Comonadicity for Localizations

by

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Abstract

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The Barr-Beck-Lurie comonadicity theorem characterizes when an adjunction $\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{D}$ can be used to present $\mathscr{C}$ as the category of comodules in $\mathscr{D}$ for the coalgebra object $L \circ R \in \mathrm{End}(\mathscr{D})$; loosely speaking, the comodule structure supplies the instructions for how to assemble object in $\mathscr{C}$ from an object in $\mathscr{D}$. This thesis explores the proofs and applications in a number of contexts of this method for endowing categories of interest $\mathscr{C}$ with this tautological algebraically-flavored description, in the hopes of being a kind of handbook and toolkit for someone looking to demonstrate a comandicity result.

The toolkit grew out of an investigation of a fundamental comonadicity result: Zariski descent for quasicoherent sheaves. Our main effort, joint with Peng Zhou, is in (1) presenting comonadicity statements in the case where $\mathscr{C} \xrightarrow{L} \mathscr{D}$ is a product of reflective localizations, and (2) applying it to deduce a descent statement for those closed covers of Lagrangian skeleta which are locally modeled on ones that arise in the coherent-constructible correspondence of Fang-Liu-Treumann-Zaslow [4].

To mom and dad, with gratitude hard to capture.

Contents

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Chapter 1

Overview

1.1 What is this all about?

The goal of this thesis is to present some comonadicity results for a collection of localizations, with an eye towards applications in microlocal sheaf theory. Loosely speaking, a *comonadicity result* is a result saying that a category of interest $\mathscr{C}$ is equivalent to a category of "comodules" built out of another category $\mathscr{D}$ that we somehow consider easier to understand:

$$\underbrace{\mathscr{C}}_{\text{hard}} \quad \simeq \quad \underbrace{{}_{\Omega}\,\mathrm{coMod}(\mathscr{D})}_{\text{easier/more concrete?}} \ ,$$

$$c \longleftrightarrow (d, d \xrightarrow{\Delta} \Omega d)$$

where part of the notation is a "comonad" Ω acting on $\mathscr{D}$. To get a feel for what these results look like, let us now turn to a familiar result and see how it is a comonadicity result.

Sheaves and open covers

Suppose we wish to understand $\mathscr{C} = \mathrm{Sh}(X)$ the 1-category of sheaves of sets on a topological space X. We ask:

Question 1.1. *How to describe a sheaf of sets $\mathscr{F}$ on X?*

One way is to take some open cover $\{U_i\}_{i\in I}$ of X and to use it to present $\mathscr{F}$ as the data of

1. a sheaf $\mathscr{F}_i := j_i^* \mathscr{F}$ on each U_i;

2. a morphism $\mathscr{F}_i|_{U_i \cap U_j} \xrightarrow{\phi_{ij}} \mathscr{F}_j|_{U_i \cap U_j}$ over each $U_i \cap U_j$

subject to the *cocycle condition*

$$\begin{tikzcd} \mathscr{F}_i|_{U_i\cap U_j\cap U_k} \arrow[rr, "\phi_{ik}|_{U_i\cap U_j\cap U_k}"] \arrow[dr, "\phi_{ij}|_{U_i\cap U_j\cap U_k}"'] & & \mathscr{F}_k|_{U_i\cap U_j\cap U_k} \\ & \mathscr{F}_j|_{U_i\cap U_j\cap U_k} \arrow[ur, "\phi_{jk}|_{U_i\cap U_j\cap U_k}"'] & \end{tikzcd}$$

over each $U_i \cap U_j \cap U_k$. We might not know it now, but the "easier" category $\mathscr{D}$ in the notation above that we have thus chosen is $\mathrm{Sh}(U) = \prod_{i\in I} \mathrm{Sh}(U_i)$. Before getting to comonadicity, let us examine ways we can package the above data.

We start by organizing it. The cover $\{U_i\}$ builds a space $U := \coprod_i U_i$ and a surjective map $f : U \to X$, and therefore also an augmented simplicial diagram of topological spaces

$$\left[\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} U\times_X U \times_X U \;\substack{\xrightarrow{\pi_{12}}\\\xrightarrow{\pi_{13}}\\\xrightarrow{\pi_{23}}}\; U\times_X U \;\substack{\xrightarrow{\pi_1}\\\xrightarrow{\pi_2}}\; U\right] \xrightarrow{f} X$$

which, by applying the functor $\mathrm{Sh}(-)^*$, becomes the augmented cosimplicial diagram of categories abbreviated as $\mathrm{Sh}(U^{\bullet+1}/X) \xleftarrow{f^*} \mathrm{Sh}(X)$, and unrollable into

$$\left[\mathrm{Sh}(U\times_X U\times_X U) \;\substack{\xleftarrow{\pi_{12}^*}\\\xleftarrow{\pi_{13}^*}\\\xleftarrow{\pi_{23}^*}}\; \mathrm{Sh}(U\times_X U) \;\substack{\xleftarrow{\pi_1^*}\\\xleftarrow{\pi_2^*}}\; \mathrm{Sh}(U)\right] \xleftarrow{f^*} \mathrm{Sh}(X)$$

$$\{\text{cocycle condition}\}_{i,j,k\in I} \qquad \{\phi_{ij}\}_{i,j\in I} \qquad \textstyle\prod_{i\in I}\mathscr{F}_i \qquad \mathscr{F}$$

The data is stratified by the depth of intersection:

1. $\prod_{i\in I} \mathscr{F}_i := f^*\mathscr{F} \in \mathrm{Sh}(U)$;

2. $\pi_1^* \prod_{i\in I} \mathscr{F}_i \xrightarrow{\phi} \pi_2^* \prod_{i\in I} \mathscr{F}_i$, which separates into components $\phi = \prod_{i,j\in I} \phi_{ij}$ for ϕ_{ij} a morphism in $\mathrm{Sh}(U_i \cap U_j)$,

and terminates with a *condition* over triple intersections:

3 $\pi_{12}^*\phi \circ \pi_{23}^*\phi = \pi_{13}^*\phi$, which separates into the cocycle condition $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ for $i, j, k \in I$.

By the way we wrote it, this data is determined by a sheaf $\mathscr{F}$ on X by restrictions. But we can abstract (1)-(3) above into a definition of objects in a new category, detached from any "origin" sheaf $\mathscr{F}$ on X. Call such data that respects the cocycle condition a *descent* *datum* $\{\mathscr{F}_i, \phi_{ij}\}$; these descent data organize into a category $\mathrm{Desc}_{\mathscr{U}}\,\mathrm{Sh}(X)$ with appropriate morphisms that preserve this structure, which is related to $\mathrm{Sh}(X)$ by functors

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[rr, bend left, "\widetilde{f^*}"] & & \mathrm{Desc}_{\mathscr{U}}\,\mathrm{Sh}(X) \arrow[ll, bend left, "f_*^{\mathrm{recon}}"] \end{tikzcd}$$

where the top functor $\widetilde{f^*}$ is the obvious functor that builds such data out of any sheaf $\mathscr{F}$ on X, and the functor going the opposite way as a functor that attempts to "reconstruct" a sheaf on $\mathscr{F}$ from a descent datum by building the Čech (truncated) cosimplicial diagram and taking the limit:

$$f_*^{\mathrm{recon}}(\{\mathscr{F}_i, \phi_{ij}\}) :=\!=\!= \varprojlim \left(f_*\mathscr{G} \rightrightarrows (f_*f^*)f_*\mathscr{G} \mathrel{\substack{\textstyle\longrightarrow\\[-0.6ex]\textstyle\longrightarrow\\[-0.6ex]\textstyle\longrightarrow}} (f_*f^*)^2 f_*\mathscr{G} \right)$$

Here, $\mathscr{G} := \prod\limits_{i\in I} \mathscr{F}_i \in \prod_{i\in I} \mathrm{Sh}(U_i)$ is the object in the product category, and therefore e.g. $f_*\mathscr{G} = \prod_{i\in I} j_{i*}\mathscr{F}_i$ is the product of sheaves in $\mathrm{Sh}(X)$. A fundamental result is:

Theorem 1.2. *Given an open cover $\{U_i\}$ of a topological space X,*

1. $\mathrm{Desc}_{\mathscr{U}}\,\mathrm{Sh}(X) \simeq ho\varprojlim \mathrm{Sh}(U^{\bullet+1}/X)^*$*;*
2. *the functors $\widetilde{f^*}, f_*^{\mathrm{recon}}$ are inverse equivalences.*

In other words, a sheaf on X is precisely the data of a "coherent" system of sheaves on the cover U_i and transition maps ϕ_{ij} on overlaps $U_i \cap U_j$ subject to the cocycle condition, and a morphism of sheaves on X is the data of a compatible system of sheaf maps on the cover U_i.

We come now to the comonadic reformulation of the problem. Both $\mathrm{Sh}(X)$ and $\mathrm{Desc}_{\mathscr{U}}\,\mathrm{Sh}(X)$ have obvious functors to $\mathscr{D} := \mathrm{Sh}(U)$:

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[rr, bend left, "\widetilde{f^*}"] \arrow[dr, bend left, "f^*"] & & \mathrm{Desc}_{\mathscr{U}}\,\mathrm{Sh}(X) \arrow[ll, "f_*^{\mathrm{recon}}"'] \arrow[dl, bend left, "\mathrm{fgt}"] \\ & \mathrm{Sh}(U) \arrow[ul, bend left, "f_*"] \arrow[ur, bend left, "\mathrm{free}"] \arrow[loop below, "\Omega"] & \end{tikzcd}$$

and owing to a certain *base change result*, they induce the same endofunctor

$$\Omega := f^* f_* \simeq \mathrm{fgt} \circ \mathrm{free}$$

of $\mathrm{Sh}(U)$. It turns out in this case that the counit and unit of the adjunction give Ω the structure of a *comonad*. The data of a *comodule* for Ω is by definition

1. an object M of $\mathrm{Sh}(U)$; that is, a collection of sheaves $\{\mathscr{F}_i \in \mathrm{Sh}(U_i)\}_{i\in I}$;
2. a *coaction* map $M \xrightarrow{\psi} \Omega M$; that is, a collection of maps to restrictions $\mathscr{F}_i \xrightarrow{\psi_{ij}} j_i^* j_{j*}\mathscr{F}_j \cong \mathscr{F}_j|_i$;
3. subject to a *coassociativity condition* that picks up compatibility for triples of indices ijk.

The details are unimportant at the moment, but the key point is that, in the presence of the same base change result that identifed Ω, this comodule data (M, ψ) can be translated to identify with the descent data $(\mathscr{G}, \phi)$ using the following dictionary:

$$
\begin{array}{ccc}
M & = & \mathscr{G} \\
\{\psi_{ij}\} & \longleftrightarrow & \{\phi_{ij}\} \\
\text{coassociativity condition} & \longleftrightarrow & \text{cocycle condition}
\end{array}
$$

Here is the comonadicity statement:

Theorem 1.3. *For the open cover $U \xrightarrow{f} X$, the functor f^* induces an equivalence of categories*

$$\mathrm{Sh}(X) \simeq {}_{\Omega}\operatorname{coMod}\mathrm{Sh}(U)$$

The earlier theorem one might call a *limit* descent theorem, while this one one might call a *comonadic* descent theorem. It is not obvious, but the primordial result here is in some sense the one on comonadic descent, and limit descent is a consequence of it in the presence of base change:

$$\text{comonadic descent} \xrightarrow{\text{base change}} \text{limit descent}$$

We will continue to examine the interplay between these concepts throughout the document.

Regardless, the main point is that both kinds of descent statements give presentations of objects and morphisms in $\mathrm{Sh}(X)$ in terms of collections of data and conditions on a cover, and it might be easier to carry out certain computations for $\mathrm{Sh}(X)$ in the equivalent categories $\mathrm{Desc}_{\mathscr{U}}\mathrm{Sh}(X)$ or ${}_{\Omega}\operatorname{coMod}\mathrm{Sh}(U)$.

Comonadicity for microlocal sheaves?

One's familiarity with sheaves of sets might make both of the above results seem silly, basic, or tautological: after all, we know that sheaves of sets are *local*, by which we mean exactly that they obey these results. So to appreciate the content of similar kinds of results, it helps to sidestep examples in which they look familiar and see if they hold in a context that looks a bit more strange.

Here is such a context of interest, and the focal one for this document. Consider a closed conic Lagrangian $\Lambda \subset T^*M$ for some real manifold M, and suppose $\{\Lambda_i\}_{i \in I}$ is a closed conic cover of Λ; this results in the functors $L_i : \mathrm{Sh}_\Lambda(M) \to \mathrm{Sh}_{\Lambda_i}(M)$ that are left adjoint to the inclusions $\mathrm{Sh}_{\Lambda_i}(M) \hookrightarrow \mathrm{Sh}_\Lambda(M)$, assembling into an adjunction that builds a comonad $\Omega := LR$

$$\begin{tikzcd} \mathrm{Sh}_\Lambda(M) \arrow[r, shift left, "L"] & \prod\limits_{i \in I} \mathrm{Sh}_{\Lambda_i}(M) \arrow[l, shift left, "R"] \arrow[loop right, "\Omega"] \end{tikzcd}$$

This adjunction sits within the following larger diagram of categories

$$\begin{tikzcd} \mathrm{Sh}_\Lambda(M) \arrow[rr, bend left, "L^{\mathrm{can}}"] \arrow[dr, bend left, "L"] & & \lim\limits_{\longleftarrow} \prod\limits_{J \subseteq I} \mathrm{Sh}_{\Lambda_J}(M) \arrow[ll, dashed] \arrow[dl, bend left, "\mathrm{ev}_0"] \\ & \prod\limits_{i \in I} \mathrm{Sh}_{\Lambda_i}(M) \arrow[ul, bend left, "R"] \arrow[ur, dashed, bend left] \arrow[loop below, "\Omega"] & \end{tikzcd}$$

Functorially, the L_i on $\mathrm{Sh}_\Lambda(M)$ are left adjoints like f_i^* on $\mathrm{Sh}(X)$ above. But unlike the f_i^*, it turns out that they are wildly *non-local* operations on sheaves. Still, the categorical set-up is analogous, and so in analogy to the questions for the categories $\mathrm{Sh}(-)^*$ and open covers that were answered affirmatively by the theorems above, we can ask:

Question 1.4. *In the above set-up,*

1. *under which conditions on the cover $\{\Lambda_i\}_{i \in I}$ is the map to the limit L^{can} an equivalence?*
2. *under which conditions on the cover $\{\Lambda_i\}_{i \in I}$ is L comonadic, i.e. is the lift of L to comodules*

$$\mathrm{Sh}_\Lambda(M) \xrightarrow{L^{\mathrm{enh}}} {}_\Omega\mathrm{coMod}\left(\prod_i \mathrm{Sh}_{\Lambda_i}(M)\right)$$

an equivalence?

These are the central questions for this document.

It turns out that neither question has a generally affirmative answer; some restrictions on the cover $\{\Lambda_i \subseteq \Lambda\}_{i \in I}$ are necessary to produce descent theorems. In this thesis, we offer some basic results on when this is true, together with some tools that can be useful for producing comonadicity statements.

1.2 Guide to the present document

The principal part of this thesis is meant to be an expository account of theorems related to (co)monadicity, and a compilation of many examples in which (co)monadicity is established, using various tools. Here is what you can hope to find:

1. Chapter 1 concludes with a brief categorical exposition, and a list of simple but useful categorical tools.

2. Chapter 2 states and proves two versions of the Barr-Beck monadicity theorem for 1-categories, suggests some perspectives on the hypotheses and methods for showing monadicity, and explores a few standard and non-stable examples.

3. Chapter 3 introduces the Beck-Chevalley conditions, and an alternative flavor of descent statement to comonadic descent, which we may call "limit" descent. It states the main theorems intertwining monadicity and limit descent, and explains how the extended investigations in the coming chapters will follow the blueprints supplied by these theorems.

4. Chapter 4 is an extended discussion of monadicity and comonadicity statements for categories of local systems. The monadicity result is easy and general, and the comonadicity result is more refined and requires a restriction of categories, yielding a family of comonadicity statements that go under the name of Koszul duality.

5. Chapter 5 is an extended discussion of comonadicity statements in topology, for various kinds of covers of spaces. The two main results are a Koszul-duality type comonadicity result for any surjective map, and a comonadicity result for a cover by a stratification.

6. Chapter 6 is an extended discussion of the most important descent result for this thesis: Zariski descent. It proves it in two ways. The second way uses the presence of semi-orthogonal decompositions that were examined by Dwyer-Greenlees, and is the most adaptable to the ultimate purpose of this document.

7. Chapter 7 is a preparatory interlude to the final chapter, and describes the world of sheaves with singular support in which we hope to build comonadicity statements.

8. Chapter 8 is an account of our first attempts at proving comonadicity results for pairs of localizations. Its main content is a characterization of pullback squares of localizations in terms of orthogonality of kernels.

9. Chapter 9 contains what we believe to be our main contributions, joint with Peng Zhou. Two criteria for comonadicity and one criterion for limit descent in the world of sheaves with singular support are established, and are used to show that a mild generalization of FLTZ skeleta and their covers admit both comonadic and limit descent.

Confession 1.5 (On the proofs you will find)**.** *The vast majority of the results in this document are certainly not my original musings. However, unless cited, the proofs are my often clumsy and misguided efforts to teach myself the subjects that I had been trying to use all these years. While I have tried to point out where possible mistakes or gaps lie, there must still be plenty of both that are unannounced. To the extent I got things right, I was likely inspired by a beautiful explanation I learned from someone long ago whom I have forgotten to credit. To the extent I got things wrong, that is my own fault. But, though you must read with skepticism, I hope it is still at least a fraction as useful as writing this thing was for me.*

1.3 Categorical synopsis

We provide a brief categorical rundown, mostly to set notation, but also to point out several results that we will use repeatedly.

With the exception of several introductory sections, this thesis is about the world of ∞-categories, as opposed to 1-categories. In particular, it is about ∞-categories that are both (1) *stable* and (2) *presentable*. We will define these terms soon. Before doing that, we briefly say why we work in this new context:

1. Traditional triangulated categories, such as the derived category $D\,\mathrm{QCoh}(X)$ of quasi-coherent sheaves on a variety X, lack an adequate supply of limits and colimits, even for finite diagrams. For example, while a morphism $X \xrightarrow{f} Y$ has a cone $\mathrm{Cone}(f)$, there are many to choose from: cones are not universal constructions, and are thus neither unique nor functorial. Stable ∞-categories correct this by giving functorial cones; distinguished triangles thus become not a structure, but a property of the stable category.

2. More externally, the category of all triangulated categories also lacks an adequate supply, or theory, of limits and colimits, even for finite diagrams. For example, the following commuting square of triangulated categories is not a fiber product square:

$$\begin{tikzcd} D\,\mathrm{QCoh}(\mathbb{P}^1) \arrow[r, "j_+^*"] \arrow[d, "j_-^*"] & D\,\mathrm{QCoh}(\mathbb{C}_+) \arrow[d, "j_{+-}^*"] \\ D\,\mathrm{QCoh}(\mathbb{C}_-) \arrow[r, "j_{-+}^*"] & D\,\mathrm{QCoh}(\mathbb{C}^\times) \end{tikzcd}$$

By moving to their stable ∞-categorical enrichments, the property of this being a fiber product square is restored.

3. This thesis is about tautology and universal constructions, and this objective justifies operating in a setting of *big* ∞-categories, i.e. categories that do not have a (small) set of objects; these include $\mathrm{Set}, {}_k\mathrm{Mod}, \mathrm{QCoh}(X)$. This is to ensure that there is ready access to a variety of universal constructions both within the categories (e.g. taking colimits of diagrams of objects) and among categories (e.g. taking colimits of diagrams of categories, and producing adjoints to functors). The downside to working with big categories is that they are big, and complicated to "present." By restricting our purview to just those big categories which are *presentable*, we situate ourselves in a happy medium: presentable categories are big enough to still enjoy the full course of universal inter- and intra-categorical constructions, and are simultaneously small enough to be governed by small subcategories.

We now give a bevy of standard definitions:

Definition 1.6. *Let $\mathscr{C}$ be an ∞-category.*

1. *$\mathscr{C}$ is* stable *if (i) it has a zero object $0 \in \mathscr{C}$, (ii) it admits all finite limits and colimits, and (iii) every square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

in $\mathscr{C}$ is a pullback square precisely when it is a pushout square.

2. *$\mathscr{C}$ is* presentable *if there exist a (small) category $\mathscr{C}^0$ and a regular cardinal κ such that (1) $\mathscr{C}^0$ admits all κ-small colimits, and (2) every object of $\mathscr{C}$ is equivalent to a formal κ-filtered colimit of objects in $\mathscr{C}^0$:*

$$\mathrm{Ind}_\kappa \mathscr{C}^0 \xrightarrow{\simeq} \mathscr{C}.$$

If κ can be chosen to be the cardinality of the natural numbers ω, then $\mathscr{C}$ is compactly generated.

Let now $\mathscr{C} \xrightarrow{F} \mathscr{D}$ be a functor of ∞-categories.

1. *If $\mathscr{C}, \mathscr{D}$ are stable categories, then F is called* exact *if it preserves all finite limits and colimits.*

2. *If $\mathscr{C}, \mathscr{D}$ are presentable categories, then F is called* accessible *if there is a regular cardinal κ so that F preserves all κ-small colimits.*

We now use this to define a collection of useful categories of categories. Let Cat_∞ denote the ∞-category of ∞-categories. This has many useful non-full subcategories, which we lay out in the following diagram:

$$\begin{array}{ccccc}
\mathrm{Pr}^{L,\mathrm{st}} & \longrightarrow & \mathrm{Pr}^{L} & \longrightarrow & \mathrm{Pr} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Cat}_\infty^{L,\mathrm{st}} & \longrightarrow & \mathrm{Cat}_\infty^{L} & \longrightarrow & \mathrm{Cat}_\infty
\end{array}$$

They are:

1. Cat_∞^L is the category whose objects are all ∞-categories, and whose morphisms are only those functors which are left adjoints;

2. $\mathrm{Cat}_\infty^{L,\mathrm{st}}$ is the category whose objects are all *stable* ∞-categories, and whose morphisms are only those functors which are left adjoint and exact;

3. Pr is the category whose objects are all presentable ∞-categories, and whose morphisms are the accessible functors;

4. Pr^L is the category whose objects are all presentable ∞-categories, and whose morphisms are those functors which are left adjoint (hence accessible);

5. $\mathrm{Pr}^{L,\mathrm{st}}$ is the category whose objects are all stable presentable ∞-categories, and whose morphisms are those functors which are exact and left adjoint;

We will be most interested in $\mathrm{Pr}^{L,\mathrm{st}}$for applications. Abstract discussion might sometimes only invoke Pr^L.

The adjoint functor theorem

Though briefly mentioned earlier, the following result is the reason for working with presentable categories. We will use it many times, possibly without explicit reference:

Theorem 1.7 (Adjoint Functor Theorem, [14] Corollary 5.5.2.9)**.** *Let*

$$F : \mathscr{C} \to \mathscr{D}$$

be a functor between presentable ∞-categories.

