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Representation Theory, Algebraic Geometry and Supersymmetric Field Theories in Low Dimensions

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Representation Theory, Algebraic Geometry and Supersymmetric Field Theories in Low Dimensions

By WENJUN NIU

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I dedicate this paper to my best friend Don Manuel.

Contents

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Representation Theory, Algebraic Geometry and Supersymmetric Field Theories in Low Dimensions

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Abstract

We study the category of line operators in specific topological, or holomorphic-topological twists of supersymmetric quantum field theories in dimension 3 and 4. More specifically, we focus on topological A and B twist of 3d $\mathcal{N}=4$ gauge theories and holomorphic-topological twists of 4d $\mathcal{N}=2$ theories. We use geometric and representation-theoretic tools to define and study these categories, and prove physical conjectures about these cartegories and their relations.

We study the category of line operators in the topological twist of a 3d $\mathcal{N}=4$ abelian gauge theory $\mathcal{T}_\rho$. We complete the analysis of the boundary vertex operator algebras of Costello-Gaiotto, which results in boundary VOAs $V_{A,\rho}$ and $V_{B,\rho}$. We obtain explicit free field realizations of these boundary VOAs and use the free field realizations to prove the isomorphism $V_{A,\rho} \cong V_{B,\rho^\vee}$, which we interpret as the mirror symmetry statement in terms of the boundary VOAs. We then use the theory of logarithmic intertwining operators to define braided tensor categories $\mathcal{L}_{A,\rho}$ and $\mathcal{L}_{B,\rho}$ of modules of $V_{A,\rho}$ and $V_{B,\rho}$ (as derived categories). We propose that these are the categories of line operators for the A and B twist of the 3d $\mathcal{N}=4$ theory $\mathcal{T}_\rho$. Using the isomorphism of VOAs, we prove equivalence of braided tensor categories $\mathcal{L}_{A,\rho} \simeq \mathcal{L}_{B,\rho^\vee}$, which we interpret as the mirror symmetry statement in terms of the category of line operators. Finally, we show that $V_{B,\rho}$ admit a sheafification over the Higgs branch $\mathcal{M}_{H,\rho}$, whose construction is related to the tangent Lie algebra.

We study the category of line operators in the holomorphic-topological twist of a 4d $\mathcal{N}=2$ gauge theory $T_{HT}[G,V]$. This category is given a geometric description by Cautis-Williams, following the work of Kapustin, as $\mathrm{Coh}(G(\mathcal{O}) \setminus \mathcal{R}_{G,V})$. Using the idea of formal geometry, we compute the derived endomorphism of the unit object $\mathbb{1}$ and show that it is quasi-isomorphic to the Poisson vertex algebra of Oh-Yagi, and that its graded super-trace reproduces the Schur index. Using

the same method, we compute the derived homomorphism between line bundles supported on the miniscule orbits of Gr_G, in the case when $G = PSL(2)$. We compare the graded super-trace of the results with the defect Schur indices of Cordova-Gaiotto-Shao.

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CHAPTER 1

Introduction

1.1. Quantum Field Theory and Mathematics

The theme of this thesis is the study of mathematical structures rising from *topological* or *holomorphic topological* quantum field theories (QFT) in 3 and 4 dimensions. The goal is to use representation theory and algebraic geometry to pin down the precise mathematical structure of line operators and local operators, and via the study of these line and local operators, with the help of physical intuitions, gain new insights into the representation theory and algebraic geometry involved.

Historically, the advancement of mathematics informs and leads to advancements in physics, and in this sense, the flow of information had been one-sided. In the past few decades, however, the mathematical community have seen a surge of information flowing in from the study of physics. Constructions from quantum field theories and string theories have led to very deep conjectures and results in mathematics.

One of the most fruitful area of such connections is that of topological quantum field theories (TQFT). Since the ground-breaking work of Witten [**Wit82**], there has been an intensive development in the study of such theories, and their relations to topology, geometry and representation theory. In this context, the physics intuition leads to the precise mathematical definition of TQFT [**Ati88**], which becomes an essential tool for the study of low-dimensional topology. On the other hand, the mathematical definition of TQFT allows one to express physical quantities (correlation functions, Hilbert spaces, etc) very precisely [**RT91**].

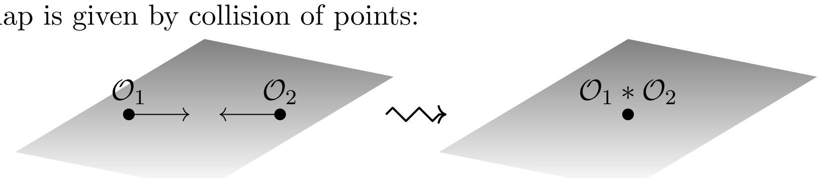
Another beautiful feature of quantum field theory is the existence of *dualities*, which is a complex network of relations (and many times equivalences) between different quantum field theories. One of the first of such examples is the Electric-Magnetic duality [**MO77**]. Although these dualities are usually proved in physics using non-rigorous methods (e.g., Feymann integrals, SUSY localization, etc), they suggest corresponding relations between the rigorous mathematical structures that one can extract from them, which in most cases are far beyond reach without the intuition of physics. Some such examples include:

- T-duality of 2d CFT [**Bus88**] and mirror symmetry [**SYZ96**].
- 3d mirror symmetry [**IS96**] and symplectic duality [**BLPW16**].
- S-duality of 4d $\mathcal{N} = 4$ theories [**MO77**] and geometric Langlands program [**KW06**].

It is only expected that connections as such will be more and more fruitful and benefit both mathematicians and physicists. This thesis serves as a small drop of contribution into the ocean of works and ideas devoted to this area of research. We now come to introduce the central object of study of this thesis: line observables in topological or holomorphic-topological field theories. Since this is a mathematics thesis, a complete review of the theory of observables in QFT is beyond the scope of this work. Instead, we will give a rather heuristic introduction to them, focusing on the physical intuitions involved.

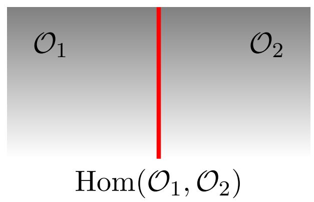
1.2. Line Operators and Local Operators

A TQFT $\mathcal{T}$ whose space-times manifold M is d-dimensional gives rise to a series of algebraic structures on the set of observables in $\mathcal{T}$. These algebraic structures come in different layers. The set of local observables $\mathrm{Ops}(m)$ at a point $m \in M$ will possess the structure of an algebra, whose multiplication map is given by collision of points:



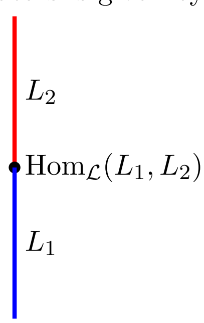
Since the theory $\mathcal{T}$ is topological, such an operation is well-defined, and defines an algebra structure on $\mathrm{Ops}(m)$. Physics in fact remembers more than the multiplication, it also remembers that there are d different directions in which the two local operators can collide. The structure of such d different multiplications that behave coherently with respect to each other leads to the definition of an $\mathbb{E}_d$ algebra. In a word, local observables in a TQFT has the structure of an $\mathbb{E}_d$ algebra. When the ground field is $\mathbb{C}$, such an algebra is shown to be equivalent to a commutative shifted Poisson algebra with a degree $1-d$ Poisson bracket. If d is odd and one only consider the underlying algebra, then this is a Poisson algebra that is the central object in the study of symplectic algebraic geometry.

For higher-dimensional defects, the structure of collision still exists, although there will be less directions of collision. On the other hand, higher-dimensional observables will have higher categorical structure. The Hom between two k-dimensional observable is the set of $k-1$ dimensional observables at the intersection:

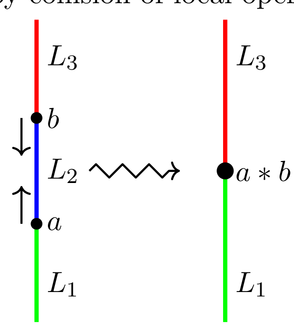


In general, if $S \subset M$ is a k-dimensional subspace of M, then the set of observables $\mathrm{Ops}(S)$ supported on S will have the structure of a k-category, and a compatible $\mathbb{E}_{d-k}$ multiplication, which leads to the mathematical definition of a $\mathbb{E}_{d-k}$ k-category.

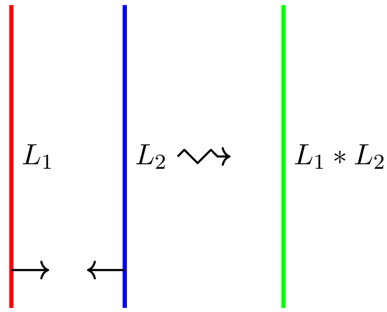
For example, when $d = 2$ and $k = 1$, the set $\mathrm{Ops}(S)$ is the category of line operators $\mathcal{L}$ in a 2d TQFT. The Hom between two line operators is given by adjunctions of local operators:



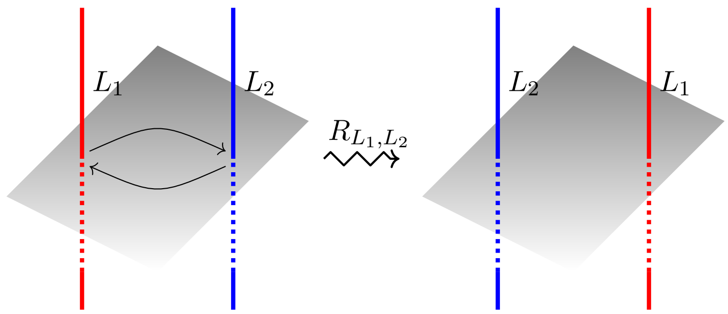
Composition of morphisms is given by collision of local operators on the line:



The $\mathbb{E}_1$ structure, which is a monoidal structure, comes from collision:



When $d = 3$ and $k = 1$, the category of line operators will not only have a monoidal structure, but multiplication in two different directions. The mathematical equivalence of this structure is a *braided tensor category* (BTC), where the braiding comes from interchanging the location of lines:



In this thesis, we will consider also *holomorphic-topological* (HT) theories defined on $M \times \mathbb{C}$ where the theory is topological on M and holomorphic on $\mathbb{C}$. The observables $\mathrm{Ops}(S \times \{z\})$ supported on $S \subseteq M$ and point-like along $\mathbb{C}$ will have its usual $\mathbb{E}_{d-k}$ k-category structure, but also a chiral structure from $\mathbb{C}$, which is a multiplication depending holomorphically on insertions on $\mathbb{C}$, with possible singularities forming as the insertions collide. When M is zero-dimensional, this is the structure of a *vertex operator algebra* (VOA), and is the central object for a conformal field theory.

These observables of different dimensions can be related to each other. For each k, the category $\mathrm{Ops}(\mathbb{R}^k)$ has a distinguished object $\mathbb{1}_k$, which is the trivial k-dimensional observable put on $\mathbb{R}^k$. The interface between two trivial k-dimensional observables is simply the set of observables supported on $\mathbb{R}^{k-1}$. In a word, we have:

$$\mathrm{Hom}_{\mathrm{Ops}(\mathbb{R}^k)}(\mathbb{1}_k, \mathbb{1}_k) \simeq \mathrm{Ops}(\mathbb{R}^{k-1}). \tag{1.2.0.1}$$

Therefore, one can obtain structures of lower dimensional observables from higher dimensional ones, and in principle, for a d-dimensional TQFT, the entire quantum field theory will be encoded in a d category. However, it is in general difficult to determine the correct d category. On the other hand, if we understand the $k-1$ dimensional observables, then we can recover part of the higher dimensional data since the functor $\mathrm{Hom}_{\mathrm{Ops}(\mathbb{R}^k)}(\mathbb{1}_k, -)$ gives an equivalence between the subcategory of $\mathrm{Ops}(\mathbb{R}^k)$ generated by $\mathbb{1}_k$ with $\mathrm{Ops}(\mathbb{R}^{k-1})-\mathrm{Mod}$. This however is usually not enough. For example, in 3d Chern-Simons theory, the category of line operators is $\mathrm{Rep}_q(G)$, and when G is reductive, the category generated by the identity line operator is a category of vector spaces, and does not contain other Wilson line operators in the theory. There are examples in which this functor gives rise to interesting representations of the set of lower-dimensional observables, which lead to the bottom-up approach of TQFT outlined in [**But21**]. In this thesis, we will take a top-down approach, namely we will use the category of line operators to compute the space of local operators.

In this thesis, we consider the following two class of theories:

(1) A and B twist of 3d $\mathcal{N} = 4$ theories, which are TQFT whose category of line operators form a BTC.

(2) Kapustin (HT) twist of 4d $\mathcal{N} = 2$ theories, which are HT QFT whose category of line operators form a monoidal chiral category.

We now proceeed to introduce these theories and the approach we will take in studying the category of line operators in them.

1.3. Line Operators in 3d $\mathcal{N} = 4$ Gauge Theories

Given a complex group G and a complex representation V of G, physicists have defined a 3 dimensional quantum field theory with $\mathcal{N} = 4$ supersymmetry, which will be denoted by $\mathcal{T}[G, V]$. The theory has an HT twist, which require the spacetime manifold to be locally of the form $\mathbb{R} \times \mathbb{C}$. The HT twisted theory $\mathcal{T}_{HT}[G, V]$ can be further deformed to two topological theories, the topological A twist $\mathcal{T}_A[G, V]$ and the B twist $\mathcal{T}_B[G, V]$, also called the Rozansky-Witten twist.

(1.3.0.1)
$$\begin{array}{ccc} & \mathcal{T}_{HT}[G,V] & \\ \swarrow & & \searrow \\ \mathcal{T}_A[G,V] & & \mathcal{T}_B[G,V] \end{array}$$

These topological theories do not yet admit a full TQFT description, but some part of their structures has now been unveiled and many beautiful mathematical statements followed.

The space of local operators in both $\mathcal{T}_A[G,V]$ and $\mathcal{T}_B[G,V]$ form (-2)-shifted Poisson algebras. For the B-twist, this algebra is the algebra of functions on a Poisson variety called the Higgs branch, which will be denoted by $\mathcal{M}_{H,G,V}$. This variety is defined as the symplectic reduction of T^*V by G. For the A-twist, the space of local operators is the algebra of functions on another Poisson variety called the Coulomb branch, denoted by $\mathcal{M}_{C,G,V}$. For a long time, physicists have predicted much of its properties, for example, [**GMN13a**, **GMN13b**, **BDGH16**, **BDG17**, **DGGH20**], but not a precise mathematical definition, since the non-perturbative analysis is difficult. Recently, this space has been given a precise definition in [**Nak16**,**BFN18**] using the Borel-Moore homology of an infinite-dimensional variety, and the result there is inspired by the physical analysis of [**DGGH20**].

The idea of [**DGGH20**] is to derive the space of local operators from the category of line operators. Let us denote by $\mathcal{L}_A[G,V]$ and $\mathcal{L}_B[G,V]$ the category of line operators for the two topological theories. In each twisted theory, there is a distinguished object $\mathbb{1}$, the tensor identity of $\mathcal{L}$. The space of local operators, namely $\mathbb{C}[\mathcal{M}]$, can be realized as the endomorphism:

$$\mathbb{C}[\mathcal{M}] \cong \mathrm{End}_{\mathcal{L}}(\mathbb{1}). \tag{1.3.0.2}$$

It was argued based on physical ground [**DGGH20**] that:

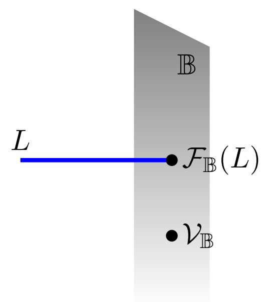
(1) $\mathcal{L}_A[G,V] \sim \text{D-Mod}\,(V(\mathcal{K})/G(\mathcal{K})) = \mathrm{Coh}(\mathrm{Maps}(\mathbb{D}^*, V/G)_{dR})$.

(2) $\mathcal{L}_B[G,V] \sim \mathrm{Coh}(\mathrm{Maps}(\mathbb{D}^*_{dR}, V/G))$.

Here X_{dR} denotes the de-Rham stack of X, which can be thought of as the algebra of algebraic de-Rham complex of X. Un-packing the above definitions, the category $\mathcal{L}_A[G,V]$ is the category of strongly $G(\mathcal{K})$-equivariant modules of the algebra of differential operators on $V(\mathcal{K})$, or perverse sheaves on $V(\mathcal{K})/G(\mathcal{K})$, while the category $\mathcal{L}_B[G,V]$ is the category of coherent sheaves on the moduli space of G-local systems on $\mathbb{D}^*$ with a section on the associated V-bundle. The problem is that it is difficult to utilize these definitions as the spaces involved are infinite-dimensional, and that it is almost impossible to understand the braided-tensor structure of the categories, which is something very crucial to the Poisson geometry of the branches $\mathcal{M}$.

In this thesis, we focus on the case when $G=(\mathbb{C}^\times)^r$ a torus, and V is a representation defined by a charge matrix $\rho:\mathbb{Z}^r\to\mathbb{Z}^n$. The approach in this thesis uses holomorphic boundary conditions of [**CG19**]. In this work, the authors defined boundary conditions of 3d $\mathcal{N}=4$ gauge theories that are compatible with the topological A and B twists, such that the twisted theory on the boundary is holomorphic. In this case, local operators on the boundary form a VOA.

The way to use the boundary VOA to access the category of line operators is as follows. Given a 3d topological QFT on $\mathbb{C}_z\times\mathbb{R}_{t\geq 0}$ with a holomorphic boundary condition $\mathbb{B}$ at $t=0$, supporting a VOA $\mathcal{V}_{\mathbb{B}}$, and a line operator L of the bulk theory, positioned perpendicular to the plane $t=0$ and supported at $z=0$. A picture is as follows:



Let $\mathcal{F}_{\mathbb{B}}(L)$ be the space of local operators that can be inserted at the junction of L and the boundary condition $\mathbb{B}$. It is acted upon by other boundary local operators, and thus is a module for $\mathcal{V}_{\mathbb{B}}$. Collision and braiding of line operators in the bulk are expected to be compatible with collision and braiding of modules of the boundary VOA, which are given by intertwining operators. Thus, one expects there to be a functor of BTC:

$$\mathcal{F}_{\mathbb{B}}:\mathcal{L}\to D^b(\mathcal{V}_{\mathbb{B}}\text{-Mod}). \tag{1.3.0.3}$$

This basic setup arises in Chern-Simons theory, with a holomorphic boundary condition supporting the WZW VOA [**Wit89**, **EMSS89**]. In good cases, it is expected that this is an equivalence of categories. According to the computations in [**CCG19**], it is expected that the boundary conditions of [**CG19**] are good in this sense so that one can describe line operators using the boundary VOA modules.

In this thesis, we will carefully define and study the boundary VOA $\mathcal{V}_{\mathbb{B}}$ supported on the holomorphic boundary conditions of [**CG19**]. We will define a category of modules that has the structure of a braided tensor category via logarithmic intertwining operators. This will lead to the definition of the category $\mathcal{L}_{A,\rho}$ and $\mathcal{L}_{B,\rho}$ as braided tensor categories.

1.4. Line Operators in 4d $\mathcal{N}=2$ Gauge Theories

The set-up is similar to 3d $\mathcal{N}=4$ theories. Let G be a complex Lie group and V a representation of G. Associated to this data is a 4 dimensional quantum field theory $T[G,V]$ with $\mathcal{N}=2$ supersymmetry. In [**Kap06a**], the author introduced a holomorphic-topological (HT) twist, which requires spacetime to locally take the form $\mathbb{R}^2\times\mathbb{C}$, and depends topologically on $\mathbb{R}^2$ and holomorphically on $\mathbb{C}$. The HT twisted theory $T_{HT}[G,V]$ is related to the 3d theory $\mathcal{T}_{HT}[G,V]$ via dimensional reduction. More precisely, if one put the theory $T_{HT}[G,V]$ on a spacetime of the form $\mathbb{R}\times S^1\times\mathbb{C}$ and treat the theory as a 3d theory on $\mathbb{R}\times\mathbb{C}$, then it is equivalent to $\mathcal{T}_{HT}[G,V]$.

The vacuum of the theory $T_{HT}[G,V]$ goes by the name of K-theoretic Coulomb branch, and is very important to the study of representations of affine Yangians [**BFM05**,**BFN19**,**CW19**,**FT19**]. The space of local operators in $T_{HT}[G,V]$ is a Poisson vertex algebra [**OY20**,**But21**], and in special cases, can be further deformed to a VOA [**BLL$^+$15**].

In this thesis, we derive this Poisson algebra using the category of line operators. This category is given a precise mathematical definition in [**CW19**] for pure gauge theory ($V=0$) and [**CWar**] for general V, following the physical predictions of [**Kap06a**,**Kap06b**]. It is shown in [**CW19**,**CWar**] that the category of line operators of $T_{HT}[G,V]$ has the structure of a chiral monoidal category. We will show that the derived endomorphism algebra:

$$\mathrm{End}\,(\mathbb{1}) \tag{1.4.0.1}$$

is quasi-isomorphic to the Poisson vertex algebra of [**OY20**, **But21**] as an algebra. The main tool in this computation is formal geometry in the derived setting, which was established in the work of [**GR14**, **GR19**, **GR17**]. The same method allows us to also compute the derived endomorphism between objects supported on other miniscule orbits of Gr_G, and we will give one example of this for $G=\mathrm{PGL}(2)$.

1.5. Organizations and Results

1.5.1. Overview of Chapter 2. In Chapter 2, we will recall known facts, and results to be studied about the two class of twisted theories that are considered in this thesis. For 3d $\mathcal{N}=4$ theories, we will focus on abelian gauge theories, and in particular, we will recall the content of 3d mirror symmetry for abelian $\mathcal{N}=4$ gauge theories. We will also recall known facts about Kapustin (HT) twist of 4d $\mathcal{N}=2$ theories, and the algebro-geometric formulation of the category of line operators. The content of this chapter is based on [**BCDN23**, **Niu21**].

1.5.2. Overview of Chapter 3. In Chapter 3, we will define boundary VOA $V_{A,\rho}$ and $V_{B,\rho}$ for both A and B twist of 3d $\mathcal{N}=4$ abelian gauge theory defined by a charge matrix ρ. They are roughly defined as following:

- The VOA $V_{A,\rho}$ is a BRST reduction of many copies of symplectic bosons $V_{\beta\gamma}^{\otimes n}$.
- The VOA $V_{B,\rho}$ is an extension of an affine Lie superalgebra $V(\mathfrak{g}_*(\rho))$.

We construct free field realizations of the above VOAs, namely embeddings of them into extension of Heisenberg VOA by Fock modules. Using such free field realization, we show:

Theorem 1.5.1 (Theorem 3.1.10). *When ρ and $\rho^\vee$ define 3d mirror dual gauge theories, then there are isomorphisms of VOA:*

$$V_{A,\rho} \cong V_{B,\rho^\vee} \qquad V_{B,\rho} \cong V_{A,\rho^\vee}. \tag{1.5.2.1}$$

We will then define categories of modules of $V_{A,\rho}$ and $V_{B,\rho}$, which will be denoted by $\mathcal{C}_{A,\rho}$ and $\mathcal{C}_{B,\rho}$. The bounded derived categories $D^b\mathcal{C}_{A,\rho}$ and $D^b\mathcal{C}_{B,\rho}$ will be proposed as the category of line operators. A feature of these categories is that they are highly non-semisimple, and therefore hard to deal with in VOA theory. We apply the idea of simple current extensions of [**CKM17**, **CMY22a**] to show that $\mathcal{C}_{A,\rho}$ and $\mathcal{C}_{B,\rho}$ have the structure of braided tensor categories via the theory of logarithmic intertwining operators. We prove:

Theorem 1.5.2 (Theorem 3.2.10). *When ρ and $\rho^\vee$ define 3d mirror dual gauge theories, then there are equivalences of braided tensor categories:*

$$\mathcal{C}_{A,\rho} \simeq \mathcal{C}_{B,\rho^\vee}, \qquad \mathcal{C}_{B,\rho} \simeq \mathcal{C}_{A,\rho^\vee} \tag{1.5.2.2}$$

Analyzing the content of the category $\mathcal{C}_{B,\rho}$ via simple current extensions, we will derive a quantum group $U_q(\mathfrak{g}_*(\rho))$, whose representation theory is equivalent to $\mathcal{C}_{B,\rho}$. We show that $U_q(\mathfrak{g}_*(\rho))-\mathrm{Mod}_{\mathrm{fin}}$, the category of finite-dimensional modules, has the structure of a braided tensor category, and present the following conjecture:

CONJECTURE 1.5.3 (Conjecture 3.2.12). *There is an equivalence of braided tensor categories:*

$$\mathcal{C}_{B,\rho} \simeq U_q(\mathfrak{g}_*(\rho))-\mathrm{Mod}_{\mathrm{fin}}. \tag{1.5.2.3}$$

Using this quantum group, we show that we can obtain the correct algebra of local operators from $D^b\mathcal{C}_{B,\rho}$:

THEOREM 1.5.4 (Theorem 3.3.1). *Let $\mathbb{1}$ be the identity object in $\mathcal{C}_{B,\rho}$, then there is a quasi-isomorphism of algebras:*

$$\mathrm{End}^*_{D^b\mathcal{C}_{B,\rho}}(\mathbb{1}) \cong \mathbb{C}[\mathcal{M}_{H,\rho}]. \tag{1.5.2.4}$$

Finally, we show that the VOA $V_{B,\rho}$ (and $V_{A,\rho}$) admits filtered version $V_{B,\rho}^{\hbar}$, which is naturally a sheaf of VOA on the symplectic quotient $T^*V/\!/\!/G$. The parameter $\hbar$ is interpreted as the cohomological grading in the category $\mathrm{QCoh}(T^*V/\!/\!/G)$. The limit $\hbar \to 0$ of $V_{B,\rho}^{\hbar}$, or in other words, $V_{B,\rho}^{\hbar}/\hbar V_{B,\rho}^{\hbar}$, is the boundary vertex algebra for the Dirichlet boundary condition for the HT twist $\mathcal{T}_{HT,\rho}$, and $V_{A,\rho}^{\hbar}/\hbar V_{A,\rho}^{\hbar}$ is the boundary vertex algebra for the Neumann boundary condition for the HT twist $\mathcal{T}_{HT,\rho}$. The sheaf of VOA unveils the Lie superalgebra $\mathfrak{g}_*(\rho)$ as the shifted tangent Lie algebra of $T^*V/\!/\!/G$. Combined with the work of [**Kuw21**], we prove the following:

THEOREM 1.5.5 (Theorem 3.3.23). *The $\hbar$-adic VOA $V_{B,\rho}^{\hbar}$ is naturally a sheaf over the product:*

$$\mathcal{M}_{H,\rho} \times \mathcal{M}_{H,\rho^\vee} \cong \mathcal{M}_{H,\rho} \times \mathcal{M}_{C,\rho}. \tag{1.5.2.5}$$

The content of this chapter is partially based on [**BCDN23**].

1.5.3. Overview of Chapter 4. In Chapter 4, we will compute the algebra of local operators of HT twist of 4d $\mathcal{N}=2$ gauge theory defined by G and V using the category of line operators. The category of line operators in this theory is the category of $G(\mathcal{O})$ equivariant coherent sheaves on $\mathcal{R}_{G,V}$, a space defined by the following Cartesian diagram:

$$\begin{array}{ccc} \mathcal{R}_{G,V} & \longrightarrow & V(\mathcal{O}) \\ \downarrow & & \downarrow i \\ G(\mathcal{K}) \times_{G(\mathcal{O})} V(\mathcal{O}) & \xrightarrow{m} & V(\mathcal{K}) \end{array} \tag{1.5.3.1}$$

The category $\mathrm{Coh}_{G(\mathcal{O})}(\mathcal{R}_{G,V})$ is a chiral monoidal category, whose chiral structure follows from the famous Beillinson-Drinfeld grassmannian [**BD**], and whose monoidal structure is defined via some convolution diagrams (see equation (2.1.1.20)). The space of local operators in this case has been computed based on physical arguments by [**OY20**,**But21**]. It is a Poisson vertex algebra $\mathcal{V}_{G,V}$ built from BRST cohomology of copies of degenerate symplectic bosons, and can be alternatively defined as functions on the infinite jet space $J_\infty T^*V/\!/\!/G$. Using derived algebraic geometry, especially the relation between formal groups and Lie algebras [**GR14**, **GR19**, **GR17**], we prove:

THEOREM 1.5.6 (Theorem 4.2.16). *Let* $\mathbb{1}$ *be the identity object in* $\mathrm{Coh}_{G(\mathcal{O})}(\mathcal{R}_{G,V})$*, then there is a quasi-isomorphism of algebras:*

$$\mathrm{End}^*_{\mathrm{Coh}_{G(\mathcal{O})}(\mathcal{R}_{G,V})}(\mathbb{1}) \cong \mathcal{V}_{G,V}. \tag{1.5.3.2}$$

Using the same method, we consider the computation of derived Hom between more general line operators. We will focus on the case when $G = \mathrm{PGL}(2)$ and compute the derived endomorphism of the structure sheaf of the miniscule orbit of Gr_G, and compare it to the physical computation of [**CGS16**]. This chapter is based on [**Niu21**].

CHAPTER 2

The Twisted Theories, Dualities and Their Mathematical Significance

In this chapter, we introduce the mathematical structures in the twisted QFT that will be the focus of the rest of the thesis. The structure of this chapter is as follows.

- In Section 2.1, we focus on the topological twists of $\mathcal{T}[G, V]$. We will introduce the Higgs and Coulomb branches as algebraic varieties, whose space of functions are the algebra of local operators in the topological twists. We then introduce the braided tensor category of line operators, and the boundary VOA approach that we will take to define them mathematically. Finally, we will introduce certain well-known VOAs that are building blocks of the VOAs in this thesis.
- In Section 2.2, we focus on the HT twist of 4d $\mathcal{N} = 2$ theory $T_{HT}[G, V]$. We will introduce the category of line operators and the space of local operators, as suggested from physical analysis.

2.1. 3d $\mathcal{N} = 4$ Theories, Boundary VOA and Braided Tensor Categories

2.1.1. Higgs and Coulomb Branches. Let G be a complex Lie group and V a complex finite dimensional representation of G. As we have discussed in the introduction, associated to this data is a 3d $\mathcal{N} = 4$ supersymmetric field theory, whose gauge group is G, and whose matter fields valued in the representation T^*V of G. This theory has an HT twist, and can be further deformed to two topological twists, as represented by the following picture:

(2.1.1.1)
$$\begin{array}{ccc} & \mathcal{T}_{HT}[G, V] & \\ \swarrow & & \searrow \\ \mathcal{T}_A[G, V] & & \mathcal{T}_B[G, V] \end{array}$$

The space of local operators $\mathrm{Ops}_A[G,V]$ and $\mathrm{Ops}_B[G,V]$ are $\mathbb{E}_3$ algebras, or (-2)-shifted Poisson algebras. Forgetting the grading, these are commutative Poisson algebras, and give rise to two (singular) Poisson varieties:

$$\mathcal{M}_{C,G,V} := \mathrm{Spec}(\mathrm{Ops}_A[G,V]), \qquad \mathcal{M}_{H,G,V} := \mathrm{Spec}(\mathrm{Ops}_B[G,V]). \tag{2.1.1.2}$$

The variety $\mathcal{M}_{C,G,V}$ is called the Coulomb branch and $\mathcal{M}_{H,G,V}$ is called the Higgs branch. These varieties, especially $\mathcal{M}_{H,G,V}$ and its resolutions, have been on the radar of mathematicians for a long time, and by now, their precise definitions as complex varieties are known. We will present their definitions in the following, and then focus on examples when $G=(\mathbb{C}^\times)^r$ is a torus.

The space $\mathcal{M}_{H,G,V}$ is more easily defined. The variety $T^*V=V\oplus V^*$ is symplectic with the standard symplectic form. The action of G on T^*V preserves the symplectic form, and there is a moment map:

$$\mu : T^*V \to \mathfrak{g}^* \tag{2.1.1.3}$$

such that:

$$\mu(v,v^*)(X) = (Xv, v^*), \quad \text{for all } X, v, v^*. \tag{2.1.1.4}$$

The map μ is a G-equivariant and generically flat, but in general not smooth. The zero fibre $\mu^{-1}(0)$ is a G-invariant subspace.

Definition 2.1.1. *The Higgs branch $\mathcal{M}_{H,G,V}$ is defined by:*

$$\mathcal{M}_{H,G,V} := \mu^{-1}(0)/\!\!/G, \tag{2.1.1.5}$$

where $\mu^{-1}(0)/\!\!/G$ denotes the geometric-invariant-theory (GIT) quotient. We also denote the above quotient by $T^*V/\!\!/\!\!/G$, called the hyper-Kähler quotient.

Remark 2.1.2. *In other words, $\mathrm{Ops}_B[G,V]=\mathbb{C}[\mu^{-1}(0)]^G$ is a Poisson algebra. Physics imposes some extra gradings on $\mathbb{C}[\mu^{-1}(0)]^G$ so that this Poisson structure is (-2)-shifted under this grading. As we will see in Section 3.3.1, it is natural that linear functions on T^*V are in degree 1, and the*

Poisson structure will be (-2)-shifted. Since we have no use for this grading in this thesis, we will not mention this grading again.

Example. *Consider the case when $G = \mathbb{C}^\times$ and $V = \mathbb{C}^2$ with weights 1 and 1. Denote by x_1 and x_2 functions on $\mathbb{C}^2$ and y_i the dual coordinates, then the map $\mu : T^*V \to \mathfrak{g}^*$ is defined by:*

$$\mu(x, y) = \sum x_i y_i. \tag{2.1.1.6}$$

Thus the subspace $\mu^{-1}(0)$ is the spectrum of the following ring:

$$A = \mathbb{C}[x_i, y_i]/(\sum x_i y_i). \tag{2.1.1.7}$$

The GIT quotient is the spectrum of $A^{\mathbb{C}^\times}$, the invariant part of A. The invariant part is clearly generated by $x_i y_j$ subjected to the condition $\sum x_i y_i = 0$. Thus the ring is generated by $e_1 = x_1 y_2, e_2 = x_2 y_1$ and $e_3 = i x_1 y_1$ subjected to:

$$e_1 e_2 = x_1 y_1 x_2 y_2 = -x_1^2 y_1^2 = e_3^2, \tag{2.1.1.8}$$

and no other conditions. The spectrum of this ring is recognized as the GIT quotient $\mathbb{C}^2/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action on $\mathbb{C}^2$ is given by $(x, y) \to (-x, -y)$. Indeed, under this isomorphism, e_1 is identified with x^2, e_2 with y^2 and e_3 with xy, all of which are $\mathbb{Z}_2$ invariant.

The space $\mathcal{M}_{H,G.V}$ has a Poisson structure inherited from T^*V. However, the space $\mathcal{M}_{H,G,V}$ is usually not smooth. One usually defines a variantion of this space via the choice of a stability condition, and under certain conditions, the variation will be smooth. More precisely, let $\xi \in \mathfrak{g}^*$ be a character of G, namely:

$$\xi \in \mathrm{Hom}(G, \mathbb{C}^\times), \tag{2.1.1.9}$$

A point $p \in T^*V$ is called ξ-**semistable** if there exists $m \in \mathbb{Z}_{>0}$ and a function $f \in \mathbb{C}[T^*V]^{G,m\xi}$ such that $f(p) \neq 0$. Here $f \in \mathbb{C}[T^*V]^{G,m\xi}$ means that f transforms under the action of G as $m\xi$. Denote by $(T^*V)^{ss}_\xi$ the subset of all semi-stable points, which is an open subvariety of T^*V. Now for any subset $S \subseteq T^*V$, we define an equivalence relation on S by declaring that $p \sim q$:

$$\overline{G \cdot p} \bigcap \overline{G \cdot q} \bigcap S \neq \emptyset. \tag{2.1.1.10}$$

DEFINITION 2.1.3. *We define the hypertoric variety $\mathcal{M}^{\xi}_{H,G,V}$ associated to the stability condition ξ by the quotient:*

$$\mathcal{M}^{\xi}_{H,G,V} := \left(\mu^{-1}(0) \cap (T^*V)^{ss}_{\xi}\right) / \sim . \tag{2.1.1.11}$$

REMARK 2.1.4. *We can alternatively define $\mathcal{M}^{\xi}_{H,G,V}$ as the following projective variety, which is much more user-friendly:*

$$\mathcal{M}^{\xi}_{H,G,V} := \mathrm{Proj}(\bigoplus_{n \geq 0} \mathbb{C}[\mu^{-1}(0)]^{G,m\xi}). \tag{2.1.1.12}$$

Localization of global sections give a map $\mathcal{M}^{\xi}_{H,G,V} \to \mathcal{M}_{H,G,V}$. Again, there is a Poisson structure on $\mathcal{M}^{\xi}_{H,G,V}$ that is induced from T^*V.

Example. *In our previous example, let us choose $\xi = 1$. In this case, the graded algebra*

$$\bigoplus_{n \geq 0} \mathbb{C}[\mu^{-1}(0)]^{G,m\xi} \tag{2.1.1.13}$$

is generated in degree 0 and degree 1 by:

$$x_i y_j \text{ in degree } 0, \;\; x_i \text{ in degree } 1. \tag{2.1.1.14}$$

The graded algebra generated by x_i is the graded algebra defining the projective variety $\mathbb{P}^1$, and one can show using local charts that the above projective variety is nothing but the variety $T^*\mathbb{P}^1$. The elements $x_i y_j$ are the global linear functions on $T^*\mathbb{P}^1$, or global sections of $O(2)$. From the perspective of semi-stability, we can see that $(T^*V)^{ss}_{\xi}$ consists of points where $x_i \neq 0$ for some i, and so:

$$\mu^{-1}(0) \cap (T^*V)^{ss}_{\xi} = \mu^{-1}(0) \setminus \{x_i = 0\}. \tag{2.1.1.15}$$

The action of $\mathbb{C}^*$ is free on this and the corresponding quotient is nothing but $T^*\mathbb{P}^1$. This is a minimal resolution of singularities of $\mathbb{C}^2/\mathbb{Z}^2$.

The variety $\mathcal{M}^{\xi}_{H,G,V}$ has conical symplectic singularities, and is therefore a nice variety to study from the point of view of symplectic geometry. In this case, one can completely characterize its deformation quantizations, and use it to study representation theory of non-commutative algebras. See for example, [**BPW16**, **BLPW16**, **Los16**]. When $G = T$ is a torus, the following result also characterizes in which situation $\mathcal{M}^{\xi}_{H,G,V}$ is a smooth variety. Define $(T^*V)^s_\xi$ the subset of $(T^*V)^{ss}_\xi$ where the stabilizer of p is a finite subgroup. Then there is a cone $\Delta(G, T^*V)$ in $\mathfrak{g}^*$ defined by the property that for any $\xi \in \Delta(G, T^*V)$, $(T^*V)^{ss}_\xi = (T^*V)^s_\xi$, namely, stability and semi-stability agree.

Theorem 2.1.5 ([**HS02**] Proposition 6.2; see also [**BK12**] Corollary 4.13). *If ξ is in the interior of $\Delta(G, T^*V)$ and the action of G on T^*V is defined by a unimodular matrix over $\mathbb{Z}$, then $\mathcal{M}^{\xi}_{H,G,V}$ is smooth. In this case, the map $\mathcal{M}^{\xi}_{H,G,V} \to \mathcal{M}_{H,G,V}$ is a resolution of singularities.*

The definition of the Coulomb branch $\mathcal{M}_{C,G,V}$ is more involved. Its existence was predicted by physics, and physicists have predicted much of its properties, for example, [**GMN13a**, **GMN13b**, **BDGH16**, **BDG17**, **DGGH20**]. Its mathematically precise definition is given in [**Nak16**, **BFN18**]. Let us recall the definitions here. Denote by $\mathcal{O}$ the ring of power series $\mathbb{C}[\![z]\!]$ and $\mathcal{K}$ the field of Laurent series $\mathbb{C}(\!(z)\!)$. We will denote by $V(\mathcal{K})$ (respectively $V(\mathcal{O})$) the space of Laurent series (respectively, power series) valued in V. Similarly, we can define $G(\mathcal{K})$ and $G(\mathcal{O})$. The spaces $V(\mathcal{O})$ and $G(\mathcal{O})$ are affine schemes of infinite dimensions, and are called pro-schemes (see [**Ras20**]), and the space $V(\mathcal{K})$ and $G(\mathcal{K})$ are ind-pro-schemes. The action of G on V naturally extends to an action on the loop spaces. We define the BFN space $\mathcal{R}_{G,V}$ by the following Cartesian diagram:

(2.1.1.16)
$$\begin{array}{ccc} \mathcal{R}_{G,V} & \longrightarrow & V(\mathcal{O}) \\ \downarrow & & \downarrow i \\ G(\mathcal{K}) \times_{G(\mathcal{O})} V(\mathcal{O}) & \xrightarrow{m} & V(\mathcal{K}) \end{array}$$

Namely, $\mathcal{R}_{G,V}$ is defined as the derived subscheme of $G(\mathcal{K}) \times_{G(\mathcal{O})} V(\mathcal{O})$ subjected to the relation that $g(z) \cdot v[z] \in V(\mathcal{O})$. The space $\mathcal{R}_{G,V}$ can be alternatively described as the moduli space of triples $(\mathcal{P}, \varphi, s)$ where $\mathcal{P}$ is a principal G torsor over $\mathbb{D}$, φ is a trivialization of $\mathcal{P}$ over $\mathbb{D}^*$, and s is a section of the associated V bundle over $\mathbb{D}$ that is sent, under φ, to a regular section of the trivial V bundle. It is an indscheme and a $G(\mathcal{O})$-equivariant fiber bundle over Gr_G. Roughly speaking, the Coulomb branch $\mathcal{M}_{C,G,V}$ is defined as the spectrum of the equivariant Borel-Moore homology $H^{BM}_{G(\mathcal{O})}(\mathcal{R}_{G,V})$. This is only rough because the space $\mathcal{R}_{G,V}$ is infinite-dimensional and so is the group $G(\mathcal{O})$. The authors of [**Nak16**, **BFN18**] use the properties of Borel-Moore homology to essentially cut the space into finite-dimensional ones. More precisely, for each $n > 0$, we have a compatible diagram whose limit is the diagram in equation (2.1.1.16):

(2.1.1.17)
$$\begin{array}{ccc} \mathcal{R}_{G,V,n} & \longrightarrow & V(\mathcal{O}) \\ \downarrow & & \downarrow i \\ G(\mathcal{K})_n \times_{G(\mathcal{O})} V(\mathcal{O}) & \xrightarrow{m} & V(\mathcal{K})_n \end{array}$$

Here the subspace with integer n are defined by requiring the orders of poles to be at most n. For each n, we can find m large enough that this diagram factors through the following:

(2.1.1.18)
$$\begin{array}{ccc} \mathcal{R}_{G,V,n,m} & \longrightarrow & V(\mathcal{O}/z^m) \\ \downarrow & & \downarrow i \\ G(\mathcal{K})_n \times_{G(\mathcal{O}/z^m)} V(\mathcal{O}/z^m) & \xrightarrow{m} & V(\mathcal{K})_n/z^{m-n}V(\mathcal{O}) \end{array}$$

Moreover, the action of $G(\mathcal{O})$ on $\mathcal{R}_{G,V,n,m}$ factors through a quotient $G_k := G(\mathcal{O}/z^k\mathcal{O})$. The Borel-Moore homology is defined as the injective limit:

(2.1.1.19)
$$H^{BM}_{G(\mathcal{O})}(\mathcal{R}_{G,V}) := \varinjlim_{n,m,k} H^{BM}_{G_k}(\mathcal{R}_{G,V,n,m}).$$

Here the first limit of k stabilizes as the kernel $G_k \to G_{k-1}$ is unipotent. The limit over m uses the pullback functor of Borel-Moore homology and the limit over n uses the push-forward. For a precise definition of these functors, see for example, the beautiful book [**CG97**]. The Borel-Moore homology defined above has the structure of an associative algebra, defined by the following convolution diagram (and its finite cut-off):

(2.1.1.20)
$$\mathcal{R}_{G,V} \times \mathcal{R}_{G,V} \xleftarrow{p} G(\mathcal{K}) \times \mathcal{R}_{G,V} \times_{V(\mathcal{K})} V(\mathcal{O}) \xrightarrow{q} G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{R}_{G,V} \times_{V(\mathcal{K})} V(\mathcal{O}) \xrightarrow{m} \mathcal{R}_{G,V}$$

Here the space $G(\mathcal{K}) \times \mathcal{R}_{G,V} \times_{V(\mathcal{K})} V(\mathcal{O})$ is the (derived) subspace of $G(\mathcal{K}) \times \mathcal{R}_{G,V}$ consisting of $(g(z), [g'(z), v[z]])$ such that $gg'v \in V(\mathcal{O})$. The maps in the above diagram is given by:

(2.1.1.21)

$$[g(z), g'(z)v[z]] \times [g'(z), v(z)] \longleftarrow (g(z), [g'(z), v[z]]) \longrightarrow [g(z), [g'(z), v[z]]] \longrightarrow [g(z)g'(z), v[z]] \,.$$

The finite cut-off of this diagram induces an algebra structure on the Borel-Moore homology $H^{BM}_{G(\mathcal{O})}(\mathcal{R}_{G,V})$. For the precise definition and computation, see [**BFN18**]. It turns out that this algebra structure is commutative, and the Coulomb branch is defined to be the spectrum of this algebra.

Definition 2.1.6 ([**BFN18**]). *The Coulomb branch $\mathcal{M}_{C,G,V}$ is defined to be an affine variety:*

$$\mathcal{M}_{C,G,V} := \operatorname{Spec}\left(H^{BM}_{G(\mathcal{O})}(\mathcal{R}_{G,V})\right). \tag{2.1.1.22}$$

In the case $G = T$ is a torus, the (reduced scheme of the) affine grassmannian $T(\mathcal{K})/T(\mathcal{O})$ is nothing but a set of points, labelled by the cocharacters of T. The Coulomb branch in this case is computed explicitly in [**BFN18**]. Assume the action of T on V is defined by a set of characters $\{\xi_i\}_{1\leq i\leq n}$. Denote by $\mathfrak{t}$ the Lie algebra of T and $\mathfrak{t}^*$ the dual, and let Λ be the set of co-characters, which is a lattice whose rank is equal to the dimension of T. Define, in addition, a function $d(m,n)$ on integers by:

$$d(m,n) = \begin{cases} 0 & \text{if } m, n \text{ have the same sign} \\ \min(|m|,|n|) & \text{otherwise} \end{cases} \tag{2.1.1.23}$$

Theorem 2.1.7 ([**BFN18**], Theorem 4.1). *The Coulomb branch $\mathcal{M}_{C,T,V}$ is the spectrum of an algebra over $\operatorname{Sym}(\mathfrak{t}^*)$ generated by symbols r^λ for $\lambda \in \Lambda$ with relations:*

$$r^\lambda r^\mu = \prod_{i=1}^n \xi_i^{d(\xi_i(\lambda),\xi_i(\mu))} r^{\lambda+\mu}. \tag{2.1.1.24}$$

For quiver gauge theories, the Coulomb branches has been identified with slices in the affine Grassmannian of G [**BFN19**]. For general G and V, the algebra structure of $\mathbb{C}[\mathcal{M}_{C,G,V}]$ is complicated. However, the geometric construction implies some immediate structures of $\mathbb{C}[\mathcal{M}_{C,G,V}]$.

For example, when G is a reductive Lie group and T the maximal torus, then there is a localization map from $\mathbb{C}[\mathcal{M}_{C,G,V}]$ to a localization of $\mathbb{C}[\mathcal{M}_{C,T,V}]$. Moreover, this localization maps to the Weyl-invariant part of the abelian Coulomb branch. This in particular can be used to show that the Borel-Moore homology is a commutative algebra.

Just as the case of the Higgs branch, the Coulomb branch is a Poisson variety, though the Poisson structure is less explicit. In [**BFN18**], the authors constructed a filtered deformation of $\mathbb{C}[\mathcal{M}_{C,G,V}]$, which from the point of view of $\mathcal{R}_{G,V}$, is constructed by considering the loop group equivariant Borel-Moore homology. More precisely, consider the $\mathbb{C}^\times$ action on $\mathcal{R}_{G,V}$ given by loop rotation. The homology group $H^{BM}_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R}_{G,V})$ has the structure of a non-commutative algebra, and the $\mathbb{C}^\times$ equivariants parameter exhibits $H^{BM}_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R}_{G,V})$ as a filtered deformation, whose associated graded is $H^{BM}_{G(\mathcal{O})}(\mathcal{R}_{G,V})$.

(2.1.1.25)
$$\pi : H^{BM}_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R}_{G,V}) \xrightarrow{\text{assoc. gr.}} H^{BM}_{G(\mathcal{O})}(\mathcal{R}_{G,V})$$

This induces a Poisson structure on $H^{BM}_{G(\mathcal{O})}(\mathcal{R}_{G,V})$ by $\{\pi(a),\pi(b)\} = \pi[a,b]$, or $\lim_{q\to 1}\frac{[a,b]_q}{q-1}$ where q is the $\mathbb{C}^\times$-equivariant parameter. The Poisson variety $\mathcal{M}_{C,G,V}$ is not smooth in general, but one can also construct resolutions of $\mathcal{M}_{C,G,V}$ using Borel-Moore homology. Suppose the action of G on V can be extended to a group $\widetilde{G}$ that fits into a short exact sequence:

(2.1.1.26)
$$1 \longrightarrow G \longrightarrow \widetilde{G} \longrightarrow T_F \longrightarrow 1$$

where T_F is a torus, usually called the flavor group. Choose a character λ_F of T_F. Let $\mathcal{R}_{\widetilde{G},V}$ be the BFN space associated to $\widetilde{G}$. Since this is a fiber bundle over $\mathrm{Gr}_{\widetilde{G}}$, there is a map:

(2.1.1.27)
$$\pi; \mathcal{R}_{\widetilde{G},V} \longrightarrow \mathrm{Gr}_{T_F},$$

where Gr_{T_F} is isomorphic to the cocharacter lattice of T_F. Denote by $\mathcal{R}^{n\lambda_F}_{\widetilde{G},V}$ the preimage of $\pi^{-1}(n\lambda_F)$, then one can define the partially-resolved Coulomb branch:

(2.1.1.28)
$$\mathcal{M}^{\lambda_F}_{C,G,V} := \mathrm{Proj}\left(\lim_{n\geq 0} H^{BM}_{G(\mathcal{O})}(\mathcal{R}^{n\lambda_F}_{\widetilde{G},V})\right).$$

Since $\mathcal{R}^{0\lambda_F}_{\widetilde{G},V} = \mathcal{R}_{G,V}$, there is a morphism:

$$\mathcal{M}^{\lambda_F}_{C,G,V} \longrightarrow \mathcal{M}_{C,G,V}. \tag{2.1.1.29}$$

2.1.2. The Category of Line Operators. As mentioned in the introduction, the categories of line operators $\mathcal{L}_A[G,V]$ and $\mathcal{L}_B[G,V]$ in the topological twists should have the structure of braided tensor categories, and can be used to obtain the corresponding algebras of local operators:

$$\mathrm{End}^*_{\mathcal{L}_A[G,V]}(\mathbb{1}) \cong \mathbb{C}[\mathcal{M}_{C,G,V}], \qquad \mathrm{End}^*_{\mathcal{L}_B[G,V]}(\mathbb{1}) \cong \mathbb{C}[\mathcal{M}_{H,G,V}], \tag{2.1.2.1}$$

such that the Poisson structure is induced from the braided tensor structure. In [**DGGH20**], the authors argued on physical ground that these categories should have the following form:

(1) $\mathcal{L}_A[G,V] \sim \text{D-Mod}\,(V(\mathcal{K})/G(\mathcal{K})) = \mathrm{Coh}(\mathrm{Maps}(\mathbb{D}^*, V/G)_{dR})$.
(2) $\mathcal{L}_B[G,V] \sim \mathrm{Coh}(\mathrm{Maps}(\mathbb{D}^*_{dR}, V/G))$.

Here $\mathbb{D}^* := \mathrm{Spec}(\mathcal{K})$ is the formal punctured disk, and $\mathbb{D} := \mathrm{Spec}(\mathcal{O})$ the formal disk. Let us try to un-pack these definitions. The space $\mathrm{Maps}(\mathbb{D}^*, V/G)$ can be thought of as the space $V(\mathcal{K})/G(\mathcal{K})$. Coherent sheaves on the de-Rham stack of $V(\mathcal{K})/G(\mathcal{K})$ is also known as the category of D-modules, modules for the algebra of differential operators on $V(\mathcal{K})/G(\mathcal{K})$. Therefore, the expectation is that $\mathcal{L}_A[G,V]$ is the category $D{-}\mathrm{Mod}(V(\mathcal{K})/G(CK))$.

The space $\mathrm{Maps}(\mathbb{D}^*_{dR}, V/G)$ is the space of G-integrable systems on $\mathbb{D}^*$ with an associated section. A G-integrable system is the choice of a connection, namely an element $A(z) \in \mathfrak{g}(\mathcal{K})$, and a section of the associated V system is an element $v(z) \in V(\mathcal{K})$ such that $(\partial_z + A(z))v(z) = 0$. Therefore, the space $\mathrm{Maps}(\mathbb{D}^*_{dR}, V/G)$ is given by the following Cartesian product:

$$\begin{array}{ccc} \mathrm{Maps}(\mathbb{D}^*_{dR}, V/G) & \longrightarrow & G(\mathcal{K}) \setminus 0 \\ \downarrow & & \downarrow \\ G(\mathcal{K}) \setminus (\mathfrak{g}(\mathcal{K}) \times V(\mathcal{K})) & \longrightarrow & G(\mathcal{K}) \setminus V(\mathcal{K}) \end{array} \tag{2.1.2.2}$$

Therefore, the expectation is that $\mathcal{L}_B[G,V]$ is the category $\mathrm{Coh}_{G(\mathcal{K})}\big((\mathfrak{g}(\mathcal{K}) \times V(\mathcal{K})) \times_{V(\mathcal{K})} \{0\}\big)$.

The above definitions are not mathematically rigorous, partly due to a lack of technological tools for inifnite-dimensional quotient spaces. There are, however, alternative definitions and specific examples in which these categories are given a rigorous definition. Before reviewing them, let us comment on how these two categories are used to derive the Higgs and Coulomb branches.

The derivation of Coulomb branch is more straight-forward. With the identification $\mathcal{L}_A[G,V] \sim D{-}\mathrm{Mod}(V(\mathcal{K})/G(\mathcal{K}))$, the identity line operator is the structure sheaf of $V(\mathcal{O})/G(\mathcal{O})$, and we would like to compute:

$$\mathrm{End}^*_{D-\mathrm{Mod}(V(\mathcal{K})/G(\mathcal{K}))}\left(\mathcal{O}_{V(\mathcal{O})/G(\mathcal{O})}\right). \tag{2.1.2.3}$$

Let Y be a finite-dimensional smooth variety and $i : X \to Y$ a morphism, then the work of [**CG97**] establishes the following:

$$\mathrm{End}^*_{D-\mathrm{Mod}(Y)}\left(i_*\mathcal{O}_X\right) \cong H^*_{BM}\left(X \times_Y X\right). \tag{2.1.2.4}$$

If we assume that this holds in the infinite-dimensional setting, then there will be a quasi-isomorphism:

$$\mathrm{End}^*_{D-\mathrm{Mod}(V(\mathcal{K})/G(\mathcal{K}))}\left(\mathcal{O}_{V(\mathcal{O})/G(\mathcal{O})}\right) \cong H^*_{BM}\left(V(\mathcal{O})/G(\mathcal{O}) \times_{V(\mathcal{K})/G(\mathcal{K})} V(\mathcal{O})/G(\mathcal{O})\right). \tag{2.1.2.5}$$

We recognize that the fibre product $V(\mathcal{O})/G(\mathcal{O}) \times_{V(\mathcal{K})/G(\mathcal{K})} V(\mathcal{O})/G(\mathcal{O})$ can be defined alternatively as $G(\mathcal{O}) \setminus \mathcal{R}_{G,V}$, which leads to Definition 2.1.6.

It is less obvious how $\mathcal{L}_B[G,V]$ leads to the symplectic reduction. It becomes more obvious if we replace $\mathbb{D}^*_{dR}$ by S^1. This is related to two different ways of understanding local systems: the de-Rham point-of-view and the Betti point-of-view. In de-Rham setting, one remembers the connection modulo gauge transformations, while in the Betti setting, one only remembers the monodromy. The algebraic space $\mathrm{Maps}(\mathbb{D}^*_{dR}, V/G)$ is from the de-Rham perspective. In Betti prespective, one replace $\mathrm{Maps}(\mathbb{D}^*_{dR}, V/G)$ by $\mathrm{Maps}(S^1, V/G) = \mathcal{L}_{top}V/G$, where $\mathcal{L}_{top}$ is the topological loop space [**BZN12**]. This leads to the expectation $\mathcal{L}_B[G,V] \sim \mathrm{Coh}\left(\mathcal{L}_{top}V/G\right)$. Koszul duality implies an equivalence $\mathrm{Coh}\left(\mathcal{L}_{top}V/G\right) \simeq \mathrm{Coh}\left(T^*[2]V/G\right)$ [**Rie19**], under which the identity line operator is simply the structure sheaf of $T^*[2]V/G$. The endomorphism is simply the derived global section of the structure sheaf of $T^*[2]V/G$, otherwise known as the G.I.T quotient:

$$\mathrm{End}_{\mathcal{L}_B[G,V]}\left(\mathcal{O}_{T^*[2]V/G}\right) \cong \Gamma\left(T^*[2]V/G\right) \cong \mathbb{C}[\mu^{-1}(0)]^G, \tag{2.1.2.6}$$

which leads to Definition 2.1.1.

It is possible to understand the resolution and deformation quantization from the perspective of line operators as well. The choice of a co-characters for Coulomb side (or a characters for Higgs side) gives rise to a set of objects $\{A_n\}$ in the category of line operators $\mathcal{L}$ labelled by $\mathbb{Z}$ such that $A_0 = \mathbb{1}$, and isomorphisms:

$$A_n \times A_m \cong A_{n+m} \in \mathcal{L} \tag{2.1.2.7}$$

that makes $\bigoplus A_n$ into a commutative algebra object in $\mathcal{L}$. With this algebra object, one can construct a projective variety via:

$$\operatorname{Proj}\left(\bigoplus_{n\geq 0} \operatorname{Hom}(\mathbb{1}, A_n)\right). \tag{2.1.2.8}$$

Here the algebra structure on $\bigoplus\limits_{n\geq 0} \operatorname{Hom}(\mathbb{1}, A_n)$ is defined using the multiplication structure on the second factor. This is commutative thanks to the commutative algebra structure, which states that the following diagram commute:

$$\begin{array}{ccc} A_n * A_m & \xrightarrow{m} & A_{m+n} \\ {\scriptstyle c_{A_n,A_m}}\downarrow & \nearrow_{m} & \\ A_m * A_n & & \end{array} \tag{2.1.2.9}$$

Here m is the multiplication map and c is the braiding. This will be a projective variety with a morphism onto $\mathcal{M} = \operatorname{Spec}(\operatorname{End}(A_0))$. The resolutions in equation (2.1.1.12) and equation (2.1.1.28) arise from this manner. In this construction, it is essential to have a braided monoidal category structure on $\mathcal{L}$, in order for the algebra object and proj to make sense.

