.2 and Corollary 6.5.4 for complete versions of the statements above.

The proof of the theorem uses the ideas and the tools, developed by many authors in symplectic geometry, notably including the aforementioned results by Arnold, Eliasson, and Vũ Ngọc and a gluing technique in [31]. I have attempted to make the proof as self-contained as possible and accessible to a general audience of geometers, not necessarily specialists on integrable systems, as focus-focus singularities appear in many parts of symplectic geometry and topology, algebraic geometry (where they are called *nodal singularities*), and mathematical physics. I would like to point out a related recent work by Bosinov-Izosimov [23] where the authors give a *smooth* classification of semiglobal germs at compact focus-focus leaves. Their

smooth invariants can be represented in terms of the quotient of the symplectic ones by an explicitly given group action but I do not know how to describe the smooth invariants in a constructive way.

6.2 Travelling along the Hamiltonian flow

6.2.1 Translation forms in fiber-transitive integrable systems

Recall Definitions 2.2.1 and 2.2.2 the definition of (flow-complete) integrable systems.

Definition 6.2.1. A flow-complete integrable system (M, ω, F) is *fiber-transitive* if the T_b^*B acts transitively on $F^{-1}(b)$ for any $b \in B$. Let $B_1 \subset B$, we say (M, ω, F) is *fiber-transitive over B_1* if $(F^{-1}(B_1), \omega|_{F^{-1}(B_1)}, F|_{F^{-1}(B_1)})$ is fiber-transitive. In this case, for any open U subset B , the action of $\Omega^1(U)$ on $F^{-1}(U)$ has each fiber of F as an orbit.

Let $P, Q, R: B \rightarrow M$ be smooth sections of F . Suppose $B_1 \subset B$ is a dense subset over which (M, ω, F) is fiber-transitive. Consider the sheaf $\Omega^1/2\pi\check{\Lambda}$ of abelian groups on B . For any open set $U \subset B_1$, $(\Omega^1/2\pi\check{\Lambda})(U) = \Omega^1(U)/2\pi\check{\Lambda}(U)$, and the following

$$\tau^{PQ}|_U = \{\beta \in \Omega^1(U) \mid \Psi_\beta \circ P|_U = Q|_U\}$$

is a coset of $2\pi\check{\Lambda}(U)$ in $\Omega^1(U)$. One can verify that $\tau^{PQ}|_U$ glues to a global section $\tau^{PQ} \in (\Omega^1/2\pi\check{\Lambda})(B_1)$. We call τ^{PQ} the *translation form* from P to Q . The translation forms satisfy the additivity property $\tau^{PQ} + \tau^{QR} = \tau^{PR}$.

Definition 6.2.2. Suppose there is a dense subset $B_1 \subset B$ over which (M, ω, F) is fiber-transitive. A section $\tau \in (\Omega^1/2\pi\check{\Lambda})(B_1)$ is *smoothable* if it has a smooth representative in $\Omega^1(B)$, namely, there is $\tilde{\tau} \in \Omega^1(B)$ such that $\tilde{\tau}|_U \in \tau|_U + 2\pi\check{\Lambda}(U)$ for any open $U \subset B_1$.

Since $2\pi\check{\Lambda}(U) \subset Z^1(U)$, the space of closed 1-forms on U for any open $U \subset B$, we can naturally define d in $(\Omega^1/2\pi\check{\Lambda})(B_r)$ and $(Z^1/2\pi\check{\Lambda})(B)$ to be the kernel of d .

6.2.2 Isomorphisms of integrable systems

Lemma 6.2.1. *Let (M, ω, F) be a flow-complete integrable system, and let $U \subset B$ be an open subset and let $\tau \in \Omega^1(U)$. Then $\Psi_\tau: F^{-1}(U) \rightarrow F^{-1}(U)$ is a symplectomorphism if and only if τ is closed.*

Proof. By Cartan's formula, $\mathcal{L}_{X_\tau}\omega = d(X_\tau \lrcorner \omega) = -d(F^*\tau)$. So for any $t \in \mathbb{R}$ we have

$$\frac{d}{dt}\Psi_{t\tau}^*\omega = \Psi_{t\tau}^*\mathcal{L}_{X_\tau}\omega = -d(\Psi_{t\tau}^*F^*\tau) = -d(F^*\tau).$$

Integrating for $t \in [0, 1]$, we obtain $\Psi_\tau^*\omega = \omega - d(F^*\tau)$. □

Lemma 6.2.2. *Let (M, ω, F) and (M', ω', F') be flow-complete integrable systems. Let $B = F(M)$ and $B' = F'(M')$. Let $\varphi: M \rightarrow M'$ and $G: B \rightarrow B'$ be diffeomorphisms such that $F' \circ \varphi = G \circ F$. Then φ is a symplectomorphism if and only if for any $\tau' \in \Omega^1(B)$, $\varphi \circ \Psi_{G^*\tau'} = \Psi_{\tau'} \circ \varphi$, and φ sends some Lagrangian section of F to a Lagrangian section of F' .*

Proof. Fix $\tau' \in \Omega^1(B')$, and let $\tau = G^*\tau' \in \Omega^1(B)$. Suppose that for any $t \in \mathbb{R}$

$$\Psi_{t\tau} = \varphi^{-1} \circ \Psi_{t\tau'} \circ \varphi.$$

Taking the t -derivative,

$$\begin{aligned} X_\tau &= \varphi_*^{-1}X_{\tau'}, \\ \omega^{-1}(F^*\tau) &= \varphi_*^{-1}(\omega')^{-1}((F')^*\tau') \\ &= (\varphi^*\omega')^{-1}(\varphi^*(F')^*\tau') \\ &= (\varphi^*\omega')^{-1}(F^*\tau). \end{aligned} \tag{6.2.1}$$

Let $P: B \rightarrow M$ be a smooth Lagrangian section of F , $b \in B$, and $x = P(b) \in M$. Let $Y_1, Y_2 \in T_x M$. If Y_1 is vertical, namely $F_* Y_1 = 0$, we can define $\tau_x \in T_x^* M$ be the unique one satisfying $\tau_x(F_* Z) = \omega(Y_1, Z)$ for any $Z \in T_x M$. If Y_2 is also vertical, let $\tau \in \Omega^1(B)$ extends τ_x and then (6.2.1) implies that

$$\varphi^* \omega'(Y_1, Y_2) = \omega(Y_1, Y_2). \quad (6.2.2)$$

Suppose the image of $\varphi^{-1} \circ P$ is Lagrangian in M' , then if either of Y_1, Y_2 is tangent to $P(B)$, both sides of (6.2.2) vanish. Hence (6.2.2) always holds and $\varphi^* \omega' = \omega$.

If φ is a symplectomorphism, then φ preserves Lagrangian sections and (6.2.1) holds for any $\tau \in \Omega^1(B)$. By multiplying (6.2.1) by t and integrating for $t \in [0, 1]$, we obtain $\Psi_{G^* \tau} = \varphi^{-1} \circ \Psi_\tau \circ \varphi$. \square

Lemma 6.2.3. *Let (M, ω, F) and (M', ω', F') be fiber-transitive integrable systems. Let $B = F(M)$, $B' = F'(M')$. Suppose φ is either a diffeomorphism from a Lagrangian section P of F to a Lagrangian section P' of F' , or an isomorphism of integrable systems $(V, \omega, F) \rightarrow (V', \omega', F')$ where $V \subset M, V' \subset M'$ are open subsets such that $F(V) = B, F'(V') = B'$. Let $G: B \rightarrow B'$ be the diffeomorphism such that $F' \circ \varphi = G \circ F$. If $G^* \check{\Lambda}^{(M', \omega', F')} = \check{\Lambda}^{(M, \omega, F)}$, then φ has a unique extension as an isomorphism $\tilde{\varphi}: (M, \omega, F) \rightarrow (M', \omega', F')$.*

Proof. Since the two integrable systems are fiber-transitive, and by the conditions of φ , for any smooth section $Q: B \rightarrow M$ of B , there is an open subset $U \subset B$ and $\tau \in \Omega^1(U)$ such that $\Psi_{-\tau} \circ Q(U)$ is in the domain of φ . Let $b = F(x) \in B$. Let $\tilde{\varphi} \circ Q = \Psi_{(G^{-1})^* \tau} \circ \varphi \circ \Psi_{-\tau} \circ Q$.

If φ is a diffeomorphism between Lagrangian sections, since (M, ω, F) is fiber-transitive, for any smooth section $Q: B \rightarrow M$ of B , there is a open subset $U \subset B$ and $\tau \in \Omega^1(U)$ such that $\Psi_{-\tau} \circ Q(U) = P(U)$. A different choice $\tau_1 \in \Omega^1(U)$ from τ will satisfy $\tau_1 - \tau \in 2\pi \check{\Lambda}^{(M, \omega, F)}(U)$. Since $G^* \check{\Lambda}^{(M', \omega', F')} = \check{\Lambda}^{(M, \omega, F)}(U)$, we have $\Psi_{(G^{-1})^* \tau_1} = \Psi_{(G^{-1})^* \tau}$. So $\tilde{\varphi}$ is independent of the choice of τ . Similarly, $\tilde{\varphi}$ is injective. Since (M', ω', F') is fiber-transitive, $\tilde{\varphi}$ is surjective. By Lemma 6.2.2, $\tilde{\varphi}$ is a symplectomorphism with $F' \circ \varphi = G \circ F$.

If φ is an isomorphism between integrable systems, for any $\tau \in Z^1(B)$ we can define $\tilde{\varphi}$ on $\Psi_\tau(V)$ as $\tilde{\varphi} = \Psi_{(G^{-1})^*\tau} \circ \varphi \circ \Psi_{-\tau}$ which is a symplectomorphism $\Psi_\tau(V) \rightarrow \Psi_{(G^{-1})^*\tau}(V')$. For similar reasons as the case of Lagrangian sections, $\tilde{\varphi}$ is uniquely defined and is a bijection. Hence $\tilde{\varphi}$ is a symplectomorphism with $F' \circ \varphi = G \circ F$. \square

6.3 Local symplectic structure near focus-focus singularities

6.3.1 Local normal form

Definition 6.3.1. Let (x_1, ξ_1, x_2, ξ_2) be the coordinates of \mathbb{R}^4 . Let $\omega_0 = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2$ be the standard symplectic form on \mathbb{R}^4 . Let $q = (q_1, q_2): \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be

$$q_1 = x_1 \xi_2 - x_2 \xi_1, \quad q_2 = x_1 \xi_1 + x_2 \xi_2.$$

We call it the *local normal form of non-degenerate focus-focus singularities*.

Now we compute the action Ψ associated with $(\mathbb{R}^4, \omega_0, q)$. Let $z = x_1 + ix_2$, $\zeta = \xi_2 + i\xi_1$, then $q_1 + iq_2 = z\zeta$, and

$$\begin{aligned} X_{q_1} &= -\omega_0^{-1} dq_1 = x_2 \partial_{x_1} - x_1 \partial_{x_2} + \xi_2 \partial_{\xi_1} - \xi_1 \partial_{\xi_2}, \\ X_{q_2} &= -\omega_0^{-1} dq_2 = -x_1 \partial_{x_1} - x_2 \partial_{x_2} + \xi_1 \partial_{\xi_1} + \xi_2 \partial_{\xi_2}. \end{aligned}$$

Let $c = (c_1, c_2)$ be the coordinates of \mathbb{R}^2 , and let $(t_1, t_2) \in \mathbb{R}^2$, so the action of $\Omega^1(\mathbb{R}^2)$ is

$$\Psi_{t_1 dc_1 + t_2 dc_2}(z, \zeta) = \left(e^{-t_2 - it_1} z, e^{t_2 + it_1} \zeta \right).$$

Then $(\mathbb{R}^4, \omega_0, q)$ is a flow-complete integrable system whose period sheaf $\check{\Lambda}^{(\mathbb{R}^4, \omega_0, q)}$ is, for any

open set $U \subset \mathbb{R}^2$, $\check{\Lambda}^{(\mathbb{R}^4, \omega_0, q)}(U) = (dc_1)\mathbb{Z}$. We will use the identifications

$$\begin{aligned} \mathbb{R}^4 &\rightarrow \mathbb{C}^2, & \mathbb{R}^2 &\rightarrow \mathbb{C}, \\ (x_1, \xi_1, x_2, \xi_2) &\mapsto (z, \zeta), & (c_1, c_2) &\mapsto c_1 + ic_2 \end{aligned}$$

throughout this chapter.

Let $P, Q: \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be two Lagrangian sections of q that $P(c) = (1, c)$, $Q(c) = (c, 1)$.

