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UNIVERSITY OF CALIFORNIA
RIVERSIDE

A Combined Boundary Integral Method for Crack Problems
in Multilayered Elastic Media Under the Plane Strain Condition

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mechanical Engineering

by

Gongbo Long

December 2013

Dissertation Committee:

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The Dissertation of Gongbo Long is approved:

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ABSTRACT OF THE DISSERTATION

A Combined Boundary Integral Method for Crack Problems
in Multilayered Elastic Media Under the Plane Strain Condition

by

Gongbo Long

Doctor of Philosophy, Graduate Program in Mechanical Engineering
University of California, Riverside, December 2013
Dr. Guanshui Xu, Chairperson

The crack problems in multilayered elastic media are of considerable interest in many engineering applications, particularly in hydraulic fracturing treatments for increasing hydrocarbon production. Various computational methods have been developed for the analysis of crack problems, and boundary integral methods stand out for their broad applicability and accuracy. Among various boundary integral methods the standard boundary integral method, such as the direct method, can be used for general elastic analysis of multilayered media, while the displacement discontinuity method based on fundamental dislocation solution is more suitable for the analysis of crack problems.

The displacement discontinuity method, however, cannot be applied directly to the general crack problems in multilayered media due to the lack of fundamental dislocation solutions. In this dissertation study, we therefore address this issue by developing a new approach that combines the displacement discontinuity method and the direct method. The combined method shares both the efficiency of the displacement discontinuity method and the applicability of the direct method. In this combined method, the displacement discontinuity method is implemented to construct the fracture matrix in each layer, while the direct method is used to characterize the effects of the interfaces. As a consequence, all variables on the interfaces can be eliminated through continuity conditions, leading to the final equation which only consists of variables on crack surfaces. The concept of the crack tip element is also adopted and extended for better treatment of the crack tip singularity. The example studies have demonstrated that the combined method has comparable accuracy but far more efficiency for practical applications as compared to the traditional direct method.

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Chapter 1

Introduction

1.1 The crack problems in the multilayered elastic media

The crack problems in multilayered elastic media are of considerable interest in many engineering applications, particularly in hydraulic fracturing treatments for increasing hydrocarbon production ^{[1][2][3]}, especially in extracting unconventional oil ^{[4][5]} and gas resources ^{[6][7]}. One critical component of the application is to model the elastic response of a pressurized crack intersecting a number of layers ^[8], as shown in Figure 1.1.

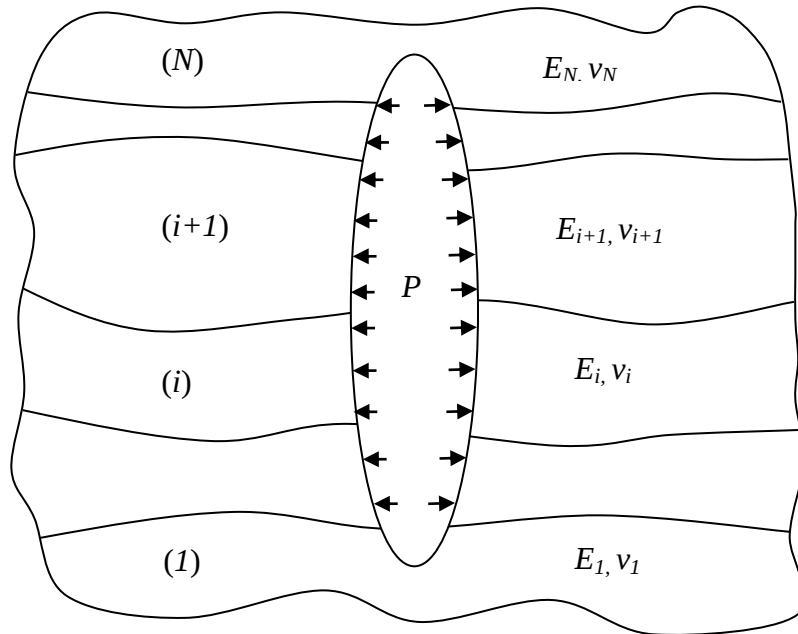


Figure 1. 1: A schematic configuration of the crack problems in the multilayered elastic media

Our objective is to develop a new efficient and accurate method based on the Boundary Integral Method (BIM). The method solves crack problems by determining the opening displacement of a crack subjected to the known compressive pressure going through the interfaces in a multilayered linear elastic media. We shall focus on plane-strain conditions as shown in Figure 1.1. In the figure, the index in the parenthesis and the one as subscript denote the layer number ranging from 1 to N ; the symbols E and ν represent the Young's modulus and Poisson ratio, respectively; and P denotes the known compressive pressure applied on the crack. Each layer is assumed individually homogeneous and isotropic but with distinct material properties.

1.2 The boundary integral method^{[9][10]}

The boundary integral method, also known as boundary element method (BEM) is based on the assumption that the governing partial differential equations are linear, which implies that certain solutions can be added together to construct new solutions in the region R .

Analytical solutions have been found in many disciplines for the case corresponding to some sort of disturbance at a point in an *infinite* homogeneous region. This disturbance could represent a heat source or sink in a heat flow problem, or a point force applied within an elastic solid in solid mechanics problem.

These solutions are usually called singular solutions because, mathematically speaking, they are well behaved everywhere in R , except at the point of disturbance, where there is a mathematical anomaly, or singularity. The solutions can be summed (in a linear problem) to find solutions for cases in which more than one point disturbance is present. However, when the analytical solutions cannot be found for the problem, for example an inhomogeneous elastic media of irregular geometry, an approximate solution then must be found by using a numerical method.

The numerical methods for solving boundary value problems can be divided into two distinct classes, those that require approximations to be made throughout the region R , and those that require approximations to be made only on the boundary contour C . Finite difference and element methods fall in the first class. The boundary integral

method constitutes the second class.

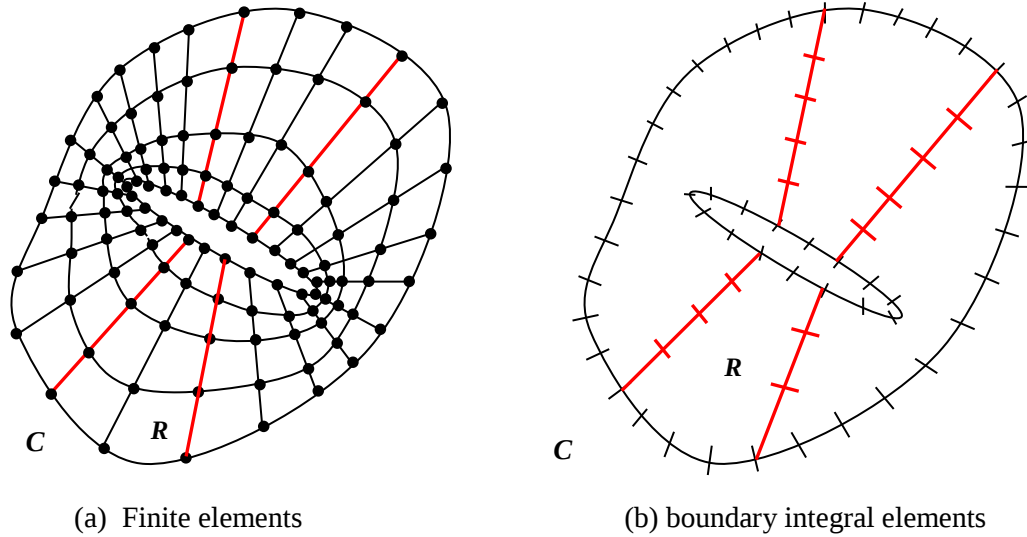


Figure 1. 2 : Finite element, boundary integral element idealizations ^[9]

For the sake of simplicity, the region R is represented in the Figure 1.2 as a two-dimensional plane region bounded by a contour C . The finite element method requires that the whole region R be divided into a network of elements, as depicted in Figure 1.2(a). The objective is to evaluate the solution to the problem at the nodes or mesh points of the network; the solution between nodes is expressed in a simple, approximate form in terms of the values at the nodes. Relating this approximate solution to the original PDE eventually leads to a system of linear algebraic equations in which the unknown parameters, the nodal values in R , are expressed in terms of the known nodal values at mesh points on the boundary C . This system of equations is large but sparse, i.e. although there is a large number of unknown parameters, and hence a correspondingly a large number of linear equations, each equation contains

only a few of the unknown parameters explicitly.

In boundary integral method, only the boundary C is divided into elements, as in Figure 1.2(b). The numerical solution builds on the analytical solutions that have already been obtained for simple singular problems in such a way as to satisfy, the specified boundary conditions at each element on C , approximately. Because each of the singular solutions satisfies the governing PDEs, there is no need to divide R itself into a network of elements. The system of equations to be solved is much smaller than the system needed to solve the same boundary value problem by the finite element method but the equations are no longer sparse.

The boundary integral technique can be explained more fully with reference to Figure 1.3. Figure 1.3 (a) represents a region R bounded by a contour C . Figure 1.3 (b) represents an *infinite* plane and C' is a tracing of the contour C onto this plane. It often is easier to find analytical solutions to the relevant PDEs in the infinite region of Figure 1.3 (b) than in the actual region R of Figure 1.3 (a). In particular, we should be able to find a singular solution for a point disturbance, for example a point force, at some point P in the infinite region. Let us suppose that it just so happened that the singular solution produced precisely the same conditions on the auxiliary curve C' in the infinite plane as those prescribed on the boundary C in Figure 1.3 (a). If this were the case then, because solutions to well posed problems are unique, we would have solved the problem of Figure 1.3 (a) by addressing the problem of Figure 1.3 (b).

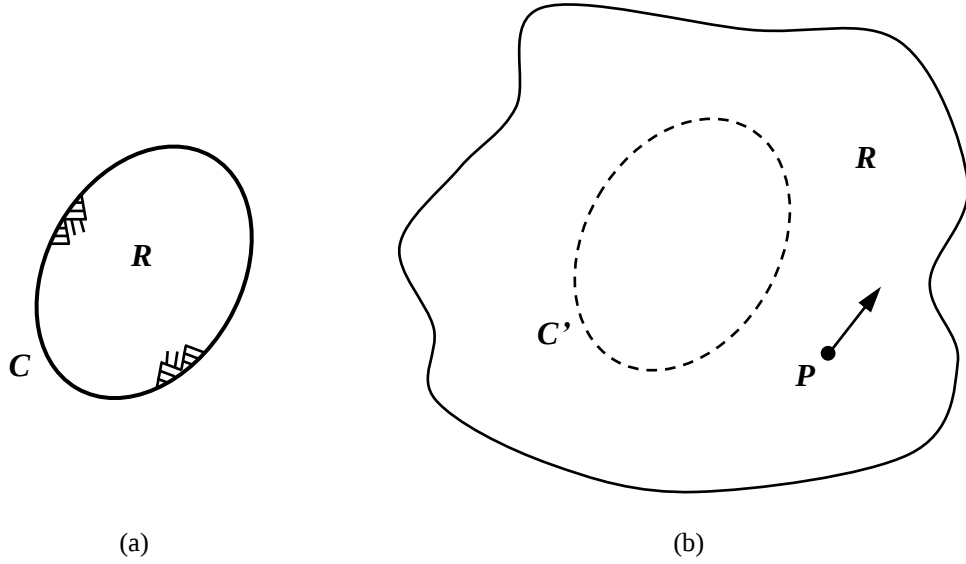


Figure 1. 3: The contour C (a) and auxiliary contour C' (b) ^[9]

Therefore, we can develop a numerical technique for finding a number of singular solutions which, when added together, produce approximately the correct conditions on C' . Note that we are assuming that the governing P.D.E is linear by adding solutions.

The technique is developed as follows:

First, we divide C' into a number (for example, N) of elements and agree that we will be satisfied with an approximate solution that can match the prescribed conditions on C only at the midpoints of the element of C' . Then we seek N singular solutions which, when superimposed, will give the required conditions at the midpoint of each element. Each singularity is positioned at the midpoint of each element on C' . The combined effects of all the N singularities at any one element can then be expressed in terms of the strengths of the singularities. Although we do not know these strengths, we do know

what their combined effects should be, via the boundary conditions on C . Therefore, we can write down a system of N linear algebraic equations in which the unknowns are the strengths of the singularities.

	FEM	BIM
<i>Mesh</i>	Throughout the whole region R	Only on the boundary C
<i>System of equations</i>	Large but sparse	Smaller, not sparse
<i>Solving the unknowns</i>	Simultaneously	First boundary, then the region, if necessary (Somigliana's formulae applied)
<i>Accuracy</i>		Better

Table 1. 1: Comparison of the FEM and BIM

From the computational point of view, a boundary integral method leads to a much smaller system of algebraic equations than a finite element solution for the same problem. According to the above discussion, however, the smaller system of equations is no longer sparse since each singularity plays a part in every equation. Once these equations have been solved, the solution *any* point in R can be constructed. In the boundary integral method, one can thus deliberately choose the points (*and only those points*) where one wishes to compute the solution, instead of automatically generating the results at a number of fixed locations (the interior mesh points) as in FEM. Because

it exploits an analytical solution that holds true throughout R , a BIM is potentially more accurate than the FEM, where approximations are made in every subdivision of R . In addition, the peculiar form of the region of interest deteriorates the FEM results since it establishes the mesh throughout the whole formation while the BIM does not suffer from this drawback because only the boundaries need discretizing. The foregoing content is summarized in the Table 1.1.

1.3 Historical review of the numerical methods for crack problems

All the numerical methods for crack problems in multilayered elastic media without body force start with the classic governing partial differential equation (P.D.E.) of theory of elasticity:

$$\sigma_{ij,j} = 0 \quad (1.1)^{[11]}$$

defined in terms of the stress gradients $\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$, where the two subscripts i and j denote the two directional component of the plane Cartesian coordinate and the symbols σ stands for the elastic parameter stress.

And the boundary condition:

$$p_i = \sigma_{ij} n_j \quad (1.2)^{[11]}$$

where n denotes the unit vector of a given boundary; and p represents the elastic parameter traction, respectively.

The objective of the methods is to solve P.D.E. (1.1) associated with the equation (1.2)

and then any unknown tractions or displacements can be determined consequently.

A number of researchers have developed and implemented various methods to solve the P.D.E. (1.1) for a multilayered elastic media with parallel interfaces but no crack inside as shown in the Figure 1.4 at first. In the figure, the bottom layer is assumed as a half space so that the displacement at the very bottom is zero and the top layer is traction free. The symbol d_i represents the thickness of the i th layer.

Burmister^[12] used Bessel functions to study single layer, i.e. homogenous and two layer, i.e. bimaterial elastic media; Cook and Erdogan^{[13][14]} adopted Mellin transforms to treat two bonded half planes within a crack going through the interface. While Thomson^[15] introduced Fourier transforms as the first systematic approach to layered elastic materials method^[16], leading to the so-called stiffness matrix method^[17].

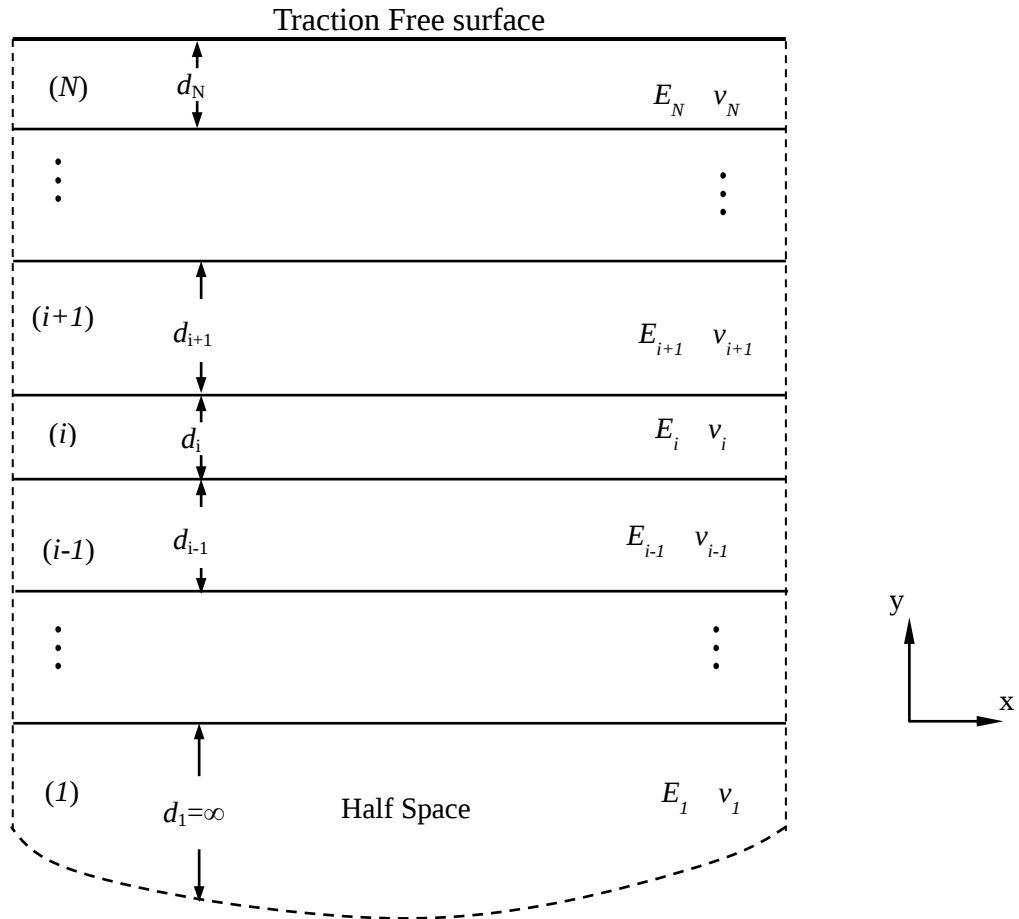


Figure 1. 4: A schematic configuration of a multilayered media with parallel interfaces

In order to determine the Green's function of the multilayered media, the Fourier transforms is applied to the governing equations of an elastic media, i.e. the system of

P.D.E. (1.1), so that the P.D.E. can then be reduced to a system of ordinary differential equations (O.D.E.) with respect to the independent variable y for each of the stress and the displacement components in each layer ^[8]. The general solution to the homogeneous O.D.E. for a given layer can be determined since any given layer is uniform and therefore has constant coefficients ^[16]. The general solution in each layer can be expressed in terms of a small number of free constants, which is referred as spectral coefficients ^{[8] [16]}. Then the appropriate Green's function can then be constructed by stitching together the solutions within each layer by applying the continuity conditions across the interfaces ^[8]. The expression of the stiffness matrix involves terms of the form $e^{k\Delta y}$ and $e^{-k\Delta y}$, where k is the wavenumber and Δy is the distance from the source point to the layer interface ^[8]. When the thick layer is involved, Δy will be very large even for relatively moderate k so that the stiffness matrix will become ill-conditioned due to the exponentially large and exponentially small terms in the same matrix so that this method is not widely used ^[8].

Gilbert and Backus then introduced the propagator matrix method^[18] to take advantage of the “chain-like” feature by eliminating the spectral coefficient at the top of a given layer so that a new propagator matrix can be introduced to transfer the stresses and displacements from the bottom interface to the top interface of a given layer. However, the method still suffers from serious ill-condition drawback^{[19][20]}, although some schemes have been proposed to scale out the exponentially growing and decaying

terms so that they can be canceled analytically^[8].

Buffler then developed the flexibility matrix method^[21] by separating the tractions and displacements apart and then solving them with the interface continuity condition to remedy the ill-conditioning problem reported by many papers^{[8] [19] [20]} when the propagator method deals with thick layers, while Linkov and Filippov found out some restrictions of the flexibility matrix method due to finite digits in the computer for extremely thin layers situation^[20]. Nevertheless, the two researchers also reported that the method can be sufficiently accurate if double precision is adopted^[20]. Based on the flexibility matrix method, A. Pierce and E. Siebrits have developed a “rescaled flexibility matrix method” by using power expansion to extract the dominant singular terms and keep the regular ones so that the all the remaining terms of the singular flexibility matrices are rescaled and lead to a well-conditioned system of equations. What is more important, this method is able to solve the situation that a crack intersecting the interfaces by introducing a displacement discontinuity^{[8] [16]}.

However, all the methods above share a common drawback: they can only deal with the parallel multilayered media since the start of them is the general solution derived from Fourier transform. Therefore, some researchers have developed other numerical methods especially associated with the Finite Element Method (FEM) and Boundary Integral Method (BIM) to overcome the drawback, while the others have been developing the methods based on Fourier transforms.

The FEM has been found less appealing compared to the BIM gradually due to the reasons discussed in the last section. Then a customary general “zoning” technique^{[22][23]} has been developed for the BIM implementation to construct the system of algebraic equations with boundary integral equations and solve all the unknowns on the boundaries simultaneously, but it is quite unrealistic to solve all the unknowns at the same time when the number of layer becomes large. According to Linkov and Filippov, the maximum layer number for this approach is 11, or the accuracy decreases as the number of layers increase^[20].

Therefore, a variant of the propagator matrix method named the transfer matrix method has been considered and developed further^{[24][25]}. The transfer matrix method was applied with the BIM to utilize the chain-like characteristic by applying the continuity condition on the interfaces so that the elastic parameters on the very bottom can be expressed in terms of the ones on the very top and chain-like multiplication coefficient matrices^[19]. However, Maier and Novati reported the ill-conditioning problem when the ratio of the layer thickness d to the mesh size h becomes large^[19].

To remedy the drawback, the two researchers have developed a method called successive stiffness method by separating tractions and displacements apart in the system equations to express the traction in terms of the displacement on the same interface of a given layer^[26]. Then the traction or displacement can be obtained by carrying out a sequential computation from top to bottom with the continuity

condition and the boundary conditions. If double precision is adopted like the flexibility matrix method as mentioned above, the accuracy meets the practical requirements even for 1000 thin layers ^[20].

When determining the opening displacement of a crack subjected to a known compressive pressure intersecting the interfaces in a multilayered elastic media, the successive stiffness method has to calculate the displacements on each crack surface since it adopts the direct method (DM) of BIM ^[26] and the effects of elements placed along one surface are indistinguishable from the ones along the other surface that coincide the former ^[9] with DM implementation. However, it is the crack opening displacement, i.e. the relative displacement of the two surfaces, rather than the respective displacement on each one that attract the practical industrial interest. Thus, the displacement discontinuity method (DDM) arises because it obtains the opening displacement by distributing a series of displacement discontinuities or dislocation dipoles throughout the region of interest^[27]. In addition, all the foregoing methods based on DM implementation solve all the interfaces, resulting in more computation. On the contrary, the DDM only solves the crack but it cannot be implemented in multilayered media directly since only the dislocation field of an infinite isotropic homogeneous plane^[28] or bonded half-planes ^{[28][29][30]} is accessible. Therefore, the work described in this dissertation combines the DM and the DDM to determine the crack opening displacement so that the method shares both the efficiency of the

displacement discontinuity method and the applicability of the direct method. In this combined method, the displacement discontinuity method is implemented to construct the fracture matrix in each layer, while the direct method is used to characterize the effects of the interfaces. As a consequence, all variables on the interfaces can be eliminated through continuity conditions, leading to the final equation which only consists of variables on crack surfaces. The concept of the crack tip element is also adopted and extended for better treatment of the crack tip singularity.

Chapter 2

The displacement discontinuity method

2.1 Abstract

The boundary integral method (BIM) has been developed in two approaches historically. One of these is an intuitive, physical way and the other is a more mathematical treatment based on concepts of classical potential theory ^[9]. In this chapter, the displacement discontinuity method (DDM), categorized in the former one, will be introduced at first and then implemented to solve the crack problem.

The displacement discontinuity method (DDM) is developed specifically to solve problems in solid mechanics involving bodies containing thin, slit-like openings or cracks, while other methods cannot be applied directly. There are two different ways to implement this method: one is based on Kelvin's solution applied on a constant displacement discontinuity; the other is based on the dislocation theory. Both the two approaches yield the exact same mathematical formulae. The introduction and implementation of the latter will be the focus in this chapter and the former is applied to revise the original one dimensional crack tip element approach for the plane strain condition. A result of the DDM implementation with different approaches in a homogeneous medium case will be presented and compared at the last section.

2.2 Introduction

2.2.1 The displacement discontinuity method starting with constant displacement discontinuity ^[9]

The problem of a constant displacement discontinuity over a finite line segment in the x, y plane of an infinite elastic solid is specified by the condition that the displacement s be continuous everywhere except over the line segment in question. The line segment may be chosen to occupy a certain portion of the x axis, say the portion $|x| \leq a, y = 0$, as shown in Figure 2.1. Then the surface on the positive side of $y=0$ is denoted as $y=0_+$, and the other one on the negative side is $y=0_-$. In crossing from one side of the line segment to the other, the displacements must undergo a constant specified change in value $D_i = (D_x, D_y)$.

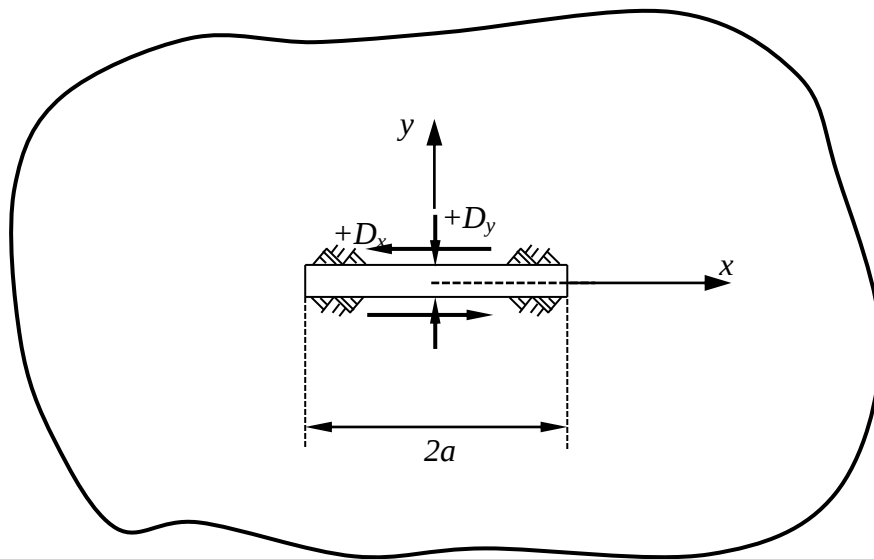


Figure 2. 1: Constant displacement discontinuity components D_x and D_y

The displacement discontinuity D_i will be defined as the difference in displacement between the two sides of the segment as follows:

$$D_i = u_i(x_i, 0_-) - u_i(x_i, 0_+) \quad (2.1)$$

Or

$$D_x = u_x(x, 0_-) - u_x(x, 0_+) \quad (2.2-a)$$

$$D_y = u_y(x, 0_-) - u_y(x, 0_+) \quad (2.2-b)$$

The solution to such a problem, known as Kelvin's solution, can be expressed in terms of a function $f(x, y)$, defined as:

$$\begin{aligned} f(x, y) &= \frac{-1}{4\pi(1-\nu)} \int_{-a}^a \ln[(x-\xi)^2 + y^2]^{\frac{1}{2}} d\xi \\ &= \frac{-1}{8\pi(1-\nu)} \int_{-a}^a \ln[(x-\xi)^2 + y^2] d\xi \end{aligned} \quad (2.3)$$

Since the stresses only involve derivatives of function f , we can get the explicit expression of function f by omitting a non-essential constant:

$$\begin{aligned} f(x, y) &= \frac{-1}{8\pi(1-\nu)} \{ (x+a) \ln[(x+a)^2 + y^2] - (x-a) \ln[(x-a)^2 + y^2] \\ &\quad + 2y \left[\arctan \frac{y}{x-a} - \arctan \frac{y}{x+a} \right] \} \end{aligned} \quad (2.4)$$

The stress field of this constant displacement discontinuity is given based on the derivatives of the function f :

$$\begin{cases} \sigma_{xx} = 2GD_x(2f_{,xy} + yf_{,xyy}) + 2GD_y(f_{,yy} + yf_{,yyy}) \\ \sigma_{yy} = 2GD_x(-yf_{,xyy}) + 2GD_y(f_{,yy} - yf_{,yyy}) \\ \sigma_{xy} = 2GD_x(f_{,yy} + yf_{,yyy}) + 2GD_y(-yf_{,xyy}) \end{cases} \quad (2.5)$$

Function $f(x, y)$ and its derivatives are obtained from Kelvin's solution:

$$\left\{ \begin{aligned} f_{,x} &= \frac{+1}{4\pi(1-\nu)} \{ \ln \sqrt{[(x-a)^2 + y^2]} - \ln \sqrt{[(x+a)^2 + y^2]} \} \\ f_{,y} &= \frac{-1}{4\pi(1-\nu)} \left(\arctan \frac{y}{x-a} - \arctan \frac{y}{x+a} \right) \\ f_{,xy} &= \frac{+1}{4\pi(1-\nu)} \left[\frac{y}{(x-a)^2 + y^2} - \frac{y}{(x+a)^2 + y^2} \right] \\ f_{,xx} = -f_{,yy} &= \frac{+1}{4\pi(1-\nu)} \left[\frac{x-a}{(x-a)^2 + y^2} - \frac{x+a}{(x+a)^2 + y^2} \right] \\ f_{,xyy} = -f_{,xxx} &= \frac{+1}{4\pi(1-\nu)} \left\{ \frac{(x-a)^2 - y^2}{[(x-a)^2 + y^2]^2} - \frac{(x+a)^2 - y^2}{[(x+a)^2 + y^2]^2} \right\} \\ f_{,yyy} = -f_{,xyx} &= \frac{+2y}{4\pi(1-\nu)} \left\{ \frac{x-a}{[(x-a)^2 + y^2]^2} - \frac{x+a}{[(x+a)^2 + y^2]^2} \right\} \end{aligned} \right. \quad (2.6)$$

2.2.2 The displacement discontinuity method starting with the dislocation theory

A constant displacement discontinuity over a finite line segment in the x, y plane of an infinite elastic solid can be treated as two pairs of the edge dislocation dipoles placed at the ends of the segment, as depicted in Figure 2.2. The displacement discontinuity components D_x and D_y are replaced by the Burgers vector b_x and b_y , respectively.

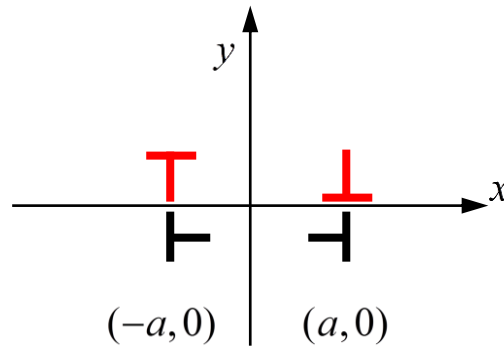


Figure 2. 2: Two pairs of dislocation dipoles

Consider an edge dislocation located at $(\xi, 0)$ in an infinite homogeneous medium as depicted in Figure 2.3:

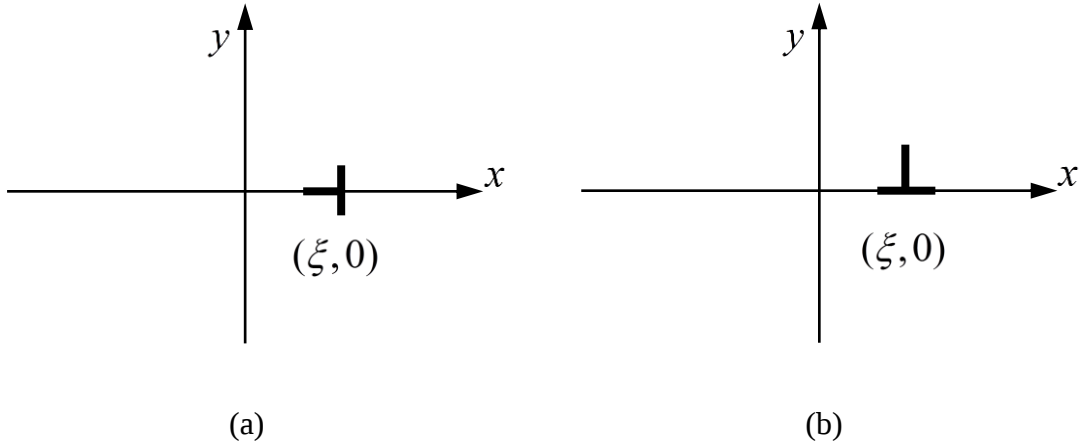


Figure 2. 3: An edge dislocation with Burgers vector b_y (a) and b_x (b)

By introducing the constant $k = \frac{G}{2\pi(1+\nu)}$, the stress field of this dislocation with

Burgers vector b_y located in $(\xi, 0)$ in the x, y plane of an infinite elastic solid is:

$$\begin{cases} \sigma_{xx}^{b_y} = b_y(-k)(x-\xi) \frac{(x-\xi)^2 - y^2}{[(x-\xi)^2 + y^2]^2} \\ \sigma_{yy}^{b_y} = b_y(-k)(x-\xi) \frac{(x-\xi)^2 + 3y^2}{[(x-\xi)^2 + y^2]^2} \\ \sigma_{xy}^{b_y} = b_y(-k)y \frac{(x-\xi)^2 - y^2}{[(x-\xi)^2 + y^2]^2} \end{cases} \quad (2.7)^{[31]}$$

G and ν represent shear modulus and Poisson ratio of the solid, respectively (See Appendix A and B for details).

Similarly, the stress field of a dislocation with Burgers vector b_x located in $(\xi, 0)$ in the x, y plane of an infinite elastic solid, as shown in Figure 2.3(b), is:

$$\begin{cases} \sigma_{xx}^{b_x} = b_x k y \frac{3(x-h)^2 + y^2}{[(x-h)^2 + y^2]^2} \\ \sigma_{yy}^{b_x} = b_x (-k) y \frac{(x-h)^2 - y^2}{[(x-h)^2 + y^2]^2} \\ \sigma_{xy}^{b_x} = b_x (-k)(x-h) \frac{(x-h)^2 - y^2}{[(x-h)^2 + y^2]^2} \end{cases} \quad (2.8) \quad [31]$$

The equation (2.7) and (2.8) can be obtained easily by letting the two materials in a bimaterial the same and substituting into the equation C.1 and C.2 in the Appendix C.

Thus, the stress field of two pairs of dislocation dipoles as shown in the Figure 2.2 is:

$\sigma = \sigma_{(\xi,0)} + \sigma_{(-\xi,0)}$, which yields the equations:

$$\begin{cases} \sigma_{xx} = b_x k y (A_1 - A_2) + b_y k [(x + \xi)A_4 - (x - \xi)A_3] \\ \sigma_{yy} = b_x k y (A_4 - A_3) + b_y k [(x + \xi)A_6 - (x - \xi)A_5] \\ \sigma_{xy} = b_x k [(x + \xi)A_4 - (x - \xi)A_3] + b_y k y (A_4 - A_3) \end{cases} \quad (2.9)$$

The 6 coefficients can be found in the Equation (2.10).

By comparing the Equations (2.5) with (2.6) and (2.9) with (2.10), a conclusion can

be drawn that the two starting point yield the same mathematical formulae.

$$\begin{aligned} A_1 &= \frac{3(x-\xi)^2 + y^2}{[(x-\xi)^2 + y^2]^2}, & A_2 &= \frac{3(x+\xi)^2 + y^2}{[(x+\xi)^2 + y^2]^2} \\ A_3 &= \frac{(x-\xi)^2 - y^2}{[(x-\xi)^2 + y^2]^2}, & A_4 &= \frac{(x+\xi)^2 - y^2}{[(x+\xi)^2 + y^2]^2} \\ A_5 &= \frac{(x-\xi)^2 + 3y^2}{[(x-\xi)^2 + y^2]^2}, & A_6 &= \frac{(x+\xi)^2 + 3y^2}{[(x+\xi)^2 + y^2]^2} \end{aligned} \quad (2.10)$$

2.3 The implementation of the displacement discontinuity method

2.3.1 The implementation for a tensile crack in an infinite homogeneous elastic medium

Consider a tensile crack, ranging from $[-l, l]$, subjected to a constant traction P_0 is shown in Figure 2.4. Since both the loading and geometry are symmetric respect to the x axis so that only the elastic parameters in the y direction need considering. In such a case the crack can be treated as a series of dislocation distribution with Burgers vector b_y as depicted in Figure 2.4.

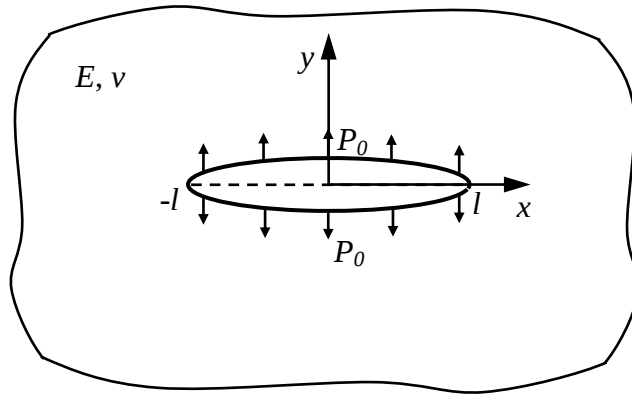


Figure 2. 4: A tensile crack in homogeneous material

Let $B(\xi)d\xi$ represent a continuous density of dislocation dipoles distributed between ξ and $\xi + d\xi$ (Figure 2.5). The coordinate ξ is on the x -axis, and between $[-l, l]$. We can set up an integral equation:

$$\frac{G}{2\pi(1-\nu)} \int_{-l}^l \frac{B(\xi)d\xi}{(x-\xi)} = P_0 \quad (2.11)$$

where G is the shear modulus of the material (see Appendix A and B for details).

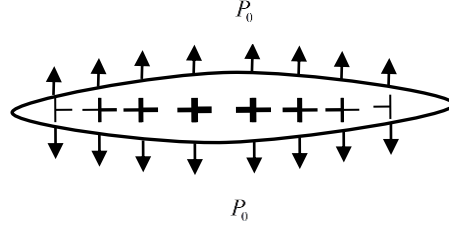


Figure 2. 5: The distribution of dislocation dipoles

To solve the integral equation (2.11) with displacement discontinuity method (DDM), we first divide the crack line $[-l, l]$ into n parts evenly, resulting in n segments, i.e. elements of equal length along $[-l, l]$. In each element, there is one left end point, one right end point and one midpoint, denoted by x_l and x_r , and x_m , respectively with the subscript denoting the element index. Therefore, $x_{l_i}, x_{r_i}, x_{m_i}$ denote the coordinate of the left end point, right end point and the midpoint of the i th element. Then, we assume the dislocation dipole at the end points of each element, which results in the stress at each midpoint, as shown in Figure 2.6.