The Poisson structure and quantization of $\mathcal{M}$ can also be deduced from $\mathcal{L}$. A braided tensor category has a natural action of the homotopy group S^1, or in other words, a morphism $S^1 \to \operatorname{Aut}(\mathcal{L})$, from the homotopy group S^1 to the category of automorphisms of $\mathcal{L}$. The equivariant category $\mathcal{L}^{S^1}$ is naturally a category fibred over the formal disk $\mathbb{D}$, since $\mathcal{O}_{BS^1} \cong \mathbb{C}[\![x]\!]$ for x a commutative variable of homological degree 2 [**Pre11**]. The fibre of this category at 0 is the original category $\mathcal{L}$ and the fibre over $\mathbb{D}^*$ is the category of equivariant objects. This gives rise to the deformation family

$$\text{End}^*_{\mathcal{L}^{S^1}}(\mathbb{1}) \tag{2.1.2.10}$$

of $\text{End}^*_{\mathcal{L}}(\mathbb{1})$. The fact that this deformation is related to the Poisson structure can be seen from the Higgs side through the work of [**Rie19**], who showed that the S^1 action on $\text{Coh}(T^*[2]X)$ for a smooth X is given by the exponential of the Poisson bivector, and the work of [**BZN12**] suggests that the deformation upon taking S^1-equivariants deforms $\mathbb{C}[T^*[2]X]$ to differential operators on X. Although in the present case $X = V/G$ is not smooth, it is still plausible that such a statement generalizes.

It is also possible to obtain the Poisson structure and quantization of the partial resolutions from the S^1-equivariant category. However, this requires much more explanation and is beyond the scope of this thesis. We wish to take on this part of the property in a future work. Nevertheless, these observations suggest that the knowledge of $\mathcal{L}$ is a great advantage in the study of $\mathcal{M}$ and its symplectic geometry. Another advantage of the knowledge of $\mathcal{L}$ is that it helps understand the statement of mirror symmetry, which we now recall.

2.1.2.1. *3d Mirror Symmetry and Line Operators.* 3d mirror symmetry is the statement that 3d quantum field theories can come in pair $(\mathcal{T}, \mathcal{T}^\vee)$, such that the two theories are equivalent, but with a non-trivial equivalence that swaps certain sets of observables. In the context of 3d $\mathcal{N} = 4$ theories and these associated varieties, it is the statement that for certain (G, V), there is a $(G^\vee, V^\vee)$ such that:

$$\mathcal{T}_{HT}[G,V] \simeq \mathcal{T}_{HT}[G^\vee, V^\vee], \qquad \mathcal{T}_A[G,V] \simeq \mathcal{T}_B[G^\vee, V^\vee], \qquad \mathcal{T}_B[G,V] \simeq \mathcal{T}_A[G^\vee, V^\vee]. \tag{2.1.2.11}$$

This leads to the following remarkable property of Higgs and Coulomb branches:

$$\mathcal{M}_{C,G,V} \cong \mathcal{M}_{H,G^\vee,V^\vee}, \ \mathcal{M}_{H,G,V} \cong \mathcal{M}_{C,G^\vee,V^\vee}. \tag{2.1.2.12}$$

Namely, the Coulomb branch, which was originally defined using Borel-Moore homology, is equal to Higgs branch of a dual theory, defined by sympletic reduction. This is known in many cases already, some of which include quiver gauge theories and abelian gauge theories. Let us recall here the full detail of abelian gauge theories. For more examples, see [**TGGH20**]. Let $T = (\mathbb{C}^\times)^r$ be a torus acting on $V = \mathbb{C}^n$, whose action is defined by a charge matrix ρ. We view ρ as a map:

$$\mathbb{Z}^r \xrightarrow{\rho} \mathbb{Z}^n \tag{2.1.2.13}$$

We will assume that ρ defines an embedding and can be completed into a short exact sequence:

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{\rho} \mathbb{Z}^n \xrightarrow{\tau} \mathbb{Z}^{n-r} \longrightarrow 0 \tag{2.1.2.14}$$

Let $\rho^\vee = \tau^\mathsf{T}$ be the transpose. Denote by $T^\vee$ the torus $(\mathbb{C}^\times)^{n-r}$ which has an action on V via the matrix $\rho^\vee$. It is predicted that the following two theories are mirror:

$$\mathcal{T}[T, V] \longleftrightarrow \mathcal{T}[T^\vee, V] \tag{2.1.2.15}$$

If we denote the Higgs and Coulomb branch of $\mathcal{T}[T, V]$ by $\mathcal{M}_{H,\rho}$ and $\mathcal{M}_{C,\rho}$, then this leads to the following identification:

$$\mathcal{M}_{H,\rho} \cong \mathcal{M}_{C,\rho^\vee},\ \mathcal{M}_{C,\rho} \cong \mathcal{M}_{H,\rho^\vee}, \tag{2.1.2.16}$$

namely, the Coulomb branch defined using Borel-Moore homology of the BFN space $\mathcal{R}_{T,V}$ is identified as a hypertoric variety. This identification is proved in [**BFN18**], Proposition 3.18. Moreover, since we have a short exact sequence of groups:

$$1 \longrightarrow T^\vee \longrightarrow (\mathbb{C}^\times)^n \longrightarrow T \longrightarrow 1 \tag{2.1.2.17}$$

the data of a stability parameter for $\mathcal{M}_{H,\rho}$, namely $\xi \in \mathrm{Hom}(T, \mathbb{C}^\times)$ is in one-to-one correspondence with a flavor parameter. There is an isomorphism:

$$\mathcal{M}^\xi_{H,\rho} \cong \mathcal{M}^\xi_{C,\rho} \tag{2.1.2.18}$$

and vice versa.

Example. *Consider the case when $T = \mathbb{C}^\times$ and $V = \mathbb{C}$ such that the weight of V under T is 1. The dual theory has $T^\vee = 1$ and $V^\vee = \mathbb{C}$. The Higgs branch for (T, V) is trivial:*

$$\mathcal{M}_{H,T,V} = \mathrm{pt} \tag{2.1.2.19}$$

and so is the Coulomb branch for $(T^\vee, V^\vee)$ *since the Borel-Moore homology of a vector space is trivial. On the other hand, from Theorem 2.1.7 we can compute the Coulomb branch of* (T, V)*. It is generated over* $\mathrm{Sym}(\mathfrak{t}^*) = \mathbb{C}[E]$ *by* r^1 *and* r^{-1} *with relation:*

$$r^1 r^{-1} = E. \tag{2.1.2.20}$$

In the meanwhile, the Higgs branch of $(T^\vee, V^\vee)$ *is simply* $T^*\mathbb{C}$*, whose algebra of functions is* $\mathbb{C}[x, y]$*. We can identify the two via identifying* r^1 *with* x *and* r^{-1} *with* y*, and* E *is identified with* xy*.*

The advantage of the isomorphism as in equation (2.1.2.16) is that one can pass from one description to another when studying the properties of these spaces. For example, if one would like to study the Poisson geometry of $\mathcal{M}_C$, it is easier to go to the equivalent space $\mathcal{M}_H$ since the Higgs branch is defined by Hamiltonian reduction. On the other hand, if one would like to study the quantization of $\mathcal{M}_H$, then one can move to the description of $\mathcal{M}_C$, and consider the $\mathbb{C}^\times$-equivariant homology, where the $\mathbb{C}^\times$ acts by loop rotation $z \mapsto qz$. This coincides with the canonical quantization of $\mathcal{M}_H$ under mirror symmetry. Moreover, one can then construct modules of the quantization by constructing Borel-Moore homology of spaces with an action of $\mathcal{R}_{G,V}$. This is the BFN Springer theory [**HKW20**], which is analogous to the usual Springer theory.

In terms of the category of line operators, 3d mirror symmetry would imply an equivalence of braided tensor categories:

$$\mathcal{L}_A[G, V] \simeq \mathcal{L}_B[G^\vee, V^\vee], \qquad \mathcal{L}_B[G, V] \simeq \mathcal{L}_A[G^\vee, V^\vee]. \tag{2.1.2.21}$$

The isomorphisms from equation (2.1.2.12) is then a consequence of the equivalence of the category of line operators. In this thesis, we focus on $G = T$ a torus with a representation V defined by a charge matrix ρ. Denote by $\mathcal{L}_{A,\rho}$ and $\mathcal{L}_{B,\rho}$ the corresponding category of line operators, we would like to define them as braided tensor cateogries and prove equivalences:

$$\mathcal{L}_{A,\rho} \simeq \mathcal{L}_{B,\rho^\vee}, \qquad \mathcal{L}_{B,\rho} \simeq \mathcal{L}_{A,\rho^\vee}. \tag{2.1.2.22}$$

Many mathematically rigorous approaches exist for the definition of this category of line operators. In [**HR22**], the authors rigorously defined the categories for abelian gauge theories, and proved mirror symmetry statement as an equivalence of derived categories. A more combinatorial approach of the A side categories was worked out in [**Web19**, **Web22**], directly inspired by [**DGGH20**, **BFN18**]. This combinatorial approach was used in [**Web16**] to understand mirror symmetry from the point of view of symplectic duality [**BPW16**, **BLPW16**]. A similar construction was contained in [**HKW20**]. The B side categories can be defined by viewing the B twist as a Rozansky-Witten theory on the stack $T^*(V/G)$, which leads to the category of coherent sheaves on Poisson varieties $\mathcal{M}_{H,G,V}$ and $\mathcal{M}^{\xi}_{H,G,V}$, whose $\mathbb{E}_2$ structure is related to the Poisson geometry of the Higgs branch [**KRS09**, **KR10**, **BZFN10**, **RW10**, **Rie19**]. Recently, the work of [**GHMG22**, **GH22**] used a 2-categorical approach to derive both the A and B side categories for abelian gauge theories.

The approach using combinatorics [**Web19**, **Web22**] is easy to work with, but lacks braided tensor structure. The approach using derived geometry [**KRS09**, **BZFN10**, **RW10**, **GH22**] gives an $\mathbb{E}_2$ structure, but it is very abstract and difficult to work with. The de-Rham approach of [**HR22**] is very impressive and most directly relates to [**DGGH20**], but it is unlikely any $\mathbb{E}_2$ structure exists in this context. In the following, we will describe an algebraic approach to this problem for abelian gauge theories, where both the categorical content and braided tensor structure can be expressed explicitly. We hope to compare this approach to other approaches in a future work.

2.1.3. Boundary Conditions and VOA. The approach we will take for this problem uses holomorphic boundary conditions of [**CG19**]. In this work, the authors defined boundary conditions of 3d $\mathcal{N}=4$ gauge theories that are compatible with the topological A and B twists, such that the twisted theory on the boundary is holomorphic. In this case, local operators on the boundary form a vertex operator algebra. The work of [**CG19**] then partially analyzed these boundary VOAs, and in [**CCG19**], they were used to derive the Higgs and Coulomb branch algebras.

For a 3d $\mathcal{N}=4$ gauge theory defined by G and V, [**CG19**] considered two types of boundary conditions, one compatible with each twist.

(1) There is a Neumann-like boundary condition **N** which can be deformed to be compatible with the bulk A twist. In order to cancel a boundary gauge anomaly, the Neumann boundary condition must be enriched with extra boundary matter. There may be multiple ways to do this One canonical choice is to add 2d complex, chiral fermions transforming

in the representation V. The corresponding boundary VOA for the A-twisted $\mathcal{T}_A[G,V]$ is then described as the BRST reduction of a beta-gamma system valued in V tensored with a V-valued bc system. Schematically,

$$\mathcal{V}^A_{G,V} := H^*_{\mathrm{BRST}}\big(\mathfrak{g}, (\beta\gamma)_V \otimes (bc)_V\big) = H^*\big((\beta\gamma)_V \otimes (bc)_V \otimes (bc)_{\mathfrak{g}}, Q_{\mathrm{BRST}}\big)\,. \tag{2.1.3.1}$$

(2) There is a Dirichlet-like boundary condition **D** which can be deformed to be compatible with the topological B twist. The corresponding VOA has a *perturbative* description as an affine Kac-Moody algebra

$$\mathcal{V}^{B,\mathrm{pert}}_{G,V} := \widehat{\mathfrak{g}_{G,V}} \tag{2.1.3.2}$$

based on a Lie superalgebra $\mathfrak{g}_{G,V}$ whose even part is $T^*\mathfrak{g}$ and whose odd part is T^*V. There is no boundary gauge anomaly to worry about, since the Dirichlet boundary condition breaks gauge symmetry to a global symmetry. There is instead a boundary 't Hooft anomaly [**DGP18**], which plays a role in determining the level of the Kac-Moody algebra (2.1.3.2) (see [**Gar22**] for a derivation).

This description is only perturbative, because it does not take into account the contribution of boundary monopole operators. It is still not known how monopole operators modify the boundary VOA in general, but in the case of abelian gauge theories, we will show that it amounts to a simple current extension of the affine Lie superalgebra $\widehat{\mathfrak{g}_{G,V}}$. We will justify this by comparing the index of the extended VOA with the index formula found in [**DGP18**].

As stated in the introduction, a boundary condition of this form gives rise to a functor:

$$\mathcal{F}_{\mathbb{B}} : \mathcal{L} \to D^b(\mathcal{V}_{\mathbb{B}}\text{-Mod}). \tag{2.1.3.3}$$

This basic setup arises in Chern-Simons theory, with a holomorphic boundary condition supporting the WZW VOA [**Wit89, EMSS89**]. In that case, the functor (2.1.3.3) is known to be an equivalence of braided tensor categories. (Moreover, since the categories involved are semisimple, it is not necessary to take the derived category on the RHS.)

It is expected that the holomorphic boundary conditions of [**CG19**] in 3d $\mathcal{N}=4$ gauge theories are also rich enough for the functor (2.1.3.3) to be an equivalence. However, making sense of such a statement requires being more precise about what models of bulk line operator categories one intends to consider, as well as what categories of VOA modules one intends to consider. In the following sections, we will specify the later, namely, we will define the vertex operator algebra explicitly and define a category of modules for the boundary VOA which will have the structure of a braided tensor category via intertwining operators. We will give a quick comment on what physical line operators the objects in this category corresponds to, thus partially justifying our definition using boundary VOAs. We will justify our definition further by showing that the derived endomorphisms of objects in this category correctly reproduces the Higgs and Coulomb branch algebras.

2.1.3.1. *Braided Tensor Categories from VOA Representations.* Let us begin by recalling the definition of a *vertex algebra* [**FBZ04**]. A vertex algebra is a $\frac{1}{2}\mathbb{Z}$-graded (called *conformal grading*) vector space V together with the following set of data:

- A *vacuum vector* $\Omega \in V$.
- A state-operator correspondence $Y : V \otimes V \to V(\!(z)\!)$ that respects conformal grading.
- A translation operator $T : V \to V$ of degree 1.

that satisfies the following conditions:

(1) The vacuum acts as identity $Y(\Omega, z)v = v$.

(2) Locality condition, which is the statement that for $v_1, v_2, v_3 \in V$, the series:

$$Y(v_1, z)Y(v_2, w)v_3 - Y(v_2, w)Y(v_1, z)v_3 \tag{2.1.3.4}$$

is a linear combination of derivatives of $\delta(z-w)$. Alternatively, this is usually expressed in terms of *operator product expansion* (OPE):

$$Y(v_1, z)Y(v_2, w) \sim \sum_{n \geq -N} \frac{Y(u_n, w)}{(z-w)^n}. \tag{2.1.3.5}$$

(3) The translation operator T acts as derivatives on Y:

$$[T, Y(v_1, z)]v_2 = \partial_z Y(v_1, z)v_2. \tag{2.1.3.6}$$

A vertex algebra is a *vertex operator algebra* (VOA) if it has a *conformal element* $\omega \in V$, such that $\oint \mathrm{d}z Y(\omega, z)$ is the translation operator T and $\oint \mathrm{d}z z Y(\omega, z)$ acts as multiplication by conformal grading. Here and in what follows, the notation $\oint \mathrm{d}z A(z)$ of a formal Laurent series $A(z) = \sum A_n z^{-n-1}$ is equal to the residue $\mathrm{Res}_{z=0} A(z) = A_0$. We usually denote $Y(\omega, z)$ by $L(z)$. This in particular means the following OPE:

$$L(z)v(w) \sim \frac{\deg(v)v(w)}{(z-w)^2} + \frac{\partial v(w)}{z-w} + \cdots \tag{2.1.3.7}$$

where $\deg(v)$ is the conformal degree of v.

Let V be a VOA. A module (usually called a *generalized module*) of V is a vector space M together with a map $Y_M : V \otimes M \to M(\!(z)\!)$ that satisfies locality with Y and compatibility condition with the action of L_0 and L_{-1}. We will not repeat it here. A more down-to-earth way to think about this is as follows. Given a VOA V, one can define an algebra $U(V)$ called the universal enveloping algebra of V, that is generated by Fourier modes of $Y(v, z)$ (namely $v_n \in \mathrm{End}(V)$ such that $Y(v, z) = \sum v_n z^{-n-1}$) whose commutation relation is determined by the OPE (see [**FBZ04**] for details). One can show that a module of the VOA is the same as a smooth module of the algebra $U(V)$, where by smooth we mean that for every $m \in M$ there eixsts $N \in \mathbb{N}$ such that the larger than N Fourier modes act trivially on m.

Let V be a VOA and $\mathcal{C}$ be an abelian category of generalized modules of V. By the work of Huang-Lepowsky-Zhang [**HLZ14**, **HLZ10a**, **HLZ10b**, **HLZ10c**, **HLZ10d**, **HLZ10e**, **HLZ11a**, **HLZ11b**], summarized nicely in [**ALSW21**], that under certain conditions on V and $\mathcal{C}$, there is a "tensor-product" structure on $\mathcal{C}$, where tensor product of two modules is defined as the universal object of logarithmic intertwining operators. A logarithmic intertwining operator from $M \otimes N$ to P is a map:

$$\mathcal{Y} : M \otimes N \to P\{z\}[\log z], \tag{2.1.3.8}$$

where $P\{z\}[\log z]$ is the space of formal series of the form $\sum_{s \in \mathbb{C}, t \geq 0} p_{s,t} z^{-s-1} \log(z)^t$, such that $\mathcal{Y}$ satisfies roughly the same set of properties as the state-operator correspondence Y. The logarithmic terms are usually a sign that the category of modules of V is non-semisimple, and the action of the conformal element L_0 has Jordan blocks. The fusion product $M \times_V N$ is a (unique, if exists) module of V, together with a universal intertwining operator $\mathcal{Y}: M \otimes N \to M \otimes_V N\{z\}[\log z]$ such that for any logarithmic intertwiner as above, there is a module map $M \times_V N \to P$ such that the diagram commutes:

$$\begin{array}{ccc} M \otimes N & \longrightarrow & M \times_V N\{z\}[\log z] \\ & \searrow & \downarrow \\ & & P\{z\}[\log z] \end{array} \tag{2.1.3.9}$$

The following is an important theorem of HLZ regarding the braided tensor category structure from intertwining operators.

Theorem 2.1.8 (See Proposition 2.1 of [**ALSW21**]). *Let V be a VOA and $\mathcal{C}$ be an abelian category of modules of V containing V. Suppose the following conditions hold:*

(1) *For any two objects M, N in $\mathcal{C}$, a universal object $M \times_V N$ exists in $\mathcal{C}$, with intertwining operator $\mathcal{Y}_{M,N}$.*
(2) *For any three objects M, N and P, the following two expressions converge in $|z_1| > |z_2| > |z_1 - z_2| > 0$*

$$\mathcal{Y}_{M,N\times_V P}(m, z_1)\mathcal{Y}_{N,P}(n, z_2)p, \qquad \mathcal{Y}_{M\times_V N,P}(\mathcal{Y}_{M,N}(m, z_1 - z_2)n, z_2)p \tag{2.1.3.10}$$

regardless of the choice of branch-cuts for $\log(z_1), \log(z_2)$ in the completion of $(M \times_V N) \times_V P$ and $M \times_V (N \times_V P)$ respectively, and either one of them can be extended to the domain of the other.

Then $\mathcal{C}$ has the structure of a braided tensor category, such that:

- *The vacuum module V is the unit object, and $M \times_V N$ is the monoidal product.*
- *The braiding is given by the map $c_{M,N} : M \times_V N \to N \times_V M$ induced from the following correspondence of intertwining operators:*

$$\mathcal{Y}(m, z)n \mapsto \mathcal{Y}'(n, z)m := e^{zL_{-1}}\mathcal{Y}(m, e^{\pi i}z)n. \tag{2.1.3.11}$$

- *There is a twist given by $\theta = e^{2\pi i L_0}$.*

- *The associativity morphism is given by a uniquely defined* $A^{z_1,z_2}_{M,N,P}$ *such that:*

$$A^{z_1,z_2}_{M,N,P}\mathcal{Y}_{M,N\times_V P}(m,z_1)\mathcal{Y}_{N,P}(n,z_2)p = \mathcal{Y}_{M\times_V N,P}(\mathcal{Y}_{M,N}(m,z_1-z_2)n,z_2)p. \tag{2.1.3.12}$$

In general, it is very hard to verify that a category $\mathcal{C}$ satisfies the above two properties. However, many stronger properties have been established such that if a category of VOA modules satisfies them then it satisfies the properties of HLZ. We mention the following criterion from [**CY21**], that is relevant for the discussion of this paper.

THEOREM 2.1.9 (See [**CY21**] Theorem 3.3.4). *Let* $\mathcal{C}$ *the category of generalized modules of* V *that are of finite length. Assume that:*

(1) *Every object in* $\mathcal{C}$ *is* C_1 *co-finite (see [**CY21**], Definition 2.1.5).*
(2) $\mathcal{C}$ *is closed under taking restricted-dual (see [**CY21**], Remark 2.1.3 and the following discussions).*

Then $\mathcal{C}$ *has the structure of a braided tensor category.*

This is a powerful theorem that ensures a large class of VOA representation theories have the structure of braided tensor categories. This will include the affine Lie superalgebra $V(\mathfrak{g}_*(\rho))$ that we will introduce in Chapter 3. However, in practice, especially in the applications considered in this thesis, the category $\mathcal{C}$ will not satisfy the above, especially the first item. To resolve this issue, we will use the idea of vertex operator algebra extensions [**CKM17**, **CMY22a**]. We will recall the idea in Chapter 3. Roughly speaking, in many cases, the VOA V contains a subVOA W whose category of modules satisfies the above properties and therefore has the structure of a BTC. If we can realize V as an algebra object in the category $W-$Mod, then we obtain a category of modules of V, namely $V-$Mod($W-$Mod), that has the structure of a braided tensor category. We will apply this on $V_{A,\rho}$ and $V_{B,\rho}$, and use this VOA machinery to define BTC of line operators.

2.2. 4d $\mathcal{N}=2$ Theories, Poisson Vertex Algebra and the Category of Line Operators

2.2.1. 4d $\mathcal{N}=2$ Theories and K-theoretic Coulomb Branch. Let G be a reductive Lie group and V a finite-dimensional representation of G. Associated to this data is a 4 dimensional $\mathcal{N}=2$ theory $T[G,V]$. In [**Kap06a**], the author introduced a holomorphic-topological (HT) twist, which requires spacetime to locally take the form $\mathbb{R}^2 \times \mathbb{C}$, and depends topologically on $\mathbb{R}^2$ and holomorphically on $\mathbb{C}$. The twisted theory $T_{HT}[G,V]$ can be further deformed to a fully topological A twist (also called the Donaldson twist). This theory is closely related to the 3d $\mathcal{N}=4$ theory $\mathcal{T}[G,V]$ through dimensional-reduction, which we will give a quick comment later.

The theory $T_{HT}[G,V]$ has many beautiful connections to mathematics, including representation theory of affine algebras, topology, integrable systems and enumerative geometry. In this thesis, we focus on the aspect of representation theory. The space of vacua of the HT twist of $T_{HT}[G,V]$ is known mathematically as the K-theoretic Coulomb branch, an object defined very similarly to the Coulomb branch $\mathcal{M}_{C,G,V}$, and studied in [**BFM05**, **BFN19**, **CW19**, **FT19**] and many others. As the nomenclature suggests, the K-theoretic Coulomb branch $M_{C,G,V}$ is defined as the $G(\mathcal{O})$-equivariant K theory of $\mathcal{R}_{G,V}$. More precisely, let $\mathcal{R}_{G,V,m,n}$ and G_k be as in Section 2.1.1, which are finite-dimensional derived schemes and finite type affine group schemes. Denote by $K_{G_k}(\mathcal{R}_{G,V,m,n})$ the vector space of K_0 group of the category of coherent sheaves $\mathrm{Coh}(\mathcal{R}_{G,V,m,n}/G_k)$. Since the maps connecting the spaces $\mathcal{R}_{G,V,m,n}$ are flat morphisms or closed embeddings, the corresponding maps (pullback for m, push-forward for n, and restriction functor for k) all preserves coherence and are in fact exact. One obtain the K group $K_{G(\mathcal{O})}(\mathcal{R}_{G,V})$ as the colimit:

$$K_{G(\mathcal{O})}(\mathcal{R}_{G,V}) := \varinjlim_{m,n,k} K_{G_k}(\mathcal{R}_{G,V,m,n}). \tag{2.2.1.1}$$

The same convolution diagram in equation (2.1.1.20) can be applied to $K_{G(\mathcal{O})}(\mathcal{R}_{G,V})$, giving the K group $K_{G(\mathcal{O})}(\mathcal{R}_{G,V})$ the structure of a commutative algebra. The K-theoretic Coulomb branch $M_{C,G,V}$ is defined as the spectrum of this ring:

$$\mathbb{C}[M_{C,G,V}] \cong K_{G(\mathcal{O})}(\mathcal{R}_{G,V}). \tag{2.2.1.2}$$

This variety as a Kähler manifold was first studied in [**SW94**] and goes under the name of "Seiberg-Witten curve", and the algebraic formulation above uses a specific complex structure compatible with the HT twist. The Coulomb branch $M_{C,G,V}$ admits a family of deformation by considering the $\mathbb{C}^\times$-equivariant K theory $\mathcal{A}^q_{G,V} := K_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R}_{G,V})$, where q is the parameter keeping track of $\mathbb{C}^\times$-equivariance. These are associative algebras and when $q=1$, the algebra $\mathcal{A}^q_{G,V}|_{q=1}$ is commutative and is $\mathbb{C}[M_{C,G,V}]$. The deformation to $\mathcal{A}^q_{G,V}$ corresponds to, in physical languages, turning on the symmetry for the rotation in the complex plane.

When $G = T$ is a torus and $V = 0$, the ring $\mathcal{A}^q_{G,V} = \mathcal{A}^q_{T,0}$ is easy to describe. It is generated by $D_i^{\pm}$ and $\Lambda_i^{\pm}$ for $1 \leq i \leq \dim(T)$ with commtuation relation:

$$D_i \Lambda_j = q^{\delta_{ij}} \Lambda_j D_i. \tag{2.2.1.3}$$

This is called the algebra of abelian difference operators. Here $D_i^{\pm}$ corresponds to structure sheaf of the $T(\mathcal{O})$ orbits in $T(\mathcal{K})/T(\mathcal{O})$ which are labelled by co-characters $\mathrm{Hom}(\mathbb{C}^\times, T)$, and Λ_i are characters of T, labelling representations of T. When $q = 1$, the above becomes commutative and the algebra $\mathcal{A}^q_{T,0}/(q-1)$ is the algebra of functions on the variety $T \times T^\vee$, where $T^\vee$ is the dual group of T. Note that in this case the Coulomb branch for the 3d theory $\mathcal{M}_{C,T,0}$ is the variety $\mathfrak{t} \times T^\vee = T^*(T^\vee)$. The algebra $\mathcal{A}^q_{T,0}$ is thus a universal deformation of the Poisson variety $T \times T^\vee$. When G is a reductive Lie group and T the maximal torus, just like the case of $\mathcal{M}_{C,G,V}$, there is a localization map:

$$\mathcal{A}^q_{G,V} \longrightarrow \left(\mathcal{A}^q_{T,0}\right)_{loc} \tag{2.2.1.4}$$

where $\left(\mathcal{A}^q_{T,0}\right)_{loc}$ is the localization of $\mathcal{A}^q_{T,0}$ along root hyperplanes of G. This in particular implies that $\mathcal{A}^q_{G,V}/(q-1)$ is commutative. For classical groups, the map above is explicitly described in [**FT19**].

For a general reductive group G, the non-commutative ring $\mathcal{A}^q_{G,V}$ is called a shifted affine Yangian, whose representation theory is closely related to representations of quantum groups and affine Lie algebras. Just as in the case of 3d Coulomb branch, one can construct modules of $\mathcal{A}^q_{G,V}$ by constructing spaces with an action by the convolution space $G(\mathcal{O}) \setminus \mathcal{R}_{G,V}$. This is an analog of BFN Springer theory [**HKW20**] in the K-theory setting. We hope to exploit this more in future works.

2.2.2. Poisson Vertex Algebra and Line Operators. Let $\mathrm{Ops}_{G,V}$ denote the space of local operators in the HT twist $T_{HT}[G, V]$. It is physically defined as the cohomology of the

HT supercharge acting on the full space of local operators in the untwisted theory. It has a $\mathbb{Z}$-valued cohomological grading 'F', and an additional non-cohomological $\frac{1}{2}\mathbb{Z}$-valued grading 'J', corresponding to rotation in the holomorphic plane (mixed with an $SU(2)$ R-symmetry). This space has been well studied from other perspectives. In particular:

- Its graded Euler character is the "Schur index" of $T_{G,V}$,

$$\chi_q \, \mathrm{Ops}_{G,V} := \mathrm{Tr}_{\mathrm{Ops}_{G,V}}(-1)^F q^J \;=\; I_{\mathrm{Schur}}[T_{G,V}]\,. \tag{2.2.2.1}$$

  The Schur index, introduced in [**GRR$^+$11**, **GRRY13**], is a particular specialization of the 4d $\mathcal{N}=2$ superconformal index [**KMMR07**, **Röm06**]; though it makes sense even when a 4d $\mathcal{N}=2$ theory is not conformal. Here and what follows, we will use q as a formal variable counting the weight of the loop rotation.

- The space $\mathrm{Ops}_{G,V}$ itself is the vacuum module of a Poisson vertex algebra $\mathcal{V}_{G,V}$. This Poisson vertex algebra was constructed for general 4d $\mathcal{N}=2$ theories from a more physical perspective by Oh and Yagi [**OY20**], and from a mathematical perspective by Dylan Butson [**But21**], as a BRST reduction of classical beta-gamma algebras valued in T^*V. When $T_{G,V}$ is superconformal — meaning quadratic indices satisfy $C_2(N) = C_2(\mathfrak{g})$ — the vertex algebra can be further quantized by introducing an Omega background, yielding a VOA $\mathcal{V}^{\hbar}_{G,V}$. These VOA's were first introduced in superconformal theories by Beem-Lemos-Liendo-Peelaers-Rastelli-Rees [**BLL$^+$15**]. However, the work of [**BLL$^+$15**] did not define this vertex algebra from the point of view of local operators in $T_{G,V}$. The fact that the two pictures coincide is a nontrivial result, and can be explained using a "cigar-like" reduction. This is explained in [**OY19**] and [**Jeo19**] from a physical perspective and [**But21**] from a mathematical perspective. This deformation does not alter the underlying vector space of the vacuum module, so

$$\mathrm{Ops}_{G,V} \simeq \mathcal{V}_{G,V} \simeq \mathcal{V}^{\hbar}_{G,V}\,. \tag{2.2.2.2}$$

In essence, the Poisson vertex algebra $\mathcal{V}_{G,V}$ is the algebra of functions on the jet space $J_\infty(T^*(V/G))$, and the Poisson structure of the jet space induces the Poisson structure on $\mathcal{V}_{G,V}$. The relation between this Poisson vertex algebra and the Poisson geometry of T^*V/G is explained by the "cigar-reduction" mentioned above: that the cigar reduction of $T_{HT}[G,V]$ is $\mathcal{T}_{HT}[G,V]$ with the Neumann boundary condition, and that under the HT twist, the space of boundary local operators is the Poisson vertex algebra of $J_\infty(T^*(V/G))$. This is of course non-rigorous mathematically as it is difficult to analyze the jet space $J_\infty(T^*(V/G))$. Nevertheless, these relations between physical theories give guidance to the study of the Poisson vertex algebra $\mathcal{V}_{G,V}$.

We would like to understand this algebra of local operators $\mathcal{V}_{G,V}$ from the point of view of the category of line operators in the HT twist of a 4d $\mathcal{N}=2$ theory. Physically, the objects of this category are line operators supported on a line in the topological $\mathbb{R}^2$ plane and the origin of the holomorphic $\mathbb{C}$ plane. The category contains half-BPS Wilson-'t Hooft lines, as well as more general quarter-BPS line operators. The category was given a geometric description by Cautis and Williams, in [**CW19**] for pure gauge theory ($V=0$) and [**CWar**] for general V, following the physical predictions of [**Kap06a**, **Kap06b**]. This category is described as the category of $G(\mathcal{O})$-equivariant coherent sheaves on $\mathcal{R}_{G,V}$, denoted by $\mathrm{Coh}(G_\mathcal{O}\setminus\mathcal{R}_{G,V})$. As we have seen before, the K theory of this category, together with its convolution product, gives the algebra of functions on the Coulomb branch of $T_{HT}^{4d}[G,V]$. The multiplication on the K group should be the de-categorification of a monoidal structure on $\mathrm{Coh}(G_\mathcal{O}\setminus\mathcal{R}_{G,V})$, defined again using the convolution diagram of $\mathcal{R}_{G,V}$, such that if $\mathbb{1}$ is the tensor identity of this category, then:

(2.2.2.3)
$$\mathrm{Ops}_{G,V}\simeq \mathrm{End}_{\mathrm{Coh}(G_\mathcal{O}\setminus\mathcal{R}_{G,V})}(\mathbb{1})\,.$$

This expectation is in fact not easy to work with. It is of course not too complicated to describe the limit of the K group as in equation (2.2.1.1), since one only concerns with the limit as a vector space. On the other hand, if one wants to replace the entries in equation (2.2.1.1) by categories:

(2.2.2.4)
$$\mathrm{Coh}(G_\mathcal{O}\setminus\mathcal{R}_{G,V}):=\varinjlim_{m,n,k}\mathrm{Coh}(G_k\setminus\mathcal{R}_{G,V,m,n})$$

then one needs to specify in which sense the limit is taken. Over the last few decades, much effort has been made towards using the techniques of derived algebraic geometry to rigorously define aspects of the physical quantum field theories, and this is one example of this. The analysis is made possible with the invention of ∞-categories and derived algebraic geometry [**Lur04**, **Lur09**]. This is not a work in which we could give the full definitions of a ∞-category and algebraic geometry in this context, and we are nowhere close to being an expert at this subject. We will simply comment that the rough idea of ∞-category as in [**Lur09**] is to view a category with all the higher data (functors and natural transformations, etc.) as a topological space built out of simplices. Objects are understood as points, morphisms as lines (1-simplex), natural transformation of morphisms as triangles (2-simplex) and so on. Then a functor between infinity categories is simply a functor between these simplices, and a continuous functor is a functor that is continuous as a map between spaces. In this context, one can simply define the limit of categories as limit of spaces, thus encompassing all the higher structures in one go. Limit in this sense behave much better than the limit of triangulated categories or abelian categories. One can recover the abelian or triangulated category by taking the heart with respect to a t structure.

The space $\mathcal{R}_{G,V}$ and the quotient $G(\mathcal{O})\setminus\mathcal{R}_{G,V}$ are examples of DG ind-schemes: namely stacks that can be represented by a colimit of schemes. Using derived geometry, one can define the category of sheaves as a colimit. The structure of a DG indscheme and its category of sheaves are studied in [**GR14**, **GR19**, **GR17**] for locally almost finite type, and [**Ras20**] in general. In Chapter 4, we will use the machinery of [**GR14**, **GR19**, **GR17**, **Ras20**] to define the category of ind-coherent sheaves on $\mathcal{R}_{G,V}/G(\mathcal{O})$. The monoidal structure is carefully defined in [**CW19**, **CWar**], which will indicates to us the identity object $\mathbb{1}$ in $\mathrm{Coh}(G(\mathcal{O})\setminus\mathcal{R}_{G,V})$. We will then compute explicitly the space $\mathrm{End}_{\mathrm{Coh}(G_{\mathcal{O}}\setminus\mathcal{R}_{G,V})}(\mathbb{1})$ and show that it coincides with $\mathcal{V}_{G,V}$ as a commutative algebra.

In [**But21**], the author showed that the category of line operators in HT twist has the structure of a factorization $\mathbb{E}_1$-category ([**But21**, Section 5.11]). This structure should give rise to the structure of an $\mathbb{E}_2$ factorization algebra to $\mathrm{Ops}_{G,V}$, which is the aforementioned Poisson vertex algebra structure. We hope that the explicit computation done in this thesis, together with the procedure outlined in [**But21**] can help rigorously produce factorization algebras from the category of line operators.

2.2.3. Line Operators and Schur Index. The relation between local operators and line operators goes beyond the unit object in $\mathrm{Coh}(G(\mathcal{O})\setminus\mathcal{R}_{G,V})$. It is expected that for any pair of objects L_1, L_2 in $\mathrm{Coh}(G(\mathcal{O}) \setminus \mathcal{R}_{G,V})$, there is a graded vector space $\mathrm{Hom}_{\mathcal{R}_{G,V}/G(\mathcal{O})}(L_1, L_2)$, graded by F and J with finite-dimensional graded pieces, that is quasi-isomorphic to the space of local operators at the junction between the two line operators L_1 and L_2. Namely, we expect:

$$\mathrm{Ops}_{G,V}(L_1, L_2) \cong \mathrm{Hom}_{\mathcal{R}_{G,V}/G(\mathcal{O})}(L_1, L_2) \tag{2.2.3.1}$$

such that the natural morphism $\mathrm{Hom}_{\mathcal{R}_{G,V}/G(\mathcal{O})}(L_1, L_2) \otimes \mathrm{Hom}_{\mathcal{R}_{G,V}/G(\mathcal{O})}(L_2, L_3) \to \mathrm{Hom}_{\mathcal{R}_{G,V}/G(\mathcal{O})}(L_1, L_3)$ corresponds to collision of line operators $\mathrm{Ops}_{G,V}(L_1, L_2) \otimes \mathrm{Ops}_{G,V}(L_2, L_3) \to \mathrm{Ops}_{G,V}(L_1, L_3)$.

Although the space of local operators $\mathrm{Ops}_{G,V}(L_1, L_2)$ is generally not known, physicists have computed its graded Euler character using supersymmetric localization technique [**CGS16**]. More specifically, let G be a reductive Lie group with maximal torus T. As we have seen, there is a localization map $K_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R}_{G,V}) \to K_{T(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_T)_{loc}$, which is an algebra embedding. Recall here subscript "loc" denotes the localization along root hyperplanes. The idea of the computation in [**CGS16**] is that there exists a specific function $\Pi(q, s, m)$ called the *half index*, which is a formal series in q and s that depends on m, where s is a coordinate on T and m is a co-character of T. Moreover, the localized ring $K_{T(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_T)_{loc}$, and therefore $K_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R}_{G,V})$, acts on $\Pi(q, s, m)$ as abelian difference operators. Using this action, one obtain a half index of a line operator L as $(L\Pi)(q, s, m)$. The defect Schur index, namely the graded Euler character of $\mathrm{Ops}_{G,V}(L_1, L_2)$, is computed as follows:

$$\chi_q(L_1, L_2) := \mathrm{Tr}_{\mathrm{Ops}_{G,V}(L_1,L_2)}(-1)^F q^J = \sum_m \int_T [\mathrm{d}s]_m (L_1\Pi)(q, s, -m)(L_2\Pi)(q, s, m). \tag{2.2.3.2}$$

Here on the RHS, the integral measure $[\mathrm{d}s]_m$ is a certain shifted Haar measure w.r.t the co-character m. Unfortunately the mathematical origin of this formula is unknown to the author, we therefore leave the precise definition of the above formula, as well as its mathematical explanation, to a future work. In the present work however, we apply the method that we developed to a special class of line operators.

More specifically, when G is semi-simple, the affine grassmannian Gr_G is reduced, and its connected components are labelled by the miniscule dominant coweights of G (see Section 4.1.1 for more details). For each miniscule co-weight μ, the connected component Gr_G^μ has a unique closed smooth $G(\mathcal{O})$ orbit, which we denote by $\mathrm{Gr}_{G,\mu}$. This is called the miniscule orbit corresponding to μ. The method developed in this thesis allows us to compute $\operatorname{Hom}(L_1, L_2)$ when L_1 and L_2 are vector bundles over the miniscule orbit $\operatorname{Gr}_{G,\mu}$. In Section 4.2.3, we will focus on the case when $G = \operatorname{PSL}(2)$ and μ the unique non-zero miniscule orbit, and compute the Hom space $\operatorname{Hom}(L_1, L_2)$ for specific line bundles L_1 and L_2. We compare our results to the results of [**CGS16**].

CHAPTER 3

From Vertex Operator Algebra to Geometry of Branches in 3d $\mathcal{N}=4$ Abelian Gauge Theories

In this chapter, we study the category of line operators for A and B twist of 3d $\mathcal{N}=4$ abelian gauge theories, using boundary VOA approach. For a charge matrix $\rho : \mathbb{Z}^r \to \mathbb{Z}^n$ which we assume throughout to induce a short exact sequence as in equation (2.1.2.14). We will denote by $\mathcal{T}_\rho$ the 3d $\mathcal{N}=4$ theory $\mathcal{T}[T,V]$ defined by ρ, and by $\mathcal{T}_{A,\rho}$ and $\mathcal{T}_{B,\rho}$ the topological twists. The structure of this chapter is as follows:

- In Section 3.1, we give mathematically-rigorous definition of $V_{A,\rho}$ and $V_{B,\rho}$, and provide free-field realizations of them, i.e., embeddings of these VOAs into lattice VOAs. We also define Morita-equivalent VOAs $\widetilde{V}_{A,\rho}$ and $\widetilde{V}_{B,\rho}$. We prove the isomorphism $V_{A,\rho} \cong V_{B,\rho^\vee}$ and $\widetilde{V}_{A,\rho} \cong \widetilde{V}_{B,\rho^\vee}$ which is the mirror symmetry statement for the boundary VOAs.
- In Section 3.2, we give a definition of abelian categories $\mathcal{C}_{A,\rho}$ and $\mathcal{C}_{B,\rho}$ which are full subcategories of generalized modules of $V_{A,\rho}$ and $V_{B,\rho}$. We show that they have the structure of braided tensor categories with exact fusion. The bounded derived category $D^b\mathcal{C}_{A,\rho}$ and $D^b\mathcal{C}_{B,\rho}$ are proposed as the category of line operators for $\mathcal{T}_{A,\rho}$ and $\mathcal{T}_{B,\rho}$ respectively. We prove an equivalence of BTC $\mathcal{C}_{A,\rho} \simeq \mathcal{C}_{B,\rho^\vee}$, which is the mirror symmetry statement for the category of line operators. We also derive a quantum group $U_q(\mathfrak{g}_*(\rho))$ whose representation theory is equivalent to $\mathcal{C}_{B,\rho}$ as an abelian category, and conjecture that this equivalence upgrades to the equivalence of BTCs.
- In Section 3.3, we show that derived extension group in $D^b\mathcal{C}_{B,\rho}$ correctly reproduces the space $\mathcal{M}_{H,\rho}$, and that the resolution of $\mathcal{M}^\xi_{H,\rho}$ can be constructed in $D^b\mathcal{C}_{B,\rho}$ by choosing an algebra object determined by ξ. We show that the VOA $V_{B,\rho}$ admit a $\hbar$-adic version $V^\hbar_{B,\rho}$ which is naturally a sheaf over the stack $\mu^{-1}(0)/T$ where $\mu : T^*V \to \mathfrak{t}^*$ is the moment

map. We show that the localization of $V^{\hbar}_{B,\rho}$ to $\mathcal{M}^{\xi}_{H,\rho}$ is the affine Lie algebra associated to the shifted tangent Lie algebra of $\mathcal{M}^{\xi}_{H,\rho}$.

3.1. Vertex Operator Algebra on the Boundary of Twisted Abelian Gauge Theories

3.1.1. Charge Lattice and its Decompositions. This section serves a technical purpose for the study of these boundary VOAs. Denote now by Λ the sublattice of $\mathbb{Z}^n$ defined by the image of ρ. As in equation (2.1.2.14), we will assume that ρ is faithful, namely there exists $\tau : \mathbb{Z}^{n-r} \to \mathbb{Z}^n$ such that the following is a short exact sequence:

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{\rho} \mathbb{Z}^n \xrightarrow{\tau} \mathbb{Z}^{n-r} \longrightarrow 0 \tag{3.1.1.1}$$

We denote by $\rho^\vee = \tau^{\mathsf{T}}$. Since $\rho^{\mathsf{T}}\rho$ may not be invertible over $\mathbb{Z}$, Λ may not have an orthogonal complement. The orthogonal set of Λ in $\mathbb{Z}^n$, which we denote by $\Lambda^\perp$, is simply $\mathrm{Im}(\rho^\vee)$, since the matrix $(\rho, \rho^\vee)$ is full rank. The sublattice

$$\Lambda \oplus \Lambda^\perp \tag{3.1.1.2}$$

is full rank in $\mathbb{Z}^n$ and therefore the quotient $\mathbb{Z}^n/(\Lambda \oplus \Lambda^\perp)$ is torsion with index (or cardinality) $\det(\rho^{\mathsf{T}}\rho)$. Denote by Λ' the linear dual of Λ, which is defined as the $\mathbb{Z}$ submodule of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ that has integer inner product with Λ. Similarly, we can define $(\Lambda^\perp)'$. By orthogonality, we can write:

$$\mathbb{Q}^n = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \bigoplus \Lambda^\perp \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{3.1.1.3}$$

This will give us natural maps:

$$\mathbb{Z}^n \longrightarrow \Lambda', \ \mathbb{Z}^n \longrightarrow (\Lambda^\perp)' \tag{3.1.1.4}$$

given by orthogonal projections. We will denote the projection maps as $\mathbb{Q}$ valued matrices $\Pi : \mathbb{Z}^n \to \mathbb{Q}^r$ and $\Pi^\vee : \mathbb{Z}^n \to \mathbb{Q}^{n-r}$, such that $\rho\Pi$ is the projection onto Λ and $\rho^\vee\Pi^\vee$ is the projection onto $\Lambda^\vee$. These matrices will satisfy:

$$\Pi\rho = \mathrm{Id}_r, \qquad \Pi^\vee\rho^\vee = \mathrm{Id}_{n-r}, \qquad \Pi\rho^\vee = \Pi^\vee\rho = 0, \qquad \rho\Pi + \rho^\vee\Pi^\vee = \mathrm{Id}_n. \tag{3.1.1.5}$$

One can write $\Pi = \rho(\rho^{\mathsf{T}}\rho)^{-1}$ and $\Pi^{\vee} = \tau^{\mathsf{T}}(\tau\tau^{\mathsf{T}})^{-1}$. Since $\mathbb{Z}^n$ is self dual, the above maps induce isomorphisms:

$$\Lambda' \cong \mathbb{Z}^n/(\Lambda^{\perp}),\ \mathbb{Z}^n/\Lambda \cong (\Lambda^{\perp})'. \tag{3.1.1.6}$$

This leads to the isomorphisms:

$$\Lambda'/\Lambda \cong \mathbb{Z}^n/(\Lambda \oplus \Lambda^{\perp}) \cong (\Lambda^{\perp})'/(\Lambda^{\perp}). \tag{3.1.1.7}$$

We denote the above quotient group by H. The natural embedding:

$$\mathbb{Z}^n \to \Lambda' \oplus (\Lambda^{\perp})' \subseteq \mathbb{Q}^n \tag{3.1.1.8}$$

maps $\mathbb{Z}^n$ to the subgroup of $\Lambda' \oplus (\Lambda^{\perp})'$ whose image under the quotient by $\Lambda \oplus \Lambda^{\perp}$ lives in the diagonal subgroup H. Namely, we have the following base-change diagram:

$$\begin{array}{ccccc} & & \mathbb{Z}^n & \longrightarrow & H \\ & & \downarrow & & \downarrow \Delta \\ \Lambda \oplus \Lambda^{\perp} & \longrightarrow & \Lambda' \oplus (\Lambda^{\perp})' & \xrightarrow{\text{proj}} & H \oplus H \end{array} \tag{3.1.1.9}$$

This leads to the following decomposition of $\mathbb{Z}^n$:

$$\mathbb{Z}^n = \bigcup_{\substack{\lambda \in \Lambda',\ \lambda^{\perp} \in (\Lambda^{\perp})' \\ \overline{\lambda} = \overline{\lambda^{\perp}} \in H}} \{\lambda + \lambda^{\perp}\}. \tag{3.1.1.10}$$

When considering gauging operations, we will repeatedly use equation (3.1.1.10) to decompose and simplify the lattice VOA.

Example. Consider the case:

$$\rho = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.1.1.11}$$

Namely the gauge group is $U(1)$ and the representation is $\mathbb{C}^2$ with weight $1, 1$. In this case, $\rho^{\vee}$ is given by $(1, -1)^{\mathsf{T}}$. The lattice Λ' is generated by $\frac{1}{2}\rho$ while the lattice $(\Lambda^{\perp})'$ is generated by $\frac{1}{2}\rho^{\vee}$, and $H = \mathbb{Z}_2$. For any element $v = (a, b)^\intercal \in \mathbb{Z}^2$, we can write:

$$v = a(\frac{1}{2}\rho + \frac{1}{2}\rho^\vee) + b(\frac{1}{2}\rho - \frac{1}{2}\rho^\vee) = \frac{a+b}{2}\rho + \frac{a-b}{2}\rho^\vee. \tag{3.1.1.12}$$

Notice that $\frac{a+b}{2}$ and $\frac{a-b}{2}$ differ by an integer, so they descend to the same element in $H = \mathbb{Z}_2$.

We will use this to decompose a Fock module of a Heisenberg VOA into tensor products of Fock modules of Heisenberg subVOAs. Let us exhibit an example here to illustrate the point. Given lattices Λ and $\Lambda^\perp$ as above, defined by matrices ρ and $\rho^\vee$. Let H_ϕ be the Heisenberg VOA generated by $\partial\phi^i$ with OPE:

$$\partial\phi^i(z)\partial\phi^j(w) \sim \frac{\delta^{ij}}{(z-w)^2}. \tag{3.1.1.13}$$

It is clear that the Heisenberg VOA H_ϕ decomposes nicely (using the short-exact sequence in equation (2.1.2.14)):

$$H_\phi = H_{\rho^\intercal\phi} \otimes H_{\tau\phi}. \tag{3.1.1.14}$$

Here we will understand $H_{\rho^\intercal\phi}$ as the Heisenberg VOA generated by $\sum \rho_{ia}\partial\phi^i$. Similarly for $H_{\tau\phi}$. Given $\lambda \in \mathbb{Z}^n$, if we rewrite $\lambda = \mu + \mu^\perp$ where $\mu \in \Lambda'$ and $\mu^\perp \in \Lambda^\perp$, then the Fock module $\mathcal{F}_{\lambda\cdot\phi}$ admits a similar decomposition into modules of the Heisenberg subVOAs:

$$\mathcal{F}_{\lambda\cdot\phi} = \mathcal{F}_{\mu\cdot\phi} \otimes \mathcal{F}_{\mu^\perp\cdot\phi}. \tag{3.1.1.15}$$

Here we understand $\mathcal{F}_{\mu\cdot\phi}$ as the Fock module of $H_{\rho^\intercal\phi}$. Technically speaking, we should write $\sum \mu_i\phi^i$ as $\sum \widetilde{\mu}_a {\rho_i}^a\phi^i$ and write $\mathcal{F}_{\widetilde{\mu}\rho^\intercal\phi}$ as the Fock module. We write $\mu\cdot\phi$ for simplicity.

We will also use the following splitting of the exact sequence (2.1.2.14):

$$0 \longrightarrow \mathbb{Z}^r \underset{\rho}{\overset{\widetilde{\tau}}{\leftrightarrows}} \mathbb{Z}^n \underset{\tau}{\overset{\widetilde{\rho}}{\leftrightarrows}} \mathbb{Z}^{n-r} \longrightarrow 0 \tag{3.1.1.16}$$

namely we choose matrices $\widetilde{\rho}$ such that $\tau\widetilde{\rho} = \mathrm{Id}_{n-r}$, and choose the co-splitting $\widetilde{\tau}$. Note that this is always possible over $\mathbb{Z}$. More concretely, such splitting means that the matrix:

$$\left(\begin{array}{c|c} \rho & \widetilde{\rho} \end{array}\right) \tag{3.1.1.17}$$

is invertible with inverse:

$$\begin{pmatrix} \widetilde{\tau} \\ \tau \end{pmatrix}. \tag{3.1.1.18}$$

In other words, the following equations are satisfied:

$$\tau \cdot \rho = 0,\ \widetilde{\tau} \cdot \rho = \mathrm{Id}_r,\ \widetilde{\tau} \cdot \widetilde{\rho} = 0,\ \tau \cdot \widetilde{\rho} = \mathrm{Id}_{n-r}. \tag{3.1.1.19}$$

Consequently $\rho\widetilde{\tau} + \widetilde{\rho}\tau = \mathrm{Id}_n$. This will be used in the following sections to perform field redefinitions in order to identify different free field realizations.

3.1.2. The A Side Boundary VOA.

3.1.2.1. *Definition of A Side Boundary VOA.* As we have stated in Section 2.1.3, the VOA living on the Neumann boundary condition is the BRST reduction of symplectic bosons. We will use the relative BRST cohomology introduced in Appendix A.

More specifically, consider n copies of symplectic bosons, also known as the $\beta\gamma$ VOA:

$$V_{\beta\gamma}^{\otimes n}. \tag{3.1.2.1}$$

This has $\mathfrak{gl}(1)^n$ symmetry generated by the currents:

$$-:\beta^i\gamma^i:,\ 1 \le i \le n. \tag{3.1.2.2}$$

Neumann boundary conditions for the vector multiplets will introduce ghosts valued in the Lie algebra $\mathfrak{gl}(1)^r$ together with a BRST operator that gauges the following subset of currents, which are generators of the $\mathfrak{gl}(1)^r$ action on the symplectic bosons:

$$J'_a = \sum_i -\rho_{ia}:\beta^i\gamma^i:. \tag{3.1.2.3}$$

However, this BRST operation will not be correct because the $\mathfrak{gl}(1)^r$ currents have nonzero anomaly [**CDG20**]. This is reflected in the fact that the $\mathfrak{gl}(1)^r$ currents J'_a satisfy the relation of an affine Kac-Moody algebra of $\mathfrak{gl}(1)$ at level $-\sum_i \rho_{ia}\rho^i{}_b$:

$$J'_a(z)J'_b(w) \sim \frac{-\sum_i \rho_{ia}\rho^i{}_b}{(z-w)^2}. \tag{3.1.2.4}$$

The computation of the boundary anomaly for general Lie group was listed in [**CDG20**]. These are simply the matrix elements of $-\rho^{\mathsf{T}}\rho$. In order to cancel this anomaly, we need to couple this VOA with a boundary CFT with level given by the opposite of the above anomaly [**CDG20**]. To do so, we first tensor $V_{\beta\gamma}^{\otimes n}$ with a set of bc ghosts:

$$V_{\beta\gamma}^{\otimes n} \otimes V_{bc}^{\otimes n}. \tag{3.1.2.5}$$

This VOA now has $\mathfrak{gl}(1)^{2n}$ symmetry, whose currents are given by:

$$-:\beta^i\gamma^i:, \qquad :b^i c^i:. \tag{3.1.2.6}$$

Instead of the J'_a, we will consider the $\mathfrak{gl}(1)^r$ currents given by:

$$J_a = \sum_i \rho_{ia}\left(-:\beta^i\gamma^i: + :b^i c^i:\right),\ 1 \leq i \leq r. \tag{3.1.2.7}$$

This $\mathfrak{gl}(1)^r$ action will be anomaly free, which is reflected by the fact that J_i satisfy the relation of an affine Kac-Moody at level 0. We can now consider the Neumann boundary condition for vector multiplets, which introduces another set of bc ghosts valued in the Lie algebra $\mathfrak{gl}(1)^r$ (whose fields are denoted by b^a and c^a):

$$V_{\beta\gamma}^{\otimes n} \otimes V_{bc}^{\otimes n} \otimes V_{bc}^{\otimes r} \tag{3.1.2.8}$$

with a BRST differential given by:

$$Q_{BRST} = \sum_a \oint \mathrm{d}z\, c^a J_a. \tag{3.1.2.9}$$

$Q^2 = 0$ since now the JJ OPE is trivial. We now arrive at the boundary VOA for the A twist as suggested in [**CG19**]:

Definition 3.1.1. *The boundary VOA on a Neumann boundary condition in $\mathcal{T}_{A,\rho}$ is defined by:*

$$V_{A,\rho} := H_{BRST}\left(\mathfrak{gl}(1)^r, V_{\beta\gamma}^{\otimes n} \otimes V_{bc}^{\otimes n}\right) = H^*\left(\left(V_{\beta\gamma}^{\otimes n} \otimes V_{bc}^{\otimes n} \otimes V_{bc}^{\otimes r}\right)^{\mathrm{rel}}, Q_{BRST}\right). \tag{3.1.2.10}$$

Remark 3.1.2. *One can obtain a conformal element for $V_{A,\rho}$ by starting with a conformal element $|\omega\rangle$ in the BRST complex such that the conformal degree of c^a is 0 and that $\omega_1 c^a_0 J_{a,-1}|0\rangle = 0$. In that case, we have:*

$$
\begin{aligned}
QY(\omega, w) &= \sum \oint \mathrm{d}(z-w) c^a(z) J_a(z) Y(\omega, w) = \sum \oint \mathrm{d}(z-w) Y(\omega, w) c^a(z) J_a(z) \\
&= \sum \oint \mathrm{d}(z-w) \frac{c^a(z)J_a(z)}{(z-w)^2} + \frac{\partial_z(c^a(z)J_a(z))}{w-z} = 0.
\end{aligned} \tag{3.1.2.11}
$$

Thus the class of $|\omega\rangle$ in the cohomology is a conformal element.