Then let $\kappa \in (\Omega^1/2\pi\check{\Lambda})(\mathbb{R}_r^2)$ denote the translation form

$$\kappa = \tau^{PQ} = -\Im \ln c \, dc_1 - \Re \ln c \, dc_2. \quad (6.3.1)$$

Define subsets of $\mathbb{R}^4 \simeq \mathbb{C}^2$ as follows:

$$\begin{aligned} \mathbb{R}_u^4 = \mathbb{C}_u^2 &= \{(z, \zeta) \in \mathbb{C}^2 \mid z = 0\}, & \mathbb{R}_s^4 = \mathbb{C}_s^2 &= \{(z, \zeta) \in \mathbb{C}^2 \mid \zeta = 0\}, \\ \mathbb{R}_{\text{nu}}^4 = \mathbb{C}_{\text{nu}}^2 &= \{(z, \zeta) \in \mathbb{C}^2 \mid z \neq 0\}, & \mathbb{R}_{\text{ns}}^4 = \mathbb{C}_{\text{ns}}^2 &= \{(z, \zeta) \in \mathbb{C}^2 \mid \zeta \neq 0\}, \\ \mathbb{R}_r^4 = \mathbb{C}_r^2 &= \{(z, \zeta) \in \mathbb{C}^2 \mid q(z, \zeta) \neq 0\}, & \mathcal{F}_0 &= \{(z, \zeta) \in \mathbb{C}^2 \mid q(z, \zeta) = 0\}. \end{aligned}$$

Here \mathbb{R}_u^4 and \mathbb{R}_s^4 are respectively, the unstable and the stable manifolds of 0 in under the flow of X_{dc_2} . For any $(t_1, t_2) \in \mathbb{R}^2$ with $t_2 > 0$, the origin is the only α -limit point for the flow lines of $X_{t_1 \, dc_1 + t_2 \, dc_2}$ in \mathbb{R}_u^4 , and the ω -limit point for the flow lines in \mathbb{R}_s^4 . Let $\mathbb{R}_r^2 \simeq C_r = \{c \in \mathbb{C} \mid c \neq 0\}$. Let $\text{pr}_1, \text{pr}_2: \mathbb{R}^2 \rightarrow \mathbb{R}$ be respectively the projection onto the first and the second component.

6.3.2 Eliasson local chart

Definition 6.3.2. Let X, Y be subsets of smooth manifolds, x, y interior points of X, Y . A *local smooth function* $f: (X, x) \rightarrow Y$ is a germ of smooth functions from a neighborhood of $x \in X$ to Y ,

at x . A *local diffeomorphism* $f: (X, x) \rightarrow (Y, y)$ is a local smooth function $f: (X, x) \rightarrow Y$ whose representatives are diffeomorphisms between neighborhoods of x and y , sending x to y . A *local differential form* β on (X, x) is a germ of differential forms on a neighborhood of $x \in X$ at x . We denote by $\Omega^1(X, x)$ and $Z^1(X, x)$, respectively, the space of local 1-forms and closed local 1-forms on (X, x) .

If moreover (X, ω_1) , (Y, ω_2) are symplectic manifolds, then a *local symplectomorphism* $f: (X, \omega_1, x) \rightarrow (Y, \omega_2, y)$ is a local diffeomorphism $f: (X, x) \rightarrow (Y, y)$ whose representatives are symplectomorphisms. If moreover (X, ω_1, F_1) , (Y, ω_2, F_2) are integrable systems, $\mathcal{F}_1 \subset X$ is a fiber of F_1 , and $\mathcal{F}_2 \subset Y$ is a fiber of F_2 , then a *semiglobal isomorphism* $f: (X, \omega_1, \mathcal{F}_1) \rightarrow (Y, \omega_2, \mathcal{F}_2)$ is a germ of symplectomorphisms between saturated neighborhoods of \mathcal{F}_1 and those of \mathcal{F}_2 , sending \mathcal{F}_1 to \mathcal{F}_2 , at \mathcal{F}_1 . Here a subset $W \subset X$ is *saturated* under F_1 if $F_1^{-1}(F_1(W)) = W$, similarly for Y and F_2 .

A local or semiglobal symplectomorphism f between two integrable systems (X, ω_1, F_1) , (Y, ω_2, F_2) is a *local or semiglobal isomorphism* near points in the fibers over, or near the fibers over $b_1 \in F_1(X)$, $b_2 \in F_2(X)$ if there is a local diffeomorphism $G: (F_1(X), b_1) \rightarrow (F_2(Y), b_2)$ where such that $F_2 \circ f = G \circ F_1$.

Definition 6.3.3. A singularity $m \in M$ of F is of *non-degenerate focus-focus type* if (M, ω, F) near m is locally isomorphic to $(\mathbb{R}^4, \omega_0, q')$ near the origin, for some smooth map $q': \mathbb{R}^4 \rightarrow \mathbb{R}^2$ such that $(q' - q)(x) \in \mathcal{O}(|x|^3)$.

Theorem 6.3.1 (Eliasson's theorem [14]). *Near any singularity $m \in M$ of non-degenerate focus-focus type, (M, ω, F) is locally isomorphic to $(\mathbb{R}^4, \omega_0, q)$ near the origin.*

That is, there is a local symplectomorphism ϕ and a local diffeomorphism G , such that

the following diagram commutes:

$$\begin{array}{ccc}
 (M, \omega, m_j) & \xrightarrow{\varphi_j} & (\mathbb{R}_4, \omega_0, 0) \\
 \downarrow F & & \downarrow q \\
 (B, 0) & \xrightarrow{G_j} & (\mathbb{R}^2, 0)
 \end{array}$$

The pair (φ, G) is called an *Eliasson local chart* at 0.

6.3.3 Flat functions

Definition 6.3.4. Let B be a subset of \mathbb{R}^2 and $b \in B$ an interior point. Let $\mathbb{R}[[T_b^*B]]$ be the space of formal power series generated by the elements of a basis of T_b^*B , or equivalently, $\mathbb{R}[[T_b^*B]]$ is the direct sum of symmetric tensor products of T_b^*B . Let $f: (B, b) \rightarrow \mathbb{R}$ and $E: (B, b) \rightarrow \mathbb{R}^2$ be local smooth maps. The *Taylor series* $\text{Taylor}_b[f]$ of f at b may be viewed as an element in $\mathbb{R}[[T_b^*B]]$, and $\text{Taylor}_b[E] \in \mathbb{R}[[T_b^*B]]^2$. We call f a *flat function* at b if $\text{Taylor}_b[f] = 0$. Denote by $\mathcal{O}(c^\infty)$ the space of flat functions f . Note that, by the Faà di Bruno's formula, the Taylor series of the composition of smooth maps is the composition of their Taylor series.

We will use the multi-index notations in Lemmas 6.3.2 and 6.3.3. A multi-index j is a pair (j_1, j_2) where $j_1, j_2 \in \mathbb{N}$. We use $|j| = j_1 + j_2$. If $c = (c_1, c_2) \in \mathbb{R}^2$ then $c^j = c_1^{j_1} c_2^{j_2}$. If $f: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ is a germ of functions at 0 then $\partial^j f = \frac{\partial^{|j|} f}{\partial c_1^{j_1} \partial c_2^{j_2}}$.

Lemma 6.3.2. For $m \in \mathbb{N}$, let $g_j: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ be local smooth functions for multi-index j with $|j| = m$. Let $g(c) = \sum_{|j|=m} g_j(c) c^j$. Then for $m \geq 1$ the function $g \ln|\cdot|$ can be extended to a C^{m-1} -function near 0, while for $m \geq 0$, that function is of class C^m if and only if $g_j(0) = 0$ for any j .

Proof. For $s \in \mathbb{N}$, let Q_s be the \mathbb{R} -vector space spanned by functions of the form $c \mapsto h(c) \frac{c^j}{|c|^{j_0}}$, defined in a deleted neighborhood of 0, where $h: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ is a local smooth function, j is a multi-index, $j_0 \in \mathbb{N}$, and $|j| \geq j_0 + s$. In particular, functions in Q_s can be extended to

C^{s-1} functions in a neighborhood of 0 for $s \geq 1$ and are locally bounded for $s = 0$. We use the multi-index notation. Note that

$$\begin{aligned}\frac{\partial}{\partial c_1}(c^j \ln|c|) &= j_1 c_1^{j_1-1} c_2^{j_2} \ln|c| + \frac{c_1^{j_1+1} c_2^{j_2}}{|c|^2}, \\ \frac{\partial}{\partial c_1} \left(\frac{c^j}{|c|^{j_0}} \right) &= j_1 \frac{c_1^{j_1-1} c_2^{j_2}}{|c|^{j_0}} - j_0 \frac{c_1^{j_1+1} c_2^{j_2}}{|c|^{j_0+2}}.\end{aligned}$$

We have

$$\partial^k(g(c) \ln|c|) \in \sum_{|j|=m} \sum_{l \leq j, \ell \leq k} \binom{k}{l} \left(\partial^{k-l} g_j \right)(c) \frac{j!}{(j-l)!} c^{j-l} \ln|c| + Q_{m-|k|}.$$

If $|k| < m$, we have $\partial^k(g \ln|\cdot|) \in C^0 + Q_1 \subset C^0$, hence $g \ln|\cdot| \in C^{m-1}$. If $|k| = m$, we have

$$\partial^k(g(c) \ln|c|) \in k! g_k(c) \ln|c| + C^0 + Q_0.$$

So $g(c) \in C^m$ implies for every multi-index k with $|k| = m$, $g_k \ln|\cdot|$ is bounded, hence $g_k(0) = 0$. □

Lemma 6.3.3. *Let $f: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ be a local smooth function, then $f \ln|\cdot|$ can be extended to a local smooth function $(\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ only when f is flat. If f is flat, $f \ln|\cdot|$ is flat.*

Proof. By Taylor expansion of f , for any $m \in \mathbb{N}$, there are local smooth functions $g_j: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ for any multi-index j with $|j| = m+1$ such that

$$f(c) = \sum_{|j|=0}^m \frac{1}{j!} \partial^j f(0) c^j + \sum_{|j|=m+1} \frac{1}{j!} g_j(c) c^j. \quad (6.3.2)$$

By Lemma 6.3.2 and (6.3.2), $f \ln|\cdot| \in C^m$ if and only if $\partial^j f(0) = 0$ for any multi-index j with $|j| \leq m$. Therefore, $f \ln|\cdot| \in C^\infty$ if and only if $f \in \mathcal{O}(c^\infty)$.

Note that $\ln|c| \in \mathcal{O}(c^{-1})$. If $f \in \mathcal{O}(c^\infty)$, then for any $m \in \mathbb{N}$, there exist local smooth

functions $g_j: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ for any multi-index j with $|j| = m + 1$ such that

$$f(c) \ln|c| = \sum_{|j|=m+1} \frac{1}{j!} g_j(c) c^j \ln|c| \in \mathcal{O}(c^m).$$

Hence $f \ln|\cdot| \in \mathcal{O}(c^\infty)$. □

Lemma 6.3.4. *Let $g \in \mathcal{O}(c^\infty)$ and $f: (\mathbb{C}_r, 0) \rightarrow \mathbb{R}$. If $|f| \leq |g|$ in a neighborhood of 0, then f has an extension $\tilde{f} \in \mathcal{O}(c^\infty): (\mathbb{C}, 0) \rightarrow \mathbb{R}$.*

Proof. First, we have $\lim_{c \rightarrow 0} |f(c)| \leq |g(0)| = 0$, so f has an extension $\tilde{f} \in C^0$ with $\tilde{f}(0) = 0$. Then, $\frac{\partial \tilde{f}}{\partial c_1}(0) = \lim_{\delta \rightarrow 0} \frac{f(\delta, 0)}{\delta} = 0$. So, inductively, every higher order derivative of \tilde{f} exists and vanishes at 0. □

Lemma 6.3.5. *Let $G: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a local diffeomorphism which is in the form of $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$. Then $G^* \kappa = \kappa + \mathcal{O}(c^\infty) dc_1 + \mathcal{O}(c^\infty) dc_2 \in (\Omega^1 / 2\pi\check{\Lambda})(\mathbb{R}_r^2)$.*

Proof. For $c \neq 0$,

$$\begin{aligned} G^* \kappa(c) - \kappa(c) &= -\ln|G(c)| \frac{\partial G_2}{\partial c_1}(c) dc_1 - \arg \frac{G(c)}{c} dc_1 \\ &\quad - \ln \left| \frac{G(c)}{c} \right| \frac{\partial G_2}{\partial c_2}(c) dc_2 - \ln|c| \left(\frac{\partial G_2}{\partial c_2}(c) - 1 \right) dc_2. \end{aligned} \tag{6.3.3}$$

We know the fact that for any local smooth function $f: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}^2 \simeq \mathbb{C}$, if $f \in \mathcal{O}(c^\infty)$, then both components of $\ln(1 + f)$ are flat, by explicitly calculating the partial derivatives of $\ln(1 + f)$. Thus $-\ln \left| \frac{G}{c} \right|, \arg \frac{G}{c} \in \mathcal{O}(c^\infty)$. Since $\frac{\partial G_2}{\partial c_2} - 1, \frac{\partial G_2}{\partial c_1} \in \mathcal{O}(c^\infty)$, by Lemma 6.3.3, we have $\ln|\cdot| \left(\frac{\partial G_2}{\partial c_2} - 1 \right), \ln|G| \frac{\partial G_2}{\partial c_1} \in \mathcal{O}(c^\infty)$. Hence the form in (6.3.3) is in $\mathcal{O}(c^\infty) dc_1 + \mathcal{O}(c^\infty) dc_2$. □

6.3.4 Isomorphisms of the local normal form

Let $\varphi_X, \varphi_Y: (\mathbb{R}^4, \omega_0) \rightarrow (\mathbb{R}^4, \omega_0)$, $G_X, G_Y: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be

$$\begin{aligned}\varphi_X(z, \zeta) &= (i\bar{z}, i\bar{\zeta}), & G_X(c) &= -\bar{c}, \\ \varphi_Y(z, \zeta) &= (i\bar{\zeta}, -i\bar{z}), & G_Y(c) &= \bar{c}.\end{aligned}$$

Then φ_X, φ_Y are symplectomorphisms such that $q \circ \varphi_X = G_X \circ q$, $q \circ \varphi_Y = G_Y \circ q$. So φ_X, φ_Y are automorphisms of the standard local model $(\mathbb{R}^4, \omega_0, q)$.