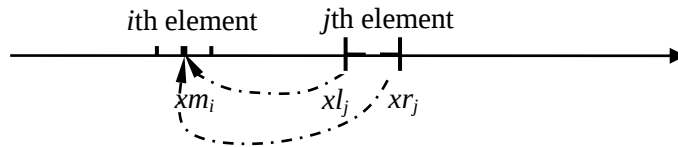


Figure 2. 6: A schematic of the DDM implementation for a tensile crack in a homogeneous medium

Let $P(xm_i)$ denotes the stress at the midpoint of the i th element, and then the equation

(2.11) can be rewritten as:

$$\sigma_{yy}(xm_i, 0) = -\frac{G}{2\pi(1-\nu)} \int_{-l}^l \frac{B(\xi)d\xi}{(xm_i - \xi)} = -P_0 = P(xm_i)$$

Thus:
$$-\frac{G}{2\pi(1-\nu)} \int_{-l}^l \frac{B(\xi)d\xi}{(xm_i - \xi)} = P(xm_i) \quad (2.12)$$

Since $B(\xi)d\xi = dw(\xi)$:

$$-\frac{G}{2\pi(1-\nu)} \int_{-l}^l \frac{dw(\xi)}{(xm_i - \xi)} = P(xm_i)$$

Then the integral is approximated with summation:

$$-\frac{G}{2\pi(1-\nu)} \sum_{j=1}^n \frac{w(\xi_j)}{(xm_i - \xi_j)} = P(xm_i) \quad (2.13)$$

For the dislocation dipole in the j th element, we assume the constant displacement discontinuity: $w(xl_j) = w(xr_j) = w_j$, the equation (2.13) can be rewritten as:

$$-\frac{G}{2\pi(1-\nu)} \sum_{j=1}^n \left[\frac{w(xl_j)}{(xm_i - xl_j)} - \frac{w(xr_j)}{(xm_i - xr_j)} \right] = P(xm_i)$$

Substitute $w(xl_j) = w(xr_j) = w_j$ and make some arrangement:

$$\frac{G}{2\pi(1-\nu)} \sum_{j=1}^n \left[\frac{xr_j - xl_j}{(xm_i - xl_j)(xm_i - xr_j)} w_j \right] = P(xm_i) \quad (2.14)$$

Note that: $xr_j - xl_j$ is just the length of the j th element. Also, since the $[-l, l]$ is divided into n parts evenly. We have: $xr_j - xl_j = \frac{2l}{n}$, and the equation (2.14) can be written as:

$$\frac{G}{\pi(1-\nu)} \frac{l}{n} \sum_{j=1}^n \left[\frac{w_j}{(xm_i - xl_j)(xm_i - xr_j)} \right] = P(xm_i) \quad (2.15)$$

Rewrite it in the format of matrix multiplication:

$$\frac{G}{\pi(1-\nu)} \frac{l}{n} \begin{bmatrix} \frac{1}{(xm_1 - xl_1)(xm_1 - xr_1)} & \frac{1}{(xm_1 - xl_2)(xm_1 - xr_2)} & \cdots & \frac{1}{(xm_1 - xl_n)(xm_1 - xr_n)} \\ \frac{1}{(xm_2 - xl_1)(xm_2 - xr_1)} & \frac{1}{(xm_2 - xl_2)(xm_2 - xr_2)} & \cdots & \frac{1}{(xm_2 - xl_n)(xm_2 - xr_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(xm_n - xl_1)(xm_n - xr_1)} & \frac{1}{(xm_n - xl_2)(xm_n - xr_2)} & \cdots & \frac{1}{(xm_n - xl_n)(xm_n - xr_n)} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{Bmatrix} = \begin{Bmatrix} P(xm_1) \\ P(xm_2) \\ \vdots \\ P(xm_n) \end{Bmatrix}$$

or:

$$K_{ij} w_j = P_i \quad (2.16)$$

where Boundary influence coefficient is given:

$$K_{ij} = \frac{Gl}{\pi(1-\nu)n} \frac{1}{(xm_i - xl_j)(xm_i - xr_j)} \quad (2.17)$$

And the traction vector is defined as:

$$P_i = P(xm_i) = -P_0 \quad (2.18)$$

Then the opening displacement can be solved by:

$$w_j = K_{ij}^{-1} P_i \quad (2.19)$$

2.3.2 The implementation for a tensile crack in an infinite elastic bimaterial

A tensile crack in a bimaterial is illustrated in Figure 2.7. The bimaterial is a combination of two bonded half-space essentially. The equation (2.19) is not valid here because it is derived from the analytical dislocation formulae in the elastic homogenous material. When dealing with the bimaterial, we need to establish the implementation based on the results from Dundurs and Mura (see Appendix C for details). Again, only the displacement and stress in y direction need considering due to

the loading and geometry symmetry.

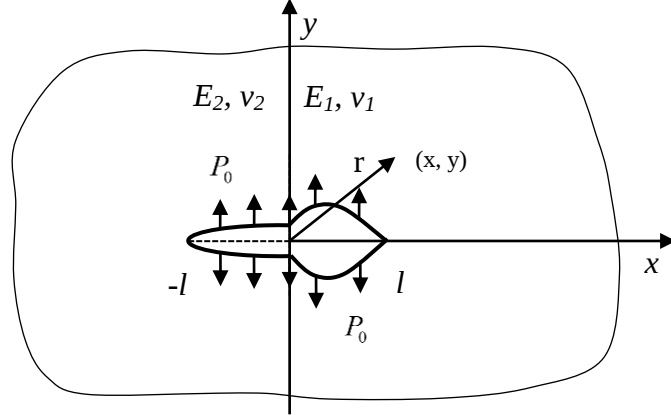


Figure 2. 7: A tensile crack in a bimaterial

Then the similar way as we did for the homogeneous medium, i.e. using the distribution of dislocation dipoles with b_y to represent the tensile crack as depicted in Figure 2.8, is adopted. The difference is that we need to use the analytical dislocation formulae (2.20) instead of (2.11).

The dislocation field formulae can be derived beginning with the definition of the three parameters as shown in the Figure 2.9:

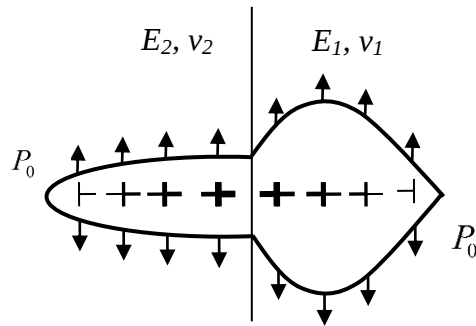


Figure 2. 8: Distribution of dislocation dipoles in Bimaterial

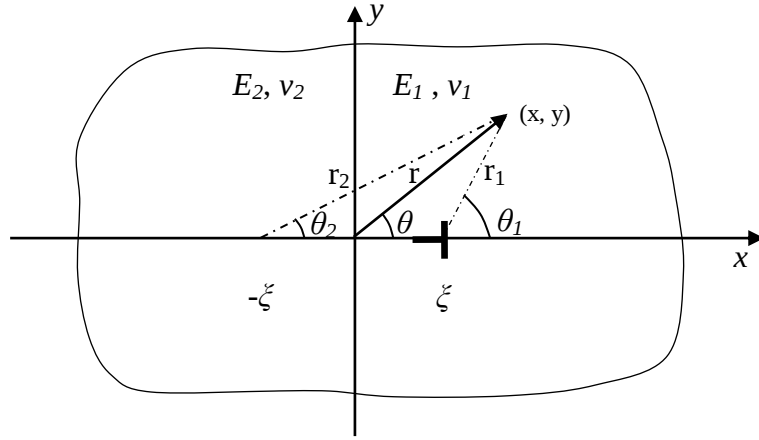


Figure 2. 9: An edge dislocation with the Burgers vector b_y ^[29]

ξ : the distance between the dislocation and the interface

r_1 : the distance between the dislocation and the field point (x, y) , $r_1 = \sqrt{(x - \xi)^2 + y^2}$

r_2 : the distance between the conjugate point of the dislocation $(-\xi, 0)$, and the field point

(x, y) , $r_2 = \sqrt{(x + \xi)^2 + y^2}$

Then Dundurs constants are then defined as follows ^[29]:

$$k_1 = \frac{G_1}{2\pi(1 + \nu_1)}, \quad \alpha = \frac{(1 - \nu_1)G_2 - (1 - \nu_2)G_1}{(1 - \nu_1)G_2 + (1 - \nu_2)G_1}, \quad \beta = \frac{(1 - 2\nu_1)G_2 - (1 - 2\nu_2)G_1}{(1 - \nu_1)G_2 + (1 - \nu_2)G_1}$$

$$a = \frac{1 + \alpha}{1 - \beta^2}, \quad q = \frac{\alpha - \beta}{1 + \beta}$$

The stress field formulae can be obtained as (see Appendix C for details):

$$\begin{aligned} \sigma_{yy}^{(1)}(x, 0) &= k_1 b_y \left\{ \frac{(x - \xi)^3}{r_1^4} + q \frac{(x + \xi)^3}{r_2^4} - 2q\xi \frac{(x + \xi)^4 - 2x(x + \xi)^3}{r_2^6} + a\beta \frac{x + \xi}{r_2^2} \right\} \\ &= b_y k_1 \left[\frac{1}{(x - \xi)} + q \frac{1}{(x + \xi)} - 2q\xi \frac{\xi - x}{(x + \xi)^3} + a\beta \frac{1}{(x + \xi)} \right] \\ &= b_y f_1(x, \xi) \end{aligned} \quad (2.20a)$$

And

$$\begin{aligned}
\sigma_{yy}^{(2)}(x, 0) &= ak_1 b_y \left[\frac{(x-\xi)^3}{r_1^4} - 2\beta \frac{\xi(x-\xi)^2}{r_1^4} \right] \\
&= b_y ak_1 \left[\frac{1}{(x-\xi)} - 2\beta \frac{\xi}{(x-\xi)^2} \right] \\
&= b_y f_2(x, \xi)
\end{aligned} \tag{2.20b}$$

And the function f is defined as:

$$f(x, \xi) = \begin{cases} f_1(x, \xi) = k_1 \left[\frac{1}{(x-\xi)} + q \frac{1}{(x+\xi)} - 2q\xi \frac{\xi-x}{(x+\xi)^3} + a\beta \frac{1}{(x+\xi)} \right], & x \geq 0 \\ f_2(x, \xi) = ak_1 \left[\frac{1}{(x-\xi)} - 2\beta \frac{\xi}{(x-\xi)^2} \right] & , \quad x < 0 \end{cases} \tag{2.20c}$$

Since the crack line is straight on the x axis, we can have the equation as we did for the homogeneous medium case for a single dislocation:

$$\sigma_{yy}(x, 0) = f(x, \xi) B(\xi) d\xi$$

The stress produced by the distribution of dislocations along the entire length of the crack is:

$$\sigma_{yy}(x, 0) = \int_{-l}^l f(x, \xi) B(\xi) d\xi = -P_0 \tag{2.21}$$

with $d\hat{u}(\xi) = B(\xi) d\xi$

In the same manner with the same definition in the last section:

$$\sum_{j=1}^n f(xm_i, \xi_j) w(\xi_j) = -P_0 \tag{2.22}$$

For the dislocation dipole in the j th element, we assume the constant displacement discontinuity: $w(xl_j) = w(xr_j) = w_j$, the equation (2.22) can be rewritten as:

$$\sum_{j=1}^n [f(xm_i, xl_j) - f(xm_i, xr_j)] w(\xi_j) = -P_0 = P(xm_i) \tag{2.23}$$

The matrix form of Equation (2.23) is:

$$\begin{bmatrix} f(xm_1, xr_1) - f(xm_1, xl_1) & f(xm_1, xr_2) - f(xm_1, xl_2) & \dots & \dots & f(xm_1, xr_n) - f(xm_1, xl_n) \\ f(xm_2, xr_1) - f(xm_2, xl_1) & f(xm_2, xr_2) - f(xm_2, xl_2) & \dots & \dots & f(xm_2, xr_n) - f(xm_2, xl_n) \\ \vdots & \vdots & \dots & \dots & \vdots \\ f(xm_n, xr_1) - f(xm_n, xl_1) & f(xm_n, xr_2) - f(xm_n, xl_2) & \dots & \dots & f(xm_n, xr_n) - f(xm_n, xl_n) \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{Bmatrix} = \begin{Bmatrix} P(xm_1) \\ P(xm_2) \\ \vdots \\ P(xm_n) \end{Bmatrix}$$

Or:

$$K_{ij}w_j = P_i \quad (2.24)$$

The boundary influence coefficients and traction vector are given as:

$$K_{ij} = f(xm_i, xr_j) - f(xm_i, xl_j) \quad (2.25)$$

$$P_i = P(xm_i) = -P_0 \quad (2.26)$$

Then the opening displacement can be solved by:

$$w_j = K_{ij}^{-1}P_i \quad (2.27)$$

However, neither Equation (2.19) nor (2.27) can be implemented for a curved crack problem since the symmetry condition breaks down. Then the local coordinate made up of the shear (*s* for short) and normal (*n* for short) direction instead of the *x* and *y* direction in the global coordinate system arises because it makes the expression more convenient. But before that we must define the shear and normal directions in the boundary integral method.

2.3.3 Convention for the shear and normal direction^[9]

In boundary integral methods, the normal direction is defined as: the outward normal to any boundary must point away from the solid. And the shear (or traversal) direction is defined as: The boundary of a finite body is traversed in the clockwise sense, whereas

the boundary of a cavity is traversed in the counterclockwise sense. Therefore, in three general cases, the normal and traversal directions are shown in the Figure 2.10:

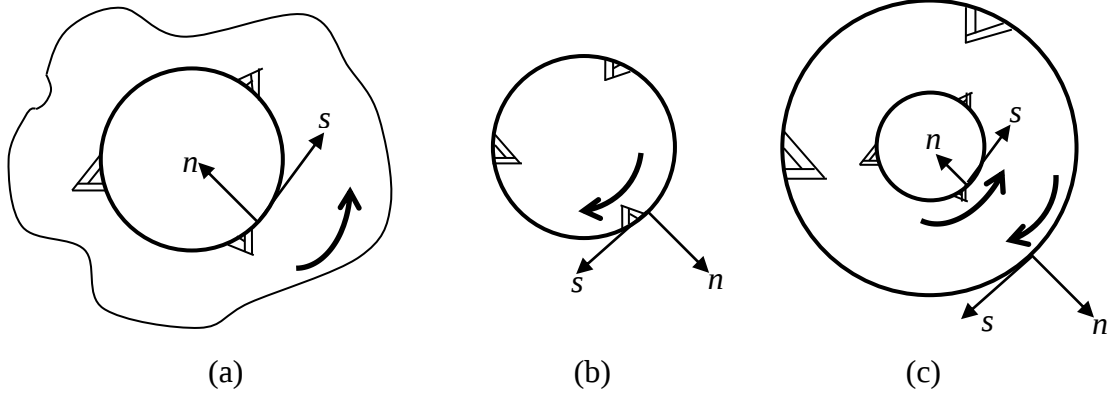


Figure 2. 10: Convention for shear and normal directions of boundaries: (a) cavity; (b) solid disk; (c) disk with hole

With the definition of this convention, we can start to figure out the boundary influence coefficients with local coordinates.

2.3.4 Boundary influence coefficients of DDM in local coordinate

When the crack line is of arbitrary form, instead of a straight line, as shown in Figure 2.11, it becomes more convenient to use the local coordinates since it is simpler to write the coefficient equation with respect to each element, respectively.

The curved crack line is discretized and approximated into N small straight line segments, each joining the ends of its adjacent ones. The local coordinate of an element is centered at the midpoint of this element with the shear and normal direction defined following the convention in the section 2.3.3 ^[9].

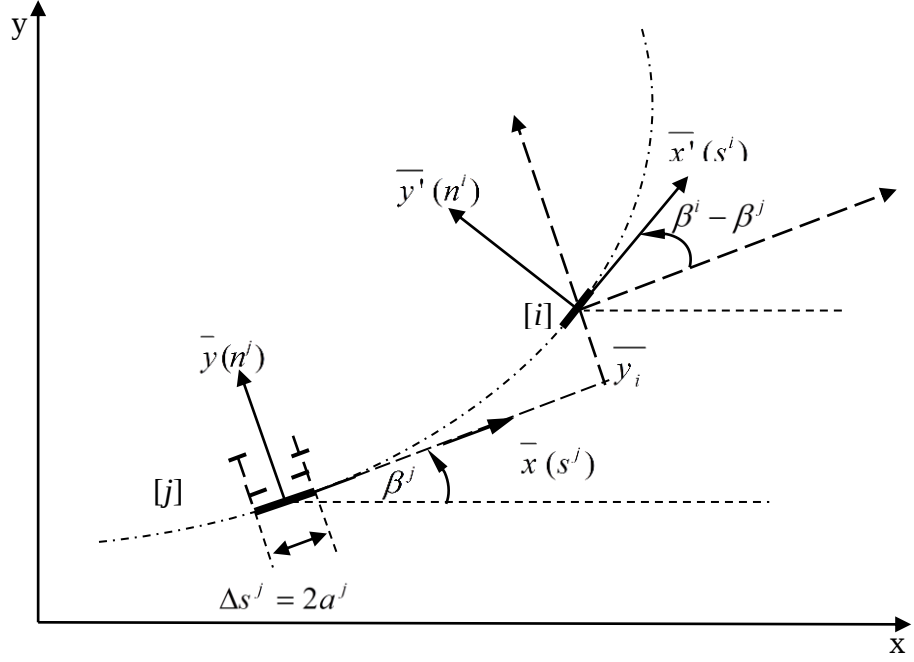


Figure 2. 11: A curved crack with local coordinate

Now we consider the stress at the center of the i th element due to the dislocation dipoles located at $(\pm a_j, 0)$, e.g. the ends of the j th element, known as the source point, in the $(\bar{x}, \bar{y})$ coordinate defined by the j th element. The center of the i th element is $(\bar{x}_i, \bar{y}_i)$ with $(\bar{x}, \bar{y})$ coordinate. In the Figure 2.11, the ζ in Equation (2.9)-(2.10) is replaced by the half length of the j th element a_j , then:

$$\begin{aligned}
 \bar{A}_1 &= \frac{3(\bar{x}_i - a_j)^2 + \bar{y}_i^2}{[(\bar{x}_i - a_j)^2 + \bar{y}_i^2]^2}, & \bar{A}_2 &= \frac{3(\bar{x}_i + a_j)^2 + \bar{y}_i^2}{[(\bar{x}_i + a_j)^2 + \bar{y}_i^2]^2} \\
 \bar{A}_3 &= \frac{(\bar{x}_i - a_j)^2 - \bar{y}_i^2}{[(\bar{x}_i - a_j)^2 + \bar{y}_i^2]^2}, & \bar{A}_4 &= \frac{(\bar{x}_i + a_j)^2 - \bar{y}_i^2}{[(\bar{x}_i + a_j)^2 + \bar{y}_i^2]^2} \\
 \bar{A}_5 &= \frac{(\bar{x}_i - a_j)^2 + 3\bar{y}_i^2}{[(\bar{x}_i - a_j)^2 + \bar{y}_i^2]^2}, & \bar{A}_6 &= \frac{(\bar{x}_i + a_j)^2 + 3\bar{y}_i^2}{[(\bar{x}_i + a_j)^2 + \bar{y}_i^2]^2}
 \end{aligned} \tag{2.28}$$

And

$$\begin{cases} \sigma_{xx}^{b_{xj}^-} = b_{xj}^- k \bar{y}_i (\bar{A}_1 - \bar{A}_2) \\ \sigma_{yy}^{b_{xj}^-} = b_{xj}^- k \bar{y}_i (\bar{A}_4 - \bar{A}_3) \\ \sigma_{xy}^{b_{xj}^-} = b_{xj}^- k [(\bar{x}_i + a_j) \bar{A}_4 - (\bar{x}_i - a_j) \bar{A}_3] \end{cases} \quad (2.29)$$

$$\begin{cases} \sigma_{xx}^{b_{yj}^-} = b_{yj}^- k [(\bar{x}_i + a_j) \bar{A}_4 - (\bar{x}_i - a_j) \bar{A}_3] \\ \sigma_{yy}^{b_{yj}^-} = b_{yj}^- k [(\bar{x}_i + a_j) \bar{A}_6 - (\bar{x}_i - a_j) \bar{A}_5] \\ \sigma_{xy}^{b_{yj}^-} = b_{yj}^- k \bar{y}_i (\bar{A}_4 - \bar{A}_3) \end{cases} \quad (2.30)$$

Notice that $w_s^j = b_{xj}^-$, $w_n^j = b_{yj}^-$:

$$\begin{cases} \sigma_{xx}^i = w_s^j k \bar{y}_i (\bar{A}_1 - \bar{A}_2) + w_n^j k [(\bar{x}_i + a_j) \bar{A}_4 - (\bar{x}_i - a_j) \bar{A}_3] \\ \sigma_{yy}^i = w_s^j k \bar{y}_i (\bar{A}_4 - \bar{A}_3) + w_n^j k [(\bar{x}_i + a_j) \bar{A}_6 - (\bar{x}_i - a_j) \bar{A}_5] \\ \sigma_{xy}^i = w_s^j k [(\bar{x}_i + a_j) \bar{A}_4 - (\bar{x}_i - a_j) \bar{A}_3] + w_n^j k \bar{y}_i (\bar{A}_4 - \bar{A}_3) \end{cases} \quad (2.31)$$

It is convenient to express the stresses in the Equation (2.31) in the local coordinate defined by the i th element, i.e. the field point.

Let $r = \beta^i - \beta^j$, and substitute the Equation (2.31) into:

$$\begin{cases} \sigma_t^i = \sigma_{x'x'}^i = \sigma_{xx}^i \cos^2 r + 2\sigma_{xy}^i \sin r \cos r + \sigma_{yy}^i \sin^2 r \\ \sigma_n^i = \sigma_{y'y'}^i = \sigma_{xx}^i \sin^2 r - 2\sigma_{xy}^i \sin r \cos r + \sigma_{yy}^i \cos^2 r \\ \sigma_s^i = \sigma_{x'y'}^i = -(\sigma_{xx}^i - \sigma_{yy}^i) \sin r \cos r + \sigma_{xy}^i (\cos^2 r - \sin^2 r) \end{cases} \quad (2.32) \quad [9]$$

We can get:

$$\begin{aligned} \frac{\sigma_t^i}{2G} &= \frac{1}{4\pi(1-\nu)} w_s^j \{ \bar{y}_i (\bar{A}_1 - \bar{A}_2) \cos^2 r + [(\bar{x}_i + a_j) \bar{A}_4 - (\bar{x}_i - a_j) \bar{A}_3] \sin(2r) \\ &\quad + \bar{y}_i (\bar{A}_4 - \bar{A}_3) \sin^2 r \} + \\ &\quad \frac{1}{4\pi(1-\nu)} w_n^j \{ [(\bar{x}_i + a_j) \bar{A}_4 - (\bar{x}_i - a_j) \bar{A}_3] \cos^2 r + \bar{y}_i (\bar{A}_4 - \bar{A}_3) \sin(2r) + \\ &\quad [(\bar{x}_i + a_j) \bar{A}_6 - (\bar{x}_i - a_j) \bar{A}_5] \sin^2 r \} \end{aligned} \quad (2.33)$$

$$\begin{aligned}
\frac{\sigma_s^i}{2G} = & \frac{1}{4\pi(1-\nu)} w_s^j \{ \bar{y}_i (\bar{A}_4 - \bar{A}_3 + \bar{A}_2 - \bar{A}_1) \frac{\sin(2r)}{2} \\
& + [(\bar{x}_i + a_j) \bar{A}_4 - (\bar{x}_i - a_j) \bar{A}_3] \cos(2r) \} + \\
& \frac{1}{4\pi(1-\nu)} w_n^j \{ [(\bar{x}_i + a_j) (\bar{A}_6 - \bar{A}_4) - (\bar{x}_i - a_j) (\bar{A}_5 - \bar{A}_3)] \frac{\sin(2r)}{2} \\
& + \bar{y}_i (\bar{A}_4 - \bar{A}_3) \cos(2r) \}
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
\frac{\sigma_n^i}{2G} = & \frac{1}{4\pi(1-\nu)} w_s^j \{ \bar{y}_i (\bar{A}_1 - \bar{A}_2) \sin^2 r + [(\bar{x}_i - a_j) \bar{A}_3 - (\bar{x}_i + a_j) \bar{A}_4] \sin(2r) \\
& + \bar{y}_i (\bar{A}_4 - \bar{A}_3) \cos^2 r \} + \\
& \frac{1}{4\pi(1-\nu)} w_n^j \{ [(\bar{x}_i + a_j) \bar{A}_4 - (\bar{x}_i - a_j) \bar{A}_3] \sin^2 r + \bar{y}_i (\bar{A}_3 - \bar{A}_4) \sin(2r) \\
& + [(\bar{x}_i + a_j) \bar{A}_6 - (\bar{x}_i - a_j) \bar{A}_5] \cos^2 r \}
\end{aligned} \tag{2.35}$$

From Equation (2.34) and (2.35), we have:

$$\frac{1}{4\pi(1-\nu)} \begin{bmatrix} B_{ss}^{ij} & B_{sn}^{ij} \\ B_{ns}^{ij} & B_{nn}^{ij} \end{bmatrix} \begin{Bmatrix} w_s^j \\ w_n^j \end{Bmatrix} = \begin{Bmatrix} \frac{\sigma_s^j}{2G} \\ \frac{\sigma_n^j}{2G} \end{Bmatrix} \tag{2.36}$$

The boundary influence coefficients are:

$$B_{ss}^{ij} = \bar{y}_i (\bar{A}_4 - \bar{A}_3 + \bar{A}_2 - \bar{A}_1) \frac{\sin(2r)}{2} + [(\bar{x}_i + a_j) \bar{A}_4 - (\bar{x}_i - a_j) \bar{A}_3] \cos(2r) \tag{2.37}$$

$$B_{sn}^{ij} = [(\bar{x}_i + a_j) (\bar{A}_6 - \bar{A}_4) - (\bar{x}_i - a_j) (\bar{A}_5 - \bar{A}_3)] \frac{\sin(2r)}{2} + \bar{y}_i (\bar{A}_4 - \bar{A}_3) \cos(2r) \tag{2.38}$$

$$\begin{aligned}
B_{ns}^{ij} = & \bar{y}_i (\bar{A}_1 - \bar{A}_2) \sin^2 r + [(\bar{x}_i - a_j) \bar{A}_3 - (\bar{x}_i + a_j) \bar{A}_4] \sin(2r) \\
& + \bar{y}_i (\bar{A}_4 - \bar{A}_3) \cos^2 r
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
B_{nn}^{ij} = & [(\bar{x}_i + a_j) \bar{A}_4 - (\bar{x}_i - a_j) \bar{A}_3] \sin^2 r + \bar{y}_i (\bar{A}_3 - \bar{A}_4) \sin(2r) \\
& + [(\bar{x}_i + a_j) \bar{A}_6 - (\bar{x}_i - a_j) \bar{A}_5] \cos^2 r
\end{aligned} \tag{2.40}$$

Let the crack be divided into N elements, so that the element index i or j range from 1

to N , the system equation can then be constructed based on the Equation (2.36):

$$\frac{1}{4\pi(1-\nu)} \begin{bmatrix} B_{ss}^{11} & B_{sn}^{11} & \dots & \dots & B_{ss}^{1j} & B_{sn}^{1j} & \dots & \dots & B_{ss}^{1N} & B_{sn}^{1N} \\ B_{ns}^{11} & B_{nn}^{11} & \dots & \dots & B_{ns}^{1j} & B_{nn}^{1j} & \dots & \dots & B_{ns}^{1N} & B_{nn}^{1N} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ B_{ss}^{i1} & B_{sn}^{i1} & \dots & \dots & B_{ss}^{ij} & B_{sn}^{ij} & \dots & \dots & B_{ss}^{iN} & B_{sn}^{iN} \\ B_{ns}^{i1} & B_{nn}^{i1} & \dots & \dots & B_{ns}^{ij} & B_{nn}^{ij} & \dots & \dots & B_{ns}^{iN} & B_{nn}^{iN} \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots & \ddots & \ddots & \dots & \dots \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots & \ddots & \ddots & \dots & \dots \\ B_{ss}^{N1} & B_{sn}^{N1} & \dots & \dots & B_{ss}^{Nj} & B_{sn}^{Nj} & \dots & \dots & B_{ss}^{NN} & B_{sn}^{NN} \\ B_{ns}^{N1} & B_{nn}^{N1} & \dots & \dots & B_{ns}^{Nj} & B_{nn}^{Nj} & \dots & \dots & B_{ns}^{NN} & B_{nn}^{NN} \end{bmatrix} \begin{Bmatrix} w_s^1 \\ w_n^1 \\ \vdots \\ \vdots \\ w_s^j \\ w_n^j \\ \vdots \\ \vdots \\ w_s^N \\ w_n^N \end{Bmatrix} = \begin{Bmatrix} \sigma_s^1 \\ \sigma_n^1 \\ \vdots \\ \vdots \\ \sigma_s^i \\ \sigma_n^i \\ \vdots \\ \vdots \\ \sigma_s^N \\ \sigma_n^N \end{Bmatrix} \frac{1}{2G}$$

Write the system of equation above in a concise form:

$$\frac{1}{4\pi(1-\nu)} B_{ij} w_j = \frac{\sigma_i}{2G} \quad (2.41)$$

2.3.6 Crack tip element approach

Crouch and Starfield proposed a crack tip element approach in their book ^[9] to treat the crack tip element for a better accuracy of a straight line crack in the homogeneous medium. The normal stress of a single straight constant displacement discontinuity in an infinite isotropic homogeneous medium is:

$$\sigma_{yy}(x,0) = \frac{-aG}{\pi(1-\nu)} D_y \frac{1}{x^2 - a^2} \quad (2.42)^{[9]}$$

where D_y stands for the normal displacement discontinuity.

This equation is obtained by assuming the opening $w_y(\xi) = D_y$ is a constant in the integral:

$$\sigma_{yy}(x,0) = \frac{-G}{2\pi(1-\nu)} \lim_{y \rightarrow 0} \int_{-a}^a D_y(\xi) \frac{1}{(x-\xi)^2 + y^2} d\xi \quad (2.43)^{[9]}$$

The special crack tip element for the crack tip element is shown in Figure 2.12:

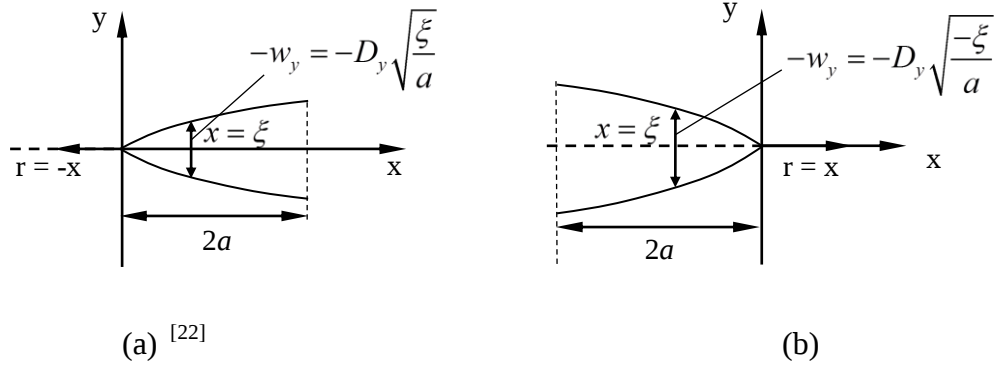


Figure 2. 12: Two crack tip elements ^[9]

For the left crack tip element, we have $w_y(\xi) = D_y \sqrt{\frac{\xi}{a}}$ in $0 \leq \xi \leq 2a$, and Equation

(2.43) can be written as:

$$\sigma_{yy}(x, 0) = \frac{-GD_y}{2\pi(1-\nu)} \lim_{y \rightarrow 0} \int_{-a}^a \sqrt{\frac{\xi}{a}} \frac{1}{(x-\xi)^2 + y^2} d\xi \quad (2.44)^{[9]}$$

Upon evaluation of the integral we obtain, for $x > 0$,

$$\sigma_{yy}(x, 0) = \frac{-GD_y}{2\pi(1-\nu)} \left[\frac{\sqrt{2}}{x-2a} + \frac{1}{2\sqrt{ax}} \ln \left| \frac{\sqrt{x}-\sqrt{2a}}{\sqrt{x}+\sqrt{2a}} \right| \right] \quad (2.45-a)^{[9]}$$

And, for $x < 0$,

$$\sigma_{yy}(x, 0) = \frac{-GD_y}{2\pi(1-\nu)} \frac{1}{\sqrt{ar}} \left[\arctan \sqrt{\frac{2a}{r}} - \frac{\sqrt{2ar}}{r+2a} \right] \quad (2.45-b)^{[9]}$$

Similar expressions can be derived for the special crack tip element on the right-hand side end of the crack as shown in the Figure 2.12(b), for $x < 0$,

$$\sigma_{yy}(x, 0) = \frac{-GD_y}{2\pi(1-\nu)} \left[\frac{\sqrt{2}}{-x-2a} + \frac{1}{2\sqrt{-ax}} \ln \left| \frac{\sqrt{-x}-\sqrt{2a}}{\sqrt{-x}+\sqrt{2a}} \right| \right] \quad (2.46-a)^{[9]}$$

And for $x > 0$,

$$\sigma_{yy}(x, 0) = \frac{-GD_y}{2\pi(1-\nu)} \frac{1}{\sqrt{ar}} \left[\arctan \sqrt{\frac{2a}{r}} - \frac{\sqrt{2ar}}{r+2a} \right] \quad (2.46-b)^{[9]}$$

Therefore, we can write the expression for both crack tip elements:

$$\sigma_{yy}(x, 0) = \frac{-GD_y}{2\pi(1-\nu)} \left[\frac{\sqrt{2}}{|x|-2a} + \frac{1}{2\sqrt{a|x|}} \ln \left| \frac{\sqrt{|x|}-\sqrt{2a}}{\sqrt{|x|}+\sqrt{2a}} \right| \right] \quad (2.47)^{[9]}$$

Notice that the $|x|$ in Equation (2.47) stands for the distance from the load point $(x, 0)$ to the crack end so that the boundary influence coefficients can be written in the *local* coordinate as follows:

$$A^{ij} = \frac{-G}{\pi(1-\nu)} \frac{a^j}{(x^i - x^j)^2 - (a^j)^2} \quad , \text{ provided } j \neq 1 \text{ and } j \neq N, \quad (2.48)^{[9]}$$

and

$$A^{ij} = \frac{-G}{2\pi(1-\nu)} \left[\frac{\sqrt{2}}{S-2a^j} + \frac{1}{2\sqrt{a^j S}} \ln \left| \frac{\sqrt{S}-\sqrt{2a^j}}{\sqrt{S}+\sqrt{2a^j}} \right| \right] \quad \text{for } j = 1 \text{ and } j = N \quad (2.49)^{[9]}$$

Inside Equation (2.49), we set $S = a^j + |x^i - x^j|$

However, the establishment of the local coordinate of the crack tip elements is different from the other elements: the local tip coordinate is based on the tip end, while other local coordinates are based on the midpoint of the elements. Then, we will change the tip local coordinate to the midpoint of the tip elements, as shown in Figure 2.13.

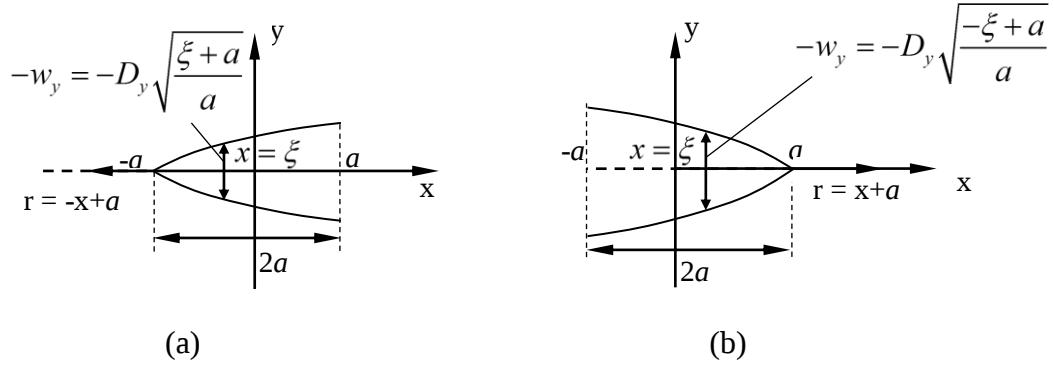


Figure 2. 13: Two crack tip elements with revised local coordinates

We can get the similar expressions by following the procedures above for the left end:

$$\sigma_{yy}(x, 0) = \frac{-Gw_y}{2\pi(1-\nu)} \begin{cases} \frac{\sqrt{2}}{x-a} + \frac{1}{2\sqrt{a(x+a)}} \ln \left| \frac{\sqrt{x+a} - \sqrt{2a}}{\sqrt{x+a} + \sqrt{2a}} \right|, & (x > -a) \\ \frac{1}{\sqrt{ar}} \left(\arctan \sqrt{\frac{2a}{r}} - \frac{\sqrt{2ar}}{r+2a} \right), & (x < -a) \end{cases} \quad (2.50)$$

And for the right end:

$$\sigma_{yy}(x, 0) = \frac{-Gw_y}{2\pi(1-\nu)} \begin{cases} -\frac{\sqrt{2}}{x+a} + \frac{1}{2\sqrt{a(a-x)}} \ln \left| \frac{\sqrt{a-x} - \sqrt{2a}}{\sqrt{a-x} + \sqrt{2a}} \right|, & (x < a) \\ \frac{1}{\sqrt{ar}} \left(\arctan \sqrt{\frac{2a}{r}} - \frac{\sqrt{2ar}}{r+2a} \right), & (x > a) \end{cases} \quad (2.51)$$

Therefore, we can write the expression for both crack tip elements, similarly:

$$\sigma_{yy}(x^i, 0) = \frac{-Gw_y}{2\pi(1-\nu)} \left[\frac{\sqrt{2}}{S'} + \frac{1}{2\sqrt{a^j(S'+2a^j)}} \ln \left| \frac{\sqrt{S'+2a^j} - \sqrt{2a^j}}{\sqrt{S'+2a^j} + \sqrt{2a^j}} \right| \right] \quad (2.52)$$

Inside Equation (2.52), we set $S' = |x^i - x^j| - a^j$

Then the boundary influence coefficients can be written in the *local* coordinate as follows:

$$A^{ij} = \frac{-G}{\pi(1-\nu)} \frac{a^j}{(x^i - x^j)^2 - (a^j)^2}, \text{ provided } j \neq 1 \text{ and } j \neq N, \quad (2.53)$$

and

$$A^{ij} = \frac{-G}{2\pi(1-\nu)} \left[\frac{\sqrt{2}}{S^j} + \frac{1}{2\sqrt{a^j(S^j + 2a^j)}} \ln \left| \frac{\sqrt{S^j + 2a^j} - \sqrt{2a^j}}{\sqrt{S^j + 2a^j} + \sqrt{2a^j}} \right| \right] \quad (2.54)$$

for $j = 1$ and $j = N$

The system equations can then be written as:

$$A^{ij} w_y^j = P_y^i \quad (2.55)$$

2.3.7 Revised crack tip element approach

In section 2.3.6, the crack is assumed to be straight to simplify the integral equation (2.43) for the following derivation of explicit formulae for the crack tip element approach implementation.

$$\sigma_{yy}(x, 0) = \frac{-G}{2\pi(1-\nu)} \lim_{y \rightarrow 0} \int_{-a}^a D_y(\xi) \frac{1}{(x - \xi)^2 + y^2} d\xi \quad (2.43)^{[9]}$$

Such formulae breaks down when the crack is curved since the y component is no longer zero in the Equation (2.43). We then have to revise the method to make it applicable for curved crack.

The DDM starting from the dislocation theory is based on the assumption that the displacement discontinuity remains constant in each element. Thus, it is better to introduce the Kelvin's solution because the crack tip opening is assumed to be proportional to $\zeta^{1/2}$ [9].

Substitute the equation $w_y(\xi) = D_y \sqrt{\frac{\xi + a}{a}}$ into the Equation (2.3) for the left-end

crack tip element depicted in the Figure 2.14(a), we have:

$$f(x, y) = \frac{-D_y}{8\pi(1-\nu)} \int_{-a}^a \ln[(x-\xi)^2 + y^2] d\xi \quad (2.3) \quad [9]$$

$$\begin{aligned} f(x, y) &= \frac{-D_y}{4\pi(1-\nu)} \int_{-a}^a \sqrt{\frac{\xi + a}{a}} \ln[(x-\xi)^2 + y^2]^{\frac{1}{2}} d\xi \\ &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \int_{-a}^a \sqrt{\xi + a} \ln[(x-\xi)^2 + y^2] d\xi \\ &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} I' \Big|_{-a}^a \end{aligned} \quad (2.56)$$

However, there is no explicit expression for $I' = \int_{-a}^a \sqrt{\xi + a} \ln[(x-\xi)^2 + y^2] d\xi$,

therefore we discretize $\sqrt{\xi + a}$ into n piece wise linear functions to approximate I' ,

then:

$$\begin{aligned} I' &= \sum_{i=1}^n \int (c_i \xi + b_i) \ln[(x-\xi)^2 + y^2] d\xi \\ &= \sum_{i=1}^n \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} + b_i(\xi - x) \right] \ln[(x-\xi)^2 + y^2] - \frac{c_i \xi^2}{2} - (c_i x + 2b_i) \xi \\ &\quad + \sum_{i=1}^n 2y(c_i x + b_i) \arctan \frac{y}{\xi - x} \end{aligned} \quad (2.57)$$

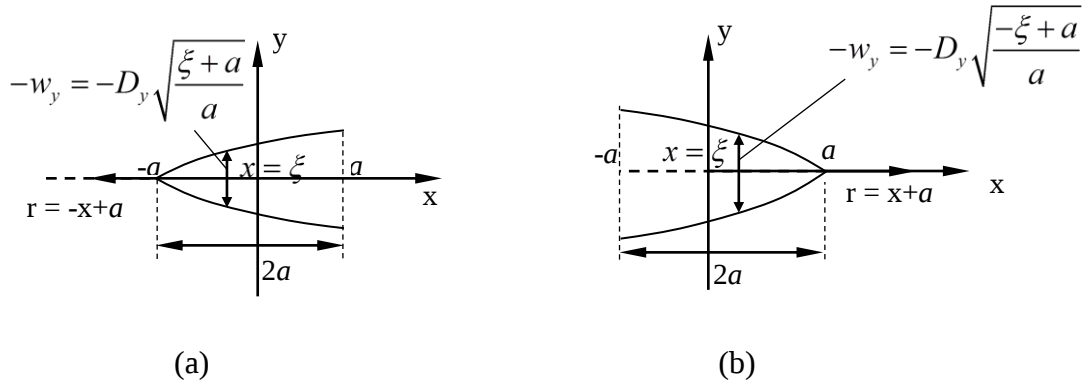


Figure 2. 14: Crack tip elements

Substitute (2.57) into (2.56):

$$f(x, y) = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} + b_i(\xi - x) \right] \ln[(x - \xi)^2 + y^2] \right. \\ \left. + 2y(c_i x + b_i) \arctan \frac{\xi - x}{y} - \frac{c_i \xi^2}{2} - (c_i x + 2b_i)\xi \right\} \Bigg|_{-a}^a \quad (2.58)$$

Omitting the constant associated with the parameter ξ only:

$$f(x, y) = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} + b_i(\xi - x) \right] \ln[(x - \xi)^2 + y^2] \right. \\ \left. + 2y(c_i x + b_i) \arctan \frac{\xi - x}{y} - c_i x \xi \right\} \Bigg|_{-a}^a \quad (2.59)$$

Since the tip element are divided into n uniform elements consequently:

$$f(x, y) = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} - b_i(x - \xi) \right] \ln[(x - \xi)^2 + y^2] \right. \\ \left. + 2y(c_i x + b_i) \arctan \frac{y}{x - \xi} - c_i x \xi \right\} \Bigg|_{a_i}^{a_{i+1}} \quad (2.60)$$

Inside Equation (2.60): $a_1 = -a$, $a_{n+1} = a$.