3.1.2.2. *Free Field Realization of $V_{A,\rho}$.* In this section, we will derive a free field realization of $V_{A,\rho}$, which is an embedding of $V_{A,\rho}$ into a lattice VOA. To begin with, let us consider the following free field realization of the symplectic boson VOA $V_{\beta\gamma}$ [**AW22**]. Let H_ϕ be the Heisenberg VOA generated by $\partial\phi^i$ for $1 \le i \le n$ with OPE:

$$
\partial\phi^i(z)\partial\phi^j(w) \sim \frac{\delta^{ij}}{(z-w)^2}. \tag{3.1.2.12}
$$

Similarly, let H_ψ be the Heisenberg VOA generated by $\partial\psi^i$ for $1 \le i \le n$ with OPE:

$$
\partial\psi^i(z)\partial\psi^j(w) \sim \frac{-\delta^{ij}}{(z-w)^2}. \tag{3.1.2.13}
$$

Consider the lattice VOA extension V_L of $H_\phi \otimes H_\psi$ by the lattice L spanned by $|\phi^i + \psi^i\rangle$. There is an embedding $V_{\beta\gamma}^{\otimes n} \hookrightarrow V_L$ given by:

$$
\beta^i(z) \mapsto {:}e^{\phi^i(z)+\psi^i(z)}{:}, \qquad \gamma^i(z) \mapsto -{:}\partial\phi^i(z) e^{-\phi^i(z)-\psi^i(z)}{:}. \tag{3.1.2.14}
$$

For each linear combination $\mu = \sum a_i\phi^i + b_i\psi^i$ where $a_i - b_i \in \mathbb{Z}$ for all i, the Fock module $\mathcal{F}_\mu$ of $H_\phi \otimes H_\psi$ can be lifted to a module of V_L, which we call $V_{L,\mu}$. When $a_i \in \mathbb{Z}$, we define operator $S^i : V_{L,\mu} \to V_{L,\mu+\phi^i}$ by:

$$
S^i = \oint \mathrm{d}z {:}\exp\left(\phi^i(z)\right){:} \tag{3.1.2.15}
$$

It is well-known that the following is true:

$$
V_{\beta\gamma}^{\otimes n} = \bigcap_{1\le i\le n} \mathrm{Ker}\left(S^i : V_L \to V_{L,\phi^i}\right). \tag{3.1.2.16}
$$

Let us use the notation of Fock modules to express $V_{\beta\gamma}^{\otimes n}$. For each vector of integers $\lambda = (\lambda_1, \dots, \lambda_n)$, we denote by $\mathcal{F}_{\lambda\cdot\phi}$ the Fock module coresponding to $\sum \lambda_i \phi^i$. Similarly we denote by $\mathcal{F}_{\lambda\cdot\psi}$ the Fock module coresponding to $\sum \lambda_i \psi^i$. The module:

$$\bigcap_i \mathrm{Ker}\left(S^i : \mathcal{F}_{\lambda\cdot\phi} \to \mathcal{F}_{\lambda\cdot\phi+\phi^i}\right) \tag{3.1.2.17}$$

will be denoted by $M_{\lambda\cdot\phi}$. One can quickly recognizes that this is a simple module of the singlet VOA $M(2)^{\otimes n}$, and is simply given by:

$$M_{\lambda\cdot\phi} = \bigotimes_i M_{\lambda_i}. \tag{3.1.2.18}$$

PROPOSITION 3.1.3. *The VOA $V_{\beta\gamma}^{\otimes n}$ is a simple current extension of $M(2)^{\otimes n} \otimes H_\psi$, and decomposes as:*

$$V_{\beta\gamma}^{\otimes n} = \bigoplus_{\lambda\in\mathbb{Z}^n} M_{\lambda\cdot\phi} \otimes \mathcal{F}_{\lambda\cdot\psi}. \tag{3.1.2.19}$$

One can write explicitly how currents from both sides correspond to each other. We will write some of them out:

$$- :\beta^i\gamma^i: \mapsto \partial\psi^i, \qquad -:\beta^i\partial\gamma^i: \mapsto \frac{1}{2}:\partial\phi^i\partial\phi^i: - \frac{1}{2}\partial^2\phi^i - \frac{1}{2}:\partial\psi^i\partial\psi^i: + \frac{1}{2}\partial^2\psi^i. \tag{3.1.2.20}$$

The conformal element we will choose will be:

$$L_{\beta\gamma}(z) = \frac{1}{2}\sum :\partial\beta_i\gamma^i: - :\beta_i\partial\gamma^i:, \tag{3.1.2.21}$$

which can be expressed in the above free field algebra as:

$$L_{\beta\gamma}(z) = \frac{1}{2}\sum :\partial\phi^i\partial\phi_i: - \frac{1}{2}\partial^2\phi^i - \frac{1}{2}:\partial\psi^i\partial\psi_i:. \tag{3.1.2.22}$$

Let us also introduce Heisenberg VOA $H_{\widetilde{\phi}}$ with OPE

$$\partial\widetilde{\phi}^i(z)\partial\widetilde{\phi}^j(w) \sim \frac{\delta^{ij}}{(z-w)^2}. \tag{3.1.2.23}$$

The Bose-Fermi correspondence [**FBZ04**] gives an isomorphism of $V_{bc}^{\otimes n}$ with the extension of Heisenberg VOA by the complete lattice spanned by $\widetilde{\phi}^i$. In other words, there is an isomorphism of VOAs:

$$V_{bc}^{\otimes n} = \bigoplus_{\lambda \in \mathbb{Z}^n} \mathcal{F}_{\lambda \cdot \widetilde{\phi}}. \tag{3.1.2.24}$$

In this correspondence, we will choose the following conformal element:

$$L_{bc} = \frac{1}{2} \sum :\partial b_i c^i: + :\partial c_i b^i: = \frac{1}{2} \sum :\partial \widetilde{\phi}^i \partial \widetilde{\phi}_i:. \tag{3.1.2.25}$$

In the BRST complex $V_{\beta\gamma}^{\otimes n} \otimes V_{bc}^{\otimes n} \otimes V_{bc}^{\otimes r}$, if we uses the following conformal element:

$$L(z) = L_{\beta\gamma} + L_{bc} + \sum :\partial c_a b^a:, \tag{3.1.2.26}$$

then this is a conformal element under which the conformal degree of $c^a J_a$ is 1 and that $L_1 | c^a J_a \rangle = 0$. Thus, $L(z)$ descends to cohomology and is a conformal element for $V_{A,\rho}$.

Now we are ready to compute the BRST cohomology of $V_{\beta\gamma}^{\otimes n} \otimes V_{bc}^{\otimes n}$. Using the free field realization, we can write:

$$V_{\beta\gamma}^{\otimes n} \otimes V_{bc}^{\otimes n} = \bigoplus_{\lambda, \mu \in \mathbb{Z}^n} M_{\lambda \cdot \phi} \otimes \mathcal{F}_{\lambda \cdot \psi} \otimes \mathcal{F}_{\mu \cdot \widetilde{\phi}}. \tag{3.1.2.27}$$

We will use the decomposition of Section 3.1.1 to decompose the above as:

$$V_{\beta\gamma}^{\otimes n} \otimes V_{bc}^{\otimes n} = \bigoplus_{\substack{\lambda, \mu \in \Lambda' \\ \lambda^\perp, \mu^\perp \in (\Lambda^\perp)' \\ \overline{\lambda} = \overline{\lambda^\perp}, \overline{\mu} = \overline{\mu^\perp}}} M_{(\lambda + \lambda^\perp) \cdot \phi} \otimes \mathcal{F}_{\lambda \cdot \psi} \otimes \mathcal{F}_{\lambda^\perp \cdot \psi} \otimes \mathcal{F}_{\mu \cdot \widetilde{\phi}} \otimes \mathcal{F}_{\mu^\perp \cdot \widetilde{\phi}}. \tag{3.1.2.28}$$

In the language of the free field realizations above, the currents J_a are given by:

$$J_a(z) = \sum \rho_{ia} (-\partial \psi^i(z) - \partial \widetilde{\phi}^i(z)), \tag{3.1.2.29}$$

and so it only acts on the part:

$$\mathcal{F}_{\lambda \cdot \psi} \otimes \mathcal{F}_{\mu \cdot \widetilde{\phi}}, \tag{3.1.2.30}$$

since $(\Lambda^{\perp})'$ is orthogonal to Λ. This makes the computation of BRST cohomology straightforward since the BRST cohomology of Fock modules are very simple (see Section A.0.5).

$$H_{BRST}(\mathfrak{gl}(1)^r, \mathcal{F}_{\lambda\cdot\psi}\otimes\mathcal{F}_{\mu\cdot\widetilde{\phi}}) = \delta_{\lambda,\mu}|\lambda\rangle\otimes|\mu\rangle, \tag{3.1.2.31}$$

one can compute the BRST cohomology easily:

$$H_{BRST}(\mathfrak{gl}(1)^r, V_{\beta\gamma}^{\otimes n}\otimes V_{bc}^{\otimes n}) = \bigoplus_{\substack{\lambda\in\Lambda' \\ \lambda^{\perp},\mu^{\perp}\in(\Lambda^{\perp})' \\ \overline{\lambda}=\overline{\lambda^{\perp}}=\overline{\mu^{\perp}}}} M_{(\lambda+\lambda^{\perp})\cdot\phi}\otimes\mathcal{F}_{\lambda^{\perp}\cdot\psi}\otimes\mathcal{F}_{\mu^{\perp}\cdot\widetilde{\phi}}. \tag{3.1.2.32}$$

Let us write this in the following way. Let $\mathcal{V}_{\rho}$ be the extension of the Heisenberg VOA $H_{\phi}\otimes H_{\tau\psi}\otimes H_{\tau\widetilde{\phi}}$ by the Fock modules:

$$\mathcal{F}_{(\lambda+\lambda^{\perp})\cdot\phi}\otimes\mathcal{F}_{\lambda^{\perp}\cdot\psi}\otimes\mathcal{F}_{\mu^{\perp}\cdot\widetilde{\phi}} \tag{3.1.2.33}$$

satisfying $\lambda\in\Lambda', \lambda^{\perp}, \mu^{\perp}\in(\Lambda^{\perp})'$ and $\overline{\lambda}=\overline{\lambda^{\perp}}=\overline{\mu^{\perp}}$. For each linear combination $\sigma=\sum\sigma_i\phi^i$ where $\sigma^i\in\mathbb{Z}$ for all i, the Fock module $\mathcal{F}_{\sigma\cdot\phi}$ can be lifted to a module of $\mathcal{V}_{\rho}$, which we denote by $\mathcal{V}_{\rho,\sigma\cdot\phi}$. The screening operators $S^i:\mathcal{V}_{\rho,\sigma}\to\mathcal{V}_{\rho,\sigma\cdot\phi+\phi^i}$ is defined by the same formula as in equation (3.1.2.15).

Theorem 3.1.4. *There is an embedding $V_{A,\rho}\hookrightarrow\mathcal{V}_{\rho}$ whose image is the kernel of the screening operators:*

$$V_{A,\rho}=\bigcap_i \mathrm{Ker}\left(S^i:\mathcal{V}_{\rho}\mapsto\mathcal{V}_{\rho,\phi^i}\right). \tag{3.1.2.34}$$

The VOA $V_{A,\rho}$ is thus a simple current extension of the following VOA:

$$M(2)^{\otimes n}\otimes H_{\tau\psi}\otimes H_{\tau\widetilde{\phi}}. \tag{3.1.2.35}$$

The VOA $H_{\tau\psi}$ is generated by $\partial\theta_{\alpha}:=\sum\tau_{\alpha i}\partial\psi^i$ with OPE:

$$\partial\theta_{\alpha}(z)\partial\theta_{\beta}(w)\sim\frac{-\sum\limits_i\tau_{\alpha i}\tau^{\beta}{}_i}{(z-w)^2}, \tag{3.1.2.36}$$

and the VOA $H_{\tau\widetilde{\phi}}$ is generated by $\partial\eta_\alpha := \sum \tau_{\alpha i}\partial\widetilde{\phi}^i$ with OPE:

(3.1.2.37)
$$\partial\eta_\alpha(z)\partial\eta_\beta(w) \sim \frac{\sum_i \tau_{\alpha i}\tau^\beta{}_i}{(z-w)^2}.$$

To write the conformal element, we need to use the matrix Π and $\Pi^\vee$ in equation (3.1.1.5). For each $1 \le i \le n$, we can write:

(3.1.2.38)
$$\psi^i = \sum_{a,j} \Pi_a{}^i \rho_j{}^a \psi^j + \sum_{\alpha,j} (\Pi^\vee)_\alpha{}^i \tau^\alpha{}_j \psi^j,$$

since $\rho n + \rho^\vee n^\vee = \mathrm{Id}_n$. Using these, we can rewrite:

(3.1.2.39)
$$\sum_i :\partial\psi^i\partial\psi_i: = \sum_{i,a,b} \Pi^{ai}\Pi^b{}_i :J_a^{\beta\gamma}J_b^{\beta\gamma}: + \sum_{i,\alpha,\beta} (\Pi^\vee)^{\alpha i}(\Pi^\vee)^\beta{}_i :\partial\theta^\alpha\partial\theta^\beta:,$$

here $J_a^{\beta\gamma} = \sum_j \rho_j{}^a \partial\psi^j$ is the generator of the $U(1)^r$ Kac-Moody algebra in $V_{\beta\gamma}^{\otimes n}$. In the cohomology, the first term is Q exact, and so the remaining part of the cohomology is:

(3.1.2.40)
$$\sum_{i,\alpha,\beta} (\Pi^\vee)^{\alpha i}(\Pi^\vee)^\beta{}_i :\partial\theta^\alpha\partial\theta^\beta:.$$

Similarly, the remaining part for $H_{\widetilde{\phi}}$ is:

(3.1.2.41)
$$\sum_{i,\alpha,\beta} (\Pi^\vee)^{\alpha i}(\Pi^\vee)^\beta{}_i :\partial\eta^\alpha\partial\eta^\beta:.$$

In conclusion, the conformal element of $V_{A,\rho}$ can be written in the free field realization as:

(3.1.2.42)
$$L(z) = \frac{1}{2}\sum :\partial\phi^i\partial\phi_i: - \frac{1}{2}\partial^2\phi^i - \frac{1}{2}\sum_{\alpha,\beta,i} (\Pi^\vee)^{\alpha i}(\Pi^\vee)^\beta{}_i :\partial\theta^\alpha\partial\theta^\beta: + \frac{1}{2}\sum_{\alpha,\beta,i} (\Pi^\vee)^{\alpha i}(\Pi^\vee)^\beta{}_i :\partial\eta^\alpha\partial\eta^\beta:$$

REMARK 3.1.5. *There is a family of conformal elements for $V_{A,\rho}$, defined by adding to the above $L(z)$ elements of the form:*

(3.1.2.43)
$$\sum_\alpha m_\alpha\partial^2\theta^\alpha + n_\alpha\partial^2\eta^\alpha,$$

*which will change the conformal degree of the Fock modules. The choice that is made here is in conformity with physics [**DGP18**], so that before taking BRST cohomology, $|\beta^i| = |\gamma^i| = |b^i| = |c^i| = \frac{1}{2}$, and $|b^a| = 1, |c^a| = 0$, and so that Q has conformal degree 0.*

3.1.3. The B Side Boundary VOA.

3.1.3.1. *The Perturbative VOA.* Let us now turn to the definition of $V_{B,\rho}$. Let T be the torus $(\mathbb{C}^\times)^r$ and $\mathfrak{t}$ its Lie algebra. In [**CG19**], the authors suggested that one should start with an affine Kac-Moody superalgebra of the Lie superalgebra $\mathfrak{g}_*(\rho) = T^*[-2](\mathfrak{t} \oplus V[-1])$, and then extend by monopole operators. More explicitly, the Lie superalgebra is the following vector space:

$$\mathfrak{t}[0] \oplus V[-1] \oplus V^*[-1] \oplus \mathfrak{t}^*[-2] \ni (N_a, \psi^{i,+}, \psi^{i,-}, E^a) \tag{3.1.3.1}$$

such that the commutator $[N_a, -]$ is given by the action of $\mathfrak{t}$ on T^*V defined by ρ, and adjoint action on $\mathfrak{t}^*$ (which is trivial for our case), and the supercommutator between V and V^* is the moment map valued in $\mathfrak{t}^*[-2]$. Namely, the commutation relation is given by:

$$[N_a, \psi^{i,\pm}] = \pm \rho^i{}_a \psi^{i,\pm}, \ \{\psi^{i,+}, \psi^{j,-}\} = \delta^{ij} \sum_a \rho^i{}_a E^a. \tag{3.1.3.2}$$

To define the Kac-Moody Lie superalgebra, we need to choose an even nondegenerate bilinear pairing and a level. There are many choices of pairings on $\mathfrak{g}_*(\rho)$, and for the setting of 3d $\mathcal{N} = 4$ theories, such a choice is determined by bulk correlation functions, and is computed explicitly in [**Gar22**].

$$\kappa(N_a, N_b) = \sum_i \rho^i{}_a \rho_{ib}, \ \kappa(N_a, E^b) = \delta^b_a, \ \kappa(\psi^{i,+}, \psi^{j,-}) = \delta^{ij}. \tag{3.1.3.3}$$

We fix the level $k = 1$, since just as for the case of $\widehat{\mathfrak{gl}(1|1)}$ [**CMY22c**], any other choice of $k \neq 0$ gives isomorphic vertex operator superalgebras. We denote by $V(\widehat{\mathfrak{g}_*(\rho)})$ the resulting vertex operator superalgebra associated to κ at level $k = 1$. This VOA has the following generators and OPEs:

$$
\begin{aligned}
&N_a(z)E^b(w) \sim \frac{\delta_a^b}{(z-w)^2},\ N_a(z)N_b(w) \sim \frac{\sum_i {\rho^i}_a \rho_{ib}}{(z-w)^2} \\
&N_a(z)\psi^{i,+}(w) \sim \frac{{\rho^i}_a \psi^{i,+}}{(z-w)},\ N_a(z)\psi^{i,-}(w) \sim \frac{-{\rho^i}_a \psi^{i,-}}{(z-w)} \\
&\psi^{i,+}(z)\psi^{j,-}(w) \sim \frac{\delta^{ij}}{(z-w)^2} + \frac{\delta^{ij}\sum_a {\rho^i}_a E^a}{z-w}.
\end{aligned}
\tag{3.1.3.4}
$$

This VOA has the following conformal element:

$$
L(z) = \frac{1}{2}\left(\sum_a :N_a E^a: + :E^a N_a: - \sum_i :\psi^{i,+}\psi_i^-: + \sum_i :\psi_i^- \psi^{i,+}:\right)
\tag{3.1.3.5}
$$

which can be constructed using a modified Sugawara construction as in [**RS92**], namely, it is the quadratic Casimir associated to the bilinear form κ_0 where $\kappa_0(N_a, E_b) = \delta_{ab}$ and $\kappa_0(\psi^{i,+}, \psi^{j,-}) = \delta^{ij}$. The VOA $V(\widehat{\mathfrak{g}_*(\rho)})$ is the perturbative boundary VOA of the B twisted gauge theory. As suggested in [**CG19**], one needs to take a suitable extension of $V(\widehat{\mathfrak{g}_*(\rho)})$ by some modules that correspond to monopole operators on the boundary. In the following, we will introduce a free field realization of $V(\widehat{\mathfrak{g}_*(\rho)})$ and construct its extensions. We will compare their indices with the calculation of indices in [**DGP18**] to identify the correct extension.

3.1.3.2. *Free Field Realizations and Monopole Operators.* Let us now introduce a free field realization of $V(\widehat{\mathfrak{g}_*(\rho)})$. For this, consider the Heisenberg VOA $H_{X,Y,Z}$ generated by $\partial X_a, \partial Y^a$ for $1 \le a \le r$ and ∂Z^i for $1 \le i \le n$, with OPE:

$$
\partial X_a \partial Y_b \sim \frac{\delta_{ab}}{(z-w)^2}, \qquad \partial Z_i \partial Z_j \sim \frac{\delta_{ij}}{(z-w)^2}.
\tag{3.1.3.6}
$$

Let V_Z be the lattice VOA extension of $H_{X,Y,Z}$ by the lattice generated by Z. The assignment:

$$
\begin{aligned}
&N_a \mapsto \partial X_a + \sum_i \rho_{ia}\partial Z^i, \\
&E^a \mapsto \partial Y^a, \\
&\psi^{i,+}(z) \mapsto :e^{Z^i}: \\
&\psi^{i,-} \mapsto :\sum_i {\rho^i}_a \partial Y^a e^{-Z^i}: + :\partial e^{-Z^i}:
\end{aligned}
\tag{3.1.3.7}
$$

defines an embedding of $V(\widehat{\mathfrak{g}_*(\rho)})$ into the lattice VOA V_Z. One can verify that the conformal element of $V(\widehat{\mathfrak{g}_*(\rho)})$ is mapped to:

$$(3.1.3.8)\qquad \frac{1}{2}\left(:\sum_a(\partial X_a\partial Y^a+\partial Y^a\partial X_a)+\sum_i(\partial Z^i)(\partial Z_i):\right)+\frac{1}{2}\sum_i\left(\sum_a\rho_{ia}\partial^2Y^a-\partial^2 Z_i\right).$$

For each linear combination $\mu=\sum_{a,i}m^aX_a+n_aY^a+t_iZ^i$, we denote by $\mathcal{F}_\mu$ the corresponding Fock module of the Heisenberg VOA $H_{X,Y,Z}$ generated by the vacuum vector $|\mu\rangle$, then by definition:

$$(3.1.3.9)\qquad V_Z=\bigoplus_{t\in\mathbb{Z}^n}\mathcal{F}_{t\cdot Z},$$

as a module of the Heisenberg VOA. Moreover, for each linear combination $\mu=\sum_a m^aX_a+n_aY^a$, the module $\mathcal{F}_\mu$ can be lifted to a module of V_Z, which we call $V_{Z,\mu}$.

Let $\widetilde{\mu}=\mu-\sum_a\rho^i{}_aY^a$. Define intertwiners $S^i(z):V_{Z,\mu}\to V_{Z,\widetilde{\mu}}(\!(z)\!)$ by the following formula:

$$(3.1.3.10)\qquad S^i(z)=:e^{Z^i(z)-\sum_a\rho^i{}_aY^a(z)}:.$$

The screening operators are defined as the residue:

$$(3.1.3.11)\qquad S^i=\oint S^i(z)\mathrm{d}z$$

Proposition 3.1.6. *The embedding $V(\widehat{\mathfrak{g}_*(\rho)})\to V_Z$ identifies the image as the kernel of the screening operators:*

$$(3.1.3.12)\qquad V(\widehat{\mathfrak{g}_*(\rho)})\cong\bigcap_i\mathrm{Ker}\left(S^i:V_Z\to V_{Z,-\sum_a\rho^i{}_aY^a(z)}\right).$$

Proof. First of all, the map $V(\widehat{\mathfrak{g}_*(\rho)})\to V_Z$ is clearly contained in the kernel of S^i for all i. We will show in Appendix B that $V(\widehat{\mathfrak{g}_*(\rho)})$ is simple, and therefore this is an embedding. To show that this is an isomorphism, we will compare indices:

$$(3.1.3.13)\qquad \mathrm{Tr}\left(\prod_a s_a^{N_{a,0}}q^{L_0}\right).$$

Here s^a and q are formal variables. This index is clearly positive and so if both sides have the same index, then they are isomorphic as vector spaces.

One can easily compute the index of $V(\widehat{\mathfrak{g}_*(\rho)})$:

$$\frac{\prod_{i=1}^{n}(-q\prod_a s_a^{\rho^i{}_a}, -q\prod_a s_a^{-\rho^i{}_a}; q)_\infty}{(q;q)_\infty^{2r}} \tag{3.1.3.14}$$

Here the Pochhammer symbols $(a, b; q)_\infty$ means $(a; q)_\infty(b; q)_\infty$. To compute the symbol of the kernel of S^i, we note that the mode algebra $U(V_Z)$ of V_Z is a filtered algebra by assigning

$$F_N U(V_Z) = \mathrm{Span}\{x_{a_1,k_1}\cdots x_{a_N,k_N} y_*^* z_*^*\}. \tag{3.1.3.15}$$

Here the $x_{*,*}$ (y_*^* and z_*^*, resp.) are the modes of $X_*(z)$ ($Y^*(z)$ and $Z^*(z)$, resp.). The associated graded:

$$\mathrm{Gr}_* F U(V_Z) = \mathbb{C}[x_{a,k}, y_k^a] \otimes U(V_{bc}^{\otimes n}). \tag{3.1.3.16}$$

Here the Fourier modes of ∂X_a and ∂Y^a are set to be commutative, and $U(V_{bc}^{\otimes n})$ is the mode algebra of n-copies of V_{bc}. For each $\mu = \sum m_a Y^a$, the module $V_{Z,\mu}$ as an $U(V_Z)$ module is also filtered with a similar filtration. In the associated graded, we have:

$$\mathrm{Gr}_* F V_{Z,\mu} \cong \mathrm{Gr}_* F V_Z = \mathbb{C}[x_{a,k}, y_k^a]_{k<0} \otimes V_{bc}^{\otimes n}, \tag{3.1.3.17}$$

and it is clear that the natural map $V_{Z,\mu} \to \mathrm{Gr}_* F V_{Z,\mu}$ is an isomorphism of vector spaces. Let $\mu_i = \sum \rho_{ia} Y^a$. Since S^i clearly preserves the filtration, the diagram:

$$\begin{array}{ccc} V_Z & \xrightarrow{S^i} & V_{Z,-\mu_i} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_* F V_Z & & \mathrm{Gr}_* F V_Z \end{array} \tag{3.1.3.18}$$

can be completed to the diagram:

$$\begin{array}{ccc} V_Z & \xrightarrow{S^i} & V_{Z,-\mu_a} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_* F V_Z & \xrightarrow{\overline{S}^i} & \mathrm{Gr}_* F V_Z \end{array} \tag{3.1.3.19}$$

One easily verifies that $\overline{S}^i$ has the expression:

$$\overline{S}^i = \oint \mathrm{d}z e^{-\sum_i \rho^i{}_a Y^a} {:} e^{Z^i} {:} \tag{3.1.3.20}$$

except that the modes of Y are now commuting with modes of X. By snake lemma, the induced map:

$$\bigcap_i \mathrm{Ker}(S^i) \to \bigcap_i \mathrm{Ker}(\overline{S}^i) \tag{3.1.3.21}$$

is an isomorphism of vector spaces. The process of taking associated graded behaves well with the conformal grading and the grading by $N_{a,0}$ since the embedding $F_N V_{Z,\mu} \subseteq F_{N+1} V_{Z,\mu}$ is one of graded vector spaces. The kernel $\bigcap_i \mathrm{Ker}(\overline{S}^i)$ can be identified easily:

$$\mathbb{C}[x_{a,k}, y^a_k]_{k<0} \otimes V_{SF}^{\otimes n}, \tag{3.1.3.22}$$

since the kernel of $\overline{S}^i$ can be easily identified with the kernel of $\oint \mathrm{d}z {:} e^{Z^i} {:}$. The piece $e^{-\sum_i \rho^i{}_a Y^a}$ in the definition of $\overline{S}^i$ commutes with everything in the associated graded. The character of this coincides with equation (3.1.3.14). This completes the proof. □

Now we can turn to the question of identifying monopole operators and constructing the correct extension of $V(\widehat{\mathfrak{g}_*(\rho)})$. This now comes down to finding extensions of V_Z, which are determined by sublattices in the lattice spanned by X_a, Y^a. Inspired by the work of [**DGP18**], we would like to identify the monopole operators as:

$${:}\exp\left(\int \sum m^a N_a\right){:} \tag{3.1.3.23}$$

for $m^a \in \mathbb{Z}$. In terms of the free field realization, this means that we would like to extend by the Fock modules corresponding to

$$\sum_a m^a X_a + \sum_{i,a} m^a \rho_{ia} \partial Z^i. \tag{3.1.3.24}$$

Consider now the VOA $\mathcal{W}_\rho$ which is the extension of V_Z by the Fock modules as in equation (3.1.3.24). As a module of $H_{X,Y,Z}$, we have:

$$\mathcal{W}_\rho = \bigoplus_{t\in\mathbb{Z}^n, s\in\mathbb{Z}^r} \mathcal{F}_{s\cdot X+t\cdot Z}. \tag{3.1.3.25}$$

For each $\sigma\in\mathbb{Z}^r$, the Fock module $\mathcal{F}_{\sigma\cdot Y}$ can be lifted to a module of $\mathcal{W}_\rho$ which we call $\mathcal{W}_{\rho,\sigma\cdot Y}$. The screening operators S^i extend to a map:

$$S^i : \mathcal{W}_\rho \to \mathcal{W}_{\rho,-\sum\limits_a \rho^i{}_a Y^a}. \tag{3.1.3.26}$$

Definition 3.1.7. *The boundary VOA on a Dirichlet boundary condition in $\mathcal{T}_{B,\rho}$ is defined by:*

$$V_{B,\rho} := \bigcap_i \mathrm{Ker}\left(S^i : \mathcal{W}_\rho \to \mathcal{W}_{\rho,-\sum\limits_a \rho^i{}_a Y^a}\right). \tag{3.1.3.27}$$

Note that $\mathcal{W}_\rho$ is a direct sum of simple Fock modules of V_Z:

$$\mathcal{W}_\rho \cong \bigoplus_{s\in\mathbb{Z}^r} V_{Z,s\cdot X}. \tag{3.1.3.28}$$

Thus $V_{B,\rho}$ is a direct sum of modules of $V(\widehat{\mathfrak{g}_*(\rho)})$:

$$V_{B,\rho} \cong \bigoplus_{s\in\mathbb{Z}^r}\bigcap_i \mathrm{Ker}\left(S^i : V_{Z,s\cdot X} \to V_{Z,s\cdot X-\sum\limits_a \rho^i{}_a Y^a}\right). \tag{3.1.3.29}$$

Each of the direct summands represents a monopole operator on the boundary. We will show in Appendix B that each of the direct summands is a simple module of $V(\widehat{\mathfrak{g}_*(\rho)})$. If we assume it for now, it is easy to write down a generator of each of the direct summands:

$$s\in\mathbb{Z}^r \leftrightarrow \left|\sum_a s^a\left(X_a + \sum_i \rho_{ia} Z^i\right)\right\rangle \mapsto :e^{\sum_a s^a\left(X_a+\sum_i \rho_{ia}Z^i\right)}:. \tag{3.1.3.30}$$

We will compare this with the monopole operator studied in [**DGP18**], and show that this generator does give the correct physical indices, justifying the definition of $V_{B,\rho}$.

Remark 3.1.8. *The VOA $V(\widehat{\mathfrak{g}_*(\rho)})$ has the following $\mathbb{Z}^r\times\mathbb{C}^r$ lattice of automorphisms:*

$$\sigma_{\lambda,\mu}(N_a) = N_a - \frac{\mu_a}{z}, \qquad \sigma_{\lambda,\mu}(E_a) = E_a - \frac{\lambda_a}{z}, \qquad \sigma_{\lambda,\mu}\psi^{i,\pm} = z^{\mp\sum \rho_{ai}\lambda^a}\psi^{i,\pm}. \tag{3.1.3.31}$$

Here $\lambda \in \mathbb{C}^r$ and $\mu \in \mathbb{C}^r$ such that $\rho(\lambda) \in \mathbb{Z}^r$. One can identify the module generated by $\sum_a s^a(X_a + \sum_i \rho_{ia} Z^i)$ as the spectral flow $\sigma_{s,\rho^\intercal \rho s} V(\widehat{\mathfrak{g}_*(\rho)})$.

3.1.3.3. *Indices of the Boundary VOA.* We compute the index of $V_{B,\rho}$, taking into account the parity/fermion number, conformal weight, and global symmetry grading. This quantity is defined by the formula

$$\mathbb{I}_{V_{B,q}} := \mathrm{Tr}_{V_{B,q}} \left((-1)^F q^{L_0} s^{N_0} \right) \tag{3.1.3.32}$$

where $s^{N_0} := \prod_{a=1}^r s_a^{N_0^a}$. Below we jot down the mode expansions of these grading operators in the free-field realization for later use:

$$N_0^a = x_0^a + \sum_{i=1}^n \rho_{ia} z_0^i \tag{3.1.3.33}$$

$$\begin{aligned} L_0 = \sum_{a=1}^r \left[\frac{1}{2}(x_0^a y_0^a + y_0^a x_0^a) + \sum_{m=1}^\infty (x_{-m}^a y_m^a + y_{-m}^a x_m^a) \right] \\ + \sum_{i=1}^n \left[\frac{1}{2}(z_0^i)^2 + \sum_{m=1}^\infty z_{-m}^i z_m^i \right] + \frac{1}{2} \sum_{i=1}^n \left[z_0^i - \sum_{a=1}^r \rho_{ia} y_0^a \right]. \end{aligned} \tag{3.1.3.34}$$

To compute the index, we compute the Verma modules generated by

$$|s\rangle := \left| \sum_{a=1}^r s^a \left(X_a + \sum_{i=1}^n \rho_{ia} Z^i \right) \right\rangle = S_{\sum_{a=1}^r s^a \left(X_a + \sum_{i=1}^n \rho_{ia} Z^i \right)} |0\rangle \tag{3.1.3.35}$$

for each $s \in \mathbb{Z}^r$. In the rest of this section, whenever we write a mode belonging to the Lie algebra associated to $V(\widehat{\mathfrak{g}_*(q)})$, the image of this mode under the free-field realization (3.1.3.7) should be implicitly understood. Thus we see that N_k^a and E_k^a act freely on $|s\rangle$ for $k < 0$ and act as (possibly vanishing) scalars for $k > 0$. Note that this parabolic decomposition is the same that we would obtain if we were analyzing the action on the vacuum $|0\rangle$; the same will not be quite true for $\psi^{i,\pm}(z)$, but the difference manifests itself as a sort of spectral flow depending on s and the charge matrix ρ_{ia}. Note that

$$\psi^{i,+}(w)|s\rangle = S_{Z^i} w^{z_0^i} e^{\sum_{k<0} -\frac{1}{k} z_k^i w^{-k}} e^{\sum_{k>0} -\frac{1}{k} z_k^i w^{-k}} S_{\sum_{a=1}^r s^a (X_a + \sum_{j=1}^n \rho_{ja} Z^j)} |0\rangle$$

$$= S_{Z^i + \sum_{a=1}^r s^a (X_a + \sum_{j=1}^n \rho_{ja} Z^j)} w^{\sum_{a=1}^r s^a \rho_{ia}} e^{\sum_{k<0} -\frac{1}{k} z_k^i w^{-k}} |0\rangle. \tag{3.1.3.36}$$

When comparing the mode expansions of the LHS and RHS, the factor $w^{\sum_i s_i q_{ai}}$ effectively shifts which modes of $\psi^{i,+}(w)$ act freely vs. act as scalars, as compared to mode splitting when acting on $|0\rangle$. Defining $L_i := \sum_{a=1}^r s^a \rho_{ia}$, we find that $\psi_k^{i,+}$ acts freely for $k < -L_i$ and as a scalar for $k \geq -L_i$. A similar analysis reveals that $\psi_k^{i,-}$ acts freely for $k < L_i$ and otherwise acts as a scalar. Thus the Verma module built upon $|s\rangle$ has PBW basis

$$\bigotimes_{i=1}^n \left[\left[\bigotimes_{k \leq -L_i - 1} (\mathbb{C} \oplus \mathbb{C}\psi_k^{i,+}) \right] \otimes \left[\bigotimes_{k \leq L_i - 1} (\mathbb{C} \oplus \mathbb{C}\psi_k^{i,-}) \right] \right] \otimes \bigotimes_{a=1}^r \left[\bigotimes_{k \leq -1} \left(\bigoplus_{m \geq 0} \mathbb{C}(N_k^a)^m \otimes \bigoplus_{m \geq 0} \mathbb{C}(E_k^a)^m \right) \right] |s\rangle. \tag{3.1.3.37}$$

The contribution of this sector of $V_{B,q}$ to the index *solely from the mode algebra* is thus, after some straightforward algebraic manipulation,

$$\frac{1}{(q)_\infty^{2r}} \prod_{i=1}^n \left(q \prod_{a=1}^r (s^a q^{s^a})^{\rho_{ia}}, q \prod_{a=1}^r (s^a q^{s^a})^{-\rho_{ia}}; q \right)_\infty . \tag{3.1.3.38}$$

But we cannot forget that $|s\rangle$ itself has non-trivial grading under L_0 and N_0. Properly taking this into account when computing the index will yield an expression equal to equation (3.1.3.38) multiplied by an overall factor consisting of the fugacities of $|s\rangle$. Let us now calculate this factor. The conformal weight of $|s\rangle$ is given by:

$$\begin{aligned} L_0|s\rangle &= \frac{1}{2} \sum_{j=1}^n \left[(z_0^j)^2 + z_0^j - \sum_{b=1}^r \rho_{jb} y_0^b \right] S_{\sum_{a=1}^r s^a \left(X^a + \sum_{i=1}^n \rho_{ia} Z^i\right)} |0\rangle \\ &= \frac{1}{2} \sum_{a,j} (s^a \rho^j{}_a)^2 |s\rangle \\ &= \frac{1}{2} s^T \rho^T \rho s |s\rangle \end{aligned} \tag{3.1.3.39}$$

The weight under N_0 is given by:

$$
\begin{aligned}
N_0^a|s\rangle &= \left[x_0^a + \sum_{i=1}^n \rho_{ia} z_0^i\right] S_{\sum_{b=1}^r s_b(X^b + \sum_{j=1}^n \rho_{jb} Z^j)}|0\rangle \\
&= \sum_{a,j} \rho_{ja}\rho_{jb} s_b |s\rangle \\
&= (\rho^T \rho s)_a |s\rangle
\end{aligned} \tag{3.1.3.40}
$$

and the parity is given by:

$$
\begin{aligned}
(-1)^F|s\rangle &= (-1)^{(\sum_{a=1}^r s^a(X^a+\sum_{i=1}^n \rho_{ia}Z^i), \sum_{b=1}^r s_b(X^b + \sum_{j=1}^n \rho_{jb}Z^j))}|s\rangle \\
&= (-1)^{\sum_{i,a,b} s^a \rho^{ia} \rho_{ib} s_b}|s\rangle \\
&= (-1)^{s^T\rho^T\rho s}|s\rangle.
\end{aligned}
$$

The missing factor is therefore

$$
(-1)^{s^T\rho^T\rho s} q^{\frac{1}{2}s^T\rho^T\rho s} \prod_{a=1}^r s_a^{(\rho^T\rho s)_a}. \tag{3.1.3.41}
$$

Summing over monopole sectors (i.e. Verma modules built upon each $|s\rangle$), we finally obtain the index

$$
\begin{aligned}
\mathbb{I}_{V_{B,\rho}} = \frac{1}{(q)_\infty^{2r}} \sum_{s\in\mathbb{Z}^r} (-1)^{s^T\rho^T\rho s} q^{\frac{1}{2}s^T\rho^T\rho s} \left[\prod_{a=1}^r s_a^{(\rho^T\rho s)_a}\right] \\
\times \prod_{i=1}^n \left(q\prod_{a=1}^r (s_a q^{s_a})^{\rho_{ia}}, q\prod_{a=1}^r (s^a q^{s^a})^{-\rho_{ia}}; q\right)_\infty .
\end{aligned} \tag{3.1.3.42}
$$

As a check, this is precisely of the form derived in [**DGP18**].

Remark 3.1.9. *The computation above reveals that the conformal degree of* $\left|X_a + \sum_i \rho_{ia} Z^i\right\rangle$ *is given by* $\frac{1}{2}\sum_{i,a} \rho_{ia}\rho^{ia}$.

3.1.4. Mirror Symmetry of Boundary VOAs. We have defined in the previous sections boundary VOAs $V_{A,\rho}$ for Neumann boundary condition of $\mathcal{T}_{A,\rho}$ and $V_{B,\rho}$ for Dirichlet boundary condition of $\mathcal{T}_{B,\rho}$. Recall the short exact sequence (2.1.2.14). The statement of 3d abelian mirror symmetry asserts that $\mathcal{T}_\rho$ and $\mathcal{T}_{\rho^\vee}$ are mirror to each other, where $\rho^\vee = \tau^\mathsf{T}$. It is hinted in

[**CG19**, **CCG19**] that these two boundary conditions are mirror dual to each other. In this section, we prove this statement:

THEOREM 3.1.10. *There is an isomorphism of VOAs:*

$$V_{A,\rho} \cong V_{B,\rho^\vee}. \tag{3.1.4.1}$$

PROOF. Our strategy is to use the free field realizations of Section 3.1.2.2 and 3.1.3.2. More specifically, we will perform a field redefinition to the free field realizations to show that the two VOA are isomorphic.

Recall in the free field realization of Section 3.1.3.2 (associated to $\rho^\vee$ instead of ρ) we have generators $\partial X^\alpha, \partial Y^\alpha$ for $1 \leq \alpha \leq n-r$ and ∂Z^i for $1 \leq i \leq n$ such that:

$$\partial X^\alpha \partial Y^\beta \sim \frac{\delta^{\alpha\beta}}{(z-w)^2}, \qquad \partial Z^i \partial Z^j \sim \frac{\delta^{ij}}{(z-w)^2}. \tag{3.1.4.2}$$

We perform a field redefinition:

$$\begin{aligned} &\widetilde{\zeta}^i = Z^i - \sum_\alpha (\rho^\vee)^i{}_\alpha Y^\alpha, \ \widetilde{\eta}^\alpha = X^\alpha + \sum_i (\rho^\vee)_i{}^\alpha Z^i, \\ &\widetilde{\theta}^\alpha = -X^\alpha + \sum_{i,\beta} (\rho^\vee)_i{}^\alpha (\rho^\vee)^i{}_\beta Y^\beta - \sum_i (\rho^\vee)_i{}^\alpha Z^i. \end{aligned} \tag{3.1.4.3}$$

They have OPE:

$$\partial\widetilde{\zeta}^i \partial\widetilde{\zeta}^j \sim \frac{\delta^{ij}}{(z-w)^2}, \ \partial\widetilde{\theta}^\alpha \partial\widetilde{\theta}^\beta \sim -\partial\widetilde{\eta}^\alpha \partial\widetilde{\eta}^\beta = -\frac{\sum_i (\rho^\vee)_i{}^\alpha (\rho^\vee)^{i\beta}}{(z-w)^2}. \tag{3.1.4.4}$$

Since $\tau^{\mathsf{T}} = \rho^\vee$, the Heisenberg VOA $H_{X,Y,Z}$ (for $\rho^\vee$) is nothing but $H_\phi \otimes H_{\tau\psi} \otimes H_{\tau\widetilde{\phi}}$, where we identify $\widetilde{\zeta}$ with ϕ, $\widetilde{\theta}$ with θ and $\widetilde{\eta}$ with η. The VOA $\mathcal{W}_{\rho^\vee}$ is the extension of this Heisenberg VOA by the Fock modules corresponding to Z^i and $X^\alpha + \sum_i (\rho^\vee)_i^\alpha Z^i$. In terms of the new generators, the second factor is nothing but $\widetilde{\eta}^\alpha$. To write the first generator, let us use the projection maps in equation (3.1.1.5), in particular the matrix $n^\vee$ which is projection onto $\Lambda^\perp$. Since $\widetilde{\theta}^\alpha + \widetilde{\eta}^\alpha = \sum_{i,\beta} (\rho^\vee)_i{}^\alpha (\rho^\vee)^i{}_\beta Y^\beta$, we have:

$$\sum_\alpha (\Pi^\vee)_{\alpha i} (\widetilde{\theta}^\alpha + \widetilde{\eta}^\alpha) = \sum_{\alpha,\beta,j} (\Pi^\vee)_{\alpha i} (\rho^\vee)_j{}^\alpha (\rho^\vee)^j{}_\beta Y^\beta = \sum_{\alpha,\beta,j} (\Pi^\vee)_{\alpha i} (\rho^\vee)_j{}^\alpha (\rho^\vee)^j{}_\beta Y^\beta + \sum_{a,\beta,j} \Pi_{ai} \rho_j{}^a (\rho^\vee)^j{}_\beta Y^\beta \tag{3.1.4.5}$$

Here we add the last term, which does not change the final result because $\tau\rho = 0$. Now we can use $\rho\Pi + \rho^\vee\Pi^\vee = \mathrm{Id}_n$ to conclude:

$$\sum_\alpha (\Pi^\vee)_{\alpha i}(\widetilde{\theta}^\alpha + \widetilde{\eta}^\alpha) = \sum_\alpha (\rho^\vee)_{i\alpha} Y^\alpha. \tag{3.1.4.6}$$

As a consequence:

$$Z^i = \widetilde{\zeta}^i + \sum_\alpha (\Pi^\vee)_{\alpha i}(\widetilde{\theta}^\alpha + \widetilde{\eta}^\alpha). \tag{3.1.4.7}$$

The VOA $\mathcal{W}_{\rho^\vee}$ is the extension of $H_{X,Y,Z}$ given by:

$$\mathcal{W}_{\rho^\vee} = \bigoplus_{t\in\mathbb{Z}^n, s\in\mathbb{Z}^{n-r}} \mathcal{F}_{t\cdot Z + s\cdot X}, \tag{3.1.4.8}$$

and it can be alternatively given by, in terms of $H_\phi \otimes H_{\tau\psi} \otimes H_{\tau\widetilde{\phi}}$ Fock modules:

$$\mathcal{W}_{\rho^\vee} = \bigoplus_{t\in\mathbb{Z}^n, s\in\mathbb{Z}^{n-r}} \mathcal{F}_{t\cdot\phi} \otimes \mathcal{F}_{n^\vee t\cdot\theta} \otimes \mathcal{F}_{(s+n^\vee t)\cdot\eta} \tag{3.1.4.9}$$

Now let us decompose t as $\lambda + \lambda^\perp$ using equation (3.1.1.10). Then it is clear that:

$$\lambda = \rho\Pi t, \qquad \lambda^\perp = \rho^\vee\Pi^\vee t \tag{3.1.4.10}$$

and so $\mathcal{F}_{n^\vee t\cdot\theta}$ is nothing but $\mathcal{F}_{\lambda^\perp\cdot\psi}$. Similarly, $\mathcal{F}_{(s+n^\vee t)\cdot\eta} = \mathcal{F}_{\mu^\perp\widetilde{\phi}}$ such that $\overline{\lambda^\perp} = \overline{\mu^\perp} \in H$ since $s\eta \in \Lambda^\perp$ and does not change the image of $\mu^\perp$ in H. We have found that there is an isomorphism of $H_\phi \otimes H_{\tau\psi} \otimes H_{\tau\widetilde{\phi}}$ modules:

$$\mathcal{W}_{\rho^\vee} = \bigoplus_{\substack{\lambda\in\Lambda' \\ \lambda^\perp,\mu^\perp\in(\Lambda^\perp)' \\ \overline{\lambda}=\overline{\lambda^\perp}=\overline{\mu^\perp}}} \mathcal{F}_{(\lambda+\lambda^\perp)\cdot\phi} \otimes \mathcal{F}_{\lambda^\perp\cdot\psi} \otimes \mathcal{F}_{\mu^\perp\cdot\widetilde{\phi}}. \tag{3.1.4.11}$$

Now the VOA $V_{B,\rho^\vee}$ is defined as the kernel of the following screening operators in $\mathcal{W}_{\rho^\vee}$:

$$\oint \mathrm{d}z {:}\exp\left(Z^i - \sum_\alpha (\rho^\vee)^i{}_\alpha Y^\alpha\right){:} = \oint \mathrm{d}z {:}\exp\left(\phi^i(z)\right){:}, \tag{3.1.4.12}$$

which implies that we can identify $V_{B,\rho^\vee}$ as an extension of $M(2)^{\otimes n} \otimes H_{\tau\psi} \otimes H_{\tau\widetilde{\phi}}$:

$$V_{B,\rho^\vee} = \bigoplus_{\substack{\lambda\in\Lambda' \\ \lambda^\perp,\mu^\perp\in(\Lambda^\perp)' \\ \overline{\lambda}=\overline{\lambda^\perp}=\overline{\mu^\perp}}} M_{(\lambda+\lambda^\perp)\cdot\phi} \otimes \mathcal{F}_{\lambda^\perp\cdot\psi} \otimes \mathcal{F}_{\mu^\perp\cdot\widetilde{\phi}}. \tag{3.1.4.13}$$

Comparing this with equation (3.1.2.32), one immediately see that there is an isomorphism of $M(2)^{\otimes n} \otimes H_{\tau\psi} \otimes H_{\tau\widetilde{\phi}}$ modules:

$$V_{B,\rho^\vee} \cong V_{A,\rho}. \tag{3.1.4.14}$$

Note that this is an isomorphism of a lattice of modules of the singlet and Heisenberg VOA, but not necessarily an isomorphism of VOA itself. However, since all direct summands involved are simple current extensions and Fock modules, the work of [**CR22**] implies that this isomorphism upgrades to an isomorphism of VOAs. This completes the proof.

□

Remark 3.1.11. *Let us rewrite X_α, Y_α and Z^i using ϕ^i, θ_α and η_α as follows:*

$$\begin{aligned} Z^i &= \phi^i + \sum_\alpha (\Pi^\vee)_{\alpha i}(\theta^\alpha + \eta^\alpha) \\ X_\alpha &= -\theta^\alpha - \sum (\rho^\vee)_{i\alpha}\phi^i \\ Y_\alpha &= \sum_{i,\beta} (\Pi^\vee)_{\alpha i}(\Pi^\vee)_\beta{}^i(\theta^\beta + \eta^\beta) \end{aligned} \tag{3.1.4.15}$$

With some tedious but conceptually simple work, we can use this to write the conformal element in equation (3.1.3.8) *as:*

$$\frac{1}{2}\sum_i :\partial\phi^i\partial\phi_i: - \frac{1}{2}\sum_i \partial^2\phi_i + \frac{1}{2}\sum(\Pi^\vee)_{\alpha i}(\Pi^\vee)_\beta{}^i :\partial\eta^\alpha\partial\eta^\beta: - \frac{1}{2}\sum(\Pi^\vee)_{\alpha i}(\Pi^\vee)_\beta{}^i :\partial\theta^\alpha\partial\theta^\beta: \tag{3.1.4.16}$$

This, of course, coincides with the conformal element in equation (3.1.2.42).

3.1.5. Morita Equivalent Constructions. In this section we will define VOAs that will be Morita equivalent to $V_{A,\rho}$ and $V_{B,\rho}$. In particular, we will define $\widetilde{V}_{A,\rho}$ and $\widetilde{V}_{B,\rho}$ whose construction are inspired by physics.

3.1.5.1. *Definition of* $\widetilde{V}_{A,\rho}$. Let us start with $\widetilde{V}_{A,\rho}$. Consider the VOA $V_{\beta\gamma}^{\otimes n}$, which is the boundary VOA for the Neumann boundary condition of the free theory. This VOA has a $\mathbb{Z}^n$ lattice of automorphism called the spectral flow, which are generated by σ_i whose action on the VOA is given by:

$$\sigma_i(\beta_j) = z^{\delta_{ij}}\beta_j, \qquad \sigma_i(\gamma_j) = z^{-\delta_{ij}}\gamma_j. \tag{3.1.5.1}$$

Using these automorphisms, one can twist the vacuum module $V_{\beta\gamma}^{\otimes n}$. For each $\lambda \in \mathbb{Z}^n$, we denote by $V_{\beta\gamma}^{\otimes n,\lambda}$ the module $V_{\beta\gamma}^{\otimes n,\lambda} := (\prod_i \sigma_i^{\lambda_i})V_{\beta\gamma}^{\otimes n}$. These are simple currents of $V_{\beta\gamma}^{\otimes n}$, and have the very simple fusion rule (which is derived, for instance, in [**AW22**]):

$$V_{\beta\gamma}^{\otimes n,\lambda} \times V_{\beta\gamma}^{\otimes n,\mu} \cong V_{\beta\gamma}^{\otimes n,\lambda+\mu}. \tag{3.1.5.2}$$

In particular, the object:

$$\bigoplus_{\lambda\in\Lambda} V_{\beta\gamma}^{\otimes n,\lambda} \tag{3.1.5.3}$$

has a unique structure of a vertex operator algebra extending $V_{\beta\gamma}^{\otimes n}$.

Definition 3.1.12. *The vertex operator algebra* $\widetilde{V}_{A,\rho}$ *is defined as the VOA:*

$$\widetilde{V}_{A,\rho} := \bigoplus_{\lambda\in\Lambda} V_{\beta\gamma}^{\otimes n,\lambda} \tag{3.1.5.4}$$

Using the free field realization of $V_{\beta\gamma}$, we can give a free field realization of $\widetilde{V}_{A,\rho}$. Indeed, consider the free field realization of $V_{\beta\gamma}^{\otimes n}$ using the Heisenberg VOA $H_\phi \otimes H_\psi$ and the lattice VOA V_L. For each $\lambda \in \mathbb{Z}^n$, there is an identification of $V_{\beta\gamma}^{\otimes n}$ modules:

$$V_{\beta\gamma}^{\otimes n,\lambda} \cong \bigcap_i \mathrm{Ker}\left(S^i : V_{L,\lambda} \to V_{L,\lambda-\phi^i}\right) \tag{3.1.5.5}$$

Thus, if we consider the extension $V_{L,\Lambda} := \bigoplus_{\lambda\in\Lambda} V_{L,\lambda}$, which is a lattice VOA extending $H_\phi \otimes H_\psi$, then there is an embedding $\widetilde{V}_{A,\rho} \hookrightarrow V_{L,\Lambda}$ of VOAs, whose image is the kernel of the screening operators S^i. We can also write the free field realization in terms of modules of the VOA $M(2)^{\otimes n} \otimes H_\psi$ as follows

$$\widetilde{V}_{A,\rho} = \bigoplus_{\mu\in\mathbb{Z}^n,\lambda\in\Lambda} M_{(\lambda+\mu)\cdot\phi}\otimes\mathcal{F}_{\mu\cdot\psi}. \tag{3.1.5.6}$$

3.1.5.2. *Definition of* $\widetilde{V}_{B,\rho}$. Let us consider the symplectic fermion VOA $V_{\chi\pm}^{\otimes n}$ and the following free field realization. Recall that Bose-Fermi correspondence identifies $V_{bc}^{\otimes n}$ with the complete lattice VOA extension of H_ϕ by the lattice spanned by all ϕ^i. There is an embedding $V_{\chi\pm}^{\otimes n}\hookrightarrow V_{bc}^{\otimes n}$ given by:

$$\chi_+^i\mapsto :e^{\phi^i}:,\ \ \chi_-^i\mapsto :\partial e^{-\phi^i}:. \tag{3.1.5.7}$$

The image of this is identified as the kernel of the screening operators $S^i := \oint \mathrm{d}z{:}e^{\phi^i}{:}$. In other words, we can identify:

$$V_{\chi\pm}^{\otimes n}\cong\bigoplus_{\lambda\in\mathbb{Z}^n} M_{\lambda\cdot\phi} \tag{3.1.5.8}$$

as a module of the singlet VOA $M(2)^{\otimes n}$. There is a $(\mathbb{C}^\times)^r$ symmetry action on $V_{\chi\pm}^{\otimes n}$ such that the subspace $M_{\lambda\cdot\phi}$ has weight $\sum\limits_i \lambda_i{\rho^i}_a$ under the a-th copy of the gauge group. As is different from $V_{\beta\gamma}$, this action is not inner. This action becomes inner when embedded into $V_{bc}^{\otimes n}$, with currents given by $\sum\rho_{ia}\partial\phi^i$. On the other hand, denote by $V_{\mathbb{Z}^n}^-$ the complete lattice VOA extension of H_ψ by the lattice spanned by ψ^i. This also has a $(\mathbb{C}^\times)^r$ action, which is inner and is generated by $\sum\rho_{ia}\partial\psi^i$.

Definition 3.1.13. *Define* $\widetilde{V}_{B,\rho}$ *to be the orbifold:*

$$\widetilde{V}_{B,\rho} := \left(V_{\chi\pm}^{\otimes n}\otimes V_{\mathbb{Z}^n}^-\right)^{(\mathbb{C}^\times)^r}, \tag{3.1.5.9}$$

where the $(\mathbb{C}^\times)^r$ *denotes the diagonal* $(\mathbb{C}^\times)^r$ *on the tensor product.*

From this definition, we immediately obtain a free field realization. We can write:

$$V_{\chi\pm}^{\otimes n}\otimes V_{\mathbb{Z}^n}^- = \bigoplus_{\lambda,\mu\in\mathbb{Z}^n} M_{\lambda\cdot\phi}\otimes\mathcal{F}_{\mu\cdot\psi}. \tag{3.1.5.10}$$

Taking $(\mathbb{C}^\times)^r$ invariant subspace, we require that $\rho^{\mathsf{T}}(\lambda-\mu)=0$, or in other words, $\lambda-\mu\in\Lambda^\perp$. Consequently, we have:

$$\widetilde{V}_{B,\rho}=\bigoplus_{\substack{\lambda,\mu\in\mathbb{Z}^n\\ \lambda-\mu\in\Lambda^\perp}} M_{\lambda\cdot\phi}\otimes\mathcal{F}_{\mu\cdot\psi}. \tag{3.1.5.11}$$

Now we can compare this free field realization with that in equation (3.1.5.6). We recognize the following:

Proposition 3.1.14. *There is an isomorphism of VOAs:*

$$\widetilde{V}_{A,\rho}\cong\widetilde{V}_{B,\rho^\vee}. \tag{3.1.5.12}$$

3.1.5.3. *An identification of $\widetilde{V}_{A,\rho}$ with $V_{B,\rho^\vee}$.* This small section is devoted to a quick and undetailed proof of the following:

Theorem 3.1.15. *There is an isomorphism of VOAs:*

$$\widetilde{V}_{A,\rho}\otimes V_{bc}^{\otimes n}\cong V_{B,\rho^\vee}\otimes V_{\overline{X},\overline{Y}}. \tag{3.1.5.13}$$

Here the VOA $V_{\overline{X},\overline{Y}}$ is a complete lattice VOA of a self-dual lattice and hence is Morita trivial.

The idea is field redefinition. From the above, we have seen that $\widetilde{V}_{A,\rho}$ has a free field realization using the Heisenberg VOA $H_\phi\otimes H_\psi$. Via Bose-Fermi correspondence, $V_{bc}^{\otimes n}$ can be realized as an extension of $H_{\widetilde{\phi}}$. This extension is given by the following Fock modules:

$$\mathcal{F}_{\lambda\cdot\phi}\otimes\mathcal{F}_{\mu\cdot\psi}\otimes\mathcal{F}_{\nu\cdot\widetilde{\phi}},\qquad \text{for } \lambda,\mu,\nu\in\mathbb{Z}^n,\ \lambda-\mu\in\Lambda. \tag{3.1.5.14}$$

Recall the splitting of the short exact sequence in equation (3.1.1.16). We perform the following field re-definition:

$$Z^i=\phi^i+\sum_{\alpha,j}\tau_\alpha{}^i\widetilde{\rho}_j{}^\alpha(\psi^j+\widetilde{\phi}^j),\qquad X_\alpha=-\sum_i\tau_{\alpha i}(\phi^i+\psi^i),\qquad Y_\alpha=\sum_i\widetilde{\rho}_{i\alpha}(\psi^i+\widetilde{\phi}^i) \tag{3.1.5.15}$$

as well as:

$$\overline{X}_a:=\sum_i\widetilde{\tau}_{ai}\widetilde{\phi}^i,\qquad \overline{Y}_a:=\sum_i\rho_{ia}(\widetilde{\phi}^i+\psi^i). \tag{3.1.5.16}$$

These will have OPE:

(3.1.5.17)
$$\partial Z^j \partial Z^j \sim \frac{\delta^{ij}}{(z-w)^2}, \qquad \partial X_\alpha \partial Y_\beta \sim \frac{\delta_{\alpha\beta}}{(z-w)^2}, \qquad \partial \overline{X}_a \partial \overline{Y}_b \sim \frac{\delta_{ab}}{(z-w)^2}, \qquad \overline{X}_a \partial \overline{X}_b \sim \frac{\sum_i \tilde{\tau}_a{}^i \tilde{\tau}_{bi}}{(z-w)^2}$$

The Heisenberg $H_{X,Y,Z}$ generated by $\partial Z^j, \partial X_\alpha$ and ∂X_β has been used in the free field realization of $V_{B,\rho}$, and we denote by $H_{\overline{X},\overline{Y}}$ the Heisenberg generated by $\partial \overline{X}_a$ and $\partial \overline{Y}_a$. Since the field re-definition above is invertible, $H_\phi \otimes H_\psi \otimes H_{\widetilde{\phi}}$ is identified with $H_{X,Y,Z} \otimes H_{\overline{X},\overline{Y}}$. The extension by Fock modules in (3.1.5.14) can be identified, in the new set of generators, as the lattice spanned by $Z^i, X^\alpha, \overline{X}_a$ and $\overline{Y}_a$. The first two variables will extend $H_{X,Y,Z}$ into $\mathcal{W}_{\rho^\vee}$, and the second two sets of variables extends $H_{\overline{X},\overline{Y}}$ into a complete lattice VOA, which we call $V_{\overline{X},\overline{Y}}$. This means that we have embeddings:

$$\widetilde{V}_{A,\rho} \otimes V_{bc}^{\otimes n} \longrightarrow \mathcal{W}_{\rho^\vee} \otimes V_{\overline{X},\overline{Y}} \longleftarrow V_{B,\rho^\vee} \otimes V_{\overline{X},\overline{Y}} \tag{3.1.5.18}$$

To finish the proof, we just need to comment that under the field re-definition, the screening operator matches:

$$\phi^i \leftrightarrow Z^i - \sum_\alpha \tau_{\alpha i} Y^\alpha. \tag{3.1.5.19}$$

This finishes the proof.

