UC Berkeley

UC Berkeley Previously Published Works

Title

On homotopy spheres of low dimension

Permalink

<https://escholarship.org/uc/item/0w8629cr>

Author

Hirsch, MW

Publication Date

1965

Peer reviewed

On Homotopy Spheres of Low Dimension

MORRIS W. HIRSCH

1. Introduction

Poincaré conjectured that every simply connected closed 3-manifold is homeomorphic to the 3-sphere S^3 . A more general conjecture is that every *homotopy n -sphere* (i.e., n -manifold having the homotopy type of S^n) is homeomorphic to S^n . This problem is still open for $n \geq 3$. However, S. Smale [6, 7] has proved that every combinatorial homotopy n -sphere M is combinatorially equivalent to S^n (symbolized by $M \equiv S^n$) if $n > 5$ and homeomorphic to S^5 if $n = 5$; moreover, every smooth homotopy 5-sphere is diffeomorphic to S^5 (symbolized by $M \approx S^5$). Zeeman [9] has proved that if M is a combinatorial homotopy 5-sphere, then $M - x \equiv$ Euclidean 5-space R^5 .

In this paper we obtain a condition on M^5 implying $M^5 \equiv S^5$ (Theorem 4, below). We show that homotopy spheres of dimension 3 and 4 exhibit behavior similar to that of S^3 and S^4 when certain operations are performed upon them. An interesting result of this sort is this:

THEOREM 1. *The suspension of a smooth homotopy 4-sphere M is homeomorphic to S^5 . The homeomorphism can be taken to be diffeomorphism on the complement of the two suspension points.*

This follows from Theorem 6 (b) below, which states that

$$M \times R^1 \approx S^4 \times R^1.$$

V. Poénaru has pointed out that a theorem of this kind leads to the study of critical points, since Theorem 1 has the following corollary.

THEOREM 2. *Every homotopy 4-sphere is diffeomorphic to a level surface of a smooth real valued function on S^5 having only two critical points.*

Theorem 1 leads to an obvious connection between Poincaré's conjecture in dimension 4 and the Hauptvermutung for arbitrary triangulations of S^5 .

In the next section Theorem 1 and related results are proved. In section 3 we prove

THEOREM 3. *Let M be a smooth homotopy 4-sphere. Then*

$$M \times M - \Delta(M) \approx S^4 \times S^4 - \Delta(S^4),$$

where Δ is the diagonal embedding.

The last section is independent from the others. In it we prove the following result.

THEOREM 4. *Let M be a combinatorial homotopy 5-sphere which bounds a compact, orientable, combinatorial 6-manifold. Then $M \equiv S^5$.*

2. Trivializations

For this section and the next we consider only *smooth* manifolds.

A homotopy k -sphere M^k is *trivialized by V* if V is a manifold such that $M^k \times V \approx S^k \times V$.

A *homotopy k -cell* is a compact contractible k -manifold A^k with simply connected boundary ∂A^k . We say A^k is *trivialized by V* if V is unbounded and $A^k \times V \approx D^k \times V$, where D^k is the closed k -disk.

A manifold is *closed* if it is compact and unbounded. The *interior* of X is $\text{Int } X = X - \partial X$.

THEOREM 5. *Let A^5 be a homotopy 5-cell and V^j a closed manifold. Then A^5 is trivialized by V^j if $j \geq 1$.*

PROOF. Let $D^5 \subset \text{Int } A^5$ be a smoothly embedded disk. It is clear that A^5 is a *simple neighborhood* of D^5 in the sense of Mazur, and that $A^5 \times V^j$ is a simple neighborhood of $D^5 \times V^j$. Unfortunately the s -cobordism theorem of Mazur [4] and Stallings (unpublished) is not valid in dimension five. It is valid in higher dimensions, however. Therefore Mazur's "simple neighborhood theorem" shows that $A^5 \times V^j \approx D^5 \times V^j$.

THEOREM 6. *A homotopy 4-sphere M^4 is trivialized by V^j if any one of the following conditions holds:*

- (a) $j \geq 1$ and V^j is closed;
- (b) $j \geq 1$ and $V^j = R^j$;
- (c) $j \geq 2$ and $V^j = D^j$.