Take derivatives with omitting non-essential constants:

$$\begin{aligned} f_{,x} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ -c_i x + 2c_i y \arctan \frac{y}{x - \xi} - (c_i x + b_i) \ln[(x - \xi)^2 + y^2] \right\} \Bigg|_{a_i}^{a_{i+1}} \\ f_{,y} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ c_i y + c_i y \ln[(x - \xi)^2 + y^2] + 2(c_i x + b_i) \arctan \frac{y}{x - \xi} \right\} \Bigg|_{a_i}^{a_{i+1}} \\ f_{,xy} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ 2c_i \arctan \frac{y}{x - \xi} - \frac{2(c_i \xi + b_i)y}{(x - \xi)^2 + y^2} \right\} \Bigg|_{a_i}^{a_{i+1}} \\ f_{,yy} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ c_i \ln[(x - \xi)^2 + y^2] + 2 \frac{(c_i x + b_i)(x - \xi) + c_i y^2}{(x - \xi)^2 + y^2} \right\} \Bigg|_{a_i}^{a_{i+1}} \\ f_{,xyy} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2 \frac{(c_i x - 2c_i \xi - b_i)(x - \xi)^2 + (c_i x + b_i)y^2}{[(x - \xi)^2 + y^2]^2} \Bigg|_{a_i}^{a_{i+1}} \\ f_{,yyy} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2y \frac{(c_i x - 3c_i \xi - 2b_i)(x - \xi) + c_i y^2}{[(x - \xi)^2 + y^2]^2} \Bigg|_{a_i}^{a_{i+1}} \end{aligned} \quad (2.61 \text{ a-f})$$

The expressions for the right tip element as shown in Figure 2.14(b) are the same as the Equation (2.61).

Notice that if n , called the number of sub-crack tip element, is an even number, the approximated piece-wise linear functions will pass the *irregular* point when $x=0$, making the integral singular. Thus, n should take odd numbers.

Then with comparison the Equation (2.9) and (2.60) with some arrangements, we get:

$$\begin{cases} \frac{\sigma_s^i}{2G} = w_s^j \{-\sin(2r)\bar{F}_1 - \cos(2r)\bar{F}_2 - \bar{y}_i[\sin(2r)\bar{F}_3 - \cos(2r)\bar{F}_4]\} + \\ \quad w_n^j \{-\bar{y}_i[\cos(2r)\bar{F}_3 + \sin(2r)\bar{F}_4]\} \\ \frac{\sigma_n^i}{2G} = w_s^j \{2\sin^2 r \bar{F}_1 + \sin(2r)\bar{F}_2 - \bar{y}_i[\cos(2r)\bar{F}_3 + \sin(2r)\bar{F}_4]\} + \\ \quad w_n^j \{-\bar{F}_2 + \bar{y}_i[\sin(2r)\bar{F}_3 - \cos(2r)\bar{F}_4]\} \end{cases} \quad (2.62)$$

Inside Equation (2.62):

$$\bar{F}_1(\bar{x}, \bar{y}) = \frac{1}{4\pi(1-\nu)} \left[\frac{\bar{y}}{(\bar{x}-a)^2 + \bar{y}^2} - \frac{\bar{y}}{(\bar{x}+a)^2 + \bar{y}^2} \right] \quad (2.63)$$

For the influence coefficients associated with the normal displacement discontinuity in the crack tip elements:

$$\begin{cases} \bar{F}_2(\bar{x}, \bar{y}) = \frac{1}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ c_i \ln[(\bar{x}-\xi)^2 + \bar{y}^2] + 2 \frac{(c_i\bar{x} + b_i)(\bar{x}-\xi) + c_i\bar{y}^2}{(\bar{x}-\xi)^2 + \bar{y}^2} \right\} \bigg|_{a_i}^{a_{i+1}} \\ \bar{F}_3(\bar{x}, \bar{y}) = \frac{1}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2 \frac{(c_i\bar{x} - 2c_i\xi - b_i)(\bar{x}-\xi)^2 + (c_i\bar{x} + b_i)\bar{y}^2}{[(\bar{x}-\xi)^2 + \bar{y}^2]^2} \bigg|_{a_i}^{a_{i+1}} \\ \bar{F}_4(\bar{x}, \bar{y}) = \frac{1}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2y \frac{(c_i\bar{x} - 3c_i\xi - 2b_i)(\bar{x}-\xi) + c_i\bar{y}^2}{[(\bar{x}-\xi)^2 + \bar{y}^2]^2} \bigg|_{a_i}^{a_{i+1}} \end{cases} \quad (2.64)$$

In other cases:

$$\begin{cases} \overline{F}_2(\bar{x}, \bar{y}) = \frac{+1}{4\pi(1-\nu)} \left[\frac{\bar{x}-a}{(\bar{x}-a)^2 + \bar{y}^2} - \frac{\bar{x}+a}{(\bar{x}+a)^2 + \bar{y}^2} \right] \\ \overline{F}_3(\bar{x}, \bar{y}) = \frac{+1}{4\pi(1-\nu)} \left\{ \frac{(\bar{x}-a)^2 - \bar{y}^2}{[(\bar{x}-a)^2 + \bar{y}^2]^2} - \frac{(\bar{x}+a)^2 - \bar{y}^2}{[(\bar{x}+a)^2 + \bar{y}^2]^2} \right\} \\ \overline{F}_4(\bar{x}, \bar{y}) = \frac{+2\bar{y}}{4\pi(1-\nu)} \left\{ \frac{\bar{x}-a}{[(\bar{x}-a)^2 + \bar{y}^2]^2} - \frac{\bar{x}+a}{[(\bar{x}+a)^2 + \bar{y}^2]^2} \right\} \end{cases} \quad (2.65)$$

Combined the Equation (2.62) through (2.65), the displacement discontinuity method can be implemented with the revised crack-tip element approach for the curved crack in an elastic homogeneous medium. The detailed derivation of the equations in this section can be found in the Appendix D.

2.4 Results and remark

Now we consider a tensile crack subjected to a constant pressure P_0 in homogeneous material as shown in Figure 2.5, which lies on the x -axis, centers at the original of the coordinate, ranges from $-l$ to l .

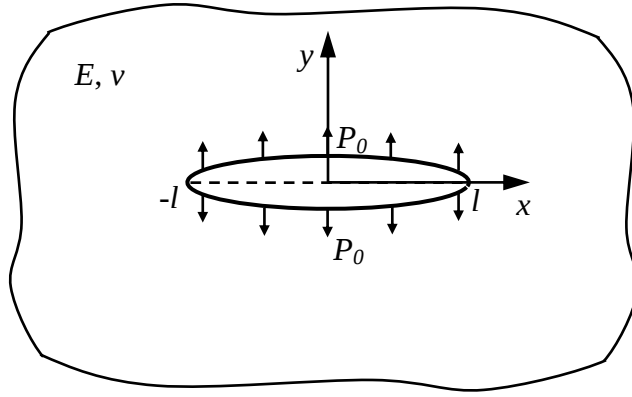


Figure 2. 4: A tensile crack in homogeneous material

The analytical solution of the crack opening is given as (see Appendix B for details):

$$w(x) = -\frac{2(1-\nu)P_0}{G} \sqrt{l^2 - x^2} \quad (2.57)$$

We take the pressure $P_0 = 0.1G$.

The crack is divided into 20 uniform elements. Since the sub-crack tip element n can only take odd numbers as explained in the last section, the errors with different at the crack tip element is listed in the Table 2.1. From the table, the error of the DDM implementation with revised crack-tip element approach is obviously larger than the one with original crack-tip element approach with small n , e.g. 1 or 3. When n is taking over 5, the error with revised one is fluctuating slightly close to the original

one with the difference value 8.59%. Thus, we can take $n=5$ (difference between the original one is 1.22%) for the following analysis.

Implementations Error at tip n	DDM with revised crack tip element	DDM with original crack tip element	DDM only
1	0.9519	0.0859	0.2614
3	0.2627		
5	0.0889		
9	0.9034		
19	0.0922		
39	0.0887		
99	0.0876		

Table 2. 1: The tip errors of different approach implementations of DDM

The average running time of 10 computations on a laptop with Intel core I7-2720QM at 2.20GHz and 12GB memory for all the cases ranges 0.0119(s) to 0.0128(s).

The detailed results with sub-crack tip element $n=5$ are further illustrated in the Figure 2.15 with increasing the number of the element of the crack. Since the loading and geometry are both symmetric about the y-axis, only the result of half the crack is necessary. The horizontal axis in the figure is normalized as x/l , ranging from 0 to 1

and the vertical one represents the errors. The figure clearly shows that the results of the two crack tip element approaches match each other quite well and both are as over twice as accurate compared to the implementation of DDM only at the tip, regardless of the element refinement. And no matter how the element is refined, the tip error of the DDM and the crack tip element approach maintain around 26% and 9%, respectively. However, the element refinement yields better result for the elements that do not located at the first two elements counting from the tip.

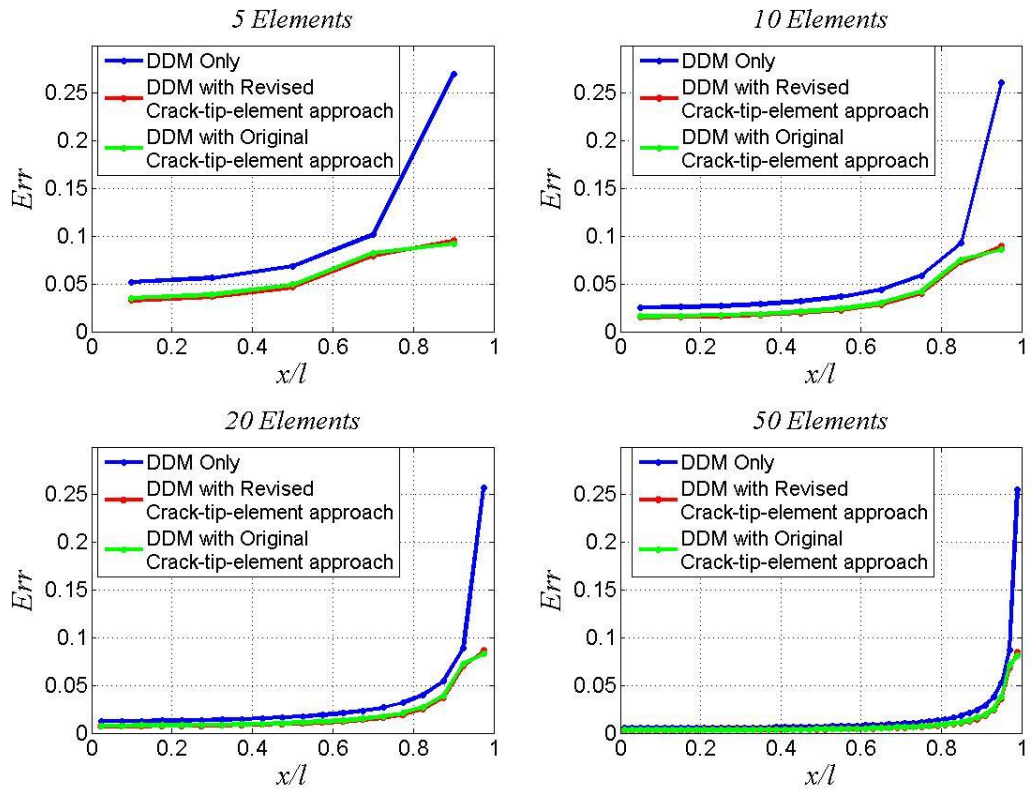


Figure 2. 15: The errors of different approach implementations of DDM with different number of uniform elements on the crack

We also consider the non-uniform mesh situation. One acceptable non-uniform mesh

is making the analytical opening difference at the midpoints of two adjacent elements remains the same. The result of the errors with this mesh is illustrated in the Figure 2.16.

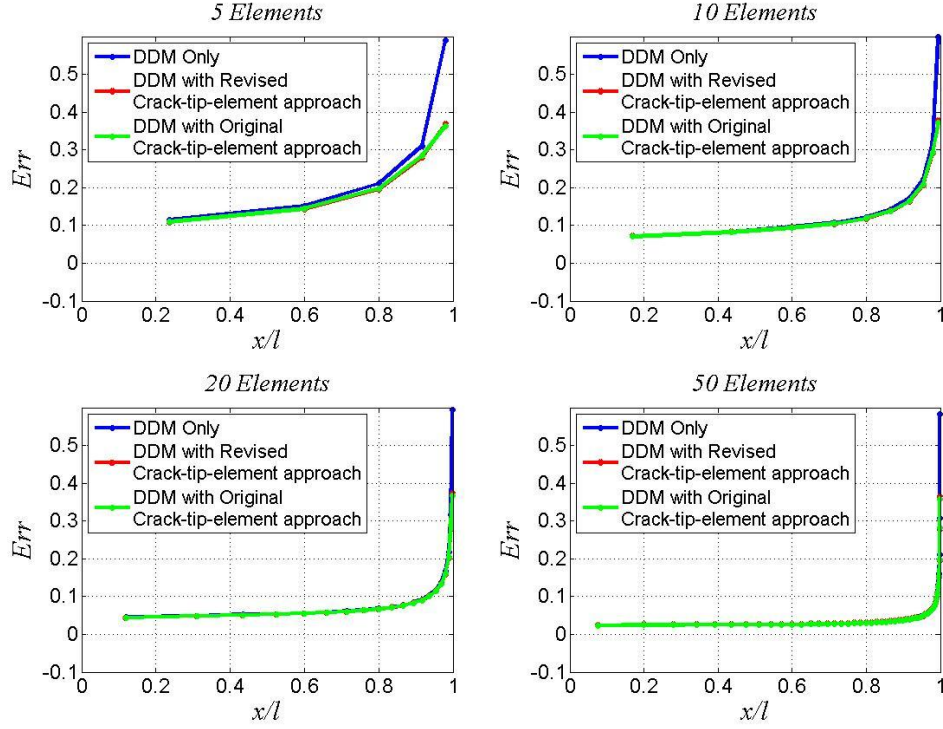


Figure 2. 16: The errors of different approach implementations of DDM with different number of non-uniform elements on the crack

The figure clearly shows that the non-uniform mesh yields worse result than the uniform one especially at the tip where the error is enlarged from around 25% to 60% and 9% to 38% for DDM only and crack tip approaches, respectively. And this peak error value varies little with element refinement as it is for the uniform mesh situation. Thus, it is better to adopt the uniform mesh for the crack problem with DDM implementation.

Having reached this step, we can conclude that the displacement discontinuity method can be implemented for the crack problem in a homogeneous medium quite accurately with the revised crack tip element approach and uniform mesh. However, the DDM cannot be implemented directly for multilayered media within over two layers due to the lack of the analytical formulae to establish the analysis. Therefore, another approach of the boundary integral method known as direct method (DM) will be introduced in the next chapter because it is applicable for the general multilayered media analysis. Then we can get the result for the bimaterial case with direct method so that a comparison can be made.

Chapter 3

The direct method

3.1 Abstract

Compared to the displacement discontinuity method based on the dislocation theory or the Kelvin's solution of a constant displacement discontinuity, the direct method is more mathematical that enable us to solve for the unknown boundary displacements or stresses directly in terms of the specified boundary conditions ^[9]. In this chapter, a detailed introduction with the reciprocal theorem and the basic implementation of the direct method (DM) with local coordinate will be presented. After that a brief historical review of the DM implementation for the multilayered media will be provided, inside of which we will focus on the successive stiffness method and revise it to make it applicable for the case with in cracks. The results for the homogeneous and bimaterial cases will be presented and compared with the one with DDM at last.

3.2 Introduction ^[9]

3.2.1 Kelvin's solution

The direct method (DM) is based on the so-called “Kelvin's solution” and the reciprocal theorem. Here, we will first introduce the Kelvin's solution.

Kelvin's problem for plane strain condition is illustrated in Figure 3. 1. The force $F_i = (F_x, F_y)$ in the figure represents a line of concentrated force applied along the z axis in an infinite elastic solid. The components $F_x > 0$ and $F_y > 0$ have dimensions of force/length (N/m).

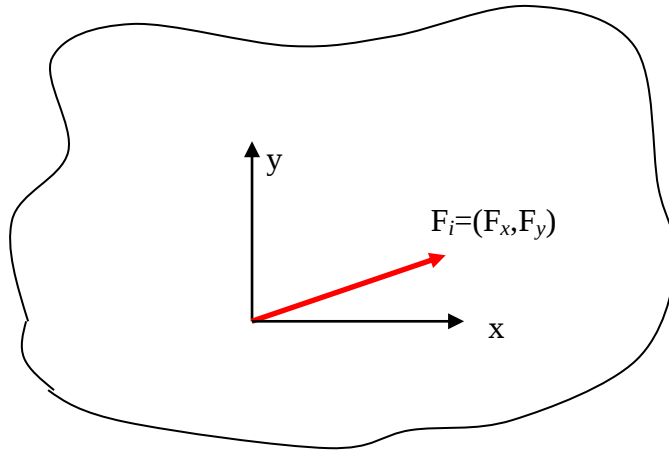


Figure 3. 1: Kelvin's Problem, plane strain

The solution to this problem can be expressed in terms of a function $g(x,y)$, defined as:

$$g(x, y) = \frac{-1}{4\pi(1-\nu)} \ln \sqrt{(x^2 + y^2)} \quad (3.1)$$

The displacements can be written as:

$$\begin{cases} u_x = \frac{F_x}{2G}[(3-4\nu)g - xg_{,x}] + \frac{F_y}{2G}[-yg_{,x}] \\ u_y = \frac{F_x}{2G}[-xg_{,y}] + \frac{F_y}{2G}[(3-4\nu)g - yg_{,y}] \end{cases} \quad (3.2)$$

And the stresses for the plane strain version are:

$$\begin{cases} \sigma_{xx} = F_x[2(1-\nu)g_{,x} - xg_{,xx}] + F_y[2\nu g_{,y} - yg_{,xx}] \\ \sigma_{yy} = F_x[2\nu g_{,x} - xg_{,yy}] + F_y[2(1-\nu)g_{,y} - yg_{,yy}] \\ \sigma_{xy} = F_x[(1-2\nu)g_{,y} - xg_{,xy}] + F_y[(1-2\nu)g_{,x} - yg_{,xy}] \end{cases} \quad (3.3)$$

where: G and ν are the shear modulus and Poisson ratio of the medium, respectively

And the derivatives of $g(x,y)$ in the above expressions are given by:

$$\begin{cases} g_{,x} = \frac{-1}{4\pi(1-\nu)} \frac{x}{x^2 + y^2} \\ g_{,y} = \frac{-1}{4\pi(1-\nu)} \frac{y}{x^2 + y^2} \\ g_{,xy} = \frac{1}{4\pi(1-\nu)} \frac{2xy}{(x^2 + y^2)^2} \\ g_{,xx} = -g_{,yy} = \frac{1}{4\pi(1-\nu)} \frac{x^2 - y^2}{(x^2 + y^2)^2} \end{cases} \quad (3.4)$$

If the force $F_i = (F_x, F_y)$ in Figure 3.1 was located at the point (c_x, c_y) , instead of the origin, the solution could be obtained by replacing coordinates x and y in (3.1)-(3.4) with transformed coordinates $x-c_x$, and $y-c_y$, respectively.

3.2.2 The reciprocal theorem and its consequences

Besides the Kelvin's solution the other key to this approach is a theorem in linear elasticity called the reciprocal theorem. The reciprocal theorem links the solutions to

two different boundary value problems for the same region $\mathbf{R}$ with the same boundary $\mathbf{C}$.

It is a direct consequence of the linearity of the equilibrium equations and of the generalized Hooke's law.

Suppose that the first boundary value problem is characterized by displacements u_s, u_n and stresses σ_s, σ_n on the boundary $\mathbf{C}$ of region $\mathbf{R}$. Suppose further that the second problem is characterized by displacements u'_s, u'_n and stresses σ'_s, σ'_n on the same boundary $\mathbf{C}$ of the same region $\mathbf{R}$. Then the reciprocal theorem states that the work done by the first of stresses (σ_s, σ_n) in moving through the second set of displacements (u'_s, u'_n) is equal to the work done by the second set of stresses in (σ'_s, σ'_n) moving through the first set of displacement (u_s, u_n) . Mathematically, it can be written as:

$$\int_C (\sigma_s u'_s + \sigma_n u'_n) ds = \int_C (\sigma'_s u_s + \sigma'_n u_n) ds \quad (3.5)$$

where the integration is to be performed along the entire boundary $\mathbf{C}$.

The first (unprimed) problem is the one that we are trying to solve, and the second (primed) problem is one for which we already know the solution, then equation (3.5) is an integral equation relating the unspecified boundary parameters of our problem to the specified boundary parameters plus the solution to another problem for the same region $\mathbf{R}$ with the same boundary $\mathbf{C}$. And the solution to the second problem is called *test solution*.

We now assume that the boundary can be approximated $\mathbf{C}$ by N straight line segments joined end to end. Equation (3.5) can be represented as:

$$\sum_{j=1}^N \int_{\Delta s_j} (\sigma_s u_s' + \sigma_n u_n') ds = \sum_{j=1}^N \int_{\Delta s_j} (\sigma_s' u_s + \sigma_n' u_n) ds \quad (3.6)$$

where Δs_j is the j th line segment, which has length $2a_j$. If we assume further that the boundary displacements and stresses for the problem we are trying to solve are constant over each segment, Equation (3.6) becomes:

$$\sum_{j=1}^N \sigma_s^j \int_{\Delta s_j} u_s' ds + \sum_{j=1}^N \sigma_n^j \int_{\Delta s_j} u_n' ds = \sum_{j=1}^N u_s^j \int_{\Delta s_j} \sigma_s' ds + \sum_{j=1}^N u_n^j \int_{\Delta s_j} \sigma_n' ds \quad (3.7)$$

Because we have N segments (boundary segments), we have a total of $4N$ boundary parameters σ_s^j , σ_n^j , u_s^j , u_n^j . Half of them are the specified boundary conditions we are trying to satisfy, and the other half are the unknowns we are trying to find. Assuming for the moment that we also know the solution to the test problem $(\sigma_s', \sigma_n', u_s', u_n')$, we see that Equation (3.7) is a single equation containing $2N$ unknowns. In order to solve for these unknowns, therefore, we will have to find another $2N-1$ equations similar to (3.7). In other words, we will have to find out a total of $2N$ different test solutions for the given region R .

An acceptable set of $2N$ test solutions can be found by applying concentrated shear and normal forces F_s^i and F_n^i at the midpoint of each element i along the boundary C .

Considering first the N test solutions generated by the concentrated shear forces F_s^i , $i=1$ to N , Equation (3.7) can be rewritten as:

$$\sum_{j=1}^N \sigma_s^j \int_{\Delta s_j} u_s'(F_s^i) ds + \sum_{j=1}^N \sigma_n^j \int_{\Delta s_j} u_n'(F_s^i) ds = \sum_{j=1}^N u_s^j \int_{\Delta s_j} \sigma_s'(F_s^i) ds + \sum_{j=1}^N u_n^j \int_{\Delta s_j} \sigma_n'(F_s^i) ds \quad (3.8)$$

Similarly to the N test solutions involving the concentrated normal forces F_n^i , $i=1$ to N ,

Equation (3.7) can be rewritten as:

$$\sum_{j=1}^N \sigma_s^j \int_{\Delta s_j} u_s'(F_n^i) ds + \sum_{j=1}^N \sigma_n^j \int_{\Delta s_j} u_n'(F_n^i) ds = \sum_{j=1}^N u_s^j \int_{\Delta s_j} \sigma_s'(F_n^i) ds + \sum_{j=1}^N u_n^j \int_{\Delta s_j} \sigma_n'(F_n^i) ds \quad (3.9)$$

Equations (3.4) and (3.5) can be put in the form:

$$\begin{aligned} \sum_{j=1}^N B_{ss}^{ij} \sigma_s^j + \sum_{j=1}^N B_{sn}^{ij} \sigma_n^j &= \sum_{j=1}^N A_{ss}^{ij} u_s^j + \sum_{j=1}^N A_{sn}^{ij} u_n^j \\ \sum_{j=1}^N B_{ns}^{ij} \sigma_s^j + \sum_{j=1}^N B_{nn}^{ij} \sigma_n^j &= \sum_{j=1}^N A_{ns}^{ij} u_s^j + \sum_{j=1}^N A_{nn}^{ij} u_n^j \end{aligned} \quad (3.10)$$

where i ranges from 1 to N , and the boundary influence coefficients are defined as follows:

(I) Displacement boundary influence coefficients:

$B_{ss}^{ij} = \int_{\Delta s_j} u_s'(F_s^i) ds$: The integral of the *shear displacement* on the j th element due to the *shear* concentrated force applied on the midpoint of the i th element

$B_{sn}^{ij} = \int_{\Delta s_j} u_n'(F_s^i) ds$: The integral of the *normal displacement* on the j th element due to the *shear* concentrated force applied on the midpoint of the i th element

$B_{ns}^{ij} = \int_{\Delta s_j} u_s'(F_n^i) ds$: The integral of the *shear displacement* on the j th element due to the *normal* concentrated force applied on the midpoint of the i th element

$B_{nn}^{ij} = \int_{\Delta s_j} u'_n(F_n^i)ds$: The integral of the *normal displacement* on the j th element due to the *normal* concentrated force applied on the midpoint of the i th element

(II) Stress boundary influence coefficients:

$A_{ss}^{ij} = \int_{\Delta s_j} \sigma'_s(F_s^i)ds$: The integral of the *shear stress* on j th element due to the *shear* concentrated force applied on the midpoint of the i th element

$A_{sn}^{ij} = \int_{\Delta s_j} \sigma'_n(F_s^i)ds$: The integral of the *normal stress* on j th element due to the *shear* concentrated force applied on the midpoint of the i th element

$A_{ns}^{ij} = \int_{\Delta s_j} \sigma'_s(F_n^i)ds$: The integral of the *shear stress* on j th element due to the *normal* concentrated force applied on the midpoint of the i th element

$A_{nn}^{ij} = \int_{\Delta s_j} \sigma'_n(F_n^i)ds$: The integral of the *normal stress* on j th element due to the *normal* concentrated force applied on the midpoint of the i th element

If the equations in (3.10) are linearly independent, we can then solve for the unknown boundary parameters *directly* from the specified boundary conditions.

3.2.3 Boundary influence coefficients

The boundary influence coefficients for the direct boundary integral method are obtained by placing a concentrated force (with components F_s^i, F_n^i) at the midpoint of the i th segment of contour C and then integrating the displacements and stresses caused by this force over the j th segment, as indicated in (3.8) and (3.9) Letting i range from 1

to N , i.e. applying N concentrated forces to the boundary, we obtain the required system of algebraic equations (3.10).

To facilitate computation of the boundary influence coefficients B_{ss}^{ij} , etc., in (3.10), we will use a local $\bar{x}, \bar{y}$ coordinate system whose origin is at the midpoint of the j th segment of contour C , as shown in Figure 3.2. These coordinates correspond to the local coordinate s^j, n^j in the usual way, with $\bar{x}$ (or s^j) pointing in the direction of traversal of the contour and $\bar{y}$ (or n^j) pointing away from the origin of interest (cf. Figure 2.10).

We want to compute displacements and stresses over the j th segment due to a concentrated force acting at the midpoint of the i th segment, point $[i]$ in Figure 3. The components of this force, respectively parallel and perpendicular to the i th segment, are F_s^i, F_n^i .

Using the geometry given in Figure 3.2, we find that the components of force in the $\bar{x} = s^j$ and $\bar{y} = n^j$ directions are related to these by the expressions:

$$\begin{aligned} F_{\bar{x}} &= F_s^i \cos \gamma - F_n^i \sin \gamma \\ F_{\bar{y}} &= F_s^i \sin \gamma + F_n^i \cos \gamma \end{aligned} \quad (3.11)$$

where $\gamma = \beta^i - \beta^j$

According to the equations (3.1)-(3.4), we can write down the expressions for the displacements and stresses due to a concentrated force $F_{\bar{x}}, F_{\bar{y}}$, at the point $\bar{x} = c_{\bar{x}}, \bar{y} = c_{\bar{y}}$, i.e. point $[i]$ in Figure 3.2.

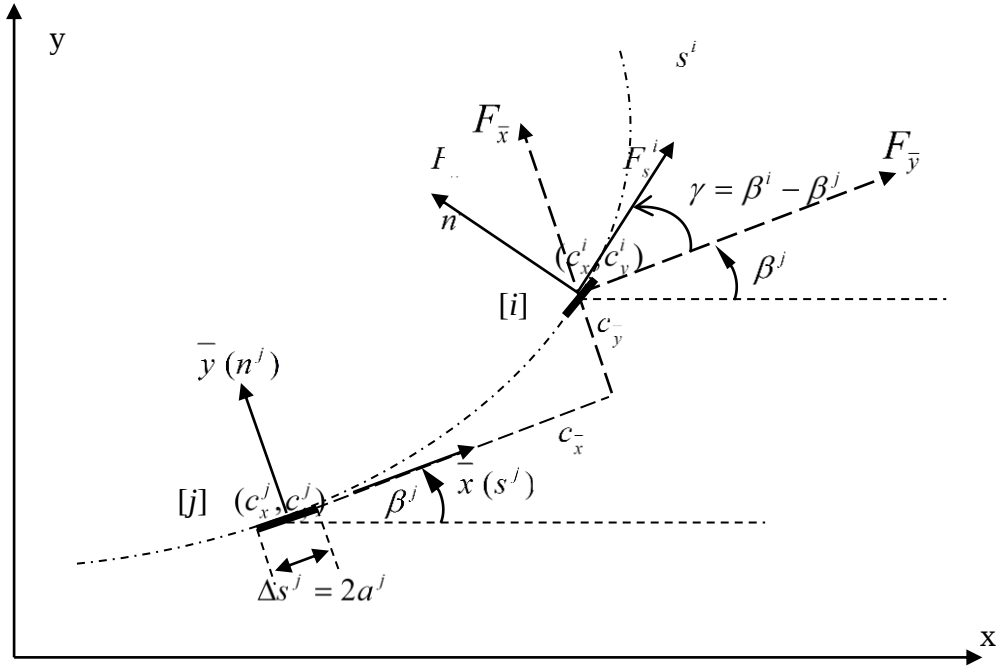


Figure 3. 2: Definition of local coordinates for boundary influence coefficients

The displacements $u_s' = u_{\bar{x}}', u_n' = u_{\bar{y}}'$ and stresses $\sigma_s' = \sigma_{\bar{x}\bar{y}}', \sigma_n' = \sigma_{\bar{y}\bar{y}}'$ over the j th segment (our test solution) are then obtained from the resulting expressions by setting $\bar{y} = 0$. Using the values of $F_{\bar{x}}'$ and $F_{\bar{y}}'$, from Equation (3.11), we find out the displacements for the test solution are:

$$\begin{aligned}
 u_s' &= \frac{F_s^i}{2G} [(3-4\nu)g \cos \gamma - (\bar{x} - c_x^-) \frac{\partial g}{\partial x} \cos \gamma + c_y^- \sin \gamma \frac{\partial g}{\partial x}] \\
 &\quad + \frac{F_n^i}{2G} [-(3-4\nu)g \sin \gamma + (\bar{x} - c_x^-) \frac{\partial g}{\partial x} \sin \gamma + c_y^- \cos \gamma \frac{\partial g}{\partial x}] \\
 u_n' &= \frac{F_s^i}{2G} [(3-4\nu)g \sin \gamma - (\bar{x} - c_x^-) \frac{\partial g}{\partial y} \cos \gamma + c_y^- \sin \gamma \frac{\partial g}{\partial x}] \\
 &\quad + \frac{F_n^i}{2G} [(3-4\nu)g \cos \gamma + (\bar{x} - c_x^-) \frac{\partial g}{\partial x} \sin \gamma + c_y^- \cos \gamma \frac{\partial g}{\partial x}]
 \end{aligned} \tag{3.12}$$

And that the stresses are:

$$\begin{aligned}
\sigma'_s &= F_s^i \left[(1-2\nu) \sin \gamma \frac{\partial g}{\partial x} + (1-2\nu) \cos \gamma \frac{\partial g}{\partial y} - (\bar{x} - c_x^-) \cos \gamma \frac{\partial^2 g}{\partial x \partial y} + c_y^- \sin \gamma \frac{\partial^2 g}{\partial x \partial y} \right] \\
&\quad + F_n^i \left[(1-2\nu) \cos \gamma \frac{\partial g}{\partial x} - (1-2\nu) \sin \gamma \frac{\partial g}{\partial y} + (\bar{x} - c_x^-) \sin \gamma \frac{\partial^2 g}{\partial x \partial y} + c_y^- \cos \gamma \frac{\partial^2 g}{\partial x \partial y} \right] \\
\sigma'_n &= F_s^i \left[2\nu \cos \gamma \frac{\partial g}{\partial x} + 2(1-\nu) \sin \gamma \frac{\partial g}{\partial y} - (\bar{x} - c_x^-) \cos \gamma \frac{\partial^2 g}{\partial y^2} + c_y^- \sin \gamma \frac{\partial^2 g}{\partial y^2} \right] \\
&\quad + F_n^i \left[-2\nu \sin \gamma \frac{\partial g}{\partial x} + 2(1-\nu) \cos \gamma \frac{\partial g}{\partial y} + (\bar{x} - c_x^-) \sin \gamma \frac{\partial^2 g}{\partial y^2} + c_y^- \cos \gamma \frac{\partial^2 g}{\partial y^2} \right]
\end{aligned} \tag{3.13}$$

where the function $g(\bar{x}, \bar{y})$ is:

$$g(\bar{x}, \bar{y}) = \frac{-1}{4\pi(1-\nu)} \ln \sqrt{(\bar{x} - c_x^-)^2 + (\bar{y} - c_y^-)^2} \tag{3.14}$$

This function and its derivatives are to be evaluated for $\bar{y} = 0$ in (3.12) and (3.13).

As noted previously, (3.12) and (3.13) actually represent two test solutions for each element i , one for the shear force F_s^i , and the other for the normal force F_n^i . The boundary influence coefficients B_{ss}^{ij} , etc., in (3.10) are obtained by taking these two solutions in turn (i.e. $F_s^i \neq 0$, $F_n^i = 0$ and then $F_s^i = 0$, $F_n^i \neq 0$), substituting (3.12) and (3.13) into (3.8) and (3.9) and performing the indicated integrations over Δs^j , i.e. integrating with respect to $\bar{x}$ between the limits $-a^j$ and a^j . In carrying out these operations, it is found out that the values of F_s^i and F_n^i are irrelevant because these quantities can be factored out on both sides of the equations during calculation. Thus we may consider that F_s^i and F_n^i both are equal to 1.

Proceeding in this way, we arrive at the following results:

$$\left\{ \begin{aligned} B_{ss}^{ij} &= \int_{-a^j}^{a^j} u'_s(F_s^i) d\bar{x} = \frac{1}{2G} [(3-4\nu) \cos \gamma \bar{T}_1 + c_{\bar{y}} (\sin \gamma \bar{T}_2 - \cos \gamma \bar{T}_3)] \\ B_{sn}^{ij} &= \int_{-a^j}^{a^j} u'_n(F_s^i) d\bar{x} = \frac{1}{2G} [(3-4\nu) \sin \gamma \bar{T}_1 + c_{\bar{y}} (\cos \gamma \bar{T}_2 + \sin \gamma \bar{T}_3)] \\ B_{ns}^{ij} &= \int_{-a^j}^{a^j} u'_s(F_n^i) d\bar{x} = \frac{1}{2G} [-(3-4\nu) \sin \gamma \bar{T}_1 + c_{\bar{y}} (\cos \gamma \bar{T}_2 + \sin \gamma \bar{T}_3)] \\ B_{nn}^{ij} &= \int_{-a^j}^{a^j} u'_n(F_n^i) d\bar{x} = \frac{1}{2G} [(3-4\nu) \cos \gamma \bar{T}_1 - c_{\bar{y}} (\sin \gamma \bar{T}_2 - \cos \gamma \bar{T}_3)] \end{aligned} \right. \quad (3.15)$$

$$\left\{ \begin{aligned} A_{ss}^{ij} &= \int_{-a^j}^{a^j} \sigma'_s(F_s^i) d\bar{x} = [(1-2\nu) \sin \gamma \bar{T}_2 + 2(1-\nu) \cos \gamma \bar{T}_3 + c_{\bar{y}} (\sin \gamma \bar{T}_4 + \cos \gamma \bar{T}_5)] \\ A_{sn}^{ij} &= \int_{-a^j}^{a^j} \sigma'_n(F_s^i) d\bar{x} = [-(1-2\nu) \cos \gamma \bar{T}_2 + 2(1-\nu) \sin \gamma \bar{T}_3 + c_{\bar{y}} (\cos \gamma \bar{T}_4 - \sin \gamma \bar{T}_5)] \\ A_{ns}^{ij} &= \int_{-a^j}^{a^j} \sigma'_s(F_n^i) d\bar{x} = [(1-2\nu) \cos \gamma \bar{T}_2 - 2(1-\nu) \sin \gamma \bar{T}_3 + c_{\bar{y}} (\cos \gamma \bar{T}_4 - \sin \gamma \bar{T}_5)] \\ A_{nn}^{ij} &= \int_{-a^j}^{a^j} \sigma'_n(F_n^i) d\bar{x} = [(1-2\nu) \sin \gamma \bar{T}_2 + 2(1-\nu) \cos \gamma \bar{T}_3 - c_{\bar{y}} (\sin \gamma \bar{T}_4 + \cos \gamma \bar{T}_5)] \end{aligned} \right. \quad (3.16)$$

The five quantities $\bar{T}_1$ to $\bar{T}_5$ in these equations represent definite integrals of the function $g(\bar{x}, \bar{y})$ and its derivatives evaluated for $\bar{y} = 0$, and are given as:

$$\begin{aligned} \bar{T}_1 &= \frac{-1}{4\pi(1-\nu)} \left[c_{\bar{y}} \left(\arctan \frac{c_{\bar{y}}}{c_{\bar{x}} - a^j} - \arctan \frac{c_{\bar{y}}}{c_{\bar{x}} + a^j} \right) \right. \\ &\quad \left. - (c_{\bar{x}} - a^j) \ln \sqrt{(c_{\bar{x}} - a^j)^2 + c_{\bar{y}}^2} + (c_{\bar{x}} + a^j) \ln \sqrt{(c_{\bar{x}} + a^j)^2 + c_{\bar{y}}^2} \right] \end{aligned} \quad (3.17-a)$$

$$\bar{T}_2 = \frac{-1}{4\pi(1-\nu)} [\ln \sqrt{(c_{\bar{x}} - a^j)^2 + c_{\bar{y}}^2} - \ln \sqrt{(c_{\bar{x}} + a^j)^2 + c_{\bar{y}}^2}] \quad (3.17-b)$$

$$\bar{T}_3 = \frac{1}{4\pi(1-\nu)} \left[\arctan \frac{c_{\bar{y}}}{c_{\bar{x}} - a^j} - \arctan \frac{c_{\bar{y}}}{c_{\bar{x}} + a^j} \right] \quad (3.17-c)$$

$$\bar{T}_4 = \frac{1}{4\pi(1-\nu)} \left[\frac{c_{\bar{y}}}{(c_{\bar{x}} - a^j)^2 + c_{\bar{y}}^2} - \frac{c_{\bar{y}}}{(c_{\bar{x}} + a^j)^2 + c_{\bar{y}}^2} \right] \quad (3.17-d)$$

$$\bar{T}_5 = \frac{1}{4\pi(1-\nu)} \left[\frac{c_{\bar{x}} - a^j}{(c_{\bar{x}} - a^j)^2 + c_{\bar{y}}^2} - \frac{c_{\bar{x}} + a^j}{(c_{\bar{x}} + a^j)^2 + c_{\bar{y}}^2} \right] \quad (3.17-d)$$

In the discussion above, we have tacitly assumed that point $[i]$ and point $[j]$ in Figure 3.2 do not coincide, i.e. we have assumed that $c_{\bar{x}}$ and $c_{\bar{y}}$ are not both equal to zero.