Corollary 3.1.16. *The VOA $V(\widehat{\mathfrak{g}_*(\rho^\vee)}) \otimes V_{\overline{X},\overline{Y}}$ is a simple current extension of $V(\widehat{\mathfrak{gl}(1|1)})^{\otimes n}$.*

Proof. This follows from the relation between $V_{\beta\gamma}$ and $V(\widehat{\mathfrak{gl}(1|1)})^{\otimes n}$ in the work of [**CR13b**], and their free field realizations [**BN22**]. ☐

3.2. Braided Tensor Category via Intertwining Operators

3.2.1. VOA Extensions and Braided Tensor Categories. The following is explained in [**CKM17**]: given a vertex operator superalgebra extension A in a VOA module category $\mathcal{C}$ where $P(z)$-intertwiners define a symmetric monoidal structure on $\mathcal{C}$, the category of local modules of A in $\mathcal{C}$ coincides with the category of generalized modules of the VOA A as braided tensor supercategories. However, in our situation, as well as in many other cases, the object A does not live in $\mathcal{C}$ but in a suitable completion of $\mathcal{C}$. Thus one needs to take a completion of $\mathcal{C}$ to allow infinite direct sums. This is explained in [**CMY22a**, Theorem 1.1]: under suitable circumstances, one can extend the braided monoidal structure from $\mathcal{C}$ to a completion called $\mathrm{Ind}(\mathcal{C})$, such that the object A is now contained in $\mathrm{Ind}(\mathcal{C})$. The authors then showed [**CMY22a**, Theorem 1.4] that the category of generalized local A-modules in $\mathcal{C}$ also has a braided tensor supercategory structure defined via $P(z)$-intertwiners. We denote this category by $A-\mathrm{Mod}_{\mathrm{loc}}(\mathrm{Ind}(\mathcal{C}))$.

For any $M \in \mathrm{Ind}(\mathcal{C})$, the object $A \times M$ has the structure of an A module, however it is not necessarily local. It was explained in [**CKL20**, **CMY22a**] that an object M in $\mathrm{Ind}(\mathcal{C})$ gives rise to a local module in the above manner if and only if the monodromy acts trivially, namely the composition:

$$A \times M \xrightarrow{R_{A,M}} M \times A \xrightarrow{R_{M,A}} A \times M \tag{3.2.1.1}$$

is identity. Thus, let $\mathrm{Ind}(\mathcal{C})^{[0]}$ be the subcategory of $\mathrm{Ind}(\mathcal{C})$ whose objects have trivial monodromy with A, the assignment:

$$\mathcal{L}(M) := A \times M \tag{3.2.1.2}$$

gives a functor:

$$\mathcal{L} : \mathrm{Ind}(\mathcal{C})^{[0]} \longrightarrow A-\mathrm{Mod}_{\mathrm{loc}}(\mathrm{Ind}(\mathcal{C})) \tag{3.2.1.3}$$

THEOREM 3.2.1. *The functor $\mathcal{L}$ is a braided tensor functor [**CMY22a**, Theore, 1.4]. If A happen to be a simple current extension, $\mathcal{C}$ has exact fusion rule, and is fixed-point free, then $\mathcal{L}$ preserves the composition series and maps simple to simple [**CMY22b**, Proposition 3.2].*

Here $\mathcal{C}$ is fixed point free means that the action of the simple currents defining A does not fix any single module M. We comment that in [**BN22**], this is used to establish a relation between the Kazhdan-Lusztig category KL of $V(\widehat{\mathfrak{gl}(1|1)})$ with a category of modules of the VOA $V_{\beta\gamma}$, which is denoted by $\mathcal{C}_{\beta\gamma}$ in *loc.cit.* More precisely, the VOA $V_{\beta\gamma} \otimes V_{bc}$ is a simple current extension of $V(\widehat{\mathfrak{gl}(1|1)})$, and so if we denote by $KL^{[0]}$ the subcategory of KL consisting of objects having trivial monodromy with $V_{\beta\gamma} \otimes V_{bc}$, then one can identify $\mathcal{C}_{\beta\gamma}$ with the image of $KL^{[0]}$ under the above lifting functor. This is used in [**BN22**] to show that $\mathcal{C}_{\beta\gamma}$ has the structure of a braided tensor category. Before moving forward, let us review the data of $KL^{[0]}$ and $\mathcal{C}_{\beta\gamma}$.

3.2.1.1. *The Kazhdan-Lusztig category of* $V(\widehat{\mathfrak{gl}(1|1)})$. In this section, we introduce the category KL, the Kazhdan-Lusztig category for the affine Lie superalgebra $V(\widehat{\mathfrak{gl}(1|1)})$. This category is characterized by satisfying certain weight constraint. For a generalized $V(\widehat{\mathfrak{gl}(1|1)})$ module W, it is called *finite-length* if it has a finite composition series of irreducible $V(\widehat{\mathfrak{gl}(1|1)})$ modules. W is called *grading restricted* if it is graded by generalized conformal weights (the generalized eigenvalues of L_0) and the generalized conformal weights are bounded from below. For more details, see [**CKM17**].

Definition 3.2.2. *The Kazhdan-Lusztig category KL is defined as the category of finite-length grading-restricted generalized $V(\widehat{\mathfrak{gl}(1|1)})$ modules.*

We will denote by $\mathcal{G}$ the category of finite-dimensional representations of $\mathfrak{gl}(1|1)$. Just like in the case of ordinary Lie algebras, there is an induction functor:

$$\mathrm{Ind} : \mathcal{C} \longrightarrow KL \tag{3.2.1.4}$$

such that $\mathrm{Ind}(M)$ is defined as the induced module

$$U(\widehat{\mathfrak{gl}(1|1)}) \otimes_{U(\widehat{\mathfrak{gl}(1|1)}_{\geq 0})} M \tag{3.2.1.5}$$

where $U(\widehat{\mathfrak{gl}(1|1)}_{\geq 0})$ is the enveloping algebra generated by the non-negative part of $\widehat{\mathfrak{gl}(1|1)}$, and M is viewed as a module where the positive part acts trivially. Such modules are called Verma modules. Since any simple module in KL is generated by the lowest conformal weight space, which is a module of $\mathfrak{gl}(1|1)$, any such module is a quotient of $\mathrm{Ind}(M)$ for some simple $\mathfrak{gl}(1|1)$ module M.

The finite-dimensional algebra $\mathfrak{gl}(1|1)$ has the following set of simple modules:

(1) $A_{n,0}$, where N acts with weight n and all other modes act as zero. This module is one-dimensional.
(2) $V_{n,e}$ where N acts with weight $n \pm \frac{1}{2}$, and E acts with weight e. This module is two-dimensional.

From the above, any simple module in KL is a quotient of $\mathrm{Ind}(A_{n,0})$ or $\mathrm{Ind}(V_{n,e})$. We will denote by $\widehat{M}$ the induced module $\mathrm{Ind}(M)$ for simplicity. The following is shown in [**CR13a**]:

- $\widehat{V}_{n,e}$ is irreducible iff $e \notin \mathbb{Z}$.
- When $e \in \mathbb{Z}$ but $e \neq 0$, $\widehat{V}_{n,e}$ is reducible, and fits into the following short exact sequence:

$$\begin{aligned} &0 \longrightarrow \widehat{A}_{n+1,e} \longrightarrow \widehat{V}_{n,e} \longrightarrow \widehat{A}_{n,e} \longrightarrow 0 \qquad (e>0) \\ &0 \longrightarrow \widehat{A}_{n-1,e} \longrightarrow \widehat{V}_{n,e} \longrightarrow \widehat{A}_{n,e} \longrightarrow 0 \qquad (e<0) \end{aligned} \tag{3.2.1.6}$$

The modules $\widehat{A}_{n,e}$ are simple currents of $V(\widehat{\mathfrak{gl}(1|1)})$ as they can be defined as the image of the following spectral flow automorphisms $\sigma_{l,\lambda}$:

$$\sigma_{l,\lambda}(N) = N - \frac{\lambda}{z} \qquad \sigma_{l,\lambda}(E) = E - \frac{l}{z} \qquad \sigma_{l,\lambda}(\psi_\pm) = z^{\mp l}\psi_\pm. \tag{3.2.1.7}$$

Introduce a function $\epsilon(l)$ on $\mathbb{Z}$ given by:

$$\epsilon(l) = \begin{cases} -\frac{1}{2} & \text{if } l<0, \\ 0 & \text{if } l=0, \\ \frac{1}{2} & \text{if } l>0. \end{cases} \tag{3.2.1.8}$$

Define $\epsilon(l,l') = \epsilon(l) + \epsilon(l') - \epsilon(l+l')$. The simple currents $\widehat{A}_{n,e}$ has the following simple fusion rules:

$$\widehat{A}_{n,l} \times \widehat{A}_{n',l'} \cong \widehat{A}_{n+n'-\epsilon(l,l'),l+l'}. \tag{3.2.1.9}$$

The category $\mathcal{C}$ and KL are both decomposed into blocks labelled by the generalized eigenvalues of E_0:

$$\mathcal{C} = \bigoplus_e \mathcal{C}_e, \qquad KL = \bigoplus_e KL_e. \tag{3.2.1.10}$$

Of course, the induction functors maps $\mathcal{C}_e$ into KL_e, and moreover, it is proven in [**BN22**] that the induction functor $\mathrm{Ind} : \mathcal{C}_e \to KL_e$ is an equivalence iff $e \notin \mathbb{Z}$ or $e = 0$. When $e \in \mathbb{Z} \setminus \{0\}$, the category KL_e are all equivalent to KL_0, with equivalences induced by fusion product with a simple current:

$$\widehat{A}_{n,e} \times - : KL_0 \to KL_e. \tag{3.2.1.11}$$

The category KL is moreover shown to be a rigid braided tensor category [**CMY22c**].

3.2.1.2. *The category $\mathcal{C}_{\beta\gamma}$.* The mode algebra of $V_{\beta\gamma}$ is identified as the algebra of differential operators on $\mathcal{K}$, the Laurent loop space of $\mathbb{C}$. Denote by x_n the n-th coordinate of the Laurent space $\mathcal{K}$, and by ∂_{x_n} the differential of x_n. We can write its mode algebra as a big tensor product:

(3.2.1.12)

$$\begin{array}{cccccccccccccc} U(V_{\beta\gamma}) & \cdots & \otimes & \mathbb{C}[x_2,\partial_{x_2}] & \otimes & \mathbb{C}[x_1,\partial_{x_1}] & \otimes & \mathbb{C}[x_0,\partial_{x_0}] & \otimes & \mathbb{C}[x_{-1},\partial_{x_{-1}}] & \otimes & \mathbb{C}[x_{-2},\partial_{x_{-2}}] & \otimes & \cdots \\ V_{\beta\gamma} & \cdots & \otimes & \mathbb{C}[x_2] & \otimes & \mathbb{C}[x_1] & \otimes & \mathbb{C}[\partial_{x_0}] & \otimes & \mathbb{C}[\partial_{x_{-1}}] & \otimes & \mathbb{C}[\partial_{x_{-2}}] & \otimes & \cdots \end{array}$$

The second line here is the vacuum module. For each $[\lambda]\in\mathbb{C}/\mathbb{Z}$, there is a module $W_{[\lambda]}$ of $V_{\beta\gamma}$ that is defined using the column picture as follows:

(3.2.1.13)

$$\begin{array}{cccccccccccccc} U(V_{\beta\gamma}) & \cdots & \otimes & \mathbb{C}[x_2,\partial_{x_2}] & \otimes & \mathbb{C}[x_1,\partial_{x_1}] & \otimes & \mathbb{C}[x_0,\partial_{x_0}] & \otimes & \mathbb{C}[x_{-1},\partial_{x_{-1}}] & \otimes & \mathbb{C}[x_{-2},\partial_{x_{-2}}] & \otimes & \cdots \\ W_{[\lambda]} & \cdots & \otimes & \mathbb{C}[x_2] & \otimes & \mathbb{C}[x_1] & \otimes & \partial_{x_0}^{\lambda}\mathbb{C}[\partial_{x_0},\partial_{x_0}^{-1}] & \otimes & \mathbb{C}[\partial_{x_{-1}}] & \otimes & \mathbb{C}[\partial_{x_{-2}}] & \otimes & \cdots \end{array}$$

The VOA $V_{\beta\gamma}$ has a spectral flow automorphism σ such that:

$$\sigma\beta = z\beta, \qquad \sigma\gamma = z^{-1}\gamma. \tag{3.2.1.14}$$

The category $\mathcal{C}_{\beta\gamma}$ is defined to be the abelian category of finite-length modules of $V_{\beta\gamma}$ generated by $\sigma^n V_{\beta\gamma}$ and $\sigma^n W_{[\lambda]}$. This category was studied in [**BN22**] and shown to have a braided tensor category structure. The way of the study is to use the relation of $V_{\beta\gamma}$ and $V(\widehat{\mathfrak{gl}(1|1)})$ derived in [**CR13b**]. More specifically, there is an embedding $V(\widehat{\mathfrak{gl}(1|1)})\hookrightarrow V_{\beta\gamma}\otimes V_{bc}$ such that $V_{\beta\gamma}\otimes V_{bc}$ is decomposed into a direct sum of simple currents:

$$V_{\beta\gamma}\otimes V_{bc} = \bigoplus_m \widehat{A}_{-m/2+\epsilon(m),m}. \tag{3.2.1.15}$$

The lifting procedure as we have recalled in Section 3.2.1 allows one to relate KL and $\mathcal{C}_{\beta\gamma}$, which was the main work of [**BN22**]. Let $KL^{[0]}$ be the subcategory of KL whose monodromy with $\widehat{A}_{-m/2+\epsilon(m),m}$ is trivial, then there is a lifting functor:

$$\mathcal{L}_0 : KL^{[0]} \longrightarrow \mathcal{C}_{\beta\gamma} \tag{3.2.1.16}$$

that is surjective and full. This functor identifies $\mathcal{C}_{\beta\gamma}$ with the $\mathbb{Z}$ quotient $KL^{[0]}/\mathbb{Z}$, or the de-equivariantization in the sense of [**EGNO16**]. This immediately give $\mathcal{C}_{\beta\gamma}$ the structure of a braided tensor category. The category $KL^{[0]}$ can be identified with the category of modules where N_0 acts semi-simply with integer eigenvalues, and the quotient identifies an object M with $\widehat{A}_{-m/2+\epsilon(m),m} \times M$. The category $\mathcal{C}_{\beta\gamma}$ also decomposes into blocks:

$$\mathcal{C}_{\beta\gamma} = \bigoplus_{[\lambda]\in\mathbb{C}/\mathbb{Z}} \mathcal{C}_{\beta\gamma,[\lambda]} \tag{3.2.1.17}$$

labelled by the generalized eigenvalue of $J_0 = :\beta\gamma:_0$. The lifting functor maps KL_e into $\mathcal{C}_{\beta\gamma,[e^{2\pi i e}]}$.

For each $\lambda \in \mathbb{C}/\mathbb{Z}$ and for each $k > 0$, the module $W_{[\lambda]}$ has an iterated self-extension $W^k_{[\lambda]}$, and it is shown in [**BN22**] that any module of $V_{\beta\gamma}$ is a quotient of a finite direct sum of $W^k_{[\lambda]}$.

3.2.2. Definition of the Category of Line Operators.

3.2.2.1. *Definition of line operators in $\mathcal{T}_{A,\rho}$.* Since we have a Morita equivalence between $V_{A,\rho}$ and $\widetilde{V}_{A,\rho}$, we will use the more convenient $\widetilde{V}_{A,\rho}$ to define the category of line operators. By definition, $\widetilde{V}_{A,\rho}$ is a simple current extension of $V_{\beta\gamma}^{\otimes n}$, and the category $\mathcal{C}_{\beta\gamma}^{\boxtimes n}$ is a braided tensor category of $V_{\beta\gamma}^{\otimes n}$ modules via $P(z)$-intertwining operators. By definition, $\mathcal{C}_{\beta\gamma}^{\boxtimes n}$ is the smallest abelian category of $V_{\beta\gamma}^{\otimes n}$ modules containing elements of the form:

$$W_1 \otimes \cdots \otimes W_n, \qquad W_i \in \mathcal{C}_{\beta\gamma} \text{ for } 1 \leq i \leq n. \tag{3.2.2.1}$$

As commented from Section 3.2.1, this gives a braided tensor category of $\widetilde{V}_{A,\rho}$ modules:

$$\widetilde{V}_{A,\rho}\text{–Mod}_{\text{loc}}(\text{Ind}(\mathcal{C}_{\beta\gamma}^{\boxtimes n})), \tag{3.2.2.2}$$

as well as a functor:

$$\mathcal{L}_A : \text{Ind}(\mathcal{C}_{\beta\gamma}^{\boxtimes n})^{[0]} \longrightarrow \widetilde{V}_{A,\rho}\text{–Mod}_{\text{loc}}(\text{Ind}(\mathcal{C}_{\beta\gamma}^{\boxtimes n})) \tag{3.2.2.3}$$

Denote by $\mathcal{C}_{\beta\gamma}^{\boxtimes n,\rho,[0]}$ the subcategory of $\mathcal{C}_{\beta\gamma}^{\boxtimes n}$ whose objects have trivial monodromy with $\widetilde{V}_{A,\rho}$.

Definition 3.2.3. *We define the category $\mathcal{C}_{A,\rho}$ to be the image of $\mathcal{C}_{\beta\gamma}^{\boxtimes n,\rho,[0]}$ under $\mathcal{L}_A$:*

$$\mathcal{C}_{A,\rho} := \mathcal{L}_A\left(\mathcal{C}_{\beta\gamma}^{\boxtimes n,\rho,[0]}\right). \tag{3.2.2.4}$$

This is a braided tensor category of $\widetilde{V}_{A,\rho}$ modules via $P(z)$-intertwiners. The category of line operators $\mathcal{L}_{A,\rho}$ of the theory $\mathcal{T}_{A,\rho}$ is defined to be the bounded derived category $\mathcal{L}_{A,\rho} := D^b\mathcal{C}_{A,\rho}$.

The following proposition gives an easy criteria for an object to be in $\mathcal{C}_{\beta\gamma}^{\boxtimes n,\rho,[0]}$:

Proposition 3.2.4. *An object $M \in \mathcal{C}_{\beta\gamma}^{\boxtimes n}$ belongs to $\mathcal{C}_{\beta\gamma}^{\boxtimes n,\rho,[0]}$ if and only if $\sum \rho_{ia} J_0^i$ acts semi-simply with integer eigenvalues, where $J^i = :\beta^i\gamma^i:$.*

The proof of this will be presented in Appendix B. Let us examine what this means to simple objects. A simple object in $\mathcal{C}_{\beta\gamma,[\lambda]}$ is of the form $\sigma^l W_{[\lambda]}$ for some $l \in \mathbb{Z}$ and $\lambda \notin \mathbb{Z}$. When $\lambda \in \mathbb{Z}$, then the simples in $\mathcal{C}_{\beta\gamma,[0]}$ are of the form $\sigma^l V_{\beta\gamma}$. With these, we see that a simple module in:

$$\boxtimes_{i=1}^n \mathcal{C}_{\beta\gamma,[\lambda_i]} \tag{3.2.2.5}$$

can be lifted to $\mathcal{C}_{A,\rho}$ if and only if $\sum \rho_{ia}\lambda^i$ are integers for all a. These give all the simple modules of $\mathcal{C}_{A,\rho}$. In particular, any simple module in $\boxtimes_{i=1}^n \mathcal{C}_{\beta\gamma,[0]}$ can be lifted to $\mathcal{C}_{A,\rho}$. There is a $\mathbb{Z}^n$ lattice of such simple modules, and upon the lift, a $\mathbb{Z}^r$ sub-lattice (the image of ρ) will be identified. These are the atypical simple modules. In conclusion, atypical simple modules in $\mathcal{C}_{A,\rho}$ are labelled by $\mathbb{Z}^n/\mathbb{Z}^r \cong \mathbb{Z}^{n-r}$ (by the property of ρ), and the quotient can be identified with the co-character lattice of the flavor symmetry group $(\mathbb{C}^\times)^{n-r}$. These atypical simples can be identified with the simple vortex lines in physical context [**BDG⁺18**, **DGGH20**]. We will show later that under mirror symmetry, these modules can be identified with the Wilson lines in the dual theory.

Moreover, the above decomposition of $\mathcal{C}_{\beta\gamma}^{\boxtimes n}$ gives a decomposition of $\mathcal{C}_{A,\rho}$, and consequently, $\mathcal{L}_{A,\rho}$. We have seen that an object in $\boxtimes_{i=1}^n \mathcal{C}_{\beta\gamma,[\lambda_i]}$ can be lifted to $\widetilde{V}_{A,\rho}$ only when $\sum \rho_{ia}\lambda^i \in \mathbb{Z}$. If we view ρ^T as inducing a map $(\mathbb{C}/\mathbb{Z})^n \to (\mathbb{C}/\mathbb{Z})^r$, then $[\lambda]$ must be in the kernel of this map, which is identified with the image of $\rho^\vee = \tau^\mathsf{T}$. In particular, we have a decomposition:

$$\mathcal{C}_{A,\rho} = \bigoplus_{[\lambda]\in(\mathbb{C}/\mathbb{Z})^{n-r}} \mathcal{C}_{A,\rho,[\lambda]}. \tag{3.2.2.6}$$

Here $\mathcal{C}_{A,\rho,[\lambda]}$ are lifts of objects from $\boxtimes_{i=1}^n \mathcal{C}_{\beta\gamma,[\sum_\alpha \tau_{i\alpha}\lambda^\alpha]}$. These blocks behave well with fusion rules:

$$\mathcal{C}_{A,\rho,[\lambda]} \times \mathcal{C}_{A,\rho,[\mu]} \longrightarrow \mathcal{C}_{A,\rho,[\lambda+\mu]}. \tag{3.2.2.7}$$

Consequently, we have a decomposition:

$$\mathcal{L}_{A,\rho} = \bigoplus_{[\lambda]\in(\mathbb{C}/\mathbb{Z})^{n-r}} \mathcal{L}_{A,\rho,[\lambda]}. \tag{3.2.2.8}$$

3.2.2.2. *Definition of line operators in $\mathcal{T}_{B,\rho}$.* Here we will use the fact that $V_{B,\rho}$ is a simple current extension of an affine Lie superalgebra $V(\widehat{\mathfrak{g}_*(\rho)})$. Denote by KL_ρ the Kazhdan-Lusztig category for $V(\widehat{\mathfrak{g}_*(\rho)})$. We need the following two statements to be able to apply the machinery of simple current extensions. The first of course shows that KL_ρ itself has a braided tensor structure, and the second states that the monopole operators generate simple currents of $V(\widehat{\mathfrak{g}_*(\rho)})$. The proof of these two statements are somewhat lengthy and will be presented in the Appendix B.

THEOREM 3.2.5. *KL_ρ is a braided tensor category defined by logarithmic intertwining operators, and tensor product is an exact functor on KL_ρ.*

PROPOSITION 3.2.6. *Let U_λ be the direct summand of $V_{B,\rho}$ corresponding to $\lambda \in \mathbb{Z}^r$, namely the summand containing the monopole operator corresponding to λ. Then U_λ are simple as $V(\widehat{\mathfrak{g}_*(\rho)})$ modules and belong to KL_ρ, and satisfy the simple fusion rules:*

$$U_\lambda \times U_{\lambda'} \cong U_{\lambda+\lambda'} \tag{3.2.2.9}$$

which can be realized by the state-operator correspondence of $V_{B,\rho}$.

The proof of these two statements, especially Theorem 3.2.5, will reveal the following statement, which is important for the proof of the mirror symmetry statement. Recall in Corollary 3.1.16, we have shown that the VOA $V(\widehat{\mathfrak{g}_*(\rho)}) \otimes V_{\overline{X},\overline{Y}}$ is a simple current extension of $V(\widehat{\mathfrak{gl}(1|1)})^{\otimes n}$. Let $KL^{\boxtimes n,\rho,[0]}$ be the full subcategory of $KL^{\boxtimes n}$ that have trivial monodromy with $V(\widehat{\mathfrak{g}_*(\rho)}) \otimes V_{\overline{X},\overline{Y}}$. There is a lifting functor of braided tensor categories:

$$\mathcal{L}_{\text{ungauge}} : KL^{\boxtimes n,\rho,[0]} \longrightarrow V(\widehat{\mathfrak{g}_*(\rho)}) \otimes V_{\overline{X},\overline{Y}}-\text{Mod}_{\text{loc}}\left(\text{Ind}(KL^{\boxtimes n})\right). \tag{3.2.2.10}$$

The proof of Theorem 3.2.5 implies the following:

THEOREM 3.2.7. *The image of $KL^{\boxtimes n,\rho,[0]}$ under $\mathcal{L}_{\text{ungauge}}$ lies in KL_ρ, and this functor is surjective onto KL_ρ. Consequently, the category KL_ρ is equivalent to the de-equivariantization of $KL^{\boxtimes n,\rho,[0]}$ by the lattice of simple currents defining $V(\widehat{\mathfrak{g}_*(\rho)}) \otimes V_{\overline{X},\overline{Y}}$.*

We now come back to defining the category of line operators. Since $V_{B,\rho}$ is a simple current extension of $V(\widehat{\mathfrak{g}_*(\rho)})$, we have the category $V_{B,\rho}-\text{Mod}_{\text{loc}}(\text{Ind}(KL_\rho))$ as well as a lifting functor

$\mathcal{L}_B$:

$$\mathcal{L}_B : \mathrm{Ind}(KL_\rho)^{[0]} \longrightarrow V_{B,\rho}\text{–}\mathrm{Mod}_{\mathrm{loc}}\left(\mathrm{Ind}(KL_\rho)\right). \tag{3.2.2.11}$$

Let $KL_\rho^{[0]}$ be the subcategory of objects having trivial monodromy with $V_{B,\rho}$. We give the following definition:

Definition 3.2.8. *The category $\mathcal{C}_{B,\rho}$ is defined to be the image of $KL_\rho^{[0]}$ under the lifting functor $\mathcal{L}_B$:*

$$\mathcal{C}_{B,\rho} := \mathcal{L}_B\left(KL_\rho^{[0]}\right). \tag{3.2.2.12}$$

The category of line operators of $\mathcal{T}_{B,\rho}$ is defined as the derived category $\mathcal{L}_{B,\rho} := D^b\mathcal{C}_{B,\rho}$.

Similar to Proposition 3.2.4, we have the following proposition, whose proof will be presented in Appendix B.

Proposition 3.2.9. *An object M of KL_ρ belong to $KL_\rho^{[0]}$ if and only if N_0^a acts semi-simply with integer eigenvalues.*

Recall the spectral flow automorphisms introduced in Section 3.1.3.2. The above proposition implies that the spectral flows $\sigma_{\lambda,\mu}V(\widehat{\mathfrak{g}_*(\rho)})$ are objects in $KL_\rho^{[0]}$ precisely when $\mu \in \mathbb{Z}^r$. These are $\mathbb{Z}^r \times \mathbb{Z}^r$ copies of simple modules, and under the lift, one identifies the sublattice given by $\{(\lambda, \rho^\mathsf{T}\rho\lambda)\}$, and the quotient lattice is isomorphic to $\mathbb{Z}^r$. The module corresponding to $\sigma_{0,\mu}V(\widehat{\mathfrak{g}_*(\rho)})$ can be identified with the Wilson line associated to the representation defined by μ, as this object is generated by:

$$\left|\int \sum_a \mu_a E^a\right\rangle. \tag{3.2.2.13}$$

on which N_a has weight μ_a. Namely, these corresponds to representations of the gauge group $(\mathbb{C}^\times)^r$. These will be identified with the vortex lines under mirror symmetry.

Similar to $\mathcal{C}_{\beta\gamma}$, the category KL_ρ also decomposes into blocks:

$$KL_\rho = \bigoplus_{\lambda\in\mathbb{C}^r} KL_{\rho,\lambda} \tag{3.2.2.14}$$

where $KL_{\rho,\lambda}$ denotes the subcategory where the generalized eigenlvaue of E_0^a is λ^a. Under the above lift, objects in $KL_{\rho,\lambda}^{[0]}$ will be identified with $KL_{\rho,\lambda+\mu}^{[0]}$ for any $\mu \in \mathbb{Z}$. Thus, the category $\mathcal{L}_{B,\rho}$ is decomposed into blocks:

$$\mathcal{L}_{B,\rho} = \bigoplus_{[\lambda]\in(\mathbb{C}/\mathbb{Z})^r} \mathcal{L}_{B,\rho,[\lambda]}. \tag{3.2.2.15}$$

This decomposition will be equivalent to the decomposition of $\mathcal{L}_{A,\rho}$ under mirror symmetry.

3.2.3. Mirror Symmetry of the Category of Line Operators. The main goal of this section is to give a short and un-detailed proof of the following theorem:

THEOREM 3.2.10. *There is an equivalence of braided tensor categories:*

$$\mathcal{C}_{A,\rho} \simeq \mathcal{C}_{B,\rho^\vee}. \tag{3.2.3.1}$$

Consequently, $\mathcal{L}_{A,\rho} \simeq \mathcal{L}_{B,\rho}$.

PROOF. The idea is to use the relation between $\widetilde{V}_{A,\rho}$ and $V_{B,\rho^\vee}$ derived in Theorem 3.1.15. We have the following diagram of VOA extensions:

$$\begin{array}{ccc} & \widetilde{V}_{A,\rho} \otimes V_{bc}^{\otimes n} \cong V_{B,\rho^\vee} \otimes V_{\overline{X},\overline{Y}} & \\ \nearrow & & \nwarrow \\ V_{\beta\gamma}^{\otimes n} \otimes V_{bc}^{\otimes n} & & V(\widehat{\mathfrak{g}_*(\rho^\vee)}) \otimes V_{\overline{X},\overline{Y}} \\ \nwarrow & & \nearrow \\ & V(\widehat{\mathfrak{gl}(1|1)})^{\otimes n} & \end{array} \tag{3.2.3.2}$$

and the corresponding lifting functors:

$$\begin{array}{ccc} & \mathcal{C}_{A,\rho} \quad \mathcal{C}_{B,\rho^\vee} & \\ \nearrow^{\mathcal{L}_A} & & \nwarrow^{\mathcal{L}_B} \\ \mathcal{C}_{\beta\gamma}^{\boxtimes n} & & KL_\rho \\ \nwarrow_{\mathcal{L}_0} & & \nearrow_{\mathcal{L}_{\text{ungauge}}} \\ & KL^{\boxtimes n} & \end{array} \tag{3.2.3.3}$$

Thus to show that $\mathcal{C}_{A,\rho}$ and $\mathcal{C}_{B,\rho}$ are the same, one needs only show that they are the image of the same lifting functor from the same subcategory of $KL^{\boxtimes n}$. The commutativity of the lifting functor follows from the uniqueness of VOA structure from simple current extension [**CR22**]. The fact that they are lifts from the same category simply follows from definition: the subcategory of local modules with respect to the lattice defining the extended VOA $\widetilde{V}_{A,\rho} \otimes V_{bc}^{\otimes n}$. Thus $\mathcal{C}_{A,\rho} \simeq \mathcal{C}_{B,\rho}$ as desired. □

We can now prove Proposition 3.2.9.

Proof of Proposition 3.2.9. Let $KL^{\boxtimes n,\rho,[0]}$ be the subcategory of $KL^{\boxtimes n}$ that has trivial monodromy with $\widetilde{V}_{A,\rho} \otimes V_{bc}^{\otimes n}$. The proof of Theorem 3.2.10 shows that the lifting functor is essentially surjective:

$$\mathcal{C}_{A,\rho} \simeq \mathcal{C}_{B,\rho} \simeq \mathcal{L}_B \circ \mathcal{L}_{\text{ungauge}} \left(KL^{\boxtimes n,\rho,[0]} \right). \tag{3.2.3.4}$$

By Proposition 3.2.4, an object belong to $\mathcal{L}_{\text{ungauge}} \left(KL^{\boxtimes n,\rho,[0]} \right)$ if and only if the zero-mode of $\sum_i \rho_{ia} {:}\beta^i \gamma^i{:}$ is semi-simple with integer eigenvalues, and this is the same as $\sum_i \rho_{ia} E_0^i$ acting semi-simply with integer eigenvalues. Since the objects that can be lifted to $\mathcal{C}_{\beta\gamma}^{\boxtimes n}$ requires that N_0^i acts semi-simply with integer eigenvalues, we deduce that an object in $KL^{\boxtimes n}$ belongs to $KL^{\boxtimes n,\rho,[0]}$ if and only if both N_0^i and $\sum_i \rho_{ia} E_0^i$ acts semi-simply with integer eigenvalues. Note that fields $\sum_i \rho_{ia} E^i$ and $\sum \widetilde{\tau}_{ai} N^i$ belong to $V_{\overline{X},\overline{Y}}$ and $\sum \rho_{i\alpha}^\vee N^i$ are identified with N^α in $V(\widehat{\mathfrak{g}_*(\rho^\vee)})$. Therefore, the image of $KL^{\boxtimes n,\rho,[0]}$ consists of precisely those objects in KL_ρ where the action of N_0^α is semi-simple with integer eigenvalues. This completes the proof.

□

Since the fields E_α are identified with the image of $\sum_i \tau_{\alpha i} {:}\beta^i \gamma^i{:}$ in $V_{A,\rho}$, we find that the above equivalence induces an equivalence:

$$\mathcal{L}_{A,\rho,[\lambda]} \simeq \mathcal{L}_{B,\rho^\vee,[\lambda]}, \text{ for any } [\lambda] \in (\mathbb{C}/\mathbb{Z})^{n-r}. \tag{3.2.3.5}$$

Let us now identify the image of the Wilson lines. The Wilson line corresponding to $\mu \in \mathbb{Z}^{n-r}$ is generated by:

$$\left| \int \sum_\alpha \mu_\alpha E^\alpha \right\rangle . \tag{3.2.3.6}$$

Using the field re-definition in the proof of Theorem 3.1.15, this is the same as the module generated by:

$$\left| \sum_{\alpha,i} \mu_\alpha \widetilde{\rho}_{i\alpha} (\psi^i + \widetilde{\phi}^i) \right\rangle . \tag{3.2.3.7}$$

Since in defining $\widetilde{V}_{A,\rho}$, we are extending by the entire lattice of $\widetilde{\phi}^i$, the lift of this module is isomorphic to the lift of the Fock module generated by:

$$\left| \sum_{\alpha,i} \mu_\alpha \widetilde{\rho}_{i\alpha} \psi^i \right\rangle . \tag{3.2.3.8}$$

Comparing this with the free field realization of the simple modules of $V_{\beta\gamma}$ [**AW22**], and using the definition of $\widetilde{\rho}$, we see that such module precisely corresponds to $(\prod_i \sigma_i^{\sum_\alpha \mu_\alpha \widetilde{\rho}_{i\alpha}}) V_{\beta\gamma}$, which is lifted to the vortex line operator in $\mathcal{L}_{A,\rho}$. We have thus shown that vortex lines in $\mathcal{L}_{A,\rho}$ correspond to Wilson lines in $\mathcal{L}_{B,\rho^\vee}$.

3.2.4. A Quantum Group Description and Kazhdan-Lusztig Correspondence. We have succeeded in defining the braided tensor categories $\mathcal{L}_{A,\rho}$ and $\mathcal{L}_{B,\rho}$ whose objects are line operators in $\mathcal{T}_{A,\rho}$ and $\mathcal{T}_{B,\rho}$ respectively. They have the right kind of simple objects, and these objects are matched under mirror symmetry. Moreover, one can compute fusion rules using the relation of these categories to the Kazhdan-Lusztig category of $V(\widehat{\mathfrak{gl}(1|1)})$, and the work of [**CMY22c**]. However, the theory of intertwining operators are not the easiest to work with, especially due to the fact that the associator is highly non-trivial. In this section, we would like to find a quantum group whose category of modules is equivalent to $\mathcal{L}_{B,\rho}$, and conjecture that this induces an equivalence of braided tensor categories. This gives a Kazhdan-Lusztig correspondence for $V_{B,\rho}$. The main result of this section is the following:

THEOREM 3.2.11. *There exists a Hopf algebra $U_q(\mathfrak{g}_*(\rho))$ and an equivalence of abelian categories:*

$$U_q(\mathfrak{g}_*(\rho))\text{−Mod}_{\text{fin}} \simeq \mathcal{C}_{B,\rho}. \tag{3.2.4.1}$$

Moreover, the category on the left has the structure of a braided tensor category. Consequently, there is an equivalence:

$$D^b U_q(\mathfrak{g}_*(\rho))\text{−Mod}_{\text{fin}} \simeq \mathcal{L}_{B,\rho}. \tag{3.2.4.2}$$

Our proof will use the extension procedure and the equivalence in the work of [**BN22**]. Let U be the algebra generated by $N, \psi^{\pm}$ and K subjected to the following relations:

$$[N, \psi^{\pm}] = \pm\psi^{\pm},\ \{\psi^+, \psi^-\} = K - 1. \tag{3.2.4.3}$$

Moreover, we impose the condition $e^{2\pi i N} = 1$ on all modules of U. With these condition, there is an equivalence of abelian categories:

$$\mathcal{C}_{\beta\gamma} \simeq U\text{-Mod}_{\text{fin}}, \tag{3.2.4.4}$$

under which $\mathcal{C}_{\beta\gamma,[\lambda]}$ corresponds to $U\text{-Mod}_{\text{fin},e^{2\pi i\lambda}}$, the subcategory where the generalized eigenvalue of K is $e^{2\pi i\lambda}$. Since the category $\mathcal{L}_{A,\rho^\vee}$ is related to $\mathcal{C}_{\beta\gamma}^{\boxtimes n}$, we will apply the procedure of lifting using the above equivalence. Let us start with the equivalence

$$\mathcal{C}_{\beta\gamma}^{\boxtimes n} \simeq U^{\otimes n}\text{-Mod}_{\text{fin}}. \tag{3.2.4.5}$$

The subcategory of modules having trivial monodromy with $V_{A,\rho^\vee}$ is equal to the subcategory where $e^{2\pi i \sum \rho_{i\alpha}^\vee J_0^i} = 1$, and translating this to $U^{\otimes n}$, this amounts to requiring:

$$\prod_i K_i^{\rho_{i\alpha}^\vee} = 1,\ \text{for all } \alpha. \tag{3.2.4.6}$$

On the other hand, the lifting will identify an object with a spectral flow:

$$M \leftrightarrow (\prod_i \sigma_i^{\rho_{i\alpha}^\vee})M, \tag{3.2.4.7}$$

and under the equivalence to $U^{\otimes n}$, this amounts to identifying M with $M[\sum_i \rho^{\vee}_{i\alpha} N_i^*]$, where the shifting means the shifts of weights of N_i, namely, the action of N_i is shifted by $N_i \mapsto N_i + \rho^{\vee}_{i\alpha}$. In conclusion, the lifting procedure has the following two effects:

(1) Taking a quotient of $U^{\otimes n}$ by the ideal generated by $\prod_i K_i^{\rho^{\vee}_{i\alpha}} - 1$ for all α. We call this quotient $\overline{U^{\otimes n}}$.

(2) Identifying modules whose action differ only by the shift $N_i \mapsto N_i + \rho^{\vee}_{i\alpha}$.

To understand the second effect, let us use the split sequence to write:

(3.2.4.8) $$\widetilde{N}_\alpha = \sum_i \widetilde{\rho}_{i\alpha} N^i, \qquad \overline{N}_a = \sum_i \rho_{ia} N^i.$$

Let us consider shifting the action of N_i by $\rho^{\vee}_{i\alpha}$, on $\widetilde{N}_\beta$, this amounts to:

(3.2.4.9) $$\widetilde{N}_\beta \mapsto \sum_i \widetilde{\rho}_{i\beta} N^i + \widetilde{\rho}_{i\beta} (\rho^{\vee})^i{}_\alpha = \sum_i \widetilde{\rho}_{i\beta} N^i + \delta_{\alpha\beta}.$$

Here we used the fact that $\tau \cdot \widetilde{\rho} = \mathrm{Id}_{n-r}$. On the other hand, on the generators $\overline{N}_a$, we have:

(3.2.4.10) $$\overline{N}_a \mapsto \sum_i \rho_{ia} N^i + \rho_{ia} (\rho^{\vee})^i{}_\alpha = \sum_i \rho_{ia} N^i = \overline{N}_a,$$

and so the shift does not change $\overline{N}_a$. Now let us define $U_q(\mathfrak{g}_*(\rho))$ to be the subalgebra of $\overline{U^{\otimes n}}$ generated by the image of $\overline{N}_a, \psi^{\pm,i}$ and K^i. With these definition, the restriction functor:

(3.2.4.11) $$\mathrm{Res} : \overline{U^{\otimes n}}\text{-Mod}_{\mathrm{fin}} \longrightarrow U_q(\mathfrak{g}_*(\rho))\text{-Mod}_{\mathrm{fin}}$$

coincides with the lifting functor, identifying an object with its spectral flow. In conclusion, we have arrived at an equivalence:

(3.2.4.12) $$U_q(\mathfrak{g}_*(\rho))\text{-Mod}_{\mathrm{fin}} \simeq \mathcal{C}_{A,\rho^{\vee}} \simeq \mathcal{C}_{B,\rho}.$$

Define $\overline{K}_a := \prod K_i^{\widetilde{\tau}_{ai}}$, then from the exact sequence, the algebra $U_q(\mathfrak{g}_*(\rho))$ is generated by $\overline{N}_a, \psi^{\pm,i}$ and $\overline{K}_a$. Moreover, we have commutation relation:

(3.2.4.13) $$[\overline{N}_a, \psi^{\pm,i}] = \pm \rho_{ia} \psi^{\pm,i}, \ \{\psi^{+,i}, \psi^{-,i}\} = K_i = \prod_j K_j^{\sum_a \rho_{ia} \widetilde{\tau}^a{}_j + \sum_\alpha \widetilde{\rho}_{i\alpha} \tau^\alpha{}_j} = \prod_a \overline{K}_a^{\rho_{ia}}.$$

Here, the second equation for K_i follows from $\rho\widetilde{\tau}+\widetilde{\rho}\tau = \mathrm{Id}_n$ and the third follows from the quotient. The above commutation relation is a quantization of $\mathfrak{g}_*(\rho)$:

$$\{\psi^{+,i}, \psi^{-,i}\} = \sum \rho_{ia} E^a \mapsto \{\psi^{+,i}, \psi^{-,i}\} = \prod_a \overline{K}_a^{\rho_{ia}}, \tag{3.2.4.14}$$

and we understand $\overline{K}_a$ as $e^{2\pi i E^a}$. Note that we also impose the condition $e^{2\pi i N_a} = 1$ on all finite-dimensional modules.

Let us now show that $U_q(\mathfrak{g}_*(\rho))$ is a Hopf algebra. Let $U_q^{N,E}(\mathfrak{gl}(1|1))$ be the unrolled-restricted quantum $\mathfrak{gl}(1|1)$ generated by $N, E, \Psi^{\pm}$ with commutator;

$$[N, \Psi^{\pm}] = \pm\Psi^{\pm}, \qquad \{\Psi^+, \Psi^-\} = \frac{q^E - q^{-E}}{q - q^{-1}}. \tag{3.2.4.15}$$

There is an embedding of U into $U_q^{N,E}(\mathfrak{gl}(1|1))$ given by:

$$N \mapsto N, \qquad \psi^+ \mapsto \Psi^+, \qquad \psi^- \mapsto q^{-E}(q^{-1} - q)\Psi^-, \qquad K \mapsto q^{-2E}. \tag{3.2.4.16}$$

This does not induce a Hopf structure on U however, since the bi-algebra map and antipode of $U_q^{N,E}(\mathfrak{gl}(1|1))$ involves q^E rather than q^{2E}. More specifically:

(3.2.4.17)

$$\Delta(\Psi^+) = \Psi^+ \otimes 1 + q^{-E} \otimes \Psi^+, \qquad \Delta(\Psi^-) = \Psi^- \otimes q^E + 1 \otimes \Psi^-, \qquad S(\Psi^+) = -q^E\Psi^+, \qquad S(\Psi^-) = -q^{-E}\Psi^-$$

which will induce the following structure for the subalgebra U:

(3.2.4.18)

$$\Delta(\psi^+) = \psi^+ \otimes 1 + q^{-E} \otimes \psi^+, \qquad \Delta(\psi^-) = \psi^- \otimes 1 + q^{-E} \otimes \psi^-, \qquad S(\psi^+) = -q^E\psi^+, \qquad S(\psi^-) = -q^E\psi^-$$

To fix this problem, we can first conjugate the bialgebra structure of $U_q^{N,E}(\mathfrak{gl}(1|1))$ by $q^{E\otimes N}$, since:

$$q^{E\otimes N}\Delta(\psi^+)q^{-E\otimes N} = \psi^+ \otimes 1 + 1 \otimes \psi^+, \tag{3.2.4.19}$$

where the equation follows from the commutation relation $[E \otimes N, f(E) \otimes \psi^+] = Ef(E) \otimes \psi^+$ for any power series $f(E)$. Similarly, the conjugation of $\Delta(\psi^-)$ becomes $\psi^- \otimes 1 + q^{-2E} \otimes \psi^-$. Since we conjugated the co-algebra structure, we need to conjugate the antipode as well, and it is easily seen that it is given by:

$$S(\psi^+) = -\psi^+, \qquad S(\psi^-) = -q^{2E}\psi^-. \tag{3.2.4.20}$$

These structures thus makes the embedding $U \to U_q^{N,E}(\mathfrak{gl}(1|1))$ into one of Hopf algebras. Since all the functors above preserve the Hopf structure, we conclude that $U_q(\mathfrak{g}_*(\rho))$ is a Hopf algebra.

Let us now show that $U_q(\mathfrak{g}_*(\rho))\text{-Mod}_{\text{fin}}$ is braided. We start by showing that $U\text{-Mod}_{\text{fin}}$ is briaded. Consider the restriction functor:

$$\text{Res} : U_q^{N,E}(\mathfrak{gl}(1|1))\text{-Mod}_{\text{fin}}^N \to U\text{-Mod}_{\text{fin}} \tag{3.2.4.21}$$

from modules of $U_q^{N,E}(\mathfrak{gl}(1|1))$ where N acts with integer eigenvalues, which is clearly a tensor functor. If we can show that the kernel of this functor is in the Drinfeld center, then we are done because this functor is clearly surjective, and we can transport the braided tensor structure on $U\text{-Mod}_{\text{fin}}$. Choose $k = 2\log(q)$. The kernel of this tensor functor is given by the tensor subcategory whose objects are direct sums of $\mathbb{C}_{n/k}$ for $n \in \mathbb{Z}$, where $\mathbb{C}_{n/k}$ is the one-dimensional module where $N, \Psi^\pm$ acts as zero and E acts as n/k. Here $q^E = e^{\pi i k n/k} = (-1)^n$. This is true since for a module to be trivial after restriction, $K = q^{-2E} = 1$ and so E acts with weight n/k. We need to show that $\mathbb{C}_{n/k}$ have trivial monodromy with modules of $U_q^{N,E}(\mathfrak{gl}(1|1))$ where N acts with integer eigenvalues. The monodromy is given by the following conjugated R matrix:

(3.2.4.22)

$$q^{N\otimes E}q^{N\otimes E+E\otimes N}(1+ \text{ some factor } \cdot \Psi^+\otimes\Psi^-)q^{-E\otimes N} = q^{2N\otimes E}(1+ \text{ some other factor } \cdot \Psi^+\otimes\Psi^-).$$

The reason we omit the factors here is that these will not contribute to the monodromy as it will act trivially on $\mathbb{C}_{n/k}$. The monodromy of M with $\mathbb{C}_{n/k}$ is easily computed from this to be:

$$q^{2N\otimes E} : M\otimes\mathbb{C}_{n/k} \to M\otimes\mathbb{C}_{n/k} \tag{3.2.4.23}$$

which is equal to $q^{2N\otimes n/k} = e^{\pi i k\cdot 2N/k} = e^{2\pi i n N} = 1$ by the assumption on M. Thus $\mathbb{C}_{n/k}$ is in the Drinfeld center and U indeed has a braided tensor structure. Now we can transport this structure to $U_q(\mathfrak{g}_*(\rho))\text{-Mod}_{\text{fin}}$, again due to the fact that the kernel of the restriction functor from $\overline{U^{\otimes n}}$ to

$U_q(\mathfrak{g}_*(\rho))$ is in the Drinfeld center, a computation that is done very easily and in a similar way as above. This completes the proof of the above theorem. We now give the following conjecture:

CONJECTURE 3.2.12 (Kazhdan-Lusztig correspondence for $V_{B,\rho}$). *The equivalence:*

$$\mathcal{C}_{B,\rho} \simeq U_q(\mathfrak{g}_*(\rho))\text{−Mod}_{\text{fin}} \tag{3.2.4.24}$$

is one of braided tensor categories.

With this conjecture, we come to the conclusion that B twist of 3d $\mathcal{N}=4$ abelian gauge theory $\mathcal{T}_{B,\rho}$ is controlled by a quantum supergroup, and thus is related to a super-group Chern-Simons theory, which was first extensively studied in [**Mik15**]. The recent work [**Gar22**, **GN23**] explores this idea further in many examples.

3.3. Hypertoric Varieties and Vertex Operator Algebras

3.3.1. Higgs and Coulomb Branches from Vertex Operator Algebras. In the previous sections, we have constructed abelian categories $\mathcal{C}_{A,\rho}$ and $\mathcal{C}_{B,\rho}$, and the derived category $\mathcal{L}_{A,\rho}$ and $\mathcal{L}_{B,\rho}$. As was predicted in [**CCG19**], these categories can be used to realize the Higgs and Coulomb branches. We start with proving the following:

THEOREM 3.3.1. *Let $\mathbb{1}$ be the identity object in $\mathcal{L}_{B,\rho}$, then there is an algebra isomorphism:*

$$\text{Ext}^*(\mathbb{1}) \cong \mathbb{C}[\mathcal{M}_{H,\rho}]. \tag{3.3.1.1}$$

By mirror symmetry statement of Theorem 3.2.10, we also have:

$$\text{Ext}^*_{\mathcal{L}_{A,\rho}}(\mathbb{1}) \cong \text{Ext}^*_{\mathcal{L}_{B,\rho^\vee}}(\mathbb{1}) \cong \mathbb{C}[\mathcal{M}_{H,\rho^\vee}] \cong \mathbb{C}[\mathcal{M}_{C,\rho}]. \tag{3.3.1.2}$$

Thus, the Coulomb branch algebra can be obtained from the category $\mathcal{L}_{A,\rho}$.

Let us prove Theorem 3.3.1. Since the identity object is in the subcategory $\mathcal{L}_{B,\rho,[0]}$, we will use the equivalence:

$$\mathcal{L}_{B,\rho,[0]} \simeq D^b U_q(\mathfrak{g}_*(\rho))\text{−Mod}^{K-1}_{fin}, \tag{3.3.1.3}$$

where the right hand side is the category where $K_a - 1$ acts nilpotently for all α. Furthermore, one can show that there is an equivalence:

$$U_q(\mathfrak{g}_*(\rho))\text{--}\mathrm{Mod}_{fin}^{K-1} \simeq \mathfrak{g}_*(\rho)\text{--}\mathrm{Mod}_{fin}^{E} \tag{3.3.1.4}$$

where the right hand side is the category of modules of $\mathfrak{g}_*(\rho)$ where E_a acts nilpotently for all a and $e^{2\pi i N_a} = 1$. This equivalence uses the fact that the power series $f(x) = (e^{2\pi i x} - 1)/x$ is invertible locally near $x = 0$, which gives a way to go between E_a and $K_a = e^{2\pi i E_a}$ when $K_a - 1$ is nilpotent.

We now treat N_a as inducing an action of $T = (\mathbb{C}^\times)^r$. Let $\mathfrak{g}_*(\rho)_{>0}$ be the subalgebra of $\mathfrak{g}_*(\rho)$ generated by $\psi^{i,\pm}$ and E_a. Then it is clear that:

$$\mathfrak{g}_*(\rho)\text{--}\mathrm{Mod}_{fin}^{E} \simeq \mathfrak{g}_*(\rho)_{>0}\text{--}\mathrm{Mod}_{fin}^{T,E}, \tag{3.3.1.5}$$

the T equivariant modules of $\mathfrak{g}_*(\rho)_{>0}$ where E_a acts nilpotently. We will do this computation using the more convenient $\mathfrak{g}_*(\rho)_{>0}$. Consider the following complex of $\mathfrak{g}_*(\rho)_{>0}$ modules:

$$C := U(\mathfrak{g}_*(\rho)_{>0}) \otimes \mathrm{Sym}(\mathfrak{g}_*(\rho)_{>0}[1]), \tag{3.3.1.6}$$

together with a differential $d = \sum((x_i)_R + \frac{1}{2}[x_i, -]) \otimes \partial_{x_i}$, where x_i is a set of basis for $\mathfrak{g}_*(\rho)_{>0}$, $(x_i)_R$ means right multiplication on $U(\mathfrak{g}_*(\rho)_{>0})$ and $[x_i, -]$ means the conjugation action on $\mathrm{Sym}(\mathfrak{g}_*(\rho)_{>0}[1])$, and ∂_{x_i} is a set of dual basis. We have:

$$d^2 = \sum_{i,j} (x_i)_R (x_j)_R \otimes \partial_{x_i}\partial_{x_j} + \frac{1}{2}\sum_{i,j}(x_i)_R \otimes \partial_{x_i}([x_j, -] \otimes \partial_{x_j}) \tag{3.3.1.7}$$

The first term here is given by:

$$\sum_{i<j,k} f_{ij}^k (x_k)_R \otimes \partial_{x_i}\partial_{x_j}, \tag{3.3.1.8}$$

while in the second term, we have $\partial_{x_i}[x_j, -] = \sum_k f_{jk}^i \partial_{x_k}$, and we obtain:

$$\frac{1}{2}\sum_{i,j}(x_i)_R \otimes \partial_{x_i}([x_j, -] \otimes \partial_{x_j}) = \frac{1}{2}\sum_{i,j,k}(x_i)_R \otimes f_{jk}^i \partial_{x_k}\partial_{x_j} = -\sum_{i<j,k} f_{ij}^k (x_k)_R \otimes \partial_{x_i}\partial_{x_j}. \tag{3.3.1.9}$$

Thus, $d^2 = 0$ and this is a cochain complex. Since left multiplication on the left of C commutes with the right multiplication, we see that C is a differential complex of $\mathfrak{g}_*(\rho)_{>0}$ modules. In fact, the complex C is the Chevalley-Eilenberg complex of the Lie superalgebra $\mathfrak{g}_*(\rho)_{>0}$, and it resolves the trivial representation $\mathbb{C}$ of $\mathfrak{g}$.

LEMMA 3.3.2. *The complex C is a projective resolution of the trivial module $\mathbb{C}$ in the category of $\mathfrak{g}_*(\rho)_{>0}$ modules. Moreover, this resolution is equiviariant with respect to T.*

PROOF. The morphism $U(\mathfrak{g}_*(\rho)_{>0}) \to \mathbb{C}$ induces a map $C \to \mathbb{C}$ that is trivial on all other homological degrees. To show that this induces a quasi-isomorphism, we comment that by PBW theorem of $\mathfrak{g}_*(\rho)_{>0}$, C is a filtered complex $C = \cup_i F_i C$ whose associated graded $\mathrm{Gr}C$ is the Koszul complex $\mathrm{Sym}(\mathfrak{g}_*(\rho)_{>0}) \otimes \mathrm{Sym}(\mathfrak{g}_*(\rho)_{>0}[1])$, and thus is quasi-isomorphic to $\mathbb{C}$. Now the standard spectral sequence argument shows that the cohomology of C is quasi-isomorphic to $\mathbb{C}$. This completes the proof.

□

Now we need the following lemma about resolutions that are T-equivariant:

LEMMA 3.3.3. *Let P^* be a T-equivariant free $U(\mathfrak{g}_*(\rho)_{>0})$ resolution of $\mathbb{C}$. Let V_* be any other T-equivariant resolution, then there exists a T-equivariant map from P^* to V_*.*

The proof of this is rudimentary and can be found in any standard algebra textbook, for instance [**Lan12**]. As a consequence, let V_* be any T-equivariant finite resolution of $\mathbb{C}$ using finite dimensional modules, then there is a T-equivariant map $C \to V_*$. We may assume that E_a acts nilpotently on all V_*. Of course this map needs to factor through $C_{\geq -N}$, the cut-off of C at degree $-N$, which is by definition the complex $\mathrm{Ker}(C_{-N}) \to C_{-N} \to \cdots \to C_0$. Moreover, there exists an integer M such that it factors through $C_{\geq -N}/(E_a^M)$, since there exists such M such that E_a^M is zero on V_* for all a. This is good because by definition, one can see that $C_{\geq -N}/(E_a^M)$ is in fact a finite complex of finite-dimensional modules of $\mathfrak{g}_*(\rho)_{>0}$. This implies in particular that the projective system $C_{N,M} := C_{\geq -N}/(E_a^M)$ is a final object in the category $D^b\mathcal{C}_{B,\rho}/\mathbb{C}$, the category of bounded complexes over $\mathbb{C}$. By the definition of Yoneda extension group, we have:

(3.3.1.10)

$$\mathrm{Ext}_{\mathcal{L}_{B,\rho}}(\mathbb{C},\mathbb{C}) \cong \mathrm{Hom}_{\mathcal{C}_{B,\rho}}(\varprojlim C_{N,M},\mathbb{C}) \cong \varinjlim \mathrm{Hom}_{\mathcal{C}_{B,\rho}}(C_{N,M},\mathbb{C}) \cong \left(\mathbb{C} \otimes \mathrm{Sym}(\mathfrak{g}_*(\rho)^*_{>0}[1])\right)^T.$$

We can show that this is actually an isomorphism of algebras, by comparing the multiplication of the generators on both sides. Let us now write $\mathrm{Sym}(\mathfrak{g}_*(\rho)^*_{>0}[1])$ explicitly. It is given by the following DG algebra:

$$\mathrm{Sym}(\mathfrak{g}_*(\rho)^*_{>0}[1]) = \mathbb{C}[x_i, y_i, b_a] \tag{3.3.1.11}$$

with a differential given by $d = \sum \rho_{ai} x^i y^i \otimes \partial b_a$. By definition, the cohomology of this complex is nothing but $\mathbb{C}[\mu^{-1}(0)]$. Taking T invariant part, we arrive at:

$$\mathrm{Ext}^*_{\mathcal{L}_{B,\rho}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}[\mu^{-1}(0)]^T = \mathbb{C}[\mathcal{M}_{H,\rho}], \tag{3.3.1.12}$$

as desired. We have now completed the proof of Theorem 3.3.1.

Corollary 3.3.4. *Let $\xi \in T^*$ be a character of T. Let $V_{n\xi}$ be the Wilson lines in $\mathcal{L}_{B,\rho}$ corresponding to the character $n\xi$ for $n \in \mathbb{Z}$, then there exists isomorphism of $\mathbb{C}[\mathcal{M}_{H,\rho}]$ modules:*

$$\mathrm{Hom}(\mathbb{1}, V_{n\xi}) \cong \mathbb{C}[\mu^{-1}(0)]^{T, n\xi} \tag{3.3.1.13}$$

where the right hand side is the subspace of functions that transform like $n\xi$ under the action of T.

Proof. The proof follows the same method as above, once one identify the Wilson line $V_{n\xi}$ as the trivial module $\mathbb{C}_{n\xi}$, the trivial module $\mathbb{C}$ of $\mathfrak{g}_*(\rho)_{>0}$ with equivariant structure shifted by $n\xi$.