PROOF. It is known that M^4 bounds a homotopy 5-cell; this is the theorem of Kervaire and Milnor [3] that $\theta^4 = 0$. Therefore (a) is implied by Theorem 5. In particular, $M^4 \times S^1 \approx S^4 \times S^1$, and consequently $M^4 \times R^1 \approx S^4 \times R^1$, proving (b).

To prove (c), observe that $\partial(M^4 \times D^j) = M^4 \times S^{j-1} \approx S^4 \times S^{j-1}$ by (a). Therefore $S^4 \times D^j$ can be diffeomorphically embedded in the interior

of $M^4 \times D^j$, and the simple neighborhood theorem [4] applies if $j \geq 2$, proving $M^4 \times D^j \approx S^4 \times D^j$.

THEOREM 7. *A homotopy 4-cell A^4 is trivialized by the closed manifold V^j if either*

- (a) $j \geq 2$, or
- (b) $j = 1$ and $\partial A^4 = S^3$.

PROOF. If $j \geq 2$, we see that $A^4 \times V^j$ is a simple neighborhood of $D^4 \times V^j$, where $D^4 \subset \text{Int } A^4$. If $j = 1$ and $\partial A^4 = S^3$, let M^4 be a homotopy 4-sphere obtained by gluing the boundary of a 4-disk B^4 onto ∂A^4 diffeomorphically. We may assume $V^j = S^1$.

By Theorem 6, we know that $M^4 \times S^1 \approx S^4 \times S^1$. Observe that

$$A^4 \times S^1 = M^4 \times S^1 - \text{Int } (B^4 \times S^1),$$

and we may think of $B^4 \times S^1$ as being a tubular neighborhood of $0 \times S^1$ in $M^4 \times S^1$. Similarly, $D^4 \times S^1$ is the complement of the interior of a tubular neighborhood of $x \times S^1 \subset S^4 \times S^1$, for some $x \in S^1$. Thus to prove (b), it suffices to prove that if $f, g: S^1 \rightarrow S^4 \times S^1$ are homotopic embeddings, there is a diffeomorphism of $S^4 \times S^1$ carrying a tubular neighborhood of $f(S^1)$ onto a tubular neighborhood of $g(S^1)$. Because of the dimensions involved, f and g are isotopic, and by the isotopy extension theorem of Thom (unpublished) and Palais [5], the isotopy can be extended to all of $S^4 \times S^1$. Now (b) follows from the uniqueness of tubular neighborhoods.

The next two theorems present similar facts about homotopy 3-spheres and 3-cells. Some of these are merely particular cases of theorems of Smale, and are included for the sake of completeness.

THEOREM 8. *A homotopy 3-sphere M^3 is trivialized by*

- (a) S^j for $j \geq 2$;
- (b) D^j for $j \geq 3$.

If M^3 bounds a homotopy 4-cell, then M^3 is trivialized by

- (c) *the closed manifold V^j for $j \geq 2$.*

PROOF. Clearly (b) implies (a). To prove (b), first embed S^3 in $M^3 \times D^j$ by a smooth embedding which is a homotopy equivalence; this can be done if $j \geq 3$ according to Haefliger [1]. Now observe that the normal bundle of S^3 in $M^3 \times D^j$ is trivial, and apply the simple neighborhood theorem [4]. Part (c) follows from Theorem 7 (a).

REMARK. It is easy to see that Poincaré's conjecture in dimension 3 is true if it is true for boundaries of homotopy 4-cells.

THEOREM 9. A homotopy 3-cell A^3 is trivialized by

- (a) a closed manifold V^j if $j \geq 3$;
- (b) S^2 .

PROOF. The proof is similar to that of Theorem 7, and is omitted.

3. Complements of manifolds

For any space Y , let $\Delta: Y \rightarrow Y \times Y$ be the diagonal embedding $\Delta(y) = (y, y)$.