1. *The functor F has a right adjoint if and only if it preserves small colimits.*

2. *The functor F has a left adjoint if and only if it is accessible and preserves small limits.*

Here is the statement explaining the way in which presentable categories carry enough universal "internal and external constructions":

Proposition 1.8. *Properties of presentable categories:*

1. *Every presentable category* $\mathscr{C}$ *is (small)* bicomplete*: that is, it admits all (small) limits and colimits.*

2. *([14] Proposition 5.5.3.13) The category* Pr^L *of presentable categories and left adjoint functors is also (small) bicomplete. Furthermore, the functor* $\mathrm{Pr}^L \xrightarrow{\mathrm{fgt}} \mathrm{Cat}_\infty$ *preserves all small limits.*

Warning 1.9. *On the other hand, the functor* $\mathrm{Pr}^L \xrightarrow{\mathrm{fgt}} \mathrm{Cat}_\infty$ *does not preserve colimits. A large class of examples comes from comparing the colimits*

$$\underset{\mathrm{Pr}^L}{\underrightarrow{\mathrm{colim}}}\left(\overset{T}{\circlearrowright} \mathscr{D} \right) \xrightarrow{\mathrm{can}} \underset{\mathrm{Cat}_\infty}{\underrightarrow{\mathrm{colim}}}\left(\overset{T}{\circlearrowright} \mathscr{D} \right)$$

for an endofunctor T *in* Pr^L*; see Lemma 2.53.*

A comment on notation

1. The notation for adjunctions will always be: functors labeled by L are left adjoints, and functors labeled by R are right adjoints, which may be abbreviated in-line by the notation (L, R). In a diagram with a horizontal adjunction like

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "L"] & \mathscr{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

   the left adjoint will always be drawn on top.

2. By the definitions above, an adjunction in Pr^L is a pair of presentable categories $\mathscr{C}, \mathscr{D}$ together with a triple of adjoint functors (L, R, R^R):

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, bend right, dashed, "R^R"'] & \mathscr{D} \arrow[l, "R"] \end{tikzcd}$$

   Only L and R are morphisms in Pr^L; we have drawn the functor R^R with a dashed arrow to remember that R^R is not actually a morphism in Pr^L. We will often omit this functor from diagrams as it does not need to be present, but sometimes will still draw it, always dashed, if we think it clarifies something.

3. We use derived (hence stable, in their stable pre-triangulated lifts) functors unless specified, so j_* for the pushforward of quasicoherent sheaves under an open embedding of schemes $U \overset{j}{\hookrightarrow} X$ in algebraic geometry will be used to denote what other texts might call the right-derived functor $\mathbb{R}j_*$.

Basic tricks

We close this introductory chapter by recording a handful of almost absurdy simple but incredibly useful categorical tricks. We might use them without mention, although usually we will try to point out their appearance.

Two of them are about how to show that an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "L"] & \mathscr{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

in Cat_∞ is an equivalence. First, here is something no one tells you:

Lemma 1.10. *Let $T := RL, \Omega := LR$ denote the monad and comonad of the adjunction, respectively:*

$$\begin{tikzcd} C \arrow[loop left, "T"] \arrow[r, shift left, "L"] & D \arrow[l, shift left, "R"] \arrow[loop right, "\Omega"] \end{tikzcd}$$

Define the full subcategories

$$\begin{aligned} C^T &:= \{c : c \xrightarrow{\eta_c} RL \text{ is an isomorphism}\} \subseteq C, \\ D^\Omega &:= \{d : LRd \xrightarrow{\epsilon_d} d \text{ is an isomorphism}\} \subseteq D \end{aligned}$$

Then:

1. *the adjunction restricts to an equivalence between C^T and D^Ω:*

$$\begin{tikzcd} C^T \arrow[r, shift left, "L"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & D^\Omega \arrow[l, shift left, "R"] \arrow[d, hook] \\ C \arrow[loop left, "T"] \arrow[r, shift left, "L"] & D \arrow[l, shift left, "R"] \arrow[loop right, "\Omega"] \end{tikzcd}$$

2. *there are no larger subcategories on which L, R could restrict to an equivalence.*

Proof. The definition of C^T is rigged to be the full subcategory on objects on which L restricts to an embedding. So it remains to check whether $L|_{C^T} : C^T \to D$ lands inside D^Ω. To that end, take a $c \in C^T$ so that $c \xrightarrow{\eta_c} RLc$ is an isomorphism, and get:

$$Lc \xrightarrow{L\eta_c} LRLc \xrightarrow{\epsilon_{Lc}} Lc$$

By one half of the Zorro adjunction property, the composite is Id_{Lc}. Since $L\eta_c$ is assumed to be an isomorphism, thus so is ϵ_{Lc}. □

As a corollary of the proof, we obtain the second trick:

Lemma 1.11. *The following are equivalent:*

1. *R is an equivalence;*

2. *R is an embedding, and L is conservative;*

3. *L and R are embeddings.*

The statements with L, R switching places are also equivalent to the above.

Proof. We show (2) $\implies$ (1). We wish to show that R is essentially surjective, so let $c \in C$. The same composite as above,

$$Lc \xrightarrow{L\eta_c} LRLc \xrightarrow{\epsilon_{Lc}} Lc,$$

is Id_{Lc}. The functor R is an embedding iff ϵ is an isomorphism, and so we learn that $Lc \xrightarrow{L\eta_c} LRLc$ is an isomorphism. Since L is conservative, this means that

$$c \xrightarrow{\eta_c} R(Lc)$$

is an isomorphism, and so R is essentially surjective. □

We will use these tautologies to unwind an even deeper tautology—the Barr-Beck-Lurie monadicity theorem.

The third and final trick concerns the calculation of colimits. Most humans find this challenging, and prefer calculating limits instead. Fortunately, if one is calculating a colimit in Pr^L, one can replace it by the calculation of a limit.

To set it up, let I be an ∞-category and $i \mapsto C_i$ a functor $I \to \mathrm{Pr}^L$. For $i \xrightarrow{\alpha} j$, let $C_i \xrightarrow{L_\alpha} C_j$ be the left adjoint functor. By definition, each L_α admits a right adjoint R_α, which determines a functor

$$I^{\mathrm{op}} \to \mathrm{Pr} \xrightarrow{\mathrm{fgt}} \mathrm{Cat}_\infty$$

Lemma 1.12 ([6], Proposition 2.5.7)**.** *If $I \to \mathrm{Pr}^L$ is a functor, then*

$$\underset{i\in I, L_\alpha}{\underrightarrow{\mathrm{colim}}} C_i \xrightarrow{\simeq} \underset{j \in I^{\mathrm{op}}, R_\alpha}{\underleftarrow{\lim}} C_j$$

where the colimit is calculated in Pr^L and the limit is calculated in Cat_∞.

Remark 1.13. *If the adjoints R_α also happened to be left adjoints, i.e. if*

$$\begin{tikzcd} C_i \arrow[r, shift left, "L_\alpha"] & C_j \arrow[l, shift left, "R_\alpha"] \end{tikzcd}$$

were a diagram in Pr^L (as it will be in our applications), then in fact the functor factors as

$$\begin{tikzcd} I^{\mathrm{op}} \arrow[rr, "R"] \arrow[dr, dashed] & & \mathrm{Cat}_\infty \\ & \mathrm{Pr}^L \arrow[ur, "\mathrm{fgt}"] & \end{tikzcd}$$

and by Proposition 1.12, we could conclude that the limit could be calculated in Pr^L *as well:*

$$\underset{i \in I,\, \mathrm{Pr}^L}{\underrightarrow{\mathrm{colim}}}\, C_i \xrightarrow{\simeq} \underset{i \in I^{\mathrm{op}},\, \mathrm{Pr}^L}{\underleftarrow{\lim}}\, C_i$$

Chapter 2

Monadicity and the Barr-Beck-Lurie Theorem

2.1 What is in this chapter?

The Barr-Beck monadicity theorem in 1-categories, and the analogous Barr-Beck-Lurie monadicity theorem in ∞-categories, is a powerful tautology, and a primary goal of this thesis is to try to understand this tautology from many perspectives and through many examples, and to compile tautological but useful observations about it. It endlessly fascinates the author that so many deep theorems follow its blueprint. He is even tempted to posit the following:

behind every interesting equivalence of categories is a monadicity statement

This itself could be either a deep or a tautological statement. In either case, hopefully it is useful as a guide for what to expect to see under the hood.

A large portion of the chapter will be devoted to detailed proofs of the Barr-Beck Theorem 2.7, and to the introduction of an alternative but highly useful partner, Theorem 2.21, which provides another characterization of monadicity. But what is the big idea of monadicity? We caught a glimpse of it in the previous chapter, but let us now begin anew and dive into the details.

2.2 A first look

It all begins with a category $\mathscr{C}$, which we imagine that we would like to understand. We probably would like to understand it in terms of some "related" category $\mathscr{D}$ that we understand better. Concretely, let us suppose that this "relation" comes in the form of an adjoint pair of functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift right, "R"'] & \mathscr{D} \arrow[l, shift right, "L"'] \end{tikzcd}$$

Our goal now is to use this set-up to construct another category that approximates $\mathscr{C}$, and which could be equivalent to $\mathscr{C}$ under some assumptions. Along the way, we would also like to build comparison functors between this candidate and $\mathscr{C}$ itself.

First guess

A natural candidate is $\mathrm{Im}(R) \subseteq \mathscr{D}$, the essential image of R. The good news is that this already comes with adjoint comparison functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift right, "R"'] & \mathrm{Im}(R) \arrow[l, shift right, "L"'] \arrow[r, hook] & \mathscr{D} \end{tikzcd}$$

Furthermore, R is already essentially surjective.

The bad news, of course, is that R may not be faithful or full. If it already was, then $\mathscr{C}$ was a full (coreflective) subcategory of $\mathscr{D}$ all along, and there is no better description we should look for.

Second guess

We first reckon with the thought there is no good way of "separating out" the images of the morphisms under R to render it faithful, short of quotienting $\mathscr{C}$ or in some way shrinking it. Since $\mathscr{C}$ is what we want to understand, we want to keep it as is, and so we immediately resign ourselves to taking faithfulness as more of an assumption on R rather than something that we can try to eke out from what is present.

So we instead focus on trying to correct the lack of fullness. To do this, we need to somehow "cut down" the number of morphisms between objects in $\mathrm{Im}(R)$. Usually we cut down things by imposing conditions, and conditions often arise as requirements to preserve structure (see Figure 2.1). So we ask:

Question 2.1. *Do objects in* $\mathrm{Im}(R)$ *have any special tautological structure?*

In fact, they do: the structure they acquire is that of a "module" over the endofunctor $T := RL \in \mathrm{End}(\mathscr{D})$:

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift right, "R"'] & \mathscr{D} \arrow[l, shift right, "L"'] \arrow[loop right, "T"] \end{tikzcd}$$

Before justifying the term "module," we first show that T is actually a kind of "algebra":

Lemma 2.2. *Given the adjunction* (L, R)*, the object* $T := RL$ *is a* monad*: that is, it is an associative unital algebra object of the (strict) monoidal category*

$$(\mathrm{End}(\mathscr{D}), \circ, \mathrm{Id})$$

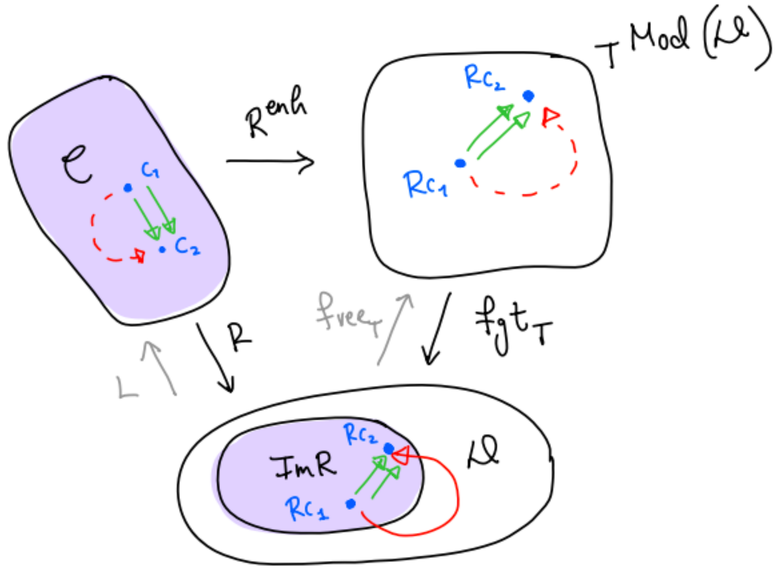


Figure 2.1: The functor R is not full, which is depicted by the presence of a red arrow in $\mathscr{D}$ that lacks a preimage in $\mathscr{C}$ (this non-existent preimage is depicted as a red dashed arrow in $\mathscr{C}$). Taking modules in $\mathscr{D}$ over the monad T, instead of just usual objects of $\mathscr{D}$, helps to correct this lack of fullness of R, because not all maps of objects in $\mathscr{D}$ are maps of T-modules in $\mathscr{D}$. The new functor R^{enh} is the "fuller" avatar of the original functor R.

Proof. The candidate "unit" map $\text{Id} \to RL$ is the unit η of the adjunction. The candidate "product" map $\mu : TT \to T$ comes from the counit ϵ via the formula

$$R(LR)L \xrightarrow{\mu := R\epsilon_{L-}} R(\text{Id})L$$

It remains to show that μ is unital and associative. This is a consequence of the two Zorro compatibility axioms between η and ϵ. □

Knowing this, we now formalize the notion of "module" structure that is present in this set-up:

Definition 2.3. *Given a monad $T \in \text{End}(\mathscr{D})$, a (left)* module *for it is an object $d \in \mathscr{D}$ together with an action morphism $Td \xrightarrow{\text{act}} d$ that satisfies unitality and associativity conditions, spelled out by the following diagrams commuting:*

$$\begin{tikzcd} TTd \arrow[r, "T\,\text{act}"] \arrow[d, "\mu_d"] & Td \arrow[d, "\text{act}"] \\ Td \arrow[r, "\text{act}"] & d \end{tikzcd}$$

and

$$\begin{tikzcd} d \arrow[r, "\eta_d"] \arrow[dr, equal, bend right, "\mathrm{Id}_d"'] & Td \arrow[d, "\mathrm{act}"] \\ & d \end{tikzcd}$$

In fact, the data and conditions of being a T-module can be tautologically organized into the following split simplicial diagram in $\mathscr{D}$,

$$\left[\cdots TTTd \begin{array}{c} \xrightarrow{TT\,\mathrm{act}} \\ \xleftarrow{TT\eta_d} \\ \xrightarrow{L\mu_d} \\ \xleftarrow{T\eta_{Td}} \\ \xrightarrow{R\epsilon_{\Omega Ld}} \end{array} TTd \begin{array}{c} \xrightarrow{T\,\mathrm{act}} \\ \xleftarrow{T\eta_d} \\ \xrightarrow{R\epsilon_{Ld}} \end{array} Td\right] \xrightarrow{\mathrm{act}} d, \qquad d \xrightarrow{\eta_d} Td, \quad Td \xrightarrow{\eta_{Td}} TTd, \quad TTd \xrightarrow{\eta_{TTd}} TTTd$$

where the face maps are bold, the degeneracy maps are dashed, and the splittings are dotted. These diagrams will become very important for understanding both proofs and applications.

Let

$${}_T\operatorname{Mod}(\mathscr{D})$$

denote the category of (left) T-modules in $\mathscr{D}$, *where the morphisms are those morphisms in $\mathscr{D}$ that are intertwined by the action maps. We use the underlined notation $\underline{d}$ for the object in ${}_T\operatorname{Mod}\mathscr{D}$, and $d := \mathrm{fgt}(\underline{d})$ for its "head" in $\mathscr{D}$.*

Not every object $d \in \mathscr{D}$ has such a structure. But:

Lemma 2.4. *Every object in* $\mathrm{Im}(R)$ *naturally acquires the structure of a T-module.*

Proof. If $d = Rc$, then $RLRc \xrightarrow{R\epsilon_c} Rc$ defines an associative unital action of T. Furthermore, R sends morphisms in $\mathscr{C}$ to morphisms of T-modules in $\mathscr{D}$. □

Thus R factors through the category of T-modules, via the functor we call the *enhanced version of* R, denoted R^{enh}:

$$\begin{tikzcd} \mathscr{C} \arrow[rr, dashed, "R^{\mathrm{enh}}"] \arrow[dr, bend right, "R"'] & & {}_T\operatorname{Mod}(\mathscr{D}) \arrow[dl, bend left, "\mathrm{fgt}_T"] \\ & \mathscr{D} & \end{tikzcd}$$

which just remembers this module structure of objects in the image, by sending $c \mapsto (Rc, TRc \xrightarrow{R\epsilon_c} Rc)$.

It is time to step back and see if we have improved things. By switching from R to R^{enh}, we have not corrected any lack of faithfulness of the original R at all; if two morphisms $f, g : c_1 \to c_2$ gave $Rf = Rg$ in $\mathscr{D}$, then they would also give the same map on the modules. It is possible that we have remedied R's lack of essential surjectivity given the above Lemma, but in fact we have not; see Example 2.37.

Statement of the theorem

The only saving grace is that the functor R^{enh} now is likelier than R to be full. Now we turn what is missing into careful assumptions, and state the main theorem. We first collect a short definition:

Definition 2.5. *A functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is* conservative *if it reflects isomorphisms. That is, if $c \xrightarrow{f} c'$ is a morphism in $\mathscr{C}$ such that $Fc \xrightarrow{Ff} Fc'$ is an isomorphism, then f must also be an isomorphism.*

Remark 2.6. *If $\mathscr{C}$ and $\mathscr{D}$ are stable categories that thus admit zero objects 0 and F is an exact functor, then F is conservative if and only if $Fc \simeq 0$ implies $c \simeq 0$. We might describe this situation by saying "F does not kill objects."*

Finally, we state the main theorem:

Theorem 2.7 (Barr-Beck Monadicity)**.** *Suppose (L, R) is an adjunction between 1-categories $\mathscr{C}, \mathscr{D}$, and $T := RL \in \mathrm{End}(\mathscr{D})$ its associated monad. If*

1. *R is conservative, and*
2. *$\mathscr{C}$ admits and R preserves the colimits[1] of R-split simplicial diagrams[2],*

then the comparison functor *R^{enh} is an equivalence of categories in the diagram below:*

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "R^{\mathrm{enh}}"] \arrow[dr, bend right, "R"'] & & {}_T\mathrm{Mod}(\mathscr{D}) \arrow[ll, dashed, bend right, "L_{\mathrm{recon}}"'] \arrow[dl, bend left, "\mathrm{fgt}_T"] \\
& \mathscr{D} \arrow[ul, bend left, "L"] \arrow[ur, bend right, "\mathrm{free}_T"'] \arrow[loop below, "T"] &
\end{tikzcd}$$

Remark 2.8. *The functor L^{recon}, a left adjoint to R^{enh}, exists as a consequence of a part of assumption (2): that $\mathscr{C}$ admits the colimits of R-split simplicial diagrams. See the proof in Lemma 2.18.*

Definition 2.9. *If R^{enh} is an equivalence, we say that R is* monadic.

We pause to summarize the main points so far:

[1]The colimit of a simplicial diagram is often called its *geometric realization*.

[2]In 1-categorical references this is often stated as "$\mathscr{C}$ admits and R preserves the coequalizers of R-split reflexive pairs." We choose the present language because (1) it is equivalent to this more standard one, and (2) it happens to be the language necessary to make the analogous ∞-categorical statement.

1. We are thinking of $\mathscr{C}$ as a category we would like to understand, and $\mathscr{D}$ as a related category that we know a lot better.

2. The *structure* of being a T-module in $\mathscr{D}$ is the descent datum that enables the underlying object in $\mathscr{D}$ to "assemble uniquely into an object of $\mathscr{C}$" under the reconstruction functor L^{recon}. This is the necessary replacement to the *property* of being in the essential image of R that corrects for the possible failure of fullness of R.

3. The content of the theorem, then, is that under some assumptions, fullness of R is "restored" by passing instead to R^{enh}, because there are fewer morphisms of T-modules than there are of just objects of $\mathscr{D}$.

Remark 2.10. *Before proceeding to some examples, let us take a moment to appreciate just how general this theorem is. It does not assume that the categories have any extra structure like a tensor product or an abelian category structure or properties like (co)completeness, and it does not assume any properties like exactness of the functors. It simply works with a completely general set-up. On the flipside, let us beware that this means that the functors* R^{enh} *and* L^{recon} *have no a priori reason to preserve any extra structure that the categories might hold and the original functors might preserve.*

In the given precise phrasing of the theorem, it is in fact an if and only if statement (see Proposition 2.22 for a proof of the converse). But the following one-way special case is more digestible, and still very useful:

Corollary 2.11 (Crude Barr-Beck)**.** *Let* (L, R) *be an adjunction as above. Then* R *is monadic if*

1. *R is conservative, and*

2. *$\mathscr{C}$ admits the colimits of simplicial diagrams and R preserves them.*

Here is the most common situation in which this version of Barr-Beck applies:

Example 2.12. *Let* (L, R) *be an adjunction where* $\mathscr{C}$ *admits the colimits of simplicial diagrams, and* R *is conservative and admits a right adjoint. So in fact* R *preserves all the colimits that* $\mathscr{C}$ *admits, and so in particular preserves the simplicial colimits. Thus* R *is monadic.*

In the following examples, the Corollary 2.11 version of Barr-Beck will suffice to show monadicity. However, in later chapters we will navigate arguments for the more minimal conditions of Theorem 2.7 in cases where the ability of R to preserve any colimits at all is very much in question.

2.3 A first example

Consider the projection $\pi : X \to Y$ from two points to one point, and the abelian categories of sheaves of k-vector spaces on these spaces for a fixed field k[3]. We ask the following descent question:

Question 2.13. *How can we monadically describe sheaves on one point in terms of sheaves on two points?*

Figure 2.2: The adjunction for the map $\pi : \mathrm{pt} \sqcup \mathrm{pt} \to \mathrm{pt}$. Note that in this simple case, $\pi^! = \pi^*$ and $\pi_! = \pi_*$.

This might sound like a silly question: after all, $\mathrm{Sh}(\mathrm{pt}) = \mathrm{Vect}_k$, so what more is there to understand? Certainly, the point of monadicity results is generally to present a complicated category in terms of modules living in a simpler one. But our goal here is simply to practice unwinding the module structure, and to then confirm with our own eyes that we are really just staring at a vector space!

To fit our set-up into our monadic framework, we examine the adjunction

$$\begin{tikzcd} \mathrm{Sh}(\mathrm{pt}) \arrow[r, bend right, "\pi^!"'] & \mathrm{Sh}(\mathrm{pt} \sqcup \mathrm{pt}) \arrow[l, bend right, "\pi_!"'] \end{tikzcd},$$

chosen so that indeed $\mathscr{C} = \mathrm{Vect}_k$. Our set-up is so simple that actually $\pi_! = \pi_*$ and $\pi^! = \pi^*$, but we will still stick to the shriek notation. This builds the diagram

[3]The results would not change if we took k to be a more general ring, and if we took derived categories of sheaves of k-modules; for the latter, some phrases in the analysis below would just have to be replaced by more categorically-savvy reasons.