□

In the category $\mathcal{L}_{B,\rho,[0]}$, one has the fusion rule $V_{n\xi} \times V_{m\xi} \cong V_{(m+n)\xi}$, which corresponds to, under the equivalence to the category $\mathfrak{g}_*(\rho)_{>0}{-}\mathrm{Mod}^{T,E}_{fin}$, the simple tensor product rule $\mathbb{C}_{n\xi} \otimes \mathbb{C}_{m\xi} \cong \mathbb{C}_{(m+n)\xi}$. The fusion product induces maps:

$$\mathrm{Hom}(\mathbb{1}, V_{n\xi}) \otimes \mathrm{Hom}(\mathbb{1}, V_{m\xi}) \longrightarrow \mathrm{Hom}(\mathbb{1}, V_{(m+n)\xi}). \tag{3.3.1.14}$$

The following space:

$$A_\xi = \bigoplus_{n \geq 0} \mathrm{Hom}(\mathbb{1}, V_{n\xi}) \tag{3.3.1.15}$$

has the structure of a $\mathbb{Z}$-graded commutative algebra. The algebra structure comes from the above fusion map, and the commutativity comes from the fact that $\mathcal{L}_{B,\rho}$ is a braided tensor category. Following Corollary 3.3.4, we arrive at the following statement.

Corollary 3.3.5. *The projective variety* $\mathrm{Proj}(A_\xi)$ *is isomorphic to* $\mathcal{M}^\xi_{H,\rho}$.

We have finally derived the desired statement, that one can obtain the Higgs and Coulomb branches of abelian gauge theories and their resolutions using boundary vertex operator algebras. We expect that this approach can give explicit understanding of the braided tensor structure on $\mathrm{Coh}(\mathcal{M}^\xi_{H,\rho})$, but will leave this for a future work.

3.3.2. Sheaf of Vertex Algebras on Hypertoric Varieties.

3.3.2.1. *Kuwabara's Sheaf of Vertex Algebras.* Fix ρ and a parameter ξ, and consider the Higgs branch $\mathcal{M}^\xi_{H,\rho}$. Following Theorem 2.1.5, we will assume that ρ is unimodular and ξ is generic so that $\mathcal{M}^\xi_{H,\rho}$ is smooth. In [**Kuw21**], following the construction of [**AKM15**], the author defined a sheaf of vertex operator algebras $\mathcal{V}_{A,\rho,\xi}$ on $\mathcal{M}^\xi_{H,\rho}$, such that over any local chart, the sheaf localizes to symplectic bosons.

More specifically, their definition of $\mathcal{V}_{A,\rho,\xi}$ is extremely similar to the definition of $V_{A,\rho}$. One start with the $\hbar$-adic version of the VOA $V_{\beta\gamma}^{\otimes n}\otimes V_{bc}^{\otimes n}$, which denote by $V_{\beta\gamma}^{\otimes n,\hbar}\otimes V_{bc}^{\otimes n}$. This is the vertex algebra over the ring $\mathbb{C}[\![\hbar]\!]$ whose OPE is:

$$\gamma^i(z)\beta^j(w)\sim\frac{\hbar\delta^{ij}}{z-w},\qquad b^i(z)c^j(w)\sim\frac{\delta^{ij}}{z-w}. \tag{3.3.2.1}$$

The formal variable $\hbar$ allow $V_{\beta\gamma}^{\otimes n,\hbar}\otimes V_{bc}^{\otimes n}$ to localize well on $T^*\mathbb{C}^n$:

Theorem 3.3.6 ([**AKM15**]). *There is a sheaf of $\hbar$-adic VOA $\mathcal{V}^\hbar_{T^*V}$ whose global sections is* $V_{\beta\gamma}^{\otimes n,\hbar}\otimes V_{bc}^{\otimes n}$.

The idea of [**Kuw21**] is then to consider the BRST complex:

$$\mathcal{V}_{BRST,\rho}:=V_{\beta\gamma}^{\otimes n,\hbar}\otimes V_{bc}^{\otimes n}\otimes V_{bc}^{\otimes r} \tag{3.3.2.2}$$

whose differential is defined in the same way as equation (3.1.2.9), except that the coefficient in front of $:b^ic^i:$ will be multiplied by $\hbar$. Note that the difference of the definition here and the one in [**Kuw21**] is that we extended the Heisenberg VOA to the full free fermion algebra, and eliminated the $\hbar$ coefficients in the fermionic OPE. Define:

$$\mathcal{V}^{\hbar}_{A,\rho} := H^0(\mathcal{V}_{BRST,\rho}, Q_{BRST}), \tag{3.3.2.3}$$

which is a sheaf of VOA on T^*V.

THEOREM 3.3.7 ([**Kuw21**]). *The cohomology sheaf $\mathcal{V}^{\hbar}_{A,\rho}$ is supported on $\mu^{-1}(0)$. Moreover, the sheaf restricted to $\mu^{-1}(0)\cap(T^*V)^{ss}_{\xi}$ descends to a sheaf on $\mathcal{M}^{\xi}_{H,\rho}$, or in other words, there exists a sheaf $\mathcal{V}_{A,\rho,\xi}$ on $\mathcal{M}^{\xi}_{H,\rho}$ such that:*

$$\pi^*\mathcal{V}_{A,\rho,\xi} \cong \mathcal{V}^{\hbar}_{A,\rho} \tag{3.3.2.4}$$

on $\mu^{-1}(0)\cap(T^*V)^{ss}_{\xi}$. Over each local chart of $\mathcal{M}^{\xi}_{H,\rho}$, the VOA $\mathcal{V}_{A,\rho,\xi}$ is a localization of $V_{\beta\gamma}^{\otimes n-r,\hbar}\otimes V_{bc}^{\otimes n,\hbar}$.

Let us now consider applying this statement to Theorem 3.1.10. The VOA $V_{B,\rho^\vee}$ also admits $\hbar$-adic version from the theorem, and we denote it by $\mathcal{V}^{\hbar}_{B,\rho^\vee}$, where the OPE of the affine Lie superalgebra $V(\mathfrak{g}_*(\rho))$ becomes:

$$\begin{aligned} &N_a(z)E^b(w)\sim\frac{\hbar\delta^b_a}{(z-w)^2},\ N_a(z)N_b(w)\sim\frac{\sum_i \rho^i{}_a\rho_{ib}}{(z-w)^2}\\ &N_a(z)\psi^{i,+}(w)\sim\frac{\rho^i{}_a\psi^{i,+}}{(z-w)},\ N_a(z)\psi^{i,-}(w)\sim\frac{-\rho^i{}_a\psi^{i,-}}{(z-w)}\\ &\psi^{i,+}(z)\psi^{j,-}(w)\sim\frac{\hbar\delta^{ij}}{(z-w)^2}+\frac{\delta^{ij}\sum_a\rho^i{}_aE^a}{z-w}. \end{aligned} \tag{3.3.2.5}$$

Moreover, the OPE of monopole operators with the fields here follow from the realization of monopole operators as $\exp\int N_a$. Note that this OPE can simply be derived from $V_{B,\rho^\vee}$ by rescaling $E_a\mapsto\hbar E_a$ and $\psi^{i,\pm}\mapsto\hbar^{1/2}\psi^{i,\pm}$, which through the isomorphism of Theorem 3.1.10, simply becomes $\beta,\gamma\mapsto\hbar^{1/2}\beta,\hbar^{1/2}\gamma$, and we recover exactly the $\hbar$-adic OPE of $V_{\beta\gamma}^{\otimes n,\hbar}$. This argument implies:

COROLLARY 3.3.8. *The $\hbar$-adic VOA $\mathcal{V}^{\hbar}_{B,\rho^\vee}$ is a sheaf of vertex operater algebra on T^*V supported on $\mu^{-1}(0)$, and that it descends to a sheaf on $\mathcal{M}^{\xi}_{H,\rho}\cong\mathcal{M}^{\xi}_{C,\rho^\vee}$. For each $r\times r$ minor of ρ, say Δ,* *localization gives a map of VOAs:*

$$\Delta : \mathcal{V}^{\hbar}_{B,\rho^\vee} \longrightarrow V^{\otimes n-r,\hbar}_{\beta\gamma} \otimes V^{\otimes n}_{bc}. \tag{3.3.2.6}$$

Example. Let $\rho = (1,1,\cdots,1)^{\mathsf{T}}$, then $\rho^\vee = \tau^{\mathsf{T}}$ where τ is the matrix:

$$\tau = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \tag{3.3.2.7}$$

The VOA $\mathcal{V}^{\hbar}_{B,\rho^\vee}$ has an affine Lie superalgebra generated by N_α, E_α and $\psi^{i,\pm}$. For each $1 \leq i \leq n$, we have an embedding:

$$\Delta_i : \mathcal{V}^{\hbar}_{B,\rho^\vee} \longrightarrow V^{\otimes n-1,\hbar}_{\beta\gamma} \otimes V^{\otimes n}_{bc}. \tag{3.3.2.8}$$

Let us write out the embedding Δ_1 explicitly:

$$\begin{aligned} &N_\alpha \mapsto \sum \tau_{\alpha i} {:}b^i c^i{:}, \qquad \psi^{+,1} \mapsto b^1,\ \psi^{-,1} \mapsto \hbar\partial c^1 + \sum_{i>1} ({:}\beta^i\gamma^i - \hbar b^i c^i{:}) c^1, \\ &\psi^{i,+} \mapsto \beta^i b^i,\ \psi^{i,-} \mapsto \gamma^i c^i,\ \text{for } i>1,\ E_\alpha = \sum_{i>\alpha} {:}\beta^i\gamma^i - \hbar b^i c^i{:} \end{aligned} \tag{3.3.2.9}$$

In conclusion, the VOA $V^{\hbar}_{A,\rho}$ is the global section of a G-equivariant sheaf of vertex algebra on T^*V and Kuwabara's sheaf of VOA on $\mathcal{M}^{\xi}_{H,\rho}$ is its localization to semi-stable points. In the following section, we will show that the VOA $\mathcal{V}^{\hbar}_{B,\rho}$ can be made into a sheaf of VOA object on the Coulomb branch, and in this context, the formal parameter $\hbar$ is the cohomological grading. But before that, let us recall the idea of shifted tangent complex.

3.3.2.2. *Shifted Tangent Complex.* Since the work of [**Kap99**], it is known that the shifted tangent complex $T_X[-1]$ of a smooth complex variety X has the structure of a Lie algebra in the symmetric monoidal category $\mathrm{Coh}(X)$. The Lie algebra structure is given by the Atiyah class. Let $\Delta : X \to X \times X$ be the diagonal embedding and let I be the sheaf of ideals defining the diagonal in $X \times X$. Since X is smooth, the coherent sheaf I/I^2 is a free module, and can be identified with the cotangent bundle L_X of X. By definition, there is an exact sequence of coherent sheaves on X:

$$L_X \cong I/I^2 \longrightarrow \mathcal{O}_{X\times X}/I^2 \longrightarrow \mathcal{O}_X. \tag{3.3.2.10}$$

Dualizing this complex gives:

$$\mathcal{O}_X \longrightarrow (\mathcal{O}_{X\times X}/I^2)^* \longrightarrow T_X. \tag{3.3.2.11}$$

Given any coherent sheaf M, tensoring with the above complex gives a short exact sequence:

$$M \longrightarrow (\mathcal{O}_{X\times X}/I^2)^* \otimes M \longrightarrow T_X \otimes M, \tag{3.3.2.12}$$

and thus an element in $\mathrm{Ext}^1(T_X \otimes M, M) = \mathrm{Hom}(T_X[-1] \otimes M, M)$, which is called the Atiyah class of M, denoted by α_M. When $M = T_X[-1]$, this map $\alpha_{T_X[-1]}$ becomes a morphism: $T_X[-1] \otimes T_X[-1] \to T_X[-1]$ and satisfies the graded anti-symmetry and Jacobi identity, making $T_X[-1]$ a Lie algebra object on X. Moreover, the Atiyah class α_M gives a canonical module structure of M as a $T_X[-1]$ module. This in fact gives a fully-faithful functor:

$$\mathrm{Coh}(X) \to T_X[-1]\text{-Mod}(\mathrm{Coh}(X)). \tag{3.3.2.13}$$

In particular, any hom is a hom of $T_X[-1]$ modules.

When X is smooth affine, the short exact sequence in equation (3.3.2.11) splits for any free module M, and in particular, $T_X[-1]$ is a trivial Lie algebra, in that there is no Lie bracket. Suppose M is a complex of free modules M^* with differential d viewed as a matrix, the morphism $T_X[-1] \otimes M \to M$ can be described as follows. For any section v of T_X, view v as a vector field on X, then we can use it to differentiate the matrix d to obtain $v(d)$. Since $d^2 = 0$, we have $0 = v(d^2) = [d, v(d)]$, and so $v(d)$ is a closed degree 1 morphism. The assignment $v \to v(d)$ gives the action of $T_X[-1]$ on M. One may worry that this definition might not satisfy the bracket of $T_X[-1]$, but this is guaranteed by the following:

$$[v(d), u(d)] = v[d, u(d)] \pm [d, vu(d)] = \pm[d, vu(d)] \tag{3.3.2.14}$$

and $[d, vu(d)]$ is trivial in the cohomology. Here the term $v[d, u(d)]$ vanish because $[d, u(d)] = 0$ which follows from $d^2 = 0$.

When X is not affine, then the short exact sequence in equation (3.3.2.11) does not split anymore. In this case, the Atiyah class is better captured by a class in Cech cohomology. Let $\{U_\alpha\}$ be an affine covering of X, then locally, a splitting of the exact sequence (3.3.2.11) amounts to the choice of a set of local coordinates, say x_i^α for U_α, since in that case the tangent complex is generated by free modules $\partial_{x_i^\alpha}$. There is a transition matrix:

$$\partial_{x_i^\alpha} = \sum G_{ij}^{\alpha,\beta} \partial_{x_{j,\beta}}, \tag{3.3.2.15}$$

such that the matrix G_{ij} is the gradient matrix of the local change of coordinate $x_i^\beta = g_i^{\alpha\beta}(x_j^\alpha)$ and $G_{ij}^{\alpha\beta} = \frac{\partial g_j}{\partial x_i^\alpha}$. These transition functions allow one to interprete the short exact sequence in equation (3.3.2.11) as a class in the Cech complex:

$$\sum_{i,j} \mathrm{d}x_i^\alpha \otimes \frac{\partial G_{ij}^{\alpha\beta}}{\partial x_j^\beta} \in T_X^* \otimes \mathcal{O}_X(U_{\alpha\beta}) \cong L_X(U_{\alpha\beta}), \tag{3.3.2.16}$$

and this class can be used then to induce a class in $\mathrm{Ext}^1(T_X \otimes M, M)$ for any coherent sheaf M.

Extending this definition of tangent complex from smooth varieties to arbitrary varieties and especially stacks is not straightforward, and is done in [**Hen18**] for derived Artin stacks, and in [**GR17**] for general pre-stacks locally almost of finite type. The definition of these can be found in [**Lur04**] and [**GR19**]. The central idea of the definition is that every nilpotent extension (formal moduli problem) of X is controlled by a Lie algebra, the relative tangent Lie algebra, and that the shifted tangent Lie algebra corresponds to the formal completion of the diagonal.

More precisely, given a stack X, denote by Lie_X the category of DG Lie algebras in $\mathrm{IndCoh}(X)$, and by PSt_X^f the category of pointed formal stacks over X. By this we mean the category whose objects are:

$$\pi : Y \leftrightarrow X : s \tag{3.3.2.17}$$

where π is an inf-schematic nil-isomorphism, namely the restriction of π on the reduced stack Y^{red} is an isomorphism, and s is a section of π such that $\pi \circ s = \mathrm{Id}$. Denote by Gr_X^f the category of formal groups over X, namely the category of group objects in PSt_X^f. The first important result in [**GR17**] is that there is a continuous functor $\Omega_X : \mathrm{PSt}_X^f \to \mathrm{Gr}_X^f$ given by:

$$\Omega_X Y = X \times_Y X, \tag{3.3.2.18}$$

the derived intersection of X in Y. This functor is an equivalence of categories, with inverse given by the Bar-complex of a group. Moreover, there is a functor $\mathcal{L}_X : \mathrm{Gr}_X^f \to \mathrm{Lie}_X$, which is essentially taking the Lie algebra of the formal group. It is proven that this is also an equivalence of categories whose inverse Exp_X is given by the formal completion of the Lie algebra at 0, and whose Lie group structure is given by Baker–Campbell–Hausdorff formula. The shifted tangent Lie algebra of X is defined by $l_X[-1] := \mathcal{L}_X \Omega_X((X \times X)^f)$, the image of the formal completion of $X \times X$ along the diagonal morphism. One can show that the underlying sheaf of $l_X[-1]$ is indeed the tangent complex, which follows from the definition of the cotangent complex as the universal object classifying derivations of X in $X \times X$. In general, the underlying sheaf of $\mathcal{L}_X \Omega_X Y$ is the relative tangent complex $T_{X/Y}$. This definition is very abstract, but it is proven in [**Hen18**] that when X is smooth, this definition coincides with the definition using Atiyah class, and moreover, this definition behaves well under pullback of open substacks.

Coming back to the hypertoric varieties, denote by T the complex torus and $\mathfrak{t}$ its Lie algebra. Let us consider the following complex of free modules over T^*V:

$$\mathcal{T} := \mathcal{O} \otimes \mathfrak{t} \longrightarrow \mathcal{O} \otimes T^*V[-1] \longrightarrow \mathcal{O} \otimes \mathfrak{t}^*[-2] \tag{3.3.2.19}$$

in which the first differential is given by the map from $\mathfrak{g}$ to the tangent space of T^*V and the second differential is given by push-forward of the tangent vector by the moment map. Suggestively, we will denote by N_a basis of $\mathcal{O} \otimes \mathfrak{t}$, $\psi^{i,\pm}$ basis of $\mathcal{O} \otimes T^*V$ and E_a basis of $\mathcal{O} \otimes \mathfrak{t}^*$, then the differential is given by $dN_a = \sum_i \rho_{ia}(x_i \psi^{i,-} - y_i \psi^{i,+})$, $d\psi^{i,-} = \sum_a \rho^i{}_a y^i E^a$, $d\psi^{i,+} = \sum_a \rho^i{}_a x^i E_a$. We think of N_a as basis of the Lie algebra, $\psi^{i,-} = \partial_{x_i}, \psi^{i,+} = \partial_{y_i}$ and $E_a = \partial_{N_a}$. This is a G-equivariant complex of free modules, if we give $\psi^{i,+}$ the opposite charge of y_i and $\psi^{i,-}$ opposite charge to x_i. We claim:

Lemma 3.3.9. *The Lie bracket in equation* (3.1.3.2) *makes* $\mathcal{T}$ *into a DG Lie algebra object in* $\mathrm{Coh}_T(T^*V)$.

Proof. Let us first comment that the differential on $\mathcal{T}$ is given by the following inner morphism $d = \sum_i x_i[\psi^{i,-}, -] + y_i[\psi^{i,+}, -]$. This immediately implies that the bracket is closed under differential, due to Jacobi identity. Moreover, $d^2 = \sum_i \rho_{ia} x^i [E^a, -] = 0$ because E_a are central. The Lie bracket is equivariant due to the grading condition, and so $\mathcal{T}$ is a DG Lie algebra object. □

The main theorem of this section is the following:

Theorem 3.3.10. *Let* $i_\xi : \mu^{-1}(\xi) \to T^*V$ *be the embedding and* $\overline{i_\xi}$ *be the corresponding embedding of* T*-quotient stack. Then the restriction* $\overline{i_\xi}^*\mathcal{T}$ *is the shifted tangent Lie algebra of* $\mu^{-1}(\xi)/T$.

We will prove this theorem in two steps. First, let us focus on the tangent complex of $\mu^{-1}(\xi)$.

Lemma 3.3.11. *Let* $i_\xi^*\mathcal{T}_{>0}$ *be the sub-complex of positive degrees. Then* $i_\xi^*\mathcal{T}_{>0}$ *is the shifted tangent Lie algebra of* $\mu^{-1}(\xi)$.

Proof. Since $\mu^{-1}(\xi)$ is affine, so is $\mu^{-1}(\xi) \times \mu^{-1}(\xi)$. Therefore, the scheme $\mu^{-1}(\xi) \times \mu^{-1}(\xi)$ as an object in $\mathrm{PSt}^f_{\mu^{-1}(0)}$ is represented by the Chevalley-Eilenberg cochain complex $\mathrm{CE}^*(T_{\mu^{-1}(\xi)}[-1])$. Since by [**Hen18**], the functor between $\mathrm{PSt}^f_{\mu^{-1}(\xi)}$ and $\mathrm{Lie}_{\mu^{-1}(\xi)}$ is an equivalence, we only need to show that the Chevalley-Eilenberg cochain complex of $i_\xi^*\mathcal{T}_{>0}$ also represents $\mu^{-1}(\xi) \times \mu^{-1}(\xi)$.

This can be computed very explicitly. Denote by X_i, Y_i the (shifted) dual of $\psi^{i,+}$ and $\psi^{i,-}$ respectively, and by c_a the dual of E_a. The Chevalley-Eilenberg cochain complex of $\mathcal{T}_{>0}$ is the following DG commutative algebra over $\mathbb{C}[T^*V]$:

$$\mathbb{C}[T^*V] \otimes \mathbb{C}[X_i, Y_i, c_a] \tag{3.3.2.20}$$

with a differential acting on the generators as:

$$dc_a = \sum \rho^i{}_a (X_i y_i + x_i Y_i + X_i Y_i). \tag{3.3.2.21}$$

We can rewrite the above into:

$$dc_a = \sum \rho^i{}_a((X_i + x_i)(y_i + Y_i) - x_i y_i). \tag{3.3.2.22}$$

Therefore, if we define the coordinates $\widetilde{x_i} = x_i + X_i$ and $\widetilde{y_i} = y_i + Y_i$, then the above is the ring of functions on the subspace of $T^*V \times T^*V$ defined by $\mu(x,y) = \mu(\widetilde{x}, \widetilde{y})$. Pulling back to $\mu^{-1}(\xi)$, this becomes the subspace of $T^*V \times T^*V$ such that:

$$\mu(x,y) = \mu(\widetilde{x}, \widetilde{y}) = \xi, \tag{3.3.2.23}$$

namely $\mu^{-1}(\xi) \times \mu^{-1}(\xi)$. Consequently, the Chevalley-Eilenberg cochain complex of $i_\xi^* \mathcal{T}_{>0}$ represents $\mu^{-1}(\xi) \times \mu^{-1}(\xi)$, and the proof is complete.

□

We now come to the part of adding the group quotient, and do so in a more general setting. Let X be an affine variety whose shifted tangent Lie algebra is a perfect complex $T_X[-1]$ concentrated in positve degrees. Assume that there is an action of G on X that induces an action of G (as well as the Lie algebra $\mathfrak{g}$) on $T_X[-1]$, and a map $\mathcal{O}_X \otimes \mathfrak{g} \to T_X$ that is G-equivariant. It is a standard result that in this case, the shifted tangent complex of X/G (as an object in $\mathrm{IndCoh}(X/G)$) is represented by the complex $\mathcal{O}_X \otimes \mathfrak{g} \to T_X[-1]$, namely the cone of the above map.

Proposition 3.3.12. *The Lie bracket of $T_{X/G}[-1]$ is given on $\mathcal{O}_X \otimes \mathfrak{g} \to T_X[-1]$ by the combination of:*

(1) *The Lie bracket of $T_X[-1]$.*
(2) *The Lie bracket of $\mathfrak{g}$.*
(3) *The action of $\mathfrak{g}$ on $T_X[-1]$.*

Proof. Let us denote by $\overline{X}$ the quotient stack X/G, and $\pi : X \to \overline{X}$ the projection. Let $\mathcal{T}_X$ be the complex $\mathcal{O}_X \otimes \mathfrak{g} \to T_X[-1]$ with the Lie bracket defined as in the proposition. We would like to show that $T_{\overline{X}}[-1] \cong \mathcal{T}_X$ as Lie algebra objects. By definition, the loop space $\Omega_{\overline{X}} \overline{X} \times \overline{X}$ is

$G \setminus H$ where H is defined by the following diagram:

$$\begin{array}{ccc} H & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ \widehat{G}_e \times X & \xrightarrow{id \times m} & \widehat{X \times X} \end{array} \tag{3.3.2.24}$$

Here $\widehat{G}_e$ is the formal completion of G at identity e and $\widehat{X \times X}$ is the formal completion of $X \times X$ at X. Let I_e be the ideal of G defining e and let G_e^n be the n-th formal neighborhood of e in G, or in other words, $G_e^n = \mathrm{Spec}(\mathbb{C}[G]/I_e^n)$. This is a formal group such that $\varinjlim_n G_e^n = \widehat{G}_e$ by definition. Let H_n be the product $G_e^n \times X \times_{\widehat{X \times X}} X$, then $G \setminus H = \varinjlim_n G \setminus H_n$. Each H_n is represented by a G-equivariant affine scheme, since both X and G_e^n are affine, and moreover, this is a limit of formal group objects over X/G. Therefore, we need to compute the Lie algebra of H_n and the limit.

By definition as in [**GR17**], the tangent Lie algebra is the relative tangent complex $T_X H_n$, and the Lie bracket is induced from the morphism $H_n \times_X H_n \to H_n$, intuitively given by $(g,h) \to ghg^{-1}h^{-1}$. We must compute H_n and the formal group structure explicitly.

Since X is affine, the diagonal $\widehat{X \times X}$ can be represented by $\mathrm{CE}^*(T_X[-1])$, the Chevalley-Eilenberg cochain complex. The diagonal embedding $X \to \widehat{X \times X}$ is of course represented by the quotient map $\mathrm{CE}^*(T_X[-1]) \to \mathbb{C}[X]$, setting $T_X^\vee$ to zero. In this context, there is a very explicit dg resolution of $\mathbb{C}[X]$ using complexes of $\mathrm{CE}^*(T_X[-1])$ (as in the case of ordinary Lie algebras, see [**Hen18**]):

$$\mathrm{CE}^*(T_X[-1]) \otimes_{\mathcal{O}_X} \mathrm{Sym}(T_X[-1]^\vee) \cong \mathbb{C}[X] \tag{3.3.2.25}$$

where we identify $\mathrm{Sym}(T_X[-1]^\vee)$ as the dual of $U(T_X[-1])$, and the above takes the Chevalley-Eilenberg cochain complex of $\mathrm{Sym}(T_X[-1]^\vee)$ as a module of $T_X[-1]$ (under conjugation action). On the other hand, the affine scheme $G_e^n \times X$ can be represented by the algebra $\mathbb{C}[\mathfrak{g}_n] \otimes \mathbb{C}[X]$ where $\mathfrak{g}_n$ is the Lie algebra of G_e^n. This is true because exponential map gives an isomorphism $\mathfrak{g}_n \cong G_e^n$ as G_e^n is formal. The map $G_e^n \times X \to \widehat{X \times X}$ is represented by the composition of the following morphisms of algebras:

$$\mathrm{CE}^*(T_X[-1]) \longrightarrow \mathbb{C}[X] \longrightarrow \mathbb{C}[\mathfrak{g}_n] \otimes \mathbb{C}[X]. \tag{3.3.2.26}$$

Consequently, the affine scheme H_n is represented by the following very explicit DG scheme:

$$\mathbb{C}[H_n] \cong \mathbb{C}[\mathfrak{g}_n] \otimes \mathrm{Sym}(T_X[-1]^\vee), \tag{3.3.2.27}$$

with a differential given by the differential on $T_X[-1]^\vee$ together with the following explicit morphism:

$$d : T_X[-1]^\vee \to T_X[-1]^\vee \otimes \mathbb{C}[\mathfrak{g}_n], \tag{3.3.2.28}$$

induced from the action $G_e^n \times X \to X \times X$. It is clear that the relative tangent complex $T_X H_n$ is given by $\mathcal{O} \otimes \mathfrak{g}_n \to T_X[-1]$, and using the above explicit DG algebra, the induced Lie bracket on $T_X H_n$ can be identified with the one in the statement of the proposition. For example, the Lie bracket of $\mathfrak{g}_n$ is given by the differential of the Lie group structure on G_e^n, and the commutator of $\mathfrak{g}_n$ with $T_X[-1]$ is given by the differential of the action of G_e^n on $T_X[-1]$.

Taking a limit as $n \to \infty$, we obtain $T_X H = \varinjlim_n T_X H_n = \mathcal{T}_X$, as desired.

□

Note that theorem 3.3.10 is a consequence of this proposition and Lemma 3.3.11. We have thus derived that the DG Lie algebra $\mathcal{T}$ built from the Lie superalgebra $\mathfrak{g}_*(\rho)$ is the shifted tangent Lie algebra of $\mu^{-1}(\xi)/T$, and in particular, $\overline{i_0}^*\mathcal{T}$ is the shifted tangent complex of the stacky quotient $\mu^{-1}(0)/T$. The work of [**Hen18**] also implies that the shifted tangent Lie algebra behave well under open pull-back, namely when $U \to X$ is open, then $T_X[-1]|_U \cong T_U[-1]$ as a Lie algebra. We thus obtain the following:

Corollary 3.3.13. *Let $j_\xi : \mathcal{M}_{H,\rho}^\xi \to \mu^{-1}(0)/G$ be the open embedding, then $j_\xi^*\mathcal{T}$ is the shifted tangent Lie algebra of the smooth variety $\mathcal{M}_{H,\rho}^\xi$, namely the localization of the Lie algebra structure is given by the Atiyah class.*

Example. Consider the case $\rho = (1, 1, \dots, 1)^\mathsf{T}$ and $\xi = 1$. In this case, $\mathcal{M}_{H,\rho}^\xi = T^*\mathbb{P}^{n-1}$, and there are n embeddings $\Delta_i : T^*\mathbb{C}^{n-1} \to \mu^{-1}(0)/G$ covering $T^*\mathbb{P}^{n-1}$. The pull-back of the complex $\mathcal{T}$ over each Δ_i is free and generated in cohomological degree 1 by ($i \neq j$):

$$v_{i,j} = \frac{1}{x_i}\psi^{j,+} - \frac{x_j}{x_i^2}\psi^{i,+}, \qquad v_{i,j}^* := x_j\psi^{j,-} - y_j\psi^{j,+}. \tag{3.3.2.29}$$

The Lie algebra structure is trivial over each chart since:

$$\{v_{i,j}, v_{i,j}^*\} = E = \frac{1}{x_i} d\psi^{i,+}. \tag{3.3.2.30}$$

The element representing the bracket $\{v_{i,j}, v_{i,j}^*\}$ in Cech cohomology is simply $-\frac{x_i}{x_j} v_{i,j}$ in $U_i \cap U_j$. One can verify that this coincides with the Atiyah class evaluated on $v_{i,j} \otimes v_{i,j}^*$.

We consider the non-degenerate bilinear form κ_0 on $\mathfrak{g}_*(\rho)$ defined by:

$$\kappa_0(N_a, E_b) = \delta_{ab}, \qquad \delta_0(\psi^{i,+}, \psi^{j,-}) = \delta^{ij}. \tag{3.3.2.31}$$

This induces, by linearity, a metric on $\mathcal{T}$: $\mathcal{T} \otimes_{\mathbb{C}[T^*V]} \mathcal{T} \to \mathbb{C}[T^*V][-2]$, which of course is invariant under T and the left action by $\mathcal{T}$. For each $\xi \in \mathfrak{g}^*$, the pullback $\overline{i_\xi^*}\mathcal{T}$ is thus a metric Lie algebra in $\mathrm{Coh}(\mu^{-1}(\xi)/T)$, and in particular, the pullback $i_0^*\mathcal{T}$ is a metric Lie algebra with metric $i_0^*\kappa_0$.

LEMMA 3.3.14. *The form $i_0^*\kappa_0$ is the symplectic form on $\mathcal{M}_{H,\rho}^{\xi}$.*

PROOF. This follows from definition. Indeed, κ_0 restricted to degree 1 is the standard symplectic form on T^*V, and so the induced form on the cohomology is the symplectic form on $\mathcal{M}_{H,\rho}^{\xi}$. □

We have now seen that the Lie superalgebra $\mathfrak{g}_*(\rho)$ together with its metric κ_0 controls the symplectic geometry of $\mathcal{M}_{H,\rho}^{\xi}$. Let us use this to construct the sheaf of VOA on hypertoric varieties.

3.3.2.3. *Sheaf of Vertex Algebras Associated to Hypertoric Varieties.* Let us start with a general statement. Let X be a smooth symplectic variety, then $T_X[-1]$ is a Lie algebra object in $\mathrm{Coh}(X)$ with a non-degenerate invariant bilinear form $\omega : T_X[-1] \otimes_{\mathcal{O}_X} T_X[-1] \to \mathcal{O}_X[-2]$. These are the ingredients we need to define an affine Lie superalgebra. The only problem here is that ω is an element of homological degree 2, and so we can't define this sheaf in the usual derived category $\mathrm{Coh}(X)$. The solution, as in [**RW10**], is to consider the $\hbar$-adic version $\mathrm{Coh}(X)^{\hbar}$. In [**RW10**], this category is defined as having the same objects as $\mathrm{Coh}(X)$, but Hom spaces between two objects become:

$$\mathrm{Hom}_{\mathrm{Coh}(X)^{\hbar}}(M, N) = \bigoplus_n \hbar^n \mathrm{Ext}^{2n}(M, N). \tag{3.3.2.32}$$

This definition is somewhat ad-hoc. To avoid this, and to be able to include monopole operators (see Lemma 3.3.22 and the discussion prior), we comment that it is more natural to define this using 2-periodic complexes. Roughly speaking, the category $\mathrm{Coh}(X)^{\hbar}$ has the same objects as $\mathrm{Coh}(X)$, but we view an object (M_*, d) as a 2-periodic complex $M_{\mathrm{even}} = \bigoplus_i M_{2i}$ and $M_{\mathrm{odd}} = \bigoplus_i M_{2i+1}$, such that the hom between M and N is the Hom:

$$\bigoplus_{i=j \bmod 2} \hbar^{j-i}\mathrm{Hom}(M_i, N_j). \tag{3.3.2.33}$$

The more geometric (∞-categorical) definition of this can be found in the work of [**Pre11**]. The idea is to consider the Cartesian product $X \times_{\mathbb{C}} 0$ where $0 \in \mathbb{C}$ is the origin, and the map $f : X \to \mathbb{C}$ is a function (which for our context, is zero). Then the category $\mathrm{Coh}(X \times_{\mathbb{C}} 0)$ has an action of $\mathrm{Coh}(0 \times_{\mathbb{C}} 0)$ via convolution, and by the equivalence $\mathrm{Coh}(0 \times_{\mathbb{C}} 0) \simeq \mathbb{C}[\![\hbar]\!]-\mathrm{Mod}$, a $\mathbb{C}[\![\hbar]\!]$-linear structure, where $\hbar$ is in homological degree 2. The 2-periodic category (or the category of matrix factorizations) is defined by:

$$\mathrm{MF}(X, f) := \mathrm{Coh}(X \times_{\mathbb{C}} 0) \otimes_{\mathbb{C}[\![\hbar]\!]} \mathbb{C}(\!(\hbar)\!) \tag{3.3.2.34}$$

or in other words, one invert the element $\hbar$. When $f = 0$, we will denote this category by $\mathrm{Coh}(X)^{\hbar}$ and $\mathrm{QCoh}(X)^{\hbar}$ the ind-completion. Heuristically, an object in $\mathrm{Coh}(X)^{\hbar}$ is a complex of coherent sheaves (M_*, d) on X together with an invertible isomorphism $\hbar : M_* \to M_{*+2}$ of homological degree 2, namely M_* is 2-periodic. The above definition makes such a definition compatible in an ∞-coherent manner. Note that these two categories have symmetric monoidal structure induced from the symmetric monoidal structure of the $\mathbb{Z}$-graded counterpart.

Note that we don't need (and will in fact avoid) the full ∞ content of this category, since it is not clear to the author how to define VOA in a homotopical setting, i.e., how to deal with the higher structures, or even what they are. In the following, we only deal with the degree 0 piece of the structure, namely we really only consider the homotopy category $\pi_0\mathrm{Coh}(X)^{\hbar}$ and $\pi_0\mathrm{QCoh}(X)^{\hbar}$, where the Hom between two objects is the degree 0 part of the Hom, which by definition is equation (3.3.2.32). In a word, we do not claim to define a VOA object in the full DG category, but rather the homotopy category. For simplicity however, we will still write $\mathrm{Coh}(X)^{\hbar}$ and $\mathrm{QCoh}(X)^{\hbar}$ and drop π_0 from all the discussion below.

With the category already set-up, let us view $T_X[-1]$ as an object in $\mathrm{Coh}(X)^\hbar$, and since $T_X[-1]\otimes_{\mathcal{O}_X} T_X[-1] \to \mathcal{O}_X$ is a genuine Hom in this category, $T_X[-1]$ becomes a metric Lie algebra with metric $\hbar\omega \in \mathrm{Hom}(T_X[-1]\otimes_{\mathcal{O}_X} T_X[-1], \mathcal{O}_X)$. Let us define the following object $\widehat{T}$ in $\mathrm{QCoh}(X)^\hbar$:

$$\widehat{T} := \bigoplus_{n\in\mathbb{Z}} z^n T_X[-1] \oplus \mathcal{O}_X. \tag{3.3.2.35}$$

Here z^n is a formal variable. There is a morphism $\widehat{\alpha} : \widehat{T} \otimes_{\mathcal{O}_X} \widehat{T} \to \widehat{T}$ given by:

$$\alpha_{T_X[-1]}(-,-) \bigoplus \hbar \oint \mathrm{d}z\omega(\partial_z -, -), \tag{3.3.2.36}$$

where we extend $\alpha_{T_X[-1]}$ to a $\mathbb{C}[z,z^{-1}]$-linear morphism. In other words, the restriction of $\widehat{\alpha}$ to $z^m T_X[-1] \otimes z^n T_X[-1]$ maps to $z^{m+n}T_X[-1] \oplus \mathcal{O}_X$, and the map to $\mathcal{O}_X$ is non-zero only when $m = -n$, in which case it is given by $m\omega$.

Lemma 3.3.15. *The morphism $\widehat{\alpha}$ gives $\widehat{T}$ the structure of a Lie algebra. We call this the affine Lie algebra associated to the symplectic variety X.*

Proof. The proof follows exactly the same way as the extension of affine Lie superalgebras. The fact that $\widehat{\alpha}$ satisfies graded skew-symmetry follows from the corresponding statement for $\alpha_{T_X[-1]}$ and the fact that ω is graded-symmetric, and so $\oint \mathrm{d}z\omega(\partial_z -, -)$ is graded skew-symmetric. The fact that $\widehat{\alpha}$ satisfy graded Jacobi identity follows from the fact that $\alpha_{T_X[-1]}$ satisfy graded Jacobi identity and that ω is invariant and graded symmetric with respect to the action of $T_X[-1]$. □

We can define the vacuum module in the same way. The object $\widehat{T}_{\geq 0} := \bigoplus_{n\geq 0} z^n T_X[-1] \oplus \mathcal{O}_X$ is a Lie algebra object and the canonical morphism $i : \widehat{T}_{\geq 0} \to \widehat{T}$ is an embedding of Lie algebra objects. Moreover, there is a quotient morphism $\widehat{T}_{\geq 0} \to T_X[-1]$, so that for each coherent sheaf M viewed as a $T_X[-1]$ module, we obtain a $\widehat{T}_{\geq 0}$ module via this morphism. We define a module of $\widehat{T}$ by induction:

$$\widehat{M} := \mathrm{Ind}_{\widehat{T}_{\geq 0}}^{\widehat{T}}(M) \tag{3.3.2.37}$$

which is by definition the left adjoint to the restriction functor from $\widehat{T}$ to $\widehat{T}_{\geq 0}$. Let $U(\widehat{T})$ be the universal enveloping algebra of $\widehat{T}$ in $\mathrm{QCoh}(X)^{\hbar}$, which is by definition the universal algebra object with a monomorphism $\widehat{T} \to U(\widehat{T})$ of Lie algebras. Also let $U(\widehat{T}_{\geq 0})$ be the universal enveloping algebra for $\widehat{T}_{\geq 0}$. By definition, U is the left-adjoint to the restriction functor from associative algebras to Lie algebras. We can thus construct $\widehat{M}$ by $\widehat{M} = U(\widehat{T}) \otimes_{U(\widehat{T}_{\geq 0})} M$, and the adjointness of this functor with restriction follows from tensor-hom adjunction.

Let $\widehat{T}_{<0} = \bigoplus\limits_{n<0} z^n T_X[-1]$, then as an $\mathcal{O}_X$ module, $\widehat{T} = \widehat{T}_{\geq 0} \oplus \widehat{T}_{<0}$, and the left adjoint action of $\widehat{T}_{<0}$ on $\widehat{T}$ induces an action on $\widehat{T}_{\geq 0}$. Similarly, there is an action of $\widehat{T}_{\geq 0}$ on $\widehat{T}_{<0}$. The Lie algebra structure on $\widehat{T}$ is totally derermined by the two subalgebra $\widehat{T}_{<0}$ and $\widehat{T}_{\geq 0}$, and their actions on each other. More precisely, denote by $A_{\geq 0} : \widehat{T}_{\geq 0} \otimes_{\mathcal{O}_X} \widehat{T}_{<0} \to \widehat{T}_{<0}$ the induced action on $\widehat{T}_{<0}$ and $A_{<0}$ vice versa, then the Lie bracket of $\widehat{T}_{\geq 0} \otimes_{\mathcal{O}_X} \widehat{T} \to \widehat{T}$ is given by $\left([-,-]_{\widehat{T}_{\geq 0}} + A_{<0} \circ \tau\right) \oplus A_{\geq 0}$, where $\tau : \widehat{T}_{\geq 0} \otimes_{\mathcal{O}_X} \widehat{T}_{<0} \to \widehat{T}_{<0} \otimes_{\mathcal{O}_X} \widehat{T}_{\geq 0}$ is the canonical isomorphism. Moreover, the data of a Lie algebra morphism $\widehat{T} \to \mathfrak{g}$ to another Lie algebra object $\mathfrak{g}$ is the datum of Lie algebra homomorphisms $f : \widehat{T}_{\geq 0} \to \mathfrak{g}$ and $g : \widehat{T}_{<0} \to \mathfrak{g}$ such that the induced bracket $\widehat{T}_{\geq 0} \otimes_{\mathcal{O}_X} \widehat{T}_{<0} \to \mathfrak{g}$ coincides with $g \circ A_{\geq 0} + f \circ A_{<0} \circ \tau$. The work of [**GR17**] has proved many general results about these Lie algebra objects and their universal enveloping algebras, and much of what is used here is proved in this work. We need the following:

LEMMA 3.3.16. *$U(\widehat{T})$ is a filtered algebra. Denote by $U^{\mathrm{gr}}(\widehat{T})$ the associated graded algebra, then there is a functorial isomorphism* $\mathrm{Sym}(\widehat{T}) \cong U^{\mathrm{gr}}(\widehat{T})$. *This is similarly true for $\widehat{T}_{\geq 0}$ and $\widehat{T}_{<0}$.*

As a consequence, the morphism $U(\widehat{T}_{<0}) \otimes U(\widehat{T}_{\geq 0}) \to U(\widehat{T})$ is an isomorphism of objects in $\mathrm{QCoh}(X)^{\hbar}$, since it is an isomorphism in the associated symmetric algebra. In fact, it is possible to define the algebra structure of $U(\widehat{T})$ using this isomorphism: that $U(\widehat{T})$ is the crossed product $U(\widehat{T}_{<0}) \bowtie U(\widehat{T}_{\geq 0})$ as defined in [**Kas12**]. This implies that $\widehat{M}$ as a module over $\mathcal{O}_X$ can be identified with $U(\widehat{T}_{<0}) \otimes_{\mathcal{O}_X} M$, since $U(\widehat{T})$ is a free right $U(\widehat{T}_{\geq 0})$ module, and by PBW theorem, this can be identified with $\mathrm{Sym}(\widehat{T}_{<0}) \otimes_{\mathcal{O}_X} M$. One can describe the action of $\widehat{T}$ on $U(\widehat{T}_{<0}) \otimes_{\mathcal{O}_X} M$ as follows. The action of $\widehat{T}_{<0}$ is the left multiplication. The action of $\widehat{T}_{\geq 0}$ is given by the conjugation action of $\widehat{T}_{\geq 0}$ on $U(\widehat{T}_{<0})$ and the action of $\widehat{T}_{\geq 0}$ on M. Here the conjugation action is given by the following composition:

(3.3.2.38)

$$\widehat{T}_{\geq 0} \otimes_{\mathcal{O}_X} U(\widehat{T}_{<0}) \longrightarrow \widehat{T}_{\geq 0} \otimes_{\mathcal{O}_X} U(\widehat{T}) \longrightarrow U(\widehat{T}) = U(\widehat{T}_{<0}) \otimes U(\widehat{T}_{\geq 0}) \longrightarrow U(\widehat{T}_{<0})$$

The first map is the canonical embedding, the second map is left multiplication, and the third map is the canonical morphism $U(\widehat{T}_{\geq 0}) \to \mathcal{O}_X$ coming from adjunction of the trivial Lie algebra morphism $\widehat{T}_{\geq 0} \to \mathcal{O}_X$. This gives a well-defined action of $\widehat{T}$ by the definition of the Lie bracket of $\widehat{T}$.

Lemma 3.3.17. *There is a filtration of $U(\widehat{T}_{<0})$ such that the action of $\widehat{T}_{\geq 0}$ on each filtered piece factors through a quotient $\widehat{T}_{\geq 0}/z^n$ for some large n.*

Proof. The Lie algebra $\widehat{T}$ is a graded Lie algebra such that the degree n part is $z^n T_X[-1]$ when $n \neq 0$ and $T_X[-1] \oplus \mathcal{O}_X$ when $n = 0$. Thus $U(\widehat{T})$ is canonically a graded algebra, and similarly, $U(\widehat{T}_{\geq 0})$ and $U(\widehat{T}_{<0})$ are graded algebra as well. Let us denote by $U(\widehat{T}_{<0})_{>i} = \bigoplus_{j>i} U(\widehat{T}_{<0})_j$, this gives a filtration to $U(\widehat{T}_{<0})$. Let $n > -i$, we show that the action of $z^n T_X[-1]$ on $U(\widehat{T}_{<0})_{>i}$ is trivial. Indeed, by degree consideration, the action of $z^n T_X[-1]$ will map $U(\widehat{T})_{>i}$ to $U(\widehat{T})_{>i+n} \subseteq U(\widehat{T})_{>0}$, which is zero after composing with the grading-preserving map $U(\widehat{T}) \to U(\widehat{T}_{<0})$.

□

Let $V_X^\hbar$ be the module $\widehat{\mathcal{O}_X}$, the module of $\widehat{T}$ associated to the trivial module of $T_X[-1]$. We prove the following main theorem of this section:

Theorem 3.3.18. *The object $V_X^\hbar$ has the structure of a vertex algebra. We call this the vertex algebra associated to the symplectic variety X with Poisson form ω.*

Recall from [**FBZ04**], we need the following structures for a vertex algebra:

(1) A vacuum element, which is a morphism $\Omega : \mathcal{O}_X \to V_X^\hbar$.
(2) A state operator correspondence $Y : V_X^\hbar \otimes_{\mathcal{O}_X} V_X^\hbar \to V_X^\hbar((t))$ such that $Y(\Omega -, t) = \mathrm{Id}$.
(3) A conformal grading and an element $T : V_X^\hbar \to V_X^\hbar$ such that $[T, Y(-,t)] = \partial Y(-,t)$.

In the following steps, we will use the fact that as an $\mathcal{O}_X$ module, $V_X^\hbar \cong \mathrm{Sym}(\widehat{T}_{<0})$ to construct the state operator correspondence. We will proceed in steps as laid out in the above.

Step 1. The vacuum element is the morphism $\Omega : \mathcal{O}_X \to \mathrm{Sym}(\widehat{T}_{<0})$, that maps as identity to $\mathrm{Sym}^0 = \mathcal{O}_X$.

Step 2. This is the difficult step. To start, we define $Y : \mathrm{Sym}^0(\widehat{T}_{<0}) \otimes_{\mathcal{O}_X} V_X^\hbar \to V_X^\hbar$ to be identity. For Sym^1,we need to define

$$Y : \mathrm{Sym}^1(\widehat{T}_{<0}) \otimes_{\mathcal{O}_X} V_X^\hbar = \widehat{T}_{<0} \otimes_{\mathcal{O}_X} V_X^\hbar \to V_X^\hbar((t)). \tag{3.3.2.39}$$

Let us denote by J_n the shifts by z^n of the identity morphism $J_n : z^m T_X[-1] \to z^{m+n} T_X[-1]$, and by m_n the action morphism $m_n : z^n T_X[-1] \otimes V_X^\hbar \to V_X^\hbar$. There is a morphism $z^{-1}T_X[-1] \to \prod_n z^n T_X[-1]$ by $\prod J_{n+1}$, we define the restriction of Y on $z^{-1}T_X[-1]$ to be:

$$z^{-1}T_X[-1] \otimes_{\mathcal{O}_X} V_X^\hbar \xrightarrow{\prod J_{n+1}} \prod_n z^n T_X[-1] \otimes V_X^\hbar \xrightarrow{\prod m_n t^{-n-1}} V_X^\hbar((t)) \tag{3.3.2.40}$$

where the first morphism is $\prod J_{n+1}$ and the second one is the action of $\widehat{T}$ on $V_X^\hbar$. The fact that the image is in $V_X^\hbar((t))$ as supposed to $\prod_n V_X^\hbar t^n$ follows from the fact that the action of $\widehat{T}_{\geq 0}$ is locally nilpotent. We define the restriction of Y to $z^{-n}T_X[-1]$ for $n > 1$ to be the composition:

$$z^{-n}T_X[-1] \otimes_{\mathcal{O}_X} V_X^\hbar \xrightarrow{J_{n-1}} z^{-1}T_X[-1] \otimes_{\mathcal{O}_X} V_X^\hbar \xrightarrow{Y} V_X^\hbar((t)) \xrightarrow{\partial_t^{n-1}/(n-1)!} V_X^\hbar((t)). \tag{3.3.2.41}$$

We must show that the morphisms thus defined satisfy locality. By [**FBZ04**], we only need to check locality for the restriction of Y on $z^{-1}T_X[-1]$, or in other words, we need to understand:

$$\sum m_n t^{-n-1} \cdot m_k s^{-k-1} \pm \sum m_k t^{-n-1} \cdot m_n s^{-k-1}. \tag{3.3.2.42}$$

Now since $V_X^\hbar$ is a module of $\widehat{T}$, commutation relation of $\widehat{T}$ implies:

$$\begin{aligned} &\sum m_n t^{-n-1} \cdot m_k s^{-k-1} \pm \sum m_k t^{-n-1} \cdot m_n s^{-k-1} \\ &\qquad = \sum_k \sum_{m+n=k} m_{k+n}(\alpha_{T_X[-1]}) t^{-n-1} s^{-k-1} + \sum_n n\hbar\omega t^{-n-1} s^{n-1} \\ &\qquad = \sum_k m_k(\alpha_{T_X[-1]}) \left(\sum_n t^{-n-1} s^n \right) s^{-k-1} + \hbar\omega \sum_n n t^{-n-1} s^{n-1} \\ &\qquad = \sum_k m_k(\alpha_{T_X[-1]}) s^{-k-1} \delta(t-s) + \hbar\omega \partial_s \delta(t-s). \end{aligned} \tag{3.3.2.43}$$

This is the locality condition. As a consequence, we can define Y on $\mathrm{Sym}^k(\widehat{T}_{<0})$ by $:Y^{\otimes k}:$, the normal-ordered product. This is well-defined thanks to locality, namely that $:Y^{\otimes k}:$ is symmetric. We have thus found the state-operator correspondence Y, and clearly by definition $Y(\Omega, t) = \mathrm{Id}$.

Step 3. The conformal degree of $V_X^\hbar$ is defined by $\deg(z^{-n}T_X[-1]) = n$ and the degree is defined multiplicatively on rest of $\mathrm{Sym}^*(\widehat{T}_{<0})$. For the definition of T, first define a derivation ∂ on $\widehat{T}$ whose restriction on $z^n T_X[-1]$ is $-nJ_{-1}$. This derivation induces a derivation on $U(\widehat{T})$. We define T by:

$$T : U(\widehat{T}_{<0}) \otimes_{\mathcal{O}_X} \mathcal{O}_X \longrightarrow U(\widehat{T}) \otimes_{\mathcal{O}_X} \mathcal{O}_X \xrightarrow{\partial\otimes\Omega} U(\widehat{T}) \otimes_{\mathcal{O}_X} V_X^\hbar \longrightarrow V_X^\hbar \tag{3.3.2.44}$$

where the third map is the action morphism. We need to show that $[T, Y(-,t)] = \partial_t Y(-,t)$, and by [**FBZ04**], we only need to show this is true when restricted to $z^{-1}T_X[-1]$. This is clear then since the action of $\widehat{T}_{<0}$ is by left multiplication and the action of $\widehat{T}_{\geq 0}$ is given by conjugation, both of which will commute with T into the action of ∂ on $\widehat{T}$, and so:

$$[T, \sum m_n t^{-n-1}] = \sum_n m_n(\partial) t^{-n-1} = \sum_n -n m_{n-1} t^{-n-1} = \partial_t \sum_n m_n t^{-n-1} \tag{3.3.2.45}$$

This completes the construction of the vertex algebra $V_X^\hbar$. Since in the definition of the structure maps of $V_X^\hbar$, all morphisms behave well under pullback of flat morphisms, namely the pullback of the morphism is the morphism of the pullback, we have the following corollary of the localization of $V_X^\hbar$.

COROLLARY 3.3.19. *Let $j : U \to X$ be an affine open subvariety. Then $j^* V_X^\hbar = V_U^\hbar$ is a symplectic fermion system.*

PROOF. Since $T_U[-1]$ is a free module generated by ∂_i for $1 \leq i \leq \dim(U)$, the Atiyah class is trivial and the commutator comes from the symplectic form. The OPE of $V_U^\hbar$ thus becomes:

$$Y(z^{-1}\partial_i, t) Y(z^{-1}\partial_j, s) \sim \frac{\hbar\omega(\partial_i, \partial_j)}{(t-s)^2}, \tag{3.3.2.46}$$

This is a symplectic fermion system restricted at each point $x \in U$.

□

We can now apply this construction to hypertoric varieties. Let $\mathcal{M}^\xi_{H,\rho}$ be the hypertoric variety with symplectic form coming from symplectic reduction. By Theorem 3.3.18, we obtain a sheaf of vertex algebras $V^\hbar_\xi$. On the other hand, consider the sheaf of vertex algebra $\mathcal{V}(\widehat{\mathfrak{g}_*(\rho)})^\hbar := \mathbb{C}[T^*V]\otimes V(\widehat{\mathfrak{g}_*(\rho)})^\hbar$, whose differential is defined in a similar way as $\mathcal{T}$: $d=\oint \mathrm{d}z \sum_i x_i\psi^{i,-}(z)+y_i\psi^{i,+}(z)$. Note that $d^2=\oint \mathrm{d}z\sum_a \xi_a E^a=0$ since E^a_0 acts as zero on the vacuum module. This is obviously a T-equivariant VOA object. From the definition of $V^\hbar_\xi$ and Theorem 3.3.10, we arrive at the following theorem:

THEOREM 3.3.20. *The pullback $\overline{i_\xi}^*\mathcal{V}(\widehat{\mathfrak{g}_*(\rho)})^\hbar$ coincides with $V^\hbar_\xi$ as a vertex algebra object over the variety $\mathcal{M}^\xi_{H,\rho}$. Consequently, the sheaf of vertex algebra $V^\hbar_\xi$ has a conformal element given by the Poisson bivector associated to a shifted bilinear form of $i_0^*\kappa_0$.*

PROOF. If we do a field redefinition:

$$N_a\mapsto N_a-\frac{1}{\hbar}\frac{\sum\rho_{ia}\rho^i{}_b}{2}E^b \tag{3.3.2.47}$$

then the VOA $\mathcal{V}(\widehat{\mathfrak{g}_*(\rho)})^\hbar$ is the affine Lie algebra associated to $\mathfrak{g}_*(\rho)$ and the bilinear form κ_0, which localize to the symplectic form. This means that the localizatoin is the affine Lie algebra of the shifted tangent Lie algebra. This completes the proof.

□

Over each point $x\in\mathcal{M}^\xi_{H,\rho}$, localization of $V^\hbar_\xi$ gives a symplectic fermion system, and by a change of basis, we may always assume that we have the standard symplectic form. On the other hand, localization of $\mathcal{V}(\widehat{\mathfrak{g}_*(\rho)})^\hbar$ simply evaluates x_i,y_i. Theorem 3.3.20 combined with Corollary 3.3.19 gives the following result:

COROLLARY 3.3.21. *For each $x\in\mu^{-1}(0)\cap(T^*V)^{ss}_\xi$, there is a differential d_x on $V(\widehat{\mathfrak{g}_*(\rho)})^\hbar$ such that:*

$$H^*(V(\widehat{\mathfrak{g}_*(\rho)})^\hbar,d_x)\cong V^{\otimes n-r,\hbar}_{\chi\pm}. \tag{3.3.2.48}$$

Moreover, the action of $g \in T$ on $\mu^{-1}(0)$ and $V(\widehat{\mathfrak{g}_*(\rho)})^\hbar$ induces a commutative diagram of isomorphisms of VOAs:

$$\begin{array}{ccc} H^*(V(\widehat{\mathfrak{g}_*(\rho)})^\hbar, d_x) & \longrightarrow & V_{\chi\pm}^{\otimes n-r,\hbar} \\ \downarrow g & & \downarrow \\ H^*(V(\widehat{\mathfrak{g}_*(\rho)})^\hbar, d_{gx}) & \longrightarrow & V_{\chi\pm}^{\otimes n-r,\hbar} \end{array} \tag{3.3.2.49}$$

Namely T acts trivially on the cohomology.

Proof. It is clear that the above are isomorphisms of vertex algebras. To show that the conformal element agrees, we comment that the conformal element of $V(\mathfrak{g}_*(\rho))$ is defined by the quadratic Casimir associated to a shift of κ_0, and so the image of this is the quadratic Casimir associated to the descent of the shift of κ_0 on the cohomology over each chart. However, since over each affine local chart the tangent complex is a free module concentrated in degree 1, the shift of κ_0 also localize to the symplectic form restricted at x and can be made into the standard symplectic form by a change of basis.

□

When $\xi \neq 0$, the sheaf $i_\xi^* \mathcal{V}(\widehat{\mathfrak{g}_*(\rho)})^\hbar$ is the best one can do, and it is not clear how to include monopole operators to the sheaf, since on those modules, $d^2 = \sum \xi_a E_0^a$ is not zero. It is zero however, when $\xi = 0$. Therefore, for $\mathcal{M}_{H,\rho}^\xi$, one can try to include those modules. Recall that as a module of $V(\mathfrak{g}_*(\rho))$, the monopole operator corresponding to $s \in \mathbb{Z}^r$ is the spectral flow $\sigma_{s,\rho^\intercal \rho s} V(\mathfrak{g}_*(\rho))$, where $\sigma_{s,\rho^\intercal \rho s}$ is defined as in Remark 3.1.8. Again, this spectral flow has an $\hbar$-adic version, where the flow of E_a is $E_a - \frac{\hbar s_a}{z}$. Let us denote by $\mathcal{V}_{B,\rho}^\hbar$ the sheaf of VOA $\mathbb{C}[T^*V] \otimes V_{B,\rho}^\hbar$, whose differential is defined in the same way as $\mathcal{V}(\widehat{\mathfrak{g}_*(\rho)})^\hbar$. This is not a complex of quasi-coherent sheaves since $d^2 \neq 0$. However, the pullback $i_0^* \mathcal{V}_{B,\rho}^\hbar$ is a well-defined sheaf of VOA, and is T-equivariant.