However, equations (3.15) – (3.17) are applicable to this case provided that we follow

the usual limiting procedure in computing the multiple-valued arctangent function.

Setting $c_x^- = 0$, and c_y^- goes zero from the positive direction (i.e. from outside region

$\mathbf{R}$), we find that only non-zero quantities are $\overline{T}_1$, $\overline{T}_3$ and $\overline{T}_5$, and they have the values:

$$\begin{cases} \overline{T}_1 = \frac{-a^j \ln(a^j)}{2\pi(1-\nu)} = \frac{-a^i \ln(a^i)}{2\pi(1-\nu)} \\ \overline{T}_3 = \frac{1}{4(1-\nu)} \\ \overline{T}_5 = \frac{-1}{2\pi a^j(1-\nu)} = \frac{-1}{2\pi a^i(1-\nu)} \end{cases} \quad (3.18)$$

The diagonal terms of the boundary influence coefficients are obtained by introducing

these results into (3.15) and (3.16), noting that $\gamma = \beta^i - \beta^j = 0$:

$$\begin{cases} B_{sn}^{ii} = B_{ns}^{ii} = 0; & B_{ss}^{ii} = B_{nn}^{ii} = \frac{-(3-4\nu)}{4\pi G(1-\nu)} a^i \ln(a^i) \\ A_{sn}^{ii} = A_{ns}^{ii} = 0; & A_{ss}^{ii} = A_{nn}^{ii} = \frac{1}{2} \end{cases} \quad (3.19)$$

Then, all the boundary influence coefficients are evaluated. After the substitution these coefficients into (3.10), we can then solve (3.10) with the known boundary parameters.

Notice that all the B coefficients have a factor $1/(2G)$, which may be of several orders

while all the A coefficients do not. To avoid ill-conditioning in the matrix operation, we

use normalized parameters for the direct method applied in our own work, i.e. we

extract the factor $1/(2G)$ out of the expressions for all the B s, and then the

corresponding stresses or tractions are $\frac{\sigma}{2G}$.

3.2.4 Coordinate transformation

In some literature, the direct method are formulated in terms of global x, y components of these quantities, i.e. $u_i = (u_x, u_y)$ and $t_i = (t_x, t_y)$. The connection between a local and a global formulation of the direct method is easy to establish using simple coordinate transformations:

$$\begin{Bmatrix} u_x^j \\ u_y^j \end{Bmatrix} = \begin{bmatrix} \cos \beta^j & -\sin \beta^j \\ \sin \beta^j & \cos \beta^j \end{bmatrix} \begin{Bmatrix} u_s^j \\ u_n^j \end{Bmatrix}, \quad \begin{Bmatrix} u_s^j \\ u_n^j \end{Bmatrix} = \begin{bmatrix} \cos \beta^j & \sin \beta^j \\ -\sin \beta^j & \cos \beta^j \end{bmatrix} \begin{Bmatrix} u_x^j \\ u_y^j \end{Bmatrix}$$

And the same transform matrices for traction transforms.

For boundary influence coefficients:

$$\begin{Bmatrix} B_{xx}^{ij} \\ B_{xy}^{ij} \end{Bmatrix} = \begin{bmatrix} \cos \beta^j & -\sin \beta^j \\ \sin \beta^j & \cos \beta^j \end{bmatrix} \begin{Bmatrix} B_{ss}^{ij} \\ B_{sn}^{ij} \end{Bmatrix}, \quad \begin{Bmatrix} B_{yx}^{ij} \\ B_{yy}^{ij} \end{Bmatrix} = \begin{bmatrix} \cos \beta^j & -\sin \beta^j \\ \sin \beta^j & \cos \beta^j \end{bmatrix} \begin{Bmatrix} B_{ns}^{ij} \\ B_{nn}^{ij} \end{Bmatrix}$$

And so force for A_{xx}^{ij} , etc.

Then Equation (3.10) can be rewritten as:

$$\begin{aligned} \sum_{j=1}^N B_{xx}^{ij} t_x^j + \sum_{j=1}^N B_{xy}^{ij} t_y^j &= \sum_{j=1}^N A_{xx}^{ij} u_x^j + \sum_{j=1}^N A_{xy}^{ij} u_y^j \\ \sum_{j=1}^N B_{yx}^{ij} t_x^j + \sum_{j=1}^N B_{yy}^{ij} t_y^j &= \sum_{j=1}^N A_{yx}^{ij} u_x^j + \sum_{j=1}^N A_{yy}^{ij} u_y^j \end{aligned} \tag{3.20}$$

3.2.5 Somigliana's formulas

So far we have only used the direct method to calculate the unspecified displacements or tractions on the boundary $\mathbf{C}$ of an arbitrary $\mathbf{R}$. If we want to compute the solution at points within region $\mathbf{R}$, we need to make use of a set of integral identities known as Somigliana's formulas.

For a given point p , not on the boundary C , with global coordinate (x, y) , its tractions and displacement and can be calculated in terms of the elements along the boundary C :

$$\begin{aligned}
u_x(p) = & \sum_{j=1}^N [(1-2\nu) \sin \beta^j \bar{T}_2 - 2(1-\nu) \cos \beta^j \bar{T}_3 + c_y (\sin \beta^j \bar{T}_4 - \cos \beta^j \bar{T}_5)] u_s^j \\
& + \sum_{j=1}^N [(1-2\nu) \cos \beta^j \bar{T}_2 + 2(1-\nu) \sin \beta^j \bar{T}_3 - c_y (\cos \beta^j \bar{T}_4 + \sin \beta^j \bar{T}_5)] u_n^j \\
& + \sum_{j=1}^N [(3-4\nu) \cos \beta^j \bar{T}_1 - c_y (\sin \beta^j \bar{T}_2 + \cos \beta^j \bar{T}_3)] \frac{\sigma_s^j}{2G} \\
& + \sum_{j=1}^N [-(3-4\nu) \sin \beta^j \bar{T}_1 + c_y (\cos \beta^j \bar{T}_2 - \sin \beta^j \bar{T}_3)] \frac{\sigma_n^j}{2G}
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
u_y(p) = & \sum_{j=1}^N [-(1-2\nu) \cos \beta^j \bar{T}_2 - 2(1-\nu) \sin \beta^j \bar{T}_3 - c_y (\cos \beta^j \bar{T}_4 + \sin \beta^j \bar{T}_5)] u_s^j \\
& + \sum_{j=1}^N [(1-2\nu) \sin \beta^j \bar{T}_2 - 2(1-\nu) \cos \beta^j \bar{T}_3 - c_y (\sin \beta^j \bar{T}_4 - \cos \beta^j \bar{T}_5)] u_n^j \\
& + \sum_{j=1}^N [(3-4\nu) \sin \beta^j \bar{T}_1 + c_y (\cos \beta^j \bar{T}_2 - \sin \beta^j \bar{T}_3)] \frac{\sigma_s^j}{2G} \\
& + \sum_{j=1}^N [(3-4\nu) \cos \beta^j \bar{T}_1 + c_y (\sin \beta^j \bar{T}_2 + \cos \beta^j \bar{T}_3)] \frac{\sigma_n^j}{2G}
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
\sigma_{xx}(p) = & 2G \sum_{j=1}^N [2 \cos^2 \beta^j \bar{T}_4 + \sin 2\beta^j \bar{T}_5 - c_y (\cos 2\beta^j \bar{T}_6 - \sin 2\beta^j \bar{T}_7)] u_s^j \\
& + 2G \sum_{j=1}^N [-\bar{T}_5 - c_y (\sin 2\beta^j \bar{T}_6 + \cos 2\beta^j \bar{T}_7)] u_n^j \\
& + \sum_{j=1}^N [-\bar{T}_2 - 2(1-\nu)(\cos 2\beta^j \bar{T}_2 - \sin 2\beta^j \bar{T}_3) + c_y (\cos 2\beta^j \bar{T}_4 + \sin 2\beta^j \bar{T}_5)] \sigma_s^j \\
& + \sum_{j=1}^N [-\bar{T}_3 + (1-2\nu)(\sin 2\beta^j \bar{T}_2 + \cos 2\beta^j \bar{T}_3) + c_y (\sin 2\beta^j \bar{T}_4 - \cos 2\beta^j \bar{T}_5)] \sigma_n^j
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\sigma_{yy}(p) = & 2G \sum_{j=1}^N [2 \sin^2 \beta^j \bar{T}_4 - \sin 2\beta^j \bar{T}_5 + c_y (\cos 2\beta^j \bar{T}_6 - \sin 2\beta^j \bar{T}_7)] u_s^j \\
& + 2G \sum_{j=1}^N [-\bar{T}_5 + c_y (\sin 2\beta^j \bar{T}_6 + \cos 2\beta^j \bar{T}_7)] u_n^j \\
& + \sum_{j=1}^N [-\bar{T}_2 + 2(1-\nu)(\cos 2\beta^j \bar{T}_2 - \sin 2\beta^j \bar{T}_3) - c_y (\cos 2\beta^j \bar{T}_4 + \sin 2\beta^j \bar{T}_5)] \sigma_s^j \\
& + \sum_{j=1}^N [-\bar{T}_3 - (1-2\nu)(\sin 2\beta^j \bar{T}_2 + \cos 2\beta^j \bar{T}_3) - c_y (\sin 2\beta^j \bar{T}_4 - \cos 2\beta^j \bar{T}_5)] \sigma_n^j
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
\sigma_{xy}(p) = & 2G \sum_{j=1}^N [\sin 2\beta^j \overline{T_4} - \cos 2\beta^j \overline{T_5} - c_y^-(\sin 2\beta^j \overline{T_6} + \cos 2\beta^j \overline{T_7})] u_s^j \\
& + 2G \sum_{j=1}^N [c_y^-(\cos 2\beta^j \overline{T_6} - \sin 2\beta^j \overline{T_7})] u_n^j \\
& + \sum_{j=1}^N [-2(1-\nu)(\sin 2\beta^j \overline{T_2} + \cos 2\beta^j \overline{T_3}) + c_y^-(\sin 2\beta^j \overline{T_4} - \cos 2\beta^j \overline{T_5})] \sigma_s^j \\
& + \sum_{j=1}^N [-(1-2\nu)(\cos 2\beta^j \overline{T_2} - \sin 2\beta^j \overline{T_3}) - c_y^-(\cos 2\beta^j \overline{T_4} + \sin 2\beta^j \overline{T_5})] \sigma_n^j
\end{aligned} \tag{3.25}$$

Functions $\overline{T_1}$ through $\overline{T_5}$ in these expressions are given by (3.17) and functions $\overline{T_6}$

and $\overline{T_7}$ are defined as:

$$\begin{cases} \overline{T_6} = -\frac{1}{4\pi(1-\nu)} \left[\frac{(c_x^- - a^j)^2 - c_y^{-2}}{\{(c_x^- - a^j)^2 + c_y^{-2}\}^2} - \frac{(c_x^- + a^j)^2 - c_y^{-2}}{\{(c_x^- + a^j)^2 + c_y^{-2}\}^2} \right] \\ \overline{T_7} = -\frac{1}{4\pi(1-\nu)} \left[\frac{c_x^- - a^j}{\{(c_x^- - a^j)^2 + c_y^{-2}\}^2} - \frac{c_x^- + a^j}{\{(c_x^- + a^j)^2 + c_y^{-2}\}^2} \right] \end{cases} \tag{3.26}$$

3.3 Different approaches of the Direct Method implementation

Many practical problems in solid mechanics involve bodies containing thin, slit-like openings or cracks. A crack has two surfaces or boundaries one effectively coinciding with the other.^[9] The direct method cannot be applied to such problems directly since according to the Kelvin's solution, the effect of elements placed along one crack surface are indistinguishable from the effects of elements placed along the other surface. Therefore, some treatment must be adopted for the direct method application. Then the customary zoning technique turns up.

3.3.1 Zoning technique

The customary general zoning technique for Bimaterial has been illustrated in the Figure 3.3. The two subregions, labeled $\mathbf{R}_1$ and $\mathbf{R}_2$ are each assumed to be homogeneous, isotropic and linearly elastic, with elastic constants E_1, ν_1 and E_2, ν_2 . The boundary contours $\mathbf{C}_1$ of subregion of $\mathbf{R}_1$ and $\mathbf{C}_2$ of subregion of $\mathbf{R}_2$, are traversed in a clockwise sense, in accordance with the convention adopted previously. The local s, n -coordinate systems associated with the two contours then appear as shown in the figure. The common portions of the two boundary contours define the interface between the subregions. The local coordinate s_1, n_1 and s_2, n_2 are oppositely directed all along the interface, i.e. $s_1 = -s_2, n_1 = -n_2$.

A boundary value problem for the inhomogeneous body of Figure 3.3 is defined by the

usual displacement and stress conditions along the “free” portions of the boundary contours C_1 and C_2 , as well as by continuity conditions for the displacements and tractions along the interface between the subregions. The continuity conditions for a point Q of the interface can be written as:

$$\begin{cases} G^{[1]} \sigma_s^{[1]}(Q) = G^{[2]} \sigma_s^{[2]}(Q) \\ G^{[1]} \sigma_n^{[1]}(Q) = G^{[2]} \sigma_n^{[2]}(Q) \end{cases} \quad (3.27) \quad [9]$$

$$\begin{cases} u_s^{[1]}(Q) = -u_s^{[2]}(Q) \\ u_n^{[1]}(Q) = -u_n^{[2]}(Q) \end{cases} \quad (3.28) \quad [9]$$

The minus sign in Equation (3.28) are a consequence of the opposite directions of the local coordinate s_1, n_1 and s_2, n_2 along the interface.

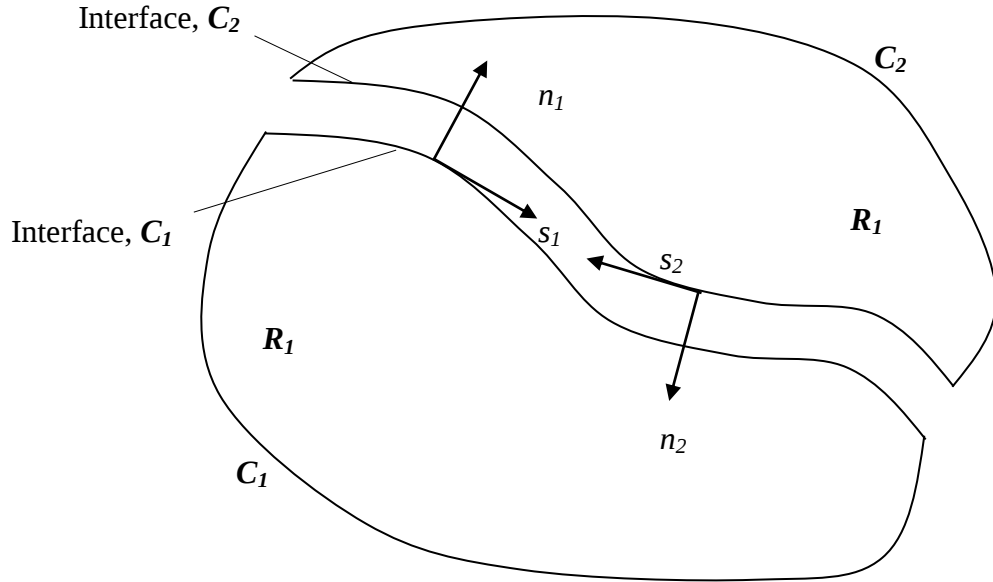


Figure 3. 3: Inhomogeneous body consisting of two isotropic, linearly elastic subregions ^[9]

Then we consider the composite problem has two separate boundary value problems,

linked by the interface continuity conditions (3.27) and (3.28). If boundary element 1 to N_1 lie along contour C_1 , and elements N_1+1 to $N_1+N_2=N$ lie along contours C_2 , then (3.10) can be used to write the following “boundary constraint equations” for the two subregions R_1 and R_2 :

$$\begin{aligned} \sum_{j=1}^{N_1} B_{ss}^{ij[1]} \sigma_s^{j[1]} + \sum_{j=1}^{N_1} B_{sn}^{ij[1]} \sigma_n^{j[1]} &= \sum_{j=1}^{N_1} A_{ss}^{ij[1]} u_s^{j[1]} + \sum_{j=1}^{N_1} A_{sn}^{ij[1]} u_n^{j[1]} \\ \sum_{j=1}^{N_1} B_{ns}^{ij[1]} \sigma_s^{j[1]} + \sum_{j=1}^{N_1} B_{nn}^{ij[1]} \sigma_n^{j[1]} &= \sum_{j=1}^{N_1} A_{ns}^{ij[1]} u_s^{j[1]} + \sum_{j=1}^{N_1} A_{nn}^{ij[1]} u_n^{j[1]} \end{aligned} \quad (3.29), \text{ where } i=1 \sim N_1^{[9]}$$

$$\begin{aligned} \sum_{j=1}^{N_1} B_{ss}^{ij[2]} \sigma_s^{j[2]} + \sum_{j=1}^{N_1} B_{sn}^{ij[2]} \sigma_n^{j[2]} &= \sum_{j=1}^{N_1} A_{ss}^{ij[2]} u_s^{j[2]} + \sum_{j=1}^{N_1} A_{sn}^{ij[2]} u_n^{j[2]} \\ \sum_{j=1}^{N_1} B_{ns}^{ij[2]} \sigma_s^{j[2]} + \sum_{j=1}^{N_1} B_{nn}^{ij[2]} \sigma_n^{j[2]} &= \sum_{j=1}^{N_1} A_{ns}^{ij[2]} u_s^{j[2]} + \sum_{j=1}^{N_1} A_{nn}^{ij[2]} u_n^{j[2]} \end{aligned} \quad (3.30), \text{ where } i= N_1 \sim N^{[9]}$$

For an element on a free portion of one of the boundary contours, two of the four parameters ($\sigma_s, \sigma_n, u_s, u_n$) will be specified as the boundary conditions, leaving the other two ones unknown. All four of the boundary parameters are unknown for an element on the interface. By using continuity conditions (3.27) and (3.28), we can associate two of the parameters with an element on one side of the interface and the other two with the corresponding element on the opposite side ^[9].

However, when the problem involves a crack goes through the interfaces. Besides cutting the whole composite along each interface, we also need to cut the material along the crack line and treat the crack surface as the free portion of the contour of its subregion.

Here, we take the Bimaterial within a crack penetrating its interface as an example. The

whole material then, according to the “cutting” policy stated previously, will be divided into 4 parts, as illustrated in the Figure 3.4.

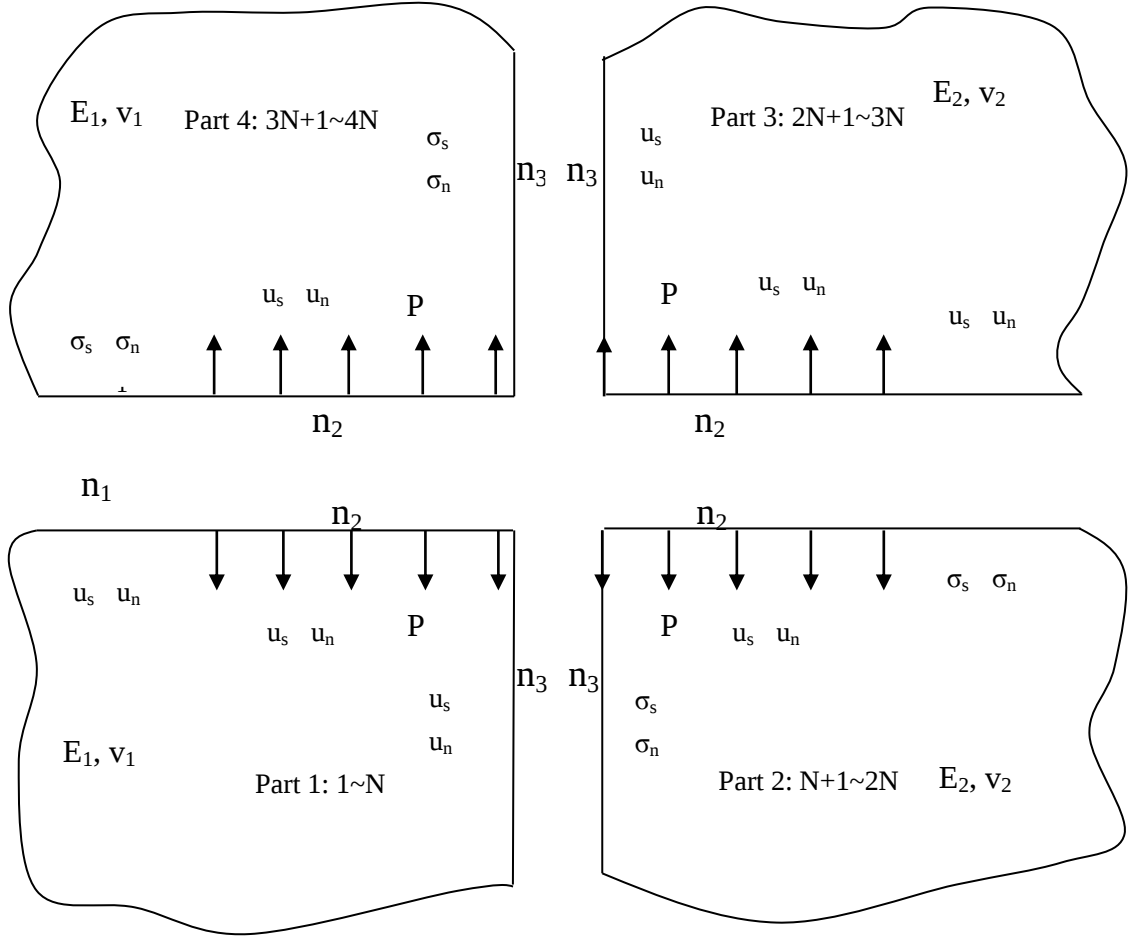


Figure 3. 4: A schematic DM implementation for the bimaterial case

In the local coordinate view defined in the procedure of the direct method, the continuity conditions can be written as (stresses components are normalized in the following calculations):

$$\begin{aligned} u_s^j &= -u_s^{j*}, \quad u_n^j = -u_n^{j*} \\ \sigma_s^j &= \sigma_s^{j*}, \quad \sigma_n^j = \sigma_n^{j*} \end{aligned} \quad (3.31)$$

Then the number of the independent unknowns in the lower or upper side is:

$$4n_1 + 2n_2 + 4n_3 + 2n_2 = 4N$$

And the total number of the independent unknowns is: $8N$.

Also, one can introduce F_s and F_n on each element to set up the linear equations system for direct method, thus, we have $8N$ linear equations. Then we can see the number of linear equations is just the same as the number of the independent unknowns. The problem can be solved by direct method in the “cutting” way.

Before setting up the linear equations, we treat the displacements u_s and u_n as the unknown boundary parameters on the interfaces of Part 1 and 3, and σ_s and σ_n as the unknown ones on the interfaces of Part 2 and 4. Then our objective is to find out those unknowns as wells as the u_s and u_n on the crack line. And the u_n on the crack line is just the opening displacement of the crack which we are seeking for.

Recall the general the basic equation (3.10) of the direct method (total $4N$ elements):

$$\begin{aligned} \sum_{j=1}^{4N} B_{ss}^{ij} \sigma_s^j + \sum_{j=1}^{4N} B_{sn}^{ij} \sigma_n^j &= \sum_{j=1}^{4N} A_{ss}^{ij} u_s^j + \sum_{j=1}^{4N} A_{sn}^{ij} u_n^j \\ \sum_{j=1}^{4N} B_{ns}^{ij} \sigma_s^j + \sum_{j=1}^{4N} B_{nn}^{ij} \sigma_n^j &= \sum_{j=1}^{4N} A_{ns}^{ij} u_s^j + \sum_{j=1}^{4N} A_{nn}^{ij} u_n^j \end{aligned} \quad (3.32)$$

Since the region of interest has been cut into 4 parts, then the Equation (3.10) associated with the known boundary conditions ($\sigma_s = 0$, $\sigma_n = -P$) for our problem should be written for 4 parts, respectively:

When $1 \leq i \leq N$:

$$\begin{aligned} & \sum_{j=1}^{n_1} (2G_1 B_{ss}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \sum_{j=n_1+n_2+1}^N (2G_1 B_{ss}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \\ & \sum_{j=1}^{n_1} (2G_1 B_{sn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_1} + \sum_{j=n_1+1}^{n_1+n_2} (2G_1 B_{sn}^{ij[1]}) \frac{-P}{2G_1} + \sum_{j=n_1+n_2+1}^N (2G_1 B_{sn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_1} \end{aligned} \quad (3.33)$$

$$\begin{aligned} & = \sum_{j=1}^N A_{ss}^{ij[1]} u_s^{j[1]} + \sum_{j=1}^N A_{sn}^{ij[1]} u_n^{j[1]} \\ & \sum_{j=1}^{n_1} (2G_1 B_{ns}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \sum_{j=n_1+n_2+1}^N (2G_1 B_{ns}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \\ & \sum_{j=1}^{n_1} (2G_1 B_{nn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_1} + \sum_{j=n_1+1}^{n_1+n_2} (2G_1 B_{nn}^{ij[1]}) \frac{-P}{2G_1} + \sum_{j=n_1+n_2+1}^N (2G_1 B_{nn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_1} \end{aligned} \quad (3.34)$$

$$= \sum_{j=1}^N A_{ns}^{ij[1]} u_s^{j[1]} + \sum_{j=1}^N A_{nn}^{ij[1]} u_n^{j[1]}$$

When $N+1 \leq i \leq 2N$:

$$\begin{aligned} & \sum_{j=N+1}^{N+n_1} (2G_2 B_{ss}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \sum_{j=N+n_1+n_2+1}^{2N} (2G_2 B_{ss}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \\ & \sum_{j=N+1}^{N+n_1} (2G_2 B_{sn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} + \sum_{j=N+n_1+1}^{N+n_1+n_2} (2G_2 B_{sn}^{ij[2]}) \frac{-P}{2G_2} + \sum_{j=N+n_1+n_2+1}^{2N} (2G_2 B_{sn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} \end{aligned} \quad (3.35)$$

$$\begin{aligned} & = \sum_{j=N+1}^{N+n_3} A_{ss}^{ij[2]} u_s^{j[2]} + \sum_{j=N+n_3+1}^{N+n_3+n_2} A_{ss}^{ij[2]} u_s^{j[2]} + \sum_{j=N+n_3+n_2+1}^N A_{ss}^{ij[2]} u_s^{j[2]} + \\ & \sum_{j=N+1}^{N+n_3} A_{sn}^{ij[2]} u_n^{j[2]} + \sum_{j=N+n_3+1}^{N+n_3+n_2} A_{sn}^{ij[2]} u_n^{j[2]} + \sum_{j=N+n_3+n_2+1}^N A_{sn}^{ij[2]} u_n^{j[2]} \\ & \sum_{j=N+1}^{N+n_1} (2G_2 B_{ns}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \sum_{j=N+n_1+n_2+1}^{2N} (2G_2 B_{ns}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \\ & \sum_{j=N+1}^{N+n_1} (2G_2 B_{nn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} + \sum_{j=N+n_1+1}^{N+n_1+n_2} (2G_2 B_{nn}^{ij[2]}) \frac{-P}{2G_2} + \sum_{j=N+n_1+n_2+1}^{2N} (2G_2 B_{nn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} \end{aligned} \quad (3.36)$$

$$= \sum_{j=N+1}^{N+n_3} A_{ns}^{ij[2]} u_s^{j[2]} + \sum_{j=N+n_3+1}^{N+n_3+n_2} A_{ns}^{ij[2]} u_s^{j[2]} + \sum_{j=N+n_3+n_2+1}^N A_{ns}^{ij[2]} u_s^{j[2]} +$$

$$\sum_{j=N+1}^{N+n_3} A_{nn}^{ij[2]} u_n^{j[2]} + \sum_{j=N+n_3+1}^{N+n_3+n_2} A_{nn}^{ij[2]} u_n^{j[2]} + \sum_{j=N+n_3+n_2+1}^N A_{nn}^{ij[2]} u_n^{j[2]}$$

When $2N+1 \leq i \leq 3N$:

$$\begin{aligned} & \sum_{j=2N+1}^{2N+n_1} (2G_2 B_{ss}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \sum_{j=2N+n_1+n_2+1}^{3N} (2G_2 B_{ss}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \\ & \sum_{j=2N+1}^{2N+n_1} (2G_2 B_{sn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} + \sum_{j=2N+n_1+1}^{2N+n_1+n_2} (2G_2 B_{sn}^{ij[2]}) \frac{-P}{2G_2} + \sum_{j=2N+n_1+n_2+1}^{3N} (2G_2 B_{sn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} \end{aligned} \quad (3.37)$$

$$\begin{aligned} & = \sum_{j=1}^{3N} A_{ss}^{ij[2]} u_s^{j[2]} + \sum_{j=1}^{3N} A_{sn}^{ij[2]} u_n^{j[2]} \\ & \sum_{j=2N+1}^{2N+n_1} (2G_2 B_{ns}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \sum_{j=2N+n_1+n_2+1}^{3N} (2G_2 B_{nn}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \\ & \sum_{j=2N+1}^{2N+n_1} (2G_2 B_{nn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} + \sum_{j=2N+n_1+1}^{2N+n_1+n_2} (2G_2 B_{nn}^{ij[2]}) \frac{-P}{2G_2} + \sum_{j=2N+n_1+n_2+1}^{3N} (2G_2 B_{nn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} \end{aligned} \quad (3.38)$$

$$= \sum_{j=1}^{3N} A_{ns}^{ij[2]} u_s^{j[2]} + \sum_{j=1}^{3N} A_{nn}^{ij[2]} u_n^{j[2]}$$

When $3N+1 \leq i \leq 4N$:

$$\begin{aligned} & \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{ss}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{ss}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \\ & \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{sn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_2} + \sum_{j=3N+n_1+1}^{3N+n_3+n_2} (2G_1 B_{sn}^{ij[1]}) \frac{-P}{2G_1} + \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{sn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_1} \end{aligned} \quad (3.39)$$

$$\begin{aligned} & = \sum_{j=3N+1}^{3N+n_3} A_{ss}^{ij[1]} u_s^{j[1]} + \sum_{j=3N+n_3+1}^{3N+n_3+n_2} A_{ss}^{ij[1]} u_s^{j[1]} + \sum_{j=3N+n_3+n_2+1}^{3N} A_{ss}^{ij[1]} u_s^{j[1]} + \\ & \sum_{j=3N+1}^{3N+n_3} A_{sn}^{ij[1]} u_n^{j[1]} + \sum_{j=3N+n_3+1}^{3N+n_3+n_2} A_{sn}^{ij[1]} u_n^{j[1]} + \sum_{j=3N+n_3+n_2+1}^{3N} A_{sn}^{ij[1]} u_n^{j[1]} \\ & \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{ns}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{ns}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \\ & \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{nn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_1} + \sum_{j=3N+n_3+1}^{3N+n_3+n_2} (2G_1 B_{nn}^{ij[1]}) \frac{-P}{2G_1} + \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{nn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_1} \end{aligned} \quad (3.40)$$

$$\begin{aligned} & = \sum_{j=3N+1}^{3N+n_3} A_{ns}^{ij[1]} u_s^{j[1]} + \sum_{j=3N+n_3+1}^{3N+n_3+n_2} A_{ns}^{ij[1]} u_s^{j[1]} + \sum_{j=3N+n_3+n_2+1}^{3N} A_{ns}^{ij[1]} u_s^{j[1]} + \\ & \sum_{j=3N+1}^{3N+n_3} A_{nn}^{ij[1]} u_n^{j[1]} + \sum_{j=3N+n_3+1}^{3N+n_3+n_2} A_{nn}^{ij[1]} u_n^{j[1]} + \sum_{j=3N+n_3+n_2+1}^{3N} A_{nn}^{ij[1]} u_n^{j[1]} \end{aligned}$$

We have introduced the interface continuity conditions in (3.31), and we define the unknowns for each part. Here we illustrate it for all the 4 parts in a detailed way, notice that three are 2 interfaces in each part, which means each part shares the interfaces with 2 different other parts. And j means the index of the independent unknowns, j^* and j^Δ represent the index of the reference of the independent ones:

When $1 \leq i \leq N$:

$$\left\{ \begin{array}{ll} n_1 \geq j^* \geq 1 & \text{and} \quad 3N + n_3 + n_2 + 1 \leq j \leq 4N \\ & \sigma_s^{j^*[1]} = \sigma_s^{j[1]} \\ & \sigma_n^{j^*[1]} = \sigma_n^{j[1]} \\ N \geq j^\Delta \geq n_1 + n_2 + 1 & \text{and} \quad N + 1 \leq j \leq N + n_3 \\ & \sigma_s^{j^\Delta[1]} = \sigma_s^{j[2]} \\ & \sigma_n^{j^\Delta[1]} = \sigma_n^{j[2]} \end{array} \right. \quad (3.41)$$

When $N + 1 \leq i \leq 2N$:

$$\left\{ \begin{array}{ll} N + n_3 \geq j^* \geq N + 1 & \text{and} \quad n_1 + n_2 + 1 \leq j \leq N \\ & u_s^{j^*[2]} = -u_s^{j[1]} \\ & u_n^{j^*[2]} = -u_n^{j[1]} \\ 2N \geq j^\Delta \geq N + n_3 + n_2 + 1 & \text{and} \quad 2N + 1 \leq j \leq 2N + n_1 \\ & u_s^{j^\Delta[2]} = -u_s^{j[2]} \\ & u_n^{j^\Delta[2]} = -u_n^{j[2]} \end{array} \right. \quad (3.42)$$

When $2N + 1 \leq i \leq 3N$:

$$\left\{ \begin{array}{ll} 2N + n_1 \geq j^* \geq 2N + 1 & \text{and} \quad N + n_3 + n_2 + 1 \leq j \leq 2N \\ & \sigma_s^{j^*[2]} = \sigma_s^{j[2]} \\ & \sigma_n^{j^*[2]} = \sigma_n^{j[2]} \\ 3N \geq j^\Delta \geq 2N + n_1 + n_2 + 1 & \text{and} \quad 3N + 1 \leq j \leq 3N + n_3 \\ & \sigma_s^{j^\Delta[2]} = \sigma_s^{j[1]} \\ & \sigma_n^{j^\Delta[2]} = \sigma_n^{j[1]} \end{array} \right. \quad (3.43)$$

When $3N+1 \leq i \leq 4N$:

$$\left\{ \begin{array}{l} 3N+n_3 \geq j^* \geq 3N+1 \quad \text{and} \quad 2N+n_1+n_2+1 \leq j \leq 3N \\ \quad \quad \quad u_s^{j*[1]} = -u_s^{j[2]} \\ \quad \quad \quad u_n^{j*[1]} = -u_n^{j[2]} \\ 4N \geq j^\Delta \geq 3N+n_3+n_2+1 \quad \text{and} \quad 1 \leq j \leq n_1 \\ \quad \quad \quad u_s^{j\Delta[1]} = -u_s^{j[1]} \\ \quad \quad \quad u_n^{j\Delta[1]} = -u_n^{j[1]} \end{array} \right. \quad (3.44)$$

Then, we substitute interface continuity conditions (3.41) - (3.44) into (3.34) - (3.40)

and rearrange the basic formulae, i.e. put the unknown items to the left side and known

ones to the right side:

When $1 \leq i \leq N$, $n_1 \geq j^* \geq 1$ and $N \geq j^\Delta \geq n_1+n_2+1$:

$$\begin{aligned} & \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{ss}^{ij*[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \sum_{j=N+1}^{N+n_3} (2G_2 B_{ss}^{ij\Delta[1]}) \frac{\sigma_s^{j[2]}}{2G_2} + \\ & \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{sn}^{ij*[1]}) \frac{\sigma_n^{j[1]}}{2G_1} + \sum_{j=N+1}^{N+n_3} (2G_2 B_{sn}^{ij\Delta[1]}) \frac{\sigma_n^{j[2]}}{2G_2} \\ & - \sum_{j=1}^N A_{ss}^{ij[1]} u_s^{j[1]} - \sum_{j=1}^N A_{sn}^{ij[1]} u_n^{j[1]} \\ & = \sum_{j=n_1+1}^{n_1+n_2} (2G_1 B_{sn}^{ij[1]}) \frac{P}{2G_1} \end{aligned} \quad (3.45)$$

$$\begin{aligned} & \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{ns}^{ij*[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \sum_{j=N+1}^{N+n_3} (2G_2 B_{ns}^{ij\Delta[1]}) \frac{\sigma_s^{j[2]}}{2G_2} + \\ & \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{nn}^{ij*[1]}) \frac{\sigma_n^{j[1]}}{2G_1} + \sum_{j=N+1}^{N+n_3} (2G_2 B_{nn}^{ij\Delta[1]}) \frac{\sigma_n^{j[2]}}{2G_2} \\ & - \sum_{j=1}^N A_{ns}^{ij[1]} u_s^{j[1]} - \sum_{j=1}^N A_{nn}^{ij[1]} u_n^{j[1]} \\ & = \sum_{j=n_1+1}^{n_1+n_2} (2G_1 B_{nn}^{ij[1]}) \frac{P}{2G_1} \end{aligned} \quad (3.46)$$

When $N+1 \leq i \leq 2N$, $N+n_3 \geq j^* \geq N+1$ and $2N \geq j^\Delta \geq N+n_3+n_2+1$:

$$\begin{aligned}
& \sum_{j=N+1}^{N+n_1} (2G_2 B_{ss}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \sum_{j=N+n_1+n_2+1}^{2N} (2G_2 B_{ss}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \\
& \sum_{j=N+1}^{N+n_1} (2G_2 B_{sn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} + \sum_{j=N+n_1+n_2+1}^{2N} (2G_2 B_{sn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} \\
& + \sum_{j=n_1+n_2+1}^N A_{ss}^{ij*[2]} u_s^{j[1]} - \sum_{j=N+n_3+1}^{N+n_3+n_2} A_{ss}^{ij[2]} u_s^{j[2]} + \sum_{j=2N+1}^{2N+n_1} A_{ss}^{ij\Delta[2]} u_s^{j[2]} \\
& + \sum_{j=n_1+n_2+1}^N A_{sn}^{ij*[2]} u_n^{j[1]} - \sum_{j=N+n_3+1}^{N+n_3+n_2} A_{sn}^{ij[2]} u_n^{j[2]} + \sum_{j=2N+1}^{2N+n_1} A_{sn}^{ij\Delta[2]} u_n^{j[2]} \\
& = \sum_{j=N+n_1+1}^{N+n_1+n_2} (2G_2 B_{sn}^{ij[2]}) \frac{P}{2G_2}
\end{aligned} \tag{3.47}$$

$$\begin{aligned}
& \sum_{j=N+1}^{N+n_1} (2G_2 B_{ns}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \sum_{j=N+n_1+n_2+1}^{2N} (2G_2 B_{ns}^{ij[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \\
& \sum_{j=N+1}^{N+n_1} (2G_2 B_{nn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} + \sum_{j=N+n_1+n_2+1}^{2N} (2G_2 B_{nn}^{ij[2]}) \frac{\sigma_n^{j[2]}}{2G_2} \\
& + \sum_{j=n_1+n_2+1}^N A_{ns}^{ij*[2]} u_s^{j[1]} - \sum_{j=N+n_3+1}^{N+n_3+n_2} A_{ns}^{ij[2]} u_s^{j[2]} + \sum_{j=2N+1}^{2N+n_1} A_{nn}^{ij\Delta[2]} u_s^{j[2]} \\
& + \sum_{j=n_1+n_2+1}^N A_{nn}^{ij*[2]} u_n^{j[1]} - \sum_{j=N+n_3+1}^{N+n_3+n_2} A_{nn}^{ij[2]} u_n^{j[2]} + \sum_{j=2N+1}^{2N+n_1} A_{nn}^{ij\Delta[2]} u_n^{j[2]} \\
& = \sum_{j=N+n_1+1}^{N+n_1+n_2} (2G_2 B_{nn}^{ij[2]}) \frac{P}{2G_2}
\end{aligned} \tag{3.48}$$

When $2N+1 \leq i \leq 3N$, $2N+n_1 \geq j^* \geq 2N+1$ and $3N \geq j^\Delta \geq 2N+n_1+n_2+1$:

$$\begin{aligned}
& \sum_{j=N+n_3+n_2+1}^{2N} (2G_2 B_{ss}^{ij*[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{ss}^{ij\Delta[2]}) \frac{\sigma_s^{j[1]}}{2G_2} \\
& + \sum_{j=N+n_3+n_2+1}^{2N} (2G_2 B_{sn}^{ij*[2]}) \frac{\sigma_n^{j[2]}}{2G_2} + \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{sn}^{ij\Delta[2]}) \frac{\sigma_n^{j[1]}}{2G_2} \\
& - \sum_{j=1}^{3N} A_{ss}^{ij[2]} u_s^{j[2]} - \sum_{j=1}^{3N} A_{sn}^{ij[2]} u_n^{j[2]} \\
& = \sum_{j=2N+n_1+1}^{2N+n_1+n_2} (2G_2 B_{sn}^{ij[2]}) \frac{P}{2G_2}
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
& \sum_{j=N+n_3+n_2+1}^{2N} (2G_2 B_{ns}^{ij*[2]}) \frac{\sigma_s^{j[2]}}{2G_2} + \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{ns}^{ij\Delta[2]}) \frac{\sigma_s^{j[1]}}{2G_2} \\
& + \sum_{j=N+n_3+n_2+1}^{2N} (2G_2 B_{nn}^{ij*[2]}) \frac{\sigma_n^{j[2]}}{2G_2} + \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{nn}^{ij\Delta[2]}) \frac{\sigma_n^{j[1]}}{2G_2} \\
& - \sum_{j=1}^{3N} A_{ns}^{ij[2]} u_s^{j[2]} - \sum_{j=1}^{3N} A_{nn}^{ij[2]} u_n^{j[2]} \\
& = \sum_{j=2N+n_1+1}^{2N+n_1+n_2} (2G_2 B_{nn}^{ij[2]}) \frac{P}{2G_2}
\end{aligned} \tag{3.50}$$

When $3N+1 \leq i \leq 4N$, $3N+n_3 \geq j^* \geq 3N+1$ and $4N \geq j^\Delta \geq 3N+n_3+n_2+1$:

$$\begin{aligned}
& \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{ss}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{ss}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \\
& \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{sn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_2} + \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{sn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_1} \\
& \sum_{j=2N+n_1+n_2+1}^{3N} A_{ss}^{ij*[1]} u_s^{j[2]} - \sum_{j=3N+n_3+1}^{3N+n_3+n_2} A_{ss}^{ij[1]} u_s^{j[1]} + \sum_{j=1}^{n_1} A_{ss}^{ij[1]} u_s^{j[1]} \\
& + \sum_{j=2N+n_1+n_2+1}^{3N} A_{sn}^{ij*[1]} u_n^{j[2]} - \sum_{j=3N+n_3+1}^{3N+n_3+n_2} A_{sn}^{ij[1]} u_n^{j[1]} + \sum_{j=1}^{n_1} A_{sn}^{ij[1]} u_n^{j[1]} \\
& = \sum_{j=3N+n_1+1}^{3N+n_1+n_2} (2G_1 B_{sn}^{ij[1]}) \frac{P}{2G_1}
\end{aligned} \tag{3.51}$$

$$\begin{aligned}
& \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{ns}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{ns}^{ij[1]}) \frac{\sigma_s^{j[1]}}{2G_1} + \\
& \sum_{j=3N+1}^{3N+n_3} (2G_1 B_{nn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_2} + \sum_{j=3N+n_3+n_2+1}^{4N} (2G_1 B_{nn}^{ij[1]}) \frac{\sigma_n^{j[1]}}{2G_1} \\
& \sum_{j=2N+n_1+n_2+1}^{3N} A_{ns}^{ij*[1]} u_s^{j[2]} - \sum_{j=3N+n_3+1}^{3N+n_3+n_2} A_{ns}^{ij[1]} u_s^{j[1]} + \sum_{j=1}^{n_1} A_{ns}^{ij[1]} u_s^{j[1]} \\
& + \sum_{j=2N+n_1+n_2+1}^{3N} A_{nn}^{ij*[1]} u_n^{j[2]} - \sum_{j=3N+n_3+1}^{3N+n_3+n_2} A_{nn}^{ij[1]} u_n^{j[1]} + \sum_{j=1}^{n_1} A_{nn}^{ij[1]} u_n^{j[1]} \\
& = \sum_{j=3N+n_1+1}^{3N+n_1+n_2} (2G_1 B_{nn}^{ij[1]}) \frac{P}{2G_1}
\end{aligned} \tag{3.52}$$

Then we set up the linear system equations for all the 4 parts based on the Equation

(3.45) through (3.52):

$$\begin{aligned}
Y_s^i &= \sum_{j=1}^{4N} C_{ss}^{ij} X_s^j + \sum_{j=1}^{4N} C_{sn}^{ij} X_n^j \\
Y_n^i &= \sum_{j=1}^{4N} C_{ns}^{ij} X_s^j + \sum_{j=1}^{4N} C_{nn}^{ij} X_n^j
\end{aligned} \tag{3.53}$$

where Y_s are known items in the equations, X_s are the independent unknowns and C_s are the boundary influence coefficients.