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{pt}) \arrow[rr, "\pi^{!\,\mathrm{enh}}"] \arrow[dr, bend right, "\pi^!"'] & & {}_T\,\mathrm{Mod}(\mathrm{Sh}(\mathrm{pt}\sqcup\mathrm{pt})) \arrow[ll, bend right=40, dashed, "\widetilde{\pi}_!"'] \arrow[dl, bend left, "\mathrm{fgt}_T"] \\
& \mathrm{Sh}(\mathrm{pt}\sqcup\mathrm{pt}) \arrow[ul, bend right, "\pi_!"'] \arrow[ur, bend left, "\mathrm{free}_T"] &
\end{tikzcd}$$

A quick warning about notation: the category $\mathrm{Sh}(\mathrm{pt}\sqcup\mathrm{pt}) \simeq \mathrm{Sh}(\mathrm{pt}) \sqcup \mathrm{Sh}(\mathrm{pt})$ has general objects denoted by $A \sqcup B$, where A is a sheaf on one point and B is a sheaf on the other point. It also has the usual coproduct $\oplus$ of sheaves, which behaves like

$$(A_1 \sqcup B_1) \oplus (A_2 \sqcup B_2) = (A_1 \oplus A_2) \sqcup (B_1 \oplus B_2)$$

The functor $\pi^!$ in our example is monadic:

Lemma 2.14. $\mathrm{Vect}_k \simeq {}_T\,\mathrm{Mod}(\mathrm{Vect}_k \sqcup \mathrm{Vect}_k)$.

Proof. We verify the hypotheses of Theorem 2.7:

1. since $\pi^!(V \xrightarrow{f} W) = (V \sqcup V \xrightarrow{f \sqcup f} W \sqcup W)$, if $f \sqcup f$ is an isomorphism, f must have been as well. Thus $\pi^!$ is conservative;

2. since $\mathscr{C} := \mathrm{Sh}(\mathrm{pt}) = \mathrm{Vect}_k$ is abelian, it admits all finite limits and colimits, in particular the coequalizers of R-split pairs. And $\pi^!$ is exact, so in particular preserves all finite colimits.

Therefore, $\pi^!$ is monadic. $\square$

Notice that $\pi^!$, though faithful, is not itself full.

This is good. But it would be nice to know:

(1) what is the monad (T, μ_T, η)?

(2) what does ${}_T\,\mathrm{Mod}(\mathrm{Vect}_k \sqcup \mathrm{Vect}_k)$ look like?

(3) what is the reconstruction functor ${\pi_!}^{\mathrm{recon}}$?

We look at each part separately:

Part (1): understand the monad. On objects, our functors are $\pi^!(V) = V \sqcup V$ and $\pi_!(A \sqcup B) = A \oplus B$, and so the composite $T := \pi^!\pi_!$ has the effect

$$T(A \sqcup B) = (A \oplus B) \sqcup (A \oplus B).$$

The unit $\eta : \mathrm{Id} \to \pi^!\pi_!$ looks like

$$A \sqcup B \xrightarrow{\eta_{A\sqcup B} := 1_A \oplus 0 \sqcup 0 \oplus 1_B} (A \oplus B) \sqcup (A \oplus B)$$

while the counit $\epsilon : \pi_!\pi^! \to \mathrm{Id}$ looks like

$$\epsilon_V : V \oplus V \xrightarrow{+} V.$$

We also need to know the product map $(\mu_T)_{A\sqcup B} := \pi^!\epsilon_{\pi_!(A\sqcup B)}$. Well, we know $\epsilon_{\pi_!(A\sqcup B)}$ is just addition

$$(A \oplus B)^2 \xrightarrow{+} A \oplus B,$$

and so the product map is pointwise addition:

$$(A \oplus B)^2 \sqcup (A \oplus B)^2 \xrightarrow{+\sqcup+} (A \oplus B) \sqcup (A \oplus B).$$

Guess 2.15. *Before honestly getting to parts (2) and (3), let us make a guess: what* should *the equivalence be? Well, the essential image of $\pi^!$ is those objects $V \sqcup W$ where $V \cong W$, so T-modules are in particular such objects. Maybe we can even expect that the* structure *of being a T-module is exactly a choice of such an isomorphism. If this is true, then the reconstruction functor $\pi_!^{\mathrm{recon}}$ should be the act of "taking half" of a T-module. Indeed, this will turn out to be the case!*

Part (2): understand T-modules A T-module is an object X together with (1) the data of an action map $TX \xrightarrow{\alpha} X$, subject to the conditions that (2) α has η_X as a section, and (3) α fits into a commutative square certifying associativity:

$$\begin{tikzcd} TTX \arrow[r, "T\alpha"'] \arrow[d, "(\mu_T)_X"'] & TX \arrow[d, "\alpha"] \arrow[l, bend right, dashed, "T\eta_X"'] \\ TX \arrow[r, "\alpha"] & X \arrow[u, bend right, dashed, "\eta_X"'] \end{tikzcd}$$

Here, the dashed lines denote sections, and are just included here for flavor.

So, what do the α's look like? They are maps

$$T(A \sqcup B) = (A \oplus B) \sqcup (A \oplus B) \xrightarrow{\alpha = \alpha_+ \sqcup \alpha_-} A \sqcup B,$$

with section $\eta_{A\sqcup B}$. This section condition shows that $\alpha_+|_A = \mathrm{Id}_A, \alpha_-|_B = \mathrm{Id}_B$. Thus, α is determined by two maps $\alpha_B : B \to A$ and $\alpha_A : A \to B$, fitting into α as

$$\alpha := \alpha_+ \sqcup \alpha_- = (\mathrm{Id}_A + \alpha_B) \sqcup (\alpha_A + \mathrm{Id}_B) \quad : \quad (A \oplus B) \sqcup (A \oplus B) \to A \sqcup B.$$

The only conditions they must satisfy come from the associativity square above. By our informal reasoning above, we are looking for associativity to simply yield the conditions "α_A and α_B are inverses."

Does it? Time to really dive into it. The only remaining mystery in the associativity diagram is what $T\alpha$ is in this example, and it is

$$(A \oplus B)^2 \sqcup (A \oplus B)^2 \xrightarrow{T\alpha = (\alpha_+ \oplus \alpha_-) \sqcup (\alpha_+ \oplus \alpha_-)} (A \oplus B) \sqcup (A \oplus B),$$

where we remember that $\alpha_+ : A \oplus B \to A$, and $\alpha_- : A \oplus B \to B$. We follow an element

$$(a_1 + b_1 + a_1' + b_1') \sqcup (a_2 + b_2 + a_2' + b_2')$$

along both composites to $A \sqcup B$. The northeast path gives

$$\begin{aligned}
&\alpha\Big((a_1 + \alpha_B b_1) \oplus (\alpha_A a_1' + b_1') \sqcup (a_2 + \alpha_B b_2) \oplus (\alpha_A a_2' + b_2')\Big)\\
=&\big(a_1 + \alpha_B b_1 + \alpha_B \alpha_A a_1' + \alpha_B b_1'\big) \sqcup \big(\alpha_A a_2 + \alpha_A \alpha_B b_2 + \alpha_A a_2' + b_2'\big)
\end{aligned}$$

while the southwest path gives

$$\begin{aligned}
&\alpha\Big((a_1 + a_1') \oplus (b_1 + b_1') \sqcup (a_2 + a_2') \oplus (b_2 + b_2')\Big)\\
=&(a_1 + a_1' + \alpha_B b_1 + \alpha_B b_1') \sqcup (\alpha_A a_2 + \alpha_A a_2' + b_2 + b_2').
\end{aligned}$$

Comparing terms, we see that associativity holds if and only if $\alpha_A \alpha_B = \mathrm{Id}_B$ and $\alpha_B \alpha_A = \mathrm{Id}_A$. Thus, a T-module in $\mathrm{Sh}(k_X)$ is precisely a pair

$$(A \sqcup B, \phi : A \xrightarrow{\cong} B).$$

Part (3): calculate $\pi_!^{\mathrm{recon}}$ Let us take a T-module $X := (A \sqcup B, \phi : A \xrightarrow{\cong} B)$ and compute $\pi_!^{\mathrm{recon}}(X)$. We should be expecting this to just be A.

Filling out the defining diagram gives

$$\begin{tikzcd} LTX \arrow[r, bend left, "L\alpha"] \arrow[r, bend right, "\epsilon_{LX}"'] & LX \end{tikzcd} \quad = \quad \begin{tikzcd} (A \oplus B) \oplus (A \oplus B) \arrow[r, bend left, "(\mathrm{Id}_A + \phi^{-1}) \oplus (\phi + \mathrm{Id}_B)"] \arrow[r, bend right, "+"'] & A \oplus B \end{tikzcd}$$

The coequalizer of the above is the same as the cokernel of the difference of these two maps. This difference sends

$$\begin{aligned}
(a_1 + b_1) \oplus (a_2 + b_2) &\mapsto (a_1 + \phi^{-1} b_1 - (a_1 + a_2)) \oplus (\phi a_2 + b_2 - (b_1 + b_2))\\
&= (\phi^{-1} b_1 - a_2) \oplus (\phi a_2 - b_1)\\
&= c \oplus -\phi c
\end{aligned}$$

for $c := \phi^{-1}b_1 - a_2$. Thus the image is the graph $\Gamma_\phi = \Gamma_{\phi^{-1}} \subseteq A \oplus B$, and so

$$\tilde{\pi}^!(A \sqcup B, \phi) := (A \oplus B)/\Gamma_\phi$$

which is a very choice-free way of saying "take half of the T-module"—as expected!

2.4 A second example

We used the above adjunction $(\pi_!, \pi^!)$ to monadically describe $\mathrm{Sh}(\mathrm{pt})$ in terms of $\mathrm{Sh}(\mathrm{pt} \sqcup \mathrm{pt})$. Can we now turn the tables, and monadically describe $\mathrm{Sh}(\mathrm{pt} \sqcup \mathrm{pt})$ in terms of $\mathrm{Sh}(\mathrm{pt})$?

Indeed, we can. For this, we shift to the adjunction (π^*, π_*), with $T := \pi_*\pi^*$, and build the diagram

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{pt} \sqcup \mathrm{pt}) \arrow[rr, "\pi_*^{\mathrm{enh}}"] \arrow[dr, bend right, "\pi_*"'] & & {}_T\,\mathrm{Mod}(\mathrm{Sh}(\mathrm{pt})) \arrow[ll, bend right, dashed, "\widetilde{\pi}^*"'] \arrow[dl, bend left, "\mathrm{fgt}_T"] \\
& \mathrm{Sh}(\mathrm{pt}) \arrow[ul, bend right, "\pi^*"'] \arrow[ur, bend left, "\mathrm{free}_T"] &
\end{tikzcd}$$

It is equally easy to verify the hypotheses of Theorem 2.7 to see that π_* is also monadic: $\pi_*(A \sqcup B) := A \oplus B$ is conservative, and exact. But again, we would like to unwind the meaning of the module category, and of the reconstruction functor. In this case, the maps are a bit easier:

Part (1): understand the monad. The endofunctor T is

$$T(V \xrightarrow{f} W) := \left(V^{\oplus 2} \xrightarrow{\begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}} W^{\oplus 2} \right)$$

The unit and counit are

$$\begin{aligned}
\mathrm{Id} \xrightarrow{\eta} \pi_*\pi^* : \qquad & V \xrightarrow{\Delta} V \oplus V \quad \text{the diagonal map,} \\
\pi^*\pi_* \xrightarrow{\epsilon} \mathrm{Id} : \qquad & (A \oplus B) \sqcup (A \oplus B) \xrightarrow{\pi_A \sqcup \pi_B} A \sqcup B \quad \text{the projections}
\end{aligned}$$

and the product map is

$$V^{\oplus 4} \xrightarrow{\mu_T|_V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} V \oplus V$$

Part (2): understand T-modules. A T-module is a vector space V together with an action map $V \oplus V \xrightarrow{\alpha=[\alpha_1\,\alpha_2]} V$, where $\alpha_i : V \to V$, which is unital and satisfies associativity. The unitality says that $\alpha_1 + \alpha_2 = \mathrm{Id}$. The associativity condition

$$\begin{tikzcd} V^{\oplus 4} \arrow[r, "\alpha\oplus\alpha"] \arrow[d, "\mu_T|_V"'] & V^{\oplus 2} \arrow[d, "\alpha"] \\ V^{\oplus 2} \arrow[r, "\alpha"] & V \end{tikzcd}$$

says that the northeast path

$$\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\alpha_1 & \alpha_2 & 0 & 0\\ 0 & 0 & \alpha_1 & \alpha_2\end{bmatrix} = \begin{bmatrix}\alpha_1^2 & \alpha_1\alpha_2 & 0 & 0\\ 0 & 0 & \alpha_2\alpha_1 & \alpha_2^2\end{bmatrix}$$

is equal to the southwest path

$$\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 0 & 0 & \alpha_2\end{bmatrix}$$

and so we see that a T-module is precisely the data $(V, \mathrm{Id} = \alpha_1 + \alpha_2)$ of a vector space together with a decomposition of the identity into two commuting idempotents.

Part (3): calculate $\pi^{*\mathrm{recon}}$. This functor is defined to be the coequalizer of

$$\begin{tikzcd} V^{\oplus 2} \sqcup V^{\oplus 2} \arrow[r, bend left, "\alpha\sqcup\alpha"] \arrow[r, bend right, "\pi_1\sqcup\pi_2"'] & V \sqcup V \end{tikzcd}$$

In other words,

$$\pi^{*\mathrm{recon}}(V, \alpha_1, \alpha_2) := \mathrm{Coker}(\alpha - \pi_1) \sqcup \mathrm{Coker}(\alpha - \pi_2)$$

Since

$$(\alpha - \pi_1)(v, w) = \alpha_1 v + \alpha_2 w - \underbrace{v}_{\alpha_1 v + \alpha_2 v} = \alpha_2(w - v),$$

we conclude that $\mathrm{Coker}(\alpha - \pi_1) = V/\operatorname{Im}(\alpha_2) = \ker(\alpha_1)$, and so

$$\pi^{*\mathrm{recon}}(V, \alpha_1, \alpha_2) = \ker(\alpha_1) \sqcup \ker(\alpha_2).$$

In summary, we have learned that monadicity for this adjunction is the statement that the category of pairs of vector spaces is equivalent to the category of vector spaces with the structure of a decomposition into a direct sum of two subspaces.

We will return to looking at more examples soon. But now that we have some practice unwinding the structure of modules over monads, we turn to justifying the monadicity theorem.

2.5 Proof of Barr-Beck monadicity

The standard monadic phrasing of Barr-Beck is Theorem 2.7. In this section, we also phrase a second monadicity theorem (which we have not seen stated or proven in the literature), and prove both versions concretely in the 1-categorical context; we hope that an ∞-yogi can give us the correct incantations to say that would lift the given proofs to that context as well. Both versions, in their ∞-analogs, will be useful for the chapters that follow.

Editorial 2.16. *Before beginning, I wish to say, earnestly but likely to the great frustration of a reader who is about to see pages of diagrams, that this is all meant to be understandable, and actually very simple.*

Barr-Beck is a beautifully well-organized tautology, and this can be seen from multiple angles, all of which will be useful. The details that follow are provided for concreteness, but the spirit and strategy—I continue to hope—can be felt intuitively. Any progress I made on a descent problem came as a result of getting a better understanding of the proof of Barr-Beck. If that anecdote is any guide, then I think understanding its proof would be worth it for anyone looking to use the theorem statement.

Start with the adjunction

$$\begin{tikzcd} C \arrow[loop left, "\Omega"] \arrow[r, shift right, "R"'] & D \arrow[l, shift right, "L"'] \arrow[loop right, "T"] \end{tikzcd}$$

decorated with the associated monad and comonad. The preliminary observation to the theorem, made above, is that R lifts to a functor R^{enh} that lands in T-modules:

$$\begin{tikzcd} C \arrow[rr, shift right, "R^{\mathrm{enh}}"'] \arrow[dr, "R"'] & & {}_T\operatorname{Mod} D \arrow[ll, dashed, shift right, "L^{\mathrm{recon}}"'] \arrow[dl, "\mathrm{fgt}"] \\ & D & \end{tikzcd}$$

We wish to study the question of when R^{enh} is an equivalence, i.e. when R is monadic.

Strategy 2.17. *Here is a blueprint for examining when R^{enh} is an equivalence:*

1. *See when a left adjoint L^{recon} exists.*

2. *Once it does, C admits a second comonad, $\Omega^{\mathrm{enh}} := L^{\mathrm{recon}} \circ R^{\mathrm{enh}}$, in addition to $\Omega := L \circ R$, and ${}_T\operatorname{Mod} D$ admits a second monad, $T^{\mathrm{enh}} := R^{\mathrm{enh}} \circ L^{\mathrm{recon}}$, in addition to* $\mathrm{free} \circ \mathrm{fgt}$.

3. *Apply Lemma 1.10 to conclude that the adjoint functors $(L^{\mathrm{recon}}, R^{\mathrm{enh}})$ mediate an equivalence on full subcategories*

$$\begin{tikzcd}
& C^{L^{\mathrm{recon}}\circ R^{\mathrm{enh}}} \arrow[d, hook] \arrow[rr, hook, shift right, "\sim"'] & & \left({}_T\operatorname{Mod} D\right)^{R^{\mathrm{enh}}\circ L^{\mathrm{recon}}} \arrow[d, hook] \arrow[ll, hook', shift right] & \\
L^{\mathrm{recon}}\circ R^{\mathrm{enh}} & C \arrow[loop left] \arrow[rr, shift right, "R^{\mathrm{enh}}"'] \arrow[dr, bend right, "R"'] & & {}_T\operatorname{Mod} D \arrow[ll, shift right, "L^{\mathrm{recon}}"'] \arrow[loop right] \arrow[dl, bend left, "\mathrm{fgt}"] & R^{\mathrm{enh}}\circ L^{\mathrm{recon}} \\
& & D & &
\end{tikzcd}$$

4. *Identify these subcategories more concretely, and look for conditions under which they coincide with C and ${}_T\operatorname{Mod} D$, respectively.*

Here is the workhorse result:

Lemma 2.18. *A pursuit of the above strategy leads to the following results:*

1. *L^{recon} exists, and is a left adjoint to R^{enh}, if C admits the colimits of R-split simplicial diagrams;*

2. *the subcategory $C^{L^{\mathrm{recon}}\circ R^{\mathrm{enh}}}$ is the full subcategory on objects of C that are the colimits of their Ω-bar diagrams:*

$$\underset{\longrightarrow}{\operatorname{colim}}\left[\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \Omega\Omega c \substack{\longrightarrow\\ \longrightarrow} \Omega c\right] \xrightarrow{\simeq} c$$

3. *the subcategory $\left({}_T\operatorname{Mod} D\right)^{R^{\mathrm{enh}}\circ L^{\mathrm{recon}}}$ is the full subcategory on modules $\underline{d} \in {}_T\operatorname{Mod} D$ whose L-image yields R-split simplicial diagrams whose colimit R preserves.*

Proof. For part (1), let $\underline{d} \in {}_T\operatorname{Mod} D$ be a module, with $\mathrm{fgt}(\underline{d}) =: d$, which determines the following diagram in D[4] (dashed arrows will denote degeneracies):

$$\left[\begin{tikzcd}[ampersand replacement=\&]
\cdots TTd \arrow[r, shift left=2, "T\,\mathrm{act}"] \arrow[r, shift right=2, "\mu_d"'] \& Td \arrow[l, dashed, "T\eta_d" description] \arrow[l, dotted, bend left=60, "\eta_{Td}"] \arrow[r, "\mathrm{act}"] \& d \arrow[l, dotted, bend left=40, "\eta_d"]
\end{tikzcd}\right]$$

The purpose of the brackets is to delineate a chunk of diagram that will migrate through applications of multiple functors. Applying L and adding in counits $\epsilon_{\Omega^\bullet Ld}$ builds the simplicial

[4]The solid arrows represent morphisms in ${}_T\operatorname{Mod} D$, where the structures on the modules $\underline{T^\bullet d}$, for $\bullet = 1, 2, \ldots$, are the free module structures. The dotted arrows are morphisms only in D.

diagram in C

$$\begin{tikzcd}[column sep=huge]
\Big[\cdots \underbrace{LTTd}_{\Omega\Omega Ld} \arrow[r, shift left=4, "LT\,\mathrm{act}"] \arrow[r, leftarrow, dashed, shift left=2, "LT\eta_d" description] \arrow[r, "L\mu_d" description] \arrow[r, leftarrow, dashed, shift right=2, "L\eta_{Td}" description] \arrow[r, shift right=4, "\epsilon_{\Omega Ld}"'] & \underbrace{LTd}_{\Omega Ld} \arrow[r, shift left=2, "L\,\mathrm{act}"] \arrow[r, leftarrow, dashed, "L\eta_d" description] \arrow[r, shift right=2, "\epsilon_{Ld}"'] & Ld\Big]
\end{tikzcd}$$

The colimit of this diagram, were it to exist, should and would be the definition of $L^{\mathrm{recon}}(\underline{d})$. This diagram is well-structured in the sense that it is an example of an R-split simplicial diagram, because applying R gives exactly the bold part of the starting diagram that was determined by $\underline{d}$:

$$\begin{tikzcd}[column sep=huge]
\Big[\cdots TTTd \arrow[r, shift left=4, "TT\,\mathrm{act}"] \arrow[r, leftarrow, dashed, shift left=2, "TT\eta_d" description] \arrow[r, "L\mu_d" description] \arrow[r, leftarrow, dashed, shift right=2, "T\eta_{Td}" description] \arrow[r, shift right=4, "R\epsilon_{\Omega Ld}"'] & TTd \arrow[r, shift left=2, "T\,\mathrm{act}"] \arrow[r, leftarrow, dashed, "T\eta_d" description] \arrow[r, shift right=2, "R\epsilon_{Ld}"'] \arrow[l, dotted, bend left=60, "\eta_{Td}"] & Td\Big] \arrow[r, dotted, "\mathrm{act}"] \arrow[l, dotted, bend left=50, "\eta_{Td}"] & d \arrow[l, dotted, bend left=40, "\eta_d"]
\end{tikzcd}$$

where we have dotted the new arrows that are available to the R-image. Thus, as long as C admits the colimits of R-split simplicial diagrams, L^{recon} is well-defined. Once it is well-defined, it automatically becomes the left adjoint to R^{enh} because the

1. counit of the adjunction $(L^{\mathrm{recon}}, R^{\mathrm{enh}})$ is the natural map out of a colimit, induced by the counit ϵ_c of the adjunction (L, R) on the last term of the simplicial diagram:

$$\begin{tikzcd}[column sep=large]
 & \Big[\cdots \Omega^2 c \arrow[r, shift left=2, "\Omega\epsilon_c"] \arrow[r, leftarrow, dashed, "L\eta_{Rc}" description] \arrow[r, shift right=2, "\epsilon_{\Omega c}"'] & \Omega c\Big] \arrow[r, "\epsilon_c"] \arrow[d] & c \\
L^{\mathrm{recon}} \circ R^{\mathrm{enh}}(c) \arrow[rr, equal] & & \varinjlim_{\Delta^{\mathrm{op}}} \Omega^{\bullet} c \arrow[ur, dashed, "\mathrm{can}"'] &
\end{tikzcd}$$

2. the unit of the adjunction $(L^{\mathrm{recon}}, R^{\mathrm{enh}})$ is gotten by first applying R to the colimit diagram

$$\begin{tikzcd}[column sep=large]
\Big[\cdots LTd \arrow[r, shift left=2, "L\,\mathrm{act}"] \arrow[r, leftarrow, dashed, "L\eta_d" description] \arrow[r, shift right=2, "\epsilon_{Ld}"'] & Ld\Big] \arrow[r, "\mathrm{can}"] \arrow[d, Rightarrow] & L^{\mathrm{recon}}(\underline{d}) \\
\Big[\cdots TTd \arrow[r, shift left=2, "T\,\mathrm{act}"] \arrow[r, leftarrow, dashed, "T\eta_d" description] \arrow[r, shift right=2, "R\epsilon_{Ld}"'] & Td\Big] \arrow[r, "R\,\mathrm{can}"] \arrow[d, shift right, "\mathrm{act}"'] & R \circ L^{\mathrm{recon}}(\underline{d}) \\
 & d \arrow[u, dotted, shift right, "\eta_d"'] \arrow[ur, bend right, "R\,\mathrm{can} \circ \eta_d"'] &
\end{tikzcd}$$

and then taking the map induced by the splitting to $R \circ L^{\mathrm{recon}}(\underline{d})$. One should check that this upgrades to a morphism of T-modules

$$\underline{d} \xrightarrow{R\,\mathrm{can}\,\circ \eta_d} R^{\mathrm{enh}} \circ L^{\mathrm{recon}}(\underline{d}),$$

yielding the unit, and that the Zorro compatibility axioms also hold.