Lemma 3.3.22. *Working over $\mathbb{C}[\hbar, \hbar^{-1}]$. For each $x \in \mu^{-1}(0) \cap (T^*V)_\xi^{ss}$, the induced differential on $\sigma_{s,\rho^\intercal \rho s} V(\widehat{\mathfrak{g}_*(\rho)})^\hbar = U_s^\hbar$ has trivial cohomology, unless $s = 0$. In other words, $H^*(V_{B,\rho}^\hbar, d_x) \cong H^*(V(\widehat{\mathfrak{g}_*(\rho)})^\hbar, d_x)$ as modules of $\mathbb{C}[\hbar, \hbar^{-1}]$.*

PROOF. Denote by σ_s the spectral flow $\sigma_{s,\rho^\intercal \rho s}$. On this module, the action of the central element $E_{a,0}$ is given by $\hbar s_a$, and there must be a such that $s_a \neq 0$. In this case, let $m \in \sigma_s V(\widehat{\mathfrak{g}_*(\rho)})$ be closed under d_x, and let F_a be an element in the universal enveloping algebra of $V(\widehat{\mathfrak{g}_*(\rho)})$ such that $d_x F_a = E_{a,0}$, then $\hbar s_a m = E_{a,0} m = (d_x F_a) m = d_x(F_a m) \pm F_a d_x m = d_x(F_a m)$, namely m is also exact. Thus the cohomology must be trivial.

□

Since in the category $\mathrm{QCoh}(\mathcal{M}_{H,\rho}^{\xi})^{\hbar}$, the element $\hbar$ is invertible (as we are working with 2-periodic complexes), the above Lemma means that the monopole operators are absent when considering the resolved smooth Higgs branch, and can only be accessed at the singular point $O \in \mu^{-1}(0)$. Consequently, the sheaf of VOA $i_0^* \mathcal{V}_{B,\rho}^{\hbar}$ restricted to $\mathcal{M}_{H,\rho}^{\xi}$ coincides with the vertex algebra $V_\xi^\hbar$. Recall from Section 3.3.2.1 that $\mathcal{V}_{A,\rho}^{\hbar}$ is a sheaf of VOA on the Higgs branch $\mathcal{M}_{H,\rho}^{\xi}$, and the isomorphism $\mathcal{V}_{A,\rho}^{\hbar} \cong \mathcal{V}_{B,\rho^\vee}^{\hbar}$ means that $\mathcal{V}_{A,\rho}^{\hbar}$ can be made into a vertex algebra object on the Higgs branch $\mathcal{M}_{H,\rho^\vee}^{\xi^\vee}$, which is isomorphic to the Coulomb branch $\mathcal{M}_{C,\rho}^{\xi^\vee}$, we arrive at the following:

THEOREM 3.3.23. *The sheaf of VOA $\mathcal{V}_{A,\rho}^{\hbar}$ on $\mathcal{M}_{H,\rho}^{\xi}$ can be made into a sheaf of VOA in* $\mathrm{QCoh}(\mathcal{M}_{H,\rho}^{\xi})\left(\mathrm{QCoh}(\mathcal{M}_{C,\rho}^{\xi^\vee})^{\hbar}\right)$*, namely, it is a sheaf of VOA on $\mathcal{M}_{H,\rho}^{\xi}$ valued in the symmetric monoidal category* $\mathrm{QCoh}(\mathcal{M}_{C,\rho}^{\xi^\vee})^{\hbar}$.

REMARK 3.3.24. *The computation of the cohomology above is valid only over $\mathbb{C}(\!(\hbar)\!)$, and it is not sure to the author how to compute it over $\mathbb{C}[\![\hbar]\!]$. The proof of Lemma 3.3.22 shows however that $\hbar$ has to act trivially, and so one can first quotient out $\hbar$ and then compute the cohomology. As we will see the next section, taking $\hbar = 0$ is related to going from the topological twists to HT twist of 3d $\mathcal{N} = 4$ theories.*

3.3.3. HT twist and boundary Vertex Algebra. The $\hbar$-adic VOA $V_{A,\rho}^{\hbar}$ is defined in [**AKM15**] without natural motivation: it is rather a tool to localize chiral differential operators on smooth varieties (to render OPE finite after inverting fields in the VOA). The isomorphism of Theorem 3.1.10 in fact reveals the role of $\hbar$ through the realization of $V_{B,\rho^\vee}^{\hbar}$: that $\hbar$ is the homological shift in $\mathrm{QCoh}(\mathcal{M}_{C,\rho})$. There is another natural interpretation of the $\hbar$-adic VOA: the deformation from HT twist to topological twists. The following two flat deformations:

$$V_{A,\rho}^{\hbar}, \qquad V_{B,\rho}^{\hbar} \tag{3.3.3.1}$$

are the boundary VOA representation of the deformation:

$$\mathcal{T}_{A,\rho} \longleftarrow \mathcal{T}_{HT,\rho} \longrightarrow \mathcal{T}_{B,\rho} \tag{3.3.3.2}$$

More specifically, the theory $\mathcal{T}_{HT,\rho}$ admits both Neumann and Dirichlet boundary conditions [**CDG20**]. The Neumann boundary condition (with extra degree of freedom to cancel anomaly), which we denote by $\mathbf{N}$, is compatible with the A twist, and gives the Neumann boundary condition for $\mathcal{T}_{A,\rho}$. Similarly, the Dirichlet boundary condition, which we denote by $\mathbf{D}$, is compatible with the B twist, and gives the Dirichlet boundary condition for $\mathcal{T}_{B,\rho}$. If we denote the two boundary vertex algebra by $V_{\mathbf{N},\rho}$ and $V_{\mathbf{D},\rho}$, then one expects:

$$V_{\mathbf{N},\rho} \cong V_{A,\rho}^{\hbar}/(\hbar), \qquad V_{\mathbf{D},\rho} \cong V_{B,\rho}^{\hbar}/(\hbar). \tag{3.3.3.3}$$

Instead of proving these statements, we take these as the definition of $V_{\mathbf{N},\rho}$ and $V_{\mathbf{D},\rho}$. The justification of these definitions can be seen from the index computation of Section 3.1.3.2. The mirror symmetry statement of Theorem 3.1.10 immediately implies the mirror symmetry statement for these vertex algebras:

Corollary 3.3.25. *There are isomorphisms of vertex algebras:*

$$V_{\mathbf{N},\rho} \cong V_{\mathbf{D},\rho^\vee}, \qquad V_{\mathbf{D},\rho} \cong V_{\mathbf{N},\rho^\vee}. \tag{3.3.3.4}$$

Let us look at the vertex algebra $V_{\mathbf{D},\rho}$ more closely. By definition, this is an extension of the affine Lie superalgebra $V_0(\widehat{\mathfrak{g}_*(\rho)})$, at level 0. The extension essentially uses the spectral flow automorphism $\sigma_{s,\rho^\intercal\rho s}$, which is associated to the action of $N^a(z)$. Therefore, the associated vertex operator is:

$$\mathcal{U}_s(z) := \exp\left(\int s\cdot N(z)\right). \tag{3.3.3.5}$$

The OPE of this vertex operator with fields in $V_0(\widehat{\mathfrak{g}_*(\rho)})$ is given by:

(3.3.3.6)

$$N_a(z)\mathcal{U}_s(w)\sim \frac{\rho^{\mathsf{T}}\rho(s)_a}{z-w},\qquad E_a(z)\mathcal{U}_s(w)\sim 0,\qquad \psi^{i,\pm}(z)\mathcal{U}_s(w)\sim (z-w)^{\pm\rho(s)^i}:\psi^{i,\pm}(w)\mathcal{U}_s(w):$$

Under the equivalence $V_{\mathbf{D},\rho}\cong V_{\mathbf{N},\rho^\vee}$, the operators $\mathcal{U}_s(z)$ is mapped to the image of:

$$\mathcal{U}_s(z)\mapsto :\prod_{i:\rho(s)_i>0} b_i^{\rho(s)_i}\prod_{i:\rho(s)_i<0} c_i^{-\rho(s)_i}: \tag{3.3.3.7}$$

under BRST cohomology, and the element E_a are mapped to:

$$E_a\mapsto -\sum_i \widetilde{\tau}_{ai}:\beta^i\gamma^i:. \tag{3.3.3.8}$$

Note that at $\hbar=0$, the fields β^i and γ^i are commutative, and generate functions on the infinite jet space $J^\infty\mathcal{M}_{H,\rho^\vee}\cong J^\infty\mathcal{M}_{C,\rho}$. We have the following corollary:

Corollary 3.3.26. *There is an embedding:*

$$\mathbb{C}[J_\infty\mathcal{M}_{C,\rho}]\hookrightarrow V_{\mathbf{D},\rho}. \tag{3.3.3.9}$$

For the remainder of this section, we will spell out this in more detail. The idea is the identification:

$$\psi^{i,+}\mapsto :\beta^i b^i:,\qquad \psi^{i,-}\mapsto :\gamma^i c^i:. \tag{3.3.3.10}$$

The space $\mathbb{C}[J_\infty\mathcal{M}_{C,\rho}]$ is generated by the following fields:

$$r_\lambda(z)=\prod_{\rho(\lambda)_i>0}(\beta^i)^{\rho(\lambda)_i}\prod_{\rho(\lambda)_i<0}(-\gamma^i)^{-\rho(\lambda)_i},\qquad \lambda\in\mathbb{Z}^r \tag{3.3.3.11}$$

then $r_\lambda(z)$ survives the BRST cohomology and gives well-defined elements in $V_{\mathbf{N},\rho^\vee}$. They have the following OPE:

$$r_\lambda(z)r_\mu(z)=\prod_i(-:\beta_i\gamma_i:)^{d(\rho(\lambda)_i,\rho(\mu)_i)}r_{\lambda+\mu}(z). \tag{3.3.3.12}$$

Due to the splitting of the short exact sequence in equation (3.1.1.16), we have that $\rho\widetilde{\tau} + \widetilde{\rho}\tau = \mathrm{Id}$. Applying this to $-{:}\beta^i\gamma^i{:}$ we find:

$$-:\beta^i\gamma^i: = -\sum \rho_{ia}\widetilde{\tau}^a{}_j:\beta^j\gamma^j: - \sum \widetilde{\rho}_{i\alpha}\tau^\alpha{}_j:\beta^j\gamma^j:. \tag{3.3.3.13}$$

Note that the second term on the right hand side is exact in the cohomology (recall $\rho^\vee = \tau^\mathsf{T}$), and consequently in the cohomology, $-\beta_i\gamma_i = \sum \rho_{ia}E^a$. Therefore, we have the following OPE in the cohomology:

$$r_\lambda(z)r_\mu(z) = \prod_i \left(\sum \rho_{ia}E^a(z)\right)^{d(\rho(\lambda)_i,\rho(\mu)_i)} r_{\lambda+\mu}(z). \tag{3.3.3.14}$$

This perfectly matches the formula of $\mathbb{C}[\mathcal{M}_{C,\rho}]$ in Theorem 2.1.7, if we identify E_a as basis elements of $\mathfrak{t}^*$, and $\sum \rho_{ia}E^a$ as ξ_i. In terms of $V_{\mathbf{D},\rho}$, we can identify:

$$\begin{aligned} r_\lambda(z) = \pm &\prod_{\rho(\lambda)_i>0} \oint (z_i - z)^{\rho(\lambda)_i^2-1}\mathrm{d}(z_i - z):(\psi^{i,+}(z_i))^{\rho(\lambda)_i}: \\ &\cdot \prod_{\rho(\lambda)_i<0} \oint (z_i - z)^{\rho(\lambda)_i^2-1}\mathrm{d}(z_i - z):(\psi^{i,-}(z_i))^{-\rho(\lambda)_i}:\mathcal{U}_{-\lambda}(z), \end{aligned} \tag{3.3.3.15}$$

Here the extra $\pm$ sign is to account for the sign change in the ordering of $\psi^{i,\pm}$. To see how the formula is true, consider trying to obtain powers of β^i in r_λ. We need the powers of $\psi^{i,+}$ because $\psi^{i,+} = \beta^i b^i$, which will contribute extra powers of b^i. These contribution of b^i will cancel out nicely with $\mathcal{U}_{-\lambda}(z)$ under residue because:

$$\oint (z - w)^{n^2-1}\mathrm{d}(z - w):b_i(z)^n::c_i(w)^n: = 1. \tag{3.3.3.16}$$

In next chapter, we will see that the vacuum module of $V_{\mathbf{D},\rho}$ is related to the Hochschild homology of the category $\mathrm{Coh}(G(\mathcal{O})\backslash\mathcal{R}_{G,V})$. Of course, there is much more to study about the representation theory of $V_{\mathbf{N},\rho}$ and $V_{\mathbf{D},\rho}$, as well as its relation to the Higgs and Coulomb branches. Unfortunately, such an endeavor is beyond the scope of this thesis, and will be left for a future work.

CHAPTER 4

Category of Line Operators in the Holomorphic Twists of 4d $\mathcal{N} = 2$ Theories

In this chapter, we study the category of line operators and the algebra of local operators in the Kapustin twist of 4d $\mathcal{N} = 2$ gauge theory $T_{HT}[G, V]$. As has been mentioned in Section 2.2, the space of local operators is computed in physics to be a Poisson vertex algebra, and the category of line operators is proposed to be $\mathrm{Coh}(G(\mathcal{O}) \setminus \mathcal{R}_{G,V})$. The central statement of this chapter is that the Poisson vertex algebra can be obtained from the category $\mathrm{Coh}(G(\mathcal{O}) \setminus \mathcal{R}_{G,V})$, as the derived endomorphism of the unit object. The structure of this chapter is as follows:

- In Section 4.1, we recall the geometry of the BFN space $\mathcal{R}_{G,V}$, especially its stratification, and the definition of the category of coherent sheaves. We will also introduce the Poisson vertex algebra $\mathcal{V}_{G,V}$ as BRST cohomology.
- In Section 4.2, we compute the derived endomorphism of the uni object in $\mathrm{Coh}(G(\mathcal{O}) \setminus \mathcal{R}_{G,V})$, and prove that it is quasi-isomorphic to $\mathcal{V}_{G,V}$ as an algebra. This statement will be the content of Theorem 4.2.13 and Theorem 4.2.16. We also compute the derived endomorphism between line bundles supported on the miniscule orbits, focusing on the case when $G = PSL(2)$ and $V = 0$. We compare our results with the physical results computed from supersymmetric localization.

4.1. The Category of Line Operators and the Poisson Vertex Algebra

4.1.1. Geometry of the BFN Spaces. Let G be a reductive Lie group. The affine Grassmannian of G is the quotient:

$$\mathrm{Gr}_G := \mathcal{R}_{G,0} = G(\mathcal{K})/G(\mathcal{O}). \quad (4.1.1.1)$$

It turns out that Gr_G is a classical ind-scheme; its geometry is well studied in the literature. We will in this section recall some basic facts about this space. For details, see [**Zhu16**]. In particular, we note that the study of the geometry of this space has two complications, one is that it is an ind-scheme; the other is that it is not always reduced.

4.1.1.1. *The Affine Grassmannian of* GL_n. Let us first consider the case when $G = \mathrm{GL}_n$. The affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n}$ can be defined alternatively as the moduli space of lattices in $\mathcal{K}^n$. More precisely, if R is an algebra over $\mathbb{C}$, then an R family of lattices in $\mathcal{K}^n$ is a finitely-generated projective $R[\![z]\!]$-submodule Λ of $R(\!(z)\!)^n$ such that $\Lambda \otimes_{R[\![z]\!]} R(\!(z)\!) = R(\!(z)\!)^n$. The affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n}$ can be defined as the presheaf assigning to R the set of R families of lattices in $\mathcal{K}^n$. We have:

PROPOSITION 4.1.1. $\mathrm{Gr}_{\mathrm{GL}_n}$ *is represented by a classical ind-projective ind-scheme. Namely, it can be written as a colimit of classical projective schemes under closed embeddings.*

Moreover, $\mathrm{Gr}_{\mathrm{GL}_n}$ is formally smooth in the following sense:

DEFINITION 4.1.2. *An ind-scheme* $X = \varinjlim X_n$ *is formally smooth if for any algebra* R *and nilpotent ideal* $I \subseteq R$*, the map* $X(R) \to X(R/I)$ *is surjective.*

4.1.1.2. *The Affine Grassmannian of General* G. Let now G be an arbitrary smooth affine reductive group. Choosing a faithful representation $G \to \mathrm{GL}_n$ one obtains a closed embedding $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$, and from Proposition 4.1.1 one concludes that Gr_G is an ind-projective ind-scheme, and moreover, one can show that it is formally smooth. There is a canonical isomorphism $\pi_0(\mathrm{Gr}_G) \cong \pi_1(G)$, and the connected components of Gr_G are labeled by the fundamental group of G, which is also the quotient of the co-weight lattice of G by its co-root lattice. All connected components are isomorphic to each other, with an isomorphism given by left multiplication by an element in $G(\mathcal{K})$. It turns out, however, this space is not always reduced, as can be seen from the following example.

Example. Consider $T = \mathbb{C}^*$. Then $\mathrm{Gr}_T = \mathcal{K}^*/\mathcal{O}^*$. The $\mathbb{C}$ points of this space is a disjoint union of infinitely many copies of $\mathrm{Spec}(\mathbb{C})$. They can be represented by $\{z^n | n \in \mathbb{Z}\}$, since any nonzero element in $\mathcal{K}$ is of the form $z^n g[z]$ for some $g[z] \in \mathcal{O}^*$. However, if we evaluate Gr_G on the algebra $\mathbb{C}[\epsilon]$ with $\epsilon^2 = 0$, then an element in $\mathbb{C}[\epsilon] \otimes \mathcal{K}$ is invertible iff its image under the map $\mathbb{C}[\epsilon] \otimes \mathcal{K} \to \mathcal{K}$

setting $\epsilon \mapsto 0$ is invertible. This means that the set of invertible elements in $\mathbb{C}[\epsilon] \otimes \mathcal{K}$ is $\mathcal{K}^* \oplus \epsilon\mathcal{K}$, while the set of invertible elements in $\mathcal{O}$ is $\mathcal{O}^* \oplus \epsilon\mathcal{O}$. The quotient is not a discrete set anymore, but rather an infinite-dimensional vector bundle over $\mathrm{Gr}_T(\mathbb{C})$. The fibre of this bundle at a point z^n is $\mathcal{K}/z^n\mathcal{O}$, and should be interpreted as the tangent space of Gr_T at z^n. This presents difficulty in considering the category of coherent sheaves, as in this case, the category of sheaves on Gr_T is different from the category of sheaves on its $\mathbb{C}$ points, even though the K_0 groups of the two categories are isomorphic.

In the case when G is semi-simple, Gr_G is in fact reduced. In general, we denote by $\mathrm{Gr}_{G,red}$ the reduced ind-scheme of Gr_G. In the following, we will present a different stratification of Gr_G from the one obtained through an embedding $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$.

4.1.1.3. *A Stratification for* $\mathrm{Gr}_{G,red}$. Let T be a maximal torus of G and B a Borel subgroup of G containing T. Denote the associated weight lattice by $X^*(T)$ and coweight lattice by $X_*(T)$. The choice of a Borel subgroup determines a set of dominant weights $X^*(T)_+$ and dominant coweights $X_*(T)_+$. Each $\lambda^\vee \in X_*(T)$ determines an element in $T(\mathcal{K})$ given by $t^{\lambda^\vee}$. The assignment $\lambda^\vee \to G(\mathcal{O})t^{\lambda^\vee}$ is a bijection between $X_*(T)_+$ and $G(\mathcal{O})$ orbits of $G(\mathcal{K})/G(\mathcal{O})$. Denoting by $\mathrm{Gr}_{\lambda^\vee}$ the associated orbit, then it is a smooth quasi-projective variety since it is the quotient of an affine algebraic group by an algebraic subgroup. The reduced locus $\mathrm{Gr}_{G,red}$ has a stratification:

$$\mathrm{Gr}_{G,red} = \bigcup_{\lambda^\vee \in X_*(T)_+} \mathrm{Gr}_{\lambda^\vee} \tag{4.1.1.2}$$

Let $\overline{\mathrm{Gr}}_{\lambda^\vee}$ be the Zariski closure of $\mathrm{Gr}_{\lambda^\vee}$. Then for $\lambda^\vee \leq \mu^\vee$ in $X_*(T)_+$, $\overline{\mathrm{Gr}}_{\lambda^\vee}$ is a closed subscheme of $\overline{\mathrm{Gr}}_{\mu^\vee}$. This gives $\mathrm{Gr}_{G,red}$ an ind-scheme structure:

$$\mathrm{Gr}_{G,red} = \varinjlim_{\lambda^\vee \in X_*(T)_+} \overline{\mathrm{Gr}}_{\lambda^\vee}. \tag{4.1.1.3}$$

Each $\overline{\mathrm{Gr}}_{\lambda^\vee}$ is a projective variety, though usually it's very singular. In general, $\overline{Gr}_{\lambda^\vee}$ is a normal projective variety, and it is smooth if and only if $\lambda^\vee$ is miniscule, in which case $\overline{\mathrm{Gr}}_{\lambda^\vee} = \mathrm{Gr}_{\lambda^\vee}$. These are called miniscule orbits, and are in one-to-one correspondence with the fundamental group of G, as well as with the number of connected components of Gr_G. Note that when G is semi-simple, $\mathrm{Gr}_G = \mathrm{Gr}_{G,red}$. Thus in this case, equation (4.1.1.3) gives an explicit stratification of Gr_G.

4.1.1.4. *The BFN space $\mathcal{R}_{G,V}$.* Now fix a finite dimensional representation V of G, and denote by $\widetilde{G}_{\mathcal{O}}$ the extended group $G(\mathcal{O}) \ltimes \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the two-fold cover of the group acting as loop rotation.[1] The Cartesian diagram in equation (2.1.1.16) defines $\mathcal{R}_{G,V}$ as a derived stack. In contrast to the affine Grassmannian, $\mathcal{R}_{G,V}$ is not a classical ind-scheme, but a DG-indscheme. This means that it is not determined solely by its value on classical rings. Now fix an ind-scheme structure of Gr_G, say $\mathrm{Gr}_G = \varinjlim \mathrm{Gr}_{G,n}$ such that each $\mathrm{Gr}_{G,n}$ is a projective scheme closed under the action of $\widetilde{G}_{\mathcal{O}}$. Let $G(\mathcal{K})_n$ be the pre-image of $\mathrm{Gr}_{G,n}$ under the projection $G(\mathcal{K}) \to \mathrm{Gr}_G$. For each n, choose N (that depends on n) large enough so that the action of $G(\mathcal{K})_n$ maps $V(\mathcal{O})$ to $z^{-N}V(\mathcal{O})$. We have the following base-change diagram:

$$\begin{array}{ccc} \mathcal{R}_{G,V,n} & \longrightarrow & V(\mathcal{O}) \\ \downarrow & & \downarrow \\ G(\mathcal{K})_n \times_{G(\mathcal{O})} V(\mathcal{O}) & \longrightarrow & z^{-N}V(\mathcal{O}) \end{array} \tag{4.1.1.4}$$

Since the bottom line of the Cartesian square in equation (2.1.1.16) is an inductive limit of the bottom line from equation (4.1.1.4), $\mathcal{R}_{G,V}$ has the following presentation as an ind-scheme:

$$\mathcal{R}_{G,V} = \varinjlim \mathcal{R}_{G,V,n}. \tag{4.1.1.5}$$

Each $\mathcal{R}_{G,V,n}$ is a coconnective DG scheme, as $V(\mathcal{O})$ is a finite codimensional vector subspace in $z^{-N}V(\mathcal{O})$, and for $n \leq m$, the map $\mathcal{R}_{G,V,n} \to \mathcal{R}_{G,V,m}$ is a closed embedding.

We will also need a local description of $\mathcal{R}_{G,V}$. Let L^-G be the group ind-scheme associating to an algebra R the set $L^-G(R) = G(R[z^{-1}])$, and let $L^{<0}G$ be the kernel of $L^-G \to G$ sending $z^{-1} \mapsto 0$. Then according to [**BL94**, **Zhu16**], the map:

$$L^{<0}G \times G(\mathcal{O}) \to G(\mathcal{K}) \tag{4.1.1.6}$$

is an open embedding. Thus $L^{<0}G$ is an open neighborhood of identity coset in Gr_G. Over $L^{<0}G$, the vector bundle $G(\mathcal{K}) \times_{G(\mathcal{O})} V(\mathcal{O})$ trivializes to $L^{<0}G \times V(\mathcal{O})$, and so over this local chart, $\mathcal{R}_{G,V}$ can be represented by a pro-DG-algebra whose underlying pro-algebra is the pro-algebra of

[1]We will use the two fold cover of the group of loop rotations here, since this will allow us to shift gradings by $q^{1/2}$, which is necessary for matching with the physical Schur indices.

functions on the following ind-scheme:

$$L^{<0}G \times V(\mathcal{O}) \times V(\mathcal{K})/V(\mathcal{O})[-1], \tag{4.1.1.7}$$

and whose differential D is induced from the action map:

$$L^{<0}G \times V(\mathcal{O}) \to V(\mathcal{K}) \to V(\mathcal{K})/V(\mathcal{O}). \tag{4.1.1.8}$$

4.1.2. Poisson Vertex Algebra. As explained in [**OY20**] and [**But21**], the algebra of local operators of the HT twist of a 4d $\mathcal{N} = 2$ gauge theory has the structure of a Poisson vertex algebra, which we denote by $\mathcal{V}_{G,V}$. Here we will recall their construction. Consider the commutative Poisson vertex algebra $\mathcal{V}_{\beta\gamma-bc}$ generated by bosonic fields (β, γ) with conformal weight $\frac{1}{2}$ and cohomological degree 0, valued in the representations V and V^*, as well as fermionic fields (b, c) with conformal weight $(1, 0)$ and cohomological degree $(-1, 1)$, valued in the Lie algebra $\mathfrak{g}$ of G. The nontrivial Poisson brackets are given by:

$$\{\beta, \gamma\} \propto \mathrm{id}_V, \ \{b, c\} \propto C_2(\mathfrak{g}). \tag{4.1.2.1}$$

There is a BRST operator Q defined by the current:

$$J_{BRST} = \mathrm{Tr}(bcc) - \beta c\gamma. \tag{4.1.2.2}$$

The action of Q is given by $Q = \{J_{BRST}, -\}$ and satisfies $Q^2 = 0$. The Poisson algebra $\mathcal{V}_{G,V}$ is defined as the Q-cohomology of $\mathcal{V}_{\beta\gamma-bc}$.

Let us now describe the vacuum module of the vertex algebra $\mathcal{V}_{G,V}$ in more detail. In fact, we will describe the vacuum module of the DG Poisson vertex algebra $(\mathcal{V}_{\beta\gamma-bc}, Q)$. The vacuum module of $(\mathcal{V}_{\beta\gamma-bc}, Q)$ is generated by a vacuum vector $|0\rangle$ such that the positive modes acts trivially, and non-positive modes act freely. This means that, as a vector space, $\mathcal{V}_{\beta\gamma-bc}$ is given by:

$$\mathcal{V}_{\beta\gamma-bc} = \mathbb{C}[\beta_{k-1/2}, \gamma_{k-1/2}, b_{k-1}, c_k]_{k\leq 0}|0\rangle, \tag{4.1.2.3}$$

with the differential Q given as above. If we shift the loop weight of $V(\mathcal{O})$ by $q^{1/2}$, then the above can be identified as the following vector space:

$$\mathbb{C}[V(\mathcal{O})] \otimes \mathbb{C}[V^*(\mathcal{O})] \otimes \bigwedge^* \mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O}) \otimes \bigwedge^* \mathfrak{g}(\mathcal{K})/z\mathfrak{g}(\mathcal{O}). \tag{4.1.2.4}$$

To understand the differential, we identify the Lie algebra of $\mathfrak{g}$ with its dual using a killing form, and view c as valued in $\mathfrak{g}(\mathcal{K})^*$; then we have the following vector space:

$$\mathbb{C}[V(\mathcal{O})] \otimes \mathbb{C}[V^*(\mathcal{O})] \otimes \bigwedge^* \mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O}) \otimes \bigwedge^* (\mathfrak{g}(\mathcal{O}))^*, \tag{4.1.2.5}$$

such that the $\beta c \gamma$ part of the differential is induced by the moment map, and $\mathrm{Tr}(bcc)$ part of the differential is identified with the Chevalley-Eilenberg differential. The vacuum module $\mathcal{V}_{G,V}$ as a DG algebra is then identified with equation (4.1.2.5) together with a differential coming from a combination of derived symplectic reduction and Chevalley-Eilenberg differential. Here $\bigwedge^* (\mathfrak{g}(\mathcal{O}))^*$ should be understood as the direct limit of the Chevalley-Eilenberg complex of $\mathfrak{g}(\mathcal{O})/z^m\mathfrak{g}(\mathcal{O})$.

If we consider pure gauge theory, then the only differential is the Chevalley-Eilenberg differential, and we obtain the vector space:

$$\bigwedge^* \mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O}) \otimes \bigwedge^* (\mathfrak{g}(\mathcal{O}))^* \tag{4.1.2.6}$$

with the CE differential.

Physically, the space of local operators is not simply the cohomology of $(\mathcal{V}_{\beta\gamma-bc}, Q)$. This is due to the fact that when computing correlation functions, the ghosts should not appear as initial or final states. Mathematically, this amounts to, after taking cohomology, projecting to c_0-ghost-number zero. This is the same as taking invariants with respect to the Lie group G (the constant gauge transformations) by hand, instead of derived invariants of its Lie algebra, and is in essence what the relative BRST cohomology is achieving. The relative BRST complex $\mathcal{V}^{\mathrm{rel}}_{\beta\gamma-bc}$ is the subcomplex on which the action of $\{J_0, -\}$ and $\{b_0, -\}$ are trivial. The Poisson vertex algebra, or the space of local operators $\mathcal{V}_{G,V}$, is the cohomology of this:

$$\mathcal{V}_{G,V} := H^*(\mathcal{V}^{\mathrm{rel}}_{\beta\gamma-bc}, Q). \tag{4.1.2.7}$$

The subset $\mathcal{V}^{\mathrm{rel}}_{\beta\gamma-bc}$ annihilated by $\{J_0,-\}$ and $\{b_0,-\}$ is precisely the $\mathfrak{g}$-invariant subset where the degree of c_0 is zero, therefore it is not difficult to recognize that this is therefore the same as taking ordinary G invariants as supposed to $\mathfrak{g}$ invariants:

$$\mathcal{V}_{G,V} \cong \left[\mathbb{C}[V(\mathcal{O})] \otimes \mathbb{C}[V^*(\mathcal{O})] \otimes \bigwedge^* \mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O}) \otimes \bigwedge^* (z\mathfrak{g}(\mathcal{O}))^*\right]^G. \tag{4.1.2.8}$$

Note that $\mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O})$ and $(z\mathfrak{g}(\mathcal{O}))^*$ contribute the same factor to the Euler character. However, their roles are not symmetric, since one of them is used for symplectic reduction and the other is for derived group invariants. This difference will show up in the geometric computation as well.

The Poisson vertex algebra $\mathcal{V}_{G,V}$ exists for any gauge theory. However, when the gauge theory is super-conformal, which happens when $C_2(V) = C_2(G)$, this algebra has a deformation through the work of [**OY19**] and [**But21**], and the deformed algebra is identified with the conformal vertex algebra (VOA) first studied in [**BLL+15**]. Their construction is as follows: the algebra $\mathcal{V}_{\beta\gamma-bc}$ has a deformation quantization into the VOA $V^{\hbar}_{\beta\gamma-bc}$ generated by bosonic fields β, γ and fermionic fields b, c with OPE:

$$\gamma(z)\beta(w) \sim \frac{\hbar \mathrm{id}_V}{z-w},\ b(z)c(w) \sim \frac{\hbar C_2(G)}{z-w}. \tag{4.1.2.9}$$

The action of Q is promoted to the action of Q_{BRST} via:

$$Q_{BRST}\mathcal{O}(z) = \oint_w J_{BRST}(z+w)\mathcal{O}(z). \tag{4.1.2.10}$$

It squares to zero precisely when $C_2(V) = C_2(G)$, and the cohomology of Q_{BRST} gives the deformation quantization of $\mathcal{V}_{G,V}$. In this case, it is expected that in the category of line operators, the Schur functor is trivial, or in other words, the left dual of a line operator is isomorphic to its right dual, which is one direct consequence of superconformal symmetry in the category of line operators. We will, however, not prove it here.

4.1.3. Algebro-geometric formulation of the Category of Line Operators.

4.1.3.1. *Equivariant Coherent Sheaves.* A reasonable DG ind-scheme, as defined in [**Ras20**, Definition 6.8.1], is a convergent prestack X such that $X = \varinjlim X_i$ such that each X_i is quasi-compact, quasi-separated and eventually coconnective, and that $X_i \to X_j$ is almost finitely-presented closed embeddings. Gr_G and $\mathcal{R}_{G,V}$ are examples of such reasonable DG ind-schemes. Let H be a classical affine group scheme that acts on X. Then the quotient stack X/H is called a weakly renormalizable pre-stack following [**Ras20**, Definition 6.28.1], and one can define the category $\mathrm{IndCoh}^*(X/H)$ via a right Kan extension:

$$\mathrm{IndCoh}^*(X/H) := \lim_{f:S\to X/H \text{ flat}} \mathrm{IndCoh}^*(S), \tag{4.1.3.1}$$

where the limit is taken over all reasonable DG ind-schemes flat over X/H, using the functoriality of $f^{*,\mathrm{IndCoh}}$. We have the following equivalence:

$$\mathrm{IndCoh}^*(X/H) \cong \mathrm{IndCoh}^*(X)^{H,w,naive} := \mathrm{Hom}_{H-\mathrm{mod}_{weak,naive}}(\mathrm{Vect}, \mathrm{IndCoh}^*(X)), \tag{4.1.3.2}$$

where the right hand side is the naive weakly equivariant category with respect to the action of H as defined in [**Ras20**] Section 5. This in particular, may not be equivalent to the ind-completion of its compact object. This category may seem abstract, but one can unpack it using flat descent. Recall that given a flat cover $T \to S$, one can consider the associated Cech nerve:

$$T^{\times_S^{*+1}}. \tag{4.1.3.3}$$

Applying this to the flat cover $X \to X/H$, the Cech nerve is:

$$X^{\times_{X/H}^{*+1}} = X \Longleftarrow\!\!\Longrightarrow X \times H \;\substack{\longleftarrow\\ \Longrightarrow \\ \longleftarrow}\; X \times H \times H \cdots \tag{4.1.3.4}$$

By [**Ras20**, Theorem 6.25.1]:

$$\mathrm{IndCoh}^*(X/H) \cong \mathrm{Tot}_{semi}(\mathrm{IndCoh}(X^{\times_{X/H}^{*+1}})). \tag{4.1.3.5}$$

The right hand side is a semi-simplicial set of categories that only involve categories of sheaves on ind-schemes. Suppose further that H acts on each X_i such that $X/H = \varinjlim X_i/H$, we can write:

$$\mathrm{Tot}_{semi}(\mathrm{IndCoh}(X^{\times_{X/H}^{*+1}})) = \mathrm{Tot}_{semi}(\lim_{\text{upper-!}} \mathrm{IndCoh}(X_i \times_{X/H} X^{\times_{X/H}^{*+1}})). \tag{4.1.3.6}$$

Commuting the limit on the right hand side using [**Ras20**, Lemma 6.17.2], noticing that $X_i \times_{X/H} X^{\times^{*+1}_{X/H}})$ is the Cech nerve of $X_i \to X_i/H$, we get:

$$\text{Tot}_{semi}(\text{IndCoh}(X^{\times^{*+1}_{X/H}})) = \lim_{\text{upper-!}} \text{IndCoh}(X_i/H). \tag{4.1.3.7}$$

By [**Gai12**, Lemma 1.3.3], we may change the limit over upper-! to the colimit over lower-$*$:

$$\text{IndCoh}^*(X/H) \cong \varinjlim_{\text{lower-}*} \text{IndCoh}^*(X_i/H). \tag{4.1.3.8}$$

Now we specialize this story to the BFN space. Let $\mathbb{C}^*$ act as the two-fold cover of the loop rotation. Both Gr_G and $\mathcal{R}_{G,V}$ have an action of $G(\mathcal{O}) \rtimes \mathbb{C}^*$.[2] We will denote this group by $\widetilde{G}_{\mathcal{O}}$. From the above discussion, we can define categories $\text{IndCoh}(\widetilde{G}_{\mathcal{O}} \setminus \text{Gr}_G)$ and more generally, $\text{IndCoh}(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V})$. Moreover, if we fix a stratification $\{\mathcal{R}_{G,V,n}\}$ of $\mathcal{R}_{G,V}$ as in Section 4.1.1.4, then:

$$\text{IndCoh}(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V}) \cong \varinjlim \text{IndCoh}(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V,n}). \tag{4.1.3.9}$$

For each n, the category of coherent sheaves $\text{Coh}(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V,n})$ is the full subcategory of $\text{IndCoh}(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V,n})$ consisting of objects whose pull-back to $\mathcal{R}_{G,V,n}$ is coherent. The category of equivariant coherent sheaves on $\mathcal{R}_{G,V}$, $\text{Coh}(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V})$, is defined as the full subcategory of $\text{IndCoh}(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V})$ whose objects are the images of $\text{Coh}(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V,n})$ under the above colimit. This category $\text{Coh}(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V})$ is expected to be the category of line operators for the theory $T_{G,V}$, and the derived Hom between objects in this category is expected to be the space of local operators at the junction of two lines.

4.1.3.2. *Hom Spaces.* Before proceeding to the computation, we give a comment about the use of Hom spaces. To obtain the space of local operators at the junction of two line operators, we need to take the Hom space between two line operators as a DG vector space, which, intuitively speaking, is the derived Hom spaces between two coherent sheaves. Let us briefly introduce the setting in which this enriched Hom can be taken.

[2]We will use the two fold cover of the group of loop rotations here, since this will allow us to shift gradings by $q^{1/2}$, which is necessary for matching with the physical Schur indices.

Let $\mathcal{C}$ be a presentable monoidal DG category and $\mathcal{M}$ be a presentable DG module category of $\mathcal{C}$. For any pair of objects (M_1, M_2) in $\mathcal{M}$, the object

$$\mathrm{Hom}^{\mathcal{C}}(M_1, M_2) \in \mathcal{C} \tag{4.1.3.10}$$

is defined by the following adjunction property:

$$\mathrm{Hom}_{\mathcal{M}}(- \otimes M_1, M_2) = \mathrm{Hom}_{\mathcal{C}}(-, \mathrm{Hom}^{\mathcal{C}}(M_1, M_2)). \tag{4.1.3.11}$$

Let X be a reasonable DG indscheme acted on by a smooth affine group scheme H (See [**Ras20**] for a definition). Then $\mathrm{IndCoh}(X/H)$ is a module category over the monoidal category $\mathrm{IndCoh}(\mathbb{B}H) \cong \mathrm{QCoh}(\mathbb{B}H)$, where $\mathbb{B}H = \mathrm{pt}/H$ is the classifying stack of H. We thus obtain a Hom functor:

$$\mathrm{Hom}^{\mathrm{QCoh}(\mathbb{B}H)}(-,-) : \mathrm{IndCoh}(X/H)^{op} \times \mathrm{IndCoh}(X/H) \to \mathrm{QCoh}(\mathbb{B}H). \tag{4.1.3.12}$$

We will abbreviate this by $\mathrm{Hom}^{\mathbb{B}H}$. Specify this to our setting, we have the Hom functor:

$$\begin{aligned} \mathrm{Hom}^{\mathbb{BC}^*}\colon\ & \mathrm{IndCoh}^*(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V})^{op} \times \mathrm{IndCoh}^*(\widetilde{G}_{\mathcal{O}} \setminus \mathcal{R}_{G,V}) \\ & \longrightarrow \mathrm{QCoh}(\mathbb{BC}^*) \cong \mathrm{IndCoh}(\mathbb{BC}^*). \end{aligned} \tag{4.1.3.13}$$

This will be the main player of this section for many of the computations. We will write $\mathrm{End}^{\mathbb{BC}^*}$ if the two arguments of Hom are identical.

Remark 4.1.3. *This definition of* Hom *spaces seem to be abstract, but in our example it is a concrete one: first of all, it will be given by a colimit of* Hom *spaces computed on each closed orbit of $G(\mathcal{O})$; secondly, on each orbit, it is the usual dg vector space of* Hom, *which can be computed by choosing an injective resolution of the second argument.*

4.2. Geometric Computation of the Poisson Algebra

4.2.1. Computation of $\mathrm{Ops}_{G,0}$. Let us start with a computation for $\mathrm{Ops}_{G,0}$, namely when first consider pure gauge theory. In this case, the category of line operators is $\mathrm{Coh}(\widetilde{G}_{\mathcal{O}} \setminus \mathrm{Gr}_G)$. This category is a monoidal category, with monoidal unit given by $\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}}$, the structure sheaf of the identity coset $[e]$ with the trivial $\widetilde{G}_{\mathcal{O}}$ equivariant structure. Our goal is to compute the space:

$$\mathrm{End}^{\mathbb{B}\mathbb{C}^*}(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}}) \tag{4.2.1.1}$$

as a $\mathbb{C}^*$-DG vector space. The remainder of this section is devoted to the computation of this space, up to quasi-isomorphism. The idea of the computation is the following:

- First, one can factor the computation into two steps: computing $\mathrm{End}^{\mathbb{B}\widetilde{G}_{\mathcal{O}}}(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}})$; then taking the (derived-)invariant subspace with respect to the $G(\mathcal{O})$ action.
- Computing the derived $G(\mathcal{O})$ invariants using the Chevalley-Eilenberg cochain complex.
- Computing $\mathrm{End}^{\mathbb{B}\widetilde{G}_{\mathcal{O}}}(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}})$ using formal completion.

4.2.1.1. *Decomposing the Hom Functor.* Let H be a smooth affine group scheme that can be written as:

$$H = H_0 \rtimes T \tag{4.2.1.2}$$

for two smooth affine group schemes H_0 and T. Assume also that T is of finite type. Let Y_n be finite-type classical H-schemes such that $Y_n \to Y_{n+1}$ are closed embeddings of H-schemes. Denote by $Y = \varinjlim Y_n$ and $\mathcal{Y} = \varinjlim Y_n/H$. Let X be a finite-type classical H-scheme together with a closed-embedding of H-schemes $i : X \to Y$. Denote by $\mathcal{X} = X/H$. Let $(\mathcal{F}, \mathcal{G})$ be a pair of objects in $\mathrm{Coh}(\mathcal{X})$. We would like to understand

$$\mathrm{Hom}^{\mathrm{pt}/T}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G}). \tag{4.2.1.3}$$

Denote by π the natural projection $\mathcal{Y} \to \mathrm{pt}/H = \mathbb{B}H$, the classifying stack of H, and by π_0 the natural map $\mathbb{B}H \to \mathbb{B}T$. Since $\mathrm{IndCoh}(\mathcal{Y})$ is a module category of $\mathrm{IndCoh}(\mathbb{B}H)$, we have an object:

$$\mathrm{Hom}^{\mathbb{B}H}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G}) \in \mathrm{IndCoh}(\mathrm{pt}/H). \tag{4.2.1.4}$$

Now if we view $\mathrm{IndCoh}(\mathbb{B}H)$ as a module category of $\mathrm{IndCoh}(\mathbb{B}T)$ via the functor π_0^*, we will have an object:

$$\mathrm{Hom}^{\mathbb{B}T}\left(\mathcal{O}_{\mathbb{B}H}, \mathrm{Hom}^{\mathbb{B}H}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})\right). \tag{4.2.1.5}$$

LEMMA 4.2.1. *The following is a quasi-isomorphism of T modules:*

$$\mathrm{Hom}^{\mathbb{B}T}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G}) \cong \mathrm{Hom}^{\mathbb{B}T}\left(\mathcal{O}_{\mathbb{B}H}, \mathrm{Hom}^{\mathbb{B}H}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})\right). \tag{4.2.1.6}$$

PROOF. Let V be an object of $\mathrm{IndCoh}(\mathbb{B}T)$, then:

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{IndCoh}(\mathbb{B}T)}\left(V, \mathrm{Hom}^{\mathbb{B}T}\left(\mathcal{O}_{\mathbb{B}H}, \mathrm{Hom}^{\mathbb{B}H}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})\right)\right)\\
&\cong \mathrm{Hom}_{\mathrm{IndCoh}(\mathbb{B}H)}\left(\pi_0^*V, \mathrm{Hom}^{\mathbb{B}H}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})\right)\\
&\cong \mathrm{Hom}_{\mathrm{IndCoh}(\mathcal{Y})}(V \otimes i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})\\
&\cong \mathrm{Hom}_{\mathrm{IndCoh}(\mathbb{B}T)}\left(V, \mathrm{Hom}^{\mathbb{B}T}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})\right).
\end{aligned} \tag{4.2.1.7}$$

This proves the claim.

□

This statement says that we can first compute the endomorphism of $i_{*,\mathrm{IndCoh}}\mathcal{F}$ and $i_{*,\mathrm{IndCoh}}\mathcal{G}$ as an H-module, and then compute invariants with respect to H_0. However, this is not the best way to understand this Hom space, since $\mathrm{IndCoh}(\mathbb{B}H)$ is not compactly generated. In [**Ras20**, Section 5.11], the author defined another category that is compactly generated. Denote by $\mathrm{Rep}(H)^c$ the monoidal subcategory of $\mathrm{IndCoh}(\mathbb{B}H)$ consisting of objects whose images under the forgetful functor $\mathrm{IndCoh}(\mathbb{B}H) \to \mathrm{Vect}$ are compact, and $\mathrm{Rep}(H) = \mathrm{Ind}(\mathrm{Rep}(H)^c)$, the ind-completion. This category is compactly generated, and if H is a smooth affine algebraic group, then it is equivalent to $\mathrm{IndCoh}(H)$. In particular, $\mathrm{IndCoh}(\mathbb{B}T) \cong \mathrm{Rep}(T)$.

Moreover, by [**Ras20**, Lemma 5.16.2], if $H = \lim H_i$ for H_i finite dimensional smooth algebraic groups, then $\mathrm{Rep}(H) = \varinjlim \mathrm{Rep}(H_i)$, and so the understanding of $\mathrm{Rep}(H)$ can be reduced to understanding representations of finite-dimensional algebraic groups.

Since the action of $\mathrm{Rep}(T)$ on both $\mathrm{IndCoh}(\mathbb{B}H)$ and $\mathrm{IndCoh}(\mathcal{Y})$ factors through an action of $\mathrm{Rep}(H)$, we can modify Lemma 4.2.1 into the following:

LEMMA 4.2.2. *The following is a quasi-isomorphism of T modules:*

$$\mathrm{Hom}^{\mathbb{B}T}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G}) \cong \mathrm{Hom}^{\mathbb{B}T}\left(\mathcal{O}_{\mathbb{B}H}, \mathrm{Hom}^{\mathrm{Rep}(H)}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})\right). \tag{4.2.1.8}$$

The object $\mathrm{Hom}^{\mathrm{Rep}(H)}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})$ behaves better with colimit:

Proposition 4.2.3. *Denote by $\mathcal{F}_k$ and $\mathcal{G}_k$ the pushforward of $\mathcal{F}$ and $\mathcal{G}$ to Y_k/H. There is a qausi-isomorphism in* Rep(H)*:*

$$\mathrm{Hom}^{\mathrm{Rep}(H)}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G}) \cong \varinjlim \mathrm{Hom}^{\mathrm{Rep}(H)}(\mathcal{F}_k, \mathcal{G}_k). \tag{4.2.1.9}$$

Proof. Given $V \in \mathrm{Rep}(H)^c$, we have:

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Rep}(H)}\left(V, \mathrm{Hom}^{\mathrm{Rep}(H)}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})\right)\\
&\cong \mathrm{Hom}_{\mathrm{IndCoh}(\mathcal{Y})}(V \otimes i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})\\
(\text{by equation (4.1.3.8)}) &\cong \varinjlim \mathrm{Hom}_{\mathrm{IndCoh}(Y_k/H)}(V \otimes \mathcal{F}_k, \mathcal{G}_k) \qquad (4.2.1.10)\\
&\cong \varinjlim \mathrm{Hom}_{\mathrm{Rep}(H)}\left(V, \mathrm{Hom}^{\mathrm{Rep}(H)}(\mathcal{F}_k, \mathcal{G}_k)\right)\\
(\text{since } V \text{ is compact}) &\cong \mathrm{Hom}_{\mathrm{Rep}(H)}\left(V, \varinjlim \mathrm{Hom}^{\mathrm{Rep}(H)}(\mathcal{F}_k, \mathcal{G}_k)\right)
\end{aligned}$$

Since Rep(H) is compactly generated, this proves the claim. □

The object $\mathrm{Hom}^{\mathrm{Rep}(H)}(\mathcal{F}_k, \mathcal{G}_k)$ may seem to be abstract at first, but we can show that this is a familiar object: the underlying vector space of this object is the derived Hom between $\mathcal{F}_k$ and $\mathcal{G}_k$ as sheaves over Y_k. Denote by **Oblv** the forgetful functor Rep$(H) \to$ Vect. This is the composition of $\Psi : \mathrm{Rep}(H) \to \mathrm{IndCoh}(\mathbb{B}H)$ with the forgetful functor $\mathrm{IndCoh}(\mathbb{B}H) \to \mathrm{Vect}$. Denote also by p_k the projection $X_k \to X_k/H$. We claim:

Proposition 4.2.4. *There is a quasi-isomorphism:*

$$\mathbf{Oblv}\mathrm{Hom}^{\mathrm{Rep}(H)}(\mathcal{F}_k, \mathcal{G}_k) \cong \mathrm{Hom}^{\mathrm{Vect}}(p_k^*\mathcal{F}_k, p_k^*\mathcal{G}_k). \tag{4.2.1.11}$$

Here the left hand side of the above equation is the underlying DG vector space of $\mathrm{Hom}^{\mathrm{Rep}(H)}(\mathcal{F}_k, \mathcal{G}_k)$.

We need the following Lemma:

Lemma 4.2.5. *Let $p_0 : \mathrm{pt} \to \mathbb{B}H$ be the projection, then:*

$$p_0^*\mathrm{Hom}^{\mathbb{B}H}(\mathcal{F}_k, \mathcal{G}_k) \cong \mathrm{Hom}^{\mathrm{Vect}}(p_k^*\mathcal{F}_k, p_k^*\mathcal{G}_k). \tag{4.2.1.12}$$

Proof. Denote by π_k the projection $Y_k/H \to \mathbb{B}H$. We know that $\mathrm{IndCoh}(Y_k/H)$ is a module category of $\mathrm{QCoh}(Y_k/H)$, and this is compatible with the monoidal functor:

$$\pi_k^* : \mathrm{IndCoh}(\mathbb{B}H) \cong \mathrm{QCoh}(\mathbb{B}H) \to \mathrm{QCoh}(Y_k/H). \tag{4.2.1.13}$$

Using adjunction property, we have:

$$\mathrm{Hom}^{\mathbb{B}H}(\mathcal{F}_k, \mathcal{G}_k) = \mathrm{Hom}^{\mathbb{B}H}\left(\mathcal{O}_{Y_k/H}, \mathrm{Hom}^{\mathrm{QCoh}(Y_k/H)}(\mathcal{F}_k, \mathcal{G}_k)\right). \tag{4.2.1.14}$$

The right hand side of the above equation can be identified with:

$$(\pi_k)_* \mathrm{Hom}^{\mathrm{QCoh}(Y_k/H)}(\mathcal{F}_k, \mathcal{G}_k). \tag{4.2.1.15}$$

We are thus interested in $p_0^*(\pi_k)_* \mathrm{Hom}^{\mathrm{QCoh}(X_k/H)}(\mathcal{F}_k, \mathcal{G}_k)$. Consider now the Cartesian diagram:

$$\begin{array}{ccc} X_k & \xrightarrow{\widetilde{\pi}_k} & \mathrm{pt} \\ \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p_0} \\ X_k/H & \xrightarrow{\pi_k} & \mathrm{pt}/H \end{array} \tag{4.2.1.16}$$

Using base-change property of QCoh, we obtain:

$$p_0^*(\pi_k)_* \mathrm{Hom}^{\mathrm{QCoh}(X_k/H)}(\mathcal{F}_k, \mathcal{G}_k) \cong (\widetilde{\pi}_k)_* p_k^* \mathrm{Hom}^{\mathrm{QCoh}(X_k/H)}(\mathcal{F}_k, \mathcal{G}_k). \tag{4.2.1.17}$$

Now by [**Lur18**, Proposition 9.5.3.3]:

$$p_k^* \mathrm{Hom}^{\mathrm{QCoh}(X_k/H)}(\mathcal{F}_k, \mathcal{G}_k) \cong \mathrm{Hom}^{\mathrm{QCoh}(X_k)}(p_k^*\mathcal{F}_k, p_k^*\mathcal{G}_k). \tag{4.2.1.18}$$

Putting this into equation (4.2.1.17) we obtain the desired result.

□

Proof of Proposition 4.2.4. By adjunction property, there is a quasi-isomorphism:

$$\mathrm{Hom}^{\mathrm{Rep}(H)}(\mathcal{F}_k, \mathcal{G}_k) \cong \mathrm{Hom}^{\mathrm{Rep}(H)}\left(\mathcal{O}_{\mathbb{B}H}, \mathrm{Hom}^{\mathbb{B}H}(\mathcal{F}_k, \mathcal{G}_k)\right). \tag{4.2.1.19}$$

Since Y_k is a classical finite-type scheme and $\mathcal{F}_k$ and $\mathcal{G}_k$ are coherent, by Lemma 4.2.5, the Hom space $\mathrm{Hom}^{\mathbb{B}H}(\mathcal{F}_k, \mathcal{G}_k)$ is an object in $\mathrm{IndCoh}(\mathbb{B}H)^+$, which is equivalent to $\mathrm{Rep}(H)^+$ via Ψ. Thus:

$$\Psi \mathrm{Hom}^{\mathrm{Rep}(H)}(\mathcal{F}_k, \mathcal{G}_k) \cong \mathrm{Hom}^{\mathbb{B}H}(\mathcal{F}_k, \mathcal{G}_k). \tag{4.2.1.20}$$

Since $\mathbf{Oblv} = p_0^* \circ \Psi$, this and Lemma 4.2.5 gives the desired result. □

REMARK 4.2.6. *The above discussions suggest that the sheaf* $\mathrm{Hom}^{\mathrm{QCoh}(Y_k/H)}(\mathcal{F}_k, \mathcal{G}_k)$ *is the usual Hom sheaf between* $\mathcal{F}_k$ *and* $\mathcal{G}_k$ *on* Y_k *with the canonical* H *equivariant structure. The (derived) global section of this sheaf over* Y_k *as an* H *module is identified with* $\mathrm{Hom}^{\mathrm{Rep}(H)}(\mathcal{F}_k, \mathcal{G}_k)$*. The* H *module* $\mathrm{Hom}^{\mathrm{Rep}(H)}(i_{*,\mathrm{IndCoh}}\mathcal{F}, i_{*,\mathrm{IndCoh}}\mathcal{G})$ *is the colimit of* $\mathrm{Hom}^{\mathrm{Rep}(H)}(\mathcal{F}_k, \mathcal{G}_k)$*.*

We will apply this to the affine Grassmannian Gr_G. Fix a stratification $\mathrm{Gr}_G = \varinjlim \mathrm{Gr}_{G,n}$ such that $\mathrm{Gr}_{G,n}$ is a projective scheme closed under the action of $\widetilde{G}_{\mathcal{O}}$. Take $\mathcal{F}$ and $\mathcal{G}$ to be objects in $\mathrm{Coh}(\widetilde{G}_{\mathcal{O}} \setminus \mathrm{Gr}_{G,n})$, viewed as objects in $\mathrm{Coh}(\widetilde{G}_{\mathcal{O}} \setminus \mathrm{Gr}_G)$. Lemma 4.2.2 implies:

$$\mathrm{Hom}^{\mathbb{B}\mathbb{C}^*}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}^{\mathbb{B}\mathbb{C}^*}\left(\mathcal{O}_{\mathbb{B}\widetilde{G}_{\mathcal{O}}}, \mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{F}, \mathcal{G})\right). \tag{4.2.1.21}$$

Proposition 4.2.3 shows that the $\widetilde{G}_{\mathcal{O}}$ module $\mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{F}, \mathcal{G})$ is a colimit of Hom spaces on finite-dimensional strata $\mathrm{Gr}_{G,k}$, namely $\mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{F}_k, \mathcal{G}_k)$. These are bounded from below independent of k by Proposition 4.2.4, and so can be identified with $\mathrm{Hom}^{\mathbb{B}\widetilde{G}_{\mathcal{O}}}(\mathcal{F}_k, \mathcal{G}_k)$. The functor $\mathrm{Hom}^{\mathbb{B}\mathbb{C}^*}(\mathcal{O}_{\mathbb{B}\widetilde{G}_{\mathcal{O}}}, -)$ thus computes the derived invariants of these $\widetilde{G}_{\mathcal{O}}$ modules with respect to the normal subgroup $G(\mathcal{O})$. Let $G_{>n}$ be the normal subgroup defined by $G(1 + z^n\mathcal{O})$, then $\widetilde{G}_{\mathcal{O}} \cong \lim \widetilde{G}_{\mathcal{O}}/G_{>n}$, and so by [**Ras20**, Lemma 5.16.2], $\mathrm{Rep}(\widetilde{G}_{\mathcal{O}}) \cong \varinjlim \mathrm{Rep}(\widetilde{G}_{\mathcal{O}}/G_{>n})$, and so taking $G(\mathcal{O})$ invariants of modules in $\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})$ can be calculated by analyzing invariants of finite algebraic groups. This is what we turn to next.

4.2.1.2. *Equivariance with Respect to* $G(\mathcal{O})$*.* Let us now deal with the second item, namely equivariance with respect to $G(\mathcal{O})$. Let V be an algebraic representation of $\widetilde{G}_{\mathcal{O}}$, there is an associated Chevalley-Eilenberg cochain complex:

$$V \otimes \mathrm{Sym}^{\bullet}((z\mathfrak{g}(\mathcal{O}))^* [-1]), \tag{4.2.1.22}$$

in which the differential $V \to V \otimes (z\mathfrak{g}(\mathcal{O}))^*$ is induced by the action of $z\mathfrak{g}(\mathcal{O})$ on V. We claim:

Proposition 4.2.7. *Let V be an algebraic $\widetilde{G}_{\mathcal{O}}$ representation, there is a quasi-isomorphism:*

$$\mathrm{Hom}^{\mathbb{B}\mathbb{C}^*}(\mathcal{O}_{\mathbb{B}\widetilde{G}_{\mathcal{O}}}, V) \cong \left[V \otimes \mathrm{Sym}^{\bullet}((z\mathfrak{g}(\mathcal{O}))^* [-1])\right]^G . \tag{4.2.1.23}$$

Here $[-]^G$ means taking the G invariant part of a representation.