Based on (3.45) ~ (3.52), we can find out easily the explicit expressions for Y , X and C :

$$C^{ij} = \begin{cases} 1 \leq i \leq N \left\{ \begin{array}{ll} -A^{ij[1]}, & 1 \leq j \leq N \\ 2G_2 B^{ij\Delta[1]}, & N+1 \leq j \leq N+n_3 \text{ and } N \geq j^\Delta \geq n_1+n_2+1 \\ 2G_1 B^{ij*[1]}, & 3N+n_3+n_2+1 \leq j \leq 3N+n_3 \text{ and } n_1 \geq j^* \geq 1 \\ 0, & \text{elsewhere} \end{array} \right. \\ \\ N+1 \leq i \leq 2N \left\{ \begin{array}{ll} A^{ij*[2]}, & n_1+n_2+1 \leq j \leq N \text{ and } N+n_3 \geq j^* \geq N+1 \\ 2G_2 B^{ij[2]}, & N+1 \leq j \leq N+n_3 \text{ or } N+n_3+n_2+1 \leq j \leq 2N \\ -A^{ij[2]}, & N+n_3+1 \leq j \leq N+n_3+n_2 \\ A^{ij\Delta[2]}, & 2N+1 \leq j \leq 2N+n_1 \text{ and } 2N \geq j^\Delta \geq N+n_3+n_2+1 \\ 0, & \text{elsewhere} \end{array} \right. \\ \\ 2N+1 \leq i \leq 3N \left\{ \begin{array}{ll} 2G_2 B^{ij*[2]}, & N+n_3+n_2+1 \leq j \leq 2N \text{ and } 2N+n_1 \geq j^* \geq 2N+1 \\ -A^{ij[2]}, & 2N+1 \leq j \leq 3N \\ 2G_1 B^{ij\Delta[2]}, & 3N+1 \leq j \leq 3N+n_3 \text{ and } 3N \geq j^\Delta \geq 2N+n_1+n_2+1 \\ 0, & \text{elsewhere} \end{array} \right. \\ \\ 3N+1 \leq i \leq 4N \left\{ \begin{array}{ll} A^{ij\Delta[1]}, & 1 \leq j \leq n_1 \text{ and } 4N \geq j^\Delta \geq 3N+n_3+n_2+1 \\ A^{ij*[2]}, & 2N+n_1+n_2+1 \leq j \leq 3N \text{ and } 3N+n_3 \geq j^* \geq 3N+1 \\ 2G_1 B^{ij[1]}, & 3N+1 \leq j \leq 3N+n_3 \text{ or } 3N+n_3+n_2+1 \leq j \leq 4N \\ -A^{ij[1]}, & 3N+n_3+1 \leq j \leq 3N+n_3+n_2 \\ 0, & \text{elsewhere} \end{array} \right. \end{cases}$$

The subscript of C can be either ss , sn , ns or nn .

$$Y_s^i = \begin{cases} \sum_{j=n_1+1}^{n_1+n_2} B_{sn}^{ij[1]} P, & 1 \leq i \leq N \\ \sum_{j=N+n_1+1}^{N+n_1+n_2} B_{sn}^{ij[2]} P, & N+1 \leq i \leq 2N \\ \sum_{j=N+n_1+1}^{N+n_1+n_2} B_{sn}^{ij[2]} P, & 2N+1 \leq i \leq 3N \\ \sum_{j=N+n_1+1}^{N+n_1+n_2} B_{sn}^{ij[2]} P, & 3N+1 \leq i \leq 4N \end{cases} \quad \text{and} \quad Y_n^i = \begin{cases} \sum_{j=n_1+1}^{n_1+n_2} B_{nn}^{ij[1]} P, & 1 \leq i \leq N \\ \sum_{j=N+n_1+1}^{N+n_1+n_2} B_{nn}^{ij[2]} P, & N+1 \leq i \leq 2N \\ \sum_{j=N+n_1+1}^{N+n_1+n_2} B_{nn}^{ij[2]} P, & 2N+1 \leq i \leq 3N \\ \sum_{j=N+n_1+1}^{N+n_1+n_2} B_{nn}^{ij[2]} P, & 3N+1 \leq i \leq 4N \end{cases}$$

$$X = \begin{cases} u^{j[1]}, & 1 \leq j \leq N \\ \sigma^{j[2]} / 2G_2, & N+1 \leq j \leq N+n_3 \\ u^{j[2]}, & N+n_3+1 \leq j \leq N+n_3+n_2 \\ \sigma^{j[2]} / 2G_2, & N+n_3+n_2+1 \leq j \leq 2N \\ u^{j[2]}, & 2N+1 \leq j \leq 3N \\ \sigma^{j[1]} / 2G_1, & 3N+1 \leq j \leq 3N+n_3 \\ u^{j[1]}, & 3N+n_3+1 \leq j \leq 3N+n_3+n_2 \\ \sigma^{j[1]} / 2G_1, & 3N+n_3+n_2+1 \leq j \leq 4N \end{cases}$$

Then we substitute the expressions of X , Y and C into Equation (3.53) and rewrite it in the matrix form:

$$\begin{bmatrix} C_{ss} & C_{sn} \\ C_{ns} & C_{nn} \end{bmatrix} \begin{Bmatrix} X_s \\ X_n \end{Bmatrix} = \begin{Bmatrix} Y_s \\ Y_n \end{Bmatrix} \quad (3.54)$$

After solving (3.54), we can get the exact displacement on each crack surface and the opening displacement by $D=u_- - u_+$ consequently.

However, as mentioned in the first chapter, the number of unknowns increases with the increment of the number of layer. It is then not realistic at all to solve all the unknowns simultaneously with zoning technique. Therefore, Transfer matrix method and Successive stiffness matrix method were introduced to solve the unknowns part by part, leading to smaller matrices operation.

3.3.2 Transfer Matrix Method (TMM) ^{[19] [24] [25]}

A typical figure layered system for the TMM application is sketched as in Fig 3.5 with assuming no crack or openings in it.

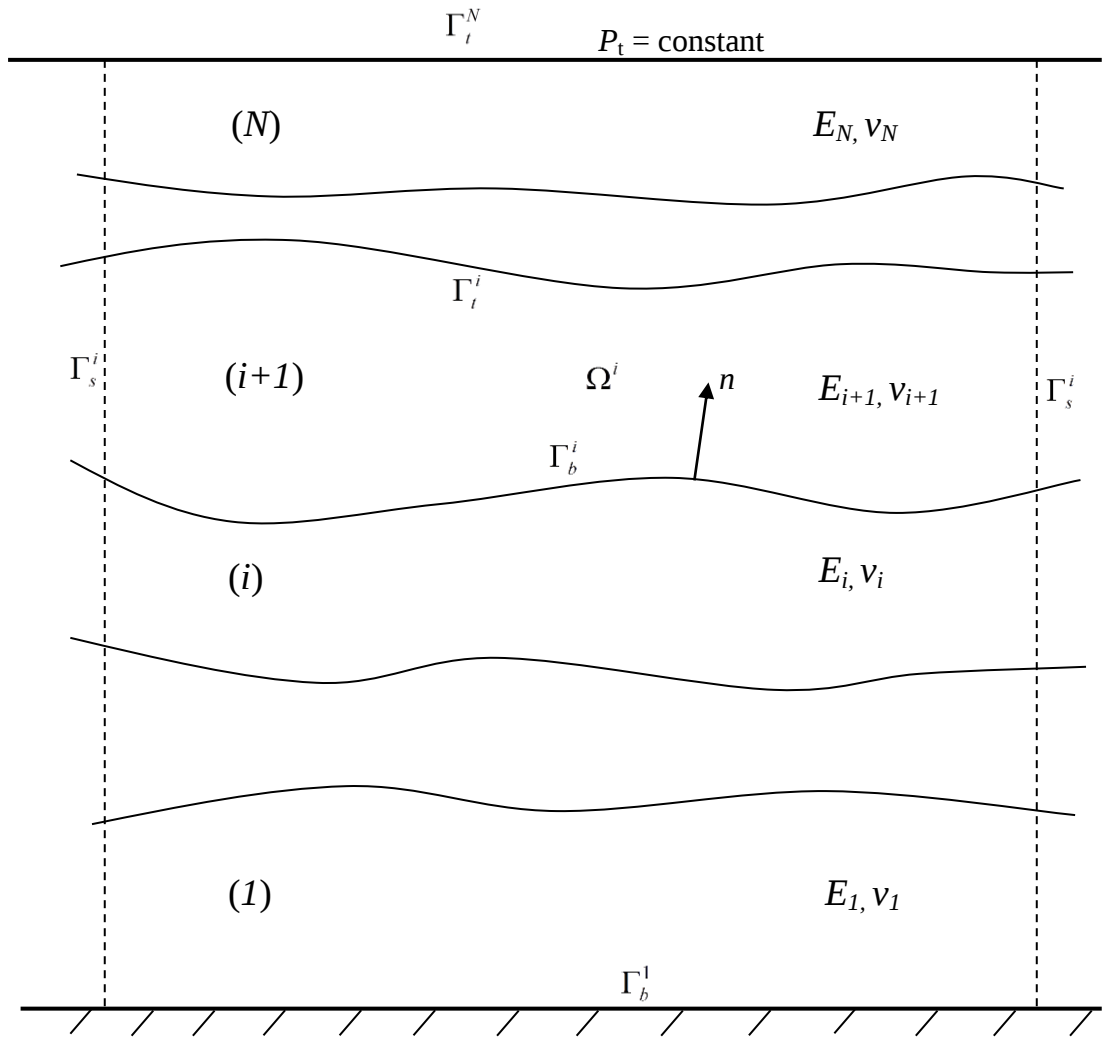


Figure 3. 5: A typical multilayered system

The multilayered system in Fig 3.5 exhibits a “chain-pattern”, namely the boundary of each layer has parts in common with at most two other layers, besides possibly with the

boundary Γ of the considered domain Ω . This assumption makes it possible to define, for any (i th) layer, two disjointed surfaces: the upper one Γ_t^i and the lower one Γ_s^i . If the domain Ω is bounded, then Γ_t^i and Γ_s^i are separated by a side surface Γ_b^1 which will be assumed here as far from the loaded “disturbed” region and, therefore belonging to the fixed portion of boundary Γ .

At every node of each interface, a single unit normal vector is assumed, specifically directed outwards for the lower (i th) and inwards for the upper ($i+1$) layer. As a consequence, the interface compatibility and equilibrium at nodes are (tractions are not normalized):

$$u_t^i = u_b^{i+1}, \quad p_t^i = p_b^{i+1} \quad (3.55)$$

The bottom of the system Γ_b^1 is assumed to be part of the portion of Γ where the displacements are assigned (usually zero to simulate a rigid underlying stratum or ‘undisturbed’ deep region). The top of the system Γ_t^N is usually assigned with known constant traction P_t because of the high flexibility of the foundation.

After the definition of the system, we establish the algebraic equations by direct method, i.e. Equation (3.10) and rewrite it in the matrix form with collocation at the nodes:

$$\mathbf{A}^i \mathbf{u}^i = \mathbf{B}^i \mathbf{p}^i \text{ or: } \begin{bmatrix} A_{tt}^i & A_{tb}^i & A_{ts}^i \\ A_{bt}^i & A_{bb}^i & A_{bs}^i \\ A_{st}^i & A_{sb}^i & A_{ss}^i \end{bmatrix} \begin{Bmatrix} u_t^i \\ u_b^i \\ u_s^i \end{Bmatrix} = \begin{bmatrix} B_{tt}^i & B_{tb}^i & B_{ts}^i \\ B_{bt}^i & B_{bb}^i & B_{bs}^i \\ B_{st}^i & B_{sb}^i & B_{ss}^i \end{bmatrix} \begin{Bmatrix} p_t^i \\ p_b^i \\ p_s^i \end{Bmatrix} \quad (3.56)$$

With reference to the i th layer, $\mathbf{A}^i$ being always non-singular, we obtain:

$$\begin{aligned} \begin{Bmatrix} p_t^i \\ p_b^i \\ p_s^i \end{Bmatrix} &= (A^i)^{-1} B^i u^i \\ &= \begin{bmatrix} K_{tt}^i & K_{tb}^i & K_{ts}^i \\ K_{bt}^i & K_{bb}^i & K_{bs}^i \\ K_{st}^i & K_{sb}^i & K_{ss}^i \end{bmatrix} \begin{Bmatrix} u_t^i \\ u_b^i \\ u_s^i \end{Bmatrix} \end{aligned} \quad (3.57)$$

where $K^i = (A^i)^{-1} B^i$

Taking $u_s = 0$, since it belongs to the “undisturbed zone”, and taking the same number of nodes on the upper Γ_t^i and lower Γ_s^i , we can derive from (3.57) the following “transfer relation” from Γ_t^i to Γ_s^i :

$$\begin{Bmatrix} u_t^i \\ p_t^i \end{Bmatrix} = T^i \begin{Bmatrix} u_b^i \\ p_b^i \end{Bmatrix} = \begin{bmatrix} T_{uu}^i & T_{up}^i \\ T_{pu}^i & T_{pp}^i \end{bmatrix} \begin{Bmatrix} u_b^i \\ p_b^i \end{Bmatrix} \quad (3.58)$$

where the blocks of the partitioned transfer matrix T^i are:

$$\begin{aligned} T_{uu}^i &= -(K_{bt}^i)^{-1} K_{bb}^i; & T_{up}^i &= (K_{bt}^i)^{-1} \\ T_{pu}^i &= K_{tb}^i - K_{tt}^i (K_{bt}^i)^{-1} K_{bb}^i; & T_{pp}^i &= K_{tt}^i (K_{bt}^i)^{-1} \end{aligned} \quad (3.59)$$

The Equation (3.58) and the interface continuity condition (3.55) give rise by chain substitutions to a transfer relation from the top surface Γ_t^N to the bottom one Γ_b^1 , across the whole layer set:

$$\begin{Bmatrix} u_t^N \\ p_t^N \end{Bmatrix} = T \begin{Bmatrix} u_b^1 \\ p_b^1 \end{Bmatrix} = \begin{bmatrix} T_{uu} & T_{up} \\ T_{pu} & T_{pp} \end{bmatrix} \begin{Bmatrix} u_b^1 \\ p_b^1 \end{Bmatrix} \quad (3.60)$$

where: $T = \prod_{i=1}^N T^i$ (3.61)

Having known boundary condition $u_b^1 = 0$ and p_t^N , we then can solve (3.60) for u_t^1 and p_b^N . Substituting those 4 known vectors into (3.60), we can finally determine all the

displacements and stresses component on each interface.

However, one can easily find out that one drawback of the TMM is the requirement that the number of nodes be the same on all interfaces and top and bottom surfaces. What is more, for a single layer of given thickness, the transfer matrix becomes both significant erroneous and ill-conditioned as the mesh is refined beyond a certain threshold, as shown in Figure 3.6, which was extracted from Ref.19.

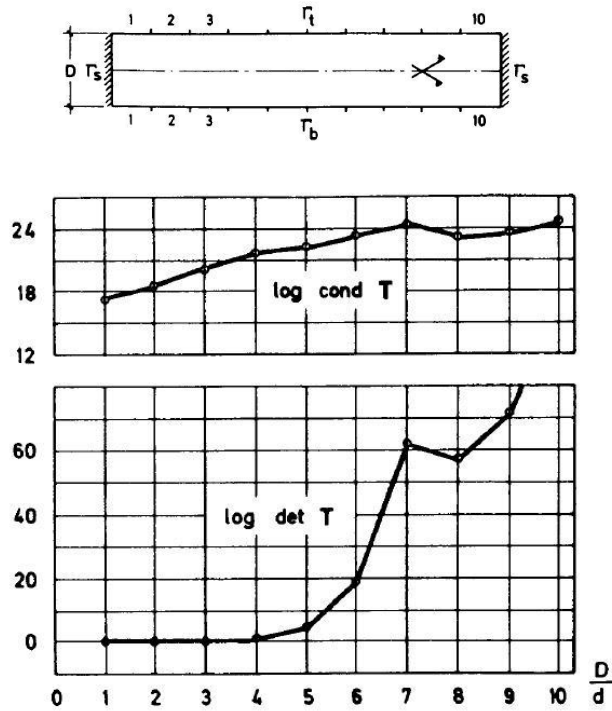


Figure 3. 6: Condition number and determinant of matrix T versus geometry ratio D/d

(thickness over BE length for the layer of the upper part of this figure)

In addition, according to Ref.19, the ill-conditioning of the transfer matrices of individual layers entails that of the overall transfer matrix with an amplification increasing with the layer number. Those limitations of the TMM make it worthwhile to

search for an alternative method to avoid above drawbacks, but still exploiting the chain pattern of layered systems. Then the successive stiffness method was developed.

3.3.3 Successive Stiffness Method (SSM) ^[26]

The successive stiffness method begins with the same sketched figure as Figure 3.5 and the same linear equations according to the direct method and interface continuity condition as (3.55), with unnormalized tractions:

$$u_t^i = u_b^{i+1}, \quad p_t^i = p_b^{i+1} \quad (3.61)$$

$$\begin{Bmatrix} p_t^i \\ p_b^i \\ p_s^i \end{Bmatrix} = \begin{bmatrix} K_{tt}^i & K_{tb}^i & K_{ts}^i \\ K_{bt}^i & K_{bb}^i & K_{bs}^i \\ K_{st}^i & K_{sb}^i & K_{ss}^i \end{bmatrix} \begin{Bmatrix} u_t^i \\ u_b^i \\ u_s^i \end{Bmatrix} \quad (3.62)$$

Since $u_s^i = 0$, then for the any layer, Equation (3.62) can be split into two parts:

$$\begin{Bmatrix} p_t^i \\ p_b^i \end{Bmatrix} = \begin{bmatrix} K_{tt}^i & K_{tb}^i \\ K_{bt}^i & K_{bb}^i \end{bmatrix} \begin{Bmatrix} u_t^i \\ u_b^i \end{Bmatrix} \quad (3.63)$$

$$p_s^i = K_{st}^i u_t^i + K_{sb}^i u_b^i \quad (3.64)$$

The same boundary condition is adopted, i.e. $u_b^1 = 0$, and $p_t^i = \text{constant}$. Thus, substitute the former one into the bottom layer ($i=1$):

$$p_t^1 = K_{tt}^1 u_t^1 \equiv \hat{K}^1 u_t^1 \quad (3.65)$$

This can be interpreted as a relation, which characterizes the upper surface of the bottom layer in terms of the “surface stiffness” matrix $\hat{K}^1$. $\hat{K}^1 \equiv K_{tt}^1$ defines the stiffness of the “elastic bed” provided by the first layer to the subsequent ones.

Similarly, we can construct $\hat{K}^2$ by imposing Equation (3.61):

$$p_b^2 = \hat{K}^1 u_b^2 \quad (3.66)$$

Substitute Equation (3.66) into (3.63) for $i=2$:

$$u_b^2 = [(\hat{K}^1 - K_{bb}^2)^{-1} K_{bt}^2] u_t^2 \quad (3.67)$$

Substitute Equation (3.67) back into Equation (3.63):

$$p_t^2 = [K_{tt}^2 - K_{tb}^2 (\hat{K}^1 - K_{bb}^2)^{-1} K_{bt}^2] u_t^2 \equiv \hat{K}^2 u_t^2 \quad (3.68)$$

Turning now to generic, we can define the relevant stiffness relation in the recursive form:

$$p_t^i = [K_{tt}^i - K_{tb}^i (\hat{K}^{i-1} - K_{bb}^i)^{-1} K_{bt}^i] u_t^i \equiv \hat{K}^i u_t^i \quad (3.69)$$

For the top layer $i=N$, with the boundary condition $p_t^i = \text{constant}$, we can get the consequent displacements:

$$u_t^N = (\hat{K}^N)^{-1} p_t^N \quad (3.70)$$

where $\hat{K}^N$ defines the overall stiffness of the system with respect to external tractions acting on its top surface.

Having obtaining the tractions and displacements on the top of the system, a sequential computation can be done over all interfaces in a descending sequence, by using:

$$u_b^i = [(\hat{K}^{i-1} - K_{bb}^i)^{-1} K_{bt}^i] u_t^i \quad (3.71)$$

$$p_b^i = \hat{K}^{i-1} u_t^i \quad \text{for } i=N \dots 2 \quad (3.72)$$

Equation (3.41) and (3.42) can be obtained by simply replace the superscript 2 into Equation (3.66) and (3.67) with i .

As indicated previously, the direct method cannot “direct” applied for crack problems,

because of the indistinguishability. We need to cut the material along the crack line, as illustrated in Figure 3.6, which is the result of the inclination 90° of Figure 3.5., into upper and lower parts and then revise the original successive stiffness method to be applicable for crack problems.

3.3.4 Revised successive stiffness method

In Figure 3.7, besides the top, bottom, side boundary components, there are two more boundary components denoted as Γ_I and Γ_c , which represents the horizontal interface boundary only existing in the bottom and top layer ($i=1$ or N) and the crack surface, respectively. According to the zoning technique, the Γ_I is a “free” surface on which all the four elastic parameters remains unknown and Γ_c acts as a boundary on which two of the four parameters are known: $P_s = 0$, $P_n = -P$. The superscript L or U stands for the block is either lower one or upper one. And we will ignore side boundary in the following calculations.

According to the successive stiffness method mentioned above, we need to first combine the lower region and its corresponding upper part to form a whole layer, i.e. put Ω_i^L and Ω_i^U into Ω_i . Then the successive method can be applied on the combined layers. And for this modified successive method, we adopted the interface continuity in zoning technique and normalized traction components:

$$u_t^i = -u_b^{i+1} \quad \text{and} \quad G^i p_t^i = G^{i+1} p_b^{i+1} \quad (3.73)$$

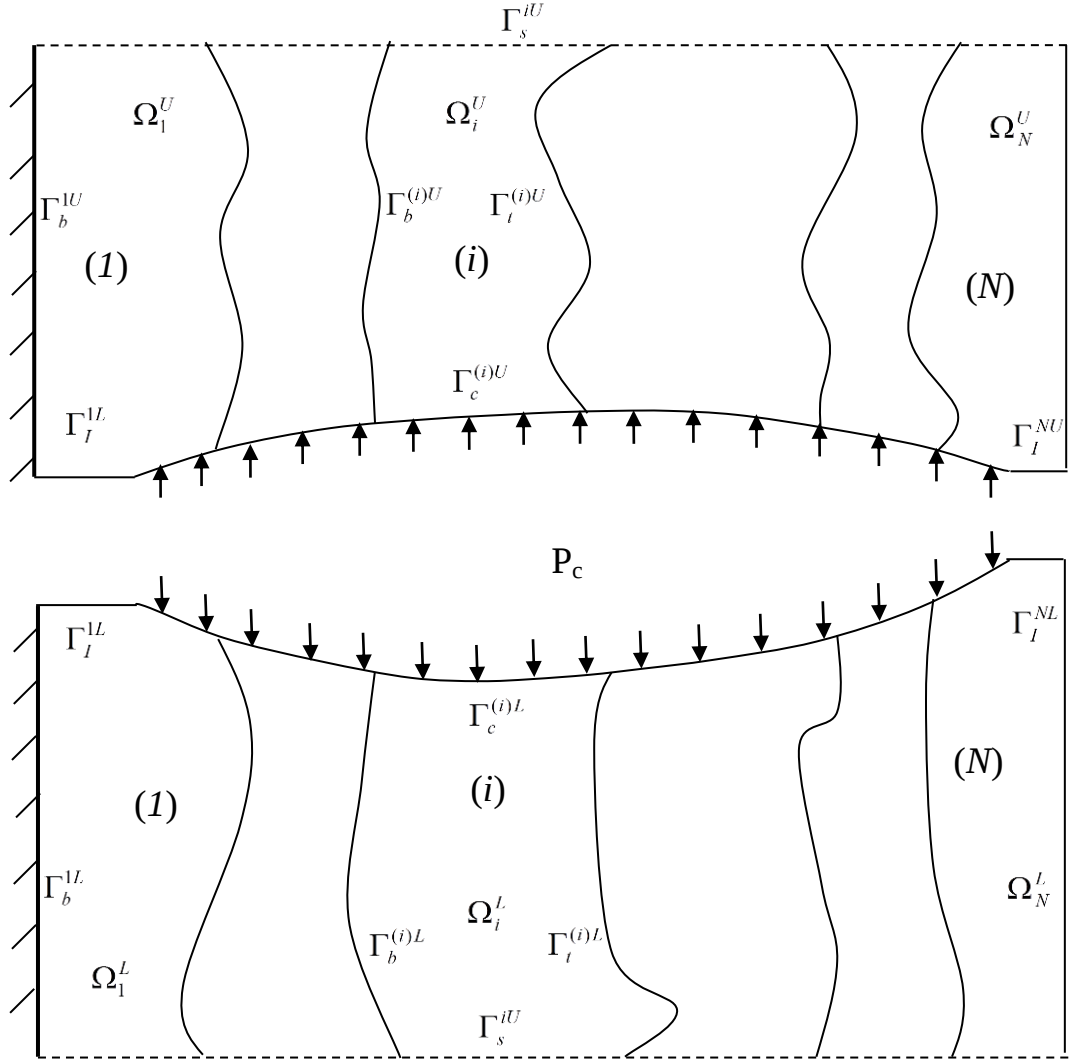


Figure 3. 7: Cutting along the crack line in the multilayered system

Then, we first write the linear equations, according to (3.61) and (3.62), for Γ_s^{iU} and Γ_b^{1U} , separately. We define α represents the non-interface boundary, i.e. top, crack and bottom boundary components. And prime denotes the boundary of the upper region and non-prime denotes the one of the lower region. We can get:

$$\begin{cases} P_\alpha^1 = K_{\alpha\alpha}^1 u_\alpha^1 + K_{\alpha I}^1 u_I^1 \\ P_I^1 = K_{I\alpha}^1 u_\alpha^1 + K_{II}^1 u_I^1 \end{cases} \text{ and } \begin{cases} P_{\alpha'}^1 = K_{\alpha'\alpha'}^1 u_{\alpha'}^1 + K_{\alpha'I}^1 u_I^1 \\ P_{I'}^1 = K_{I'\alpha'}^1 u_{\alpha'}^1 + K_{I'I'}^1 u_{I'}^1 \end{cases} \quad (3.74)$$

Notice that: $u_I^1 = -u_{I'}^1$ and $p_I^1 = p_{I'}^1$

We have:
$$K_{I\alpha}^1 u_\alpha^1 + K_{II}^1 u_I^1 = K_{I'\alpha'}^1 u_{\alpha'}^1 + K_{I'I'}^1 u_{I'}^1,$$

and then
$$u_I^1 = (K_{II}^1 + K_{I'I'}^1)^{-1} (K_{I'\alpha'}^1 u_{\alpha'}^1 - K_{I\alpha}^1 u_\alpha^1) \quad (3.75)$$

Substitute Equation (3.75) into (3.74), we can get:

$$\begin{Bmatrix} p_\alpha^1 \\ p_{\alpha'}^1 \end{Bmatrix} = \begin{bmatrix} H_{\alpha\alpha}^1 & H_{\alpha\alpha'}^1 \\ H_{\alpha'\alpha}^1 & H_{\alpha'\alpha'}^1 \end{bmatrix} \begin{Bmatrix} u_\alpha^1 \\ u_{\alpha'}^1 \end{Bmatrix} \quad (3.76)$$

where:

$$\begin{cases} H_{\alpha\alpha}^1 = K_{\alpha\alpha}^1 - K_{\alpha I}^1 (K_{II}^1 + K_{I'I'}^1)^{-1} K_{I\alpha}^1 \\ H_{\alpha\alpha'}^1 = K_{\alpha I}^1 (K_{II}^1 + K_{I'I'}^1)^{-1} K_{I'\alpha'}^1 \\ H_{\alpha'\alpha}^1 = K_{\alpha'I}^1 (K_{II}^1 + K_{I'I'}^1)^{-1} K_{I\alpha}^1 \\ H_{\alpha'\alpha'}^1 = K_{\alpha'\alpha'}^1 - K_{\alpha'I}^1 (K_{II}^1 + K_{I'I'}^1)^{-1} K_{I'\alpha'}^1 \end{cases} \text{ and } H^1 \equiv \begin{bmatrix} H_{\alpha\alpha}^1 & H_{\alpha\alpha'}^1 \\ H_{\alpha'\alpha}^1 & H_{\alpha'\alpha'}^1 \end{bmatrix}$$

As $u_b^i = 0$, similarly to (3.63), the related columns and rows will be extracted out from the matrix H^1 , leading to a submatrix $\hat{H}^1$ involving only the top boundary and crack surface:

$$\begin{Bmatrix} p_t^1 \\ p_c^1 \end{Bmatrix} = \begin{bmatrix} H_{tt}^1 & H_{tc}^1 \\ H_{ct}^1 & H_{cc}^1 \end{bmatrix} \begin{Bmatrix} u_t^1 \\ u_c^1 \end{Bmatrix} \quad (3.77)$$

From Equation (3.77): $u_c^1 = (H_{cc}^1)^{-1} (p_c^1 - H_{ct}^1 u_t^1)$

Substitute u_c^1 into the first row of (3.77) along with the interface continuity (3.73):

$$\begin{aligned} p_t^1 &= [(H_{tt}^1)^{-1} - H_{tc}^1 (H_{cc}^1)^{-1} H_{ct}^1] u_t^1 + H_{tc}^1 (H_{cc}^1)^{-1} p_c^1 \\ &= \hat{K}_t^1 u_t^1 + \hat{p}_t^1 \\ &= -\hat{K}_t^1 u_b^2 + \hat{p}_t^1 \\ &= \frac{G_2}{G_1} p_b^2 \end{aligned} \quad (3.78)$$

For the second layer, the crack surfaces have split it into two disjointed part. We then also need to write the linear equations $Ku=p$ for each of them:

$$p^2 = K^2 u^2 \quad \text{and} \quad p^{2'} = K^{2'} u^{2'}$$

or in a explicit form:

$$\begin{Bmatrix} p_t^2 \\ p_b^2 \\ p_c^2 \end{Bmatrix} = \begin{bmatrix} K_{tt}^2 & K_{tb}^2 & K_{tc}^2 \\ K_{bt}^2 & K_{bb}^2 & K_{bc}^2 \\ K_{ct}^2 & K_{cb}^2 & K_{cc}^2 \end{bmatrix} \begin{Bmatrix} u_t^2 \\ u_b^2 \\ u_c^2 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} p_{t'}^2 \\ p_{b'}^2 \\ p_{c'}^2 \end{Bmatrix} = \begin{bmatrix} K_{t't'}^2 & K_{t'b'}^2 & K_{t'c'}^2 \\ K_{b't'}^2 & K_{b'b'}^2 & K_{b'c'}^2 \\ K_{c't'}^2 & K_{c'b'}^2 & K_{c'c'}^2 \end{bmatrix} \begin{Bmatrix} u_{t'}^2 \\ u_{b'}^2 \\ u_{c'}^2 \end{Bmatrix}$$

Since the two parts are separate, they can be simply patched together as:

$$\begin{Bmatrix} p^2 \\ p^{2'} \end{Bmatrix} = \begin{bmatrix} K^2 & 0 \\ 0 & K^{2'} \end{bmatrix} \begin{Bmatrix} u^2 \\ u^{2'} \end{Bmatrix} \quad (3.79)$$

Equation (3.3.79) clearly shows that the disjointed part of the inner layers (i runs over 2 to $N-1$) do not share a common interface so that one cannot directly “interact” with the other.

Let $H^2 \equiv \begin{bmatrix} K^2 & 0 \\ 0 & K^{2'} \end{bmatrix}$, and it will be partitioned according to the division of the

boundaries along the whole second layer, e.g. both t and t' will be redeemed as t . Then

Equation (3.79) can be rewritten as:

$$\begin{Bmatrix} p_t^2 \\ p_b^2 \\ p_c^2 \end{Bmatrix} = \begin{bmatrix} H_{tt}^2 & H_{tb}^2 & H_{tc}^2 \\ H_{bt}^2 & H_{bb}^2 & H_{bc}^2 \\ H_{ct}^2 & H_{cb}^2 & H_{cc}^2 \end{bmatrix} \begin{Bmatrix} u_t^2 \\ u_b^2 \\ u_c^2 \end{Bmatrix} \quad (3.80)$$

From the third row:

$$u_c^2 = (H_{cc}^2)^{-1} (p_c^2 - H_{ct}^2 u_t^2 - H_{cb}^2 u_b^2) \quad (3.81)$$

Substitute Equation (3.81) and (3.78) into the 2nd row of (3.80):

$$\frac{G_2}{G_1} p_b^2 = \frac{G_2}{G_1} (H_{bt}^2 u_t^2 + H_{bb}^2 u_b^2 + H_{bc}^2 u_c^2) = -\hat{K}_{tt}^1 u_b^2 + \hat{p}_t^1 = p_t^1$$

And we can get:

$$u_b^2 = -(A^2 - \hat{K}_{tt}^1)^{-1} B^2 u_t^2 + (A^2 - \hat{K}_{tt}^1)^{-1} \left[\frac{G_2}{G_1} H_{bc}^2 (H_{cc}^2)^{-1} p_c^2 - \hat{p}_t^1 \right] \quad (3.82)$$

Substitute Equation (3.82) into (3.81), we can get:

$$u_c^2 = -(H_{cc}^2)^{-1} [H_{ct}^2 - H_{cb}^2 (A^2 - \hat{K}_{tt}^1)^{-1} B^2] u_t^2 + (H_{cc}^2)^{-1} \left\{ \left[I - \frac{G_2}{G_1} H_{cb}^2 (A^2 - \hat{K}_{tt}^1)^{-1} H_{bc}^2 (H_{cc}^2)^{-1} \right] p_c^2 + H_{cb}^2 (A^2 - \hat{K}_{tt}^1)^{-1} \hat{p}_t^1 \right\} \quad (3.83)$$

Substitute Equation (3.82) and (3.83) into 1st row of (3.80):

$$\begin{aligned} p_t^2 &= \hat{K}_{tt}^2 u_t^2 + C_{tc}^2 p_c^2 + C_{tt}^{21} \hat{p}_t^1 \\ &= \hat{K}_{tt}^2 u_t^2 + \hat{p}_t^2 \\ &= -\hat{K}_{tt}^2 u_b^3 + \hat{p}_t^2 = \frac{G_3}{G_2} p_b^3 \end{aligned} \quad (3.84)$$

where:

$$\begin{aligned} A^2 &= \frac{G_2}{G_1} [H_{bc}^2 (H_{cc}^2)^{-1} H_{cb}^2 - H_{bb}^2] \\ B^2 &= \frac{G_2}{G_1} [H_{bc}^2 (H_{cc}^2)^{-1} H_{ct}^2 - H_{bt}^2] \\ \hat{p}_t^2 &= C_{tc}^2 p_c^2 + C_{tt}^{21} \hat{p}_t^1 \\ \hat{K}_{tt}^2 &= H_{tt}^2 - H_{tb}^2 (A^2 - \hat{K}_{tt}^1)^{-1} B^2 - H_{tc}^2 (H_{cc}^2)^{-1} [H_{ct}^2 - H_{cb}^2 (A^2 - \hat{K}_{tt}^1)^{-1} B^2] \\ C_{tt}^{21} &= H_{tc}^2 (H_{cc}^2)^{-1} H_{cb}^2 (A^2 - \hat{K}_{tt}^1)^{-1} - H_{tb}^2 (A^2 - \hat{K}_{tt}^1)^{-1} \\ C_{tc}^2 &= H_{tc}^2 (H_{cc}^2)^{-1} \left[I - \frac{G_2}{G_1} H_{cb}^2 (A^2 - \hat{K}_{tt}^1)^{-1} H_{bc}^2 (H_{cc}^2)^{-1} \right] + \frac{G_2}{G_1} H_{tb}^2 (A^2 - \hat{K}_{tt}^1)^{-1} H_{bc}^2 (H_{cc}^2)^{-1} \end{aligned} \quad (3.85)$$

I stands for the identity matrix.