For part (2), we simply note by a diagram above that the counit $L^{\mathrm{recon}} \circ R^{\mathrm{enh}}(c) \to c$ being an equivalence is precisely the statement that the Ω-bar resolution of c is a colimit diagram.

For part (3), observe that $\underline{d}$ belongs to this subcategory if and only if morphism

$$Td \xrightarrow{R\,\mathrm{can}} R \circ L^{\mathrm{recon}}(\underline{d})$$

is an isomorphism, which happens to be true if and only if R preserves the colimit of the diagram

$$\left[\begin{tikzcd} \cdots LTTd \arrow[r, shift left=4, "LT\,\mathrm{act}"] \arrow[r, dashed, leftarrow, shift left=2, "LT\eta_d"] \arrow[r, "L\mu_d"] \arrow[r, dashed, leftarrow, shift right=2, "L\eta_{Td}"] \arrow[r, shift right=4, "\epsilon_{\Omega Ld}"] & LTd \arrow[r, shift left=2, "L\,\mathrm{act}"] \arrow[r, dashed, leftarrow, "L\eta_d"] \arrow[r, shift right=2, "\epsilon_{Ld}"] & Ld \end{tikzcd}\right]$$

which is the statement of the claim.

□

This gives the immediate corollary:

Corollary 2.19. *Given an adjunction (L, R) such that L^{recon} exists,*

1. *R^{enh} is an embedding if and only if each object $c \in C$ is the colimit of its Ω-bar diagram;*
2. *L^{recon} is an embedding if R preserves the colimits of simplicial diagrams that it splits.*

Two statements of Barr-Beck

As per the outlined strategy, there are actually two statements of Barr-Beck. We reproduce the basic diagram

$$\begin{tikzcd} \Omega{:=}LR \circlearrowright \mathscr{C} \arrow[rr, "R^{\mathrm{enh}}"] \arrow[dr, bend right, "R"'] & & {}_T\mathrm{Mod}(\mathscr{D}) \arrow[ll, dashed, bend right, "L^{\mathrm{recon}}"'] \arrow[dl, bend left, "\mathrm{fgt}_T"] \\ & \mathscr{D} \arrow[ul, bend left, "L"] \arrow[ur, bend right, "\mathrm{free}_T"'] \arrow[loop below, "T{:=}RL"] & \end{tikzcd}$$

and start with the most common statement, which already appeared as Theorem 2.7:

Theorem 2.20 (Barr-Beck, version 1). *Suppose that*

1. *R is conservative; and*

2. *C admits and R preserves the colimits of R-split simplicial diagrams.*

Then R is monadic.

Proof. The assumption that C admits the colimits of R-split simplicial diagrams ensures that L^{recon} exists. From then on, the argument proceeds by assembling pieces from the discussion above:

1. R^{enh} is conservative: given that fgt is conservative and $R = \text{fgt} \circ R^{\text{enh}}$, this is true if and only if R is conservative, which is assumed;

2. L^{recon} is an embedding: this holds by assumption, via Corollary 2.19.

We conclude by Lemma 1.11. For another proof, see Lecture 3 of [2]. $\square$

Here is a less common statement, which will also be useful to us:

Theorem 2.21 (Barr-Beck, version 2). *Suppose that*

1. *C admits the colimits of R-split simplicial diagrams (and therefore L^{recon} exists);*

2. *L^{recon} is conservative; and*

3. *every Ω-bar diagram in C is a colimit diagram.*

Then R is monadic.

Proof. As before, the assumption that C admits the colimits of R-split simplicial diagrams is necessary to ensure that L^{recon} exists. But now again, the argument is simple:

1. L^{recon} is conservative: this is simply assumed;

2. R^{enh} is an embedding: by Corollary 2.19, this is equivalent to the assumption that every Ω-bar diagram in C is a colimit diagram.

We again conclude by Lemma 1.11. $\square$

Barr-Beck is an if and only if statement

As mentioned earlier, Barr-Beck is an if and only if statement:

Proposition 2.22. *Theorem 2.7 is an if and only if statement.*

Proof. Suppose that $(L^{\mathrm{recon}}, R^{\mathrm{enh}})$ are inverse equivalences. We need to show three things:

R is conservative. This is the easiest. Since R is the composite $R = \mathrm{fgt} \circ R^{\mathrm{enh}}$ of conservative functors, R itself is conservative.

$\mathscr{C}$ admits the colimits of R-split simplicial diagrams. We already know that $\mathscr{C}$ admits some such colimits:

1. For L^{recon} to be defined, $\mathscr{C}$ must already admit those colimits coming from L-images of diagrams defining T-module structures;

2. Since R^{enh} is an embedding, Corollary 2.19 says that Ω-bars on $c \in \mathscr{C}$ are all colimit diagrams. These diagrams are R-split.

We wish to use this to show that in fact any R-split simplicial diagram admits a colimit. So take $\Delta^{\mathrm{op}} \xrightarrow{c_\bullet} \mathscr{C}$ a simplicial diagram

$$\left[\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} c_1 \rightrightarrows c_0\right]$$

which is R-split, and look at its split R-image

$$\begin{tikzcd} \Big[\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Rc_1 \arrow[l, bend left] \arrow[r, shift left] \arrow[r, shift right] & Rc_0\Big] \arrow[l, bend left] \arrow[r, "\theta"] & d \arrow[l, bend left] \end{tikzcd}$$

Our strategy will be to show that (1) the object $d \in \mathscr{D}$ inherits the structure of a T-module $\underline{d}$, and (2) the colimit of $c_\bullet$ is $L^{\mathrm{recon}}(\underline{d})$. We now spell this out:

1. *The object $d \in \mathscr{D}$ inherits a T-module structure.* Iterate T on the diagram

$$\left[\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} Rc_1 \rightrightarrows Rc_0\right] \xrightarrow{\theta} d$$

and use the fact that d is a colimit to build the (bi?)augmented bisimplicial diagram

$$\begin{tikzcd}
\left[\ddots\right. \arrow[d, shift left=2] \arrow[d, shift left=0.7] \arrow[d, shift right=0.7] \arrow[d, shift right=2] & \vdots \arrow[d, shift left=2] \arrow[d, shift left=0.7] \arrow[d, shift right=0.7] \arrow[d, shift right=2] & \left.\vdots\right] \arrow[d, shift left=2] \arrow[d, shift left=0.7] \arrow[d, shift right=0.7] \arrow[d, shift right=2] & \vdots \arrow[d, shift left=2] \arrow[d, shift left=0.7] \arrow[d, shift right=0.7] \arrow[d, shift right=2] \\
\left[\cdots\right. \arrow[r, shift left=2] \arrow[r, shift left=0.7] \arrow[r, shift right=0.7] \arrow[r, shift right=2] \arrow[d, shift left=1] \arrow[d] \arrow[d, shift right=1] & R\Omega^2 c_1 \arrow[r, shift left=1] \arrow[r] \arrow[r, shift right=1] \arrow[d, shift left=1] \arrow[d] \arrow[d, shift right=1] & \left.R\Omega^2 c_0\right] \arrow[r, "T^2\theta"] \arrow[d, shift left=1] \arrow[d] \arrow[d, shift right=1] & T^2 d \arrow[d, shift left=1] \arrow[d] \arrow[d, shift right=1] \\
\left[\cdots\right. \arrow[r, shift left=1.5] \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] \arrow[r, shift right=1.5] \arrow[d, shift left=0.7] \arrow[d, shift right=0.7] & R\Omega c_1 \arrow[r, shift left=1] \arrow[r] \arrow[r, shift right=1] \arrow[d, shift left=0.7] \arrow[d, shift right=0.7] & \left.R\Omega c_0\right] \arrow[r, "T\theta"] \arrow[d, shift left=0.7] \arrow[d, shift right=0.7] & Td \arrow[d, "\mathrm{can}"] \\
\left[\cdots\right. \arrow[r, shift left=1.5] \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] \arrow[r, shift right=1.5] \arrow[d] & R c_1 \arrow[r, shift left=1] \arrow[r] \arrow[r, shift right=1] \arrow[d] & \left.R c_0\right] \arrow[r, "\theta"] \arrow[d] & d
\end{tikzcd}$$

where the rows are (split) colimit diagrams by the R-split hypothesis, and the columns are (split) colimit diagrams by the hypothesis that Ω-bars are colimit diagrams. It remains to check that the right-most column admits $\{\eta_{T^{\star+1}d}\}$ as a splitting. But these are inherited through the functoriality of the colimits $T^{\star+1}d$ from the splittings $\{\eta_{R\Omega^{\star+1}c_\bullet}\}$ on the columns. Thus d admits a canonical T-module structure $\underline{d} := (Td \xrightarrow{\mathrm{can}} d)$ which completes the R^{enh} image

$$\left[\cdots \rightrightarrows \underline{Rc_1} \rightrightarrows \underline{Rc_0}\right] \xrightarrow{\theta} \underline{d}$$

in ${}_T\operatorname{Mod}\mathscr{D}$ to a colimit diagram.

2. *The colimit of $c_\bullet$ is $L^{\mathrm{recon}}(\underline{d})$.* Consider the bisimplicial diagram in $\mathscr{C}$

$$\begin{tikzcd}
\Big[\ \ddots & \vdots \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] & \vdots\ \Big] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & & \vdots \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] \\
\Big[\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left=1] \arrow[d, shift right=1] & \Omega^2 c_1 \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \Omega^2 c_0 \Big] \arrow[rr] \arrow[d, shift left=1] \arrow[d, shift right=1] & & LTd \arrow[d, shift left=2] \arrow[d, "L\,\mathrm{can}"'] \arrow[d, shift right=2] \\
\Big[\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \Omega c_1 \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d] & \Omega c_0 \Big] \arrow[rr] \arrow[d] & & Ld \arrow[d] \\
\Big[\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & c_1 \arrow[r, shift left=1] \arrow[r, shift right=1] & c_0 \Big] \arrow[rr, dashed, "\theta"] & & L^{\mathrm{recon}}(\underline{d})
\end{tikzcd}$$

All rows but the bottom row are (split) colimit diagrams, and all columns are colimit diagrams (the right-most one by the definition of L^{recon}). The morphism θ making the bottom-right square commute is canonical, being defined by the existence of splittings of the columns.

We now argue that the bottom row is a colimit diagram by considering any given $c \in \mathscr{C}$, together with the corresponding constant simplicial diagram $\Delta^{\mathrm{op}} \xrightarrow{c_{\mathrm{cst}}} \mathscr{C}$ that c determines, and calculating:

$$\begin{aligned}
\hom_{\Delta^{\mathrm{op}}_\bullet}(c_\bullet, c_{\mathrm{cst}}) \quad &= \quad \hom_{\Delta^{\mathrm{op}}_\bullet}(\varinjlim_{\Delta^{\mathrm{op}}_\star} \Omega^{\star+1} c_\bullet, c_{\mathrm{cst}}) \quad = \quad \varprojlim_{\Delta^{\mathrm{op}}_\star} \hom_{\Delta^{\mathrm{op}}_\bullet}(\Omega^{\star+1} c_\bullet, c_{\mathrm{cst}}) \\
= \quad \varprojlim_{\Delta^{\mathrm{op}}_\star} \hom_{\mathscr{C}}(L^{\star+1} d, c_{\mathrm{cst}}) \quad &= \quad \hom_{\mathscr{C}}(\varinjlim_{\Delta^{\mathrm{op}}_\star} L^{\star+1} d, c) \quad = \quad \hom_{\mathscr{C}}(L^{\mathrm{recon}}(\underline{d}), c)
\end{aligned}$$

Thus, $L^{\mathrm{recon}}(\underline{d})$ is the colimit of the diagram $c_\bullet$.

This concludes the proof that $\mathscr{C}$ admits the colimits of R-split simplicial diagrams.

R preserves geometric realizations of R-split simplicial diagrams. Finally, we must show that R preserves the colimit diagram

$$\begin{tikzcd}
\Big[\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & c_1 \arrow[r, shift left=1] \arrow[r, shift right=1] & c_0 \Big] \arrow[r, "\theta"] & L^{\mathrm{recon}}(\underline{d})
\end{tikzcd}$$

This is now immediate given the fact that $R \circ L^{\mathrm{recon}}(\underline{d}) = d$, because the R-image of this is therefore the assumed (split) colimit diagram

$$\begin{tikzcd}
\Big[\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Rc_1 \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[l, bend left] & Rc_0 \Big] \arrow[r, "\theta"] \arrow[l, bend left] & \underbrace{R \circ L^{\mathrm{recon}}(\underline{d})}_{\simeq d} \arrow[l, bend left]
\end{tikzcd}$$

We have therefore seen that the assumption that R^{enh} is an equivalence implies the assumptions of Barr-Beck. This concludes our proof of Proposition 2.22. ☐

As a corollary, we also get:

Corollary 2.23. *Theorem 2.21 is also an if and only if statement.*

Proof. If R is monadic, then certainly (2) and (3) are true. And (1) follows by Proposition 2.22. ☐

Example 2.24. *We briefly pause to use this information to make a simple deduction about a tautological example:* ${}_T\mathrm{Mod}\,\mathscr{D} \xrightarrow{\mathrm{fgt}} \mathscr{D}$*, yielding the diagram*

$$\begin{tikzcd}
{}_T\mathrm{Mod}\,\mathscr{D} \arrow[loop left, "\Omega:=\mathrm{free}\circ\mathrm{fgt}"] \arrow[rr, shift right, "\mathrm{fgt}^{\mathrm{enh}}"'] \arrow[dr, "\mathrm{fgt}"'] & & {}_T\mathrm{Mod}\,\mathscr{D} \arrow[ll, shift right, "\mathrm{free}^{\mathrm{recon}}"', "\sim"] \arrow[dl, "\mathrm{fgt}"] \\
& \mathscr{D} \arrow[loop below, "T\simeq\mathrm{fgt}\circ\mathrm{free}"] &
\end{tikzcd}$$

It is tautologically monadic, since $\mathrm{fgt}^{\mathrm{enh}} = \mathrm{Id}$ *is an equivalence. The implication of Corollary 2.23 is the fact that the* Ω*-bars on* $\underline{d} \in \mathscr{D}$*,*

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3, "T\,\mathrm{act}"] \arrow[r, shift right=3, "\mu_d"'] & \underline{Td} \arrow[l, dashed, "T\eta_d"'] \arrow[r, "\mathrm{act}"] & \underline{d}
\end{tikzcd}$$

i.e. the "resolutions" by free modules $\underline{T^\bullet d}$*, are actually resolutions, i.e. colimit diagrams. Note that these bars are* fgt*-split:*

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3, "T\,\mathrm{act}"] \arrow[r, shift right=3, "\mu_d"'] & Td \arrow[l, dashed, "T\eta_d"'] \arrow[r, "\mathrm{act}"] \arrow[l, dotted, bend left=50, "\eta_{Td}"] & d \arrow[l, dashed, bend left=40, "\eta_d"]
\end{tikzcd}$$

Similarly, inside the full subcategory ${}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D} \hookrightarrow {}_T\mathrm{Mod}\,\mathscr{D}$ *of free* T*-modules,* Ω*-cobars are also colimit diagrams.*

Statement of comonadicity theorem

In our applications, we will actually be most interested in the comonadic version of Barr-Beck. As before, there will be two statements.

Suppose now that the adjunction is

$$\begin{tikzcd}
\mathscr{C} \arrow[r, shift left, "L"] & \mathscr{D} \arrow[l, shift left, "R"]
\end{tikzcd}$$

giving rise to the diagram

$$
\begin{tikzcd}
\mathscr{C} \arrow[rr, bend left, "L^{\mathrm{enh}}"] \arrow[dr, bend left, "L"] \arrow["T:=RL"', loop left] & & {}_{\Omega}\mathrm{coMod}(\mathscr{D}) \arrow[ll, dashed, "R^{\mathrm{recon}}"'] \arrow[dl, bend right, "\mathrm{fgt}"'] \\
& \mathscr{D} \arrow[ul, bend left, "R"] \arrow[ur, bend right, "\mathrm{cofree}"'] \arrow["\Omega:=LR"', loop below] &
\end{tikzcd}
$$

Theorem 2.25 (Barr-Beck, comonadic version 1)**.** *If*

1. *L is conservative, and*
2. *$\mathscr{C}$ admits and L preserves the limits*[5] *of L-split cosimplicial diagrams,*

then L is comonadic, i.e. L^{enh} is an equivalence.

Theorem 2.26 (Barr-Beck, comonadic version 2)**.** *If*

1. *$\mathscr{C}$ admits the limits of L-split cosimplicial diagrams,*
2. *R^{recon} is conservative, and*
3. *the T-cobar on each object $c \in \mathscr{C}$,*

$$
c \longrightarrow \left[Tc \;\substack{\longrightarrow \\ \longrightarrow}\; TTc \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \cdots \right]
$$

is a limit diagram,

then L is comonadic, i.e. L^{enh} is an equivalence.

Finally, as in the monadic version:

Theorem 2.27. *Both statements are if and only if statements.*

The proofs are entirely mirror to those of their monadic analogs.

[5] The limit of a cosimplicial diagram is often called its *totalization*.

The comonadicity 1-800 hotline

Here we conclude our abstract discussion of comonadicity. We first recapitulate the landscape of the comonadaicity question with the tautological diagram below, which assumes that $\mathscr{C}$ admits enough limits for R^{recon} to exist:

$$\begin{tikzcd}
R\mathscr{D} \arrow[rr, equal] \arrow[d, hook] & & {}_{\Omega}\operatorname{coMod}^{\mathrm{cofree}}\mathscr{D} \arrow[d, hook] \\
\mathscr{C}^{T^{\mathrm{enh}}} \arrow[rr, equal] \arrow[d, hook] & & \left({}_{\Omega}\operatorname{coMod}\mathscr{D}\right)^{\Omega^{\mathrm{enh}}} \arrow[d, hook] \\
\mathscr{C} \arrow[loop left, "T^{\mathrm{enh}}:=R^{\mathrm{recon}}\circ L^{\mathrm{enh}}"] \arrow[rr, shift left, "L^{\mathrm{enh}}"] \arrow[dr, bend right, "L"'] & & {}_{\Omega}\operatorname{coMod}\mathscr{D} \arrow[ll, shift left, "R^{\mathrm{recon}}"] \arrow[loop right, "\Omega^{\mathrm{enh}}:=L^{\mathrm{enh}}\circ R^{\mathrm{recon}}"] \arrow[dl, bend left, "\mathrm{fgt}"] \\
& \mathscr{D} &
\end{tikzcd}$$

Here, $R\mathscr{D}$ is the full subcategory on the objects in the image of R.

Next, we summarize what we have learned:

1. To prove that an adjunction is a pair of inverse equivalences, it is the same as showing that one functor is an embedding and the other is conservative.
2. Comonadicity has two characterizations, each of which approaches the question of when L^{enh} is an equivalence from the perspective of when either L^{enh} or R^{recon} is an embedding.
3. From the perspective of the R^{recon}-centric Barr-Beck version 1, the main kinds of L-split cosimplicial diagrams to consider in proving comonadicity are those coming from the structure diagrams of objects in ${}_{\Omega}\operatorname{coMod}\mathscr{D}$. From the perspective of the L^{enh}-centric version 2, the main kinds of L-split cosimplicial diagrams to consider in proving comonadicity are the T-cobars on objects in $\mathscr{C}$.

Finally, we offer an FAQ for when things just don't seem to work.

Question 2.28. *Hello, I found a functor L, but it is not conservative! What do I do?*

You can try restricting L to a subcategory on which L *is* conservative, and work from there. Let us work in $\mathrm{Pr}_k^{L,\mathrm{st}}$ of k-linear presentable stable categories and left adjoint exact functors, where every category admits a zero object and the reconstruction functor R^{recon} always exists. There are two canonical choices for such a subcategory, and we only discuss one:

$$\operatorname{Ker}(L)^{\perp} := \{c \,:\, \hom(\operatorname{Ker}(L), c) = 0\}$$

We now prove the following:

Lemma 2.29. *In* $\mathrm{Pr}_k^{L,\mathrm{st}}$,

1. $L|_{\mathrm{Ker}(L)^\perp}$ *is conservative;*
2. $\mathrm{Ker}(L)$ *and* $\mathrm{Ker}(L)^\perp$ *are in* $\mathrm{Pr}^{L,\mathrm{st}}$*; and*
3. R^{recon} *lands inside* $\mathrm{Ker}(L)^\perp$.

Proof. To see (1), take any $c \in \mathrm{Ker}(L) \cap \mathrm{Ker}(L)^\perp$ and look at its identity endomorphism

$$\underbrace{c}_{\in \mathrm{Ker}(L)^\perp} \xrightarrow{\mathrm{Id}_c} \underbrace{c}_{\in \mathrm{Ker}(L)}$$

which is therefore $\mathrm{Id}_c \simeq 0$, whence $c \simeq 0$. To see (2), note that both categories are the fiber products of diagrams in Pr^L:

$$\begin{tikzcd} \mathrm{Ker}(L) \arrow[r, hook] \arrow[d] & \mathscr{C} \arrow[d, "L"] \\ \mathrm{pt} \arrow[r, "0"] & \mathscr{D} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Ker}(L)^\perp \arrow[r, hook] \arrow[d] & \mathscr{C} \arrow[d, "\prod\limits_{c \in \mathrm{Ker}(L)} \hom_{\mathscr{C}}(c-)"] \\ \mathrm{pt} \arrow[r, "0"] & {}_k\,\mathrm{Mod} \end{tikzcd}$$

and conclude by Proposition 1.8. Finally, to see (3), note first that R lands inside $\mathrm{Ker}(L)^\perp$ because if $c \in \mathrm{Ker}(L)$ then

$$\hom_{\mathscr{C}}(c, Rd) = \hom_{\mathscr{C}}(\underbrace{Lc}_{\simeq 0}, d) \simeq 0.$$

Since $\mathrm{Ker}(L)^\perp$ admits limits, it follows that R^{recon} lands inside $\mathrm{Ker}(L)^\perp$ as well. ☐

Thus we can draw the following diagram

$$\begin{tikzcd} \mathrm{Ker}(L)^\perp \arrow[rr, shift left, "L^{\mathrm{enh}}|_{\mathrm{Ker}(L)^\perp}"] \arrow[dr, bend right, "L|_{\mathrm{Ker}(L)^\perp}"] \arrow[dd, hook] & & {}_\Omega\,\mathrm{coMod}\,\mathscr{D} \arrow[ll, shift left, "R^{\mathrm{recon}}"] \arrow[dl, bend left, "\mathrm{fgt}"'] \arrow[dd, equal] \\ & \mathscr{D} \arrow[dd, equal] & \\ \mathscr{C} \arrow[rr, shift left, "L^{\mathrm{enh}}"] \arrow[dr, bend right, "L"] & & {}_\Omega\,\mathrm{coMod}\,\mathscr{D} \arrow[ll, shift left, "R^{\mathrm{recon}}"] \arrow[dl, bend left, "\mathrm{fgt}"'] \\ & \mathscr{D} & \end{tikzcd}$$

and instead study comonadicity of the top triangle, where $L|_{\mathrm{Ker}(L)^\perp}$ is conservative. This is the perspective taken in arguing for comonadicity for local systems in Theorem 4.37.