Proof. We have a short exact sequence of groups:

$$1 \to G_{>0} \to G(\mathcal{O}) \to G \to 1, \tag{4.2.1.24}$$

which gives a natural equivalence of functors:

$$\mathrm{Hom}^{\mathbb{B}\mathbb{C}^*}(\mathcal{O}_{\mathbb{B}\widetilde{G}_{\mathcal{O}}}, V) \cong \mathrm{Hom}^{\mathbb{B}\mathbb{C}^*}\Big(\mathcal{O}_{\mathbb{B}(G\times\mathbb{C}^*)}, \mathrm{Hom}^{\mathbb{B}(G\times\mathbb{C}^*)}(\mathcal{O}_{\mathbb{B}\widetilde{G}_{\mathcal{O}}}, V)\Big) \tag{4.2.1.25}$$

Since G is reductive, the category of algebraic representations of G is semi-simple, which implies that:

$$\mathrm{Hom}^{\mathbb{B}\mathbb{C}^*}\Big(\mathcal{O}_{\mathbb{B}(G\times\mathbb{C}^*)}, \mathrm{Hom}^{\mathbb{B}(G\times\mathbb{C}^*)}(\mathcal{O}_{\mathbb{B}\widetilde{G}_{\mathcal{O}}}, V)\Big) = \Big[\mathrm{Hom}^{\mathbb{B}(G\times\mathbb{C}^*)}(\mathcal{O}_{\mathbb{B}\widetilde{G}_{\mathcal{O}}}, V)\Big]^G, \tag{4.2.1.26}$$

where $[-]^G$ is taking ordinary G invariants. Thus we only need to understand $\mathrm{Hom}^{\mathbb{B}(G\times\mathbb{C}^*)}(\mathcal{O}_{\mathbb{B}\widetilde{G}_{\mathcal{O}}}, V)$, which is the(derived) $G_{>0}$ invariants of V. To understand this, we need the following lemma:

Lemma 4.2.8. *Let K be a finite-dimensional simply-connected unipotent Lie group and V be an algebraic representation of K, then* $\mathrm{RHom}(\mathbb{C}, V) \cong \mathrm{H}^*\left(V \otimes \mathrm{Sym}^{\bullet}(\mathfrak{k}^*[-1])\right) =: \mathcal{H}^*(\mathfrak{k}, V)$*, where* $V \otimes \mathrm{Sym}^{\bullet}(\mathfrak{k}^*[-1])$ *is the Chevalley Eilenberg cochain complex of V as a $\mathfrak{k}$ module.*

Let us assume this for now and apply it to $G_{>0}$. From $\mathrm{Rep}(G_{>0}) = \varinjlim_k \mathrm{Rep}(G_{>0}/z^k)$, we see that for any finite-dimensional representation V:

$$\mathrm{RHom}_{G_{>0}}(\mathbb{C}, V) = \varinjlim_k \mathrm{RHom}_{G_{>0}/z^k}(\mathbb{C}, V). \tag{4.2.1.27}$$

Now the Lie group $G_{>0}/z^k$ is unipotent simply-connected, whose Lie algebra is $z\mathfrak{g}(\mathcal{O})/z^k\mathfrak{g}(\mathcal{O})$, so for finite-dimensional V, one has:

$$\mathrm{RHom}_{G_{>0}/z^k}(\mathbb{C}, V) \cong V \otimes \mathrm{Sym}^{\bullet}\Big(\Big(z\mathfrak{g}(\mathcal{O})/z^k\mathfrak{g}(\mathcal{O})\Big)^* [-1]\Big) \tag{4.2.1.28}$$

Taking co-limit over k, one obtain, for any finite-dimensional(and more generally algebraic) representation V:

$$\mathrm{RHom}_{G_{>0}}(\mathbb{C}, V) \cong V \otimes \mathrm{Sym}^{\bullet}((z\mathfrak{g}\,(\mathcal{O}))^{*}\,[-1])\,. \tag{4.2.1.29}$$

This completes the proof.

□

For completeness, we present the proof of Lemma 4.2.8 here:

Proof of Lemma 4.2.8. Clearly $\mathcal{H}^0 = \mathrm{Hom}_K(\mathbb{C}, V)$, so by the usual idea of homological algebra(for instance, in [**L**$^+$**02**]), we need only show that the functors $\mathcal{H}^i$ are erasable for $i > 0$. This is done by induction and a use of the function ring $\mathcal{O}_K$. We claim that $\mathcal{H}^i(\mathfrak{k}, \mathcal{O}_K)$ is zero for $i > 0$. When $\mathfrak{k} = \mathbb{C}$ and $K = \mathbb{C}$, $\mathcal{O}_K = \mathbb{C}[x]$ and the action of $\mathfrak{k}$ is given by taking derivatives. Thus $\mathcal{H}^1(\mathbb{C}, \mathbb{C}[x]) = 0$ since taking derivative is a surjective map.

Now for general $\mathfrak{k}$, by nilpotency, we have a short exact sequence of Lie algebras $0 \to \mathfrak{h} \to \mathfrak{k} \to \mathbb{C} \to 0$. This must split since $\mathbb{C}$ is one dimensional and so we have a covering map $H \rtimes \mathbb{C} \to K$ where H is simply connected. By assumption K is simply connected so the map is an isomorphism. Thus we have an exact sequence of Lie groups $0 \to H \to K \to \mathbb{C} \to 0$. Let us consider $\mathcal{H}^*(\mathfrak{k}, \mathcal{O}_K)$. By Hochschild-Serre spectral sequence [**HS53**], there is a spectral sequence whose second term is given by $E_2^{*,*} = \mathcal{H}^*(\mathbb{C}, \mathcal{H}^*(\mathfrak{h}, \mathcal{O}_K))$, that converges to $E_\infty^* = \mathcal{H}^*(\mathfrak{k}, \mathcal{O}_K)$. Since $\mathbb{C}$ is one dimensional, E_2 is supported on two columns, the spectral sequence terminates and $\mathcal{H}^n(\mathfrak{k}, \mathcal{O}_K) = \oplus_{p+q=n}\mathcal{H}^p(\mathbb{C}, \mathcal{H}^q(\mathfrak{h}, \mathcal{O}_K))$. Consider $\mathcal{H}^q(\mathfrak{h}, \mathcal{O}_K)$, we need to understand the module structure of $\mathcal{O}_K$ as an H module. From the isomorphism $H \rtimes \mathbb{C} \cong K$ of Lie groups, we see that there is an isomorphism of algebras

$$\mathcal{O}_K = \mathcal{O}_H \otimes \mathbb{C}[x], \tag{4.2.1.30}$$

which is described by the following: for $g \in K$, we write $g = h_g c_g$ with $h_g \in H$ and $c_g \in \mathbb{C}$, then the map is given by mapping function f on K to $f(h_g c_g)$ on $H \rtimes \mathbb{C}$. Now to understand the module structure, if we take an object $f_1 \otimes f_2$ where $f_1 \in \mathcal{O}_H$ and $f_2 \in \mathbb{C}[x]$, for any $h \in H$, $h(f_1 \otimes f_2)(g) = f_1 \otimes f_2(h^{-1}g) = f_1 \otimes f_2(h^{-1}h_g c_g) = f_1(h^{-1}h_g) \otimes f_2(c_g) = ((hf_1) \otimes f_2)(g)$. All the equations use the fact that the decomposition of $g = h_g c_g$ is unique. Thus under the above isomorphism (4.2.1.30) , $\mathcal{O}_K$ as an H module is nothing but a direct sum of $\mathcal{O}_H$, hence $\mathcal{H}^q(\mathfrak{h}, \mathcal{O}_K) = 0$ for $q > 0$, and $\mathcal{H}^0(\mathfrak{h}, \mathcal{O}_K) = \mathcal{O}_K^{\mathfrak{h}}$, the invariant part of $\mathcal{O}_K$. Again from the identification (4.2.1.30) this is isomorphic to $\mathbb{C}[x]$. But what is the module structure? Let $f = f_1 \otimes f_2$ where f_1 is H invariant(it is a constant function in this case), let $c \in \mathbb{C}$, then $cf(g) = f(c^{-1}h_g c_g) = f(c^{-1}hcc^{-1}c_g)$, now since H is a normal subgroup($\mathfrak{h}$ is an ideal), $c^{-1}hc \in H$, and so by the uniqueness of the above decomposition, $cf(g) = f_1(c^{-1}hc)f_2(c^{-1}c_g) = f_1(h)(cf_2)(c_g)$, where we used that f_1 is a constant function on H. Thus the action on $\mathbb{C}[x]$ is taking derivative and we already see that the cohomology is zero for positive degree. This completes the inductive hypothesis.

Since every K module has an injective resolution by $\mathcal{O}_K$, we conclude that $\mathcal{H}^i$ are indeed erasable for $i > 0$.

□

By Proposition 4.2.7, there is a quasi-isomorphism of algebras:

$$\text{(4.2.1.31)} \qquad \mathrm{End}^{\mathbb{B}\mathbb{C}^*}(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}}) \cong \left[\mathrm{End}^{\mathbb{B}\widetilde{G}_{\mathcal{O}}}(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}}) \otimes \mathrm{Sym}^{\bullet}((z\mathfrak{g}\,(\mathcal{O}))^*\,[-1])\right]^G .$$

We are thus left to understand the algebra $\mathrm{End}^{\mathbb{B}\widetilde{G}_{\mathcal{O}}}(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}})$ as a $\widetilde{G}_{\mathcal{O}}$-equivariant module. This is the last step and uses the idea of formal completion and formal geometry of [**GR17**].

4.2.1.3. *Formal Completion.* We are left with computing $\mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}})$ as an algebraic representation of $\widetilde{G}_{\mathcal{O}}$. Before going into any details, we would like to comment that this computation may seem complicated, but it is rooted on this simple observation: if R is smooth and I is a complete intersection ideal, then R/I is quasi-isomorphic to its Koszul resolution, and $\mathrm{End}_{R-\mathrm{Mod}}(R/I)$ is an exterior algebra over R/I generated by $(I/I^2)^*$. This is not quite obvious when we replace R by a formally smooth indscheme, since each of the strata may be very singular. In this section, we will introduce formal completion introduced in [**GR14**] to render the situation amenable.

Let $\mathcal{X}$ be a prestack; then its de-Rham stack is defined by:

$$\text{(4.2.1.32)} \qquad \mathcal{X}_{dR}(S) = \mathcal{X}(S_{red})$$

where S_{red} is the reduced scheme of S. Given a morphism of prestacks $\mathcal{X} \to \mathcal{Y}$, the formal completion is defined by([**GR14**, Section 6.1]):

$$\widehat{\mathcal{Y}_{\mathcal{X}}} := \mathcal{Y} \times_{\mathcal{Y}_{dR}} \mathcal{X}_{dR}. \tag{4.2.1.33}$$

This operation behaves well with filtered colimit as explained in [**GR14**, 6.1.3]: if $\mathcal{X} = \varinjlim \mathcal{X}_n$ and $\mathcal{Y} = \varinjlim \mathcal{Y}_n$ such that the map $\mathcal{X} \to \mathcal{Y}$ comes from a system of maps $\mathcal{X}_n \to \mathcal{Y}_n$, then:

$$\widehat{\mathcal{Y}_{\mathcal{X}}} = \varinjlim \widehat{\mathcal{Y}_{n\mathcal{X}_n}}. \tag{4.2.1.34}$$

Now assume that X is a locally almost finite type DG scheme and Y an almost finite type DG indscheme, and an embedding $i : X \to Y$, then by [**GR14**, Proposition 6.3.1], $\widehat{Y_X}$ is a DG indscheme. More-over, from the above we see that:

$$\widehat{Y_X} \cong \varinjlim \widehat{Y_{nX}}, \tag{4.2.1.35}$$

which in particular means that:

$$\mathrm{IndCoh}(\widehat{Y_X}) \cong \varinjlim \mathrm{IndCoh}(\widehat{Y_{nX}}). \tag{4.2.1.36}$$

Denote by $\widehat{i}$ the embedding $\widehat{Y_X} \to Y$, and by $\widehat{i_n}$ the embedding of $\widehat{Y_{nX}} \to Y_n$, then by [**GR14**, Proposition 7.4.5], the adjunction $\mathrm{Id} \to \widehat{i_n}^! \widehat{i}_{n*,\mathrm{IndCoh}}$ is an equivalence. Taking colimit, we see that $\mathrm{Id} \to \widehat{i}^! \widehat{i}_{*,\mathrm{IndCoh}}$ is an equivalence. If we now consider the sequence of maps:

$$X \xrightarrow{j} \widehat{Y_X} \xrightarrow{\widehat{i}} Y, \tag{4.2.1.37}$$

then $i = \widehat{i} \circ j$, and so we have an equivalence of continuous endo-functors of $\mathrm{IndCoh}(X)$:

$$i^! i_{*,\mathrm{IndCoh}} \cong j^! j_{*,\mathrm{IndCoh}}. \tag{4.2.1.38}$$

Now let us take $Y = \mathrm{Gr}_G$ and X a miniscule orbit, denote by $\mathcal{X} = X/\widetilde{G}_{\mathcal{O}}$ and $\mathcal{Y} = Y/\widetilde{G}_{\mathcal{O}}$. The formal completion $\widehat{Y_X}$ is a DG-indscheme with an $\widetilde{G}_{\mathcal{O}}$ action, we denote by $\widehat{\mathcal{Y}}_{\mathcal{X}}$ the quotient stack

$\widehat{Y_X}/\widetilde{G}_{\mathcal{O}}$. We have the following diagram of maps:

(4.2.1.39)
$$\begin{array}{ccccc} X & \xrightarrow{j} & \widehat{Y_X} & \xrightarrow{\widehat{i}} & Y \\ \downarrow p & & \downarrow p & & \downarrow p \\ \mathcal{X} & \xrightarrow{\bar{j}} & \widehat{\mathcal{Y}}_{\mathcal{X}} & \xrightarrow{\bar{\widehat{i}}} & \mathcal{Y} \end{array}$$

LEMMA 4.2.9. *There is an equivalence of continuous endo-functors of* $\mathrm{IndCoh}(\mathcal{X})$*:*

(4.2.1.40)
$$\bar{i}^! \bar{i}_{*,\mathrm{IndCoh}} \cong \bar{j}^! \bar{j}_{*,\mathrm{IndCoh}}.$$

PROOF. Since p is conservative and t-exact, we need only show that:

(4.2.1.41)
$$p^* \bar{i}^! \bar{i}_{*,\mathrm{IndCoh}} \cong p^* \bar{j}^! \bar{j}_{*,\mathrm{IndCoh}}.$$

By definition of IndCoh^* as well as the definition of functors involved, we have $p^* \bar{i}^! \bar{i}_{*,\mathrm{IndCoh}} \cong i^! i_{*,\mathrm{IndCoh}} p^*$, as well as $p^* \bar{j}^! \bar{j}_{*,\mathrm{IndCoh}} = j^! j_{*,\mathrm{IndCoh}} p^*$. These two functors are equivalent as seen from the above discussion. This completes the proof. □

Recall that we would like to compute $\mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\bar{i}_{*,\mathrm{IndCoh}} \mathcal{O}_{\mathcal{X}})$. By adjunction:

(4.2.1.42)
$$\mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\bar{i}_{*,\mathrm{IndCoh}} \mathcal{O}_{\mathcal{X}}) \cong \mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{O}_{\mathcal{X}}, \bar{i}^! \bar{i}_{*,\mathrm{IndCoh}} \mathcal{O}_{\mathcal{X}}).$$

By Lemma 4.2.9 we have:

(4.2.1.43)
$$\mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{O}_{\mathcal{X}}, \bar{i}^! \bar{i}_{*,\mathrm{IndCoh}} \mathcal{O}_{\mathcal{X}}) \cong \mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{O}_{\mathcal{X}}, \bar{j}^! \bar{j}_{*,\mathrm{IndCoh}} \mathcal{O}_{\mathcal{X}}).$$

Thus we have transfered the computation onto the formal completion. In the next section, we will specialize to the case when $X = [e]$ and $Y = \mathrm{Gr}_G$, and explicitly understand this formal completion using the idea of formal geometry studied in [**GR17**].

4.2.1.4. *Formal Groups and Lie Algebras.* In [**GR17**, Chapter 7], the authors studied formal groups, and showed that the category of formal groups over a prestack $\mathcal{X}$ is equivalent to that of Lie algebra objects in $\mathrm{IndCoh}(\mathcal{X})$. In Chapter 3, Section 3.3.2, we have used this idea to compute the tangent Lie algebra of a Hamiltonian reduction. Let us recall the important notations here. Denote by $\mathrm{FormMod}_{/\mathcal{X}}$ the category of locally almost finite type stacks $\mathcal{Z}$ over $\mathcal{X}$ such that the map $\mathcal{Z} \to \mathcal{X}$ is inf-schematic and induces an equivalence $\mathcal{Z}_{red} \cong \mathcal{X}_{red}$ ([**GR17**, Chapter 5, 1.1.1]).

A formal group over $\mathcal{X}$ is a group object in $\mathrm{FormMod}_{/\mathcal{X}}$. This category is denoted by $\mathrm{Gr}^f_{\mathcal{X}}$. On the other hand, consider the category of Lie algebra objects in $\mathrm{IndCoh}(\mathcal{X})$, which we denote by $\mathrm{Lie}_{\mathcal{X}}$. The result of [**GR17**, Chapter 7], more specifically Theorem 3.1.4, states that there is an equivalence:

$$\mathrm{Gr}^f_{\mathcal{X}} \simeq \mathrm{Lie}_{\mathcal{X}}. \tag{4.2.1.44}$$

The idea of this is that given a formal group $\mathcal{Y}$ over $\mathcal{X}$, the object $\pi_{*,\mathrm{IndCoh}}(\omega_{\mathcal{Y}})$, the pushforward of the dualizing sheaf, has the structure of a cocommutative Hopf algebra. This is the universal enveloping algebra of the Lie algebra associated to $\mathcal{Y}$.

When $\mathcal{X} = \mathrm{pt}$, then the category LieAlg(IndCoh(pt)) is the category of DG Lie algebras in Vect studied in [**Lur11**]. In the special case when $\mathfrak{g}$ is a Lie algebra concentrated in degree 0, the formal moduli problem is simply $\widehat{\mathfrak{g}_0}$, the formal completion of $\mathfrak{g}$ at 0([**Lur11**, Construction 2.2.13.]). The formal group structure is given by the Baker–Campbell–Hausdorff formula.

Let us now apply this to the case when $X = [e]$ and $Y = \mathrm{Gr}_G$, we have:

Lemma 4.2.10. *The formal completion $\widehat{Y_X}$ is a formal group whose Lie algebra is $z^{-1}\mathfrak{g}[z^{-1}]$.*

Proof. From the discussion of Section 4.1.1, the group ind-scheme $L^{<0}G$ is an open neighborhood of X in Y, and so $\widehat{Y_X} \cong \widehat{L^{<0}G_X}$. Now $\widehat{L^{<0}G_X}$ is a formal group whose associated Lie algebra is $z^{-1}\mathfrak{g}[z^{-1}]$.

∎

Denote by $L^{<0}\mathfrak{g}$ the Lie algebra of $L^{<0}G$. By [**GR17**, Chapter 7, Theorem 3.1.4], we see that $\widehat{Y_X}$ is equivalent to $\widehat{L^{<0}\mathfrak{g}_0}$, the formal completion of $L^{<0}\mathfrak{g}$ at 0. The action of $\widetilde{G}_{\mathcal{O}}$ is given by conjugation on $L^{<0}\mathfrak{g} \cong \mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O})$. Again denote by $\mathcal{X} = X/\widetilde{G}_{\mathcal{O}}$ and $\mathcal{Y} = Y/\widetilde{G}_{\mathcal{O}}$. Recall the morphism $\overline{j} : \mathcal{X} \to \widehat{\mathcal{Y}}_{\mathcal{X}}$ and $\overline{i} : \mathcal{X} \to \mathcal{Y}$. We claim:

Proposition 4.2.11. *There is an equivalence of continuous endofunctors on* $\mathrm{IndCoh}(\mathcal{X})$

$$\overline{i}^!\overline{i}_{*,\mathrm{IndCoh}} \cong \mathrm{Sym}^\bullet\big(L^{<0}\mathfrak{g}[-1]\big) \otimes - \tag{4.2.1.45}$$

where $L^{<0}\mathfrak{g}$ is understood as a $G(\mathcal{O})$ module under conjugation action.

PROOF. By Lemma 4.2.9, we can replace the left hand side of equation (4.2.1.45) by $\bar{j}^!\bar{j}_{*,\mathrm{IndCoh}}$. Consider the following diagram:

$$\mathcal{X} \xrightarrow{\bar{j}} \widetilde{G}_{\mathcal{O}} \setminus \widehat{L^{<0}\mathfrak{g}_0} \longrightarrow G(\mathcal{O}) \setminus L^{<0}\mathfrak{g} \tag{4.2.1.46}$$

Denote by $\bar{i}_g$ the inclusion $\mathcal{X} \to G(\mathcal{O}) \setminus L^{<0}\mathfrak{g}$, Lemma 4.2.9 again implies:

$$\bar{j}^!\bar{j}_{*,\mathrm{IndCoh}} \cong \bar{i}_g^!\bar{i}_{g,*,\mathrm{IndCoh}}. \tag{4.2.1.47}$$

The latter can be computed explicitly using a Koszul resolution, and the result follows.

□

We can now prove:

COROLLARY 4.2.12. *There is a quasi-isomorphism of $\widetilde{G}_{\mathcal{O}}$ vector spaces:*

$$\mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}}) \cong \mathrm{Sym}^\bullet(\mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O})[-1]) \,. \tag{4.2.1.48}$$

PROOF. By Proposition 4.2.11:

$$\mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}}) \cong \mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}}, \mathrm{Sym}^\bullet(\mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O})[-1]) \otimes \mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}}\right). \tag{4.2.1.49}$$

Since $\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}}$ is simply the trivial representation of $\widetilde{G}_{\mathcal{O}}$, the right hand side of equation (4.2.1.49) can be identified as the right hand side of equation (4.2.1.48). This completes the proof. □

Using Proposition 4.2.7 and Corollary 4.2.12, we obtain the following theorem:

THEOREM 4.2.13. *There is a quasi-isomorphism of $\mathbb{C}^*$ vector spaces:*

$$\mathrm{End}^{\mathbb{BC}^*}(\mathcal{O}_{[e]/\widetilde{G}_{\mathcal{O}}}) \cong \left[\mathrm{Sym}^\bullet\left((\mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O}) \oplus (z\mathfrak{g}(\mathcal{O}))^*\right)[-1]\right)\right]^G, \tag{4.2.1.50}$$

where $[-]^G$ is taking ordinary G invariants. This space coincides with $\pi_0\mathcal{V}_{G,0}$ of equation (4.1.2.6) *after shifting the degree of $\mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O})$ to -1.*

REMARK 4.2.14. *As remarked in [**OY20**], the character of the above space is given by:*

$$\frac{1}{|W|}\oint_T \frac{\mathrm{d}s}{2\pi i s} \prod_{\alpha \text{ roots}} (1-s^\alpha)\left[(q)_\infty^{2\mathrm{rank}(G)} \prod_{\alpha \text{ roots}} (qs^\alpha;q)^2_\infty\right], \tag{4.2.1.51}$$

which reproduces Schur index of a pure gauge theory.

4.2.1.5. *Other Miniscule Orbits.* We can in fact use this technique for other miniscule orbits of Gr_G. Let us now take X to be a miniscule orbit and $Y = \mathrm{Gr}_G$. Denote by $\mathcal{X}$ and $\mathcal{Y}$ the quotients of X and Y by $\widetilde{G}_{\mathcal{O}}$. Choose $[g]$ a point in $\mathcal{X}$, let $\widetilde{P}$ be the stabilizer of $[g]$ in $\widetilde{G}_{\mathcal{O}}$, then the there is an equivalence of prestacks:

$$\mathcal{X} \cong \widetilde{G}_{\mathcal{O}} \setminus \widetilde{G}_{\mathcal{O}}/\widetilde{P} \cong \mathbb{B}\widetilde{P}. \tag{4.2.1.52}$$

Under this, the map $\bar{i} : \mathcal{X} \to \mathcal{Y}$ corresponds to the map of schemes:

$$\widetilde{P} \setminus \mathrm{pt} \xrightarrow{j} \widetilde{P} \setminus \mathrm{Gr}_G \xrightarrow{m} \widetilde{G}_{\mathcal{O}} \setminus \mathrm{Gr}_G \tag{4.2.1.53}$$

where the map j is the embedding of pt as $[g^{-1}]$. Let V be the $\widetilde{P}$ module given by:

$$\mathfrak{g}(\mathcal{K})/(\mathfrak{g}(\mathcal{O}) + g\mathfrak{g}(\mathcal{O})g^{-1}). \tag{4.2.1.54}$$

We prove:

Proposition 4.2.15. *There is a quasi-isomorphism of objects in* $\mathrm{IndCoh}(\mathcal{X})$*:*

$$\bar{i}^! \bar{i}_{*,\mathrm{IndCoh}}(\mathcal{O}_{\mathcal{X}}) \cong \widetilde{G}_{\mathcal{O}} \times_{\widetilde{P}} \mathrm{Sym}^\bullet(V[-1]). \tag{4.2.1.55}$$

Proof. Using the presentation of $\mathcal{X}$ in equation (4.2.1.53), we would like to show that:

$$j^! m^! m_{*,\mathrm{IndCoh}} j_{*,\mathrm{IndCoh}}(\mathcal{O}_{\mathcal{X}}) = \mathrm{Sym}^\bullet(V[-1]) \tag{4.2.1.56}$$

as a module of $\widetilde{P}$. Let us understand the composition $m^! m_{*,\mathrm{IndCoh}} j_{*,\mathrm{IndCoh}}(\mathcal{O}_{\mathcal{X}})$, consider the following Cartesian diagram:

$$\begin{array}{ccc} \widetilde{P} \setminus \widetilde{G}_{\mathcal{O}}/\widetilde{P} & \xrightarrow{\widetilde{m}} & \widetilde{P} \setminus \mathrm{pt} \\ \downarrow{\scriptstyle \widetilde{\bar{i}}} & & \downarrow{\scriptstyle \bar{i}} \\ \widetilde{P} \setminus \mathrm{Gr}_G & \xrightarrow{m} & \widetilde{G}_{\mathcal{O}} \setminus \mathrm{Gr}_G \end{array} \tag{4.2.1.57}$$

Here $\widetilde{m}$ is the projection of $\widetilde{G}_{\mathcal{O}}/\widetilde{P}$ to a point, and $\widetilde{\overline{i}}$ is induced by the embedding $X \to \mathrm{Gr}_G$. By base-change property [**GR14**, Proposition 2.9.2], we have:

$$m^!\overline{i}_{*,\mathrm{IndCoh}} \cong \widetilde{\overline{i}}_{*,\mathrm{IndCoh}}\widetilde{m}^!. \tag{4.2.1.58}$$

Thus the object $m^!\overline{i}_{*,\mathrm{IndCoh}}(\mathcal{O}_{\mathcal{X}})$ is $\widetilde{\overline{i}}_{*,\mathrm{IndCoh}}\omega_X$, where ω_X is the dualizing sheaf of X with the canonical $\widetilde{P}-$equivariant structure. In our case, since $X = \widetilde{G}_{\mathcal{O}}/\widetilde{P}$, ω_X is the line bundle over X associated to the one dimensional $\widetilde{P}$ representation:

$$L_{\mathrm{top}} = \mathrm{Sym}^{\mathrm{top}}(\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}[1]) \tag{4.2.1.59}$$

Here $\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}[1]$ is a finite dimensional vector space in cohomological degree -1, and so the exterior algebra has finite cohomological degree. The representation L_{top} is the top degree part of the exterior algebra, and is in cohomological degree $-\dim(X)$. Let us now employ the idea of formal completion. Consider the Cartesian diagram:

$$\begin{array}{ccc} \widetilde{P}\backslash \widehat{X_{[g]}} & \longrightarrow & \widetilde{P}\backslash \widetilde{G}_{\mathcal{O}}/\widetilde{P} \\ \big\downarrow & & \big\downarrow \\ \widetilde{P}\backslash \widehat{Y_{[g]}} & \longrightarrow & \widetilde{P}\backslash \mathrm{Gr}_G \end{array} \tag{4.2.1.60}$$

By base-change property [**GR14**, Proposition 2.9.2], the shriek pullback of $\widetilde{\overline{i}}_{*,\mathrm{IndCoh}}\omega_X$ to $\widehat{Y_{[g]}}$ is the pushforward of the dualizing sheaf of $\widehat{X_{[g]}}$ to $\widehat{Y_{[g]}}$. The advantage is that these local completions have very explicit descriptions. Indeed, by [**GR17**, Chapter 7, Theorem 3.1.4], the space $\widehat{X_{[g]}}$ is equivalent to the completion of $\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}$ at 0. Similarly, the space $\widehat{Y_{[g]}}$ is equivalent to the completion of $L^{<0}\mathfrak{g}$ at 0. The map $\widehat{X_{[g]}} \to \widehat{Y_{[g]}}$ corresponds to the embedding of the following $\widetilde{P}$ modules:

$$\psi : \mathfrak{g}_{\mathcal{O}}/\mathfrak{p} \to \mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O}),\ H \to gHg^{-1}. \tag{4.2.1.61}$$

We can thus transfer to the following diagram:

$$\begin{array}{ccccc} & & \widetilde{P}\backslash \widehat{X_{[g]}} & \xrightarrow{\widehat{\varphi}} & \widetilde{P}\backslash \mathfrak{g}_{\mathcal{O}}/\mathfrak{p} \\ & & \big\downarrow \widehat{\psi} & & \big\downarrow \psi \\ \widetilde{P}\backslash \mathrm{pt} & \xrightarrow{\widehat{j}} & \widetilde{P}\backslash \widehat{Y_{[g]}} & \xrightarrow{\widehat{\phi}} & \widetilde{P}\backslash \mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O}) \end{array} \tag{4.2.1.62}$$

with which we can derive:

$$j^! \widetilde{\widetilde{i}}_{*,\mathrm{IndCoh}} \omega_X \cong \widehat{j}^! \widehat{\phi}^! \psi_{*,\mathrm{IndCoh}}(\omega_{\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}}). \tag{4.2.1.63}$$

Here the sheaf $\omega_{\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}}$ is the structure sheaf of $\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}$ tensored with the representation L_{top}. We now have:

$$\widehat{j}^! \widehat{\phi}^! \psi_{*,\mathrm{IndCoh}}(\omega_{\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}}) \cong \mathrm{Hom}^{\mathbb{B}\widetilde{P}} \left((\widehat{j} \circ \widehat{\phi})_{*,\mathrm{IndCoh}}(\mathcal{O}_{\mathbb{B}\widetilde{P}}), \psi_{*,\mathrm{IndCoh}}(\omega_{\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}}) \right). \tag{4.2.1.64}$$

The right hand side can be computed using a Koszul resolution of $(\widehat{j} \circ \widehat{\phi})_{*,\mathrm{IndCoh}}(\mathcal{O}_{\mathbb{B}\widetilde{P}})$, and the result is the following complex:

$$\mathrm{Sym}^{\bullet}(\mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O})[-1]) \otimes \mathbb{C}[\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}] \otimes L_{\mathrm{top}}, \tag{4.2.1.65}$$

together with a differential induced from the Koszul resolution. Here $\mathbb{C}[\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}]$ denotes the algebra of functions on $\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}$. The nonzero part of the Koszul differential lies in:

$$\mathrm{Sym}^{\bullet}(\mathfrak{g}(\mathcal{O})/\mathfrak{p}[-1]) \otimes \mathbb{C}[\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}] \otimes L_{\mathrm{top}} \cong \mathrm{Sym}^{\bullet}((\mathfrak{g}(\mathcal{O})/\mathfrak{p})^* [1]) \otimes \mathbb{C}[\mathfrak{g}_{\mathcal{O}}/\mathfrak{p}]. \tag{4.2.1.66}$$

The quasi-isomorphism is due to tensoring with L_{top}, which makes this into a usual Koszul complex. The cohomology of this complex is $\mathbb{C}$ in degree 0, and so the cohomology of the complex in equation (4.2.1.65) is thus identified with $\mathrm{Sym}^{\bullet}(V[-1])$. This completes the proof. □

4.2.2. Computation of $\mathrm{Ops}_{G,V}$ for General V. In this section, we will generalize the computation above to $\mathrm{Ops}_{G,V}$ for General V. Let V be a representation of G. Recall that the BFN space is defined by the base change diagram:

$$\begin{array}{ccc} \mathcal{R}_{G,V} & \longrightarrow & V(\mathcal{O}) \\ \downarrow & & \downarrow \\ G(\mathcal{K}) \times_{G(\mathcal{O})} V(\mathcal{O}) & \longrightarrow & V(\mathcal{K}) \end{array} \tag{4.2.2.1}$$

We add to this another base-change diagram:

$$\begin{array}{ccccc} Z & \longrightarrow & \mathcal{R}_{G,V} & \longrightarrow & V(\mathcal{O}) \\ \downarrow & & \downarrow & & \downarrow \\ e \times V(\mathcal{O}) & \longrightarrow & G(\mathcal{K}) \times_{G(\mathcal{O})} V(\mathcal{O}) & \longrightarrow & V(\mathcal{K}) \end{array} \tag{4.2.2.2}$$

Here $Z = V(\mathcal{O}) \times_{V(\mathcal{K})} V(\mathcal{O})$ can be described as $V(\mathcal{O}) \times V(\mathcal{K})/V(\mathcal{O})[-1]$. The identity line is the pushforward of structure sheaf of $V(\mathcal{O})$ along the embedding $i : V(\mathcal{O}) \to \mathcal{R}_{G,V}$. Note that this is a classical scheme embedded into a derived scheme. We will label the maps:

$$\begin{array}{ccccccc} V(\mathcal{O}) & \xrightarrow{l} & Z & \xrightarrow{m} & \mathcal{R}_{G,V} & \longrightarrow & V(\mathcal{O}) \\ & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow \\ & & e \times V(\mathcal{O}) & \xrightarrow{j} & G(\mathcal{K}) \times_{G(\mathcal{O})} V(\mathcal{O}) & \longrightarrow & V(\mathcal{K}) \end{array} \tag{4.2.2.3}$$

The Schur index is then the graded Euler character of:

$$\mathrm{End}^{\mathbb{BC}^*}\left(i_{*,\mathrm{IndCoh}}(\mathcal{O}_{V(\mathcal{O})/\widetilde{G}_{\mathcal{O}}})\right). \tag{4.2.2.4}$$

In the following, we will write $\overline{X}$ for the quotient stack $X/\widetilde{G}_{\mathcal{O}}$, in order to avoid clustering of notations. We will also omit the IndCoh for all the push-forward functors. To do this computation, fix again an ind-scheme structure of Gr_G and $\mathcal{R}_{G,V}$ that are compatible with the action of $\widetilde{G}_{\mathcal{O}}$. We make use of the following diagram:

$$\begin{array}{ccccccc} V(\mathcal{O}) & \xrightarrow{l_n} & Z_n & \xrightarrow{m_n} & \mathcal{R}_{G,V,n} & \longrightarrow & V(\mathcal{O}) \\ & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow \\ & & e \times V(\mathcal{O}) & \xrightarrow{j_n} & G(\mathcal{K})_n \times_{G(\mathcal{O})} V(\mathcal{O}) & \longrightarrow & z^{-N}V(\mathcal{O}) \end{array} \tag{4.2.2.5}$$

such that $m_n \circ l_n = i_n$. Since $\mathcal{R}_{G,V}$ is a colimit of $\mathcal{R}_{G,V,n}$, by equation (4.2.1.9):

$$\mathrm{End}^{\mathbb{BC}^*}\left(i_*(\mathcal{O}_{\overline{V(\mathcal{O})}})\right) = \varinjlim_n \mathrm{End}^{\mathbb{BC}^*}\left(i_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right). \tag{4.2.2.6}$$

Lemma 4.2.2 implies that $\mathrm{End}^{\mathbb{BC}^*}\left(i_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right)$ this is the $G(\mathcal{O})$ invariants of:

$$\mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(i_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right). \tag{4.2.2.7}$$

Let us compute this vector space using adjunctions. Since $i_n = m_n \circ l_n$, from the adjunction pair $(m_{n,*}, m_n^!)$, one has:

$$\mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(i_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right) \cong \mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(l_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}}), m_n^! i_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right). \tag{4.2.2.8}$$

As Z_n is a very explicit DG scheme with a very explicit action of $\widetilde{G}_{\mathcal{O}}$, one can write an explicit projective resolution of $l_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})$ given by the Koszul complex:

$$l_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}}) \cong \mathcal{O}_{\overline{Z_n}} \otimes \mathrm{Sym}^{\bullet}\big((z^{-N}V(\mathcal{O})/V(\mathcal{O}))^*[2]\big). \tag{4.2.2.9}$$

together with the differential given by the usual Koszul differential. This is a quasi-isomorphism of $\widetilde{G}_{\mathcal{O}}$ equivariant sheaves. Substituting the resolution of equation (4.2.2.9) into the above equation, one has:

$$\begin{aligned} &\mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(l_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}}), m_n^! i_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right) \\ &\quad \cong \mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{\overline{Z_n}}, m_n^! i_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right) \otimes \mathrm{Sym}^{\bullet}\big(z^{-N}V(\mathcal{O})/V(\mathcal{O})[-2]\big). \end{aligned} \tag{4.2.2.10}$$

By definition, $\mathcal{O}_{\overline{Z_n}} = (p_1)^*\mathcal{O}_{\overline{V(\mathcal{O})}}$, using push-pull adjunction, one has:

$$\mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{\overline{Z_n}}, m_n^! i_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right) \cong \mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{\overline{V(\mathcal{O})}}, (p_1)_* m_n^! i_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right). \tag{4.2.2.11}$$

We then apply the base-change property established in [**Ras20**] Lemma 6.16.1, namely that $(p_1)_* m_n^! \cong j_n^!(p_2)_*$, which implies:

$$\mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{\overline{V(\mathcal{O})}}, (p_1)_* m_n^! i_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right) \cong \mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{\overline{V(\mathcal{O})}}, j_n^! j_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right). \tag{4.2.2.12}$$

To make contact with the affine Grassmannian, we now consider the following Cartesian diagram:

$$\begin{array}{ccc} e \times V(\mathcal{O}) & \xrightarrow{j_n} & G(\mathcal{K})_n \times_{G(\mathcal{O})} V(\mathcal{O}) \\ \big\downarrow{\scriptstyle q_1} & & \big\downarrow{\scriptstyle q_2} \\ e & \xrightarrow{k_n} & \mathrm{Gr}_{G,n}. \end{array} \tag{4.2.2.13}$$

Since $\mathcal{O}_{\overline{V(\mathcal{O})}} \cong q_1^*\mathcal{O}_{\overline{[e]}}$, by pull-push adjunction:

$$\mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{\overline{V(\mathcal{O})}}, j_n^! j_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right) \cong \mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{\overline{[e]}}, (q_1)_* j_n^! j_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right). \tag{4.2.2.14}$$

By base-change formula again, $(q_1)_* j_n^! \cong k_n^! (q_2)_*$, we obtain:

$$\mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{\overline{[e]}}, (q_1)_* j_n^! j_{n,*}(\mathcal{O}_{\overline{V(\mathcal{O})}})\right) \cong \mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(k_{n,*}\mathcal{O}_{\overline{[e]}}) \otimes \mathbb{C}[V(\mathcal{O})]. \tag{4.2.2.15}$$

Here $V(\mathcal{O})$ is in cohomological degree 0. By taking the colimit and applying Proposition 4.2.3, we find that the underlying $\widetilde{G}_{\mathcal{O}}$ representation of $\mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(i_*(\mathcal{O}_V))$ can be identified with:

$$\mathbb{C}[V(\mathcal{O})] \otimes \mathrm{Sym}\,(V(\mathcal{K})/V(\mathcal{O})) \otimes \mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{O}_{\overline{[e]}}). \tag{4.2.2.16}$$

Here $\mathrm{End}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}(\mathcal{O}_{\overline{[e]}})$ is computed in equation (4.2.1.48).

As far as the character is concerned, the above computation thus gives us the desired $\widetilde{G}_{\mathcal{O}}$ module. However, the differential is kept obscured in this computation. To analyze the differential, we will use formal completion. Recall the following diagram:

$$\begin{array}{ccc} \mathcal{R}_{G,V} & \longrightarrow & V(\mathcal{O}) \\ \downarrow & & \downarrow \\ G(\mathcal{K}) \times_{G(\mathcal{O})} V(\mathcal{O}) & \longrightarrow & V(\mathcal{K}) \end{array} \tag{4.2.2.17}$$

Denote by $\widehat{\mathcal{T}}$ the formal completion of $G(\mathcal{K}) \times_{G(\mathcal{O})} V(\mathcal{O})$ along $[e] \times V(\mathcal{O})$. This is a $\widetilde{G}_{\mathcal{O}}$-equivariant formal scheme over $V(\mathcal{O})$. It is clear that it is isomorphic to $\widehat{\mathrm{Gr}_{G,[e]}} \times V(\mathcal{O})$ where $\widehat{\mathrm{Gr}_{G,[e]}}$ is the formal completion of Gr_G along $[e]$. As already discussed in Lemma 4.2.10, the space $\widehat{\mathrm{Gr}_{G,[e]}}$ is a formal group, and thus by [**GR17**, Theorem 3.1.4], it is isomorphic, as a formal group, to the formal completion of its Lie algebra at 0, namely $\widehat{L^{<0}\mathfrak{g}_0}$. We define $\widehat{\mathcal{R}}$ by the following diagram:

$$\begin{array}{ccccc} \widehat{\mathcal{R}} & \longrightarrow & \mathcal{R}_{G,V} & \longrightarrow & V(\mathcal{O}) \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{L^{<0}\mathfrak{g}_0} \times V(\mathcal{O}) & \longrightarrow & G(\mathcal{K}) \times_{G(\mathcal{O})} V(\mathcal{O}) & \longrightarrow & V(\mathcal{K}) \end{array} \tag{4.2.2.18}$$

By [**GR14**, Section 6.1.3 (iv)], $\widehat{\mathcal{R}}$ can be identified as the formal completion of $\mathcal{R}_{G,V}$ along $V(\mathcal{O}) \times_{V(\mathcal{K})} V(\mathcal{O})$. Let $\hat{i}$ be the embedding $V(\mathcal{O}) \to \widehat{\mathcal{R}}$, then just as in Lemma 4.2.9, we have:

$$\mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{\overline{V(\mathcal{O})}}, \hat{i}^! \hat{i}_* \mathcal{O}_{\overline{V(\mathcal{O})}}\right) \cong \mathrm{Hom}^{\mathrm{Rep}(\widetilde{G}_{\mathcal{O}})}\left(\mathcal{O}_{\overline{V(\mathcal{O})}}, i^! i_* \mathcal{O}_{\overline{V(\mathcal{O})}}\right). \tag{4.2.2.19}$$

The advantage of this construction is the following: the space $\widehat{\mathcal{R}}$ is an explicit DG ind-scheme whose underlying pro-algebra is represented by the pro-algebra of functions on the following ind-scheme:

$$\widehat{L^{<0}\mathfrak{g}_0} \times V(\mathcal{O}) \times V(\mathcal{K})/V(\mathcal{O})[-1], \tag{4.2.2.20}$$

and has a differential D as described in Section 4.1.1, induced by the formal group action. We will denote by A the pro-algebra defining this DG ind-scheme. It is worth writing down this differential explicitly here. Choose a basis v^i for V, let ρ_i^j be the matrix elements of $\mathfrak{g}$ action on V, namely:

$$Xv^j = \sum_i \rho_i^j(X)v^i. \tag{4.2.2.21}$$

Denote by $\rho_{i,n}^j$ the corresponding linear function on $L^{<0}\mathfrak{g}$, by $v_{i,n}^*$ the corresponding linear functions on $V(\mathcal{O})$, and by $w_{i,n}^*$ the linear functions on $V(\mathcal{K})/V(\mathcal{O})[-1]$. Note that $w_{i,n}^*$ are odd variables. The differential D can be expressed as:

$$Dw_{i,n}^* = \sum_{j,m+k=n} \rho_{i,m}^j \otimes v_{j,k}^* + \frac{1}{2} \sum_{j_1,j_2,m_1+m_2+k=n} \rho_{i,m_1}^{j_1}\rho_{j_1,m_2}^{j_2} \otimes v_{j_2,k}^* + \cdots . \tag{4.2.2.22}$$

We comment that this comes from exponentiating the action of X, and the first term vanish because functions on $V(\mathcal{K})/V(\mathcal{O})$ is zero on $V(\mathcal{O})$. This differential should be understood in the pro-algebra otherwise the summation would be infinite.

The identity line is the structure sheaf of $e \times V(\mathcal{O})$, and we would like to use a Koszul resolution:

$$\mathbb{C}[V(\mathcal{O})] \cong \mathrm{Sym}^\bullet\big((L^{<0}\mathfrak{g})^*[1]\big) \otimes A \otimes \mathrm{Sym}^\bullet\big((V(\mathcal{K})/V(\mathcal{O}))^*[2]\big). \tag{4.2.2.23}$$

We comment that this should be understood as a projective system of resolutions. To make this a DG resolution, we need to include the usual Koszul differential d_1 coming from the pair $\widehat{L^{<0}\mathfrak{g}_0}$ and $(L^{<0}\mathfrak{g})^*$, as well as d_2 coming from the pair $V(\mathcal{K})/V(\mathcal{O})$ and $(V(\mathcal{K})/V(\mathcal{O}))^*$. However, these are not enough, since $\{D, d_2\} \neq 0$. One can in fact compute this commutator explicitly: let $u_{i,n}^*$ be the linear function on $V(\mathcal{K})/V(\mathcal{O})$ corresponding to $w_{i,n}^*$ in the Koszul resolution, then:

$$\{D, d_2\}u_{i,n}^* = Dw_{i,n}^* = \sum_{j,m+k=n} \rho_{i,m}^j \otimes v_{j,k}^* + \sum_{j_1,j_2,m_1+m_2+k=n} \rho_{i,m_1}^{j_1}\rho_{j_1,m_2}^{j_2} \otimes v_{j_2,k}^* + \cdots . \tag{4.2.2.24}$$

And this commutator acts trivially on other generators of the pro-algebra. To make this into a DG resolution, we need to include another differential $\widetilde{D}$: let $\epsilon^j_{i,n}$ be the linear function on $L^{<0}\mathfrak{g}$ corresponding to $\rho^j_{i,n}$ in the Koszul resolution. Then we define $\widetilde{D}$ by:

$$\widetilde{D}u^*_{i,n} = \sum_{j,m+k=n} \epsilon^j_{i,m} \otimes v^*_{j,k} + \frac{1}{2} \sum_{j_1,j_2,m_1+m_2+k=n} \epsilon^{j_1}_{i,m_1} \rho^{j_2}_{j_1,m_2} \otimes v^*_{j_2,k} + \cdots . \tag{4.2.2.25}$$

This differential is of course $\widetilde{G}_{\mathcal{O}}$ invariant, since $\epsilon^j_{i,n}$ transforms in the same way as $\rho^j_{i,n}$, and $u^*_{i,n}$ transforms in the same way as $w^*_{i,n}$. After introducing this new differential, $\{D, d_2\} = \{\widetilde{D}, d_1\}$ and the combination of the four differentials will be a differential, and the above indeed becomes a projective system of free resolutions. Now if we take endomorphism with $\mathbb{C}[V(\mathcal{O})]$, we obtain the space:

$$\mathrm{Sym}^\bullet\big(L^{<0}\mathfrak{g}[-1]\big) \otimes \mathbb{C}[V(\mathcal{O})] \otimes \mathrm{Sym}^\bullet(V(\mathcal{K})/V(\mathcal{O})[-2])\,, \tag{4.2.2.26}$$

and the only nonzero differential is that induced from $\widetilde{D}$. Examining the definition of $\widetilde{D}$, we find that the higher order terms all drop off, and the linear term maps $L^{<0}\mathfrak{g}$ to $\mathbb{C}[V(\mathcal{O})] \otimes \mathrm{Sym}^\bullet(V(\mathcal{K})/V(\mathcal{O})[-2])$, and is identified with the differential induced by the moment map.

Combining the above steps, we obtain the following:

THEOREM 4.2.16. *There is a quasi-isomorphism of DG-$\mathbb{C}^*$ modules:*

$$\mathrm{End}^{\mathbb{B}\mathbb{C}^*}(i_*(\mathcal{O}_{\overline{V(\mathcal{O})}})) \cong \left[\mathbb{C}[V(\mathcal{O})] \otimes \mathrm{Sym}^\bullet(V(\mathcal{K})/V(\mathcal{O})[-2]) \otimes \mathrm{Sym}^\bullet((\mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O}) \oplus (z\mathfrak{g}(\mathcal{O}))^*)\,[-1])\right]^G . \tag{4.2.2.27}$$

If we shift the loop grading of $V(\mathcal{K})$ by $q^{1/2}$, the cohomological degree of $V(\mathcal{K})/V(\mathcal{O})$ to 0*, and the cohomological degree of $\mathfrak{g}(\mathcal{K})/\mathfrak{g}(\mathcal{O})$ to -1, then the cohomology of this space coincides with $\mathcal{V}_{G,V}$ in equation* (4.1.2.7)*.*

PROOF. After the grading shift, $\mathrm{Sym}^\bullet(V(\mathcal{K})/V(\mathcal{O}))$ can be identified with $\mathbb{C}[V^*(\mathcal{O})]$. Since $\mathcal{V}_{G,V}$ restricts to taking the Lie group invariants, comparing equation (4.2.2.27) with (4.1.2.7), we conclude that $\mathcal{V}_{G,V} \cong \mathrm{End}^{\mathbb{B}\mathbb{C}^*}(i_*(\mathcal{O}_{\overline{V(\mathcal{O})}}))$.

□

Remark 4.2.17. *The character of the space in equation* (4.2.2.27) *is given by:*

$$\frac{1}{|W|}\oint_T \frac{\mathrm{d}s}{2\pi i s}\prod_{\alpha \text{ roots}}(1-s^\alpha)\left[(q)_\infty^{2\mathrm{rank}(G)}\frac{\prod_{\alpha \text{ roots}}(qs^\alpha;q)_\infty^2}{\prod_{\beta \text{ weights of } N\oplus N^*}(-q^{1/2}s^\beta,q)_\infty}\right]. \tag{4.2.2.28}$$

*This is the Schur index for the gauge theory with matter as stated in [**OY20**].*

4.2.3. The Insertion of Fundamental 't Hooft Lines in Pure $PSL(2)$. Using the technique developed in the previous sections, one can consider the space of local operators at the junction of two half-BPS Wilson-'t Hooft line operators. As stated in the introduction, these operators correspond to vector bundles on the reduced $\tilde{G}_{\mathcal{O}}$ orbits of $\mathcal{R}_{G,V}$. Among these, the 't Hooft line operators are certain line bundles on the $\tilde{G}_{\mathcal{O}}$ orbits, and are labelled by the dominant coweight of G. These Wilson-'t Hooft line operators are the perverse coherent sheaves appearing in the work of [**CW19**, **CWar**]. For a full dictionary of correspondences between line operators and coherent sheaves, see [**Kap06a**, **CW19**, **CWar**].

Given two line operators L_1 and L_2, the space of local operators at their adjunction is given by:

$$\mathrm{Ops}_{G,V}(L_1,L_2)=\mathrm{Hom}^{\mathbb{BC}^*}_{\mathrm{Coh}(\tilde{G}_{\mathcal{O}}\backslash\mathcal{R}_{G,V})}(L_1,L_2). \tag{4.2.3.1}$$

The space $\mathrm{Ops}_{G,V}(L_1,L_2)$ should give rise to a module of the Poisson vertex algebra $\mathrm{Ops}_{G,V}=\pi_0\mathcal{V}_{G,V}$. Indeed, since $\mathbb{1}*L_i\cong L_i$ for $i=1,2$, the Poisson algebra $\mathrm{Ops}_{G,V}$ acts on $\mathrm{Ops}_{G,V}(L_1,L_2)$ through convolution. By the work of [**But21**], this action is also compatible with the factorization structure. The structure of these spaces as $\mathrm{Ops}_{G,V}$ modules has not been carefully described in literature; however, the Euler character of these spaces $\chi_q\mathrm{Ops}_{G,V}(L_1,L_2)$ are computed in [**CGS16**].

We will look at the simplest non-trivial example: the space of local operators at the junction of fundamental 't Hooft lines and basic dyonic Wilson-'t Hooft lines in pure $PSL(2)$ ($PSU(2)$ in physics notation) theory. The fundamental t'Hooft line here is the structure sheaf of the miniscule orbit $\mathrm{Gr}_{1/2}\cong\mathbb{P}^1$, corresponding to the minimal dominant coweight $\frac{1}{2}$ of $PSL(2)$. We wiil not try to identify these as representations of $\mathrm{Ops}_{G,V}$, but only compute the vector spaces and their indices. We will match the indices with the indices of [**CGS16**].

We will keep using the notation $\overline{X}$ for the quotient stack $X/\tilde{G}_{\mathcal{O}}$. Consider now $L_1 = L_2 = \mathcal{O}_{\overline{\mathrm{Gr}_{1/2}}}$, we would like to compute:

$$\mathrm{Ops}_{G,V}(L_1, L_2) = \mathrm{End}^{\mathbb{B}\mathbb{C}^*}(\mathcal{O}_{\overline{\mathrm{Gr}_{1/2}}}) \tag{4.2.3.2}$$

Proposition 4.2.15 reduces the computation of Endomorphism algebra to computing the global sections of an associated vector bundle. In the computations below, we will drop all the quotients by $\tilde{G}_{\mathcal{O}}$ in order to simlify the notations, although all the discussions below is carried in the equivariant settings. Fix a fixed point $z^{1/2}$, let $\tilde{P}$ be the stabilizer. The module V is given by:

$$\mathfrak{g}(\mathcal{K})/\left(z^{1/2}\mathfrak{g}(\mathcal{O})z^{-1/2} + \mathfrak{g}(\mathcal{O})\right) \tag{4.2.3.3}$$

The computation then requires that we understand the associated vector bundle as a bundle over $\mathbb{P}^1$.

Since $z^{1/2}\mathfrak{g}(\mathcal{O})z^{-1/2} + \mathfrak{g}(\mathcal{O}) = H(\mathcal{O}) \oplus E(\mathcal{O}) \oplus z^{-1}F(\mathcal{O})$, the representation:

$$\mathfrak{g}(\mathcal{K})/\left(z^{1/2}\mathfrak{g}(\mathcal{O})z^{-1/2} + \mathfrak{g}(\mathcal{O})\right) \tag{4.2.3.4}$$

falls into the following exact sequence:

$$0 \to z^{-1}\mathfrak{b} \to \mathfrak{g}(\mathcal{K})/\left(z^{1/2}\mathfrak{g}(\mathcal{O})z^{-1/2} + \mathfrak{g}(\mathcal{O})\right) \to \mathfrak{g}(\mathcal{K})/z^{-1}\mathfrak{g}(\mathcal{O}) \to 0, \tag{4.2.3.5}$$

where $\mathfrak{b}$ is the Lie algebra of $B \subseteq G$. This short exact sequence split as a representation of B, and since $G(\mathcal{O})/\tilde{P} = G/B$, we have an isomorphism of vector bundles:

$$\tilde{G}_{\mathcal{O}} \times_{\tilde{P}} V \cong G \times_B z^{-1}\mathfrak{b} \bigoplus \mathcal{O}_{\mathrm{Gr}_{1/2}} \otimes \mathfrak{g}(\mathcal{K})/z^{-1}\mathfrak{g}(\mathcal{O}). \tag{4.2.3.6}$$

Taking exterior power on both sides, we have an isomorphism:

$$\tilde{G}_{\mathcal{O}} \times_{\tilde{P}} \mathrm{Sym}^\bullet(V[1]) \cong G \times_B \mathrm{Sym}^\bullet(z^{-1}\mathfrak{b}[-1]) \otimes \mathrm{Sym}^\bullet\!\left(\mathfrak{g}(\mathcal{K})/z^{-1}\mathfrak{g}(\mathcal{O})[-1]\right). \tag{4.2.3.7}$$

The global section of $G\times_B \mathrm{Sym}^\bullet(z^{-1}\mathfrak{b}[-1])$ can be computed easily:

$$
\begin{aligned}
&H^*(G\times_B \mathrm{Sym}^0(z^{-1}\mathfrak{b}[-1]))=\mathbb{C}[0],\\
&q^{-1}H^*(G\times_B \mathrm{Sym}^1(z^{-1}\mathfrak{b}[-1]))=\mathbb{C}[-1]\oplus\mathfrak{g}[-1],\\
&q^{-2}H^*(G\times_B \mathrm{Sym}^2(z^{-1}\mathfrak{b}[-1]))=\mathfrak{g}[-2].
\end{aligned}
\tag{4.2.3.8}
$$

The index of $H^*(G\times_B \mathrm{Sym}^\bullet(z^{-1}\mathfrak{b}[-1]))$ is equal to $(1-q)(1-q-qs^2-qs^{-2})$. The index of $\mathrm{Sym}^\bullet\big(\mathfrak{g}(\mathcal{K})/z^{-1}\mathfrak{g}(\mathcal{O})[-1]\big)$ is given by:

$$
\frac{(q)_\infty(q:qs^2)_\infty(q:qs^{-2})_\infty}{(1-q)(1-qs^2)(1-qs^{-2})},
\tag{4.2.3.9}
$$

so the index of $\mathrm{End}^{\mathrm{Rep}(\tilde{G}_\mathcal{O})}(\mathcal{O}_{\overline{\mathrm{Gr}_{1/2}}})$ is given by:

$$
\frac{1-q-qs^2-qs^{-2}}{(1-qs^2)(1-qs^{-2})}(q)_\infty(q:qs^2)_\infty(q:qs^{-2})_\infty.
\tag{4.2.3.10}
$$

In conclusion:

THEOREM 4.2.1. *The space* $\mathrm{End}^{\mathbb{BC}^*}(\mathcal{O}_{\overline{\mathrm{Gr}_{1/2}}})$ *is quasi-isomorphic to:*

$$
\Big[\big(\mathbb{C}\oplus q\mathbb{C}[-1]\oplus q\mathfrak{g}[-1]\oplus q^2\mathfrak{g}[-2]\big)\otimes \mathrm{Sym}^\bullet\Big(\big(\mathfrak{g}(\mathcal{K})/z^{-1}\mathfrak{g}(\mathcal{O})\oplus(z\mathfrak{g}(\mathcal{O}))^*\big)[-1]\Big)\Big]^G.
\tag{4.2.3.11}
$$

The character of this space is:

$$
\frac{1}{2}(q)_\infty^2\oint_s \mathrm{d}s(1-s^2)(1-s^{-2})\frac{(1-q-qs^2-qs^{-2})}{(1-qs^2)(1-qs^{-2})}(q:qs^2)_\infty^2(q:qs^{-2})_\infty^2.
\tag{4.2.3.12}
$$

We expect (4.2.3.11) to be the space of local operators supported at a point on a single straight 't Hooft line. The resulting character does not match the index obtained in [**CGS16**]. The computation that matches their result is the following: consider now $L_1=\mathcal{O}_{\overline{\mathrm{Gr}_{1/2}}}$ the fundamental 't Hooft line, and $L_2=\Omega_{\overline{\mathrm{Gr}_{1/2}}}$ a dyonic Wilson-'t Hooft line, where $\Omega_{\overline{\mathrm{Gr}_{1/2}}}$ is the dualizing sheaf of $\mathrm{Gr}_{1/2}$. Physically, $\Omega_{\overline{\mathrm{Gr}_{1/2}}}$ corresponds to the dyonic Wilson-'t Hooft line with fundamental magnetic charge and $+1$ electric charge. Consider the junction:

$$
\mathrm{Ops}_{G,V}(L_1,L_2)=\mathrm{Hom}^{\mathbb{BC}^*}(\mathcal{O}_{\overline{\mathrm{Gr}_{1/2}}},\Omega_{\overline{\mathrm{Gr}_{1/2}}}),
\tag{4.2.3.13}
$$

In this case, Proposition 4.2.15 still applies, with the associated bundle of V twisted by the canonical sheaf of $\mathrm{Gr}_{1/2} = G/B$. The cohomology of the twisted $G \times_B z^{-1}\mathfrak{b} \otimes O(-2)$ is:

$$
\begin{aligned}
&H^*(G \times_B \mathrm{Sym}^0(z^{-1}\mathfrak{b} \otimes O(-2)[-1])) = \mathbb{C}[-1],\\
&q^{-1}H^*(G \times_B \mathrm{Sym}^1(z^{-1}\mathfrak{b} \otimes O(-2)[-1])) = \mathbb{C}[-1] \oplus \mathbb{C}[-2],\\
&q^{-2}H^*(G \times_B \mathrm{Sym}^2(z^{-1}\mathfrak{b} \otimes O(-2)[-1])) = \mathbb{C}[-2].
\end{aligned}
\tag{4.2.3.14}
$$

This space has index $-(1-q)(1+q)$. The contribution from $\mathfrak{g}(\mathcal{K})/z^{-1}\mathfrak{g}(\mathcal{O})$ remains the same. Hence the space $\mathrm{Hom}^{\mathbb{BC}^*}(\mathcal{O}_{\overline{\mathrm{Gr}_{1/2}}}, \Omega_{\overline{\mathrm{Gr}_{1/2}}})$ has index:

$$
-\frac{1}{2}(q)_\infty^2 \oint_s \mathrm{d}s(1-s^2)(1-s^{-2})\frac{(1+q)}{(1-qs^2)(1-qs^{-2})}(q:qs^2)_\infty^2(q:qs^{-2})_\infty^2,
\tag{4.2.3.15}
$$

Shifting by $q^{1/2}$, this exactly matches the formula in [**CGS16**]. In this paper, the authors are implicitly using a Serre functor to rotate the line operators, or in other words, they took the dual of the line operators in the monoidal category. This subtle operation was described explicitly in [**CW19**], and an important feature is that the left dual of a line operator is **not** necessarily equivalent to the right dual, unless the theory is superconformal, cf. the end of Section 4.1.2. As explained in [**CW19**], the difference between left dual and right dual is due to the fact that the dualizing sheaf Ω of $\tilde{G}_{\mathcal{O}} \setminus \mathrm{Gr}_G$ is not isomorphic to its involution $s^*\Omega$, where the involution $s : \tilde{G}_{\mathcal{O}} \setminus \mathrm{Gr}_G \to \tilde{G}_{\mathcal{O}} \setminus \mathrm{Gr}_G$ is defined by $s([g]) = [g^{-1}]$. Thus, after rotating by 2π, a line operator receives contribution from the dualizing sheaf of Gr_G. In the case of the structure sheaf of a miniscule orbit, this is simply the dualizing sheaf of the orbit. This explains the presence of $\Omega_{\overline{\mathrm{Gr}_{1/2}}}$ in the above formula.