Having obtained Equation (3.84) and (3.85), we then treat the remaining inner layers (i runs over 2 to $N-1$) in the same way as we did for the second one. Thus, it is quite simple to get the successive stiffness equations for the i th layer (i runs over 2 to $N-1$).

Simply replace all the superscript “1” with “i-1”, “2” in (3.84) and (3.85) with “i”, and “3” with “i+1”.

After dealing with the bottom and inner layers, we come to the top one. The top layer has one common feature as the bottom one does: the upper and lower region share a common horizontal interface. Thus, we need to use an analogous way as used in the equations (3.74) - (3.76), with replacing the superscript “1” with “N”, to get the matrix H^N . Since all the three boundary parts, i.e. top, bottom, crack surface, are not displacement-assigned, then the stiffness matrix of top layer $\hat{H}^N$ is H^N itself.

As mentioned in last paragraph, all the three boundary parts of the top layer are not displacement-assigned, which is then different from the bottom layer with fixed bottom. Hence, we need to deal with it as we did with the inner layer and get similar formulae as Equation (3.84) and (3.85), replacing all the superscript “1” with “N-1”, “2” in (3.84) and (3.85) with “N”.

To sum up:

1) For the bottom layer ($i=1$):

$$\left\{ \begin{array}{l} H_{\alpha\alpha}^1 = K_{\alpha\alpha}^1 - K_{\alpha d}^1 (K_{\parallel}^1 + K_{\Gamma\Gamma'}^1) K_{I\alpha}^1 \\ H_{\alpha\alpha'}^1 = K_{\alpha d}^1 (K_{\parallel}^1 + K_{\Gamma\Gamma'}^1) K_{\Gamma'\alpha'}^1 \\ H_{\alpha'\alpha}^1 = K_{\alpha'\Gamma'}^1 (K_{\parallel}^1 + K_{\Gamma\Gamma'}^1) K_{I\alpha}^1 \\ H_{\alpha'\alpha'}^1 = K_{\alpha'\alpha'}^1 - K_{\alpha'\Gamma'}^1 (K_{\parallel}^1 + K_{\Gamma\Gamma'}^1) K_{\Gamma'\alpha'}^1 \end{array} \right. \quad \text{and} \quad H^1 \equiv \begin{bmatrix} H_{\alpha\alpha}^1 & H_{\alpha\alpha'}^1 \\ H_{\alpha'\alpha}^1 & H_{\alpha'\alpha'}^1 \end{bmatrix} \quad \text{provides a}$$

submatrix $\hat{H}^1$, including H_{tt}^1 , etc. then

$$u_c^1 = (H_{cc}^1)^{-1} (p_c^1 - H_{ct}^1 u_t^1) \quad (3.86)$$

$$\begin{aligned}
p_t^1 &= [(H_{tt}^1)^{-1} - H_{tc}^1 (H_{cc}^1)^{-1} H_{ct}^1] u_t^1 + H_{tc}^1 (H_{cc}^1)^{-1} p_c^1 \\
&= \hat{K}_{tt}^1 u_t^1 + \hat{p}_c^1 \\
&= -\hat{K}_{tt}^1 u_b^2 + \hat{p}_c^1 \\
&= \frac{G_2}{G_1} p_b^2
\end{aligned} \tag{3.87}$$

2) For the inner layer ($i=2 \sim N-1$):

After partitioning $H^i \equiv \begin{bmatrix} K^i & 0 \\ 0 & K^{i'} \end{bmatrix}$ according to the division of the boundaries along

the whole i th layer, we get $\hat{H}^i = \begin{bmatrix} H_{tt}^i & H_{tb}^i & H_{tc}^i \\ H_{bt}^i & H_{bb}^i & H_{bc}^i \\ H_{ct}^i & H_{cb}^i & H_{cc}^i \end{bmatrix}$ and:

$$u_b^i = -(A^i - \hat{K}_{tt}^{i-1})^{-1} B^i u_t^i + (A^i - \hat{K}_{tt}^{i-1})^{-1} \left[\frac{G_i}{G_{i-1}} H_{bc}^i (H_{cc}^i)^{-1} p_c^i - \hat{p}_c^{i-1} \right] \tag{3.88}$$

$$\begin{aligned}
u_c^i &= -(H_{cc}^i)^{-1} [H_{ct}^i - H_{cb}^i (A^i - \hat{K}_{tt}^{i-1})^{-1} B^i] u_t^i + \\
&\quad (H_{cc}^i)^{-1} \left\{ \left[I - \frac{G_i}{G_{i-1}} H_{cb}^i (A^i - \hat{K}_{tt}^{i-1})^{-1} H_{bc}^i (H_{cc}^i)^{-1} \right] p_c^i + H_{cb}^i (A^i - \hat{K}_{tt}^{i-1})^{-1} \hat{p}_c^{i-1} \right\}
\end{aligned} \tag{3.89}$$

$$\begin{aligned}
p_t^i &= \hat{K}_{tt}^i u_t^i + C^i p_c^i + C^{i(i-1)} \hat{p}_c^{i-1} \\
&= \hat{K}_{tt}^i u_t^i + \hat{p}_c^i \\
&= -\hat{K}_{tt}^i u_b^{i+1} + \hat{p}_c^i = \frac{G_{i+1}}{G_i} p_b^{i+1}
\end{aligned} \tag{3.90}$$

where:

$$\begin{aligned}
A^i &= \frac{G_i}{G_{i-1}} [H_{bc}^i (H_{cc}^i)^{-1} H_{cb}^i - H_{bb}^i] \\
B^i &= \frac{G_i}{G_{i-1}} [H_{bc}^i (H_{cc}^i)^{-1} H_{ct}^i - H_{bt}^i] \\
\hat{p}_t^i &= C_{tc}^i p_c^i + C_{tt}^{i(i-1)} \hat{p}_t^{i-1} \\
\hat{K}_{tt}^i &= H_{tt}^i - H_{tb}^i (A^i - \hat{K}_{tt}^{i-1})^{-1} B^i - H_{tc}^i (H_{cc}^i)^{-1} [H_{ct}^i - H_{cb}^i (A^i - \hat{K}_{tt}^{i-1})^{-1} B^i] \\
C_{tt}^{i(i-1)} &= H_{tc}^i (H_{cc}^i)^{-1} H_{cb}^i (A^i - \hat{K}_{tt}^{i-1})^{-1} - H_{tb}^i (A^i - \hat{K}_{tt}^{i-1})^{-1} \\
C_{tc}^i &= H_{tc}^i (H_{cc}^i)^{-1} \left[I - \frac{G_i}{G_{i-1}} H_{cb}^i (A^i - \hat{K}_{tt}^{i-1})^{-1} H_{bc}^i (H_{cc}^i)^{-1} \right] + \frac{G_i}{G_{i-1}} H_{tb}^i (A^i - \hat{K}_{tt}^{i-1})^{-1} H_{bc}^i (H_{cc}^i)^{-1}
\end{aligned} \tag{3.91}$$

3) For the top layer

$$\begin{cases} H_{\alpha\alpha}^N = K_{\alpha\alpha}^N - K_{\alpha I}^N (K_{II}^N + K_{I'I'}^N) K_{I\alpha}^N \\ H_{\alpha\alpha'}^N = K_{\alpha I}^N (K_{II}^N + K_{I'I'}^N) K_{I'\alpha'}^N \\ H_{\alpha'\alpha}^N = K_{\alpha'I'}^N (K_{II}^N + K_{I'I'}^N) K_{I\alpha}^N \\ H_{\alpha'\alpha'}^N = K_{\alpha'\alpha'}^N - K_{\alpha'I}^N (K_{II}^N + K_{I'I'}^N) K_{I'\alpha'}^N \end{cases} \quad \text{and} \quad H^N \equiv \begin{bmatrix} H_{\alpha\alpha}^N & H_{\alpha\alpha'}^N \\ H_{\alpha'\alpha}^N & H_{\alpha'\alpha'}^N \end{bmatrix} = \hat{H}^N \quad \text{provides } H_{tt}^N,$$

etc.

$$u_b^N = -(A^N - \hat{K}_{tt}^{N-1})^{-1} B^N u_t^N + (A^N - \hat{K}_{tt}^{N-1})^{-1} \left[\frac{G_N}{G_{N-1}} H_{bc}^N (H_{cc}^N)^{-1} p_c^N - \hat{p}_c^{N-1} \right] \quad (3.92)$$

$$\begin{aligned} u_c^N = & -(H_{cc}^N)^{-1} [H_{ct}^N - H_{cb}^N (A^N - \hat{K}_{tt}^{N-1})^{-1} B^N] u_t^N + \\ & (H_{cc}^N)^{-1} \left\{ \left[I - \frac{G_N}{G_{N-1}} H_{cb}^N (A^N - \hat{K}_{tt}^{N-1})^{-1} H_{bc}^N (H_{cc}^N)^{-1} \right] p_c^N \right. \\ & \left. + H_{cb}^N (A^N - \hat{K}_{tt}^{N-1})^{-1} \hat{p}_c^{N-1} \right\} \end{aligned} \quad (3.93)$$

$$\begin{aligned} p_t^N &= \hat{K}_{tt}^N u_t^N + C^N p_c^N + C^{N(N-1)} \hat{p}_c^{N-1} \\ &= \hat{K}_{tt}^N u_t^N + \hat{p}_c^N \end{aligned} \quad (3.94)$$

The coefficients in Equations (3.92) – (3.94) can be obtained from Equation (3.91) by replacing the superscript i with N .

3.4 Results and remark

3.4.1 A tensile crack in a Homogeneous medium

Here, we consider the same problem with same parameters presented in the Section 2.4, i.e. a tensile crack in a homogeneous medium as shown in Figure 2.4.

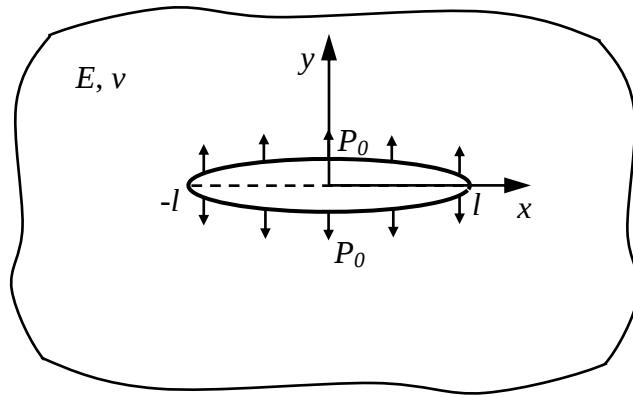


Figure 2. 4: A tensile crack in homogeneous material

The analytical solution of the crack opening is given as:

$$w(x) = -\frac{2(1-\nu)P_0}{G} \sqrt{l^2 - x^2} \quad (2.57)$$

In this case, zoning tech with one cut is enough: cut along the x-axis and discretize along the interface as well as the two crack surfaces and then the opening displacement is obtained by using the equation $w=u_- - u_+$. The interface is discretized with adaptive elements: the 5 elements adjacent to the crack are uniformly distributed with the same size as the crack ones and the remaining ones are distributed to the remote with the same size ratio to the previous one. Since a very big ratio results in

more inaccuracy ^[9], we adopt an acceptable ratio 1.2 in all necessary situations. In addition, a big ratio of the largest element to the smallest one also deteriorates the result ^[9], thus we make the maximum allowable ratio $s_{\max}$ is 10 so that if an element is more than 10 times larger than the smallest one, this element will be just 10 times larger of the smallest one and then the upcoming elements share the same size. We take $l = 0.5$ so that the crack lies between -0.5 to 0.5 and the errors at crack tip with different interface element number are listed in the Table 3.1.

Interface Element Number	Interface length	Running time (s)	Error
80	-1.8930 to 1.8930	0.0143	0.5694
140	-4.7399 to 4.7399	0.0351	0.1461
300	-14.7399 to 14.7399	0.0709	0.1192
460	-24.7399 to 24.7399	0.1625	0.1188
2040	-123.4899 to 123.4899	4.4009	0.1186

Table 3. 1: The crack tip errors of different interface number with implementations of DM

The running time is the average value of 10 computations on the same laptop and such time has the same meaning in all the following discussions. From the table, the

accuracy becomes over 4 times better from 80 to 140; 22.5671% from 140 to 300 but only 0.7576% and 0.1686% from 300 to 460 and 460 to 2040, respectively. Considering the both effect of running time and accuracy, 300 elements case is an acceptable choice in this case and the detailed result is illustrated in the Figure 3.8:

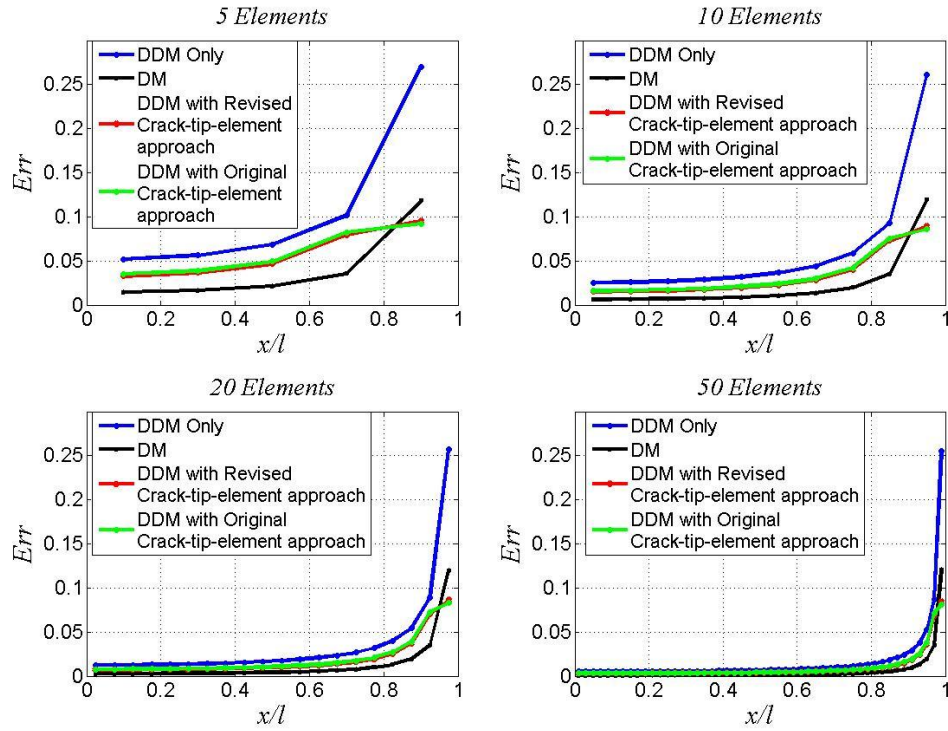


Figure 3. 8: The errors of different approach implementations of DDM and DM with different number of uniform elements on the crack

From the figure, we can see that the DM is less accurate than DDM with crack tip element approach at the tip but more accurate in other elements. That is because the analytical integrals are adopted, e.g. Equation (3.15) and (3.16), for DM implementation. If the problem is extended to non-plane strain case, such analytical integrals are not available while the dislocation formulae are still valid, which

requires numerical integral for the DM implementation, yielding additional inaccuracy. And all the methods are getting more accurate and closer to each other at the elements that do not locate at the first two elements counting from the tip with the element refinement. As for the efficiency, the DM discretizes the interfaces while DDM does not, making the DDM implementation about 6 times (0.0119s:0.0709s) faster than the DM in the 5-uniform-element case.

3.4.2 A tensile crack in a bimaterial

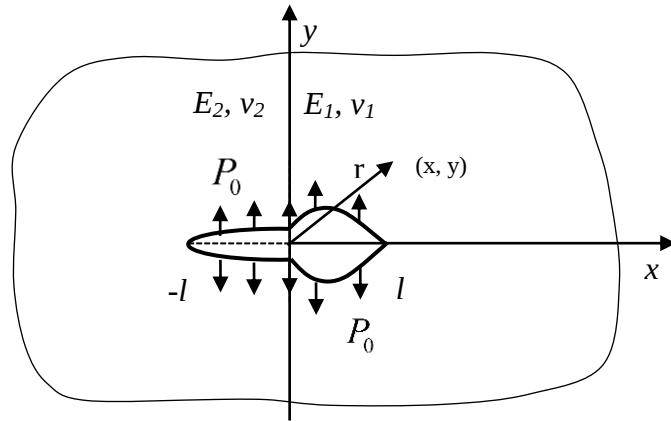


Figure 2. 7: A tensile crack in a bimaterial

A second example will be the tensile crack in a bimaterial as shown in Figure 2.7. The implementation of DDM and DM follow the Section 2.3.2 and 3.3.1. The crack has been divided into 20 uniform elements and the DM implementation follows the foregoing the adaptive discretization in Section 3.4.1. We first take the ratio of the material property is: $E_1 : E_2 = 1:50$; $\nu_1 : \nu_2 = 1:1.2$ and the pressure is one-tenth of the minimum Shear modulus, i.e. $P_0 = 0.1G_{\min}$. Then the opening displacement of this

problem obtained by DM and DDM is illustrated, respectively in the Figure 3.9. The y-axis is normalized by dividing the opening displacement with the half length of crack, represented as w/l in the figure.

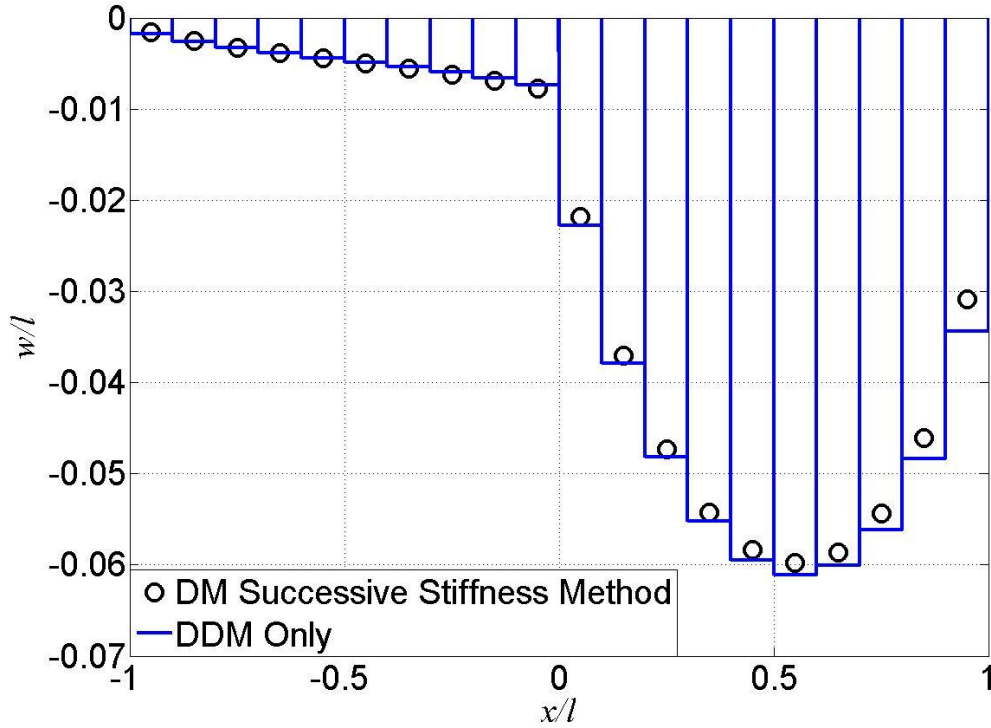


Figure 3. 9:The normalized opening displacement of a tensile crack in a bimaterial with Young's Moudlulus ratio 50: 1 obtained by DDM and DM

A detailed result of the difference between the DDM and DM with different material property contrast is presented in the Figure 3.10. We set the Poisson ratio constant as 1.2:1 and change the Young's modulus ratio since it plays the dominant role in material property contrast. Neither the ratio 25:1 or 50:1 is not quite common in the practical engineering situation but still presented here to test the algorithm as extreme cases. The “difference of **A** and **B**” in this dissertation is defined as the result obtained by method **A** minus the one with method **B** and then divided by the one with **B**.

From the result in Figure 3.8, we can assume that the DM is more accurate than DDM only due to the adoption of the analytical integrals if crack tip element approach is not implemented. And the crack tip element approach cannot be implemented here since the material becomes heterogeneous in this bimaterial case. Thus, the DM is treated as a benchmark for the following analysis, i.e. the method **B**.

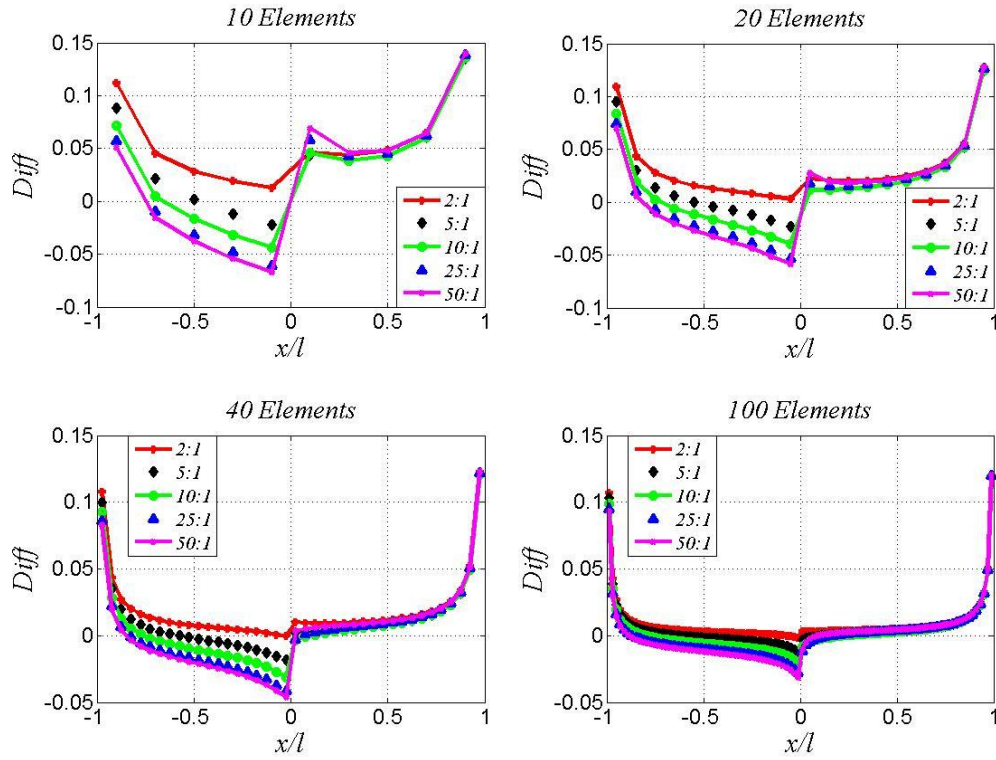


Figure 3. 10: The difference of the DDM and DM with different number of uniform element on the crack

From Figure 3.10, the DDM based on Dundur's formulation is quite close to the DM-based successive method with the difference less than 6% except at the tip elements, where the differences rise to around 10.00% and around 12.66% even regardless of the material property contrast. The phenomenon that higher difference

locates at tips and less difference at other elements conforms to the results in Figure 3.8. With the element refinement, the results obtained by DDM are getting close to the ones obtained by DM while the crack tip difference increases. This increase is getting larger with bigger material contrast. The two methods match each other better in the soft material than in the hard one. As for the efficiency, the DM implementation needs to treat the interfaces while DDM implementation does not, making the DDM implementation more efficient than the DM. For example, it takes about 1.86 seconds to run the DM implementation while it takes only 0.021 second for DDM in the 20-uniform-element case.

From the results and remarks above, the advantage and disadvantage of DDM and DM are quite obvious: the DDM lacks the general applicability but can be implemented efficiently in homogeneous and bimaterial situations while DM is applicable for the general multilayered media but takes more time and more treatments for the appropriate implementation. Then, we consider of combining the two methods so that the new method shares both the efficiency of the displacement discontinuity method and the applicability of the direct method. The new combined method is going to be elaborated in the upcoming chapter.

Chapter 4

The combined method

4.1 Abstract

In this chapter, a new boundary integral analysis that combines the displacement discontinuity method (DDM) and the direct method (DM) is elaborated to determine the opening displacement of a pressurized crack passing through a number of layers in a multilayered elastic media. The intersecting crack can be either straight or curved as both the methods are able to deal the cases as explained in the last two chapters. The displacement discontinuity method is implemented to construct the fracture matrix in each layer, while the direct method is used to characterize the effects of the interfaces. As a consequence, all variables on the interfaces can be eliminated through continuity conditions, leading to the final equation which only consists of variables on crack surfaces. The concept of the crack tip element is also adopted and extended for better treatment of the crack tip singularity. The results will be illustrated and compared with the foregoing two methods at last.

4.2 Preliminaries

The Figure 4.1 shows a detailed schematic configuration of a pressurized crack going through a number of layers. Besides the crack in the media, the difference of this figure from the Figure 3.5 is that the fixed bottom of the multilayered media is replaced by a half space. However, both the two features share the same boundary condition $u_b^1 = 0$ so that the physics still remains the same.

The notation to be used is specified as follows. Quantities on the lower(bottom), upper(top), crack and side boundaries of each separate layer will be denoted by the subscripts b , t , c and s and the layer index denoted by the superscript runs over the set of the strata from bottom ($i=1$) to top ($i=N$). Identity and null matrices and column vectors are indicated by bold-face symbols $\mathbf{I}$ and $\mathbf{0}$, respectively.

The long dash line stands for the original crack status that is a slit-like opening with two surfaces, one effectively coinciding with the other. Subjected to the known pressure, the crack will open as shown in the figure.

At the center of every element of each layer, a local coordinate is defined by following the definition in the Section 2.3.3 from S.L.Crouch and A.M.Starfield's book ^[9]. Consequently, the continuity conditions are:

$$u_t^i = -u_b^{i+1}, p_t^i = p_b^{i+1} \quad (4.1)$$

where u and p denotes the vectors of displacements and tractions, respectively. The bottom layer is half space so that the boundary conditions are assumed here for

simplicity:

$$u_b^1 = 0, p_t^N = 0 \quad (4.2)$$

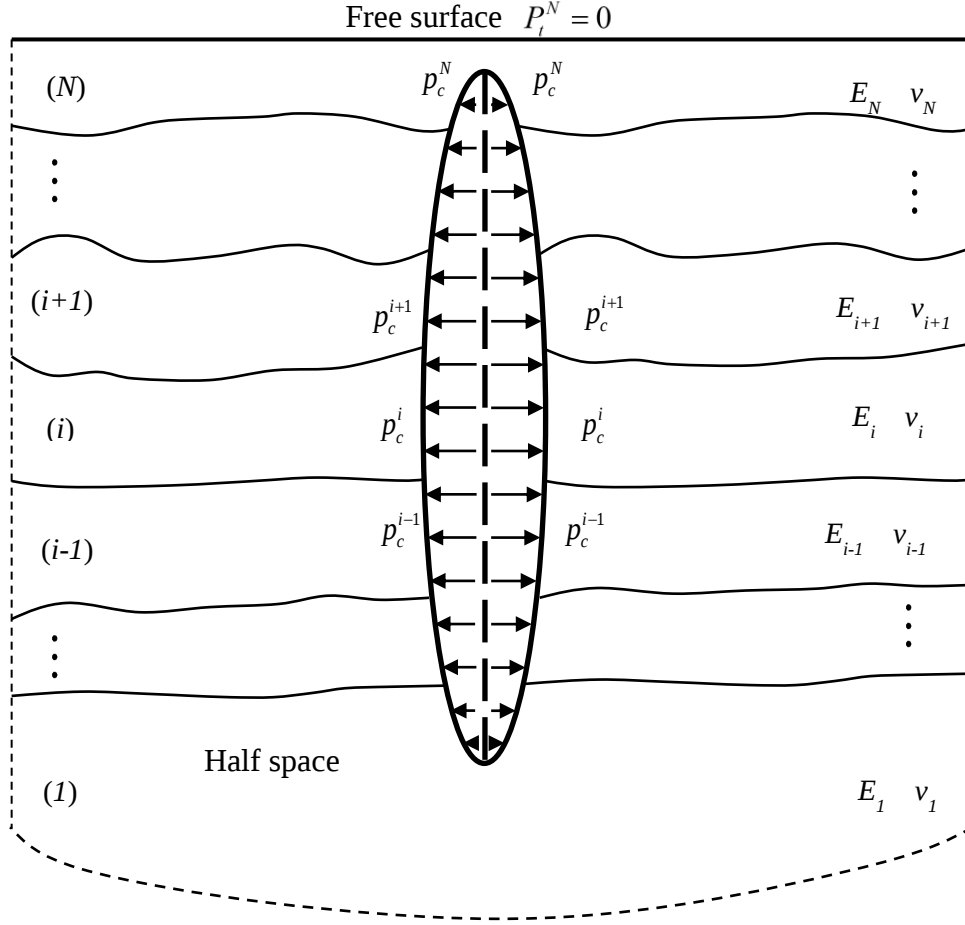


Figure 4. 1 : A detailed schematic configuration for a pressurized crack going through a number of layers

The former of Equation (4.2) often represents an underlying stiff rock formation beneath the bottom of the multilayered media and the latter stands for the flexible foundation.

The two short dash lines at left and right side represent the side boundaries located in

“undisturbed” regions on which vanishing displacements are assumed ^[26]:

$$u_s^i = 0 \quad (1 \leq i \leq N) \quad (4.3)$$

Consequently, the equations associated with side-boundary tractions can be extracted outside the equations associated with other boundaries ^[26], which is our main interest.

Then, we will elaborate the implementation of the combined boundary integral analysis with discrete layer of the multilayered medium.

4.3 The combine method implementation

Although the BIM can deal with curved cracks, we still assume parallel interfaces for the display simplicity here since we can have some benchmarks for comparison. The bottom layer of a multilayered media is shown in Figure 4.2 and then a half plane of the same material property shown in the long dash curve is patched on the interface. Then a traction with the same magnitude but in opposite direction of the one on the top of the bottom layer, denoted as p_t^1 , is applied on the interface. As a consequence, the original interface represented by the vertical long dash line deforms as shown by the solid curved line in the figure, just like a pressurized slit-like crack opens up. The displacement of the two surfaces of the interface is hence denoted as u_t^1, v_t^1 , respectively. In addition, d_1 represents the thickness of the bottom layer.

Based on the local coordinate definition above, the displacement discontinuity on the top of the bottom layer is:

$$D_t^1 = u_t^1 + v_t^1 \quad (4.4)$$

Therefore, the boundary integral equations can be formulated according to the DDM and discretized by boundary elements and by collocation:

$$\begin{bmatrix} K_{bb}^1 & K_{bc}^1 & K_{bt}^1 \\ K_{cb}^1 & K_{cc}^1 & K_{ct}^1 \\ K_{tb}^1 & K_{tc}^1 & K_{tt}^1 \end{bmatrix} \begin{Bmatrix} D_b^1 \\ D_c^1 \\ D_t^1 \end{Bmatrix} = \begin{Bmatrix} p_b^1 \\ p_c^1 \\ p_t^1 \end{Bmatrix} \quad (4.5)$$

From equation (2):

$$D_b^1 = 0 \quad (4.6)$$

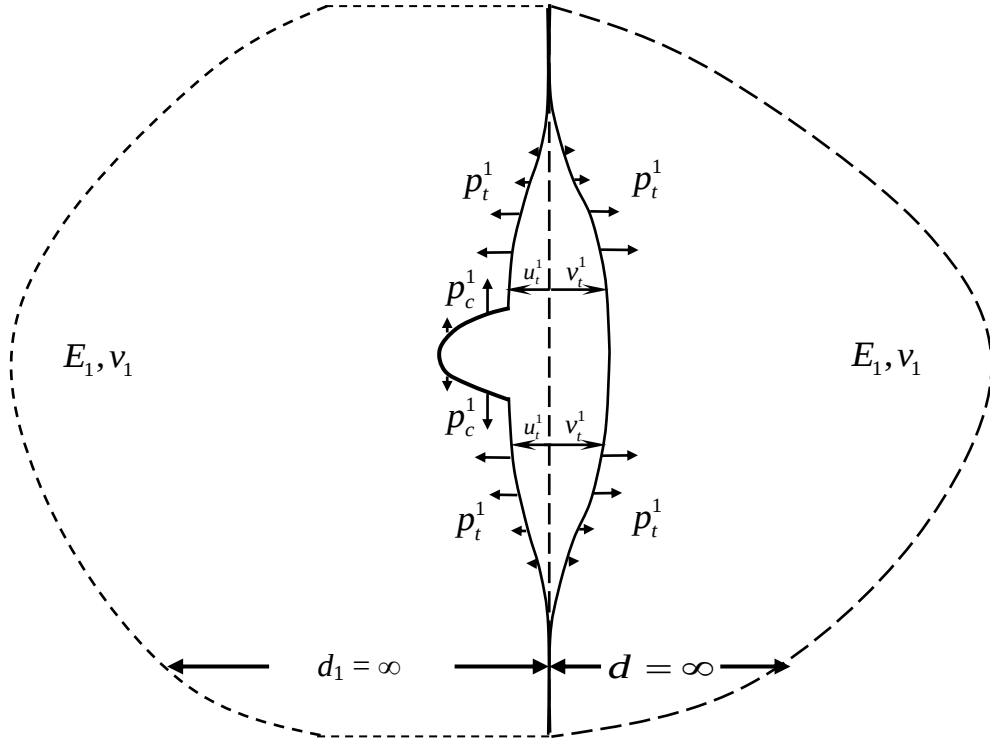


Figure 4. 2: A schematic of the bottom layer patched with a half plane of the same material
property

Substitute the Equation (4.4) and (4.6) into (4.5), (4.5) can be split into two parts:

$$\begin{bmatrix} K_{cc}^1 & K_{ct}^1 \\ K_{tc}^1 & K_{tt}^1 \end{bmatrix} \begin{Bmatrix} D_c^1 \\ u_t^1 + v_t^1 \end{Bmatrix} = \begin{Bmatrix} p_c^1 \\ p_t^1 \end{Bmatrix} \quad (4.7)$$

$$p_b^1 = K_{bc}^1 D_c^1 + K_{bt}^1 (u_t^1 + v_t^1) \quad (4.8)$$

Other boundary integral equations for the elements on the half-space interface can be formulated based on the DM:

$$H_{tt}^1 v_t^1 = p_t^1 \quad (4.9)$$

It can be rewritten as:

$$v_t^1 = (H_{tt}^1)^{-1} p_t^1 \quad (4.10)$$

Substitute (4.10) into (4.7):

$$\begin{bmatrix} K_{cc}^1 & K_{ct}^1 \\ K_{tc}^1 & K_{tt}^1 \end{bmatrix} \begin{Bmatrix} D_c^1 \\ u_t^1 \end{Bmatrix} = \begin{bmatrix} I & T_{ct}^1 \\ 0 & T_{tt}^1 \end{bmatrix} \begin{Bmatrix} p_c^1 \\ p_t^1 \end{Bmatrix} \quad (4.11)$$

Where
$$T_{ct}^1 = -K_{ct}^1 (H_{tt}^1)^{-1} \quad (4.12)$$

$$T_{tt}^1 = I - K_{tt}^1 (H_{tt}^1)^{-1} \quad (4.13)$$

Inverse the right hand side coefficient matrix so that there is only a column vector left at right:

$$\begin{bmatrix} \hat{K}_{cc}^1 & \hat{K}_{ct}^1 \\ \hat{K}_{tc}^1 & \hat{K}_{tt}^1 \end{bmatrix} \begin{Bmatrix} D_c^1 \\ u_t^1 \end{Bmatrix} = \begin{Bmatrix} p_c^1 \\ p_t^1 \end{Bmatrix} \quad (4.14)$$

where
$$\begin{cases} \hat{K}_{tt}^1 = (T_{tt}^1)^{-1} K_{tt}^1 \\ \hat{K}_{tc}^1 = (T_{tt}^1)^{-1} K_{tc}^1 \\ \hat{K}_{ct}^1 = K_{ct}^1 - T_{ct}^1 \hat{K}_{tt}^1 \\ \hat{K}_{cc}^1 = K_{cc}^1 - T_{ct}^1 \hat{K}_{tc}^1 \end{cases} \quad (4.15)$$

Let $\hat{D}_c^1 = D_c^1$, $\hat{p}_c^1 = p_c^1$, the hat here means the displacements or tractions on the crack elements placed along the crack surface from the bottom layer to the current one. The equation (4.14) is of the final format:

$$\begin{bmatrix} \hat{K}_{cc}^1 & \hat{K}_{ct}^1 \\ \hat{K}_{tc}^1 & \hat{K}_{tt}^1 \end{bmatrix} \begin{Bmatrix} \hat{D}_c^1 \\ u_t^1 \end{Bmatrix} = \begin{Bmatrix} \hat{p}_c^1 \\ p_t^1 \end{Bmatrix} \quad (4.16)$$

There are three unknowns $\hat{D}_c^1, u_t^1$ and p_t^1 in the equation (4.16) containing two equations, we need more equations to solve it. As a consequence, the upper layers need checking and the continuity conditions (4.1) are necessary.

A schematic configuration of the second layer with both bottom and top patched with the half plane of the same material property is shown in Figure 4.3. The analysis is

similar to the one of the bottom layer, while the displacement discontinuity of the bottom boundary is taken into account.