Question 2.30. *What if L is not a right adjoint, and thus does not preserve all limits?*

It can still preserve enough limits to be comonadic. However, showing such a result is usually subtle, and requires tools. We survey the two tools that, in due time, we will use:

1. Our main tool, detailed in the final chapter, is *semiorthogonality* of kernels of localizations, which sometimes appears under the guises of *orthogonality* and *adjointability*, a.k.a. *base change*.

2. Another tool comes from taking to heart Barr-Beck version 2, and looking for conditions that ensure that every T-cobar is a limit diagram in $\mathscr{C}$. This in fact generalizes the above tool. See the last chapter for a discussion and an application.

Question 2.31. *Help! My $\mathscr{C}$ does not even admit the necessary limits for R^{recon} to exist!*

You might be out of luck here. But if you work in Pr, you will not run into this problem!

Question 2.32. *Hi, I don't care about $\mathscr{C}$, or a subcategory of it! I just want monadicity! Now!*

You really are a convert. Koszul duality results are of this type; see Section 2.8 of this chapter for a discussion.

2.6 Some more examples

In this section we present a few more first examples of (co)monadicity in the 1-categorical context, as well as some non-examples. Many more detailed examples, though in the world of ∞-categories, will follow in the subsequent chapters. The point presently is to illustrate the wide jurisdiction of the monadicity theorem, which as mentioned in Remark 2.10 does not require the categories to have any structures or properties, or to admit any general (co)limits apart from very precise ones.

The free-forget adjunction for groups

Consider the forgetful functor $\mathrm{fgt} : \mathrm{Grp} \to \mathrm{Set}$ from the category of groups to the category of sets. It has a left adjoint called $\mathrm{free} : \mathrm{Set} \to \mathrm{Grp}$ which builds the free group $\mathrm{free}(X)$ on the letters in the set X:

$$\hom_{\mathrm{Grp}}(\mathrm{free}(X), G) \cong \hom_{\mathrm{Set}}(X, \mathrm{fgt}(G))$$

Is $\mathrm{fgt} : \mathrm{Grp} \to \mathrm{Set}$ monadic? That is, is the functor

$$\mathrm{Grp} \xrightarrow{\mathrm{fgt}^{\mathrm{enh}}} {}_{\mathrm{fgt}\circ\mathrm{free}}\mathrm{Mod}(\mathrm{Set})$$

an equivalence? It turns out that it is. We verify the hypotheses of Barr-Beck:

1. The functor fgt is certainly conservative: a morphism of groups $G \xrightarrow{f} H$ is an isomorphism iff it is a bijection as a map of sets $\mathrm{fgt}(G) \xrightarrow{\mathrm{fgt}(f)} \mathrm{fgt}(H)$.

2. While Grp is closed under equalizers—and in fact under all finite limits—it is not closed under all coequalizers. For example, if $i : G \hookrightarrow H$ is a subgroup, then the colimit of the pair

$$\begin{tikzcd} G \arrow[r, hook, shift left, "i"] \arrow[r, shift right, "1_H"'] & H \end{tikzcd}$$

"ought to be" G/H, but this is not a group unless H is normal.

Nonetheless, Grp admits reflexive coequalizers: limits of diagrams of the form

$$\begin{tikzcd} G \arrow[r, two heads, shift left=2, "a"] \arrow[r, two heads, shift right=2, "b"'] & H \arrow[l, dashed, hook', "i"'] \end{tikzcd}$$

where $a \circ i = b \circ i = \mathrm{Id}_H$, i.e. pairs where a and b admit a common section. This occurs because the colimit, if it were to exist, would be H/Q where Q is the subgroup generated by all elements $\{a(g)b(g^{-1}) \ : \ g \in G\}$. And indeed, the presence of the section i makes Q a normal subgroup of H: for any $h = ai(h) = bi(h) \in H$,

$$\begin{aligned} h\big(a(g)b(g^{-1})\big)h^{-1} &= (ha(g)h^{-1})(hb(g^{-1})h^{-1}) \\ &= a\big(i(h)gi(h^{-1})\big)b\big(i(h)g^{-1}i(h^{-1})\big) \\ &= a\big(i(h)gi(h^{-1})\big)b\big((i(h)gi(h^{-1}))^{-1}\big) \end{aligned}$$

which is therefore still in Q. Thus the colimit of this diagram exists in Grp, and is H/Q.

The final point is that fgt preserves these reflexive coequalizers, and so in particular preserves the coequalizers of those reflexive forks that it splits.

Therefore, fgt is monadic. This means something unsurprising: that a group is the same data as a pair of a set X and a map

$$\underbrace{\mathrm{fgt} \circ \mathrm{free}(X)}_{\text{words in } X} \xrightarrow{a} X$$

that satisfies unitality and associativity properties. The operation a is an instruction for how to produce an element of X out of any word in X, which associativity dictates must be independent of order. Since words are finite, the data of the a is equivalent to the data of its restriction to two-letter words $X \times X \xrightarrow{a|_{X \times X}} X$, which is the group multiplication.

In topology

We again examine a forgetful functor, this time from the one from the category Top of topological spaces and continuous maps to Set:

$$\mathrm{fgt} : \mathrm{Top} \to \mathrm{Set}$$

This functor has both a left adjoint $\mathrm{fgt}^L := (-)^{\mathrm{discrete}}$ and a right adjoint $\mathrm{fgt}^R := (-)^{\mathrm{coarse}}$, called taking the discrete topology on the set and the coarse topology on the set, respectively:

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "\mathrm{fgt}"] & \mathrm{Set} \arrow[l, bend right, "(-)^{\mathrm{discrete}}"'] \arrow[l, bend left, "(-)^{\mathrm{coarse}}"] \end{tikzcd}$$

The category Top is bicomplete, so it admits all the necessary colimits for Barr-Beck. And since fgt admits both adjoints, in in fact preserves all limits and colimits, so in particular preserves the colimits of reflexive pairs that it splits. Things look good!

However, fgt fails to be (co)monadic because it is not conservative: if a continuous map $X \xrightarrow{f} Y$ of topological spaces is such that $\mathrm{fgt}(f)$ is a bijection of sets, it is still not necessarily a homeomorphism—that is, f^{-1} is not necessarily continuous. Another way to see the lack of monadicity by looking directly at the monad $T = \mathrm{fgt}((-)^{\mathrm{discrete}})$: it is the identity endofunctor, with the identity natural transformation as both the multiplication $TT \Rightarrow T$ and the unit $\mathrm{Id} \Rightarrow T$. Thus ${}_T\,\mathrm{Mod}\,\mathrm{Set} = \mathrm{Set}$, which certainly do not account of all of Top. Similarly, ${}_\Omega\,\mathrm{coMod}\,\mathrm{Set} = \mathrm{Set}$ as well.

One common trick for restoring conservativity, addressed in Question 2.28, is to restrict the domain category. Recall the following result from point-set topology:

Theorem 2.33. *A continuous bijection from a compact space to a Hausdorff space has a continuous inverse.*

So restricting from Top to the full subcategory CHaus of compact Hausdorff spaces does restore conservativity of fgt on this subcategory. Furthermore, the functor fgt still has a left adjoint, U, coming from the diagram

$$\begin{tikzcd} \mathrm{CHaus} \arrow[r, shift right, "\iota"'] & \mathrm{Top} \arrow[l, shift right, "\beta"'] \arrow[r, shift right, "\mathrm{fgt}"'] & \mathrm{Set} \arrow[l, shift right, "\mathrm{discrete}"'] \arrow[ll, bend right, "U"'] \end{tikzcd}$$

Thus, the monadicity question can be posed for (U, fgt): is $\mathrm{fgt}^{\mathrm{enh}}$ an equivalence?

$$\begin{tikzcd} \mathrm{CHaus} \arrow[rr, shift right, "\mathrm{fgt}^{\mathrm{enh}}"'] \arrow[dr, bend right, "\mathrm{fgt}"'] & & {}_T\,\mathrm{Mod}\,\mathrm{Set} \arrow[ll, shift right, dashed, "U^{\mathrm{recon}}"'] \arrow[dl, bend left, "\mathrm{fgt}_T"] \\ & \mathrm{Set} \arrow[loop below, "T := \mathrm{fgt} \circ U"] & \end{tikzcd}$$

The issue now is whether CHaus admits and fgt preserves the necessary colimits. This turns out to be the case:

Theorem 2.34 ([16])**.** *The forgetful functor* $\mathrm{CHaus} \xrightarrow{\mathrm{fgt}} \mathrm{Set}$ *is monadic.*

Proof. See the Bachelor project [20] for a proof. □

Faithfully flat descent

Let $\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A$ be a morphism of affine schemes, giving rise to the adjunction of abelian categories and right-exact functors

$$\begin{tikzcd} \mathrm{QCoh}(\operatorname{Spec} A) \arrow[r, bend left, "f^*"] & \mathrm{QCoh}(\operatorname{Spec} B) \arrow[l, "f_*"] \end{tikzcd}$$

where $f^*M := M \otimes_A B$. We wish to know when f^* is comonadic.

Proposition 2.35. *If f is faithfully flat, then f^* is comonadic.*

Proof. We know that f^* is comonadic if and only if

1. f^* is conservative and
2. $\mathrm{QCoh}(\operatorname{Spec} A)$ admits and f^* preserves the limits of those reflexive pairs that it splits.

Since these categories are abelian, f^* is conservative if and only if all M for which $f^*M = 0$ must in fact be $M = 0$. This is exactly the definition of B being a faithful module over A.

Since these categories are abelian, they admit all finite limits and colimits. So it remains to check that f^* preserves the limits of f^*-split reflexive pairs. But f is a flat morphism, meaning that f^* is left-exact, and thus it preserves all equalizers, so in particular it preserves the necessary limits for comonadicity. □

Remark 2.36. *We pause to meditate on the role of conservativity, and what it does not (point 1) and does (point 2) mean:*

1. *To be (co)monadic, F does* not *need to "separate"* objects *in the sense of sending non-isomorphic objects to non-isomorphic objects. Indeed, all of the above examples fail this very stringent condition, but are nonetheless (co)monadic! The point is that, while separation might not happen at the level of F-images in $\mathscr{D}$, it still can happen at the level of F^{enh}-images in ${}_T\mathrm{Mod}\,\mathscr{D}$ or ${}_\Omega\mathrm{coMod}\,\mathscr{D}$, because these categories have "fewer" morphisms than $\mathscr{D}$.*
2. *On the other hand, conservativity of F—the milder relative of the above condition, ensuring that F can "separate"* morphisms*—is absolutely crucial. If a non-isomorphism in $\mathscr{C}$ is rendered an isomorphism by F, then it will also be rendered an isomorphism by F^{enh}, and therefore F^{enh} has no hope of being faithful.*

Non-examples of monadicity

Here we address a sobering reality: a right adjoint functor can fail to be monadic (and dually, a left adjoint can fail to be comonadic). Since Barr-Beck is a characterization of monadicity, monadicity can fail for three reasons:

1. $\mathscr{C}$ does not admit enough colimits for L^{recon} to even exist;

2. R is not conservative;

3. R does not preserve enough colimits.

For the curious, here we record a precise example of each:

Point of failure 1: $\mathscr{C}$ does not admit enough colimits.

Example 2.37. *Let us look at the tautological example where $\mathscr{C} = {}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D}$:*

$$\begin{tikzcd}
{}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D} \arrow[loop left, "\Omega=\mathrm{free}\circ\mathrm{fgt}"] \arrow[rr, hook, "\iota:=\mathrm{fgt}^{\mathrm{enh}}"] \arrow[dr, shift right, "\mathrm{fgt}"'] & & {}_T\mathrm{Mod}\,\mathscr{D} \arrow[dl, "\mathrm{fgt}"] \\
& \mathscr{D} \arrow[ul, shift right, "\mathrm{free}"'] \arrow[loop below, "T=\mathrm{fgt}\circ\mathrm{free}"] &
\end{tikzcd}$$

Here, the following is true:

1. ${}_T\mathrm{Mod}\,\mathscr{D} \xrightarrow{\mathrm{fgt}} \mathscr{D}$ *is tautologically monadic;*

2. ${}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D} \xrightarrow{\mathrm{fgt}} \mathscr{D}$ *is conservative, but generally not monadic. This is because not every T-module is isomorphic, as a T-module, to a free one; in other words,* $\mathrm{fgt}^{\mathrm{enh}}|_{{}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D}}$ *is an embedding, but it is not essentially surjective. Another way of thinking about this is through Example 2.24 and Theorem 2.21: while all Ω-bar diagrams in ${}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D}$ are colimit diagrams, it is generally not true that the category admits enough colimits for the functor L^{recon} to even exist.*

3. *If all ambient colimits were added back to enlarge ${}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D}$ into $\langle {}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D}\rangle_{colimits}$, then*
$$\langle {}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D}\rangle_{colimits} = {}_T\mathrm{Mod}\,\mathscr{D}$$
and the extended functor fgt *would tautologically preserve the necessary colimits for monadicity.*

Point of failure 2: R is not conservative The forget functor $\mathrm{Top} \xrightarrow{\mathrm{fgt}} \mathrm{Set}$, as discussed above, is not conservative. But note that the other hypotheses are satisfied: Top admits all colimits, and fgt preserves all of them since fgt is a left adjoint.

Point of failure 3: R does not preserve the colimits of R-split simplicial diagrams See Example 6.15 for a comonadic example.

2.7 In Cat_∞ and Pr^L: comonadicity vs. monadicity

The analog of Barr-Beck in the setting of ∞-categories is the Barr-Beck-Lurie theorem, which reads exactly the same as the 1-categorical statement, but all categories are ∞-categories; no further hypotheses are levied. So we do not rewrite them, but we do observe that, just like Barr-Beck for 1-categories, Barr-Beck-Lurie has completely dual versions for monadicity and comonadicity because left and right adjoint functors are treated equally.

However, instead of vanilla Cat_∞, our categorical setting for the future will be Pr^L, and its variants $\mathrm{Pr}^{L,\mathrm{st}}$ and $\mathrm{Pr}^{L,\mathrm{st}}_\omega$. This new context features the following important changes:

1. it only permits one to consider functors which are left adjoints. So all functors in Pr^L preserve all colimits, and in particular all simplicial colimits;

2. it assures that all categories considered, being presentable, are in fact bicomplete. So the reconstruction functors always exist.

These two facts have the following consequence:

Theorem 2.38 (Barr-Beck-Lurie, in Pr^L)**.** *The following are characterizations of monadicity and comonadicity in* Pr^L*:*

1. *an adjunction* (L, R) *in* Pr^L *is monadic simply if and only if* R *is conservative;*

2. *an adjunction* (L, R) *in* Pr^L *is comonadic if and only if* L *is conservative and* L *preserves the limits of* L*-split cosimplicial diagrams.*

In summary: Pr^L, showing comonadicity statements requires more effort than showing monadicity statements. For a discussion of why the (co)module categories are presentable, see Lemma 2.53.

2.8 The language of thick envelopes, and Koszul duality

Here we discuss an important construction for our future investigations into descent—that of the *thick envelope* of an object—and show how it arises in the context of looking for a monadicity result.

Morita theory as monadicity

Let $\mathscr{C}$ be a presentable k-linear (hence stable) ∞-category, and $X \in \mathscr{C}$ an object. We can try to use X to describe $\mathscr{C}$ monadically via the functor

$$\mathscr{C} \xrightarrow{\hom_{\mathscr{C}}(X,-)} {}_k\operatorname{Mod},$$

which, since it definitionally preserves limits, admits a left adjoint that we denote $X \otimes_k (-)$, with associated reconstruction functor $(X \otimes_k (-))^{\text{recon}} := X \otimes_{\mathscr{C}} (-)$:

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, shift left, "\hom_{\mathscr{C}}(X-)^{\text{enh}}"] \arrow[dr, shift right, "\hom_{\mathscr{C}}(X-)"'] & & \operatorname{Mod}_{\operatorname{End}_{\mathscr{C}}(X)} \arrow[ll, shift left, "X\otimes_k(-)"] \arrow[dl, "\text{fgt}"] \\
& {}_k\operatorname{Mod} \arrow[ul, shift right] &
\end{tikzcd}$$

We warn that this diagram is in $\Pr^{\text{st}}$ but not necessarily in $\Pr^{L,\text{st}}$, because $\hom_{\mathscr{C}}(X,-)$ need not preserve colimits. But if it did, its monadicity would be in better shape. Here is the fundamental result:

Theorem 2.39 (Morita theorem)**.** *If X is a compact generator, then $\hom_{\mathscr{C}}(X,-)$ is monadic, rendering an equivalence*

$$\mathscr{C} \simeq \operatorname{Mod}_{\operatorname{End}_{\mathscr{C}}(X)}$$

Proof. By definition, X is a generator if $\hom_{\mathscr{C}}(X,-)$ is conservative. Furthermore, if X is compact, then $\hom_{\mathscr{C}}(X,-)$ preserve colimits. Thus the functor is monadic by the crude version of Barr-Beck-Lurie. $\square$

The thick envelope

Sometimes, we are not so lucky, and X is neither compact nor a generator. But if we are willing to abandon the goal of monadically describing $\mathscr{C}$, then we may still be able to monadically describe *something*. Intuitively, we would like to engineer a category that is in some sense "generated" by X, such that the "restriction" of the functor $\hom_{\mathscr{C}}(X,-)$ to it would *have* to be both conservative and colimit-preserving.

We build this new category in two steps:

1. take the *thick envelope* of X, $\operatorname{Thick}_{\mathscr{C}}(X)$;

2. then take the ind-completion, $\operatorname{Ind}\operatorname{Thick}_{\mathscr{C}}(X)$.

We now elaborate on the first construction:

Definition 2.40. *Let $\mathscr{C}$ be a stable ∞-category.*

1. *A full subcategory $\mathscr{C}' \subseteq \mathscr{C}$ is* thick *if it contains* 0 *and is closed under finite limits and retracts.*

2. *Let $X \in \mathscr{C}$ an object. Then the* thick envelope *of X,*

$$\mathrm{Thick}_{\mathscr{C}}(X) \subset \mathscr{C},$$

is the smallest thick subcategory containing X.

Remark 2.41. *The following are equivalent for a subcategory of a stable category:*

1. *closure under fibers;*

2. *closure under cofibers;*

3. *closure under finite limits;*

4. *closure under finite limits and finite colimits.*

To our eyes, here are the important properties of the thick envelope:

Lemma 2.42. *Let $X \in \mathscr{C}$ be an object in a stable ∞-category, and consider* $\mathrm{Thick}_{\mathscr{C}}(X)$*. Then:*

1. $\mathrm{Thick}_{\mathscr{C}}(X)$ *is again stable;*

2. $X \in \mathrm{Thick}_{\mathscr{C}}(X)$ *is a generator for this subcategory.*

Proof. Part (1) follows from an equivalent characterization of $\mathrm{Thick}_{\mathscr{C}}(X)$: it is the smallest full subcategory of C containing 0 and X that is closed under finite limits and colimits. For part (2), suppose $\hom_{\mathscr{C}}(X, Y) \simeq 0$ for some $Y \in \mathscr{C}$, and consider the full subcategory $\mathscr{C}_Y \subseteq \mathrm{Thick}_{\mathscr{C}}(X)$ consisting of all those $Z \in \mathrm{Thick}_{\mathscr{C}}(X)$ for which $\hom_{\mathscr{C}}(Z, Y) \simeq 0$. If we can show that $Y \in \mathscr{C}_Y$, then $\hom_{\mathscr{C}}(Y, Y) \simeq 0$ would force $Y \simeq 0$. We do this by showing that actually, $\mathscr{C}_Y = \mathrm{Thick}_{\mathscr{C}}(X)$. Well, we know $0, X \in \mathscr{C}_Y$, and furthermore, $\mathscr{C}_Y$ is closed under finite limits and colimits. So indeed $\mathscr{C}_Y = \mathrm{Thick}_{\mathscr{C}}(X)$. $\square$

Here is a simple example of this construction:

Example 2.43. *For $\mathscr{C} = {}_A\mathrm{Mod}$ for an $\mathbb{E}_1$ algebra A, we recognize* $\mathrm{Perf}(A)$ *as* $\mathrm{Thick}_A(A)$*, and therefore*

$${}_A\mathrm{Mod} = \mathrm{Ind}\,\mathrm{Perf}(A) = \mathrm{Ind}\,\mathrm{Thick}_A(A)$$

As a consequence of Lemma 2.42, taking the ind-completion gives

$$\mathrm{Ind}\,\mathrm{Thick}_{\mathscr{C}}(X),$$

which is a compactly generated category, with compact generator X. The converse is also true: if $X \in \mathscr{C}$ is a compact generator, then $\mathscr{C} \simeq \mathrm{Ind}\,\mathrm{Thick}_{\mathscr{C}}(X)$. We may think of

$\mathrm{Thick}_{\mathscr{C}}(X)$ as a full subcategory of both $\mathrm{Ind}\,\mathrm{Thick}_{\mathscr{C}}(X)$ and $\mathscr{C}$, but where the latter is concocted to force X to be a compact generator:

$$\begin{tikzcd}
& \mathrm{Thick}_{\mathscr{C}}(X) \arrow[dl, hook'] \arrow[dr, hook] & \\
\mathrm{Ind}\,\mathrm{Thick}_{\mathscr{C}}(X) \arrow[rr, "\mathrm{can}"] \arrow[dr, "R"'] & & \mathscr{C} \arrow[dl, "\hom_{\mathscr{C}}(X-)"] \\
& \mathscr{D} &
\end{tikzcd}$$

Here, $R := \mathrm{Ind}\,\hom_{\mathrm{Thick}_{\mathscr{C}}(X)}(X, -)$ is obtained by first restricting $\hom_{\mathscr{C}}(X, -)$ to $\mathrm{Thick}_{\mathscr{C}}(X) \subseteq \mathscr{C}$, and then ind-completing. The comparison functor can of evaluating the formal colimit as a colimit in $\mathscr{C}$ is colimit-preserving, but generally is neither an embedding nor essentially surjective. However, we flag a result for future use:

Lemma 2.44. *If X is compact in $\mathscr{C}$, then* can *is an embedding*

$$\begin{tikzcd}
\mathrm{Thick}_{\mathscr{C}}(X) \arrow[r, hook] \arrow[d, hook] & {}^{\perp}(X^{\perp}) \arrow[d, hook] \\
\mathrm{Ind}\,\mathrm{Thick}_{\mathscr{C}}(X) \arrow[r, hook, "\mathrm{can}"] \arrow[ur, dashed, "\mathrm{can}"] & \mathscr{C}
\end{tikzcd}$$

that factors through the stable, presentable full subcategory ${}^{\perp}(X^{\perp})$. In fact, it is an equivalence.