APPENDIX A

Free Field Vertex Algebras and BRST Cohomology

In this appendix we introduce the basic vertex algebras and their modules that appear in this work. A free field algebra is a vertex algebra that is strongly generated by fields that have the property that only the identity appears in their operator product algebra. There are four classes of free field algebras that admit a Virasoro structure (stress tensor): the free boson/Heisenberg VOA, the free fermion, the symplectic fermions, and the symplectic bosons. These algebras are related in various way that we now recall.

A.0.1. Heisenberg VOA's and Fock modules. The basic Heisenberg VOA H_J is generated by a single even (bosonic) field $J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}$ with OPE

$$J(z)J(w) \sim \frac{1}{(z-w)^2}\,. \tag{A.0.1.1}$$

Its simple modules are Fock modules $\mathcal{F}_\lambda$ of highest weight $\lambda \in \mathbb{C}$. These are generated by a highest-weight vector $|\,\lambda\,\rangle$ on which J_0 acts by multiplication with λ, the J_n for positive n annihilate $|\,\lambda\,\rangle$, and the negative modes act freely. We will denote vector space tensor products by the usual symbol $\otimes$; while the fusion product, *i.e.* the tensor product as modules of a VOA $\mathcal{V}$, will be denoted by the symbol $\times_{\mathcal{V}}$. The Fock modules of the Heisenberg VOA satisfy the fusion rules

$$\mathcal{F}_\lambda \times_{H_J} \mathcal{F}_\mu = \mathcal{F}_{\lambda+\mu}\,. \tag{A.0.1.2}$$

The vertex operator associated to the highest-weight vector $|\,\lambda\,\rangle$ is denoted by $Y(|\,\lambda\,\rangle, z)$, and the fusion rules of the Fock modules are reflected in the following OPE of intertwining operators:

$$Y(|\,\lambda\,\rangle, z)Y(|\,\mu\,\rangle, w) \sim (z-w)^{\lambda\mu}Y(|\,\lambda+\mu\,\rangle, w) + \dots \tag{A.0.1.3}$$

More generally, let V be a finite-dimensional complex vector space, say of dimension n, with symmetric bilinear form $B : V \times V \to \mathbb{C}$, and fix a basis $\{v^1, \dots, v^n\}$ of V. Then the Heisenberg

VOA associated to (V,B), which we denote compactly as $H_{\{v^i\}}$, is strongly and freely generated by fields J^i for $i=1,\ldots,n$ with OPE

$$H_{\{v^i\}}: \quad J^i(z)J^j(w) \sim \frac{B(v^i,v^j)}{(z-w)^2}\,. \tag{A.0.1.4}$$

We also introduce formal fields $v^i(z)$, obeying $J^i(z)=\partial v^i(z)$, with a non-analytic OPE

$$v^i(z)v^j(w) \sim B(v^i,v^j)\log(z-w)\,, \tag{A.0.1.5}$$

which implies (A.0.1.4). The $v^i(z)$ themselves are not part of the Heisenberg VOA, but they provide a useful way to describe modules.

Fock modules (*a.k.a.* Verma modules) for the generalized Heisenberg VOA $H_{\{v^i\}}$ are in one-to-one correspondence with linear maps $V\to\mathbb{C}$. In all applications in this paper, the bilinear form B will be non-degenerate, and can thus be used to establish an isomorphism between linear maps $V\to\mathbb{C}$ and elements of V itself. We will mainly use the latter to describe Fock modules.

Given an element $\lambda\in V$, with associated map $B(\lambda,-):V\to\mathbb{C}$, there is a unique Fock module denoted

$$\mathcal{F}_\lambda \qquad \text{or (for clarity)} \quad \mathcal{F}_\lambda^{v^1,\ldots,v^n}\,, \tag{A.0.1.6}$$

generated by a highest-weight state $|\lambda\rangle$ that satisfies

$$J_0^i|\lambda\rangle = B(\lambda,v^i)|\lambda\rangle\,, \qquad J^i_{n>0}|\lambda\rangle = 0\,, \tag{A.0.1.7}$$

and on which the $J^i_{n<0}$ act freely. The vacuum module of the Heisenberg VOA is simply $\mathcal{F}_0$. If we expand $\lambda=\sum_i\lambda_i v^i$ and correspondingly set $\lambda(z):=\sum_i\lambda_i v^i(z)$, then we can formally express the vertex operator corresponding to the highest-weight state $|\lambda\rangle$ as

$$Y(|\,\lambda\,\rangle,z) \;=\; :e^{\lambda(z)}:. \tag{A.0.1.8}$$

The OPE between vertex operators follows from (A.0.1.5). In particular,

$$:e^{\lambda(z)}::e^{\eta(w)}: \;\sim\; (z-w)^{B(\lambda,\eta)}:e^{\lambda(w)+\eta(w)}:+\ldots\,. \tag{A.0.1.9}$$

Thus the fusion rules are $\mathcal{F}_\lambda \times_{H_{v^i}} \mathcal{F}_\eta = \mathcal{F}_{\lambda+\eta}$.

We finally note that, fixing an orthogonal basis $v^1, ..., v^n$ of V, there is a decomposition

$$H_{v^1,...,v^n} \cong \bigotimes_i H_{v^i} \,. \tag{A.0.1.10}$$

Correspondingly, letting $\lambda = \sum_i \lambda_i v^i$, there is a decomposition of Fock modules

$$\mathcal{F}_\lambda^{v^1,...,v^n} \cong \bigotimes_i \mathcal{F}^{v^i}_{\lambda_i v^i} \,. \tag{A.0.1.11}$$

A.0.2. Lattice VOA and Free fermions. Let us choose the basis $v^1, ..., v^n$ such that $B(v^i, v^j)$ is real for all pairs (v^i, v^j), and consider the subcategory of those Fock modules $\mathcal{F}_\lambda$ that have the property that all λ_i are real. This category is a braided tensor category [**CKLR19**]. Let $L \subset V$ be a lattice, meaning a $\mathbb{Z}$-submodule of V, with the property that B restricted to L is integral. Then

$$\mathcal{V}_L := \bigoplus_{\lambda \in L} \mathcal{F}_\lambda \tag{A.0.2.1}$$

is itself a vertex superalgebra, the lattice VOA of the lattice L; and if L is even, then it is actually a vertex algebra.

In particular, in rank one with $V = \mathbb{C}\langle v \rangle$ generated by a vector with $B(v, v) = 1$, choosing $L = \mathbb{Z}\langle v \rangle \simeq \mathbb{Z}$ gives rise to a vertex algebra

$$\mathcal{V}_\mathbb{Z} \simeq \mathcal{V}_{bc} \tag{A.0.2.2}$$

strongly generated by a pair of free fermions. This is the classic bose-fermi correspondence. The two fermionic generators may be chosen as

$$b(z) = Y(|1\rangle, z) = \,:e^{v(z)}:\,, \qquad c(z) = Y(|-1\rangle, z) = \,:e^{-v(z)}:\,, \tag{A.0.2.3}$$

with OPE

$$b(z)c(w) \sim \frac{1}{(z-w)} \,. \tag{A.0.2.4}$$

The free fermions $\mathcal{V}_{bc}$ are a *holomorphic* VOA, in the sense that the VOA itself is the only simple module and every module is completely reducible. Thus the module category of $\mathcal{V}_{bc}$ is isomorphic to the "trivial" category of vector spaces. (The terminology "holomorphic VOA," sometimes also called a "holomorphic CFT," originates from the fact that all spaces of conformal blocks are automatically one-dimensional, implying that the VOA itself has well-defined partition functions in any genus, and carries the structure of a full CFT. For example, the free-fermion VOA $\mathcal{V}_{bc}$ is equivalent to the well-defined physical CFT containing a free complex-valued 2d chiral fermion.) In general, when $L \subset V$ is a full-rank complete self-dual lattice, then by [**DLM97**], the VOA $\mathcal{V}_L$ is holomorphic, or it has trivial category of modules.

A.0.3. Symplectic fermions and the singlet VOA. The symplectic fermion vertex algebra $\mathcal{V}_{SF}$ may be defined as the vertex algebra strongly generated by two fermionic fields $\chi^{\pm}(z)$ with OPE

$$\mathcal{V}_{SF}: \quad \chi^+(z)\chi^-(w) \sim \frac{1}{(z-w)^2} \tag{A.0.3.1}$$

More generally, given any complex vector space W with a non-degenerate anti-symmetric bilinear form $\Omega : W \times W \to \mathbb{C}$ (a symplectic form), there is an asociated symplectic fermion VOA $\mathcal{V}_{SF}^W$. Given a basis $\{\chi^i\}$ for W, the VOA is generated by fermionic fields $\{\chi^i(z)\}$ with

$$\mathcal{V}_{SF}^W: \quad \chi^i(z)\chi^j(w) \sim \frac{\Omega(\chi^i,\chi^j)}{(z-w)^2}\,. \tag{A.0.3.2}$$

Symplectic fermions can be embedded in free fermions (and thus in a lattice VOA), as the kernel of certain screening charges. Define the screening operator $S(z) : \mathcal{V}_{bc} \to \mathcal{V}_{bc}(\!(z)\!)$ by

$$S(z) := b(z)\,, \tag{A.0.3.3}$$

and the "screening charge" $S_0 : \mathcal{V}_{bc} \to \mathcal{V}_{bc}$ by

$$S_0 := \frac{1}{2\pi i}\oint S(z)dz = b_0\,. \tag{A.0.3.4}$$

In this case, this is just the zero-mode of the fermionic field $b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1}$. The kernel of S_0 is simply the subalgebra generated by $b(z)$ and $\partial c(z)$, since b_0 commutes with all modes of $c(z)$ except its zero-mode. Letting $\chi^+(z) = b(z)$ and $\chi^-(z) = \partial c(z)$, we find that $\chi^{\pm}$ satisfy the symplectic fermion OPE (A.0.3.1). Thus

$$\mathcal{V}_{SF} \cong \ker(S_0 : \mathcal{V}_{bc} \to \mathcal{V}_{bc})\,. \tag{A.0.3.5}$$

There is an action of $\mathbb{C}^*$ on both free fermions and symplectic fermions, such that b, χ^+ have weight $+1$ and c, χ^- have weight -1. This makes $\mathcal{V}_{bc}$ and $\mathcal{V}_{SF}$ $\mathbb{Z}$-graded vertex algebras. Decomposing symplectic fermions (as a vector space) into graded components

$$\mathcal{V}_{SF} = \bigoplus_{\mu\in\mathbb{Z}} M_\mu\,, \tag{A.0.3.6}$$

we find that the degree-zero subspace $M := M_0$ (*a.k.a.* the $\mathbb{C}^*$ orbifold of $\mathcal{V}_{SF}$) is a vertex algebra itself, while the other components M_n are simple modules for M. Conversely, $\mathcal{V}_{SF}$ is an extension of M by the modules $\{M_\mu\}_{\mu\in\mathbb{Z}}$.

The vertex algebra $M = M_0$ is known as the $p = 2$ singlet algebra. It contains the fields with equal numbers of χ^+ and χ^-, such as $:\!\chi^+\chi^-\!:$, $:\!\chi^+\partial^a\chi^-\!:$, $::\!\chi^+\partial^a\chi^+\chi^-\partial^b\chi^-$, etc. The modules M_μ that appear in the decomposition of symplectic fermions are simple currents with fusion rules

$$M_\mu \times_M M_\nu = M_{\mu+\nu}\,. \tag{A.0.3.7}$$

The singlet algebra has many other modules, however; its full representation theory is rather complicated and has only been completely understood in the past year [**CMY21**, **CMY23**].

Later in the paper we will encounter multiple copies of symplectic fermions and singlet algebras/modules. We summarize some notation and relations. By combining the relation to free fermions (A.0.3.5) with the bose-fermi correspondence (A.0.2.2), we find that n symplectic fermions are embedded in a rank-n lattice VOA

$$\mathcal{V}_{SF}^{\otimes n} \hookrightarrow \mathcal{V}_{\mathbb{Z}^n}\,, \qquad \chi_+^i \mapsto\, :\!e^{v^i}\!:\,, \quad \chi_-^i \mapsto -\!:\!\partial v^i\, e^{-v^i}\!:\,, \tag{A.0.3.8}$$

where $\mathcal{V}_{\mathbb{Z}^n}$ is the extension of the Heisenberg algebra $H_{v^1,\dots,v^n}$ with $B(v^i,v^j) = \delta^{ij}$ by Fock modules $\mathcal{F}_{\mu\cdot v}$ for all $\mu \in \mathbb{Z}^n$. The embedding is the kernel of screening operators

$$\mathcal{V}_{SF}^{\otimes n} = \bigcap_{i=1}^n \ker S_0^i\big|_{\mathcal{V}_{\mathbb{Z}^n}}\,, \qquad S^i(z) = \frac{1}{2\pi i}\oint :\!e^{v^i(z)}\!:. \tag{A.0.3.9}$$

Furthermore, there is a $(\mathbb{C}^*)^n$ action on $\mathcal{V}_{SF}^{\otimes n}$, induced from the $(\mathbb{C}^*)^n$ action on $\mathcal{V}_{\mathbb{Z}^n}$ under which $:e^{\mu\cdot v}:$ has charge $\mu \in \mathbb{Z}^n$. (And so in particular $\chi^i_\pm$ have charges $(0,...,0,\underset{i}{\pm 1},0,...,0)$.) We denote the weight spaces of this action $M^{\{v^i\}}_{\mu\cdot v}$ or simply $M_{\mu\cdot v}$, with

(A.0.3.10) $$\mathcal{V}_{SF}^{\otimes n} = \bigoplus_{\mu\in\mathbb{Z}^n} M_\mu\,, \qquad M_{\mu\cdot v} = \bigcap_{i=1}^n \ker S_0^i\big|_{\mathcal{F}_{\mu\cdot v}}\,.$$

Here $M_0^{\{v^i\}} \cong \bigoplus_{i=1}^n M_0^{v^i}$ is n copies of the singlet VOA, and each of the $M_{\mu\cdot v}$'s are simple currents thereof.

A.0.4. Symplectic bosons. The basic symplectic boson VOA $\mathcal{V}_{\beta\gamma}$, *a.k.a.* a beta-gamma system, is strongly generated by two bosonic fields $\beta(z)$, $\gamma(z)$ with OPE

(A.0.4.1) $$\mathcal{V}_{\beta\gamma}: \quad \beta(z)\gamma(w) \sim \frac{-1}{z-w}\,.$$

More generally, given a symplectic vector space (W,Ω), there is an associated symplectic boson VOA $\mathcal{V}^W_{\beta\gamma}$. Given a basis $\{\beta^i\}$ for W, the VOA is generated by fields $\{\beta^i(z)\}$ with OPE

(A.0.4.2) $$\mathcal{V}_{\beta\gamma[W]}: \quad \beta^i(z)\beta^j(w) \sim \frac{\Omega(\beta^i,\beta^j)}{z-w}\,.$$

Symplectic bosons are closely related to the other free field VOA's above, in several interesting ways.

To begin, let $V = \mathbb{C}\langle\eta\rangle$ be the one-dimensional vector space with negative inner product $B(\eta,\eta) = -1$, and let $\sqrt{-1}\mathbb{Z} := \mathbb{Z}\langle\eta\rangle$ denote the integer lattice therein. Correspondingly, we have a Heisenberg VOA $\mathcal{H}_\eta$ and its lattice extension

(A.0.4.3) $$\mathcal{V}_{\sqrt{-1}\mathbb{Z}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}_n\,,$$

where each Fock module $\mathcal{F}_n$ is generated by $:e^{n\eta}:$. Now consider the $\mathbb{C}^*$ action on $\mathcal{V}_{\sqrt{-1}\mathbb{Z}}$ under which the subspace $\mathcal{F}_n$ has weight $-n$. Also recall the $\mathbb{C}^*$ action on symplectic fermions in (A.0.3.6). The invariant part of the tensor-product VOA $\mathcal{V}_{SF}\otimes\mathcal{V}_{\sqrt{-1}\mathbb{Z}}$ under the diagonal $\mathbb{C}^*$ action is

(A.0.4.4) $$(\mathcal{V}_{SF}\otimes\mathcal{V}_{\sqrt{-1}\mathbb{Z}})^{\mathbb{C}^*} = \bigoplus_{n\in\mathbb{Z}} M_n\otimes\mathcal{F}_n\,,$$

and it is generated by fields

$$\beta = \chi^{+} \otimes :e^{\eta}: \quad \text{and} \quad \gamma = -\chi^{-} \otimes :e^{-\eta}: \tag{A.0.4.5}$$

precisely satisfying the symplectic boson OPE A.0.4.1. Thus

$$(\mathcal{V}_{SF} \otimes \mathcal{V}_{\sqrt{-1}\mathbb{Z}})^{\mathbb{C}^*} \simeq \mathcal{V}_{\beta\gamma}\,. \tag{A.0.4.6}$$

Alternatively, by combining the embedding (A.0.3.5) of symplectic fermions in a lattice VOA with (A.0.4.6), we obtain a well-known free field realization of symplectic bosons, *cf.* [**AW22**]. Let $H_{\phi,\eta}$ be the rank-two Heisenberg algebra corresponding to a two-dimensional vector space with basis $\{\phi, \eta\}$ and inner product $B(\phi,\phi) = 1$, $B(\eta,\eta) = -1$, $B(\phi,\eta) = 0$. Consider the one-dimensional lattice $L = \mathbb{Z}\langle \phi + \eta \rangle$ and the corresponding lattice VOA

$$\mathcal{V}_L = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{n(\phi+\eta)}\,. \tag{A.0.4.7}$$

There is an embedding $\mathcal{V}_{\beta\gamma} \hookrightarrow \mathcal{V}_L$ given by

$$\beta \;\mapsto\; :e^{\phi+\eta}:\,, \quad \gamma \;\mapsto\; :\partial\phi\, e^{-\phi-\eta}:\,. \tag{A.0.4.8}$$

To characterize this as the kernel of a screening charge, we note that the lattice VOA $\mathcal{V}_L$ has modules $\mathcal{V}_{L,k\phi}$ defined by lifting the Fock modules $\mathcal{F}_{k\phi}$ of the Heisenberg VOA to the lattice $\mathcal{V}_L$. Explicitly,

$$\mathcal{V}_{L,k\phi} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{(k+n)\phi + n\eta}\,. \tag{A.0.4.9}$$

Consider the intertwining operator $S(z) : \mathcal{V}_{L,k\phi} \to \mathcal{V}_{L,(k+1)\phi}(\!(z)\!)$ defined by

$$S(z) \;=\; :e^{\phi(z)}: \tag{A.0.4.10}$$

and corresponding screening charges $S_0 = \frac{1}{2\pi i} \oint S(z)\, dz : \mathcal{V}_{L,k\phi} \to \mathcal{V}_{L,(k+1)\phi}$. Then the embedding of symplectic bosons in $\mathcal{V}_L$ coincides with the kernel

$$\mathcal{V}_{\beta\gamma} \;=\; \ker(S_0 : \mathcal{V}_L \to \mathcal{V}_{L,\phi})\,. \tag{A.0.4.11}$$

This follows from decomposing $\mathcal{F}_{n(\phi+\eta)} = \mathcal{F}_{n\phi} \otimes \mathcal{F}_{n\eta}$ in (A.0.4.7) as modules for two Heisenberg VOA's $H_\phi \otimes H_\eta$, and combining the decomposition with (A.0.3.5) and (A.0.4.4).

A.0.5. BRST cohomology. Conversely, we may go back from symplectic bosons to symplectic fermions using BRST cohomology. By BRST cohomology we mean relative semiinfinite Lie algebra cohomology as in [**FGZ86**]; and the version that we need here is exactly the one used in [**CGNS22**].

Let $\mathcal{V}$ be a vertex algebra with an internal $\mathbb{C}^\times$ Kac-Moody action at level zero. In other words, $\mathcal{V}$ contains a field $J(z)$ (the Kac-Moody current) that has non-singular OPE with itself, and generates an action of the loop group $\mathbb{C}(\!(z)\!)^\times$ on $\mathcal{V}$ by sending:

$$\text{(A.0.5.1)} \qquad \text{for } \alpha(z) \in \mathbb{C}(\!(z)\!), \quad v \mapsto \frac{1}{2\pi i} \oint \alpha(z) J(z) \cdot v\,.$$

Intuitively, BRST cohomology takes the symplectic quotient of $\mathcal{V}$ by the $\mathbb{C}(\!(z)\!)^\times$ action — setting the current $J(z)$ to zero and taking $\mathbb{C}(\!(z)\!)^\times$ invariants — in a derived way, *i.e.* by expressing this quotient as the zeroth cohomology of a complex. Concretely, let $\mathcal{V}_{bc}$ be a free-fermion VOA, Z-graded such that c has degree 1 and b has degree -1. The tensor product $\mathcal{V} \otimes \mathcal{V}_{bc}$ inherits this grading, and has a differential

$$\text{(A.0.5.2)} \qquad Q_{BRST} := \frac{1}{2\pi i} \oint c(z)\, J(z)\, dz$$

of degree $+1$. Let the *relative complex* $(\mathcal{V} \otimes \mathcal{V}_{bc})^{\text{rel}}$ be the subspace of $\mathcal{V} \otimes \mathcal{V}_{bc}$ annihilated by the zero-modes b_0 and J_0. Then one defines BRST cohomology as the Q_{BRST}-cohomology of the relative complex, denoted

$$\text{(A.0.5.3)} \qquad H_{BRST}(\mathcal{V}) := H^\bullet\big((\mathcal{V} \otimes \mathcal{V}_{bc})^{\text{rel}}, Q_{\text{BRST}}\big)\,.$$

Note that Q_{BRST} sends $b(z) \mapsto J(z)$ (whence $J(z)$ is effectively set to zero in cohomology); it also sends any element of $\mathcal{V}$ to its image under the $GL\big(1, \mathbb{C}(\!(z)\!)\big)$ action with generator $c(z)$ (whence cohomology also takes invariants for the action).

Similarly, if M is any module for $\mathcal{V}$, its BRST cohomology is defined by

$$\text{(A.0.5.4)} \qquad H_{BRST}(M) := H^\bullet\big((M \otimes \mathcal{V}_{bc})^{\text{rel}}, Q_{BRST}\big)\,,$$

where again $(M \otimes \mathcal{V}_{bc})^{\text{rel}}$ denotes the subspace of $M \otimes \mathcal{V}_{bc}$ annihilated by b_0 and J_0.

Going from symplectic bosons to symplectic fermions uses a particularly well-behaved instance of BRST cohomology. Let $H_{\phi,\eta}$ be the rank-two Heisenberg algebra associated to vectors ϕ, η of norms $+1, -1$, respectively. The field $J(z) = \partial(\phi + \eta)$ is a level-zero $\mathbb{C}^*$ Kac-Moody current. Consider the Fock module $\mathcal{F}_{\lambda\phi+\mu\eta}$ (here $\lambda, \eta \in \mathbb{C}$). Its BRST cohomology has the property that

$$H^i_{BRST}(\mathcal{F}_{\lambda\phi+\eta\mu}) = \delta_{i,0}\delta_{\lambda-\mu,0}\mathbb{C}[|\lambda\phi + \mu\eta\rangle]\,. \tag{A.0.5.5}$$

In other words, the cohomology vanishes unless $\lambda - \mu = 0$, in which case the cohomology is one-dimensional and given by the class of the highest-weight vector.

Now, the symplectic boson VOA $\mathcal{V}_{\beta\gamma}$ has a current $J_{\beta\gamma} = -{:}\beta\gamma{:}$ at level -1 ($J_{\beta\gamma}$ generates a Heisenberg subalgebra H_η with inner product -1). The free fermion VOA $\mathcal{V}_{bc}$ has a current $J_{bc} = {:}bc{:}$ at level 1 (generating a Heisenberg subalgebra H_ϕ with norm $+1$). Therefore, their tensor product $\mathcal{V}_{bc} \otimes \mathcal{V}_{\beta\gamma}$ has a diagonal current $J = J_{\beta\gamma} + J_{bc}$ at level zero. Using the bose-fermi correspondence and (A.0.4.4), we know that as modules for $H_{\phi,\eta} = H_\phi \otimes H_\eta$ we have

$$\mathcal{V}_{bc} \otimes \mathcal{V}_{\beta\gamma} = \bigoplus_{m,n\in\mathbb{Z}} \mathcal{F}^\phi_{n\phi} \otimes \mathcal{F}^\eta_{m\eta} \otimes M_m\,. \tag{A.0.5.6}$$

Taking BRST cohomology for the diagonal $\mathbb{C}^*$ action, we get

$$\begin{aligned} H_{\text{BRST}}(\mathcal{V}_{bc} \otimes \mathcal{V}_{\beta\gamma}) &= \bigoplus_{n,m\in\mathbb{Z}} H_{\text{BRST}}\left(\mathcal{F}^\phi_{n\phi} \otimes \mathcal{F}^\eta_{m\eta}\right) \otimes M_m \\ &= \bigoplus_{n,m\in\mathbb{Z}} \delta_{n-m,0}\mathbb{C} \otimes M_m \\ &= \bigoplus_{m\in\mathbb{Z}} M_m \\ &= \mathcal{V}_{SF}. \end{aligned} \tag{A.0.5.7}$$

A.0.6. Affine $\mathfrak{gl}(1|1)$. There is in fact one more player that we can introduce, the affine VOA of $\mathfrak{gl}(1|1)$. It is the diagonal $\mathbb{C}^*$ orbifold of $\mathcal{V}_{bc} \otimes \mathcal{V}_{\beta\gamma}$, that is

$$V(\mathfrak{gl}(1|1)) \cong (\mathcal{V}_{bc} \otimes \mathcal{V}_{\beta\gamma})^{\mathbb{C}^*} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}^\phi_{n\phi} \otimes \mathcal{F}^\eta_{n\eta} \otimes M_n \tag{A.0.6.1}$$

Here the level k can be any non-zero number and in particular it can always be set to one. It is generated by the fields $N = :bc:, \psi^+ = :\beta b:, \psi^- = -:\gamma c:$ and $E = :bc: - :\beta\gamma:$.

All the relations between the different free field algebras above have been explored in detail in [**CR09**, **CR13a**]. The representation theory of $\mathcal{V}_{\beta\gamma}$ is worked out in [**AW22**]; the one for $\widehat{\mathfrak{gl}(1|1)}$ in [**CMY22c**], and the one for the singlet in [**CMY21**, **CMY23**]. All the module categories are non-finite and non-semisimple ribbon (super)categories. Orbifolds, simple current extensions, and BRST cohomologies provide nice functors between representation categories, as explained in [**CGNS22**, **CMY22b**].

A.0.7. Spectral flow. In this final section on VOAs, we review the idea of spectral-flow automorphisms, and spectral-flow modules of a VOA, which will play a central role in many of our constructions. Spectral flow is associated with abelian Kac-Moody symmetries of VOA's; loosely speaking, it mixes the Kac-Moody symmetry with conformal symmetry. The basic idea appeared in physics in the 80's, in particular in the context of of worldsheet superstring theory. In the mathematical theory of VOA's, spectral flow is implemented by Haisheng Li's Δ-operator [**Li95**].

Let $\mathcal{V}$ be a VOA that has a Heisenberg subVOA, say of rank n with associated bilinear form B as in (A.0.1.4). (The Heisenberg subVOA is another name for an abelian current algebra, or an abelian Kac-Moody symemtry.) The Heisenberg VOA has a huge group of automorphisms. In particular, for any vector $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{C}^n$, there is a spectral automorphism σ^ℓ that acts on the modes of the Heisenberg VOA as

$$\sigma^\ell(J^i_n) = J^i_n + \delta_{n,0}\ell_i \,, \tag{A.0.7.1}$$

doing nothing but shifting the zero-modes by a scalar. Not all of these automorphisms lift from the Heisenberg subVOA to an automorphism of the mode algebra of the full VOA; this depends on the way that $\mathcal{V}$ is graded by the J^i_0. The automorphisms that do lift are called spectral flows automorphisms of $\mathcal{V}$.

For example, if $\mathcal{V}$ is a lattice VOA $\mathcal{V}_L$, there are spectral-flow automorphism so long as ℓ lies in the lattice L' that is dual to L. If $\mathcal{V}$ is an affine VOA, then spectral-flow automorphisms correspond to coweights. The example of $\mathfrak{sl}(2)$ is instructive; it is discussed in detail in section 2 of [**CKLR19**].

If σ^ℓ is an automorphism of the algebra of modes of $\mathcal{V}$, then to any $\mathcal{V}$-module M one defines the *spectral-flow module* $\sigma^\ell(M)$ as follows. The underlying vector space of $\sigma^\ell(M)$ is isomorphic to M, the isomorphism mapping $m \in M$ to $\sigma^\ell(m) \in \sigma^\ell(M)$; but an element x of the mode algebra of $\mathcal{V}$ acts as

$$x \cdot \sigma^\ell(m) = \sigma^\ell(\sigma^{-\ell}(x) \cdot m)\,. \tag{A.0.7.2}$$

Haisheng Li's Δ-operator [**Li95**], implementing spectral flow in a mathematical context, has several nice properties. By Proposition 2.11 of [**Li95**] together with skew-symmetry of intertwining operators, spectral flow respects fusion:

$$\sigma^\ell(M) \times_{\mathcal{V}} \sigma^{\ell'}(M') \cong \sigma^{\ell+\ell'}(M \times_{\mathcal{V}} M'). \tag{A.0.7.3}$$

In particular, the spectral flow image of the VOA itself, $\sigma^\ell(\mathcal{V})$ is always a simple current, with fusion rules

$$\sigma^\ell(\mathcal{V}) \times_{\mathcal{V}} \sigma^{\ell'}(\mathcal{V}) \cong \sigma^{\ell+\ell'}(\mathcal{V}), \qquad \sigma^\ell(\mathcal{V}) \times_{\mathcal{V}} M \cong \sigma^\ell(M). \tag{A.0.7.4}$$

Another property is that spectral flow is exact, i.e., it maps simple to simples, non-split short exact sequences to non-split short-exact sequences, Loewy diagrams to Loewy diagrmas and so on — see Proposition 2.5 of [**CKLR19**].

EXAMPLE A.0.1. *When $\mathcal{V}$ itself is a Heisenberg VOA, spectral flow acts on its own Fock modules. One has that $\sigma^{-\lambda}(\mathcal{F}_\mu) \cong \mathcal{F}_{\lambda+\mu}$ in particular $\sigma^{-\lambda}(H_{v^i}) \cong \mathcal{F}_\lambda$ and the well-known fusion rules*

$$\mathcal{F}_\lambda \times_{H_{v^i}} \mathcal{F}_\mu \cong \sigma^{-\lambda}(H_{v^i}) \times_{H_{v^i}} \sigma^{-\mu}(H_{v^i}) \cong \sigma^{-\lambda-\mu}(H_{v^i}) \cong \mathcal{F}_{\lambda+\mu}$$

follow immediately.

EXAMPLE A.0.2. *The example of the affine VOA of $\mathfrak{gl}(1|1)$ is found in section 3.2 of [**CR09**]. The spectral-flow automorphism acts on the modes as*

$$\sigma^\ell(N_r) = N_r, \qquad \sigma^\ell(E_r) = E_r - \ell k \delta_{r,0}, \qquad \sigma^\ell(\Psi^\pm_r) = \psi^\pm_{r\mp\ell}.$$

This illustrates that spectral flow changes the mode labels, i.e. *it doesnot leave the horizontal subalgebra of* $\widehat{\mathfrak{gl}(1|1)}$ *invariant.*

APPENDIX B

Representation Theory of Affine Lie Superalgebra $V(\mathfrak{g}_*(\rho))$

In this appendix, we study the representation theory of the affine Lie superalgebra $V(\mathfrak{g}_*(\rho))$, and present proofs of many statements found in Section 3.2. An important part of the proof of these statements will be the free field realization of $V(\mathfrak{g}_*(\rho))$.

B.0.1. Verma Modules. Recall the Kazhdan-Lusztig category KL_ρ, namely the category of finite-length grading-restricted generalized modules of $V(\mathfrak{g}_*(\rho))$. Let W be a simple object in KL_ρ, then it is generated by the lowest conformal-weight space W_0, which is a finite-dimensional (necessarily simple) module of $\mathfrak{g}_*(\rho)$. Recall the induction functor Ind, which is given by:

$$\mathrm{Ind}(W_0) := U(\widehat{\mathfrak{g}_*(\rho)}) \otimes_{U(\widehat{\mathfrak{g}_*(\rho)})_{\geq 0}} W_0. \tag{B.0.1.1}$$

Here $U(\widehat{\mathfrak{g}_*(\rho)})_{\geq 0}$ is the universal enveloping algebra of the non-negative modes. By universal property of induction functor, one obtain a morphism $\mathrm{Ind}(W_0) \to W$, which is surjective. Therefore, any simple module of KL_ρ is a quotient of $\mathrm{Ind}(W_0)$ for a simple W_0. Such modules are called the Verma modules. Let $\mathcal{G}_\rho$ be the category of finite-dimensional modules of $\mathfrak{g}_*(\rho)$, then a consequence of Theorem 2.1.9 is the following statement.

Corollary B.0.1. *If the image of $\mathcal{G}_\rho$ under* Ind *lies in KL_ρ, then the category KL_ρ has the structure of a braided tensor category.*

Our first main result is the following theorem.

Theorem B.0.2. *The image of $\mathcal{G}_\rho$ under* Ind *lies in KL_ρ. Consequently, KL_ρ has the structure of a braided tensor category.*

We will prove this statement in two steps. First, let us consider an object $M \in \mathcal{G}_{\rho,e}$, the subcategory where E_a has generalized eigenvalue e_a, and $e = (e_a)$.

Proposition B.0.3. *Assume that $\sum_a \rho_{ia} e^a \notin \mathbb{Z}$ or $\sum_a \rho_{ia} e^a = 0$. If M is simple, then so is* $\mathrm{Ind}(M)$.

Proof. We only need to show that any $w \in \mathrm{Ind}(M)$ generates the entire module. Note that an element in $\mathrm{Ind}(M)$ is always of the form:

$$\sum N^a_* \psi^{i,+}_* \psi^{i,-}_* E^a_* v \tag{B.0.1.2}$$

where v is in the lowest conformal weight space of $\mathrm{Ind}(M)$, which is simply M. The subscripts are all negative integers. We give a lexicographic order to $\mathrm{Ind}(M)$ such that $N > \psi^+ > \psi^- > E$, and

$$N^a_{-n} > N^a_{-n+1} > \cdots > N^{a+1}_{-n} > N^{a+1}_{-n+1} \tag{B.0.1.3}$$

and similarly for $\psi^\pm$ and E. Given an w, let $w = w_0 + w'$ where $w_0 = N^a_* \psi^{i,+}_* \psi^{i,-}_* E^a_* v$ is a homogeneous vector that is the biggest in the lexicographic order, such that $w_0 > w'$. Denote by $\mathcal{W}$ the sub-representation generated by w.

We perform the following procedure. If the expression of w_0 involves $(N^a_{-n})^k$, we will apply to w $(E^a_n)^k$. Since $[E^a_n, N^a_{-n}] = n$, the vector $(E^a_n)^k w_0$ will have no N^a_{-n} in its expression. Moreover, each time applying E^a_n, we obtain a non-zero vector. We repeate this process until all the N^a_* in the expression of w_0 is killed.

The next step we apply $\psi^{i,-}_n$ until there is no $\psi^{i,+}_*$ in the expression of w_0. Note that this step, we need the assumption that $\sum_i \rho_{ia} v^a \notin \mathbb{Z}$ or equal to 0, since the commutator $\{\psi^{i,-}_n, \psi^{i,+}_{-n}\} = n + \sum_i {\rho_a}^i E^a_0$ acts non-trivially on w_0 when $n + \sum_i {\rho_a}^i E^a_0 \neq 0$. By definition, $\sum_i \rho_{ia} E^a_0$ acts as $\sum_i \rho_{ia} e^a$, which is not an integer or equal to 0 by assumption. Therefore the action of $\psi^{i,-}_n$ is nontrial on w_0. We can now safely keep the procedure until all the $\psi^{i,+}_*$ in the expression of w_0 is annihilated.

We can repeat this process for all $\psi^{j,-}_*$ and E^a_*, until all the negative modes in the expression of w_0 is annihilated. Since $w_0 > w'$, this process must annihilate w' entirely, and we are left with a nonzero vector in the lowest conformal weight space. Consequently, $\mathcal{W}$ contains a nonzero vector v in M. We can now conclude the proof since M is assumed to be simple, which means $\mathcal{W}$ contains M, and consequently $\mathrm{Ind}(M)$.

□

Remark B.0.4. *In particular, the vacuum module* $V(\mathfrak{g}_*(\rho))$ *is simple, since the vacuum module is defined as* $\mathrm{Ind}(\mathbb{C})$ *where* $\mathbb{C}$ *is the trivial (and therefore simple)* $\mathfrak{g}_*(\rho)$ *module.*

Corollary B.0.5. *The monopole modules* U_s *are simple, and can be identified with the spectral flow* $\sigma_{s,\rho^\intercal\rho s}V(\mathfrak{g}_*(\rho))$.

Proof. The map $\sigma_{s,\rho^\intercal\rho s}|0\rangle \to |s\cdot X+\rho(s)\cdot Z\rangle$ clearly extends to a map of modules $\sigma_{s,\rho^\intercal\rho s}V(\mathfrak{g}_*(\rho)) \to U_s$. It is an embedding since $\sigma_{s,\rho^\intercal\rho s}V(\mathfrak{g}_*(\rho))$ is simple, and it is surjective since by [**Ada03**, **CRW14**], the module:

(B.0.1.4) $$\bigcap_i \mathrm{Ker} S_0^i\big|_{V_{Z,s\cdot X}}$$

is also a simple module of $\bigcap_i \mathrm{Ker} S_0^i\big|_{V_Z} = V(\mathfrak{g}_*(\rho))$. This completes the proof.

□

An immediate consequence of this is the fusion rule of monopole operators in Proposition 3.2.6, which follows from the general theory of spectral flow automorphism in Appendix A.0.7.

We in fact have the following theorem, whose proof is a word-to-word translation of the proof of [**BN22**, Proposition 3.2].

Theorem B.0.6. *When* $e \in \mathbb{C}^r$ *satisfies that* $\sum_a \rho_{ia}e^a \notin \mathbb{Z}$ *or* $\sum_a \rho_{ia}e^a = 0$, *then induction is an equivalence of categories:*

(B.0.1.5) $$\mathrm{Ind} : \mathcal{G}_{\rho,e} \simeq KL_{\rho,e}$$

where $KL_{\rho,e}$ *is the subcategory where the generalized eigenvalue of* E_0^a *are* e^a.

How do we deal with $KL_{\rho,e}$ when e does not satisfy the above? The answer is the spectral-flow automorphism. Recall in Section 3.1.3, Remark 3.1.8, we have introduced the following spectral flow automorphism:

(B.0.1.6) $$\sigma_{\lambda,\mu}(N_a) = N_a - \frac{\mu_a}{z}, \qquad \sigma_{\lambda,\mu}(E_a) = E_a - \frac{\lambda_a}{z}, \qquad \sigma_{\lambda,\mu}\psi^{i,\pm} = z^{\mp\sum \rho_{ai}\lambda^a}\psi^{i,\pm}.$$

Here $\lambda \in \mathbb{C}^r$ and $\mu \in \mathbb{C}^r$ such that $\rho(\lambda) \in \mathbb{Z}^r$. For any e such that $\rho(e)$ does not satisfy the above, there must be $\lambda \in \mathbb{C}^r$ with $\rho(\lambda) \in \mathbb{Z}^r$ such that $\rho(e+\lambda)$ satisfy the requirement $\sum_a \rho_{ia}(e^a+\lambda^a) \notin \mathbb{Z}$ or $\sum_a \rho_{ia}(e^a+\lambda^a) = 0$. We can now finish the proof of Theorem B.0.2.

Proof of Theorem B.0.2. Given any e and M and object in $\mathcal{G}_{\rho,e}$ that is simple, choose $\lambda \in \mathbb{C}^r$ such that $\rho(\lambda) \in \mathbb{Z}^r$, and that the entries of $\rho(e+\lambda)$ are either zero or non-integers. We just need to show that $\sigma_{\lambda,0}\mathrm{Ind}(M)$ is of finite-length, since $\sigma_{\lambda,0}$ preserves composition series. Choose $m \in M$, let I be the subset of all i where $\sum \rho_{ai}\lambda^a > 0$ and J where $\sum \rho_{ai}\lambda^a < 0$. Consider the sub-representation of $\sigma_{\lambda,0}\mathrm{Ind}(M)$ given by:

$$N := \bigoplus_{i\in I, 0\leq n<\rho_{ai}\lambda^a} < \psi_n^{i,-} m > \oplus \bigoplus_{j\in J, 0<n\leq -\rho_{aj}\lambda^a} < \psi_n^{j,+} m > . \tag{B.0.1.7}$$

Here $\langle v\rangle$ for $v \in \mathrm{Ind}(M)$ denotes the submodule of $V(\mathfrak{g}_*(\rho))$ generated by the vector v. By the definition of N, the quotient $\sigma_{\lambda,0}\mathrm{Ind}(M)/N$ is generated by a single element $\overline{m}$ (the image of m in the quotient) and that all the positive modes of $V(\mathfrak{g}_*(\rho))$ acts trivially on $\overline{m}$. Let M' be the $\mathfrak{g}_*(\rho)$ submodule of $\sigma_{\lambda,0}\mathrm{Ind}(M)/N$ generated by $\overline{m}$, which must be finite-dimensional. By universal property of the induction functor, we have a surjection $\mathrm{Ind}(M') \to \sigma_{\lambda,0}\mathrm{Ind}(M)/N$, and consequently, $\sigma_{\lambda,0}\mathrm{Ind}(M)/N$ is finite length.

We can now repeat this argument for all the summands in N, and this process must terminate because there are only finitely many positive modes that act non-trivially. We therefore obtain a finite filtration $\mathcal{F}_*\sigma_{\lambda,0}\mathrm{Ind}(M)$ such that each associated graded piece is a quotient of a finite-length Verma module. Therefore, $\sigma_{\lambda,0}\mathrm{Ind}(M)$ is finite-length, and so must be $\mathrm{Ind}(M)$. This completes the proof.

□

We have now shown that the category KL_ρ has the structure of a braided tensor category, and that the simple modules U_s satisfies the fusion rule $U_s \times U_{s'} \cong U_{s+s'}$. Of course this fusion rule can be realized by the intertwining operator of the free field algebra as in Section 3.1.3. This justifies the definition of $\mathcal{C}_{B,\rho}$ as the de-equivariantization of KL_ρ by the simple currents U_s. However, to analyze the category $\mathcal{C}_{B,\rho}$, one needs to understand the sub-category consisting of objects whose monodromy with U_s is trivial. The best way to do understand monodromy is through free-field realizations. We now turn to the free-field realization of $V(\mathfrak{g}_*(\rho))$.

B.0.2. Free Field Realizations. Consider the free field realization of Section 3.1.3, where we obtained a VOA embedding $V(\mathfrak{g}_*(\rho)) \hookrightarrow V_Z$ such that the image is equal to the kernel of the screening operators $S^i = \oint \mathrm{d}z{:}e^{Z^i-\rho(Y)^i}{:}$. For each $\lambda \cdot X + \mu \cdot Y$ where $\lambda, \mu \in \mathbb{C}^r$, we have a V_Z module $V_{Z,\lambda\cdot X+\mu\cdot Y}$, which we can restrict to obtain a module of $V(\mathfrak{g}_*(\rho))$. We would like to first understand what these modules are.

Consider a simple module M of $\mathfrak{g}_*(\rho)$. Since N^a and E^a commutes with each other, there must be at least one simultaneous eigenvector for all of them. By acting on $\psi^{i,-}$, we may assume that this eigenvector v is annihilated by $\psi^{i,-}$ for all i. Let (n_a, e_a) be its eigenvalues under N^a and E^a. It is clear then that the module M is spanned by vectors of the form:

$$\psi^{i_1,+}\cdots\psi^{i_k,+}v, \qquad i_1<i_2<\cdots<i_k, k\leq n. \tag{B.0.2.1}$$

Each of this vector is an eigenvector of N^a, E^a with eigenvalues:

$$\left(n_a + \sum_{1\leq s\leq k}\rho_{i_s a},\ e_a\right). \tag{B.0.2.2}$$

Let us define module $V_{(n,e)}$ to be the module generated by vectors of the form in equation (B.0.2.1). Then any simple module is a quotient of $V_{(n,e)}$ for some $(n,e)\in\mathbb{C}^r\times\mathbb{C}^r$.

Coming back to the module $V_{Z,\lambda\cdot X+\mu\cdot Y}$, let $b^i(z) = {:}e^{Z^i(z)}{:}$ and $c^i(z) = {:}e^{-Z^i(z)}{:}$, and consider the following grading Δ on $V_{Z,\lambda\cdot X+\mu\cdot Y}$ by:

$$\Delta(v)=\Delta(c^i_{-1}v)=0, \qquad \Delta(b^i_{-1}v)=\Delta(X^a_{-1}v)=\Delta(Y^a_{-1}v)=1. \tag{B.0.2.3}$$

Moreover, $\Delta(A_nB)=\Delta(A)+\Delta(B)-n-1$. With this grading, $V_{Z,\lambda\cdot X+\mu\cdot Y}$ is positively graded, and the minimal degree part of this is spanned by vectors of the form:

$$\prod_{i_1<i_2<\cdots<i_k} c^{i_1}_{-1}\cdots c^{i_k}_{-1}v. \tag{B.0.2.4}$$

The vector $c^1_{-1}c^2_{-1}\cdots c^n_{-1}v$ has weight (λ^a) under the action of E^a_0 and weights:

(B.0.2.5)
$$\mu_a - \sum_i \rho_{ia}.$$

under the action of N^a_0. Moreover, this vector is annihilated by positive modes of $V(\mathfrak{g}_*(\rho))$ by degree considerations as $\Delta(x) = 1$ for $x \in \mathfrak{g}_*(\rho)$, and it is killed by $\psi^{i,-}_0$ by considering the weights of N_0. Therefore, this vector, under the action of $\psi^{i,+}_0$, generates a copy of $V_{(\mu-\rho,\lambda)}$, where $\mu - \rho = (\mu_a - \sum_i \rho_{ia})$. By universal property of induction functor, there is an induced morphism:

(B.0.2.6)
$$\mathrm{Ind}(V_{(\mu-\rho,\lambda)}) \longrightarrow V_{Z,\lambda\cdot X+\mu\cdot Y}.$$

Proposition B.0.7. *When $\sum_i \rho_{ia}\lambda^a \notin \mathbb{Z}$ or $\sum_i \rho_{ia}\lambda^a = 0$ for all i, the morphism in equation* (B.0.2.6) *is an isomorphism.*

Proof. We first show that this is an embedding. To do so, by Theorem B.0.6, we only need to show that it is non-zero on any submodule of $V_{(\mu-\rho,\lambda)}$. In fact, the $\mathfrak{g}_*(\rho)$ module $V_{(\mu-\rho,\lambda)}$ has a unique simple submodule generated by:

(B.0.2.7)
$$\prod_{i,\sum \rho_{ia}\lambda^a=0} \psi^{i,+}_0 v.$$

It is very clear then that the above map $\mathrm{Ind}(V_{(\mu-\rho,\lambda)}) \to V_{Z,\lambda\cdot X+\mu\cdot Y}$ is nonzero when restricted to this unique simple.

To show that this is an isomorphism, we just need to define a positive grading on $\mathrm{Ind}(V_{(\mu-\rho,\lambda)})$ such that $\mathrm{Ind}(V_{(\mu-\rho,\lambda)}) \to V_{Z,\lambda\cdot X+\mu\cdot Y}$ is graded and that they have the same grading. We define the grading on $\mathrm{Ind}(V_{(\mu-\rho,\lambda)})$ such that $\Delta(V_{(\mu-\rho,\lambda)}) = 0$ and $\Delta(x_{-1}v) = 1$ for all $x \in \mathfrak{g}_*(\rho)$, and that $\Delta(A_nB) = \Delta(A) + \Delta(B) - n - 1$. It is clear that the map $\mathrm{Ind}(V_{(\mu-\rho,\lambda)}) \to V_{Z,\lambda\cdot X+\mu\cdot Y}$ is a map of positively graded vector spaces. It is straightforward that the graded character $\mathrm{Tr}(q^\Delta)$ agrees on the two modules, and so they must be isomorphic.

□

As a consequence, the restriction of any free-field module is an object in KL_ρ.

Corollary B.0.8. *For any $\lambda, \mu \in \mathbb{C}^r$, the module $V_{Z,\lambda\cdot X+\mu\cdot Y}$ is an object in KL_ρ.*

Proof. We have shown this when entries of $\rho(\lambda)$ are either zero or non-integer. When this is not the case, choose ν such that $\rho(\lambda+\nu)$ satisfy this, and $\rho(\nu) \in \mathbb{Z}^n$. This implies that $V_{Z,(\lambda+\nu)\cdot X+\mu\cdot Y}$ is an object in KL_ρ. We now can finish the proof since there is clearly an isomorphism:

$$V_{Z,\lambda\cdot X+\mu\cdot Y} \cong \sigma_{-\nu,-\rho^\perp\rho\nu} V_{Z,(\lambda+\nu)\cdot X+\mu\cdot Y}, \tag{B.0.2.8}$$

given by mapping $\sigma_{-\nu,-\rho^\perp\rho\nu}|(\lambda+\nu)\cdot X+\mu\cdot Y\rangle$ to $|\lambda\cdot X+\mu\cdot Y-\rho(\nu)\cdot Z\rangle$ (the free field realization). Since the spectral flow σ preserves KL_ρ, the object $V_{Z,\lambda\cdot X+\mu\cdot Y}$ is in KL_ρ, and the proof is complete. □

Namely, when entries of $\rho(\lambda)$ are either zero or non-integer, the module $V_{Z,\lambda\cdot X+\mu\cdot Y}$ is identified with the induction of the lowest-weight module $\mathrm{Ind}(V_{(\mu-\rho,\lambda)})$. Otherwise, choose ν as above, $V_{Z,\lambda\cdot X+\mu\cdot Y}$ can be identified with $\sigma_{-\nu,-\rho^\perp\rho\nu}\mathrm{Ind}(V_{(\mu-\rho,\lambda+\nu)})$. Since any simple objects in $\mathcal{G}_\rho$ is a quotient of a lowest-weight module $V_{(\mu-\rho,\lambda)}$, we see that any module in KL_ρ is a quotient of $V_{Z,\lambda\cdot X+\mu\cdot Y}$. In fact, this goes beyond simple modules. For each $\lambda,\mu \in \mathbb{C}^r$, and each $p,q \in \mathbb{N}^r$ (namely vectors with natural numbers entries), the module $V_{Z,\lambda\cdot X+\mu\cdot Y}$ has a self-extension, which we denote by $V^{p,q}_{Z,\lambda\cdot X+\mu\cdot Y}$, generated by the following free-field generator:

$$\prod_a X_a^{p_a} Y_a^{q_a} |\lambda\cdot X+\mu\cdot Y\rangle. \tag{B.0.2.9}$$

This gives the action of X_0^a and Y_0^a Jordan blocks since $[X_0^a, Y^b] = \delta^{ab} = [Y_0^a, X^b]$. It is a self-extension of $V_{Z,\lambda\cdot X+\mu\cdot Y}$ in the sense that it has a filtration whose associated graded are all isomorphic to $V_{Z,\lambda\cdot X+\mu\cdot Y}$. It turns out that any object in KL_ρ is a subquotient of such modules restricted to $V(\mathfrak{g}_*(\rho))$. The proof of this is similar to the appendix of [**GN23**], and we won't repeat here again.

Proposition B.0.9. *Any object in KL_ρ is a quotient of a sub-module of a finite direct sum of $V^{p,q}_{Z,\lambda\cdot X+\mu\cdot Y}$.*

We use this to prove Theorem 3.2.7.

Proof of Theorem 3.2.7. Recall the field redefinition of equation (3.1.5.15) and (3.1.5.16). For each $\lambda,\mu \in \mathbb{C}^r$, the module $V_{Z,\lambda\cdot X+\mu\cdot Y} \otimes V_{\overline{X},\overline{Y}}$ of $V(\mathfrak{g}_*(\rho)) \otimes V_{\overline{X},\overline{Y}}$ is clearly a lift of the

$V(\mathfrak{gl}(1|1))^{\otimes n}$ module restricted from the free-field algebra $H_{X,Y,Z} \otimes H_{\overline{X},\overline{Y}}$. This, together with Proposition B.0.9 shows that the functor $\mathcal{L}_\rho^{\mathrm{ungauge}}$ is surjective.

We now show that $\mathcal{L}_\rho^{\mathrm{ungauge}}$ maps into KL_ρ. Let W be a simple module of $V(\mathfrak{gl}(1|1))^{\otimes n}$ that has trivial monodromy with $V(\mathfrak{g}_*(\rho)) \otimes V_{\overline{X},\overline{Y}}$. By [**BN22**, Proposition 4.3], every such module W embed uniquely into a module of the free-field algebra $H_{X,Y,Z} \otimes H_{\overline{X},\overline{Y}}$, say $\mathcal{F}_\lambda$ for some λ that is a linear combination of the Heisenberg generators X, Y and $\overline{X}, \overline{Y}$. Since monodromy acts semi-simply on $\mathcal{F}_\lambda$, W has trivial monodromy with $V(\mathfrak{g}_*(\rho)) \otimes V_{\overline{X},\overline{Y}}$ if and only if $\mathcal{F}_\lambda$ does, and this is true if and only if $\lambda(\overline{X}), \lambda(\overline{Y}) \in \mathbb{Z}$, and the lift of such $\mathcal{F}_\lambda$ can be clearly identified with $V_{Z,\lambda\cdot X+\mu\cdot Y} \otimes V_{\overline{X},\overline{Y}}$. Therefore, the lift of W must be in KL_ρ. This completes the proof.

□

B.0.3. Monodromy via Free Field Realization. In this last section of this appendix, we use free-field realization to compute monodromy. Let W be an object in KL_ρ, we show:

Proposition B.0.10. *The monodromy:*

(B.0.3.1)
$$U_s \times W \longrightarrow W \times U_s \longrightarrow U_s \times W$$

is given by $\mathrm{Id} \times e^{2\pi i \sum_a s_a N_0^a}$.

The idea of the proof is as follows. We first show that the above is true for any $W = V^{p,q}_{Z,\lambda\cdot X+\mu\cdot Y}$. Then it will follow that this is true for all W since any W is a sub-quotient of $V^{p,q}_{Z,\lambda\cdot X+\mu\cdot Y}$ and monodromy is functorial with respect to sub-quotient. We present a proof of this here.

First of all, there is an embedding $U_s \hookrightarrow V_{Z,s\cdot X}$, and the free-field intertwining operator:

(B.0.3.2)
$$\mathcal{Y} : V_{Z,s\cdot X} \times_{V_Z} V^{p,q}_{Z,\lambda\cdot X+\mu\cdot Y} \to V^{p,q}_{Z,(\lambda+s)\cdot X+\mu\cdot Y}$$

induces the universal intertwining operator $U_s \times_{V(\mathfrak{g}_*(\rho))} V^{p,q}_{Z,\lambda\cdot X+\mu\cdot Y} \to V^{p,q}_{Z,(\lambda+s)\cdot X+\mu\cdot Y}$. Therefore we only need to compute the monodromy using this intertwining operator. By definition, U_s is generated by the vector $v = |s\cdot X + \rho(s)\cdot Z\rangle$, and $V^{p,q}_{Z,(\lambda+s)\cdot X+\mu\cdot Y}$ generated by $w = \prod_a X_a^{p_a} Y_a^{q_a} |\lambda \cdot X + \mu\cdot Y\rangle$. The logarithmic intertwining operator $\mathcal{Y}$ is defined by the formula:

(B.0.3.3)
$$\mathcal{Y}(v,z)w = :\exp\left(\sum s_a(X^a + {\rho_i}^a Z^i)\right):w.$$

In this formula, the logarithmic part comes from:

$$e^{\sum_a s_a (X_0^a + \rho_i{}^a Z_0^i)\log(z)} \prod_a X_a^{p_a} Y_a^{q_a} |\lambda \cdot X + \mu \cdot Y\rangle. \tag{B.0.3.4}$$

Since $[X_0^a, Y^b] = \delta^{ab}$, the above is given by:

$$\prod_a (X_a)^{p_a} (Y_a + s_a \log(z))^{q_a} e^{\sum_a s_a (X_0^a + \rho_i{}^a Z_0^i)\log(z)} |\lambda \cdot X + \mu \cdot Y\rangle. \tag{B.0.3.5}$$

To compute the monodromy, we rotate the z coordinate by $z \mapsto e^{2\pi i} z$, which results in $\log(z) \mapsto \log(z) + 2\pi i$. The contribution of the above comes from two parts, where the first part is:

$$\prod_a (X_a)^{p_a} (Y_a + s_a \log(z) + 2\pi i s_a)^{q_a}, \tag{B.0.3.6}$$

and the second part is:

$$e^{\sum_a s_a (X_0^a + \rho_i{}^a Z_0^i)(\log(z) + 2\pi i)} |\lambda \cdot X + \mu \cdot Y\rangle. \tag{B.0.3.7}$$

The contribution of the first part can be compactly written as:

$$e^{2\pi i \sum_a s_a \frac{\partial}{\partial Y^a}}, \tag{B.0.3.8}$$

while the second part as:

$$e^{2\pi i \sum_a s_a \mu^a} = e^{2\pi i (s,\mu)}. \tag{B.0.3.9}$$

One can verify that the morphism corresponding to:

$$e^{2\pi i \sum_a s_a \frac{\partial}{\partial Y^a}} e^{2\pi i (s,\mu)} \tag{B.0.3.10}$$

is nothing but:

$$e^{2\pi i \sum_a s_a N_0^a}. \tag{B.0.3.11}$$

We have, in conclusion

$$\mathcal{Y}(v, e^{2\pi i} z) w = \mathcal{Y}(v, z) e^{2\pi i \sum_a s_a N_0^a} w, \tag{B.0.3.12}$$

which is the desired statement that the monodromy is $\mathrm{Id} \times e^{2\pi i \sum_a s_a N_0^a}$. We comment that the proof of Proposition 3.2.4 and Proposition 3.2.9 follows exactly in the same way as above, using the explicit formula of the free-field intertwining operator.

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