Lemma 6.3.6. *Let (M, ω, F) be a flow-complete integrable system and $B = F(M)$. Let $\tau \in (\Omega^1/2\pi\check{\Lambda})(B_r)$. Then $\Psi_\tau: M \rightarrow M$ is a symplectomorphism if and only if τ is closed.*

Proof. For any open subset U and any 1-form $\alpha \in \tau(U)$, we know that Ψ_α is a symplectomorphism of $F^{-1}(U)$ if and only if α is closed. However, any two such 1-forms on U_1 and U_2 differ by an element in $2\pi\check{\Lambda}(U_1 \cap U_2)$ on their common domain $U_1 \cap U_2$, so they give the same map on $F^{-1}(U_1 \cap U_2)$. Hence we get a well-defined map $\Psi_\tau: M \rightarrow M$ which is a symplectomorphism if and only if τ is closed. \square

Lemma 6.3.7. *The map Ψ_κ defined in Lemma 6.3.6 can be extended to a symplectomorphism $\tilde{\Psi}_\kappa: (\mathbb{R}_{\text{nu}}^4, \omega_0) \rightarrow (\mathbb{R}_{\text{ns}}^4, \omega_0)$.*

Proof. Since the map

$$\begin{aligned}\tilde{\Psi}_\kappa: \mathbb{R}_{\text{nu}}^4 &\rightarrow \mathbb{R}_{\text{ns}}^4, \\ (z, \zeta) &\mapsto (z^2\zeta, z^{-1}),\end{aligned}$$

coincides with Ψ_κ on \mathbb{R}_r^4 , $\tilde{\Psi}_\kappa$ is an extension of Ψ_κ as a diffeomorphism. Since $d\kappa = 0$ in \mathbb{R}_r^2 , by Lemma 6.3.6, Ψ_κ is symplectomorphism of $(\mathbb{R}_r^4, \omega_0)$. By continuity, $\tilde{\Psi}_\kappa$ is a symplectomorphism. Alternatively, one can verify $\tilde{\Psi}_\kappa^*\omega_0 = \omega_0$ by explicit computations. \square

Lemma 6.3.8. *Let $G: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a local diffeomorphism in the form of $G(c_1, c_2) = (c_1, g(c_1, c_2))$ for some local smooth function $g: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ with $\frac{\partial g}{\partial c_2} > 0$. Then we have $G^* \kappa \in (\Omega^1/2\pi\check{\Lambda})(B_r)$. The symplectomorphism $\Psi_{G^* \kappa}$ of intersections of neighborhoods of \mathcal{F} with \mathbb{R}_r^4 , can be extended to a semiglobal symplectomorphism $(\mathbb{R}_{\text{nu}}^4, \omega_0, \mathbb{R}_s^4) \rightarrow (\mathbb{R}_{\text{ns}}^4, \omega_0, \mathbb{R}_u^4)$ if and only if $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$.*

Proof. By Lemma 6.2.3, the local map $\Psi_{G^* \kappa}$ can be extended to a semiglobal map. Recall that

$$G^* \kappa(c) = -\ln|G(c)| \frac{\partial G_2}{\partial c_1}(c) dc_1 - \arg G(c) dc_1 - \ln|G(c)| \frac{\partial G_2}{\partial c_2}(c) dc_2.$$

Let $(z, \zeta) = P(c) = (1, c)$, so $c = q(z, \zeta)$. Let U be a neighborhood of 0 in \mathbb{C} . Let $h_1: U \cap \mathbb{C}_r \rightarrow \mathbb{C}$ and $h_2: (\mathbb{C}, 0) \rightarrow \mathbb{C}$ be

$$h_1(c) = \frac{G(c)}{c}, \quad h_2(c) = \frac{\partial G_2}{\partial c_2}(c) - 1 + i \frac{\partial G_2}{\partial c_1}(c).$$

Then we have, for $c \in U \cap \mathbb{C}_r$,

$$\Psi_{G^* \kappa} \circ P(c) = \left(|G(c)|^{h_2(c)} G(c), |G(c)|^{-h_2(c)} h_1(c)^{-1} \right).$$

Suppose $\Psi_{G^* \kappa}$ can be extended to $\tilde{\Psi}_{G^* \kappa}: (\mathbb{R}_{\text{nu}}^4, \omega_0, \mathbb{R}_s^4) \rightarrow (\mathbb{R}_{\text{ns}}^4, \omega_0, \mathbb{R}_u^4)$. By continuity, we have $\lim_{c \rightarrow 0} |G(c)|^{-h_2(c)} h_1(c)^{-1} = \text{pr}_2 \circ \Psi_{G^* \kappa}(1, 0) \neq 0$. For any fixed $c \in \mathbb{C} \setminus \{0\}$, $t \mapsto h_1(tc)$ is smooth at 0 and $\lim_{t \rightarrow 0} h_1(tc) \neq 0$. The map $t \mapsto |t|^{-h_2 \circ G^{-1}(tc)}$ is smooth and has nonzero limit at 0. So $t \mapsto h_2 \circ G^{-1}(tc) \ln|tc| = h_2 \circ G^{-1}(tc) \ln|t| + \mathcal{C}^\infty$ is smooth at 0. Hence by an analogous 1-dimensional version of Lemma 6.3.3, $t \mapsto h_2 \circ G^{-1}(tc)$ is flat at 0. By arbitrariness of c , we have $h_2 \circ G^{-1} \in \mathcal{O}(c^\infty)$, so $h_2 \in \mathcal{O}(c^\infty)$. Therefore $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$.

On the other hand, if it is known that $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$, then h_1 can be extended to 0 such that $h_1(0) \neq 0$ and $h_2 \in \mathcal{O}(c^\infty)$. Moreover, by Lemma 6.3.3 $h_2 \ln|G| \in \mathcal{O}(c^\infty)$, so $|G|^{h_2}$ can be extended to a local smooth function with value 1 at 0. Then $\Psi_{G^* \kappa}$ can be extended to

a semiglobal diffeomorphism $\tilde{\Psi}_{G^*\kappa}: (\mathbb{R}_{\text{nu}}^4, \mathbb{R}_{\text{s}}^4) \rightarrow (\mathbb{R}_{\text{ns}}^4, \mathbb{R}_{\text{u}}^4)$. The map $\tilde{\Psi}_{G^*\kappa}$ is a semiglobal symplectomorphism since it is a symplectomorphism on the part of its domain within \mathbb{R}_{r}^4 . \square

Lemma 6.3.9. *Let $G: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a local diffeomorphism in the form of $G(c_1, c_2) = (c_1, g(c_1, c_2))$ for some local smooth function $g: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ with $\frac{\partial g}{\partial c_2} > 0$. Then there is a unique semiglobal symplectomorphism*

$$\varphi_G: (\mathbb{R}_{\text{nu}}^4, \omega_0, \mathbb{R}_{\text{s}}^4) \rightarrow (\mathbb{R}_{\text{nu}}^4, \omega_0, \mathbb{R}_{\text{s}}^4)$$

characterized by $\varphi_G(1, c) = (1, G(c))$ on $(\mathbb{R}^2, 0)$.

If $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$, then φ_G can be uniquely extended to a semiglobal symplectomorphism

$$\tilde{\varphi}_G: (\mathbb{R}^4, \omega_0, \mathcal{F}) \rightarrow (\mathbb{R}^4, \omega_0, \mathcal{F}).$$

Proof. The first part is a result of Lemma 6.2.3, as $c \mapsto (1, c)$ and $c \mapsto (1, G(c))$ are Lagrangian sections of q , and $G^* dc_1 = dc_1$.

For the second part, consider the symplectomorphism $\varphi'_G: (\mathbb{R}_{\text{ns}}^4, \omega_0, \mathbb{R}_{\text{u}}^4) \rightarrow (\mathbb{R}_{\text{ns}}^4, \omega_0, \mathbb{R}_{\text{u}}^4)$ sending $\tilde{\Psi}_{G^*\kappa}(1, c)$ to $\Psi_\kappa(1, G(c))$ on $(\mathbb{R}^2, 0)$. By Lemma 6.3.8, in this case $\Psi_{G^*\kappa}$ is defined on $\Psi_{G^*\kappa}(1, c)$. Since φ_G and φ'_G coincide in their common domain the intersection of a neighborhood of \mathcal{F} with \mathbb{R}_{r}^4 , they glue to a symplectomorphism $\tilde{\varphi}_G$ of $(\mathbb{R}^4, \omega_0, \mathcal{F})$. \square

For the following see [35, Lemma 4.1, Lemma 5.1].

Lemma 6.3.10. *Let G be a local diffeomorphism of $(\mathbb{R}^2, 0)$. Then there is a local symplectomorphism φ of $(\mathbb{R}^4, \omega, 0)$ such that $q \circ \varphi = G \circ q$ if and only if $G(c_1, c_2) = (e_1 c_1, e_2 c_2 + \mathcal{O}(c^\infty))$, with $e_i = \pm 1$, $i = 1, 2$.*

Proof. We extend φ to $\tilde{\varphi}$, by Lemma 6.2.3, a semiglobal symplectomorphism of $(\mathbb{R}^4, \omega_0, \mathcal{F}_0)$.

Then we know that, for some saturated neighborhood W of $\mathcal{F}_0 \subset \mathbb{R}^4$,

$$\tilde{\varphi}^{-1} \circ \Psi_{\kappa} \circ \tilde{\varphi}|_{W_{\Gamma}}: W_{\Gamma} \rightarrow W_{\Gamma}$$

is a symplectomorphism, where $W_{\Gamma} = W \cap \mathbb{R}_{\Gamma}^4$. However, in W_{Γ} we have $\Psi_{G^* \kappa} = \tilde{\varphi}^{-1} \circ \Psi_{\kappa} \circ \tilde{\varphi}$. By uniqueness of the map, we require $G^*(\check{\Lambda}(U)) \subset \check{\Lambda}(U)$ for any open $U \subset q(W_{\Gamma})$. That is to say, $G(c_1, c_2) = (e_1 c_1, g(c_1, c_2))$, where $e_1 = \pm 1$, for some local smooth function $g: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$.

Since φ preserves the singular fiber \mathcal{F}_0 , and the punctured fiber $\mathcal{F}_0 \setminus \{0\}$ has two components $\mathbb{R}_{\mathfrak{u}}^4 \setminus \{0\}$ and $\mathbb{R}_{\mathfrak{s}}^4 \setminus \{0\}$, either φ preserves the two components or exchanges them. In the first case, $\tilde{\varphi}(\mathbb{R}_{\mathfrak{s}}^4) = \mathbb{R}_{\mathfrak{s}}^4$; in the latter case, $\tilde{\varphi}(\mathbb{R}_{\mathfrak{s}}^4) = \mathbb{R}_{\mathfrak{u}}^4$.

In any of the four cases above ($e_1 = \pm 1$, $\tilde{\varphi}(\mathbb{R}_{\mathfrak{s}}^4) = \mathbb{R}_{\mathfrak{s}}^4$ or $\mathbb{R}_{\mathfrak{u}}^4$), there is exactly one choice of (φ_0, G_0) from the set

$$\{(\text{id}, \text{id}), (\varphi_X, G_X), (\varphi_Y, G_Y), (\varphi_Y \circ \varphi_X, G_Y \circ G_X)\} \quad (6.3.4)$$

such that $\varphi_0 \circ \tilde{\varphi}: (\mathbb{R}_{\text{nu}}^4, \omega_0, \mathbb{R}_{\mathfrak{s}}^4) \rightarrow (\mathbb{R}_{\text{nu}}^4, \omega_0, \mathbb{R}_{\mathfrak{s}}^4)$, and $\frac{\partial(\text{pr}_2 \circ G_0 \circ G)}{\partial c_2} > 0$. Now we have $q \circ (\varphi_0 \circ \tilde{\varphi}) = (G_0 \circ G) \circ q$ and that

$$(\varphi_0 \circ \tilde{\varphi})^{-1} \circ \Psi_{\kappa} \circ (\varphi_0 \circ \tilde{\varphi}) = \Psi_{(G_0 \circ G)^* \kappa}: (\mathbb{R}_{\text{nu}}^4, \omega_0, \mathbb{R}_{\mathfrak{s}}^4) \rightarrow (\mathbb{R}_{\text{ns}}^4, \omega_0, \mathbb{R}_{\mathfrak{u}}^4)$$

is a semiglobal symplectomorphism. By Lemma 6.3.8, we have $G_0 \circ G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$. Therefore, $G(c_1, c_2) = (e_1 c_1, e_2 c_2 + \mathcal{O}(c^\infty))$, with $e_i = \pm 1$, $i = 1, 2$.

On the other hand, we assume, without loss of generality, that $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$. Otherwise, we can apply a pair of maps in (6.3.4). Let $\varphi = \tilde{\varphi}_G$ be the semiglobal symplectomorphism of $(\mathbb{R}^4, \omega_0, \mathcal{F})$ defined as in Lemma 6.3.9. Then we have $q \circ \varphi = G \circ q$. \square

6.4 Semiglobal topological structure near the focus-focus fiber

Let \mathcal{I}_{ff} be the collection of 4-dimensional integrable systems $(M, \omega, F) \in \mathcal{I}$ such that F is proper and has connected fibers¹ one of which is a singular fiber \mathcal{F} over an interior point of $B = F(M)$. On the singular fiber \mathcal{F} , F has non-degenerate focus-focus singularities, and there are no other singularities in a saturated neighborhood of \mathcal{F} . These assumptions are not too restrictive since, by the local normal form, non-degenerate focus-focus singularities are isolated. For convenience, we always assume \mathcal{F} to be $F^{-1}(0)$. Let $(M, \omega, F) \in \mathcal{I}_{\text{ff}}$. In this section, we give a detailed proof of the topological structure theorem of a focus-focus singular fiber \mathcal{F} and its neighborhood. For this reason, we often restrict M to a saturated neighborhood of \mathcal{F} for simplicity. Since F is proper, \mathcal{F} has finitely many singular points, say $k \in \mathbb{N}$, called the *multiplicity* of \mathcal{F} .