Proof. See the proof of Proposition 4.31. □

In any case, armed with the concept of the ind-thick envelope, we make the following revision to Theorem 2.39, which one may interpret as a prototype Koszul duality theorem:

Theorem 2.45. *Let $\mathscr{C}$ be a presentable k-linear category, and $X \in \mathscr{C}$ any object. Then the functor*

$$\mathrm{Ind}\,\mathrm{Thick}_{\mathscr{C}}(X) \xrightarrow{R} {}_k\mathrm{Mod}$$

is monadic, rendering an equivalence

$$\mathrm{Ind}\,\mathrm{Thick}_{\mathscr{C}}(X) \simeq \mathrm{Mod}_{\mathrm{End}_{\mathscr{C}}(X)}$$

Proof. The functor R is rigged to be conservative and bicontinuous, and therefore it admits a colimit-preserving left adjoint L, which is defined by the condition $L(k) = X$. Since both R and L preserve colimits, and in particular tensors, then for any $V \in {}_k\mathrm{Mod}$ we may rewrite

$$R \circ L(V) = R \circ L(V \otimes_k k) \simeq \underbrace{(R \circ L(k))}_{\mathrm{End}_{\mathscr{C}}(X)} \otimes_k V$$

which allows us to identify the monad $T := R \circ L$ with the algebra $\mathrm{End}_{\mathscr{C}}(X)^{\mathrm{op}}$. □

Example 2.46. *Applying this to the example above of $\mathscr{C} = {}_A\mathrm{Mod}$ in the case where A a classical ring and $X = P$ is a finitely-generated projective module recovers the basic result of Morita theory:*

$${}_A\mathrm{Mod} \simeq \mathrm{Mod}_{\mathrm{End}_A(P)}$$

Koszul duality is monadicity for $\mathrm{Ind\,Thick}$ categories

One might wonder whether, like $\mathrm{Perf}(A)$, $\mathrm{Coh}(A)$ is ever a thick envelope of anything. It turns out that it is, for particular kinds of algebras. We learned the contents of this subsection from Lecture 6 of [2].

Definition 2.47. *Let k be a field. A dg k-algebra is* small *if*

1. *A is connected, i.e. $\pi_0(A) = k$;*

2. *A is connective, i.e. $\pi_{i<0}A = 0$; and*

3. *A is of finite type, i.e. $\sum_{i\in\mathbb{Z}} \dim_k \pi_i(A) < \infty$.*

Note that if A is small, then it is automatically and canonically augmented, since the quotient $A \to \pi_0(A) = k$ is a module map.

Remark 2.48. *We may think of this definition as codifying in algebraic geometry the notion of a "small" connected topological space, with only one classical point $0 \in \mathrm{Spec}(A)$. Our monadic descent question for $\mathscr{C} = \mathrm{QCoh}(A)$ can be formulated as the hope that sheaves $\mathscr{F}$ living on $\mathrm{Spec}(A)$ can be faithfully presented by the data of their fiber $\mathscr{F}|_0$ together with an action of the'based loop space $\Omega_0\,\mathrm{Spec}(A) := \mathrm{End}_A(k)$.*

As mentioned, in the case that A is small, we can identify $\mathrm{Thick}_A(k)$:

Lemma 2.49. *If A is small, then*

$$\mathrm{Thick}_A(k) \simeq \mathrm{Coh}(A).$$

Proof. For such A, we leave it as proven that

$$\mathrm{Coh}(A) \simeq \{M \,:\, \sum_{n=0}^{\infty} \dim_k \pi_n(M) < \infty\} =: \mathrm{Perf}_{/k}(A).$$

Certainly, $\mathrm{Coh}(A)$ is thick and contains k, so $\mathrm{Thick}_A(k) \subseteq \mathrm{Coh}(A)$. Now let $M \in \mathrm{Coh}(A)$, $M \neq 0$. There is some smallest n so that $\pi_n(M) \neq 0$, and so M fits into a cofiber sequence

$$M \to \pi_n(M) \to M'$$

where $\pi_{\leq n}(M') = 0$. But for every nonzero $x \in \pi_n(M)$, we can split off a copy of the augmentation module k via the cofiber sequence $k[n] \xrightarrow{x} \pi_n(M) \to M''$, where $\dim_k \pi_n(M'') <$

$\dim_k \pi_n(M) < \infty$, and $\pi_{m\neq n}M'' = 0$. So in finitely many steps, we split $\pi_n(M)$ completely off from M, and now we proceed with the same argument on M'. This induction is finite, showing that $M \in \mathrm{Thick}_A(k)$. Note of course that $A \in \mathrm{Thick}_A(k)$ as well. □

Here is the consequent Koszul duality result, which just stitches together the previous few results:

Proposition 2.50. *[Lurie, Beilinson-Ginzburg-Soergel] For A a small $\mathbb{E}_1$-algebra over k, the* Koszul duality functor $\mathbb{D} := \hom_A(k, -)$ *induces an equivalence of stable presentable ∞-categories*

$$\mathrm{Ind}\,\mathbb{D}' : \mathrm{IndCoh}(A) \xrightarrow{\simeq} \mathrm{Mod}_{\mathrm{End}_A(k)}$$

Proof. This follows immediately from Theorem 2.45 and Lemma 2.49. The functor $\mathbb{D}'$ is the functor R from the notation in the Theorem. □

Example 2.51. *Take V a finite-dimensional vector space over $k = \mathbb{C}$, and put $\Lambda :=$ $\mathrm{Sym}^\bullet(V[1])$ and $S := \mathrm{Sym}^\bullet(V^*[-2])$. Then $A := \Lambda$ is $\mathbb{E}_1$ and small. The vanilla functor $\mathbb{D}$ is not colimit-preserving, because $k \in {}_\Lambda\mathrm{Mod}$ is not compact. But if we restrict to $\mathrm{Coh}(\Lambda) \simeq \mathrm{Perf}_{/k}(\Lambda)$, i.e. complexes with cohomology in bounded degree and each cohomology group finite-dimensional, then $k \in \mathrm{IndCoh}(\Lambda)$ becomes compact, and therefore*

$$\mathrm{IndCoh}(\Lambda) \xrightarrow{\simeq} \mathrm{Mod}_S$$

Letting $n = \dim_{\mathbb{C}} V$, we may recognize

$$\Lambda \cong C_{-\bullet}(\mathbb{T}^n; \mathbb{C}), \qquad S \cong C^\bullet(B\mathbb{T}^n; \mathbb{C})$$

for $\mathbb{T}^n = (S^1)^n$ the real n-torus.

For $n = 1$, this reads

$$\mathrm{Ind}\,\mathrm{Coh}\,k[\lambda] \simeq \mathrm{Mod}_{k\langle u\rangle},$$

where $|\lambda| = -1$ and $|u| = 2$.

2.9 Appendix: categories of T-modules as homotopy limits

Here we offer an alternative and non-essential but illuminating origin story for the category of T-modules, together with its full subcategory of free T-modules. We begin with the 1-categorical story, and then afterwards move into our ∞-categorical context of interest.

The story in Cat_1

Consider again an adjunction of 1-categories

$$\begin{tikzcd} \Omega \arrow[loop left] & \mathscr{C} \arrow[r, bend right, "R"'] & \mathscr{D} \arrow[l, bend right, "L"'] \arrow[loop right, "T"] \end{tikzcd}$$

with the monad and comonad drawn. The following proposition is the content of this section:

Proposition 2.52. *The category of T-modules, a.k.a. the* Eilenberg-Moore category*, is the category of homotopy T-invariants:*

$$_T\operatorname{Mod}\mathscr{D} \xrightarrow{\simeq} \varprojlim^{\mathrm{oplax}} \left(\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \end{tikzcd} \right)$$

The category of free T-modules, a.k.a. the Kliesli category*, is the category of homotopy T-coinvariants:*

$$\varinjlim^{\mathrm{oplax}} \left(\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \end{tikzcd} \right) \xrightarrow{\simeq} {_T\operatorname{Mod}^{\mathrm{free}}}\mathscr{D}$$

Strictly speaking, it is probably wrong to call these (co)invariants because T is not invertible and the (co)limits are oplax, but we do it poetically.

Proof. By definition, the limit of the diagram is the data of a triple $(\mathscr{L}, F, \eta)$ of a category $\mathscr{L}$, a functor $\mathscr{L} \xrightarrow{F} \mathscr{D}$, and a natural transformation $TF \overset{\eta}{\Rightarrow} F$ subject to the conditions that

1. the "associativity" square

$$\begin{tikzcd} TTF \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "\mu_F"] & TF \arrow[d, Rightarrow, "\eta"] \\ TF \arrow[r, Rightarrow, "\eta"] & F \end{tikzcd}$$

   commutes; and

2. the "unit" triangle

$$\begin{tikzcd} F \arrow[r, Rightarrow, "u_F"] \arrow[dr, Rightarrow, "\mathrm{Id}"'] & TF \arrow[d, Rightarrow, "\eta"] \\ & F \end{tikzcd}$$

   commutes

The limit also satisfies the following universality property: for all other such triples $(\mathscr{L}', F', \eta')$, there exists a unique functor $\Phi : \mathscr{L}' \to \mathscr{L}$ and natural isomorphism $h : \Phi \circ F \Rightarrow F'$, which summarized in the diagram

$$\begin{tikzcd} (\mathscr{L}', \eta') \arrow[rr, "F'"] \arrow[dr, "\Phi"'] & & \mathscr{D} \arrow[loop above, "T"] \\ & (\mathscr{L}, \eta) \arrow[ur, "F"'] \arrow[u, Rightarrow, "h"] & \end{tikzcd}$$

We can probe the objects and morphisms in $\mathscr{L}$ by taking $\mathscr{L}'$ to be the categories pt and (pt $\to$ pt), respectively:

1. the triples one can build on $\mathscr{L}' =$ pt are an object $d \in \mathscr{D}$ together with a morphism $Td \xrightarrow{a} d$ subject to the above two conditions, which is precisely a unital associative T-module;

2. the triples one can build on $\mathscr{L}' = (\text{pt} \to \text{pt})$ are an arrow $d_1 \xrightarrow{f} d_2$ in $\mathscr{D}$ together with morphisms we will call $\eta_{d_i} : Td_i \to d_i$, which constitute a natural transformation iff the following square commutes:

$$\begin{tikzcd} Td_1 \arrow[r, "Tf"] \arrow[d, "\eta_{d_1}"] & Td_2 \arrow[d, "\eta_{d_2}"] \\ d_1 \arrow[r, "f"] & d_2 \end{tikzcd}$$

   This is precisely the condition that f is a morphism of T-modules $(d_1, \eta_{d_1}) \xrightarrow{f} (d_2, \eta_{d_2})$.

This shows that the homotopy oplax limit is indeed the triple

$$({}_T\operatorname{Mod}\mathscr{D}, {}_T\operatorname{Mod}\mathscr{D} \xrightarrow{\text{fgt}} \mathscr{D}, \mu)$$

We now turn to studying the homotopy oplax colimit of $(T \curvearrowright \mathscr{D})$. Dually to the above, it is the data of a triple $(\mathscr{C}, G, \delta)$ of a category $\mathscr{C}$, functor $G : \mathscr{D} \to \mathscr{C}$, and natural transformation $\delta : GT \Rightarrow G$ which satisfy the conditions that

1. the "associativity" square

$$\begin{tikzcd} GTT \arrow[r, Rightarrow, "\delta_T"] \arrow[d, Rightarrow, "G\mu"] & GT \arrow[d, Rightarrow, "\delta"] \\ GT \arrow[r, Rightarrow, "\delta"] & G \end{tikzcd}$$

   commutes; and

2. the "unit" triangle

$$\begin{tikzcd} G \arrow[r, Rightarrow, "Gu"] \arrow[dr, Rightarrow, "\mathrm{Id}"'] & GT \arrow[d, Rightarrow, "\delta"] \\ & G \end{tikzcd}$$

commutes

subject to the universality property that for all other such triples $(\mathscr{C}', G', \delta')$, there is a unique functor $\Psi : \mathscr{C} \to \mathscr{C}'$ and natural isomorphism $k : \Psi \circ G \Rightarrow G'$, which is summarized in the diagram

$$\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \arrow[rr, "G"] \arrow[dr, "G'"'] & \arrow[d, phantom, "\Downarrow k"] & (\mathscr{C}, \delta) \arrow[dl, "\Psi"] \\ & (\mathscr{C}', \delta') & \end{tikzcd}$$

Let us simply check that this universal property holds for the valid triple

$$({}_T\operatorname{Mod}^{\text{free}} \mathscr{D}, \mathscr{D} \xrightarrow{\text{free}} {}_T\operatorname{Mod}^{\text{free}} \mathscr{D}, \mu)$$

So consider another triple $(\mathscr{C}', G', \delta')$; we wish to build a functor Ψ and a natural isomorphism k such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \arrow[rr, "\text{free}"] \arrow[dr, "G'"'] & \arrow[d, phantom, "\Downarrow k"] & ({}_T\operatorname{Mod}^{\text{free}} \mathscr{D}, \mu) \arrow[dl, dashed, "\Psi"] \\ & (\mathscr{C}', \delta') & \end{tikzcd}$$

To see what Ψ should be, take some $d \in \mathscr{D}$; its image in $\mathscr{C}'$ is $G'd$, while its image in ${}_T\operatorname{Mod}^{\text{free}} \mathscr{D}$ is Td, or rather the module it determines:

$$\left[\begin{tikzcd} \cdots \arrow[r, shift left, "\mu_{Td}"] \arrow[r, shift right, "T\mu_d"'] & TTd \arrow[r, "\mu_d"] & Td \end{tikzcd} \right]$$

Apply G' to the entire diagram to fill it out with extra morphisms (dotted) and backwards maps (dashed)

$$\left[\begin{tikzcd} \cdots \arrow[r, shift left=2, "G'\mu_{Td}"] \arrow[r, "G'T\mu_d"] \arrow[r, dotted, shift right=2, "\delta'_{TTd}"'] & G'TTd \arrow[l, dashed, bend left, "G'u_{TTd}"] \arrow[r, shift left, "G'\mu_d"] \arrow[r, dotted, shift right, "\delta'_{Td}"'] & G'Td \arrow[l, dashed, bend left, "G'u_{Td}"] \end{tikzcd} \right] \begin{tikzcd} {} \arrow[r, dotted, "\delta'_d"] & G'd \arrow[l, dashed, bend left, "G'u_d"] \end{tikzcd}$$

which in fact constitute a split simplicial diagram. Thus its colimit exists, regardless of whether $\mathscr{C}'$ admits colimits, and is $G'd$. In this way, the formula

$$\Psi(Td) := G'd$$

specifies a well-defined functor. In this case, $\Psi \circ \mathrm{free} = G$ on the nose, whence k here is the identity natural isomorphism. Therefore, free T-modules is the homotopy oplax colimit of the diagram $(T \curvearrowright \mathscr{D})$. □

What changes in Cat_∞ and Pr^L?

The main outcome of the above perspective in Cat_1 on module categories as homotopy (co)limits is that we can use this description as a definition in Cat_∞:

$$ {}_T\mathrm{Mod}\,\mathscr{D} := \varprojlim_{\mathrm{Cat}_\infty} \left(\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \end{tikzcd} \right), \qquad {}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D} := \varinjlim_{\mathrm{Cat}_\infty} \left(\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \end{tikzcd} \right)$$

Since we will be interested in working in Pr^L instead of Cat_∞, we can ask whether these categories are presentable. In fact, the move from Cat_∞ to Pr^L treats these categories differently:

Lemma 2.53. *If (L, R) is an adjunction in Pr^L, then*

1. *the category of modules ${}_T\mathrm{Mod}\,\mathscr{D}$ is presentable;*
2. *it appears as both a limit and colimit in Pr^L:*

$$\varinjlim_{\mathrm{Pr}^L} \left(\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \end{tikzcd} \right) \simeq \varprojlim_{\mathrm{Pr}^L} \left(\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \end{tikzcd} \right) \simeq {}_T\mathrm{Mod}\,\mathscr{D}$$

3. *on the other hand, the category of free modules ${}_T\mathrm{Mod}^{\mathrm{free}}\,\mathscr{D}$ is generally not presentable. In fact, ${}_T\mathrm{Mod}\,\mathscr{D}$ is often its smallest "presentable envelope."*

Proof. Part (1) follows by Remark 1.13, since

$$ {}_T\mathrm{Mod}\,\mathscr{D} := \varprojlim_{\mathrm{Cat}_\infty} \left(\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \end{tikzcd} \right) \simeq \varprojlim_{\mathrm{Pr}^L} \left(\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \end{tikzcd} \right),$$

using implicitly the closure of Pr^L under limits. Furthermore, part (2) follows by Lemma 1.12:

$$\lim_{\substack{\longleftarrow \\ \mathrm{Cat}_\infty}} \left(\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \end{tikzcd} \right) \simeq \operatorname*{colim}_{\substack{\longrightarrow \\ \mathrm{Pr}^L}} \left(\begin{tikzcd} \mathscr{D} \arrow[loop above, "T"] \end{tikzcd} \right)$$

For part (3), we assert that free modules are typically not closed under colimits, e.g. retracts, and so cannot form a presentable category. For the second part, let $\mathscr{E}$ be a presentable category sandwiched between ${}_T\operatorname{Mod}^{\mathrm{free}}\mathscr{D}$ and ${}_T\operatorname{Mod}\mathscr{D}$:

$$\begin{tikzcd}
{}_T\operatorname{Mod}^{\mathrm{free}}\mathscr{D} \arrow[r, hook] \arrow[dr, shift right, "\mathrm{fgt}"'] & \mathscr{E} \arrow[r, hook, "\iota"] \arrow[d, "\mathrm{fgt}"] & {}_T\operatorname{Mod}\mathscr{D} \arrow[dl, "\mathrm{fgt}"] \\
& \mathscr{D} \arrow[ul, shift right, "\mathrm{free}"'] &
\end{tikzcd}$$

Since $\mathscr{E}$ contains ${}_T\operatorname{Mod}^{\mathrm{free}}\mathscr{D}$, the left adjoint to $\mathscr{E} \xrightarrow{\mathrm{fgt}} \mathscr{D}$ is still the functor free. Since $\mathscr{E}$ admits all colimits, the reconstruction functor ${}_T\operatorname{Mod}\mathscr{D} \xrightarrow{\mathrm{free}^{\mathrm{recon}}} \mathscr{E}$ exists, and is a left adjoint to the embedding ι. Therefore, in the event that free is a conservative functor (which is often the case), $\mathrm{free}^{\mathrm{recon}}$ would also be conservative, rendering $\mathscr{E} \simeq {}_T\operatorname{Mod}\mathscr{D}$ by Lemma 1.11. $\square$

Chapter 3

Monadicity and Beck-Chevalley

This chapter is an account of the abstract machinery that establishes one domain of usefulness of monadicity results: as concrete realizations of homotopy limits of cosimplicial diagrams of categories. We begin with a common instance of this, and then proceed to record the general results.

3.1 Fundamental construction

Consider a simplicial diagram $X_\bullet$ in some category of "spaces"—for example, this could be Top, or Sch_k:

$$\begin{tikzcd} \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[r, shift left] \arrow[r, shift right] & X_0 \end{tikzcd}$$

Take now a theory of sheaves $\mathrm{Sh}_{\mathscr{S}}(-)^*$ which is adapted to this category of spaces in the sense that it affords adjunctions (f^*, f_*) for each map $X \xrightarrow{f} Y$ of spaces, and apply it to this diagram to obtain a cosimplicial diagram of categories:

$$\begin{tikzcd} \varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(X_\bullet) \arrow[dr, dashed, "\mathrm{ev}_0"] \arrow[drr, dashed, bend left, "\mathrm{ev}_1"] \arrow[drrr, dashed, bend left] & & & \\ & \mathrm{Sh}_{\mathscr{S}}(X_0) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Sh}_{\mathscr{S}}(X_1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \end{tikzcd}$$

The (homotopy) limit $\varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(X_\bullet)$ of this diagram always exists in Cat_∞ and is an object of fundamental importance. We will interpret it in two contexts soon. For now, we assume its importance, and ask:

Question 3.1. *Does* $\varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(X_\bullet)$ *admit a concrete description?*

One notion of "concrete description" is a comonadic one. And it is a sensible one here:

1. if the functor ev_0 admitted a right adjoint ev_0^R, we would have the basic groundwork to investigate whether or not ev_0 is comonadic;

2. ev_0 seems the most sensible functor to subject to this investigation because it determines all the other ev_i. Furthermore, the shape of the diagram suggests that its "tip" $\mathrm{Sh}_{\mathscr{S}}(X_0)$ is its most important category.

In fact, a comonadic description is available, and is afforded in two layers of concreteness by the following special case of Theorem 3.6:

Theorem 3.2. *In the above set-up,*

1. *if* ev_0 *admits a right adjoint* ev_0^R*, then the limit can be described comonadically:*

$$\begin{tikzcd}
\varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(X_\bullet) \arrow[rr, equal, "\mathrm{ev}_0^{\mathrm{enh}}"] \arrow[dr, bend right, "\mathrm{ev}_0"'] & & {}_{\mathrm{ev}_0 \circ \mathrm{ev}_0^R}\mathrm{coMod}\, \mathrm{Sh}_{\mathscr{S}}(X_0) \arrow[dl, bend left, "\mathrm{fgt}"] \\
& \mathrm{Sh}_{\mathscr{S}}(X_0) \arrow[loop below, "\mathrm{ev}_0 \circ \mathrm{ev}_0^R"] &
\end{tikzcd}$$

2. *if furthermore the simplicial diagram* $\mathrm{Sh}_{\mathscr{S}}(X_\bullet)$ *satisfies the* Beck-Chevalley condition *(see Theorem 3.6), then the comonad has a more concrete description:*

$$\mathrm{ev}_0 \circ \mathrm{ev}_0^R \simeq s_* \circ t^*$$

where s, t *are the names of the maps*

$$X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0$$

Remark 3.3. *One may call the consequence* $\mathrm{ev}_0 \circ \mathrm{ev}_0^R \simeq s_* \circ t^*$ *of the Beck-Chevalley condition a "base change" result, because it says that the square built using the dashed arrows below commutes:*

$$\begin{tikzcd}
\varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(X_\bullet) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_0"] & \mathrm{Sh}_{\mathscr{S}}(X_0) \arrow[d, "t^*"] \arrow[l, dashed, bend left, "\mathrm{ev}_0^R"] \\
\mathrm{Sh}_{\mathscr{S}}(X_0) \arrow[r, "s^*"] & \mathrm{Sh}_{\mathscr{S}}(X_1) \arrow[l, dashed, bend left, "s_*"]
\end{tikzcd}$$

So, a comonadic description exists quite generally.

Two sources of simplicial diagrams

As mentioned above, we now present two important kinds of simplicial diagrams of spaces, and give names to the resulting limits $\varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(X_\bullet)$. The project of the ensuing chapters will be to explore many examples of each, in a variety of space-sheaf theories (see the next section).