Similarly to the analysis of the bottom layer, the equations based on DDM and DM can be established as follows:

$$\begin{bmatrix} K_{bb}^2 & K_{bc}^2 & K_{bt}^2 \\ K_{cb}^2 & K_{cc}^2 & K_{ct}^2 \\ K_{tb}^2 & K_{tc}^2 & K_{tt}^2 \end{bmatrix} \begin{Bmatrix} u_b^2 + v_b^2 \\ D_c^2 \\ u_t^2 + v_t^2 \end{Bmatrix} = \begin{Bmatrix} p_b^2 \\ p_c^2 \\ p_t^2 \end{Bmatrix} \quad (4.17)$$

And

$$v_b^2 = (H_{bb}^2)^{-1} p_b^1 \quad (4.18)$$

$$v_t^2 = (H_{tt}^2)^{-1} p_t^1 \quad (4.19)$$

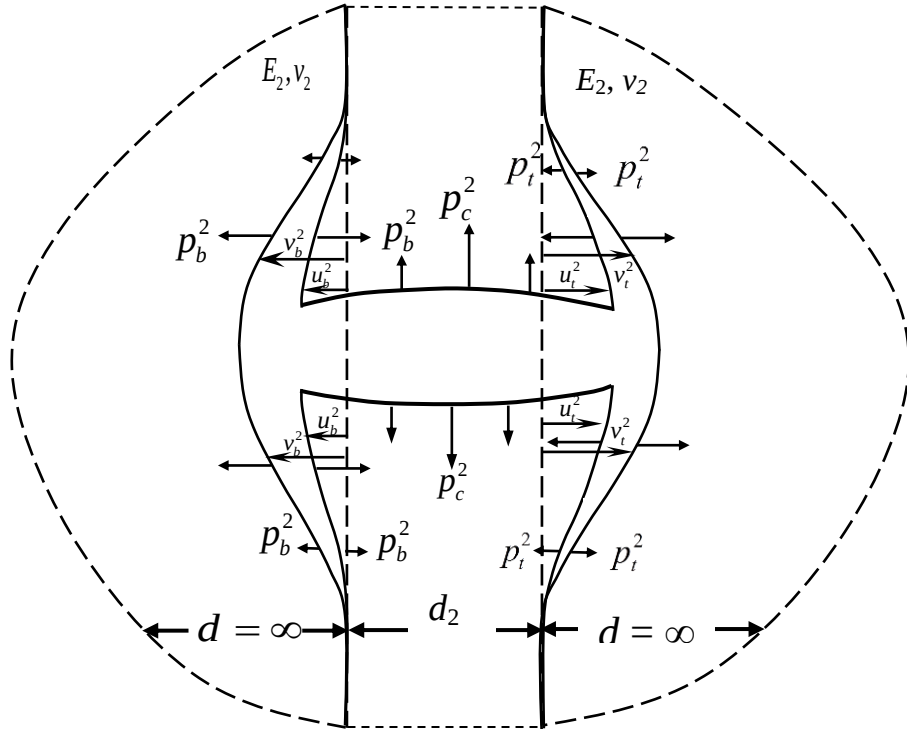


Figure 4. 3: A schematic of the second layer patched with a half plane of the same material property

Substitute equation (4.19) into (4.17):

$$\begin{bmatrix} K_{bb}^2 & K_{bc}^2 & K_{bt}^2 \\ K_{cb}^2 & K_{cc}^2 & K_{ct}^2 \\ K_{tb}^2 & K_{tc}^2 & K_{tt}^2 \end{bmatrix} \begin{Bmatrix} u_b^2 + (H_{bb}^2)^{-1} p_b^1 \\ D_c^2 \\ u_t^2 + (H_{tt}^2)^{-1} p_t^1 \end{Bmatrix} = \begin{Bmatrix} p_b^2 \\ p_c^2 \\ p_t^2 \end{Bmatrix} \quad (4.20)$$

Split the displacements and tractions in the equation (4.20) to different sides:

$$\begin{bmatrix} K_{bb}^2 & K_{bc}^2 & K_{bt}^2 \\ K_{cb}^2 & K_{cc}^2 & K_{ct}^2 \\ K_{tb}^2 & K_{tc}^2 & K_{tt}^2 \end{bmatrix} \begin{Bmatrix} u_b^2 \\ D_c^2 \\ u_t^2 \end{Bmatrix} = \begin{bmatrix} T_{bb}^2 & 0 & T_{bt}^2 \\ T_{cb}^2 & I & T_{ct}^2 \\ T_{tb}^2 & 0 & T_{tt}^2 \end{bmatrix} \begin{Bmatrix} p_b^2 \\ p_c^2 \\ p_t^2 \end{Bmatrix} \quad (4.21)$$

where

$$\begin{cases} T_{bb}^2 = I - K_{bb}^2 (H_{bb}^2)^{-1} \\ T_{bt}^2 = -K_{bt}^2 (H_{tt}^2)^{-1} \\ T_{cb}^2 = -K_{cb}^2 (H_{bb}^2)^{-1} \\ T_{ct}^2 = -K_{ct}^2 (H_{tt}^2)^{-1} \\ T_{tb}^2 = -K_{tb}^2 (H_{bb}^2)^{-1}, \\ T_{tt}^2 = I - K_{tt}^2 (H_{tt}^2)^{-1} \end{cases} \quad (4.22)$$

From Equation (4.21) and (4.22):

$$p_b^2 = (T_{bb}^2)^{-1} [K_{bb}^2 u_b^2 + K_{bc}^2 D_c^2 + K_{bt}^2 u_t^2 - T_{bt}^2 p_t^2] \quad (4.23)$$

And from Equation (4.16):

$$p_t^1 = \hat{K}_{tc}^1 D_c^1 + \hat{K}_{tt}^1 u_t^1 \quad (4.24)$$

Apply the continuity conditions: $u_b^2 = -u_t^1, p_b^2 = p_t^1$

$$\begin{aligned} p_b^2 &= (T_{bb}^2)^{-1} [K_{bb}^2 u_b^2 + K_{bc}^2 D_c^2 + K_{bt}^2 u_t^2 - T_{bt}^2 p_t^2] \\ &= \hat{K}_{tc}^1 \hat{D}_c^1 + \hat{K}_{tt}^1 u_t^1 \\ &= \hat{K}_{tc}^1 \hat{D}_c^1 - \hat{K}_{tt}^1 u_b^2 = p_t^1 \end{aligned} \quad (4.25)$$

Make some arrangement on equation (4.25), we can get:

$$u_b^2 = (D_{bb}^2)^{-1} \hat{K}_{tc}^1 \hat{D}_c^1 - (D_{bb}^2)^{-1} (T_{bb}^2)^{-1} [K_{bc}^2 D_c^2 + K_{bt}^2 u_t^2 - T_{bt}^2 p_t^2] \quad (4.26)$$

where

$$D_{bb}^2 = (T_{bb}^2)^{-1} K_{bb}^2 + \hat{K}_{tt} \quad (4.27)$$

Substitute equation (4.26) into (4.24):

$$p_t^1 = [I - \hat{K}_{tt}^1 (D_{bb}^2)^{-1}] \hat{K}_{tc}^1 \hat{D}_c^1 + \hat{K}_{tt}^1 (D_{bb}^2)^{-1} (T_{bb}^2)^{-1} K_{bc}^2 D_c^2 + \hat{K}_{tt}^1 (D_{bb}^2)^{-1} (T_{bb}^2)^{-1} K_{bt}^2 u_t^2 - \hat{K}_{tt}^1 (D_{bb}^2)^{-1} (T_{bb}^2)^{-1} T_{bt}^2 p_t^2 \quad (4.28)$$

From the first row of equation (4.16) with the continuity condition:

$$\hat{p}_c^1 = \hat{K}_{cc}^1 \hat{D}_c^1 + \hat{K}_{ct}^1 u_t^1 = \hat{K}_{cc}^1 \hat{D}_c^1 - \hat{K}_{ct}^1 u_b^2 \quad (4.29)$$

Substitute equation (4.26) into (4.29):

$$[\hat{K}_{cc}^1 - \hat{K}_{ct}^1 (D_{bb}^2)^{-1} \hat{K}_{tc}^1] \hat{D}_c^1 + J_{cb}^{12} K_{bc}^2 D_c^2 + J_{cb}^{12} K_{bt}^2 u_t^2 = \hat{p}_c^1 + 0 + J_{cb}^{12} T_{bt}^2 p_t^2 \quad (4.30)$$

where,

$$J_{cb}^{12} = \hat{K}_{ct}^1 (D_{bb}^2)^{-1} (T_{bb}^2)^{-1} \quad (4.31)$$

The two-digit superscript on the J coefficient matrix represents that the matrix J links the parameters on the crack of the layer with index of the former digit (1 here) and the one on the bottom boundary of the layer with index of the latter digit (2 here).

The upcoming matrices with two-digit superscript follow the definition here.

Then, the Equation (4.26) and (4.28) were substituted into the second row of (4.21):

$$K_{cb}^2 N_{bb}^2 \hat{K}_{tc}^1 \hat{D}_c^1 + (K_{cc}^2 - M_{cb}^2 K_{bc}^2) D_c^2 + (K_{ct}^2 - M_{cb}^2 K_{bt}^2) u_t^2 = 0 + p_c^2 + [M_{cb}^2 K_{bt}^2 - K_{ct}^2] (H_{tt}^2)^{-1} p_t^2 \quad (4.32)$$

where,

$$\begin{cases} B_{bb}^2 = I - (H_{bb}^2)^{-1} \hat{K}_{tt}^1 \\ N_{bb}^2 = (H_{bb}^2)^{-1} + B_{bb}^2 (D_{bb}^2)^{-1} \\ M_{cb}^2 = K_{cb}^2 B_{bb}^2 (D_{bb}^2)^{-1} (T_{bb}^2)^{-1} \end{cases} \quad (4.33)$$

Similarly, substitute the Equation (4.26) and (4.28) into the third row of (4.21):

$$\begin{aligned} & K_{tb}^2 N_{bb}^2 \hat{K}_{tc}^1 \hat{D}_c^1 + (K_{tc}^2 - M_{tb}^2 K_{bc}^2) D_c^2 + (K_{tt}^2 - M_{tb}^2 K_{bt}^2) u_t^2 \\ & = 0 + 0 + [(M_{tb}^2 K_{bt}^2 - K_{tt}^2)(H_{tt}^2)^{-1} + I] p_t^2 \end{aligned} \quad (4.34)$$

where,

$$M_{tb}^2 = K_{tb}^2 B_{bb}^2 (D_{bb}^2)^{-1} (T_{bb}^2)^{-1} \quad (4.35)$$

According to the Equation (4.30), (4.32) and (4.34), a matrix equation can be set up:

$$\begin{bmatrix} \hat{K}_{cc}^{11} & \hat{K}_{cc}^{12} & \hat{K}_{ct}^{12} \\ \hat{K}_{cc}^{21} & \hat{K}_{cc}^{22} & \hat{K}_{ct}^{22} \\ \hat{K}_{tc}^{21} & \hat{K}_{tc}^{22} & \hat{K}_{tt}^{22} \end{bmatrix} \begin{Bmatrix} \hat{D}_c^1 \\ D_c^2 \\ u_t^2 \end{Bmatrix} = \begin{bmatrix} I & 0 & \hat{T}_{ct}^{12} \\ 0 & I & \hat{T}_{ct}^{22} \\ 0 & 0 & \hat{T}_{tt}^{22} \end{bmatrix} \begin{Bmatrix} \hat{p}_c^1 \\ p_c^2 \\ p_t^2 \end{Bmatrix} \quad (4.36)$$

where,

$$\begin{cases} \hat{T}_{ct}^{12} = J_{cb}^{12} T_{bt}^2 \\ \hat{T}_{ct}^{22} = (M_{cb}^2 K_{bt}^2 - K_{ct}^2)(H_{tt}^2)^{-1} \\ \hat{T}_{tt}^{22} = (M_{tb}^2 K_{bt}^2 - K_{tt}^2)(H_{tt}^2)^{-1} \end{cases} \quad (4.37)$$

and

$$\begin{cases} \hat{K}_{cc}^{11} = \hat{K}_{cc}^1 - \hat{K}_{ct}^1 (D_{bb}^2)^{-1} \hat{K}_{tc}^1 \\ \hat{K}_{cc}^{12} = J_{cb}^{12} K_{bc}^2 D_c^2 \\ \hat{K}_{ct}^{12} = J_{cb}^{12} K_{bt}^2 \\ \hat{K}_{cc}^{21} = K_{cb}^2 N_{bb}^2 \hat{K}_{tc}^1 \\ \hat{K}_{cc}^{22} = K_{cc}^2 - M_{cb}^2 K_{bc}^2 \\ \hat{K}_{ct}^{22} = K_{ct}^2 - M_{cb}^2 K_{bt}^2 \\ \hat{K}_{tc}^{21} = K_{tb}^2 N_{bb}^2 \hat{K}_{tc}^1 \\ \hat{K}_{tc}^{22} = K_{tc}^2 - M_{tb}^2 K_{bc}^2 \\ \hat{K}_{tt}^{22} = K_{tt}^2 - M_{tb}^2 K_{bt}^2 \end{cases} \quad (4.38)$$

Then, we make the following combination so that the equation (36) can be expressed

in the format similar to the (11):

$$\begin{aligned}
\tilde{K}_{cc}^2 &= \begin{bmatrix} \hat{K}_{cc}^{11} & \hat{K}_{cc}^{12} \\ \hat{K}_{cc}^{21} & \hat{K}_{cc}^{22} \end{bmatrix}, \tilde{K}_{ct}^2 = \begin{bmatrix} \hat{K}_{ct}^{12} \\ \hat{K}_{ct}^{22} \end{bmatrix}, \tilde{K}_{tc}^2 = \begin{bmatrix} \hat{K}_{tc}^{21} & \hat{K}_{tc}^{22} \end{bmatrix}, \tilde{K}_t^2 = \hat{K}_t^{22}, \hat{D}_c^2 = \begin{bmatrix} \hat{D}_c^1 \\ \hat{D}_c^2 \end{bmatrix} \\
\hat{T}_{ct}^2 &= \begin{bmatrix} \hat{T}_{ct}^{12} \\ \hat{T}_{ct}^{22} \end{bmatrix}, \hat{T}_t^2 = \hat{T}_t^{22}, \hat{p}_c^2 = \begin{bmatrix} \hat{p}_c^1 \\ \hat{p}_c^2 \end{bmatrix} \\
\begin{bmatrix} \tilde{K}_{cc}^2 & \tilde{K}_{ct}^2 \\ \tilde{K}_{tc}^2 & \tilde{K}_t^2 \end{bmatrix} \begin{bmatrix} \hat{D}_c^2 \\ u_t^2 \end{bmatrix} &= \begin{bmatrix} I & \hat{T}_{ct}^2 \\ 0 & \hat{T}_t^2 \end{bmatrix} \begin{bmatrix} \hat{p}_c^2 \\ p_t^2 \end{bmatrix} \tag{4.37}
\end{aligned}$$

Inverse the right hand side coefficient matrix so that the Equation (4.37) is in the format similar to the (4.16):

$$\begin{bmatrix} \hat{K}_{cc}^2 & \hat{K}_{ct}^2 \\ \hat{K}_{tc}^2 & \hat{K}_t^2 \end{bmatrix} \begin{bmatrix} \hat{D}_c^2 \\ u_t^2 \end{bmatrix} = \begin{bmatrix} \hat{p}_c^2 \\ p_t^2 \end{bmatrix} \tag{4.38}$$

where,

$$\begin{cases} \hat{K}_t^2 = (\hat{T}_t^2)^{-1} \tilde{K}_t^2 \\ \hat{K}_{tc}^2 = (\hat{T}_t^2)^{-1} \tilde{K}_{tc}^2 \\ \hat{K}_{ct}^2 = \tilde{K}_{ct}^2 - \hat{T}_{ct}^2 \hat{K}_t^2 \\ \hat{K}_{cc}^2 = \tilde{K}_{cc}^2 - \hat{T}_{ct}^2 \hat{K}_{tc}^2 \end{cases} \tag{4.39}$$

The foregoing procedure is valid and recursive for an arbitrary layer in such a multilayered media by replacing the superscript 1 and 2 with $i-1$ and i respectively, except the simplified one for bottom layer with all coefficients with “hat” $\hat{K}$ and $\hat{T}$ equal to the ones without “hat” K and T respectively due to the fixed bottom.

Therefore, the final equation can be established as follows:

$$\begin{bmatrix} \hat{K}_{cc}^N & \hat{K}_{ct}^N \\ \hat{K}_{tc}^N & \hat{K}_t^N \end{bmatrix} \begin{bmatrix} \hat{D}_c^N \\ u_t^N \end{bmatrix} = \begin{bmatrix} \hat{p}_c^N \\ p_t^N \end{bmatrix} \tag{4.40}$$

In the final Equation (4.41), all the parameters on the interfaces are eliminated by implementing the procedure above, all the parameters involved are on the crack

surfaces and the top.

The definition of all the intermediate matrices and vectors is the same as above, except the superscript 1 and 2 are replaced by $N-1$ and N respectively. The symbol $\hat{D}_c^N$ represents all the displacement discontinuities on the element on the crack from the bottom to the top layer so that $\hat{D}_c^N$ is the solution to the crack problem here.

From the second row of (4.40):

$$u_t^N = (\hat{K}_t^N)^{-1} (p_t^N - \hat{K}_{tc}^N \hat{D}_c^N) \quad (4.41)$$

Substitute the equation (41) into the first row of (40):

$$\hat{D}_c^N = (\hat{A}_{cc}^N)^{-1} [\hat{p}_c^N - \hat{K}_{ct}^N (\hat{K}_t^N)^{-1} p_t^N] \quad (4.42)$$

where,

$$\hat{A}_{cc}^N = \hat{K}_{cc}^N - \hat{K}_{ct}^N (\hat{K}_t^N)^{-1} \hat{K}_{tc}^N \quad (4.43)$$

A conclusion can be drawn by inspecting the equation (4.42) that the method presented here is valid as long as the traction on the top is known. Since a flexible top surface is assumed for our problem, i.e. $p_t^N = 0$, then the crack opening is:

$$\hat{D}_c^N = (\hat{A}_{cc}^N)^{-1} \hat{p}_c^N \quad (4.44)$$

Notice that the revised crack tip element approach is applied for the crack tips at the bottom and top layer to construct the boundary influence coefficients, i.e. the K s in the implementation.

4.4 Results and remark

The combined method can be implemented for the homogeneous medium simply by making all the layers of the multilayered media share the same material property. However, it will be less efficient than the pure DDM implementation because it needs to characterize the interfaces with DM though not solving them, while the DDM with revised crack tip element approach does not take care of interfaces at all. Thus, the DDM with revised crack tip element approach is the best method for the homogeneous medium among all that mentioned as discussed before so that we shall begin with the bimaterial example.

4.4.1 A tensile crack in a bimaterial

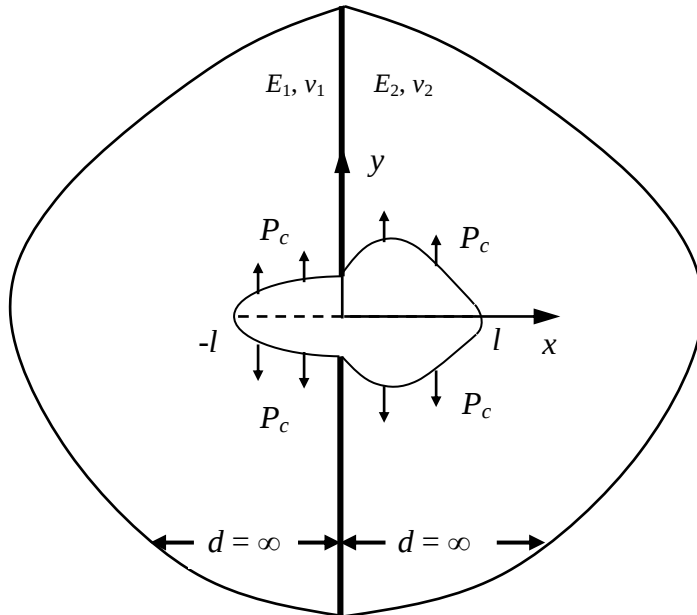


Figure 4. 4: A crack intersects the bonded half-plane

The configuration of a crack intersecting through an infinite elastic bimaterial is

depicted in the Figure 4.4. The figure here is slightly different from Figure 2.7 for emphasizing the infinite thickness of each layer, so that the bimaterial is depicted in the form of the combination of two bonded half-spaces by setting $d = 30l$ in the computation. Thus, we can assume fixed bottom at the bottom and traction-free at the top so that the combined method is applicable for this problem.

We set $v_1:v_2=1:1.2$ and change the ratio $E_1:E_2$ inside the practical application range, then display the difference of the crack opening obtained by DDM, DM-based successive stiffness method and the combined method. The crack is discretized into different number n uniform elements and the interface is discretized with the foregoing mesh in Section 3.4.2. The differences between the combined method and the DM-based successive stiffness method are presented in Figure 4.5.

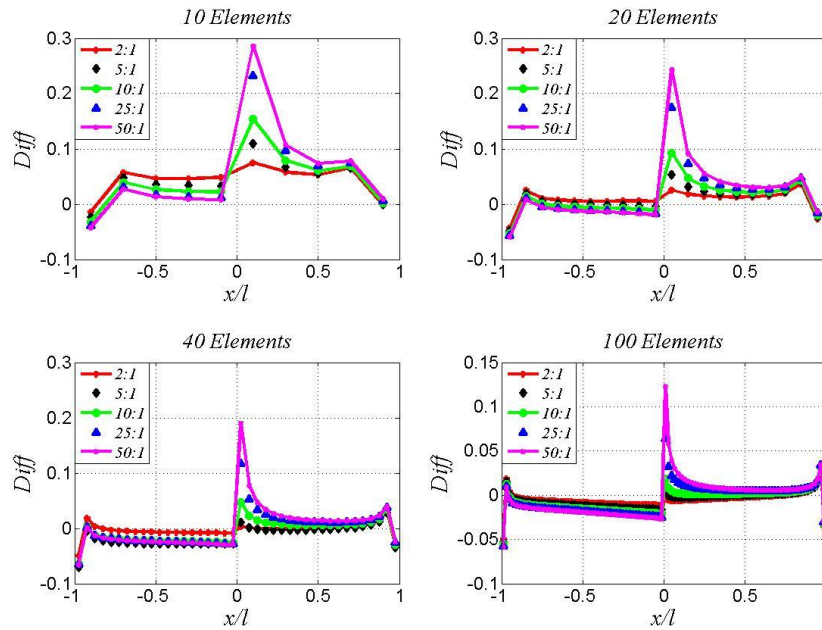


Figure 4. 5: The differences of the combined method and DM-based successive stiffness method with different numbers of uniform element on the crack

The largest difference occurs at the interface of the soft material because the combination of DDM and DM locates right there and the soft material makes it easier to open up. The refined mesh decreases it from around 29% to about 12% with refining the mesh 10 times. The adoption of the revised crack-tip element approach makes the tip difference less than 5%. The differences at other elements are all less than 5% for all the material contrasts here. The two methods provide closer results with refining the mesh.

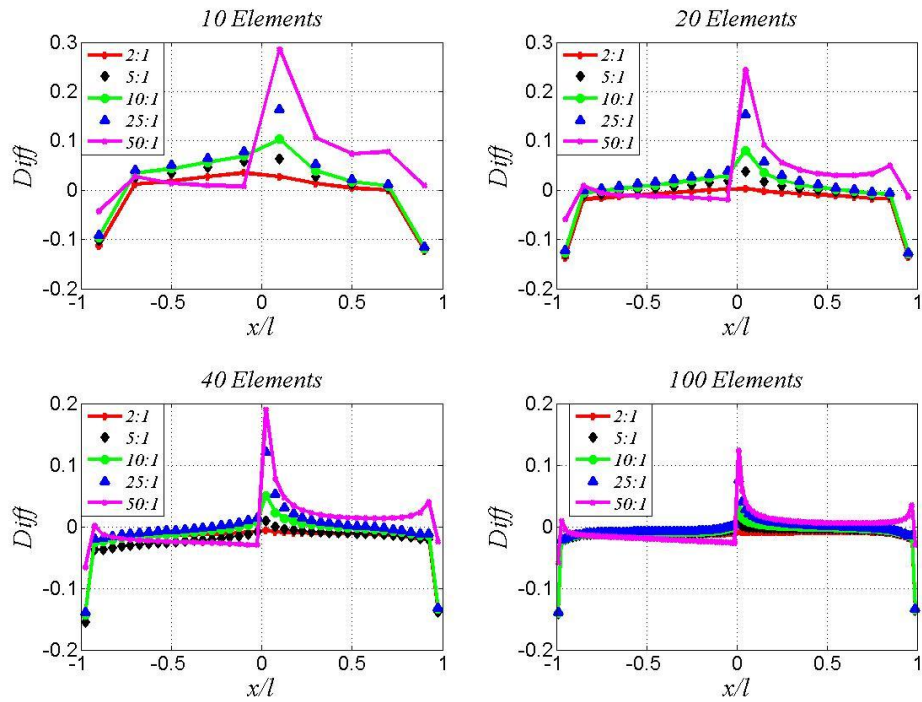


Figure 4. 6: The differences of the combined method and DDM with different numbers of uniform element on the crack

The difference between the combined method and the DDM based on Dundur's formulae are presented in the Figure 4.6. The refined mesh can decrease the largest difference at the interface elements in the soft material from about 30% to about 12%

with refining the mesh 10 times. The differences of the elements not locating at the tips or adjacent to the interface at soft side are all less than 5%. However, the tip discrepancy is over twice as the one between the combined method and DM because the original crack tip element approach cannot be applied for the DDM here.

4.4.2 A tensile crack in a three-layered media

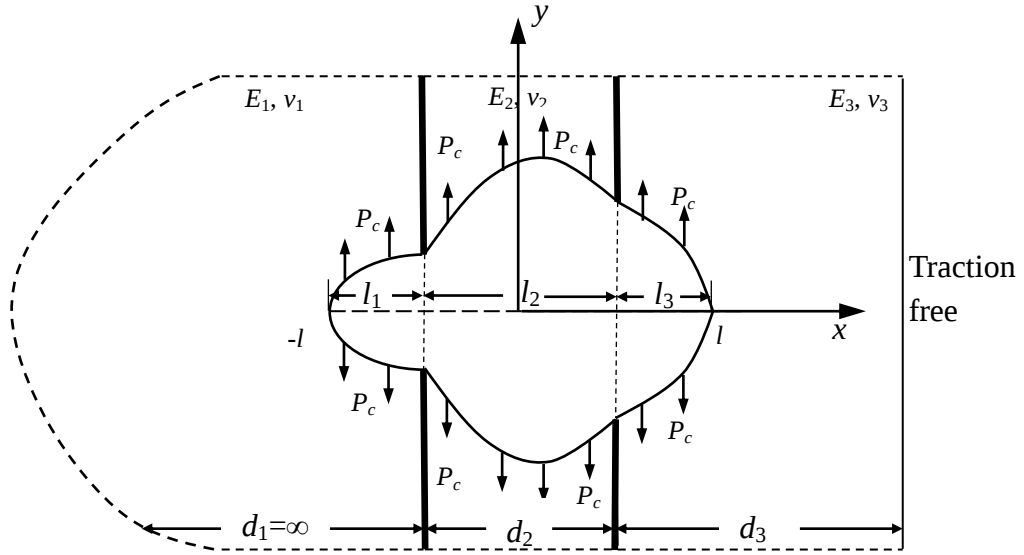


Figure 4. 7: A crack intersects a three-layered elastic media

A general multilayered system, a three-layered elastic media within a crack going through, is shown in the Figure 4.7. We approximate the half plane bottom condition by setting $d_1=30l_1$. The combined method is used to analyze two cases: the first one is the non-uniform crack segment with $l_1 = l_3 = 10l_2$, and the second one is uniform segment case: $l_1 = l_3 = l_2$. For each case, we set two sets of material contrast: $E_1:E_2:E_3 = 1:10:1$, $v_1:v_2:v_3 = 1:1.2:1$ and $E_1:E_2:E_3 = 10:1:10$, $v_1:v_2:v_3 = 1.2:1:1.2$. The uniform crack mesh is adopted for both situations.

We first consider the non-uniform segment situation. Figure 4.8 keeps the maximum size ratio $s_{\max} = 10$ for the crack element number n comparison; The Figure 4.9 and 4.10 choose different n for $s_{\max}$ comparison. The crack element number n plays a significant role of improving results as shown in Figure 4.8 by making the difference range throughout the whole crack less than 10% with refining the mesh 10 times.

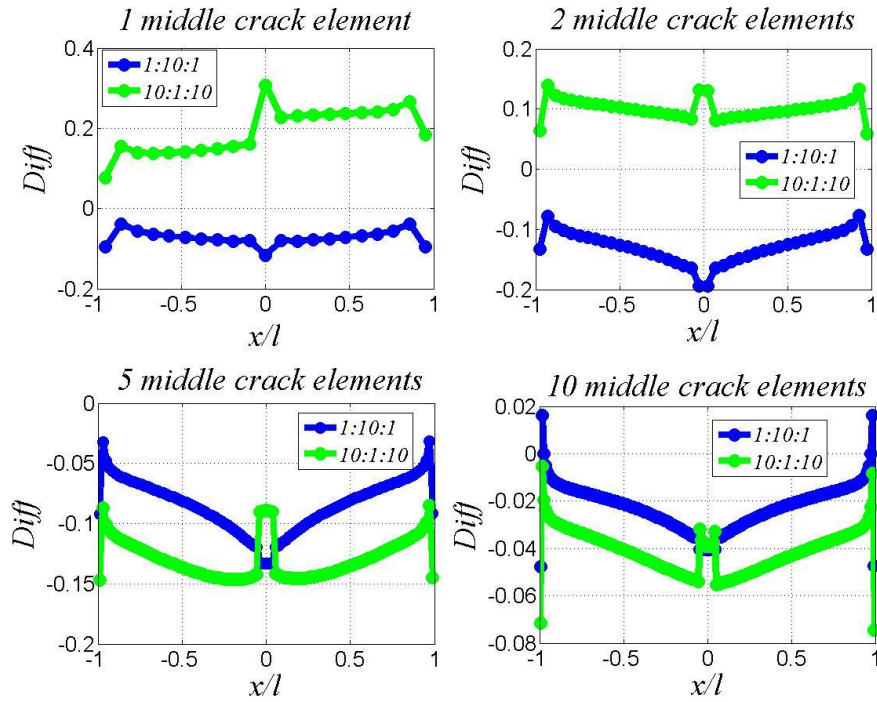


Figure 4. 8: The differences of the combined method and DM with different number of crack elements

The extreme coarse mesh with only one element crack in the middle layer with $s_{\max} = 10$ yields quite an erroneous result. The transform matrix deterioration introduced in the Section 3.3.2 reminds us that it may be because that the value of the largest element size over the thickness of a given layer $e_{\max}/d$, is large (the value is 10 here). From the transition shown in the figure, the larger the $e_{\max}/d$ produces more inaccurate

results. With the value less than 2, as shown in the two pictures in the second row, the combined method can provide a good approximate to the DM. The best approximation is made when $e_{\max}/d$ takes the minimum value 1 here. This phenomenon does not appear in the bimaterial example because each semi-infinite half plane in that case is thick enough compare to the largest interface element size so that $e_{\max}/d$ is less than 1 in all the computation.

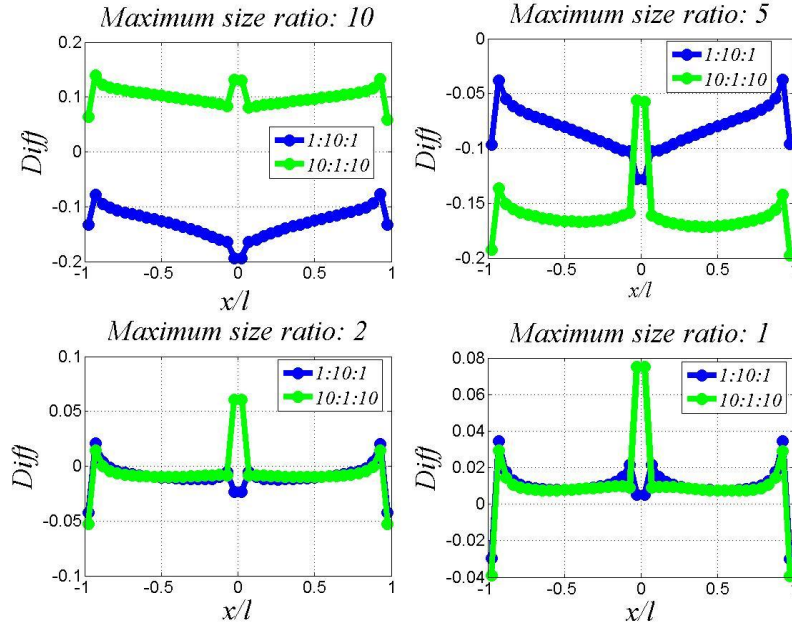


Figure 4. 9: The differences of the combined method and DM with different maximum interface element size ratio and 2 middle crack elements

Besides the crack element number n , controlling the interface element maximum size ratio $s_{\max}$ also improves the result significantly as shown in Figure 4.9 because it changes the value of $e_{\max}/d$, too. When $s_{\max} = 10$ in this situation, the result becomes inaccurate with $e_{\max}/d = 5$, while the uniform mesh is adopted throughout the whole

formation the difference decreases under 8% in the whole region with $e_{\max}/d = 0.5$, which is comparable to the most accurate one in Figure 4.8.

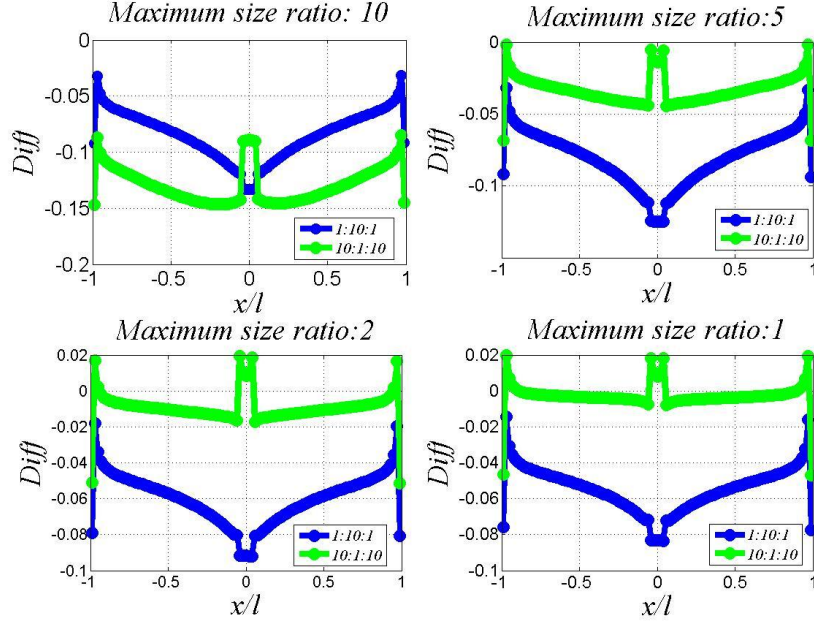


Figure 4. 10: The differences of the combined method and DM with different maximum interface element size ratio and 5 middle crack elements

Thus, the accuracy can be improved by refining crack elements or control the maximum interface element size ratio if the crack element is coarse. The latter approach carries out the improvement more efficiently than the former by saving about 20% of running time. However, if the crack element is refined enough to get relative small $e_{\max}/d$, the maximum interface element size ratio control improves the results in a moderate way as shown in Figure 4.10.

Then the uniform segment case is considered and the results are illustrated in Figure 4.11 and 4.12.

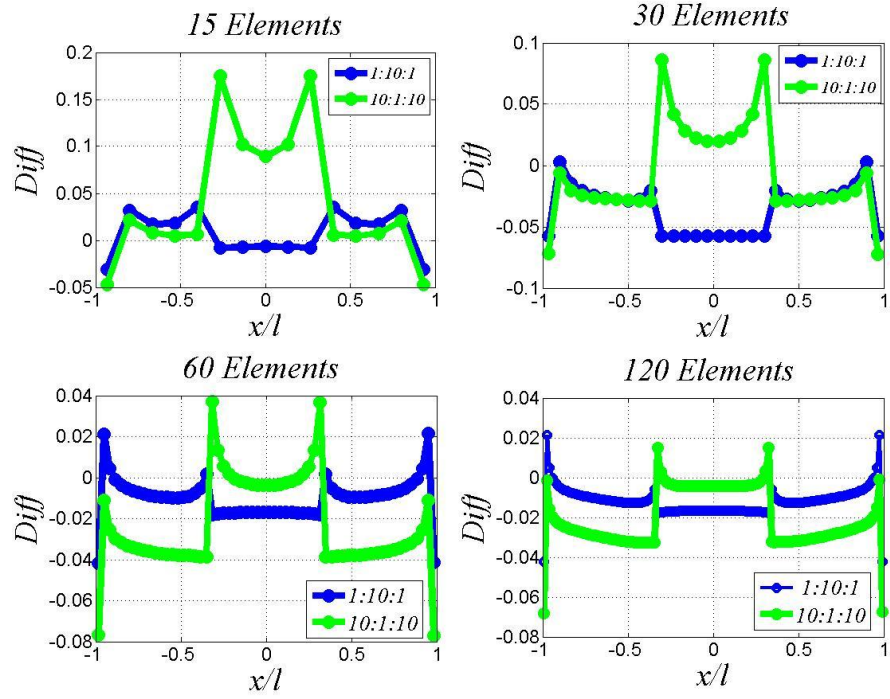


Figure 4. 11: The differences of the combined method and DM with different number of crack elements in the uniform crack segments

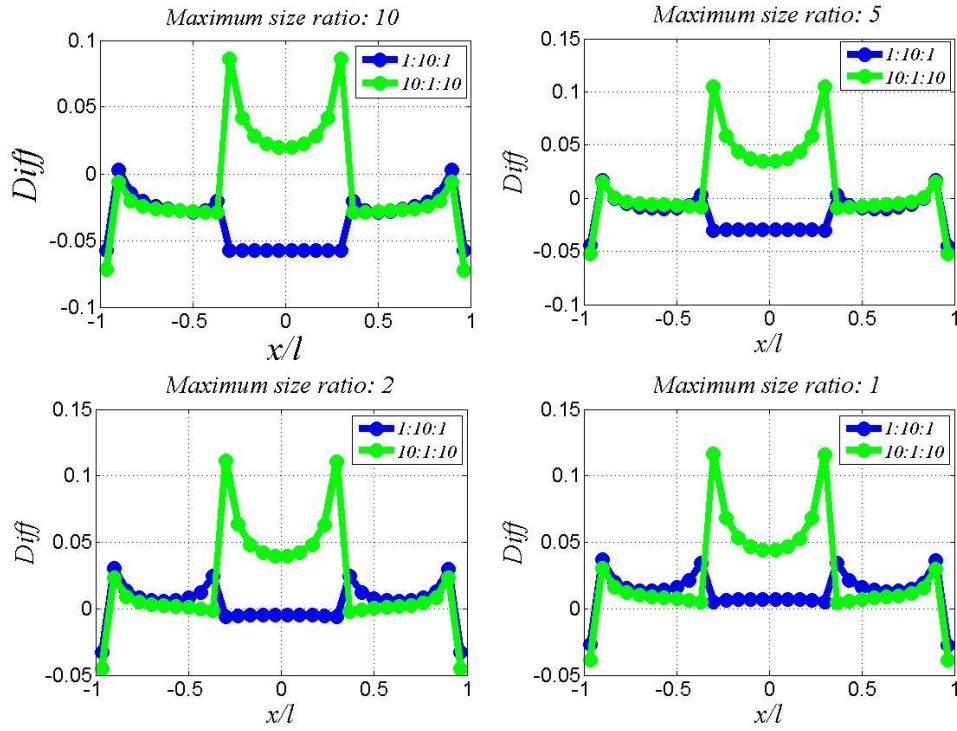


Figure 4. 12: The differences of the combined method and DM with different maximum interface element size ratio in in the uniform crack segments

The Figure 4.11 keeps the maximum interface element size ratio $s_{\max} = 10$ for crack element number n comparison and Figure 4.12 keeps $n = 30$ for $s_{\max}$ comparison. The Figure 4.11 shows that the largest difference is at the interface element of the soft material of the hard-soft-hard material combination, which can be decreased significantly from around 18% to barely less than 2% with the refining the mesh 8 times. As discussed in the previous case, the control of the maximum size ratio $s_{\max}$ does not provide the great improvement as the refining mesh does from the Figure 4.12 because the mesh is refined to get a small enough value of $e_{\max}/d$. By inspecting all the figures in the three-layered example, we can remark that the hard-soft-hard combination is easier affected than the soft-hard-soft one by different mesh.

Chapter 5

Conclusions and ongoing works

The foregoing analysis and examples yield the following conclusions:

- 1) The displacement discontinuity method can be implemented to solve crack problems in a homogeneous medium or a bimaterial with good accuracy and efficiency associated with the crack tip element approach. However, it is unable to solve the multilayered case due to the lack of the analytical formulae to establish the analysis.
- 2) The direct method can be implemented directly to solve crack problems in a multilayered media with a moderate better accuracy than the displacement discontinuity method due to the analytical integral adoption. However, this advantage is not available for non-plane strain case so that the numerical integral adoption in such cases results in additional inaccuracy, while the dislocation theory formulae still holds. In addition, its running time is about 4-5 times in homogeneous medium case and about 9-10 times in bimaterial case as the one of DDM because it discretizes both crack surfaces, due to the indistinguishability effect, as well as the interfaces, while the DDM only needs to treat the crack itself. The discretization on the interfaces does not affect the accuracy of results insignificantly until the interface size is 20 times more than the targeted

boundary (crack in this dissertation) size. The direct method-based successive stiffness method is implemented for the problem in an inefficient manner since the method needs to do some preliminary treatment to construct the formulae and then solve all the unknowns on the intermediate interfaces while only the opening displacement of the crack interests us.

- 3) The proposed combined method solves the crack problem in multilayered elastic media with comparable accuracy and better efficiency than the direct method if an appropriate mesh is adopted to make sure the value of $e_{\max}/d$ is less enough for the desired accuracy. In our analysis, each layer of the bimaterial is thick enough to simulate the semi-infinite half space so that $e_{\max}/d$ remains less than 1 in the computations. As the consequence, the results will only be affected significantly by the mesh of the crack element.

For the general multilayered media, any factor that yields larger $e_{\max}/d$ deteriorates the results. The results show that $e_{\max}/d \leq 1$ provides the difference of the combined method and DM with the order of 10^{-2} . If the crack element is coarse, the maximum interface element size control can decrease the difference as significantly as the crack element refinement but more efficiently. If the crack element is refined enough, the maximum interface element size control decreases the difference in a moderate way. The hard-soft-hard material combination will be affected easier by the mesh than the soft-hard-soft one.

The ongoing work may be extended in the following aspects:

- 1) The work presented in this dissertation is purely solid fracture mechanics. However, the real hydraulic fracturing includes the driven fluid inside, which calls for models about the fluid dynamics coupled with solid fracture mechanics, for example the Perkins-Kern-Nordgren model and Kiristonovich-Geertsma-Daneshy model ^{[32][33][34]}.
- 2) The targeted rock in real hydraulic fracturing is porous but with very small permeability resulting a volume leak-off to rock formation. The one-dimensional leak-off model is proposed by Carter in 1960s which can be improved by coupling the fluid flow inside crack with the porous media flow of non-Newtonian fluid theory ^{[2][7][35]}.
- 3) The temperature increases 25°C per kilometer underground, generally. The hydraulic fracture will face the complexity caused by existing vertical temperature gradient as a consequence. If temperature sensitive fluid is used, its properties will be changed and the difference in temperature between the fluid and rock formation in depth will create a thermal stress in rock formations. Thus, the temperature effect on fluid properties and the thermal stress should be taken in to account ^[2].