For $k \in \mathbb{N}$, let $\mathcal{I}_{\text{ff}}^k$ be the collection of $(M, \omega, F) \in \mathcal{I}_{\text{ff}}$ where \mathcal{F} has multiplicity k . Throughout this chapter, we denote by \mathbb{Z}_k , $k \in \mathbb{N}$, the quotient group $\mathbb{Z}/k\mathbb{Z}$ of residue classes modulo k with the induced operation from the addition on \mathbb{Z} .

Theorem 6.4.1 (Zung [37, Theorem 5.1]). *Let $(M, \omega, F) \in \mathcal{I}_{\text{ff}}^k$ and $B = F(M)$. Then the singular fiber \mathcal{F} is homeomorphic to the k -fold wedge sum of \mathbb{S}^2 's. The kernel of the T_0^*B -action on \mathcal{F} by Ψ is an infinite cyclic group.*

Proof. Let m_j , $j \in \mathbb{Z}_k$, be the focus-focus singular points in \mathcal{F} . The regular points may form \mathbb{R}^2 , $\mathbb{R} \times \mathbb{S}^1$, or \mathbb{T}^2 orbits with \mathbb{R}^2 -actions by translation, while any focus-focus point itself is an orbit. Let (ψ_j, E_j) , $j \in \mathbb{Z}_k$, be Eliasson local charts near m_j such that ψ_j can be defined in a neighborhood V_j of m_j in M , such that the flow of $2\pi X_{E_j^* \text{dc}_1}$ is an \mathbb{S}^1 -action on $\mathcal{F} \cap V_j$. Note that for any $j \in \mathbb{Z}_k$, $\mathcal{F} \cap V_j \setminus \{m_j\}$ has two connected components. Since F is proper, the neighborhood $\mathcal{F} \cap V_j$ of m_j in V_j for every $j \in \mathbb{Z}_k$ is compactified by two cylinders of regular

¹In this case, by the local models, F is an open map. So, that F has connected fibers implies the preimage of any connected set under F is connected.

points; on the other hand, each cylinder in \mathcal{F} has two ends to be compactified by focus-focus points. Thus \mathcal{F} is topologically the k -fold wedge sum of \mathbb{S}^2 's.

Since there is only one \mathbb{S}^1 -action by a subgroup of T_0^*B on each cylinder of regular points in \mathcal{F} which should coincide with the \mathbb{S}^1 -action on V_j , $j \in \mathbb{Z}_k$, there should be exactly one \mathbb{S}^1 -action by a subgroup of T_0^*B , so the kernel of the T_0^*B -action is isomorphic to \mathbb{Z} . \square

We want to show that, integrable systems in \mathcal{A}_{ff} are semitoric near the singular fiber.

Lemma 6.4.2. *Let $(M, \omega, F) \in \mathcal{A}_{\text{ff}}^k$ and $B = F(M)$. Then the section space $\check{\Lambda}(B)$ of the sheaf $\check{\Lambda}$ is an infinite cyclic group in $Z^1(B)$, so it can be viewed as a constant sheaf associated to \mathbb{Z} over B . The quotient sheaf restricted to B_r , $(\check{\Lambda}/\check{\Lambda}(B))|_{B_r}$, is also a constant sheaf associated to \mathbb{Z} over B , namely, there is an assignment to any simply connected open set $U \subset B_r$ a generator α_U of the infinite cyclic group $\check{\Lambda}(U)/\check{\Lambda}(B)|_U$, such that, for any such open sets U_1 and U_2 , the restrictions of α_{U_1} and α_{U_2} to $U_1 \cap U_2$ coincide.*

Proof. Let m_j , $j \in \mathbb{Z}_k$, be the focus-focus singular points in the singular fiber \mathcal{F} . Let $\beta \in T_0^*B$ such that the flow of X_β is 2π -periodic. By shrinking M to some saturated open neighborhood of \mathcal{F} , we assume that B is open and (ψ_j, E_j) , $j \in \mathbb{Z}_k$, are Eliasson local charts near m_j such that E_j can be defined in B and $E_j^* dc_1(0) = \beta$. For $j \in \mathbb{Z}_k$, let $\beta_j = E_j^* dc_1 \in \Omega^1(B)$, so then $\beta_j \in \check{\Lambda}(B)$. Note that β_j is independent of the choice of the Eliasson local chart. Fix $c \in B_r$. For $j \in \mathbb{Z}_k$, let V_j be a neighborhood of m_j in M such that they do not intersect one another. By a suitable choice of c and V_j , there is $x_j \in M_r$ and such that $\gamma_j: \mathbb{S}^1 \rightarrow F^{-1}(c), t \mapsto \Psi_{t\beta_j}(x)$ is an embedded circle in V_j . Since the images of γ_j , $j \in \mathbb{Z}_k$ do not intersect one another, they represent the same homology class in $H^1(F^{-1}(c))$ up to the sign. So β_j , $j \in \mathbb{Z}_k$, are equal up to the sign. By the arbitrariness of c and the fact that $\beta_j(0) = \beta$ for $j \in \mathbb{Z}_k$ we conclude that β_j , $j \in \mathbb{Z}_k$, are the same. Hence the section space $\check{\Lambda}(B)$ is an infinite cyclic group $\beta\mathbb{Z}$.

For any simply connected open set $U \subset B_r$, the quotient group $\check{\Lambda}(U)/\check{\Lambda}(B)|_U \simeq \mathbb{Z}^2/\mathbb{Z} \simeq \mathbb{Z}$. By Theorem 2.2.2 we know that all nonzero elements in $\check{\Lambda}(U)/\check{\Lambda}(B)|_U$ are of the form

$fE_0^*dc_2 + \check{\Lambda}(B)|_U$ for some smooth function $f: U \rightarrow \mathbb{R}$ which never vanishes. So let α_U be the only generator of $\check{\Lambda}(U)/\check{\Lambda}(B)|_U$ for which f is positive. Hence $(\check{\Lambda}/\check{\Lambda}(B))|_{B_r}$, is also a constant sheaf associated to \mathbb{Z} over B . \square

There are some further results for the smooth structure of the neighborhoods of singular fibers in [23].

6.5 Semiglobal symplectic structure near the focus-focus fiber:

Invariants

6.5.1 Orientations and singularity atlas

Fix a $k \in \mathbb{N}$. Let $(M, \omega, F) \in \mathcal{S}_{\text{ff}}^k$, $B = F(M)$, and \mathcal{F} be the singular fiber. By shrinking M to a saturated neighborhood of \mathcal{F} if necessary, by Lemma 6.4.2, Σ be as above, then $\check{\Lambda}(B)$ and $(\check{\Lambda}/\check{\Lambda}(B))|_{B_r}$ can be viewed as infinite cyclic groups.

Definition 6.5.1. A pair (α_1, α_2) is an *orientation* of (M, ω, F) if α_1 is a generator of $\check{\Lambda}(B)$ and α_2 is an generator of $(\check{\Lambda}/\check{\Lambda}(B))|_{B_r}$. We call α_1 the *J-orientation* and α_2 the *H-orientation*. We denote by $\text{Ori}(M, \omega, F)$ the set of orientations of (M, ω, F) .

Remark 6.5.1. The use of the letters *J* and *H* is inspired by the notations in semitoric systems where the momentum maps are usually written as (J, H) such that the flow of X_J is 2π -periodic.

The set $\text{Ori}(M, \omega, F)$ contains 4 different orientations. Recall from Section 6.3.1 that near any non-degenerate focus-focus singularity $m_j \in M$ there is an Eliasson local chart.

Definition 6.5.2. Let $(\alpha_1, \alpha_2) \in \text{Ori}(M, \omega, F)$. An Eliasson local chart (ψ_j, E_j) near $m_j \in M$ is *compatible with the orientation* (α_1, α_2) if $E_0^*dc_1 = \alpha_1$ and $E_0^*dc_2$ at non-origin points are linear combinations of α_1, α_2 with positive α_2 -coefficients. A collection $((\psi_j, E_j))_{j \in \mathbb{Z}_k}$ where (ψ_j, E_j) is an Eliasson local chart at $m_j, j \in \mathbb{Z}_k$, is a *singularity atlas* of (M, ω, F) .

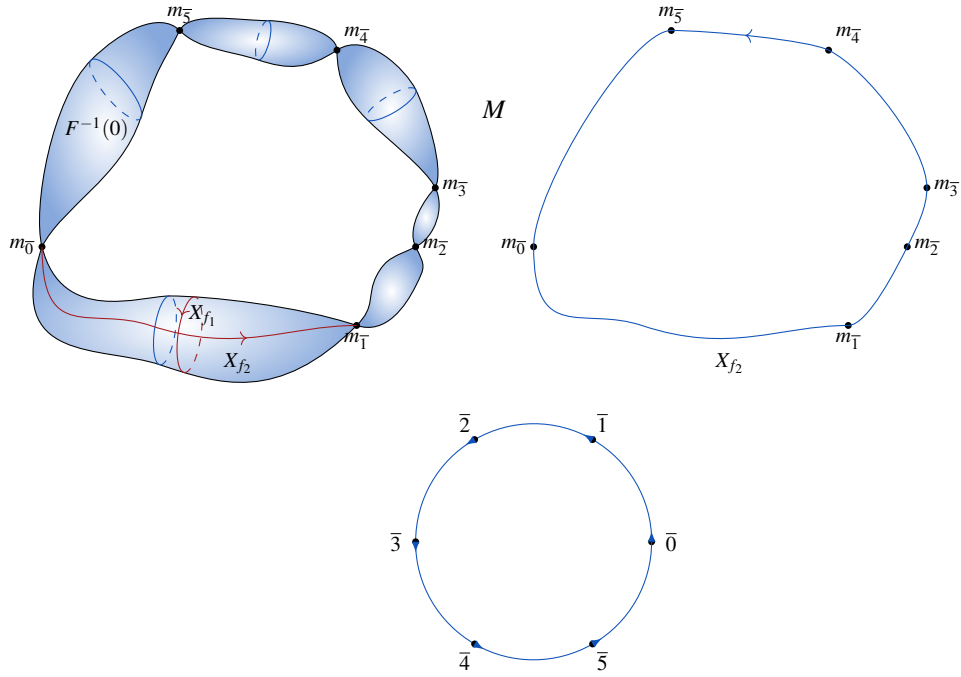


Figure 6.2: Reduction of the singular fiber. The quotient space \mathcal{F}/\mathbb{S}^1 , is a circle with k marked points. Compare \mathcal{F}/\mathbb{S}^1 with Γ_k , which is the cycle graph with vertex set \mathbb{Z}_k and set of edges $\{(j, j+1) \mid j \in \mathbb{Z}_k\}$. The automorphism group D_k of Γ_k is isomorphic to the group generated by γ_Y and θ_p . That is not a coincidence.

A singularity atlas is *compatible with the orientation* (α_1, α_2) if for every $j \in \mathbb{Z}_k$, (ψ_j, E_j) is compatible with (α_1, α_2) , and for any flow line of X_{α_1} in \mathcal{F} , whenever the α -limit point is labeled m_j , $j \in \mathbb{Z}_k$, its ω -limit point is labeled m_{j+1} .

6.5.2 Construction of the invariants

Let $(M, \omega, F) \in \mathcal{A}_{\text{ff}}^k$ and $(\alpha_1, \alpha_2) \in \text{Ori}(M, \omega, F)$. Let $((\psi_j, E_j))_{j \in \mathbb{Z}_k}$ be a singularity atlas compatible with the orientation (α_1, α_2) .

We construct the first set of the invariants: we split the period form α_1 into the singular part “across singularities” and the regular part. The regular part is an invariant.

Lemma 6.5.1. *The closed form*

$$\sigma = - \sum_{j \in \mathbb{Z}_k} E_j^* \kappa \in (\Omega^1/2\pi\check{\Lambda})(B_r) \quad (6.5.1)$$

is smoothable.

Proof. Let $P_j, Q_j: B \rightarrow M$ be Lagrangian sections of F such that $P_j(c) = E_j^{-1}(1, c)$, $Q_j(c) = E_j^{-1}(c, 1)$, for $j \in \mathbb{Z}_k$. Since ϕ_j is a symplectomorphism, by Lemma 6.2.2 and the definition of κ in (6.3.1), the translation form $\tau^{P_j Q_j} = E_j^* \kappa \in (\Omega^1/2\pi\check{\Lambda})(B_r)$.

For Q_j and $P_{j+\bar{1}}$, note that their images lie in the domain of $\tilde{\psi}_j$, an extension of ψ_j . So $\tilde{\psi}_j \circ Q_j$ and $\tilde{\psi}_j \circ P_{j+\bar{1}}$ are two smooth sections of q in \mathbb{R}_{ns}^4 . By an explicit calculation with the logarithm function, the translation form $\tau \in (\Omega^1/2\pi\check{\Lambda}^{\mathbb{R}^4, \omega, q})(\mathbb{R}^2)$ between these two section is smoothable, so $\tau^{Q_j P_{j+\bar{1}}} = G^* \tau \in (\Omega^1/2\pi\check{\Lambda}^{(M, \omega, F)})(B)$ is smoothable.