The first source of diagrams is a group action within the chosen category of spaces: a group $G = (G, 1_G, \mu)$ acting on a space X via a map $G \times X \xrightarrow{\text{act}} X$. This produces the simplicial diagram

$$\cdots \begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} G \times G \times X \underset{\mathrm{Id}_G \times \text{act}}{\overset{\mu \times \mathrm{Id}_X}{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}}} G \times X \underset{\pi_2}{\overset{\text{act}}{\rightrightarrows}} X$$

denoted by $G_\bullet \times X$.

Definition 3.4. *The G-equivariant category of $\mathscr{S}$-sheaves on X, denoted $\mathrm{Sh}_{\mathscr{S}}(X)^G$, is defined to be the associated limit:*

$$\mathrm{Sh}_{\mathscr{S}}(X)^G := \varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(G_\bullet \times X)$$

The goal for this situation is to calculate the category.

The second source of diagrams is a morphism $X \xrightarrow{f} Y$ in the chosen category of spaces: iterating fiber products gives the augmented simplicial diagram

$$\cdots \begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} X \times_Y X \times_Y X \underset{\pi_{23}}{\overset{\pi_{12}}{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}}} X \times_Y X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \overset{f}{\dashrightarrow} Y$$

denoted by $X^{\bullet+1}/Y$. This builds a comparison map

$$\mathrm{Sh}_{\mathscr{S}}(Y) \xrightarrow{f^*} \varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(X^{\bullet+1}/Y)$$

The goal for this situation is to determine under which conditions f^* is an equivalence, and pair this with comonadicity to calculate the category $\mathrm{Sh}_{\mathscr{S}}(Y)$.

3.2 Space-sheaf theories

We pause to take stock of the space-sheaf theories we will examine. Let $\mathscr{C}$ denote a category of "spaces", and $\mathscr{S}$ a theory of sheaves on it that carries an adjunction (f^*, f_*) for every morphism $f : X \to Y$ in $\mathscr{C}$. Here are our main examples:

1. $\mathscr{C}$ = the category of spaces with weak equivalences inverted, and $\mathscr{S} := \mathrm{Loc}_k(-)$ the theory of local systems of k-modules (see the next chapter for a more in-depth discussion);

2. $\mathscr{C} = \mathrm{Top}$ the category of topological spaces and continuous maps, and $\mathscr{S} := \mathrm{Sh}_k(-)$ the theory of sheaves of k-modules;

3. $\mathscr{C} = \mathrm{Sch}_k$ the category of k-schemes and qcqs morphisms between them, and $\mathscr{S} := \mathrm{QCoh}(-)$ the theory of quasicoherent sheaves of $\mathscr{O}$-modules;

4. $\mathscr{C} = \mathrm{Sch}_{\mathbb{C}}$ as above, and $\mathscr{S} := \mathrm{DMod}(-)$ the theory of D-modules.

Depending on the context, these sheaf theories take values in AbCat, $\mathrm{st}_k^{\mathrm{Rex}}$, Cat_∞, Cat_∞^L, or $\mathrm{Pr}_k^{L,\mathrm{st}}$.

3.3 All the important theorems

Having asserted that there are important families of cosimplicial diagrams of categories whose limits would be good to calculate, we now formulate in a very general setting the results that will be crucial in calculating them monadically.

Here is a basic set-up: suppose we have a cosimplicial category $\Delta \xrightarrow{\mathscr{C}^\bullet} \mathrm{Cat}_\infty$. We would like to be able to describe its limit, $\varprojlim_\Delta \mathscr{C}^\bullet$. Well, we certainly have a canonical functor

$$\varprojlim_\Delta \mathscr{C}^\bullet \xrightarrow{\mathrm{fgt}} \mathscr{C}^0.$$

The idea, as mentioned in the beginning of this chapter, is that, given the cofiltered shape of Δ and the fact that $\mathscr{C}^0$ is at the "tip" of the entire diagram, an object of $\varprojlim_\Delta \mathscr{C}^\bullet$ "ought" to just be an object of $\mathscr{C}^0$, plus some extra data coming from the deeper parts of the diagram. In other words, one roughly expects for this canonical functor fgt to be monadic for some monad T:

$$\mathrm{fgt}^{\mathrm{enh}} : \varprojlim_\Delta \mathscr{C}^\bullet \xrightarrow{\simeq} {}_T\mathrm{Mod}(\mathscr{C}^0)$$

The results in this direction are stratified as follows:

1. as long as fgt admits a left adjoint fgt^L, it is in fact automatically monadic, for the monad $T := \mathrm{fgt} \circ \mathrm{fgt}^L$;

2. under a few more, we can even more explicitly describe T;

3. there is a simple criterion for when a functor $\mathscr{C}^{-1} \to \varprojlim_\Delta \mathscr{C}^\bullet$, i.e. a coaugmentation of $\mathscr{C}^\bullet$, is in fact an equivalence.

Here are these results, in that order:

Proposition 3.5 ([13] Proposition 4.7.5.1)**.** *Suppose J is any small ∞-category that contains an object $0 \in J$ with the property that for each $j \in J$, it can be reached by a morphism $0 \to j$.*

Let now $\mathscr{C}^\bullet : J \to \mathrm{Cat}_\infty$ be a J-diagram of categories, with $0 \mapsto \mathscr{C}^0$, and put $\varprojlim_J(\mathscr{C}^\bullet) \xrightarrow{\mathrm{fgt}} \mathscr{C}^0$ the canonical map. If fgt *admits a left adjoint* fgt^L*, then* fgt *is monadic:*

$$\mathrm{fgt}^{\mathrm{enh}} : \varprojlim_J \mathscr{C}^\bullet \xrightarrow{\simeq}_{\mathrm{fgt}\circ\mathrm{fgt}^L} \mathrm{Mod}(\mathscr{C}^0)$$

This is psychologically comforting. However, since $\varprojlim_J \mathscr{C}^\bullet$ is abstractly-defined, the monad $T := \mathrm{fgt} \circ \mathrm{fgt}^L$ is also quite abstract. It would be great if it were possible to rewrite it in terms of functors within the original diagram $\mathscr{C}^\bullet$.

Here is a result about when one can do this, in the case of the favorite example of $J = N(\Delta)$ the nerve of the category of cosimplicial sets:

Theorem 3.6 ([13], Theorem 4.7.5.2)**.** *Let $\mathscr{C}^\bullet : N(\Delta) \to \mathrm{Cat}_\infty$ be a cosimplicial category with the* (left) Beck-Chevalley condition*: the functors $\mathscr{C}^m \xrightarrow{d^0} \mathscr{C}^{m+1}$ admit left adjoints $(d^0)^L$ such that, for every $\alpha : [m] \to [n]$ in Δ, the induced commuting solid square*

$$\begin{tikzcd} \mathscr{C}^m \arrow[r, "d^0"] \arrow[d, "\alpha"'] & \mathscr{C}^{m+1} \arrow[l, dashed, bend right, "(d^0)^L"'] \arrow[d, "0\star\alpha"] \\ \mathscr{C}^n \arrow[r, "d^0"] & \mathscr{C}^{n+1} \arrow[l, dashed, bend right, "(d^0)^L"'] \end{tikzcd}$$

commutes as a dashed arrow square as well (we say the solid square is left adjointable*). Put $\mathscr{C} := \varprojlim_\Delta \mathscr{C}^\bullet$, and $\mathscr{C} \xrightarrow{\mathrm{fgt}} \mathscr{C}^0$ the canonical map. Then:*

1. fgt *admits a left adjoint* fgt^L*;*

2. *the "coaugmentation" square is left adjointable:*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\mathrm{fgt}"] \arrow[d, "\mathrm{fgt}"'] & \mathscr{C}^0 \arrow[l, dashed, bend right, "\mathrm{fgt}^L"'] \arrow[d, "d^1"] \\ \mathscr{C}^0 \arrow[r, "d^0"] & \mathscr{C}^1 \arrow[l, dashed, bend right, "(d^0)^L"'] \end{tikzcd}$$

3. fgt *is monadic, where by part (2) above the monad is*

$$T := \mathrm{fgt} \circ \mathrm{fgt}^L = (d^0)^L \circ d^1$$

Remark 3.7. *To remember the diagram above in part (2), think of* fgt *as the only morphism* $C \xrightarrow{d^0} \mathscr{C}^0$*; thus we expect* $[0] \star \mathrm{fgt} = d^1$*, which indeed is the case.*

In other words: if the $N(\Delta)$-diagram satisfies the left Beck-Chevalley condition, then the diagram $N(\Delta_+)$ coaugmented by the totalization also satisfies the left Beck-Chevalley condition for coaugmented cosimplicial categories $\mathscr{C}^{-1} \to \mathscr{C}^\bullet$, where now m, n can also be -1 as well as $\{0, 1, 2, \dots\}$.

Lastly, there is a theorem about how to identify a coaugmentation of a $N(\Delta)$-diagram as the limit, which will be especially important later:

Theorem 3.8 ([13], Corollary 4.7.5.3)**.** *Let* $\mathscr{C}^\bullet : N(\Delta_+) \to \mathrm{Cat}_\infty$ *be a coaugmented cosimplicial category satisfying the (coaugmentd form of the) left Beck-Chevalley condition. Put* $\mathscr{C}^{-1} \xrightarrow{\Theta} \varprojlim_{\Delta} (\mathscr{C}^\bullet)$ *the canonical map, and* $\mathscr{C}^{-1} \xrightarrow{R} \mathscr{C}^0$ *the augmentation.*

In this set-up: if R *is monadic (and thus* R^{enh} *is an equivalence), then* Θ *is also an equivalence. In other words, we simultaneously have the following equivalences:*

$$_{RL}\mathrm{Mod}(\mathscr{C}^0) \xleftarrow{\simeq} \mathscr{C}^{-1} \xrightarrow{\simeq} \varprojlim_{\Delta} (\mathscr{C}^\bullet) = {_T}\mathrm{Mod}(\mathscr{C}^0) = {_{(d^0)^L \circ d^1}}\mathrm{Mod}(\mathscr{C}^0)$$

To summarize the above theory briefly:

1. A monadic description of the *abstract limit* of a cosimplicial diagram of categories $\mathscr{C}^\bullet$, in terms of the first category $\mathscr{C}^0$, is almost always available.

2. In the presence of some base change (for the diagram $\mathscr{C}^\bullet$), this monadic describing the abstract limit can be more explicitly identified. Furthermore, the limit participates in a further piece of base change with the terms $\mathscr{C}^0, \mathscr{C}^1$ of the diagram.

3. A concrete coaugmentation $\mathscr{C}^{-1}$ for the diagram $\mathscr{C}^\bullet$ can be identified with the abstract limit under *a little more base change* (involving $\mathscr{C}^{-1}$ and $\mathscr{C}^0, \mathscr{C}^1$), plus a comonadicity statement for the coaugmentation functor $\mathscr{C}^{-1} \to \mathscr{C}^0$.

3.4 How will we use all this?

We conclude the main part of this chapter by returning to our two main examples of cosimplicial limits of categories and recording what we would need to show in order for the theorems above to apply.

Equivariant categories Given a group action $G \curvearrowright X$ in $\mathscr{C}$, recall that the goal is to calculate the limit

$$\begin{tikzcd}
\mathrm{Sh}_{\mathscr{S}}(X)^G := \varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(G_\bullet \times X) \arrow[dr, dashed, "\mathrm{ev}_0"] \arrow[drr, dashed, bend left, "\mathrm{ev}_1"] \arrow[drrr, dashed, bend left] & & & \\
& \mathrm{Sh}_{\mathscr{S}}(X) \arrow[r, shift left, "\mathrm{act}^*"] \arrow[r, shift right, "\pi_2^*"'] & \mathrm{Sh}_{\mathscr{S}}(G \times X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}$$

We know:

1. by Proposition 3.5, if ev_0 admits a right adjoint ev_0^R, then
$$\mathrm{Sh}_{\mathscr{S}}(X)^G \xrightarrow{\mathrm{ev}_0^{\mathrm{enh}}} {}_{\mathrm{ev}_0 \circ \mathrm{ev}_0^R}\mathrm{coMod}\,\mathrm{Sh}_{\mathscr{S}}(X)$$
is an equivalence;

2. by Theorem 3.6, if furthermore a base change result holds, then we an rewrite the comonad on $\mathrm{Sh}_{\mathscr{S}}(X)$ as
$$\mathrm{ev}_0 \circ \mathrm{ev}_0^R \quad \simeq \quad \pi_{2*} \circ \mathrm{act}^*$$

Remark 3.9. *In editing this document, I decided to omit the chapter on comonadicity and equivariant categories of sheaves. Still, I think it is useful to leave this sketch of the comonadic perspective on describing equivariant categories of sheaves.*

Sheaves on covers Given a morphism $X \xrightarrow{f} Y$ in $\mathscr{C}$, recall the goal is to understand when (i.e. under which conditions on f) the canonical map can below is an equivalence:

$$\begin{tikzcd}
\varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(X^{\bullet+1}/Y) \arrow[drr, dashed, "\mathrm{ev}_0"] \arrow[drrr, dashed, bend left, "\mathrm{ev}_1"] \arrow[drrrr, dashed, bend left] & & & & \\
\mathrm{Sh}_{\mathscr{S}}(Y) \arrow[u, dotted, "\mathrm{can}"] \arrow[rr, "f^*"] & & \mathrm{Sh}_{\mathscr{S}}(X) \arrow[r, shift left, "\pi_1^*"] \arrow[r, shift right, "\pi_2^*"'] & \mathrm{Sh}_{\mathscr{S}}(X \times_Y X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}$$

We know:

1. by Proposition 3.5, if ev_0 admits a right adjoint ev_0^R, then
$$\varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(X^{\bullet+1}/Y) \xrightarrow{\mathrm{ev}_0^{\mathrm{enh}}} {}_{\mathrm{ev}_0 \circ \mathrm{ev}_0^R}\mathrm{coMod}\,\mathrm{Sh}_{\mathscr{S}}(X)$$
is an equivalence;

2. by Theorem 3.6, if furthermore a base change result for the diagram $\mathrm{Sh}_{\mathscr{S}}(X^{\bullet+1}/Y)$ holds, then we an rewrite this comonad on $\mathrm{Sh}_{\mathscr{S}}(X)$ as

$$\mathrm{ev}_0 \circ \mathrm{ev}_0^R \quad \simeq \quad \pi_{2*} \circ \pi_1^*;$$

3. by Theorem 3.8, if furthermore (1) f^* is comonadic and (2) the following *additinal* base change result holds

$$\begin{tikzcd}
\mathrm{Sh}_{\mathscr{S}}(Y) \arrow[r, "f^*"] \arrow[d, "f^*"'] & \mathrm{Sh}_{\mathscr{S}}(X) \arrow[d, "\pi_1^*"] \arrow[l, dashed, bend left, "f_*"] \\
\mathrm{Sh}_{\mathscr{S}}(X) \arrow[r, "\pi_2^*"] & \mathrm{Sh}_{\mathscr{S}}(X \times_Y X) \arrow[l, dashed, bend left, "\pi_{2*}"]
\end{tikzcd}$$

then the comonad can be rewritten again, this time in terms of f as

$$\mathrm{ev}_0 \circ \mathrm{ev}_0^R \quad \simeq \quad \pi_{2*} \circ \pi_1^* \quad \simeq \quad f^* f_*,$$

and furthermore

$$\mathrm{Sh}_{\mathscr{S}}(Y) \xrightarrow{\mathrm{can}} {}_{f^*f_*}\mathrm{coMod}\,\mathrm{Sh}_{\mathscr{S}}(X)$$

is an equivalence. That is, we can identify the desired category $\mathrm{Sh}_{\mathscr{S}}(Y)$ with a category of comodules.

Sheaves with singular support Here we adumbrate, in more detail than in the introduction, how we intend to employ the comonadicity and base change perspective to describe categories of sheaves with singular support.

Given a manifold X, a singular support Lagrangian $\Lambda \subseteq T^*X$, and a closed cover $\{\Lambda_i \subseteq \Lambda\}_{i \in I}$ of Λ, the goal is to understand when (i.e. under which conditions on the cover Λ_i) the canonical map below is an equivalence:

$$\begin{tikzcd}
\varprojlim_{\Delta} \mathrm{Sh}_{\Lambda_{\bullet+1}}(X) \arrow[dr, dashed, "\mathrm{ev}_0"] \arrow[drr, dashed, bend left, "\mathrm{ev}_1"] \arrow[drrr, dashed, bend left] & & & \\
\mathrm{Sh}_\Lambda(X) \arrow[u, dotted, "\mathrm{can}"] \arrow[r, "L"] & \prod_{i \in I} \mathrm{Sh}_{\Lambda_i}(X) \arrow[r, shift left, "\pi_1^*"] \arrow[r, shift right, "\pi_2^*"'] & \prod_{i,j \in I} \mathrm{Sh}_{\Lambda_i \cap \Lambda_j}(X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}$$

We know:

1. by Proposition 3.5, if ev_0 admits a right adjoint ev_0^R, then

$$\varprojlim_{\Delta} \mathrm{Sh}_{\Lambda_{\bullet+1}}(X) \xrightarrow{\mathrm{ev}_0^{\mathrm{enh}}} {}_{\mathrm{ev}_0 \circ \mathrm{ev}_0^R}\,\mathrm{coMod} \prod_{i \in I} \mathrm{Sh}_{\Lambda_i}(X)$$

   is an equivalence;

2. by Theorem 3.6, if furthermore a base change result holds for the diagram $\mathrm{Sh}_{\Lambda_{\bullet+1}}(X)$, then we an rewrite this comonad on $\prod_{i\in I} \mathrm{Sh}_{\Lambda_i}(X)$ as

$$\mathrm{ev}_0 \circ \mathrm{ev}_0^R \quad \simeq \quad \pi_{2*} \circ \pi_1^*;$$

3. by Theorem 3.8, if furthermore (1) L is comonadic and (2) the following *additional* base change result holds

$$\begin{tikzcd}
\mathrm{Sh}_\Lambda(X) \arrow[r, "L"] \arrow[d, "L"] & \prod_{i\in I} \mathrm{Sh}_{\Lambda_i}(X) \arrow[l, dashed, bend left, "R"] \arrow[d, "\pi_1^*"] \\
\prod_{i\in I} \mathrm{Sh}_{\Lambda_i}(X) \arrow[r, "\pi_2^*"] & \prod_{i,j\in I} \mathrm{Sh}_{\Lambda_i\cap\Lambda_j}(X) \arrow[l, dashed, bend left, "\pi_{2*}"]
\end{tikzcd}$$

   then the comonad can be rewritten again, this time in terms of L as

$$\mathrm{ev}_0 \circ \mathrm{ev}_0^R \quad \simeq \quad \pi_{2*} \circ \pi_1^* \quad \simeq \quad LR,$$

   and furthermore

$$\mathrm{Sh}_\Lambda(X) \xrightarrow{\mathrm{can}} {}_{LR}\,\mathrm{coMod} \prod_{i\in I} \mathrm{Sh}_{\Lambda_i}(X)$$

   is an equivalence. That is, we can identify the desired category $\mathrm{Sh}_\Lambda(X)$ with a category of comodules.

3.5 Appendix: bootstrapping comonadicity results

In this section, we consider the question of when comonadicity for a functor of interest can be concluded from comonadicity of related functors. We begin with an immediate result:

Observation 3.10. *Conservative functors satisfy the two-out-of-three property: given a commutative triangle*

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "L_1"] \arrow[dr, "L_2 L_1"'] & & \mathscr{B} \arrow[dl, "L_2"] \\
& \mathscr{D} &
\end{tikzcd}$$

then if any two functors are conservative, so is the third.

In contrast to conservative functors, comonadic functors only satisfy the following two-out-of-three property:

Lemma 3.11. *If $L_2 L_1$ and L_2 are comonadic, then L_1 is comonadic.*

Proof. Certainly L_1 is conservative. Suppose now that $\Delta \xrightarrow{F^\bullet} \mathscr{C}$ is such that $L_1 F^\bullet$ is split. Then $L_2(L_1 F^\bullet)$ is also split, whence by comonadicity of $L_2 L_1$, we know

$$L_2 L_1 \varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} L_2 L_1 F^\bullet$$

Furthermore, again since $L_2(L_1 F^\bullet)$ is split, by comonadicity of L_2 we know that

$$\varprojlim_{\Delta} L_2 L_1 F^\bullet \simeq L_2 \varprojlim_{\Delta} L_1 F^\bullet,$$

which we chain together to conclude that

$$L_2 L_1 \varprojlim_{\Delta} F^\bullet \simeq L_2 \varprojlim_{\Delta} L_1 F^\bullet$$

Since L_2 is conservative, we conclude that in fact

$$L_1 \varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} L_1 F^\bullet$$

as desired. □

It is a bit of a crime that the other two do not hold. What goes wrong?

Well, the composition $L_2 L_1$ is always conservative. The issue is with L_1 being able to preserve certain limits. To see it, consider an $L_2 L_1$-split cosimplicial object $\Delta \xrightarrow{F} \mathscr{C}$; we wish to show that the canonical morphism

$$L_2 L_1 \varprojlim_{\Delta} F^\bullet \xrightarrow{\text{can}} \varprojlim_{\Delta} L_2 L_1 F^\bullet$$

is an isomorphism. We can begin by noting that $L_1 F^\bullet$ is L_2-split, and therefore, since L_2 is comonadic, that

$$L_2 L_1 \varprojlim_{\Delta} F^\bullet \xrightarrow{\text{can}} \varprojlim_{\Delta} L_2 L_1 F^\bullet \simeq L_2 \varprojlim_{\Delta} L_1 F^\bullet$$

But the argument halts here, since there is no reason for L_1 to commute past the limit; indeed, $F^\bullet$ is not L_1-split, only $L_2 L_1$-split.

In spite of the above bad luck, there are certain situations in which the composite of two comonadic functors is comonadic. We continue working in Pr^L, where all categories admit all limits; thus the only issue is whether L_1 preserves enough limits. This can be achieved by strengthening either L_1 or L_2:

Lemma 3.12. *If L_1 and L_2 are comonadic, and L_1 preserves limit cosimplicial diagrams, then L_2L_1 is comonadic.*

Proof. The assumption allows L_1 to commute past the limit. $\square$

Lemma 3.13. *Let L be a functor. If S is another functor so that $SL = \mathrm{Id}$, then L is comonadic. In fact, it reflects absolute limit diagrams.*

Proof. The assumption immediately implies that L is conservative. Furthermore, if $\Delta \xrightarrow{F^\bullet} \mathscr{C}$ is such that $LF^\bullet$ is split, then $S(LF^\bullet) = F^\bullet$ is also split, whence it is an absolute limit diagram and so $L\varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} LF^\bullet$. $\square$

As an example application, we now show that, once we know that Zariski descent for $\mathrm{QCoh}(-)$ holds, then so does descent under bundle projections $P \to X$.