THEOREM 10. Let M^4 be a homotopy 4-sphere. Then

$$M^4 \times M^4 - \Delta(M^4) \approx S^4 \times S^4 - \Delta(S^4).$$

PROOF. From Theorem 6 (a) we see that there is a diffeomorphism $f: M^4 \times M^4 \rightarrow S^4 \times S^4$. The proof is complete if we can show that

$$M^4 \times M^4 - \Delta(M^4) \approx M^4 \times M^4 - f^{-1}\Delta(S^4).$$

This is a consequence of the following theorem.

THEOREM 11. Let $\varphi: M_0 \rightarrow M_1$ be a homotopy equivalence between closed m -manifolds. For $i = 0, 1$ let $g_i: M_i \rightarrow N$ be diffeomorphic embeddings in an unbounded n -manifold N , such that $g_0 \simeq g_1\varphi$. Then

$$N - g_0(M_0) \approx N - g_1(M_1),$$

provided $2n > 3(m + 1)$. Moreover, the diffeomorphism is isotopic in N to the inclusion $N - g_0(M_0) \rightarrow N$.

PROOF. Let us identify M_i with the submanifolds $g_i(M_i)$ of N , so that g_i is the inclusion $M_i \rightarrow N$. Let $X_1 \supset X_2 \supset \dots$ be a decreasing sequence of open tubular neighborhoods of M_0 , such that $\text{Cl}(X_{j+1}) \subset X_j$ for $j = 1, 2, \dots$, and $\bigcap X_j = M_0$, where Cl means closure. Let $Y_1 \supset Y_2 \supset \dots$ be a similar sequence of open tubular neighborhoods of $M_1 = \varphi(M_0)$.

We shall use Haefliger's work [1] on embeddings and isotopy. Because of the assumption on dimension, any map of M_i in an open subset of N which is a homotopy equivalence is homotopic to an embedding; any two homotopic embeddings of M_i in an open subset U of N are isotopic by an isotopy of U which is fixed outside of a compact set.

Let $\xi_0 = \eta_0 = \eta_1$ be the identity map of N .

Since $g_1\varphi: M_0 \rightarrow Y_1$ is a homotopy equivalence, there is an embedding $h: M_0 \rightarrow Y_1$ with $h \simeq g_1\varphi$. Since $h \simeq g_0$ in N , there is an isotopy of N

carrying g_0 into h . Thus we obtain a diffeomorphism $\xi_1: N \rightarrow N$ enjoying the following properties:

- (P_{ξ_1}) ξ_1 is isotopic to the identity (symbolized by $\xi_1 \cong 1$).
- (Q_{ξ_1}) $\xi_1(X_1) \subset Y_1$.
- (R_{ξ_1}) $\xi_1: M_0 \rightarrow Y_1$ and $\varphi: M_0 \rightarrow Y_1$ are homotopic in Y_1 .

Similar considerations show that there is a diffeomorphism $\eta_2: N \rightarrow N$ such that:

- (P_{η_2}) $\eta_2 \cong 1$, the isotopy being fixed outside of Y_1 .
- (Q_{η_2}) $\eta_2(Y_2) \subset \xi_1(X_2)$.
- (R_{η_2}) $\eta_2: M_1 \rightarrow \xi_1(X_1)$ and $\xi_1\varphi^{-1}: M_1 \rightarrow \xi_1(X_1)$ are homotopic in $\xi_1(X_1)$.

Continuing recursively in this manner, we obtain two sequences $\{\xi_k\}$, $\{\eta_k\}$ of diffeomorphisms of N , $k = 1, 2, \dots$. These have the following properties. Put $\xi_k \cdot \dots \cdot \xi_1 = \xi^k$ and $\eta_k \cdot \dots \cdot \eta_1 = \eta^k$. Then for $k \geq 1$:

- (P_{ξ^k}) $\xi^k \cong 1$, the isotopy being fixed outside of $\xi^{k-1}(X_k)$.
- (P_{η^k}) $\eta^k \cong 1$, the isotopy being fixed outside of $\eta^{k-1}(Y_k)$.
- (Q_{ξ^k}) $\xi^k(X_k) \subset \eta^k(Y_k)$.
- (Q_{η^k}) $\eta^k(Y_k) \subset \xi^{k-1}(X_k)$.
- (R_{ξ^k}) ξ^k and $\eta^k\varphi$ are homotopic as maps of M_0 in $\eta^k(Y_k)$.
- (R_{η^k}) η^k and $\eta^{k-1}\varphi^{-1}$ are homotopic as maps of M_1 in $\xi^{k-1}(X_k)$.