Note that α_1 is a section in $\check{\Lambda}(B)$, so

$$- \sum_{j \in \mathbb{Z}_k} E_j^* \kappa = 2\pi\alpha_1 - \sum_{j \in \mathbb{Z}_k} \tau^{P_j Q_j} = \sum_{j \in \mathbb{Z}_k} \tau^{Q_j P_{j+\bar{1}}} \in (\Omega^1/2\pi\check{\Lambda})(B_r)$$

is smoothable. The form σ is closed since κ is closed. □

Since σ defined in (6.5.1) is closed, there is a local smooth functions $S: (B, 0) \rightarrow \mathbb{R}$ such that $S(0) = 0$ and $dS = \sigma$ for each representative of σ in $\Omega^1(B)$. For different choices of representatives of σ , S differ by integer multiples of $2\pi A_1$, where A_1 is the action integral with $dA_1 = \alpha_1$. Let $A_1 = \text{Taylor}_0[A_1] \in \mathbb{R}[[T_0^*B]]_0$. Let $X = dc_1, Y = dc_2$ be the variables of the formal power series.

Definition 6.5.3. We call $\sigma \in (\Omega^1/2\pi\check{\Lambda})(B_r)$ the *desingularized period form*. We call the coset $S + 2\pi A_1 \mathbb{Z}$ the *desingularized action integral*. Let $S = \text{Taylor}_0[S] + 2\pi A_1 \mathbb{Z}$ which is in $\mathbb{R}[[T_0^*B]]_0 / (2\pi A_1) \mathbb{Z}$.

Let \mathbb{R} be the space of formal power series in two variables X, Y , without the constant term, and let $\mathbb{R}_{2\pi X} = \mathbb{R}/(2\pi X)\mathbb{Z}$. For any $m_j, j \in \mathbb{Z}_k$, let $s_j(X, Y) \stackrel{\text{def}}{=} (E_j^{-1})^*S$. We call $s_j \in \mathbb{R}_{2\pi X}$ the *action Taylor series at m_j* .

We construct the second set of the invariants: these invariants are Taylor series reflecting the difference between the Eliasson local charts at different singularities.

Definition 6.5.4. Let $\mathcal{R}^+ = \{g: \mathbb{R}^2 \rightarrow \mathbb{R} \mid \partial g/\partial c_2 > 0\}$ be a group whose product is $(g_1 \cdot g_2)(c_1, c_2) = g_1(c_1, g_2(c_1, c_2))$ for any $g_1, g_2 \in \mathcal{R}^+$. Let $g_{j,\ell} = \text{pr}_2 \circ E_\ell \circ E_j^{-1} \in \mathcal{R}^+$. Then we have $(E_\ell \circ E_j^{-1})(c_1, c_2) = (c_1, g_{j,\ell}(c_1, c_2))$. We call $(g_{j,\ell})_{j,\ell \in \mathbb{Z}_k}$ the set of *momentum transitions*.

Let $\mathbb{R}^+ = \{g(X, Y) = g^X X + g^Y Y + \mathcal{O}((X, Y)^2) \in \mathbb{R} \mid g^X \in \mathbb{R}, g^Y > 0\}$ be a group with the product $(g_1 \cdot g_2)(X, Y) = g_1(X, g_2(X, Y))$ for any $g_1, g_2 \in \mathbb{R}^+$. Let $g_{j,\ell} = \text{Taylor}_0[g_{j,\ell}] \in \mathbb{R}^+$. They follow the cocycle relation $g_{j,\ell} \cdot g_{\ell,p} = g_{j,p}$. We call $g_{j,\ell}$ the *transition Taylor series* from m_j to m_ℓ . We call $(g_{j,\ell})_{j,\ell \in \mathbb{Z}_k}$ the *transition cocycle*.

6.5.3 Moduli spaces and main theorem

Definition 6.5.5. Let $(M, \omega, F), (M', \omega', F') \in \mathcal{S}_{\text{ff}}^k$. Let $B = F(M), B' = F'(M')$. Let \mathcal{F} and \mathcal{F}' , respectively, be the singular fibers of F and F' . A *semiglobal isomorphism* between (M, ω, F) and (M', ω', F') is a semiglobal symplectomorphism $\varphi: (M, \omega, \mathcal{F}) \rightarrow (M', \omega', \mathcal{F}')$ such that there is a local diffeomorphism $G: (B, 0) \rightarrow (B', 0)$ with $F' \circ \varphi = G \circ F$. Let $\mathcal{M}_{\text{ff}}^k$ be the moduli space of integrable systems in $\mathcal{S}_{\text{ff}}^k$ up to semiglobal isomorphisms.

We list in Table 6.1 the definitions of some moduli spaces of integrable systems.

Let $(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})$ be a basepointed oriented integrable system, $((\psi_j, E_j))_{j \in \mathbb{Z}_k}$ be a singularity atlas compatible with (α_1, α_2) , and $(m_j)_{j \in \mathbb{Z}_k}$ be the k -tuple of singularities of F . Let $(s_j)_{j \in \mathbb{Z}_k}$ be the k -tuple of action Taylor series and let $(g_{j,\ell})_{j,\ell \in \mathbb{Z}_k}$ be the transition cocycle. These

²The isomorphism φ in Column 3 refers to $\varphi: (M, \omega, \mathcal{F}) \rightarrow (M', \omega', \mathcal{F}')$, G refers to $G: (B, 0) \rightarrow (B', 0)$ with $F' \circ \varphi = G \circ F$, and similarly for the other entries of the table. Here $\text{Ori}(B)$ is the set of 2 orientations of a connected subset $B \subset \mathbb{R}^2$, and $\text{Crit}(F)$ is the set of singularities of F .

Table 6.1: Moduli spaces of integrable systems in \mathcal{S}_{ff} with extra data.

² The moduli space ...	of ...	up to isomorphisms φ such that ...
$\mathcal{M}_{\text{ff,b}}^k$	<i>basepointed integrable systems</i> $(M, \omega, F, m_{\bar{0}}), m_{\bar{0}} \in \text{Crit}(F)$	$\varphi(m_{\bar{0}}) = m'_{\bar{0}}$
$\mathcal{M}_{\text{ff,o}}^k$	<i>basepointed integrable systems</i> $(M, \omega, F, (\alpha_1, \alpha_2)), (\alpha_1, \alpha_2) \in \text{Ori}(M, \omega, F)$	$(G^* \alpha'_1, G^* \alpha'_2) = (\alpha_1, \alpha_2)$
$\mathcal{M}_{\text{ff,mo}}^k$	<i>momentum-oriented integrable systems</i> $(M, \omega, F, o_B), o_B \in \text{Ori}(B)$	$G^* o_{B'} = o_B$
$\mathcal{M}_{\text{ff,o,b}}^k$	<i>basepointed oriented integrable systems</i> $(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})$ $(\alpha_1, \alpha_2) \in \text{Ori}(M, \omega, F), m_{\bar{0}} \in \text{Crit}(F)$	$\varphi(m_{\bar{0}}) = m'_{\bar{0}},$ $(G^* \alpha'_1, G^* \alpha'_2) = (\alpha_1, \alpha_2)$

series are constrained by the following relations:

$$\left. \begin{aligned} s_j &= s_\ell \cdot g_{j,\ell} \text{ for } j, \ell \in \mathbb{Z}_k; \\ g_{j,j}(X, Y) &= Y \text{ for } j \in \mathbb{Z}_k; \\ g_{j,\ell} \cdot g_{\ell,p} &= g_{j,p} \text{ for } j, \ell, p \in \mathbb{Z}_k. \end{aligned} \right\} \quad (6.5.2)$$

Theorem 6.5.2. *There is a bijection*

$$\Phi: \mathcal{M}_{\text{ff,o,b}}^k \rightarrow \mathcal{S}_{\text{ff,o,b}}^k \stackrel{\text{def}}{=} \left\{ ((s_j)_{j \in \mathbb{Z}_k}, (g_{j,\ell})_{j,\ell \in \mathbb{Z}_k}) \in \mathbb{R}_{2\pi X}^k \times \mathbb{R}_+^{k^2} \mid (6.5.2) \right\}$$

$$[(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})] \mapsto (s_{\bar{0}}, \dots, s_{\bar{-1}}, g_{\bar{0},\bar{0}}, \dots, g_{\bar{0},\bar{-1}}, \dots, g_{\bar{-1},\bar{0}}, \dots, g_{\bar{-1},\bar{-1}}).$$

We prove Theorem 6.5.2 in the remaining subsections.

6.5.4 Φ is well-defined

In this subsection, we are going to show that, the output of Φ does not depend on the choice of the singularity atlas, and we also want to know how the Taylor series will change if the orientation and the base point change.

Define bijections γ_X and γ_Y of $\mathcal{S}_{\text{ff},0,b}^k$ by

$$\begin{aligned}\gamma_X(\dots, s_j, \dots, \mathfrak{g}_{j,\ell}, \dots) &= (\dots, s'_j, \dots, \mathfrak{g}'_{j,\ell}, \dots), \\ s'_j(X, Y) &= s_j(-X, Y) + k\pi X, \\ \mathfrak{g}'_{j,\ell}(X, Y) &= \mathfrak{g}_{j,\ell}(-X, Y);\end{aligned}$$

and

$$\begin{aligned}\gamma_Y(\dots, s_j, \dots, \mathfrak{g}_{j,\ell}, \dots) &= (\dots, s''_j, \dots, \mathfrak{g}''_{j,\ell}, \dots), \\ s''_j(X, Y) &= -s_{-j}(X, -Y), \\ \mathfrak{g}''_{j,\ell}(X, Y) &= -\mathfrak{g}_{-j,-\ell}(X, -Y).\end{aligned}$$

Define a bijection θ_p of $\mathcal{S}_{\text{ff},0,b}^k$ by

$$\theta_p(\dots, s_j, \dots, \mathfrak{g}_{j,\ell}, \dots) = (\dots, s_{j+p}, \dots, \mathfrak{g}_{j+p,\ell+p}, \dots).$$

Lemma 6.5.3. *The map $\Phi: \mathcal{M}_{\text{ff},0,b}^k \rightarrow \mathcal{S}_{\text{ff},0,b}^k$ is well defined and satisfies the relations:*

$$\begin{aligned}\Phi\left([(M, \omega, F, (-\alpha_1, \alpha_2), m_{\bar{0}})]\right) &= \gamma_X\left(\Phi\left([(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})]\right)\right), \\ \Phi\left([(M, \omega, F, (\alpha_1, -\alpha_2), m_{\bar{0}})]\right) &= \gamma_Y\left(\Phi\left([(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})]\right)\right), \\ \Phi\left([(M, \omega, F, (\alpha_1, \alpha_2), m_p)]\right) &= \theta_p\left(\Phi\left([(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})]\right)\right), \quad \text{for } p \in \mathbb{Z}_k.\end{aligned}$$

Proof. Let $[(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})] \in \mathcal{M}_{\text{ff},0,b}^k$ and let $((\psi_j, E_j))_{j \in \mathbb{Z}_k}$ be a singularity atlas compatible with (α_1, α_2) . Let $((\psi'_j, E'_j))_{j \in \mathbb{Z}_k}$ be another singularity atlas and let $m'_{\bar{0}}$ be another basepoint. We may need to reorder the singularities to $(m'_j)_{j \in \mathbb{Z}_k}$ so that (ψ'_j, E'_j) is a chart near m'_j for $j \in \mathbb{Z}_k$ and $((\psi'_j, E'_j))_{j \in \mathbb{Z}_k}$ is compatible with some orientation $(\alpha'_1, \alpha'_2) \in \text{Ori}(M, \omega, F)$.

Let

$$\begin{aligned} (\dots, s_j, \dots, \mathfrak{g}_{j,\ell}, \dots) &= \Phi\left([(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})]\right), \\ (\dots, s'_j, \dots, \mathfrak{g}'_{j,\ell}, \dots) &= \Phi\left([(M, \omega, F, (\alpha'_1, \alpha'_2), m'_{\bar{0}})]\right). \end{aligned}$$

Let σ, σ' be the desingularized period forms, and $(g_{j,\ell})_{j,\ell \in \mathbb{Z}_k}, (g'_{j,\ell})_{j,\ell \in \mathbb{Z}_k}$ respectively be the set of momentum transitions of $((\psi_j, E_j))_{j \in \mathbb{Z}_k}, ((\psi'_j, E'_j))_{j \in \mathbb{Z}_k}$.

Case 1: If $(\alpha'_1, \alpha'_2) = (\alpha_1, \alpha_2)$ and $m'_{\bar{0}} = m_{\bar{0}}$, then $m'_j = m_j$ and if we let $G_j = E'_j \circ E_j^{-1}$ be a local diffeomorphism of $(\mathbb{R}^2, 0)$, then $d(\text{pr}_1 \circ G_j) = dc_1$ and $\frac{\partial(\text{pr}_2 \circ G_j)}{\partial c_2} > 0$. By Lemma 6.3.10, $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$. By Lemma 6.3.5, we have $(G_j^{-1})^* \kappa = \kappa + \mathcal{O}(c^\infty) dc_1 + \mathcal{O}(c^\infty) dc_2$. Then

$$\begin{aligned} \sigma' - \sigma &= \left(2\pi\alpha'_2 - \sum_{j \in \mathbb{Z}_k} (E'_j)^* \kappa\right) - \left(2\pi\alpha_1 - \sum_{j \in \mathbb{Z}_k} E_j^* \kappa\right) \\ &= \sum_{j \in \mathbb{Z}_k} E_j^* (\kappa - G_j^* \kappa) = \mathcal{O}(c^\infty) dc_1 + \mathcal{O}(c^\infty) dc_2 \end{aligned}$$

and

$$\mathfrak{g}'_{j,\ell} = \text{Taylor}_0[\text{pr}_2 \circ G_\ell \circ G_{j,\ell} \circ G_j^{-1}] = \text{Taylor}_0[\text{pr}_2 \circ G_\ell \circ G_j^{-1}] = \mathfrak{g}_{j,\ell}.$$

Hence we have $S' = S, s'_j = s_j$. Therefore, the map Φ is well defined.