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Appendix A: Stress field of an edge dislocation in an infinite plane^{[28] [31]}

We first consider the Airy stress function of the form in the plain-strain state:

$$\phi = Cr \ln r \cos \theta \quad (\text{A.1})$$

C is an arbitrary constant and let E , ν , G denote Young's modulus, Poisson ratio and Shear modulus, respectively.

Then:

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = C \frac{\cos \theta}{r} \quad (\text{A.2})$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = C \frac{\cos \theta}{r} \quad (\text{A.3})$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \phi}{\partial \theta} \right) = C \frac{\sin \theta}{r} \quad (\text{A.4})$$

The strains can then be deduced from stresses by Hooke's Law. The displacements are then obtained from strains by integration. The final result is:

$$u_r = \frac{C}{G} \left[(1-\nu)\theta \sin \theta - \frac{1}{4} \cos \theta + \frac{1}{2} (1-2\nu) \ln r \cos \theta \right] \quad (\text{A.5})$$

$$u_\theta = \frac{C}{G} \left[(1-\nu)\theta \cos \theta - \frac{1}{4} \sin \theta + \frac{1}{2} (1-2\nu) \ln r \sin \theta \right] \quad (\text{A.6})$$

Clearly, the displacement field in this case is nonsingle valued, because

$$u_r(\theta = 0) - u_r(\theta = 2\pi) = 0 \quad (\text{A.7})$$

and

$$u_\theta(\theta = 0) - u_\theta(\theta = 2\pi) = -\frac{2\pi C(1-\nu)}{G} \equiv b_y \quad (\text{A.8})$$

The physical scenario associated with this is shown in Figure A.1, which shows that the dislocation is created, in a heuristic manner, by first making a *cut*, and then creating a *gap* of width b_y , or removing a slab of material of thickness b_y , and then rejoining the two sides of the cut. In the case where material has been removed the *empty gap* would

be filled with material of identical properties. The body is left in the state of stress:

$$\sigma_{rr} = -\frac{Gb_y}{2\pi(1-\nu)} \frac{\cos \theta}{r} \quad (\text{A.9a})$$

$$\sigma_{\theta\theta} = -\frac{Gb_y}{2\pi(1-\nu)} \frac{\cos \theta}{r} \quad (\text{A.9b})$$

$$\sigma_{r\theta} = -\frac{Gb_y}{2\pi(1-\nu)} \frac{\sin \theta}{r} \quad (\text{A.9c})$$

The magnitude of the so-called Burgers vector of the dislocation is b . The stresses are singular at the center of dislocation ($r=0$).

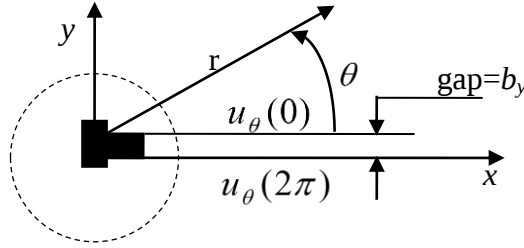


Figure A. 1: A schematic dislocation with Burgers vector b_y

Appendix B: Analytical solution of a tensile crack opening in an infinite homogeneous elastic plane ^{[9] [28]}

The tensile crack ranging $[-l, l]$ from subjected to a constant traction P_0 is shown in Figure 2.4. Let $B(\xi)d\xi$ represent a continuous density of dislocations distributed between ξ and $\xi + d\xi$ as depicted in Figure 2.5. The coordinate ξ is on the x-axis, and between $[-l, l]$. Thus from the equation (A. 9b), the density will produce the stress:

$$\sigma_{yy}(x, 0) = \sigma_{\theta\theta}(r, 0) = -\frac{GB(\xi)d\xi}{2\pi(1-\nu)(x-\xi)} \quad (\text{B.1})$$

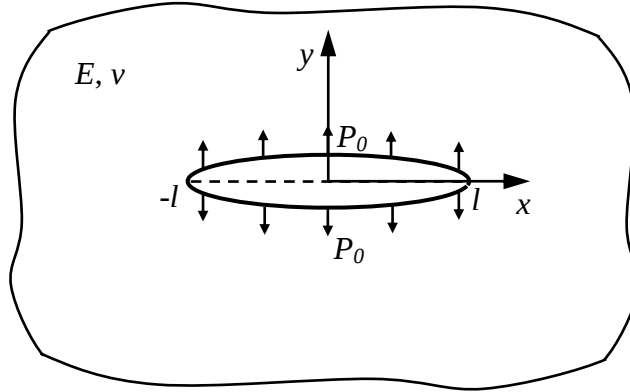


Figure 2. 4: A tensile crack in homogeneous material

The stress produced by the distribution of dislocation dipoles along the entire length of the crack is:

$$\sigma_{yy}(x, 0) = \sigma_{\theta\theta}(r, 0) = -\frac{G}{2\pi(1-\nu)} \int_{-l}^l \frac{B(\xi)d\xi}{(x-\xi)} = \frac{P_0}{n_y} = -P_0$$

Thus we have:

$$\frac{G}{2\pi(1-\nu)} \int_{-l}^l \frac{B(\xi)d\xi}{(x-\xi)} = P_0 \quad (\text{B.2})$$

where $B(\xi) = \frac{d\hat{u}}{d\xi} \Big|_{x=\xi}$, $\hat{u}$ is the opening displacement, thus $d\hat{u}(\xi) = B(\xi)d\xi$

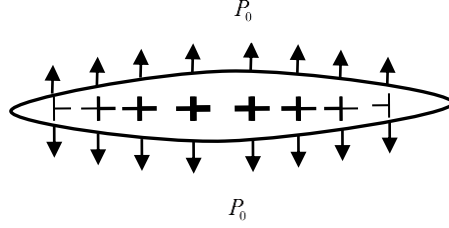


Figure 2. 5: The distribution of dislocation dipoles

By solving the equation (B.2):

$$B(\xi) = -\frac{2(1-\nu)P_0}{G} \frac{\xi}{\sqrt{l^2 - \xi^2}}$$

Then the opening displacement of the crack is:

$$\begin{aligned} \hat{u}(\xi) &= \int B(\xi)d\xi \\ &= -\frac{2(1-\nu)P_0}{G} \int \frac{\xi}{\sqrt{l^2 - \xi^2}} d\xi, \\ &= \frac{2(1-\nu)P_0}{G} \sqrt{l^2 - \xi^2} + Q \end{aligned}$$

where Q is constant and at $\xi = \pm l$, $\hat{u}(\xi) = 0$, hence $Q=0$.

$$\text{Then:} \quad \hat{u}(\xi) = \frac{2(1-\nu)P_0}{G} \sqrt{l^2 - \xi^2} \quad (\text{B.3})$$

Appendix C: The dislocation stress field in an infinite bimaterial plane ^[28-30]

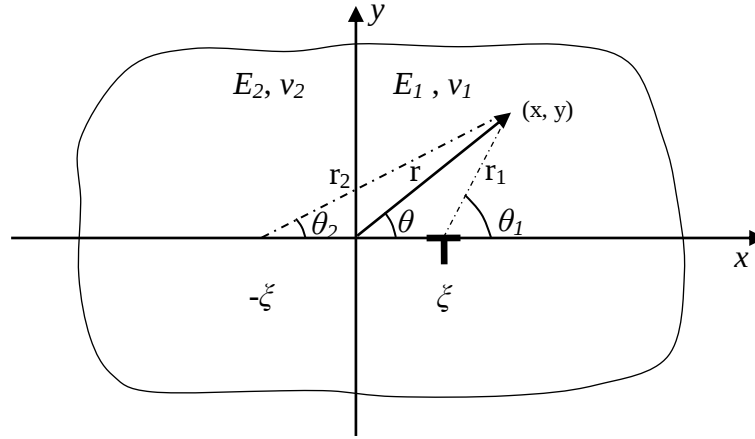


Figure C. 1: An edge dislocation with the Burgers vector b_x

First we define 3 parameters as shown in Figure C.1:

ξ : the distance between the dislocation and the interface

r_1 : the distance between the dislocation and the field point (x, y) , $r_1 = \sqrt{(x - \xi)^2 + y^2}$

r_2 : the distance between the conjugate point of the dislocation $(-\xi, 0)$, and the field point

(x, y) , $r_2 = \sqrt{(x + \xi)^2 + y^2}$

Then Dundurs constants are then defined as follows:

$$k_1 = \frac{G_1}{2\pi(1+\nu_1)}, \quad \alpha = \frac{(1-\nu_1)G_2 - (1-\nu_2)G_1}{(1-\nu_1)G_2 + (1-\nu_2)G_1}, \quad \beta = \frac{(1-2\nu_1)G_2 - (1-2\nu_2)G_1}{(1-\nu_1)G_2 + (1-\nu_2)G_1}$$

$$a = \frac{1+\alpha}{1-\beta^2}, \quad q = \frac{\alpha-\beta}{1+\beta}$$

The shear modulus is given: $G = \frac{E}{2(1+\nu)}$

The corresponding Airy stress functions are given:

$$\varphi^{(1)} = k_1 b_x [r_1 \ln r_1 \sin \theta_1 + (a-1)r_2 \ln r_2 \sin \theta_2 - q\xi(\sin 2\theta_2 - 2\xi \frac{\sin \theta_2}{r_2}) + a\beta r_2 \theta_2 \cos \theta_2]$$

$$\varphi^{(2)} = ak_1 b_x [r_1 \ln r_1 \sin \theta_1 - \beta(r_1 \theta_1 \cos \theta_1 + 2\xi \ln r_1 + 2\xi \theta_1)]$$

The superscript in a parenthesis indicates in which material (material 1 or material 2).

$$\text{According to the theory of elasticity: } \sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}$$

The Airy stress functions yields the formulations of the stress field:

$$\sigma_{xx}^{(1)}(x, y) = k_1 b_x y \left[\frac{3(x-\xi)^2 + y^2}{r_1^4} + q \frac{3(x+\xi)^2 + 3y^2}{r_2^4} + 4q\xi x \frac{3(x+\xi)^2 - y^2}{r_2^6} + a\beta \frac{1}{r_2^2} \right] \quad (\text{C.1a})$$

$$\sigma_{yy}^{(1)}(x, y) = -k_1 b_x y \left[\frac{(x-\xi)^2 - y^2}{r_1^4} + q \frac{(x+\xi)^2 - y^2}{r_2^4} + a\beta \frac{1}{r_2^2} - 4qh \frac{2(x+\xi)^3 - 3x(x+\xi)^2 + 2(x+\xi)y^2 + xy^2}{r_2^6} \right] \quad (\text{C.1b})$$

$$\sigma_{xy}^{(1)}(x, y) = -k_1 b_x \left\{ \frac{(x-\xi)[(x-\xi)^2 - y^2]}{r_1^4} + q \frac{(x+\xi)[(x+\xi)^2 - y^2]}{r_2^4} - 2q\xi \frac{(x+\xi)^4 - 2x(x+\xi)^3 + 6x(x+\xi)y^2 - y^4}{r_2^6} + a\beta \frac{x+h}{r_2^2} \right\} \quad (\text{C.1c})$$

$$\sigma_{xx}^{(2)}(x, y) = ak_1 b_x y \left[\frac{3(x-\xi)^2 + y^2}{r_1^4} + 2\beta \frac{x^2 - \xi^2}{r_1^4} \right] \quad (\text{C.1d})$$

$$\sigma_{yy}^{(2)}(x, y) = -ak_1 b_x y \left[\frac{(x-\xi)^2 - y^2}{r_1^4} + 2\beta \frac{2h(x-\xi) - y^2}{r_1^4} \right] \quad (\text{C.1e})$$

$$\sigma_{xy}^{(2)}(x, y) = -ak_1 b_x \left[\frac{(x-\xi)[(x-\xi)^2 - y^2]}{r_1^4} + 2\beta \frac{h(x-\xi)^2 - xy^2}{r_1^4} \right] \quad (\text{C.1f})$$

When dealing with an edge dislocation with the Burgers vector b_y as shown in Figure

2.9, we can get stresses field similarly.

The corresponding Airy stress functions are:

$$\varphi^{(1)} = k_1 b_y [r_1 \ln r_1 \cos \theta_1 + (a-1)r_2 \ln r_2 \cos \theta_2 - q\xi(2 \ln r_2 - \cos 2\theta_2 + 2\xi \frac{\cos \theta_2}{r_2}) - a\beta r_2 \theta_2 \sin \theta_2]$$

$$\varphi^{(2)} = ak_1 b_y [r_1 \ln r_1 \cos \theta_1 + \beta(2\xi \ln r_1 + r_1 \theta_1 \sin \theta_1)]$$

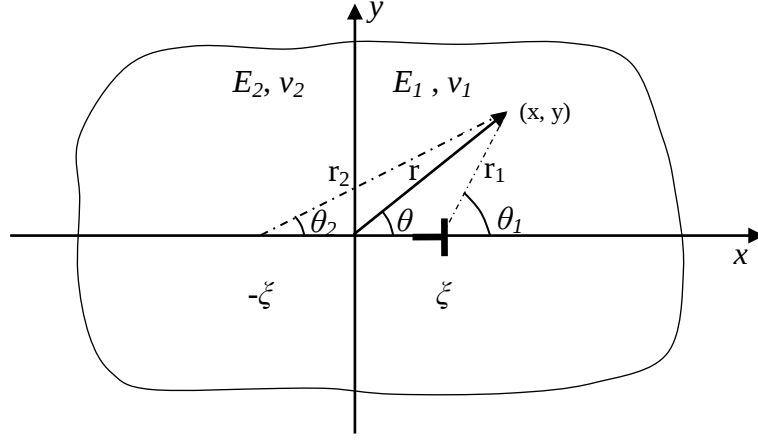


Figure 2. 9: An edge dislocation with the Burgers vector b_y ^[29]

And these two formulations yield:

$$\sigma_{xx}^{(1)}(x, y) = k_1 b_y \left\{ \frac{(x - \xi)[(x - \xi)^2 - y^2]}{r_1^4} + q \frac{(x + \xi)[(x + \xi)^2 - y^2]}{r_2^4} - 2qh \frac{(x + \xi)^4 + 2x(x + \xi)^3 - 6x(x + \xi)y^2 - y^4}{r_2^6} - a\beta \frac{x + \xi}{r_2^2} \right\} \quad (C.2a)$$

$$\sigma_{yy}^{(1)}(x, y) = k_1 b_y \left\{ \frac{(x - \xi)[(x - \xi)^2 + 3y^2]}{r_1^4} + q \frac{(x + \xi)[(x + \xi)^2 + 3y^2]}{r_2^4} - 2qh \frac{(x + \xi)^4 - 2x(x + \xi)^3 + 6x(x + \xi)y^2 - y^4}{r_2^6} + a\beta \frac{x + \xi}{r_2^2} \right\} \quad (C.2b)$$

$$\sigma_{xy}^{(1)}(x, y) = k_1 b_y \left\{ \frac{y[(x - \xi)^2 - y^2]}{r_1^4} + q \frac{y[(x + \xi)^2 - y^2]}{r_2^4} - 4q\xi xy \frac{3(x + \xi)^2 - y^2}{r_2^6} - a\beta \frac{y}{r_2^2} \right\} \quad (C.2c)$$

$$\sigma_{xx}^{(2)}(x, y) = ak_1 b_y \left[\frac{(x - \xi)[(x - \xi)^2 - y^2]}{r_1^4} + 2\beta \frac{x(x - \xi)^2 - \xi y^2}{r_1^4} \right] \quad (C.2d)$$

$$\sigma_{yy}^{(2)}(x, y) = ak_1 b_y \left[\frac{(x - \xi)[(x - \xi)^2 + 3y^2]}{r_1^4} - 2\beta \frac{\xi(x - \xi)^2 - xy^2}{r_1^4} \right] \quad (C.2e)$$

$$\sigma_{xy}^{(2)}(x, y) = ak_1 b_y \left\{ \frac{y[(x - \xi)^2 - y^2]}{r_1^4} + 2\beta y \frac{x^2 - \xi^2}{r_1^4} \right\} \quad (C.2f)$$

Only the stress terms σ_{yy} are needed to summarize the equations (C.1) and (C.2) for

the implementation of a tensile crack:

$$\begin{aligned}
\sigma_{yy}^{(1)}(x, 0) &= k_1 b_y \left\{ \frac{(x-\xi)^3}{r_1^4} + q \frac{(x+\xi)^3}{r_2^4} - 2qh \frac{(x+\xi)^4 - 2x(x+\xi)^3}{r_2^6} + a\beta \frac{x+\xi}{r_2^2} \right\} \\
&= b_y k_1 \left[\frac{1}{(x-\xi)} + q \frac{1}{(x+\xi)} - 2qh \frac{\xi-x}{(x+\xi)^3} + a\beta \frac{1}{(x+\xi)} \right] \\
&= b_y f_1(x, \xi)
\end{aligned} \tag{C.3a}$$

And

$$\begin{aligned}
\sigma_{yy}^{(2)}(x, 0) &= ak_1 b_y \left[\frac{(x-\xi)^3}{r_1^4} - 2\beta \frac{\xi(x-\xi)^2}{r_1^4} \right] \\
&= b_y ak_1 \left[\frac{1}{(x-\xi)} - 2\beta \frac{\xi}{(x-\xi)^2} \right] \\
&= b_y f_2(x, \xi)
\end{aligned} \tag{C.3b}$$

And the function f is defined as:

$$f(x, \xi) = \begin{cases} f_1(x, \xi) = k_1 \left[\frac{1}{(x-\xi)} + q \frac{1}{(x+\xi)} - 2q\xi \frac{\xi-x}{(x+\xi)^3} + a\beta \frac{1}{(x+\xi)} \right], & x \geq 0 \\ f_2(x, \xi) = ak_1 \left[\frac{1}{(x-\xi)} - 2\beta \frac{\xi}{(x-\xi)^2} \right] & , \quad x < 0 \end{cases} \tag{C.4}$$

Appendix D: The derivation of the formulae in the revised crack tip element approach

We have the references:

$$\int \frac{dx}{ax^2 + b} = \begin{cases} \frac{1}{\sqrt{ab}} \arctan \sqrt{\frac{a}{b}} x + C & (b > 0) \\ \frac{1}{2\sqrt{-ab}} \ln \left| \frac{\sqrt{ax} - \sqrt{-b}}{\sqrt{ax} + \sqrt{-b}} \right| + C & (b < 0) \end{cases}$$

$$\int \frac{xdx}{ax^2 + b} = \frac{1}{2a} \ln |ax^2 + b| + C$$

$$\int \frac{x^2 dx}{ax^2 + b} = \frac{x}{a} - \frac{b}{a} \int \frac{dx}{ax^2 + b}$$

Derivation of Equation (2.4)

$$\begin{aligned} I_1 &= \int \ln[(x - \xi)^2 + y^2] d\xi \\ &= \int \ln[(\xi - x)^2 + y^2] d(\xi - x) \\ &= (\xi - x) \ln[(\xi - x)^2 + y^2] - \int (\xi - x) \frac{2(\xi - x)}{(\xi - x)^2 + y^2} d(\xi - x) : \\ &= (\xi - x) \ln[(\xi - x)^2 + y^2] - 2 \int \frac{(\xi - x)^2}{(\xi - x)^2 + y^2} d(\xi - x) \\ &= (\xi - x) \ln[(\xi - x)^2 + y^2] - 2 \left\{ (\xi - x) - y^2 \int \frac{d(\xi - x)}{(\xi - x)^2 + y^2} \right\} \end{aligned}$$

Notice that x is a constant respect to the integral:

$$\begin{aligned} I_1 &= (\xi - x) \ln[(\xi - x)^2 + y^2] - 2 \left\{ \xi - y^2 \frac{1}{y} \arctan \frac{\xi - x}{y} \right\} \\ &= (\xi - x) \ln[(\xi - x)^2 + y^2] - 2 \left[\xi - y \arctan \frac{\xi - x}{y} \right] \end{aligned} \tag{D.1}$$

Upon evaluation of the integral range:

$$\begin{aligned}
I_1 \Big|_{-a}^a &= (\xi - x) \ln[(\xi - x)^2 + y^2] - 2[\xi - y \arctan \frac{\xi - x}{y}] \Big|_{-a}^a \\
&= (a - x) \ln[(a - x)^2 + y^2] - (-a - x) \ln[(-a - x)^2 + y^2] \\
&\quad - 2[(a - x) - (-a - x)] + 2y[\arctan \frac{a - x}{y} - \arctan \frac{-a - x}{y}]
\end{aligned}$$

Thus,

$$\begin{aligned}
I_1 \Big|_{-a}^a &= (x + a) \ln[(x + a)^2 + y^2] - (x - a) \ln[(x - a)^2 + y^2] \\
&\quad + 2y[\arctan \frac{x + a}{y} - \arctan \frac{x - a}{y}] - 4a
\end{aligned}$$

Omit the constant $-4a$:

$$\begin{aligned}
f(x, y) &= \frac{-D_y}{8\pi(1-\nu)} \{ (x + a) \ln[(x + a)^2 + y^2] - (x - a) \ln[(x - a)^2 + y^2] \\
&\quad + 2y[\arctan \frac{y}{x - a} - \arctan \frac{y}{x + a}] \}
\end{aligned}$$

Derivation of Equation (2.60)

$$\begin{aligned}
I &= \int (c\xi + b) \ln[(x - \xi)^2 + y^2] d\xi \\
&= c \int \xi \ln[(x - \xi)^2 + y^2] d\xi + b \int \ln[(x - \xi)^2 + y^2] d\xi \\
&= cI_2 + bI_1
\end{aligned} \tag{D.2}$$

$$\begin{aligned}
I_2 &= \int \xi \ln[(\xi - x)^2 + y^2] d\xi \\
&= \frac{\xi^2}{2} \ln[(\xi - x)^2 + y^2] - \int \frac{\xi^2}{2} \frac{2(\xi - x)}{(\xi - x)^2 + y^2} d\xi \\
&= \frac{\xi^2}{2} \ln[(\xi - x)^2 + y^2] - \int \frac{\xi^3}{(\xi - x)^2 + y^2} d\xi + x \int \frac{\xi^2}{(\xi - x)^2 + y^2} d\xi
\end{aligned} \tag{D.3}$$

We calculate the last integral in (D.3) at first:

$$\begin{aligned}
& \int \frac{\xi^2}{(\xi-x)^2+y^2} d\xi \\
&= \int \frac{(\xi-x)^2}{(\xi-x)^2+y^2} d(\xi-x) + 2x \int \frac{\xi}{(\xi-x)^2+y^2} d(\xi-x) - x^2 \int \frac{d(\xi-x)}{(\xi-x)^2+y^2} \\
&= (\xi-x) - y^2 \int \frac{d(\xi-x)}{(\xi-x)^2+y^2} + x \ln[(\xi-x)^2+y^2] - x^2 \int \frac{d(\xi-x)}{(\xi-x)^2+y^2}
\end{aligned}$$

Omit the constant x respect to the integral:

$$\begin{aligned}
& \int \frac{\xi^2}{(\xi-x)^2+y^2} d\xi \\
&= \int \frac{(\xi-x)^2 + 2\xi x - x^2}{(\xi-x)^2+y^2} d\xi \\
&= \int \frac{(\xi-x)^2 + 2(\xi-x)x + x^2}{(\xi-x)^2+y^2} d\xi \\
&= \int \frac{(\xi-x)^2}{(\xi-x)^2+y^2} d(\xi-x) + 2x \int \frac{(\xi-x)}{(\xi-x)^2+y^2} d\xi + 2x^2 \int \frac{d(\xi-x)}{(\xi-x)^2+y^2} - x^2 \int \frac{d\xi(-x)}{(\xi-x)^2+y^2} \\
&= \int \frac{(\xi-x)^2}{(\xi-x)^2+y^2} d(\xi-x) + 2x \int \frac{(\xi-x)}{(\xi-x)^2+y^2} d\xi + x^2 \int \frac{d(\xi-x)}{(\xi-x)^2+y^2} \\
&= \xi - y^2 \int \frac{d(\xi-x)}{(\xi-x)^2+y^2} + x \ln[(\xi-x)^2+y^2] + x^2 \int \frac{d(\xi-x)}{(\xi-x)^2+y^2} \\
&= \xi + x \ln[(\xi-x)^2+y^2] + \frac{x^2-y^2}{y} \arctan \frac{\xi-x}{y}
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_2 &= \frac{\xi^2}{2} \ln[(\xi-x)^2+y^2] - \int \frac{\xi^3}{(\xi-x)^2+y^2} d\xi + x \int \frac{\xi^2}{(\xi-x)^2+y^2} d\xi \\
&= \frac{\xi^2+2x^2}{2} \ln[(\xi-x)^2+y^2] + x \frac{x^2-y^2}{y} \arctan \frac{\xi-x}{y} + x\xi - \int \frac{\xi^3}{(\xi-x)^2+y^2} d\xi
\end{aligned} \tag{D.4}$$

Then, we calculate the remaining integral:

$$\begin{aligned}
& \int \frac{\xi^3}{(\xi-x)^2+y^2} d\xi \\
&= \int \frac{\xi[(\xi-x)^2+y^2] + 2x[(\xi-x)^2+y^2] + (3x^2\xi - 3x^3 - \xi y^2 + xy^2) + (x^3 - 3xy^2)}{(\xi-x)^2+y^2} d\xi \\
&= \int (\xi + 2x) d\xi + (3x^2 - y^2) \int \frac{(\xi-x)d(\xi-x)}{(\xi-x)^2+y^2} + x(x^2 - 3y^2) \int \frac{d(\xi-x)}{(\xi-x)^2+y^2} \\
&= \frac{\xi^2}{2} + 2x\xi + \frac{1}{2}(3x^2 - y^2) \ln[(\xi-x)^2+y^2] + x \frac{(x^2 - 3y^2)}{y} \arctan \frac{\xi-x}{y}
\end{aligned} \tag{D.5}$$

Substitute (A.5) into (A.4):

$$\begin{aligned}
I_2 &= \frac{\xi^2 + 2x^2}{2} \ln[(\xi-x)^2+y^2] + x \frac{x^2 - y^2}{y} \arctan \frac{\xi-x}{y} + x\xi - \frac{\xi^2}{2} - 2x\xi \\
&\quad - \frac{1}{2}(3x^2 - y^2) \ln[(\xi-x)^2+y^2] - x \frac{(x^2 - 3y^2)}{y} \arctan \frac{\xi-x}{y} \\
&= \frac{\xi^2 - x^2 + y^2}{2} \ln[(\xi-x)^2+y^2] + 2xy \arctan \frac{\xi-x}{y} - \frac{\xi^2}{2} - x\xi
\end{aligned} \tag{D.6}$$

Therefore, substitute (D.1) and (D.6) into (D.2):

$$\begin{aligned}
I &= cI_2 + bI_1 \\
&= c \frac{\xi^2 - x^2 + y^2}{2} \ln[(\xi-x)^2+y^2] + 2cxy \arctan \frac{\xi-x}{y} - \frac{c\xi^2}{2} - cx\xi \\
&\quad + b(\xi-x) \ln[(\xi-x)^2+y^2] - 2b[\xi - y \arctan \frac{\xi-x}{y}] \\
I &= [c \frac{\xi^2 - x^2 + y^2}{2} + b(\xi-x)] \ln[(\xi-x)^2+y^2] \\
&\quad + 2y(cx+b) \arctan \frac{\xi-x}{y} - \frac{c\xi^2}{2} - (cx+2b)\xi
\end{aligned}$$

The final expression is:

$$\begin{aligned}
I &= \sum_{i=1}^n [c_i \frac{(\xi^2 - x^2 + y^2)}{2} + b_i(\xi-x)] \ln[(x-\xi)^2+y^2] - \frac{c_i \xi^2}{2} - (c_i x + 2b_i)\xi \\
&\quad + \sum_{i=1}^n 2y(c_i x + b_i) \arctan \frac{\xi-x}{y}
\end{aligned} \tag{D.6}$$

Notice that:

$$\begin{aligned}\arctan \frac{\xi - x}{y} &= \frac{\pi}{2} - \arctan \frac{\xi - x}{y} \\ &= \frac{\pi}{2} + \arctan \frac{\xi - x}{y}\end{aligned}$$

Omit the constants:

$$I = \sum_{i=1}^n \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} - b_i(x - \xi) \right] \ln[(x - \xi)^2 + y^2] - c_i \xi x_i + 2y(c_i x + b_i) \arctan \frac{y}{x - \xi}$$

Derivation of $f_{,x}$

$$\begin{aligned}f_{,x} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} - b_i(x - \xi) \right] \frac{2(x - \xi)}{(x - \xi)^2 + y^2} - c_i \xi \right. \\ &\quad \left. - 2(c_i x + b_i) \frac{y^2}{(x - \xi)^2 + y^2} + 2y c_i \arctan \frac{y}{x - \xi} \right. \\ &\quad \left. - (c_i x + b_i) \ln[(x - \xi)^2 + y^2] \right\} \Bigg|_{a_i}^{a_{i+1}}\end{aligned}$$

Inside:

$$\begin{aligned}& \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} - b_i(x - \xi) \right] \frac{2(x - \xi)}{(x - \xi)^2 + y^2} - 2(c_i x + b_i) \frac{y^2}{(x - \xi)^2 + y^2} \\ &= \left[\frac{c_i(\xi^2 - x^2 + y^2)(x - \xi)}{2} - b_i(x - \xi)^2 - (c_i x + b_i)y^2 \right] \frac{2}{(x - \xi)^2 + y^2} \\ &= \left\{ \frac{c_i(\xi^2 - x^2 + y^2)(x - \xi) - 2c_i x y^2}{2} - b_i[(x - \xi)^2 + y^2] \right\} \frac{2}{(x - \xi)^2 + y^2} \\ &= \left\{ c_i \frac{\xi^2 x - \xi^3 - x^3 + x^2 \xi + y^2 x - y^2 \xi - 2xy^2}{2} - b_i[(x - \xi)^2 + y^2] \right\} \frac{2}{(x - \xi)^2 + y^2} \\ &= \left\{ c_i \frac{\xi^2 x - \xi^3 - x^3 + x^2 \xi - y^2 x - y^2 \xi}{2} - b_i[(x - \xi)^2 + y^2] \right\} \frac{2}{(x - \xi)^2 + y^2} \\ &= \left\{ c_i \frac{(x + \xi)[-(x - \xi)^2 - y^2]}{2} - b_i[(x - \xi)^2 + y^2] \right\} \frac{2}{(x - \xi)^2 + y^2} \\ &= -c_i(x + \xi) - 2b_i\end{aligned}$$

Thus,

$$f_{,x} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ -c_i(x + 2\xi) - 2b_i + 2c_i y \arctan \frac{y}{x - \xi} - (c_i x + b_i) \ln[(x - \xi)^2 + y^2] \right\} \Bigg|_{a_i}^{a_{i+1}}$$

Omitting the constants:

$$f_{,x} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ -c_i x + 2c_i y \arctan \frac{y}{x-\xi} - (c_i x + b_i) \ln[(x-\xi)^2 + y^2] \right\} \Big|_{a_i}^{a_{i+1}}$$

Derivation of $f_{,y}$

$$\begin{aligned} f(x, y) &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} + b_i(\xi - x) \right] \ln[(x-\xi)^2 + y^2] \right. \\ &\quad \left. + 2y(c_i x + b_i) \arctan \frac{\xi - x}{y} - c_i x \xi \right\} \Big|_{-a}^a \\ f_{,y} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ c_i y \ln[(x-\xi)^2 + y^2] + \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} - b_i(x-\xi) \right] \frac{2y}{(x-\xi)^2 + y^2} \right. \\ &\quad \left. + 2(cx + b) \arctan \frac{y}{x-\xi} + 2y(c_i x + b_i) \frac{x-\xi}{(x-\xi)^2 + y^2} \right\} \Big|_{a_i}^{a_{i+1}} \end{aligned}$$

Inside:

$$\begin{aligned} f_{,y} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ c_i y \ln[(x-\xi)^2 + y^2] + \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} - b_i(x-\xi) \right] \frac{2y}{(x-\xi)^2 + y^2} \right. \\ &\quad \left. + 2(cx + b) \arctan \frac{y}{x-\xi} + 2y(c_i x + b_i) \frac{x-\xi}{(x-\xi)^2 + y^2} \right\} \Big|_{a_i}^{a_{i+1}} \end{aligned}$$

Inside:

$$\begin{aligned} &\left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} - b_i(x-\xi) \right] \frac{2y}{(x-\xi)^2 + y^2} + 2y(c_i x + b_i) \frac{x-\xi}{(x-\xi)^2 + y^2} \\ &= \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} - b_i(x-\xi) + (c_i x + b_i)(x-\xi) \right] \frac{2y}{(x-\xi)^2 + y^2} \\ &= \left[\frac{c_i(\xi^2 - x^2 + y^2)}{2} + c_i x^2 - c_i x \xi \right] \frac{2y}{(x-\xi)^2 + y^2} \\ &= c_i \frac{\xi^2 - x^2 + y^2 + 2x^2 - 2x\xi}{2} \frac{2y}{(x-\xi)^2 + y^2} \\ &= c_i \frac{\xi^2 + x^2 + y^2 - 2x\xi}{2} \frac{2y}{(x-\xi)^2 + y^2} \end{aligned}$$

$$= c_i \frac{(x-\xi)^2 + y^2}{2} \frac{2y}{(x-\xi)^2 + y^2}$$

$$= c_i y$$

$$f_{,y} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ c_i y + c_i y \ln[(x-\xi)^2 + y^2] + 2(c_i x + b_i) \arctan \frac{y}{x-\xi} \right\} \Big|_{a_i}^{a_{i+1}}$$

Derivation of $f_{,xy}$

$$f_{,x} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ -c_i x + 2c_i y \arctan \frac{y}{x-\xi} - (c_i x + b_i) \ln[(x-\xi)^2 + y^2] \right\} \Big|_{a_i}^{a_{i+1}}$$

$$f_{,xy} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ 2c_i \arctan \frac{y}{x-\xi} + 2c_i y \frac{1}{1 + \left(\frac{y}{x-\xi}\right)^2} \frac{1}{x-\xi} - (c_i x + b_i) \frac{2y}{(x-\xi)^2 + y^2} \right\} \Big|_{a_i}^{a_{i+1}}$$

$$= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ 2c_i \arctan \frac{y}{x-\xi} + 2c_i y \frac{x-\xi}{(x-\xi)^2 + y^2} - (c_i x + b_i) \frac{2y}{(x-\xi)^2 + y^2} \right\} \Big|_{a_i}^{a_{i+1}}$$

$$= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ 2c_i \arctan \frac{y}{x-\xi} - \frac{2(c_i \xi + b_i) y}{(x-\xi)^2 + y^2} \right\} \Big|_{a_i}^{a_{i+1}}$$

Derivation of $f_{,yy}$

$$f_{,yy} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ c_i + c_i \ln[(x-\xi)^2 + y^2] + c_i y \frac{2y}{(x-\xi)^2 + y^2} \right. \\ \left. + 2(c_i x + b_i) \frac{1}{1 + \left(\frac{x-\xi}{y}\right)^2} \frac{1}{x-\xi} \right\}$$

$$= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ c_i \ln[(x-\xi)^2 + y^2] + 2 \frac{(c_i x + b_i)(x-\xi) + c_i y^2}{(x-\xi)^2 + y^2} \right\} \Big|_{a_i}^{a_{i+1}}$$

by omitting the constant c_i .

Derivation of $f_{,xyy}$

$$\begin{aligned}
 f_{,xy} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ 2c_i \arctan \frac{y}{x-\xi} - \frac{2(c_i\xi + b_i)y}{(x-\xi)^2 + y^2} \right\} \Big|_{a_i}^{a_{i+1}} \\
 f_{,yyy} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2c_i \frac{1}{1 + \left(\frac{y}{x-\xi}\right)^2} \frac{1}{x-\xi} - 2(c_i\xi + b_i) \frac{(x-\xi)^2 + y^2 - y \times 2y}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}} \\
 &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2 \frac{x-\xi}{(x-\xi)^2 + y^2} - 2(c_i\xi + b_i) \frac{(x-\xi)^2 - y^2}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}} \\
 &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2 \frac{c_i(x-\xi)[(x-\xi)^2 + y^2] - (c_i\xi + b_i)(x-\xi)^2 + (c_i\xi + b_i)y^2}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}} \\
 &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2 \frac{(c_i x - 2c_i\xi - b_i)(x-\xi)^2 + (c_i x + b_i)y^2}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}}
 \end{aligned}$$

Derivation of $f_{,xyy}$

$$\begin{aligned}
 f_{,yy} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n c_i \ln[(x-\xi)^2 + y^2] + 2 \frac{(c_i x + b_i)(x-\xi) + c_i y^2}{(x-\xi)^2 + y^2} \Big|_{a_i}^{a_{i+1}} \\
 f_{,yyy} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n c_i \frac{2y}{(x-\xi)^2 + y^2} + 2 \frac{2c_i y [(x-\xi)^2 + y^2] - [(c_i x + b_i)(x-\xi) + c_i y^2] 2y}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}} \\
 &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2y \frac{c_i}{(x-\xi)^2 + y^2} + 2y \frac{2c_i [(x-\xi)^2 + y^2] - 2[c_i x^2 + c_i y^2 - c_i x\xi + b_i(x-\xi)]}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}} \\
 f_{,yyy} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2y \frac{c_i}{(x-\xi)^2 + y^2} + 2y \frac{2c_i [(x-\xi)^2 + y^2] - 2[c_i x^2 + c_i y^2 - c_i x\xi + b_i(x-\xi)]}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}} \\
 &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2y \frac{c_i}{(x-\xi)^2 + y^2} + 2y \frac{-2c_i x\xi + 2c_i \xi^2 - 2b_i(x-\xi)}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}} \\
 &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2y \frac{c_i(x-\xi)^2 + c_i y^2 - 2c_i \xi(x-\xi) - 2b_i(x-\xi)}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}} \\
 &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2y \frac{(c_i x - 3c_i \xi - 2b_i)(x-\xi) + c_i y^2}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}} \\
 f_{,x} &= \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ -c_i x + 2c_i y \arctan \frac{y}{x-\xi} - (c_i x + b_i) \ln[(x-\xi)^2 + y^2] \right\} \Big|_{a_i}^{a_{i+1}}
 \end{aligned}$$

$$f_{,x} = \frac{-D_y}{8\pi(1-\nu)} \{-\ln[(x-\xi)^2 + y^2]\} \Big|_{-a}^a$$

$$f_{,y} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \{c_i y + c_i y \ln[(x-\xi)^2 + y^2] + 2(cx+b) \arctan \frac{y}{x-\xi}\} \Big|_{a_i}^{a_{i+1}}$$

$$f_{,y} = \frac{-D_y}{4\pi(1-\nu)} \arctan \frac{y}{x-\xi} \Big|_{-a}^a$$

$$f_{,xy} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ 2c \arctan \frac{y}{x-\xi} - \frac{2(c\xi+b)y}{(x-\xi)^2 + y^2} \right\} \Big|_{a_i}^{a_{i+1}}$$

$$f_{,xy} = \frac{-D_y}{8\pi(1-\nu)} \frac{-2y}{(x-\xi)^2 + y^2} \Big|_{-a}^a$$

$$f_{,yy} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n \left\{ c \ln[(x-\xi)^2 + y^2] + 2 \frac{(cx+b)(x-\xi) + cy^2}{(x-\xi)^2 + y^2} \right\} \Big|_{a_i}^{a_{i+1}}$$

$$f_{,yy} = \frac{-D_y}{8\pi(1-\nu)} \frac{2(x-\xi)}{(x-\xi)^2 + y^2} \Big|_{-a}^a$$

$$f_{,xyy} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2 \frac{(c_i x - 2c_i \xi - b_i)(x-\xi)^2 + (c_i x + b_i)y^2}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}}$$

$$f_{,xyy} = \frac{-D_y}{8\pi(1-\nu)} \frac{-2[(x-\xi)^2 - y^2]}{[(x-\xi)^2 + y^2]^2} \Big|_{-a}^a$$

$$f_{,yyy} = \frac{-D_y}{8\pi(1-\nu)\sqrt{a}} \sum_{i=1}^n 2y \frac{(c_i x - 3c_i \xi - 2b_i)(x-\xi) + c_i y^2}{[(x-\xi)^2 + y^2]^2} \Big|_{a_i}^{a_{i+1}}$$

$$f_{,yyy} = \frac{-D_y}{8\pi(1-\nu)} \frac{-2y(x-\xi)}{[(x-\xi)^2 + y^2]^2} \Big|_{-a}^a$$