From these properties it follows that $\xi^k = 1$ on $N - X_{k-1}$, and $\eta^k = 1$ on $N - Y_{k-1}$. Hence for $x \in N - M_0$ and $y \in N - M_1$, the sequences $\{\xi^k(x)\}$ and $\{\eta^k(y)\}$, $k = 1, 2, \dots$, are constant for k sufficiently large. Define $\xi(x) = \lim_{k \rightarrow \infty} \xi^k(x)$ and $\eta(y) = \lim_{k \rightarrow \infty} \eta^k(y)$. Then $\xi: N - M_0 \rightarrow N$ and $\eta: N - M_1 \rightarrow N$ are diffeomorphic embeddings. It is easily proved that ξ and η have the same image, namely

$$\cup_k(N - \xi^k(X_k)) = \cup_k(N - \eta^k(Y_k)).$$

Thus $\eta^{-1}\xi: N - M_0 \rightarrow N - M_1$ is the desired diffeomorphism.

4. Combinatorial homotopy 5-spheres

Throughout this section, let M be a combinatorial homotopy 5-sphere. It is known that M is homeomorphic to S^5 , but whether $M \equiv S^5$ is still unknown. In theorem 4 (see section 1) we present a sufficient (and trivially necessary) condition for this to occur, namely that M bound a compact orientable combinatorial 6-manifold V .

PROOF OF THEOREM 4. We may assume V is connected. We then make V simply connected, if necessary, by surgery [3]. (Those who feel queasy about surgery on combinatorial manifolds may wish to endow V with a compatible differential structure in a neighborhood of the 1-skeleton. This can easily be done directly, or [2] can be used.)

Assuming that V is simply connected, we proceed to build up a compatible differential structure in V over neighborhoods of successive skeletons. For this purpose the results of the author's paper [2] are used. Assuming that there is a compatible differential structure in a neighborhood of the k -skeleton, there exists one in a neighborhood of the $(k+1)$ -skeleton provided $H^{k+1}(V, \Gamma_k) = 0$. It is known that the group $\Gamma_k = 0$ for $k \leq 3$ and $k = 5$. The assumption that V is simply connected implies (by duality) that $H^5(V, \Gamma_4) = 0$. Therefore V has a compatible differential structure. Consequently M has one too. It follows from Smale [6] that $M \equiv S^5$, QED.

ADDED IN PROOF. The recent theorem $\Gamma_4 = 0$ of J. Cerf implies that always $M \equiv S^5$.

REFERENCES

- [1] A. HAEFLIGER, Plongements différentiables de variétés dans variétés, *Comm. Math. Helv.*, 36 (1961), p. 46-82.
- [2] M. HIRSCH, Obstruction theories for smoothing manifolds and maps, *Bull. Amer. Math. Soc.*, 69 (1963), p. 352-356.
- [3] M. Kervaire and J. Milnor, Groups of homotopy spheres (to appear).
- [4] B. MAZUR, Simple neighborhoods, *Bull. Amer. Math. Soc.*, 68 (1962), p. 87-92.
- [5] R. S. PALAIS, Local triviality of the restriction map for embeddings, *Comm. Math. Helv.*, 34 (1960), p. 305-312.
- [6] S. SMALE, Differentiable and combinatorial structures on manifolds, *Annals of Math.*, 73 (1961), p. 498-502.
- [7] ———, Generalized Poincaré's conjecture in dimensions greater than 4, *Annals of Math.*, 73 (1961), p. 391-406.
- [8] J. STALLINGS, Polyhedral homotopy spheres, *Bull. Amer. Math. Soc.*, 66 (1960), p. 485-488.
- [9] E. C. ZEEMAN, The generalized Poincaré conjecture, *Bull. Amer. Math. Soc.*, 67 (1961), p. 270.