Case 2: If $(\alpha'_1, \alpha'_2) = (-\alpha_1, \alpha_2)$ and $m'_{\bar{0}} = m_{\bar{0}}$, then $m'_j = m_j$. Let $G_j: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be $G_j(c) = -\bar{c}$, so we have $G_j^* \kappa = \kappa + \pi dc_1$. By Case 1, it is sufficient to assume that $E'_j = G_j \circ E_j$.

Then

$$\begin{aligned}\sigma' &= 2\pi\alpha'_2 - \sum_{j \in \mathbb{Z}_k} (E'_j)^* \kappa = 2\pi\alpha_2 - \sum_{j \in \mathbb{Z}_k} E_j^* (\kappa + \pi dc_1) \\ &= \sigma - \pi \sum_{j \in \mathbb{Z}_k} E_j^* dc_1 = \sigma - k\pi dc_1,\end{aligned}$$

and $g'_{j,\ell}(c) = \text{pr}_2 \circ G_\ell \circ G_{j,\ell} \circ G_j^{-1}(c) = g_{j,\ell}(-\bar{c})$. In this case,

$$\begin{aligned}S' &= S - k\pi[c_1], \\ s'_j(X, Y) &= s_j(-X, Y) + k\pi X, \\ g'_{j,\ell}(X, Y) &= g_{j,\ell}(-X, Y).\end{aligned}$$

Case 3: If $(\alpha'_1, \alpha'_2) = (\alpha_1, -\alpha_2)$ and $m'_0 = m_{\bar{0}}$, then $m'_j = m_{-j}$. Let $G_j: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be $G_j(c) = \bar{c}$, so we have $(G_j^{-1})^* \kappa = -\kappa$. By Case 1, it is sufficient to assume that $E'_j = G_{-j} \circ E_{-j}$. Then $\sigma' = 2\pi\alpha'_2 - \sum_{j \in \mathbb{Z}_k} (E'_j)^* \kappa = -2\pi\alpha_2 + \sum_{j \in \mathbb{Z}_k} E_{-j}^* \kappa = -\sigma$, and $g'_{j,\ell}(c) = \text{pr}_2 \circ G_{-\ell} \circ G_{-j,-\ell} \circ G_{-j}^{-1}(c) = -g_{-j,-\ell}(\bar{c})$. In this case,

$$\begin{aligned}S' &= -S, \\ s'_j(X, Y) &= -s_{-j}(X, -Y), \\ g'_{j,\ell}(X, Y) &= -g_{-j,-\ell}(X, -Y).\end{aligned}$$

Case 4: If $(\alpha'_1, \alpha'_2) = (\alpha_1, \alpha_2)$ and $m'_0 = m_p$, $p \in \mathbb{Z}_k$, then $m'_j = m_{j+p}$. By Case 1, it is sufficient to assume that $E'_j = E_{j+p}$. Then $\sigma' = \sigma$ and $g'_{j,\ell} = g_{j+p,\ell+p}$. Hence $S' = S, s'_j = s_{j+p}, g'_{j,\ell} = g_{j+p,\ell+p}$. \square

The bijections γ_X , γ_Y , and θ_p are subject to the relations

$$\gamma_X^2 = \gamma_Y^2 = \theta_p^2 = (\gamma_Y \circ \theta_p)^2 = \text{id}.$$

So they generate a $(\mathbb{Z}_2 \times D_k)$ -action on $\mathcal{S}_{\text{ff},0,b}^k$.

Corollary 6.5.4 (Corollary of Lemma 6.5.3). *There is a bijection*

$$\begin{aligned} \tilde{\Phi}: \mathcal{M}_{\text{ff}}^k &\rightarrow \mathcal{S}_{\text{ff}}^k \stackrel{\text{def}}{=} \mathcal{S}_{\text{ff},0,b}^k / (\mathbb{Z}_2 \times D_k) \\ [(M, \omega, F)] &\mapsto \Phi([(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})]) \end{aligned}$$

where $(\alpha_1, \alpha_2) \in \text{Ori}(M, \omega, F)$, $m_{\bar{0}}$ is a singularity of F , and the $(\mathbb{Z}_2 \times D_k)$ -action is generated by γ_X , γ_Y , and θ_p .

Remark 6.5.2. As pointed out in [34] the Taylor series invariant in the case that the singular fiber contains exactly one focus-focus point is defined up to a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -action, which accounts for the choices of Eliasson local charts in its construction. It becomes unique in the presence of a global S^1 -action (i.e. semitoric systems) provided one assumes everywhere that the Eliasson local charts preserve the S^1 -action and the \mathbb{R}^2 -orientation. In Corollary 6.5.4, we have the $(\mathbb{Z}_2 \times D_k)$ -action instead. When $k = 1$, $(\mathbb{Z}_2 \times D_k) \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

6.5.5 Φ is injective

Lemma 6.5.5. *The map $\Phi: \mathcal{M}_{\text{ff},0,b}^k \rightarrow \mathcal{S}_{\text{ff},0,b}^k$ is injective.*

Let $[(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})], [(M', \omega', F', (\alpha'_1, \alpha'_2), m'_{\bar{0}})] \in \mathcal{M}_{\text{ff},0,b}^k$ such that

$$\Phi\left([(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})]\right) = \Phi\left([(M', \omega', F', (\alpha'_1, \alpha'_2), m'_{\bar{0}})]\right).$$

Let $B = F(M)$, $B' = F'(M')$. Let \mathcal{F} and \mathcal{F}' , respectively, be the singular fibers of F and F' . We

want to show the two basepointed oriented systems are semiglobally isomorphic, that is, there is a semiglobal symplectomorphism $\varphi: (M, \omega, \mathcal{F}) \rightarrow (M', \omega', \mathcal{F}')$ and a local diffeomorphism $G: (B, 0) \rightarrow (B', 0)$ such that $F' \circ \varphi = G \circ F$, $(G^* \alpha'_1, G^* \alpha'_2) = (\alpha_1, \alpha_2)$ and $\varphi(m_j) = m'_j$.

Let $((\psi_j, E_j))_{j \in \mathbb{Z}_k}$ be a singularity atlas of (M, ω, F) compatible with (α_1, α_2) , and let $((\psi'_j, E'_j))_{j \in \mathbb{Z}_k}$ be a singularity atlas of (M', ω', F') compatible with (α'_1, α'_2) .

Lemma 6.5.6 (Vũ Ngọc [35, Lemma 5.1]). *Suppose $\alpha'_2 - \alpha_2 \in \mathcal{O}(c^\infty) dc_2$. Then there is a local diffeomorphism $E: (B, 0) \rightarrow (B', 0)$ isotopic to the identity such that $(E^* \alpha'_1, E^* \alpha'_2) = (\alpha_1, \alpha_2)$ and $E(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$.*

Proof. Let $\rho = \alpha'_2 - \alpha_2 \in \mathcal{O}(c^\infty) dc_2$. Throughout the proof t is a variable in $[0, 1]$. Let $\alpha_{2,t} = \alpha_2 + t\rho$, and let $R \in \mathcal{O}(c^\infty)$ be such that $dR = \rho$. Define $f_t: \mathbb{C}_1 \rightarrow \mathbb{R}$ as

$$f_t = \frac{-R}{\langle \alpha_{2,t}, \frac{\partial}{\partial c_2} \rangle} = \frac{-R}{\langle \sigma + t\rho, \frac{\partial}{\partial c_2} \rangle - (2\pi)^{-1} \sum_{j \in \mathbb{Z}_k} \ln |E_j| \frac{\partial(\text{pr}_2 \circ E_j)}{\partial c_2}}.$$

Note that $\frac{\partial(\text{pr}_2 \circ E_j)}{\partial c_2}(0) > 0$ for any $j \in \mathbb{Z}_k$. Since $|f_t| \leq |R|$ near 0, by Lemma 6.3.4, f_t has an extension $\tilde{f}_t \in \mathcal{O}(c^\infty)$. Take $E = E_1$ as E_t be the flow of $Y_t = f_t \frac{\partial}{\partial c_1} \in \mathfrak{X}(B, 0)$. Then

$$\frac{d}{dt}(E_t^* \alpha_{2,t}) = E_t^*(d\langle \alpha_{2,t}, Y_t \rangle + \rho) = E_t^*(d(\tilde{f}_t \langle \alpha_{2,t}, \frac{\partial}{\partial c_2} \rangle) + \rho).$$

Hence $E_t^* \alpha_{2,t} = \alpha_2$. By the construction $E^* \alpha'_1 = \alpha_1$ and $E(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$. \square

Proof of Lemma 6.5.5. Initialization: Let E be the local diffeomorphism in Lemma 6.5.6. Since

$$E^* \alpha'_1 = \alpha_1, E^* \alpha'_2 = \alpha_2,$$

$(M', \omega', E^{-1} \circ F', (\alpha_1, \alpha_2), m'_0)$ is a basepointed oriented integrable system semiglobally isomorphic to $(M', \omega', F', (\alpha'_1, \alpha'_2), m'_0)$ via $(\text{id}_{M'}, E)$, and $((\psi'_j, E'_j \circ E, m'_j))_{j \in \mathbb{Z}_k}$ is a singularity atlas of $(M', \omega', E^{-1} \circ F')$ compatible with (α_1, α_2) .

Define local diffeomorphisms $E_{j,\ell} = E_\ell^{-1} \circ E_j: (B, 0) \rightarrow (B, 0)$ and local symplectomorphisms $\psi_{j,\ell} = \psi_\ell^{-1} \circ \psi_j: (M, \omega, m_j) \rightarrow (M, \omega, m_\ell)$. Let

$$\begin{aligned} G &= (E'_0)^{-1} \circ E_0: (B, 0) \rightarrow (B', 0), \\ E''_j &= E_j \circ G^{-1}: (B', 0) \rightarrow (\mathbb{R}^2, 0), \\ G''_j &= E''_j \circ E^{-1} \circ (E'_j)^{-1}: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0) \end{aligned}$$

and define local symplectomorphisms

$$\tilde{\varphi}_{G''_j}: (\mathbb{R}^4, \omega_0, 0) \rightarrow (\mathbb{R}^4, \omega_0, 0)$$

be as in Lemma 6.3.9 for $j \in \mathbb{Z}_k$. Since $\text{Taylor}_0[E] = (X, Y)$ and the two integrable systems share the same transition cocycle,

$$\text{Taylor}_0[G''_j] = \text{Taylor}_0[E_j \circ E_0^{-1}] \circ \text{Taylor}_0[E'_j \circ (E'_0)^{-1}]^{-1} = (X, Y).$$

We have $G''_j(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$. Let $E''_j = E_j \circ E_0^{-1} \circ E'_0$ for $j \in \mathbb{Z}_k$. Then $E''_\ell \circ (E''_j)^{-1} = E_\ell \circ E_j^{-1}$ for $j, \ell \in \mathbb{Z}_k$. By Lemma 6.3.10, $(\tilde{\varphi}_{G''_j} \circ \psi'_j, E''_j)$ is an Eliasson local chart at m'_j for $j \in \mathbb{Z}_k$, compatible with (α_1, α_2) . Hence $((\tilde{\varphi}_{G''_j} \circ \psi'_j, E''_j, m'_j))_{j \in \mathbb{Z}_k}$ is a singularity atlas of $(M', \omega', E^{-1} \circ F')$ compatible with (α_1, α_2) .

By replacing $(M', \omega', F', (\alpha'_1, \alpha'_2), m'_0)$ with $(M', \omega', E^{-1} \circ F', (\alpha_1, \alpha_2), m'_0)$ and $((\psi'_j, E'_j, m'_j))_{j \in \mathbb{Z}_k}$ with $((\tilde{\varphi}_{G''_j} \circ \psi'_j, E''_j))_{j \in \mathbb{Z}_k}$ if necessary, we assume later, without loss of generality, that $(\alpha'_1, \alpha'_2) = (\alpha_1, \alpha_2)$ and $E'_\ell \circ (E'_j)^{-1} = E_\ell \circ E_j^{-1}$ for $j, \ell \in \mathbb{Z}_k$.

Construction of the semiglobal isomorphism: We define the semiglobal isomorphism φ . The definition of the isomorphism is made by induction as follows. Define the local symplectomorphism $\varphi_0 = (\psi'_0)^{-1} \circ \psi_0: (M, \omega, m_0) \rightarrow (M', \omega', m'_0)$. For $j \in \mathbb{Z}_k$, let M_j be a fiber-transitive

neighborhood of m_j , where ψ_j is defined, in (M, ω, F) , and $M'_j \subset M'$ is defined analogously. By Lemma 6.2.3, we can extend φ_0 to $\tilde{\varphi}_0$ a semiglobal symplectomorphism of $(M_0, \omega, \mathcal{F} \cap M_0)$.

For $j \in \mathbb{Z}_k \setminus \{-1\}$, suppose that we have defined the semiglobal symplectomorphism $\tilde{\varphi}_j: (M_j, \omega, \mathcal{F} \cap M_j) \rightarrow (M'_j, \omega', \mathcal{F}' \cap M'_j)$. We want to define $\tilde{\varphi}_{j+1}$ on $(M_{j+1}, \omega, \mathcal{F} \cap M_{j+1})$. Let λ_{j+1} be a local symplectomorphism determined by the following commutative diagram:

$$\begin{array}{ccccc}
& & (M, \omega, m_{j+1}) & \xrightarrow{\lambda_{j+1}} & (M', \omega', m'_{j+1}) \\
& \nearrow \psi_{j+1}^{-1} \circ \psi_j & \downarrow F & & \downarrow F' \\
(M, \omega, m_j) & \xrightarrow{\varphi_j} & (M', \omega', m'_j) & & \\
\downarrow F & & \downarrow F' & & \\
(B, 0) & \xrightarrow{G} & (B, 0) & \xrightarrow{G} & (B', 0) \\
& \nearrow (E_{j+1})^{-1} \circ E_j & & & \nearrow E_{j+1}^{-1} \circ E_j \\
(B, 0) & \xrightarrow{G} & (B', 0) & &
\end{array}$$

Let $M_{j,j+1} = M_j \cap M_{j+1}$ and $M'_{j,j+1} = M'_j \cap M'_{j+1}$. Then we can extend λ_{j+1} to the map $\tilde{\lambda}_{j+1}: (M_j, \omega, \mathcal{F} \cap M_j) \rightarrow (M_{j+1}, \omega, \mathcal{F} \cap M_{j+1})$. Define $\mu_{j,j+1}$ such that the diagram

$$\begin{array}{ccccc}
(M'_{j,j+1}, \omega', \mathcal{F}' \cap M'_{j,j+1}) & \xrightarrow{\mu_{j,j+1}} & (M'_{j+1}, \omega', \mathcal{F}' \cap M'_{j+1}) & & \\
\downarrow F' & \nwarrow \tilde{\varphi}_j & \nearrow \tilde{\lambda}_{j+1} & & \downarrow F' \\
& (M_{j,j+1}, \omega, \mathcal{F} \cap M_{j,j+1}) & & & \\
\downarrow F & & \downarrow F & & \\
(B', 0) & \xrightarrow{G} & (B, 0) & \xrightarrow{G} & (B', 0)
\end{array}$$

commutes.

By Lemma 6.2.3, we can extend $\mu_{j,j+1}$ to a semiglobal symplectomorphism $\tilde{\mu}_{j,j+1}$ of $(M', \omega', \mathcal{F}')$. Note that $\tilde{\lambda}_{j+1}(m_{j+1}) = m'_{j+1}$ and $\tilde{\varphi}_j(x) \rightarrow m'_{j+1}$ as $x \rightarrow m_{j+1}$ in M , so we have

$\tilde{\mu}_{j,j+\bar{1}}(m'_{j+\bar{1}}) = m'_{j+\bar{1}}$. Now let

$$\begin{aligned}\tilde{\varphi}_{j+\bar{1}} &: (M_{j+\bar{1}}, \omega, \mathcal{F} \cap M_{j+\bar{1}}) \rightarrow (M'_{j+\bar{1}}, \omega', \mathcal{F}' \cap M'_{j+\bar{1}}), \\ \tilde{\varphi}_{j+\bar{1}} &= \lambda_{j+\bar{1}}^{-1} \circ \tilde{\mu}_{j,j+\bar{1}} \Big|_{(M_{j+\bar{1}}, \mathcal{F} \cap M_{j+\bar{1}})}.\end{aligned}$$

Then $\varphi_{j+\bar{1}} = \varphi_j$ in their common domain $M_{j,j+\bar{1}}$.

For $\varphi_{-\bar{1}}$ and $\varphi_{\bar{0}}$, they coincide on regular values of F near \mathcal{F} , so by continuity, they must coincide on their common domain $M_{-\bar{1}} \cap M_{\bar{0}}$. Hence, we can glue $\tilde{\varphi}_j$, $j \in \mathbb{Z}_k$ to get a semiglobal symplectomorphism $\varphi: (M, \omega, \mathcal{F}) \rightarrow (M', \omega', \mathcal{F}')$ with the commuting diagram:

$$\begin{array}{ccc}(M, \omega, \mathcal{F}) & \xrightarrow{\varphi} & (M', \omega', \mathcal{F}') \\ \downarrow F & & \downarrow F' \\ (B, 0) & \xrightarrow{G} & (B', 0)\end{array}$$

□

6.5.6 Φ is surjective

Lemma 6.5.7. *The map $\Phi: \mathcal{M}_{\text{ff},\text{o},\text{b}}^k \rightarrow \mathcal{S}_{\text{ff},\text{o},\text{b}}^k$ is surjective.*

Let

$$(\dots, \mathbf{v}_j, \dots, \mathbf{w}_{j,\ell}, \dots) \in \mathcal{S}_{\text{ff},\text{o},\text{b}}^k.$$

We want to show that there is $(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})$ such that

$$\Phi\left([(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})]\right) = (\dots, \mathbf{v}_j, \dots, \mathbf{w}_{j,\ell}, \dots). \quad (6.5.3)$$

The local structures of the integrable system (M, ω, F) near the singularities m_j are isomorphic to the local normal form in Section 6.3.1, which we can extend to the fiber-transitive subset containing the neighborhood of m_j . We use the symplectic gluing technique similar to [31,

Section 3] to construct (M, ω, F) .

By Borel's lemma, there is a local smooth map $s_{\bar{0}}: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ and a local diffeomorphism $G_{\bar{0},j}: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ such that $\text{Taylor}_0[s_{\bar{0}}] = v_{\bar{0}}$ and $\text{Taylor}_0[G_{\bar{0},j}] = (X, w_{\bar{0},j})$. Let $w_{\bar{0},j}^Y > 0$ be the Y -coefficient of $w_{\bar{0},j}$, and $v_{\bar{0}}^Y \in \mathbb{R}$ the Y -coefficient of $v_{\bar{0}}$. Let U be an open neighborhood of $0 \in \mathbb{R}^2$ and $\tilde{G}_{\bar{0},j}: U \rightarrow \mathbb{R}^2$ for $j \in \mathbb{Z}_k$ an extension of $G_{\bar{0},j}$ to a diffeomorphism onto its image ($\tilde{G}_{\bar{0},\bar{0}} = \text{id}$). For $\delta > 0$ sufficiently small, there is an open neighborhood $U_{\bar{0}} \subset U$ of 0 , such that for $j \in \mathbb{Z}_k$, if we let $U_j = \tilde{G}_{\bar{0},j}(U_{\bar{0}})$, then for $c \in U_j$,

$$|c| < \delta < 1, \quad \frac{\partial(\text{pr}_2 \circ \tilde{G}_{\bar{0},j})}{\partial c_2}(c) > w_{\bar{0},j}^Y - \delta > 0, \quad \frac{\partial s_{\bar{0}}}{\partial c_2}(c) > v_{\bar{0}}^Y - \delta > 0. \quad (6.5.4)$$

For $j \in \mathbb{Z}_k$, let $W_j = q^{-1}(U_j)$, $W_{j,\text{nu}} = W_j \cap \mathbb{R}_{\text{nu}}^4$, $W_{j,\text{ns}} = W_j \cap \mathbb{R}_{\text{ns}}^4$, $W_{j,r} = W_j \cap \mathbb{R}_r^4$, $U_{j,r} = U_j \cap \mathbb{R}_r^2$. Note that these spaces depend on δ . For $j, \ell \in \mathbb{Z}_k$, let $\tilde{G}_{j,\ell} = \tilde{G}_{\bar{0},\ell} \circ \tilde{G}_{\bar{0},j}^{-1}: U_j \rightarrow U_\ell$.

Define, by Lemmas 6.2.1 and 6.2.2, symplectomorphisms

$$\varphi_{j,j+\bar{1}} = \varphi_{\tilde{G}_{j,j+\bar{1}}} \circ \Psi_{-\kappa}: W_{j,\text{nu}} \rightarrow W_{j+\bar{1},\text{ns}}$$

for $j \in \mathbb{Z}_k \setminus \{\bar{1}\}$, and

$$\varphi_{\bar{1},\bar{0}} = \Psi_{-\text{ds}_{\bar{0}}} \circ \varphi_{\tilde{G}_{\bar{1},\bar{0}}} \circ \Psi_{-\kappa}: W_{\bar{1},\text{nu}} \rightarrow W_{\bar{0},\text{ns}}.$$

Let \mathcal{G} be the groupoid generated by the restrictions of $\varphi_{j,j+\bar{1}}$ for $j \in \mathbb{Z}_k$ onto open subsets. Recall Γ_k is the cycle graph with k vertices. Consider its fundamental groupoid $\Pi(\Gamma_k)$ whose elements are of the form $[j_0, j_1]_p$, where $j_0, j_1 \in \mathbb{Z}_k, p \in \mathbb{Z}$ and $j_0 + \bar{p} = j_1$. The multiplication is given by concatenation $[j_1, j_2]_{p'} \cdot [j_0, j_1]_p = [j_0, j_2]_{p+p'}$. Any element $[j_0, j_1]_p$ of $\Pi(\Gamma_k)$

corresponds to an element of \mathcal{G} :

$$\left\{ \begin{array}{ll} \varphi_{[j,j]_{\bar{0}}} = \text{id}: W_j \rightarrow W_j, & p = 0; \\ \varphi_{[j,j+\bar{1}]_1} = \varphi_{j,j+\bar{1}}: W_{j,\text{nu}} \rightarrow W_{j+\bar{1},\text{ns}}, & p = 1; \\ \varphi_{[j,j-\bar{1}]_{-1}} = \varphi_{j-\bar{1},j}^{-1}: W_{j,\text{ns}} \rightarrow W_{j-\bar{1},\text{nu}}, & p = -1; \\ \varphi_{[j,j+\bar{p}]_p} = \varphi_{j+\bar{p}-\bar{1},j+\bar{p}} \circ \cdots \circ \varphi_{j+\bar{1},j+\bar{2}} \circ \varphi_{j,j+\bar{1}}: W_{j,r} \rightarrow W_{j+\bar{p},r}, & p \geq 2; \\ \varphi_{[j,j+\bar{p}]_p} = \varphi_{j+\bar{p},j+\bar{p}+\bar{1}}^{-1} \circ \cdots \circ \varphi_{j-\bar{2},j-\bar{1}}^{-1} \circ \varphi_{j-\bar{1},j}^{-1}: W_{j,r} \rightarrow W_{j+\bar{p},r}, & p \leq -2. \end{array} \right.$$

Actually, \mathcal{G} consists of restrictions of $\varphi_{[j_0,j_1]_p}$ for all $[j_0,j_1]_p \in \Pi(\Gamma_k)$ restricted to open subsets, and \mathcal{G} is a groupoid of symplectomorphisms.

Let $kW = \coprod_{j \in \mathbb{Z}_k} W_j$ and $kW_r = \coprod_{j \in \mathbb{Z}_k} W_{j,r}$. Let $kD = \coprod_{j \in \mathbb{Z}_k} D_j \subset kW$ where

$$D_{\bar{0}} = \left\{ (z, \zeta) \in W_{\bar{0}} \mid |z| \leq 1, |\zeta| \leq e^{\frac{\partial s_{\bar{0}}}{\partial c_2}(q(z, \zeta))} \right\},$$

$$D_j = \{(z, \zeta) \in W_j \mid |z| \leq 1, |\zeta| \leq 1\}, \quad \text{for } j \in \mathbb{Z}_k \setminus \{\bar{0}\}.$$

Lemma 6.5.8. *Define a smooth function*

$$\rho: kW_r \rightarrow \mathbb{R},$$

$$(z, \zeta) \in W_{j,r} \mapsto \frac{\partial(\text{pr}_2 \circ G_{\bar{0},j})}{\partial c_2}(G_{j,\bar{0}} \circ q(z, \zeta)) \ln|z|.$$

Then for any $x \in W_j \subset kW$ there is a $p \in \mathbb{Z}$ such that $\rho \circ \varphi_{[j,j+\bar{p}]_p}$ is defined and is in $D_{j+\bar{p}}$, and there is a smooth function $L: U_{\bar{0},r} \rightarrow \mathbb{R}$ such that $\rho \circ \varphi_{[\bar{0},\bar{0}]_k}(z, \zeta) - \rho(z, \zeta) = L \circ q(z, \zeta)$ for any $(z, \zeta) \in W_{\bar{0},r}$, and $\lim_{c \rightarrow 0} L(c) = \infty$.

Proof. For $(z, \zeta) \in U_{\bar{0}}$, let $c = q(z, \zeta)$. Let $L_j: U_{\bar{0},r} \rightarrow \mathbb{R}$, where

$$\begin{aligned} L_{\bar{0}}(z, \zeta) &= -\ln|c| + \frac{\partial s_{\bar{0}}}{\partial c_2}(c), \\ L_j(z, \zeta) &= -\frac{\partial(\text{pr}_2 \circ G_{\bar{0},j})}{\partial c_2}(c) \ln|G_{\bar{0},j}(c)|, \quad \text{for } j \in \mathbb{Z}_k \setminus \{\bar{0}\}. \end{aligned}$$

Then we have, for any $p \in \mathbb{Z}$,

$$\rho \circ \varphi_{[\bar{0}, \bar{p}]_p}(z, \zeta) = \begin{cases} \rho(z, \zeta) + \sum_{s=1}^p L_{\bar{s}}(c), & p \geq 0, \\ \rho(z, \zeta) - \sum_{s=p+1}^0 L_{\bar{s}}(c), & p < 0, \end{cases}$$

and by (6.5.4),

$$\begin{aligned} L_{\bar{0}}(z, \zeta) &\geq (1 - \delta)|\ln \delta| + (v_{\bar{0}}^Y - \delta), \\ L_j(z, \zeta) &\geq (w_{\bar{0},j}^Y - \delta)|\ln \delta|, \quad \text{for } j \in \mathbb{Z}_k \setminus \{\bar{0}\}. \end{aligned}$$

Hence $\lim_{c \rightarrow 0} L_j(c) = \infty$.

For any $(z, \zeta) \in W_{0,r}$, if $\rho(z, \zeta) \leq 0$, there is a $p \in \mathbb{Z}$, $p \geq 1$ such that

$$-\sum_{s=1}^p L_{\bar{s}}(c) \leq \rho(z, \zeta) \leq -\sum_{s=1}^{p-1} L_{\bar{s}}(c)$$

so $-L_{\bar{p}}(c) \leq \rho \circ \varphi_{[\bar{0}, \bar{p}]_p}(z, \zeta) \leq 0$ and $\varphi_{[\bar{0}, \bar{p}]_p}(z, \zeta) \in D_{\bar{p}}$. Similarly, if $(z, \zeta) \in W_{0,r}$ with $\rho(z, \zeta) > 0$, there is a $p \in \mathbb{Z}$, $p \leq 0$ such that $\varphi_{[\bar{0}, \bar{p}]_p}(z, \zeta) \in D_{\bar{p}}$. If $\zeta = 0$ and $|z| \leq 1$, or $z = 0$ and $|\zeta| \leq 1$, we already have $(z, \zeta) \in D_{\bar{0}}$. If $\zeta = 0$ and $|z| > 1$, then $\varphi_{[\bar{0}, \bar{-1}]_{-1}}(z, \zeta) = (0, \zeta') \in D_{-1}$ since $0 < |\zeta'| < 1$. If $z = 0$ and $|\zeta| > 1$, then $\varphi_{[\bar{0}, \bar{1}]_1}(z, \zeta) = (z', 0) \in D_1$ since $0 < |z'| < 1$. Analogously, for any $x \in W_j \subset kW$, $j \in \mathbb{Z}_k$, there is a $p \in \mathbb{Z}$ such that $\rho \circ \varphi_{[j, j+\bar{p}]_p} \in D_{j+\bar{p}}$.

Now, let $L = -\sum_{j \in \mathbb{Z}_k} L_j: U_{\bar{0},r} \rightarrow \mathbb{R}$. Then $\rho \circ \varphi_{[\bar{0}, \bar{0}]_k} = \rho + L \circ q$ on $U_{\bar{0},r}$ and $\lim_{c \rightarrow 0} L(c) = \infty$. □

We define an equivalence relation $\sim_{\mathcal{G}}$ on kW as $x \sim_{\mathcal{G}} y$ if and only if there is a $\varphi \in \mathcal{G}$ such that $y = \varphi(x)$. Let $M = kW / \sim_{\mathcal{G}}$ be the quotient space, $\lambda : kW \rightarrow M$, $\lambda_j : W_j \rightarrow M$, $j \in \mathbb{Z}_k$ be the quotient maps. Let $\Delta_{\mathcal{G}} = \{(x, y) \in kW \times kW \mid x \sim_{\mathcal{G}} y\}$.

Lemma 6.5.9. *The topological space M can be uniquely realized as a symplectic manifold with the symplectic structure ω , and a smooth function $F : M \rightarrow \mathbb{R}^2$ such that $\lambda_j : (W_j, \omega_0, x) \rightarrow (M, \omega, \lambda_j(x))$ is a local symplectomorphism for any $x \in W_j$ and $\tilde{G}_{\bar{0}, j} \circ F \circ \lambda_j = q|_{W_j}$ for $j \in \mathbb{Z}_k$.*

Proof. We want to prove that M is a topological manifold with the quotient topology.

The map λ_j is open: for any open set $V \subset W_j$, the preimage

$$\lambda_j^{-1}(\lambda_j(V)) = V \cup \varphi_{[j, j+1]_1}(V \cap W_{j, \text{nu}}) \cup \varphi_{[j, j-1]_{-1}}(V \cap W_{j, \text{ns}}) \cup \bigcup_{p \in \mathbb{Z}, |p| \geq 2} \varphi_{[j, j+p]_p}(V \cap W_{j, r})$$

is open, so λ_j is an open map.

The map λ_j is locally injective: we need to prove that, any $x \in W_j$ has a neighborhood V in W_j such that for any $p \in \mathbb{Z} \setminus \{0\}$, as long as x is in the domain, the map $\varphi_{[j, j]_{pk}}$ sends x outside of V . If $k \geq 2$, then $x \in W_{j, r}$. This is a consequence of Lemma 6.5.8. If $k = 1$ and $x \in W_{j, s} \setminus \{0\}$, we have $\varphi_{[j, j]_1}(x) \in W_{j, u}$ away from x . The case $k = 1$ and $x \in W_{j, u} \setminus \{0\}$ is analogous.

The subset $\Delta_{\mathcal{G}}$ is closed in $kW \times kW$: suppose there are points $(x_i, y_i) \in \Delta_{\mathcal{G}}$ converging to $(x_{\infty}, y_{\infty}) \in kW \times kW$. Assume, without loss of generality, that $(x_{\infty}, y_{\infty}) \in W_{\bar{0}} \times W_j$ for some fixed $j \in \mathbb{Z}_k$. Since $W_{\bar{0}}, W_j$ are open in kW , we can assume $(x_i, y_i) \in W_{\bar{0}} \times W_j$. There is $[0, \bar{p}_i]_{p_i} \in \Pi(\Gamma_k)$ such that $y_i = \varphi_{[0, \bar{p}_i]_{p_i}}(x_i)$. If there is a subsequence $\{p_{i_m}\}$ of p_i with $p_{i_m} = p_0 \in \mathbb{Z}$, then $y_{i_m} = \varphi_{[0, \bar{p}_0]_{p_0}}(x_{i_m})$. In this case, $y_{\infty} = \varphi_{[0, \bar{p}_0]_{p_0}}(x_{\infty})$, so $(x_{\infty}, y_{\infty}) \in \Delta_{\mathcal{G}}$. Otherwise, by descending to a subsequence we can assume $|p_i| \rightarrow \infty$, so for i large, $x_i \in W_{\bar{0}, r}, y_i \in W_{j, r}$. By Lemma 6.5.8, we have $|\rho(x_i) - \rho(y_i)| \rightarrow \infty$, which contradicts $(x_i, y_i) \rightarrow (x_{\infty}, y_{\infty})$.

Since λ_j is open and locally injective, λ_j is a local homeomorphism, and M is locally Euclidean. Since λ_j is open and $\Delta_{\mathcal{G}} \subset W_j \times W_j$ is closed, M is Hausdorff. Since W_i is second countable, $M = \coprod_{j \in \mathbb{Z}_k} \lambda_j(W_j)$ is second countable. We conclude that M is a topological manifold.

Noting that the maps $\varphi_{j,\ell}$, $j, \ell \in \mathbb{Z}_k$ are symplectomorphisms satisfying $q \circ \varphi_{j,\ell} = \tilde{G}_{j,\ell} \circ q$, there is a unique symplectic structure ω on M , and a smooth function $F: M \rightarrow \mathbb{R}^2$ such that $\lambda_j^* \omega = \omega_0$ and $\tilde{G}_{0,j} \circ F \circ \lambda_j = q|_{W_j}$. \square

Proof of Lemma 6.5.7. Let $m_j = \lambda_j(0)$ and $\mu_j = \lambda_j|_{(M, m_j)}^{-1}: (M, m_j) \rightarrow (W_j, 0)$ be a local symplectomorphism for $j \in \mathbb{Z}_k$. Finally, we need to show the following: the construction (M, ω, F) in Lemma 6.5.9 lies inside $\mathcal{S}_{\text{ff}}^k$, has a singularity atlas $((\mu_j, \tilde{G}_{0,j}))_{j \in \mathbb{Z}_k}$ for singularities m_j , $j \in \mathbb{Z}_k$, compatible with some $(\alpha_1, \alpha_2) \in \text{Ori}(M, \omega, F)$ such that (6.5.3) holds.

The triple (M, ω, F) is in $\mathcal{S}_{\text{ff}}^k$: The triple (M, ω, F) is an integrable system since it is locally isomorphic to integrable systems everywhere, and so the only singular points of F are m_j on \mathcal{F} , $j \in \mathbb{Z}_k$, which are of non-degenerate focus-focus type. To show that F is proper, let $K \subset U_0$ be any compact subset. By Lemma 6.5.8, $\lambda(kW) = \lambda(kD)$. Since $q^{-1}(\tilde{G}_{0,j}(K)) \cap D_j$ is compact, $F^{-1}(K) = \bigcup_{j \in \mathbb{Z}_k} \lambda_j(q^{-1}(\tilde{G}_{0,j}(K)) \cap D_j)$ is compact. The fibers of F are connected since q has connected fibers.

Computation of $\check{\Lambda}^{(M, \omega, F)}$: Let $U \subset U_{0,r}$ be a simply connected open set. Note that $\kappa|_U \in (\Omega^1/2\pi\check{\Lambda})(\mathbb{R}_r^2)$ and let $\kappa_U \in \Omega^1(U)$ be a representative of $\kappa|_U$. Let $\alpha_2|_U = ds_{\bar{0}} - \sum_{j \in \mathbb{Z}_k} \tilde{G}_{0,j}^* \kappa_U \in Z^1(U)$. We have, in $F^{-1}(U)$,

$$\begin{aligned} \varphi_{[\bar{0}, \bar{0}]_k} \Big|_{F^{-1}(U)} &= \Psi_{-ds_{\bar{0}}} \circ \varphi_{\tilde{G}_{-1, \bar{0}}} \circ \Psi_{-\kappa_U} \circ \cdots \circ \Psi_{-\kappa_U} \circ \varphi_{\tilde{G}_{1, \bar{2}}} \circ \Psi_{-\kappa_U} \circ \varphi_{\tilde{G}_{0, \bar{1}}} \circ \Psi_{-\kappa_U} \\ &= \Psi_{-ds_{\bar{0}} - \sum_{j \in \mathbb{Z}_k} \tilde{G}_{0,j}^* \kappa_U} \circ \varphi_{\tilde{G}_{-1, \bar{0}}} \circ \cdots \circ \varphi_{\tilde{G}_{1, \bar{2}}} \circ \varphi_{\tilde{G}_{0, \bar{1}}} \\ &= \Psi_{-2\pi\alpha_2|_U}. \end{aligned}$$

So $\alpha_2|_U \in \check{\Lambda}^{(M, \omega, F)}(U)$. Let $\alpha_1 = dc_1 \in \Omega^1(U_{\bar{0}})$, then $\alpha_1|_U \in \check{\Lambda}^{(M, \omega, F)}(U)$. On the other hand, for any $\tau \in Z^1(U)$ to be a period form, it has to satisfy $\Psi_{2\pi\tau} = \varphi_{[\bar{0}, \bar{0}]_{pk}}$ for some $p \in \mathbb{Z}$. Therefore, $\check{\Lambda}^{(M, \omega, F)}(U)$ is the abelian group generated by $\alpha_1|_U, \alpha_2|_U$. Similarly, we have $\check{\Lambda}^{(M, \omega, F)}(U) = \alpha_1 \mathbb{Z}$ if U is an open neighborhood of 0.

Computation of the invariants: For each $j \in \mathbb{Z}_k$, $(\mu_j, \tilde{G}_{\bar{0},j})$ is an Eliasson local chart near m_j since $q \circ \mu_j = \tilde{G}_{\bar{0},j} \circ F$. For $j = \bar{0}$, note that $\alpha_1 = dc_1$ and $\frac{\partial}{\partial c_2} \lrcorner \alpha_2 = L$, so $(\mu_{\bar{0}}, \text{id})$ is compatible with (α_1, α_2) . For $j \in \mathbb{Z}_k$, since $d\tilde{G}_{\bar{0},j}$ has positive diagonal entries near the origin, $(\mu_j, \tilde{G}_{\bar{0},j})$ is compatible with (α_1, α_2) . By the construction of M , any flow line of X_{α_2} with α -limit m_j for some $j \in \mathbb{Z}_k$ has ω -limit $m_{j+\bar{1}}$, so $((\mu_j, \tilde{G}_{\bar{0},j}))_{j \in \mathbb{Z}_k}$ is a singularity atlas compatible with (α_1, α_2) .

Now since $ds_{\bar{0}} = 2\pi\alpha_2 - \sum_{j \in \mathbb{Z}_k} \tilde{G}_{\bar{0},j}^* \kappa$ and $s_{\bar{0}}(0) = 0$, the action Taylor series $s_{\bar{0}}$ at $m_{\bar{0}}$ is, $s_{\bar{0}} = \text{Taylor}_0[s_{\bar{0}}] = v_{\bar{0}}$. The transition cocycle $(g_{j,\ell})_{j,\ell \in \mathbb{Z}_k}$ is such that $g_{j,\ell} = \text{Taylor}_0[\text{pr}_2 \circ G_{j,\ell}] = w_{j,\ell}$ for $j, \ell \in \mathbb{Z}_k$. \square

Theorem 6.5.2 follows by putting together Lemmas 6.5.3, 6.5.5 and 6.5.7.

Proof of Theorem 6.5.2. The map

$$\Phi: \mathcal{S}_{\text{ff},0,b}^k \rightarrow \mathbb{R}_{2\pi X} \times \mathbb{R}_+^{k-1}$$

$$\left(s_{\bar{0}}, \dots, s_{-\bar{1}}, g_{\bar{0},\bar{0}}, \dots, g_{\bar{0},-\bar{1}}, \dots, g_{-\bar{1},\bar{0}}, \dots, g_{-\bar{1},-\bar{1}} \right) \mapsto \left(s_{\bar{0}}, g_{\bar{0},\bar{1}}, g_{\bar{1},\bar{2}}, \dots, g_{-\bar{1},\bar{0}} \right)$$

is a bijection. \square

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