Lemma 3.14. *The functor p^* is comonadic for any bundle projection $P \xrightarrow{p} X$.*

Proof. If $P \cong X \times F$ and p is the projection, then it admits a section s, whence $s^*p^* = \mathrm{Id}$, and we conclude by the above Lemma. Otherwise, there is a Zariski open cover $U \xrightarrow{f} X$ over which $P \times_U X \cong U \times F$ admits a section s_U:

$$\begin{tikzcd} P \times_U X \arrow[r, "f'"] \arrow[d, "p'"] & P \arrow[d, "p"] \\ U \arrow[r, "f"] \arrow[u, bend left, "s_U"] \arrow[ur, dashed, "s"] & X \end{tikzcd}$$

Since Zariski covers are stable under base change, $(f')^*$ is comonadic, and therefore by Lemma 3.11 we will be done if we show that $(f')^*p^*$ is comonadic. By commutativity, this is the composite $(p')^*f^*$. We now show that this functor preserves the necessary limits. So let $\Delta \xrightarrow{F^\bullet} \mathrm{QCoh}(X)$ be a diagram such that $(p')^*f^*F^\bullet$ is split. Use the section to deduce that $s_U^*(p')^*f^*F^\bullet \simeq f^*F^\bullet$ is split. Since f^* is comonadic, $f^*\varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} f^*F^\bullet$, and thus

$$(p')^*f^*\varprojlim_{\Delta} F^\bullet \simeq (p')^*\varprojlim_{\Delta} f^*F^\bullet \simeq \varprojlim_{\Delta} (p')^*f^*F^\bullet$$

where the second isomorphism is due to $f^*F^\bullet$ being split. $\square$

Chapter 4

Examples of Descent I: Local Systems

4.1 What is in this chapter?

A pointed topological space (X, x) participates in the following silly but important diagram:

$$\mathrm{pt} \xrightarrow{x} X \xrightarrow{p} \mathrm{pt}$$

This diagram produces a list of important adjunctions on categories of local systems in Pr^L,

$$\begin{tikzcd} \mathrm{Loc}(\mathrm{pt}) \arrow[loop left, "T:=x^* \circ x_!^h"] \arrow[r, shift left=2, "x_!^h"] \arrow[r, dashed, shift right=2, "x_*^h"'] & \mathrm{Loc}(X) \arrow[l, "x^*" description] \arrow[r, shift left=2, "p_!^h"] \arrow[r, dashed, shift right=2, "p_*^h"'] & \mathrm{Loc}(\mathrm{pt}) \arrow[l, "p^*" description] \arrow[loop right, "\Omega:=p_!^h \circ p^*"] \end{tikzcd}$$

where we recall that our convention is to use dashed arrows to denote right adjoints that are not necessarily colimit-preserving. We can use these adjunctions to study the category $\mathrm{Loc}(X)$ (co)monadically in terms of the easy category $\mathrm{Loc}(\mathrm{pt}) = {}_k\mathrm{Mod}$. Since ${}_k\mathrm{Mod}$ is atomically simple, all of the complexity will lie in the (co)monads.

In particular, we might hope that these adjunctions are monadic and comonadic already, respectively. After seeing how to identify the algebra $x^*x_!^h(k) \simeq C_{-\bullet}(\Omega_x X; k)$ and the coalgebra $p_!^h p^*(k) \simeq C_{-\bullet}(X; k)$, we may therefore formulate this hope as the statement that the following is a diagram of equivalences:

$$\begin{tikzcd} & \mathrm{Loc}(X) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ {}_{C_{-\bullet}(\Omega_x X)}\mathrm{Mod} & & {}_{C_{-\bullet}(X)}\mathrm{coMod} \end{tikzcd}$$

This hope generally is only half correct:

1. The good news is that the left-hand side is an equivalence with almost no assumptions: indeed, as long as X is path-connected, the operation x^* of taking the fiber at any

point $x \in X$ will detect whether a local system is 0 or not. This assumption of path-connectivity in fact suffices for monadicity of $x_!^h$.

2. The bad news is that conservativity of $p_!^h$ is harder to arrange, and generally comes at the price of having to restrict to subcategories of just those local systems that can "see" the constant local system $p^*k \simeq k_X$. For instance, on $X = S^1$, the rank 1 local system $\mathscr{L}_{-1}$ with monodromy -1 is a nonzero object that is killed by the functor $p_!^h = C_{-\bullet}(-;k)$ of taking homology. But there, restricting to unipotent local systems fixes the bug, and gives a sub-equivalence on the right-hand side. We will see some other fixes as well, but they all feature restricting the kinds of local systems, restricting the kinds of spaces, or both.

This chapter is a pedagogical side quest, and the only goal is to practice proving descent statements, and in particular ones that involve carefully restricting the domains of functors to restore certain hypotheses required by Barr-Beck-Lurie.

4.2 Definition of ∞-local systems

Classically, a local system on a space X is a locally constant sheaf over some base ring k. If X is path-connected and semi-locally simply-connected, then the choice of any basepoint $x \in X$ gives an equivalence of categories

$$\mathrm{Loc}_k(X) \simeq \mathrm{Rep}_k(\pi_1(X,x))$$

In this case, there is an equivalence of categories $B\pi_1(X,x) \simeq \Pi_1 X$, the fundamental groupoid of X, and so we can reformulate this equivalence as

$$\begin{aligned} \mathrm{Loc}_k(X) &\xrightarrow{\quad\simeq\quad} \mathrm{Fun}(\Pi_1(X), {}_k\mathrm{Mod}^{\heartsuit}), \\ \mathscr{L} &\longmapsto \begin{cases} x \mapsto M_x, \\ (\gamma : x \rightsquigarrow y) \mapsto (\Gamma : M_x \xrightarrow{\cong} M_y) \end{cases} \end{aligned}$$

This erases the dependence on any basepoint, and thus the above equivalence holds for X not necessarily path-connected.

We now turn this observation into a definition in the setting of ∞-categories. First we make the following replacements:

1. $\Pi_1(X) \rightsquigarrow \Pi_\infty(X)$ the fundamental ∞-groupoid of X, or its simplicial set analog $\mathrm{Sing}(X)$;

2. ${}_k\mathrm{Mod}^{\heartsuit} \rightsquigarrow {}_k\mathrm{Mod}$ the stable presentable ∞-category of k-modules;

3. Fun $\rightsquigarrow$ Fun_∞ the ∞-category of ∞-functors.

We can now define:

Definition 4.1. *For a homotopy type X, the category of* ∞-local systems *of k-modules is*

$$\mathrm{Loc}_{{}_k\mathrm{Mod}}(X) := \mathrm{Fun}_\infty(\Pi_\infty X, {}_k\mathrm{Mod})$$

Informally, such a functor assigns a cochain complex of k-modules to each point of X, a roof of quasi-isomorphisms of complexes to each path in X (the "1-monodromy"), a homotopy of roofs of quasi-isomorphisms to each 1-cell between paths (the "2-monodromy"), a homotopy of homotopies to each 2-cell between 1-cells between paths (the "3-monodromy"), and so on.

The goal of this chapter is to use monadicity to describe these categories, or pieces of them.

4.3 Base change for local systems

We first collect an important tool in the theory of local systems: a Beck-Chevalley, or base change, type of result.

To do this, we momentarily switch to a more general set-up, replacing ${}_k\mathrm{Mod}$ by any fixed ∞-category $\mathscr{C}$, and considering the functor

$$\mathrm{Fun}_\infty(-, \mathscr{C}) : \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty\,.$$

Of course, we can evaluate it on spaces X via the ∞-groupoid construction:

$$\mathrm{Loc}_{\mathscr{C}}(X) := \mathrm{Fun}_\infty(\Pi_\infty X, \mathscr{C}) \in \mathrm{Cat}_\infty\,.$$

We abbreviate the latter simply by $\mathrm{Loc}(X)$ if $\mathscr{C}$ is clear from context. Knowing nothing else, one of these categories is always easily computable: $\mathrm{Loc}(\mathrm{pt}) = \mathscr{C}$.

Let us examine the functoriality of these categories. Suppose that $A \xrightarrow{f} B$ is a morphism in Cat_∞:

1. Most basically, f determines a *pullback* functor $f^* : \mathrm{Fun}(B, \mathscr{C}) \to \mathrm{Fun}(A, \mathscr{C})$, just by by precomposition with f.

2. If $\mathscr{C}$ furthermore admits all colimits, then we can also define an adjoint functor $f^h_!$, the (homotopical) shriek pushforward under f, as being given by the left Kan extension along f:

$$\begin{tikzcd}
\mathrm{pt} \underset{B}{\overset{\rightarrow}{\times}} A \arrow[r, dotted, "\mathrm{fgt}"] \arrow[d, dotted] & A \arrow[r, "\mathscr{L}"] \arrow[d, "f"] & \mathscr{C} \\
\mathrm{pt} \arrow[r, "b"] \arrow[ur, Rightarrow] & B \arrow[ur, dashed, bend right, "f^h_!\mathscr{L} := {}_f\mathscr{L}"'] &
\end{tikzcd}$$

where by definition $\mathrm{pt} \overset{\rightarrow}{\underset{B}{\times}} A =: A_{b/f}$ is the comma category, and

$$ {}_f\mathscr{L} := \underset{\longrightarrow}{\mathrm{colim}}\left(\mathrm{pt} \overset{\rightarrow}{\underset{B}{\times}} A \xrightarrow{\mathrm{fgt}} A \xrightarrow{\mathscr{L}} \mathscr{C} \right) $$

Remark 4.2. *The superscript h in the notation $f_!^h$, which stands for "homotopical," is meant to caution us that this functor from homotopy theory is very different from the proper pushforward $f_!$ in topology. To see the difference, we can compare them in the context of a concrete continuous map $f : X \to Y$ of topological spaces and the choice $\mathscr{C} = {}_k\mathrm{Mod}$. In this case, the following diagram usually does not commute:*

$$ \begin{tikzcd} \mathrm{Loc}_k(X) \arrow[r, hook] \arrow[d, "f_!^h"] & \mathrm{Sh}_k(X) \arrow[d, "f_!"] \\ \mathrm{Loc}_k(Y) \arrow[r, hook] & \mathrm{Sh}_k(Y) \end{tikzcd} $$

Indeed, $f_!$ need not even produce local systems out of local systems! But $f_!^h$ must. We will see examples in a moment.

In sum, for $\mathscr{C}$ a category with colimits over the necessary comma categories, we have an adjunction in Cat_∞

$$ \begin{tikzcd} \mathrm{Fun}(B,\mathscr{C}) \arrow[r, shift right, "f^*"'] & \mathrm{Fun}(A,\mathscr{C}) \arrow[l, shift right, "f_!^h"'] \end{tikzcd} $$

To calculate these categories $\mathrm{Fun}(A,\mathscr{C})$, the following base change result is crucial, and is the central result of this section:

Lemma 4.3 (Base change for local systems)**.** *Let $\mathscr{C}$ be a category with (small) colimits, and consider a fiber product square in* Cat_∞

$$ \begin{tikzcd} C \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] \arrow[dr, phantom, "\square"] & A \arrow[d, "f"] \\ D \arrow[r, "g"] & B \end{tikzcd} $$

with the further property that

1. *A and B are ∞-groupoids, and*

2. *C and D are small ∞-categories.*

Then the following square with dashed arrows commutes:

$$\begin{tikzcd} \mathrm{Fun}(C,\mathscr{C}) \arrow[r, dashed, bend left, "\overline{g}_!^h"] & \mathrm{Fun}(A,\mathscr{C}) \arrow[l, "\overline{g}^*"'] \\ \mathrm{Fun}(D,\mathscr{C}) \arrow[u, "\overline{f}^*"] \arrow[r, dashed, bend left, "g_!^h"] & \mathrm{Fun}(B,\mathscr{C}) \arrow[l, "g^*"'] \arrow[u, "f^*"'] \end{tikzcd}$$

i.e. $\overline{g}_!^h \overline{f}^* \simeq f^* g_!^h$.

Proof. This result appears in the proof of Proposition 4.4.3 of [8]; we learned about it from Peter Haine, and the current proof follows the proof of Proposition 1.5 in the note [7]. Let $\mathscr{L} \in \mathrm{Fun}(D, \mathscr{C})$. For $a \in A$, we simply wish to relate the following two objects of $\mathscr{C}$:

$$\begin{aligned} \overline{g}_!^h \overline{f}^* \mathscr{L}(a) &:= \varinjlim\left(C_{a/\overline{g}} \xrightarrow{\mathrm{fgt}} C \xrightarrow{\overline{f}} D \xrightarrow{\mathscr{L}} \mathscr{C}\right), \\ f^* g_!^h \mathscr{L}(a) &:= \varinjlim\left(D_{f(a)/g} \xrightarrow{\mathrm{fgt}} D \xrightarrow{\mathscr{L}} \mathscr{C}\right) \end{aligned}$$

Here, the colimits are indexed by comma categories that are rendered small by property (2), hence they exist in $\mathscr{C}$ by its assumed small co-completeness. To make sense of them, it helps to look at the diagram

$$\begin{tikzcd} C_{a/\overline{g}} \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d, "a"] \arrow[dl, Rightarrow] \\ C \arrow[r, "\overline{g}"] \arrow[d, "\overline{f}"'] \arrow[dr, phantom, "\square"] & A \arrow[d, "f"] \\ D \arrow[r, "g"] \arrow[d, "\mathscr{L}"] & B \\ \mathscr{C} & \end{tikzcd}$$

The consequences of assumption (1) now allow us to re-interpret it. Since A is a groupoid, that means that the top square is actually a fiber product, and so the overcategory $C_{a/\overline{g}}$ is $\mathrm{pt} \times_A^h C$. Since B is a groupoid, that means that the overcategory $D_{f(a)/g}$ is $\mathrm{pt} \times_B^h D$. Lastly, the 2-3 property of fiber product squares identifies the fiber products $C_{a/\overline{g}} \simeq D_{f(a)/g}$, which identifies the colimits above. □

Consequences and examples of base change

Since spaces are by definition objects of the subcategory $\mathrm{Grpd}_\infty \subseteq \mathrm{Cat}_\infty$, the following version of the above base change result holds very generally:

Corollary 4.4. *For any category $\mathscr{C}$ with colimits, $\mathscr{C}$-valued local systems on spaces satisfy base change for $f_!^h$ and f^*.*

Remark 4.5. *A parallel result holds for $\mathscr{C}$ a category with limits, and homotopical pushforward f^h_*, defined as a right Kan extension.*

In fact, we now return to picking our favorite instance of $\mathscr{C}$: the category ${}_k\,\mathrm{Mod}$ for k a classical ring. In addition to ensuring that both base change results for $\mathrm{Loc}(-)$ hold for any map $X \xrightarrow{f} Y$ of spaces, it also implies the following nice properties of $\mathrm{Loc}(X)$:

Lemma 4.6. *Let $\mathscr{C} := {}_k\,\mathrm{Mod}$ for k a classical ring, or any other bicomplete stable presentable category. Then the following are true for categories $\mathrm{Loc}(X) := \mathrm{Fun}(X, {}_k\,\mathrm{Mod})$:*

1. *the category $\mathrm{Loc}(X)$ is stable and presentable, hence in particular bicomplete;*

2. *every map of spaces $X \xrightarrow{f} Y$ yields adjoints $(f^h_!, f^*, f^h_*)$ in Cat_∞, and therefore adjoints $(f^h_!, f^*)$ in Pr^L:*

$$\begin{tikzcd} \mathrm{Loc}(X) \arrow[r, shift left=2, "f^h_!"] \arrow[r, leftarrow, "f^*" description] \arrow[r, shift right=2, dashed, "f^h_*"'] & \mathrm{Loc}(Y) \end{tikzcd}$$

Proof. For part (1), since X is a small simplicial set and ${}_k\,\mathrm{Mod}$ is presentable, [14] Proposition 5.5.3.6 implies that $\mathrm{Fun}(X, {}_k\,\mathrm{Mod})$ is also presentable. For part (2), as discussed, the adjoints exist because ${}_k\,\mathrm{Mod}$ is bicomplete and hence admit the construction of Kan extensions. Since $f^h_!$ and f^* are left adjoints, it means that they exist as morphisms in Pr^L. $\square$

Remark 4.7. *We record some immediate but useful consequence of the above Lemma, for $X \xrightarrow{f} Y$ a map of spaces:*

1. *pullbacks f^* always preserve all limits and colimits on $\mathrm{Loc}(-)$, and*

2. *$f^h_!$ is proper, i.e. preserves compact objects. Thus $x^h_! k$ is always compact for any point $\mathrm{pt} \xrightarrow{x} X$. Since $\hom(x^h_! k, \mathscr{L}) = \hom(k, x^*\mathscr{L}) = \mathscr{L}_x$ is the fiber at x of any local system $\mathscr{L}$, the object $x^h_! k$ corepresents the stalk functor x^*.*

We give a name to the objects $x^h_! k$:

Definition 4.8. *Let $\mathrm{pt} \xrightarrow{x} X$ be a point in a path-connected space X.*

1. *Call $x^h_! k$ the* couniversal local system*, and*

2. *call $x^h_* k$ the* universal local system.

As mentioned at the beginning of the chapter, the most important examples of maps for our discussions are going to be

$$\mathrm{pt} \xrightarrow{x} X \xrightarrow{p} \mathrm{pt}$$

which just specify a based space (X, x). The resulting homotopy fiber product square

$$\begin{tikzcd} \Omega_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{pt} \arrow[d, "x"] \\ \mathrm{pt} \arrow[r, "x"] & X \end{tikzcd}$$

will be the centerpiece of our monadic description of local systems on X. Before getting there, let us calculate these functors on some local systems that are always available.

Example 4.9. *Let X be any space. The local system corresponding to the constant functor $k : X \to {}_k\mathrm{Mod}$ will be denoted k_X. It factors through the map $X \xrightarrow{p} \mathrm{pt}$, and therefore $k_X \simeq p^* k_{\mathrm{pt}} = p^* k$.*

Example 4.10. *Taking k_X and applying p^h_* and $p^h_!$ to it gives the cohomology and homology of X, respectively:*

$$\begin{aligned} p^h_! k_X &\simeq C_{-\bullet}(X; k), \\ p^h_* k_X &\simeq C^\bullet(X; k) \end{aligned}$$

To see the first, recall that

$$C_{-\bullet}(-; k) : \mathrm{Space} \to {}_k\mathrm{Mod}$$

can be characterized as the unique colimit-preserving functor that is normalized by the condition that $\mathrm{pt} \mapsto k$; since every space is a colimit of a diagram of points, this defines the functor on Space. Another functor $\mathrm{Space} \to {}_k\mathrm{Mod}$ is

$$X \quad \mapsto \quad p^h_! k_X := \varinjlim\mathrm{colim}\big(X \xrightarrow{k_X} {}_k\mathrm{Mod}\big)$$

which by definition is colimit-preserving, and also maps $X := \mathrm{pt}$ to k. Thus the two coincide, hence the isomorphism. A dual argument gives the isomorphism for cochains.

Example 4.11. *Let (X, x) be a based space, and consider $x^h_! k \in \mathrm{Loc}(X)$. We can understand its fiber at a point $y \in X$ by using Lemma 4.3 applied to the homotopy fiber square*

$$\begin{tikzcd} \mathrm{Path}_{x \to y} \arrow[r, "p"] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & \mathrm{pt} \arrow[d, "y"] \\ \mathrm{pt} \arrow[r, "x"] & X \end{tikzcd}$$

to get

$$\begin{aligned} y^* x^h_! k &= p^h_! p^* k \\ &\simeq C_{-\bullet}(\mathrm{Path}_{x \to y}; k) \\ &\simeq \begin{cases} C_{-\bullet}(\Omega_x X; k) & x \simeq y, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Similarly, we can find the fiber of $x^h_* k$ by

$$\begin{aligned} y^*(x^h_* k) &= \hom_{\mathrm{pt}}(k, y^* x^h_* h) \\ &\simeq \hom_{\mathrm{pt}}(k, p^h_* p^* k) \\ &\simeq \hom_{\mathrm{Path}_{x\to y}}(p^* k, p^* k) \\ &\simeq C^\bullet(\mathrm{Path}_{x\to y}; k) \\ &\simeq \begin{cases} C^\bullet(\Omega_x X; k) & x \simeq y, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The object k_X receives and gives canonical maps

$$x^h_! k \to k_X \to x^h_* k$$

coming from the counit for $(x^h_!, x^*)$ and the unit for (x^*, x^h_*), respectively.

Recall that the couniversal local system $x^h_! k$ corepresents the functor x^*, and has trivial homology $p^h_! x^h_! k \simeq k$. Since $H_0(\Omega_x X; k) \cong k[\pi_1(X, x)]$, we might think of its classical piece

$$\mathscr{H}^0(x^h_! k) \cong c_* k_{\tilde{X}},$$

where $\tilde{X} \xrightarrow{c} X$ is the universal cover, as being the local system that is freely generated by all possible monodromies, and subject only to the condition that loop composition results in monodromy composition.

Example 4.12. *To be more concrete, take $X = S^1$. Since $\Omega_x S^1 \simeq \mathbb{Z}$ are the winding numbers, the fibers of the couniversal and universal local systems are therefore chains and cochains on $\mathbb{Z}$:*

$$\begin{aligned} x^* x^h_! k &\simeq k^{\oplus \mathbb{Z}} = k[t, t^{-1}], \\ x^* x^h_* k &\simeq k^{\times \mathbb{Z}} = k[[t, t^{-1}]] \end{aligned}$$

Example 4.13. *More generally, for $X = S^n$ for $n \geq 2$, the fibers are*

$$\begin{aligned} x^* x^h_! k_{S^2} &\simeq k\langle u\rangle, |u| = -(n-1) \\ x^* x^h_* k_{S^2} &\simeq k\langle s\rangle, |s| = (n-1) \end{aligned}$$

Aside: another category of local systems?

We make a short comment about the definition of $\mathrm{Loc}(X)$ we chose to work with.

Given that ${}_k\mathrm{Mod} \simeq \mathrm{Ind}\,\mathrm{Perf}(k)$, in defining local systems, one could choose to take Ind before or after taking functors:

$$\mathrm{Fun}(X, \mathrm{Ind}\,\mathrm{Perf}(k)) \quad \text{vs.} \quad \mathrm{Ind}\,\mathrm{Fun}(X, \mathrm{Perf}(k))$$

They are:

1. $\mathrm{Loc}(X) := \mathrm{Fun}(X, \mathrm{Ind}\,\mathrm{Perf}(k))$, the category of local systems whose fibers are k-modules;

2. $\mathrm{Loc}'(X) := \mathrm{Ind}\,\mathrm{Fun}(X, \mathrm{Perf}(k)) = \mathrm{Ind}\,\mathrm{Loc}^{\mathrm{Perf}}(X)$, the category of "ind-perfect" local systems: local systems that are formal inductive limits of finite-rank local systems.

Their relationship is established by the canonical morphism in $\mathrm{Pr}^{L,\mathrm{st}}$

$$\begin{array}{ccc} \mathrm{Loc}'(X) & \xrightarrow{\quad\Phi\quad} & \mathrm{Loc}(X) \\ & & \\ \text{“}\underset{\longrightarrow\ \alpha}{\mathrm{colim}}\text{''}\,\mathscr{L}_\alpha & \longmapsto & \underset{\longrightarrow\ \alpha}{\mathrm{colim}}\ \mathscr{L}_\alpha \end{array}$$

that sends the formal colimit to an actual colimit calculated in $\mathrm{Loc}(X)$, which is the ind-completion of the embedding $\mathrm{Loc}^{\mathrm{Perf}}(X) \subseteq \mathrm{Loc}(X)$. [1]

While it preserves colimits by definition, Φ is far from an equivalence:

1. the compact object k_X in $\mathrm{Loc}'(X)$ need not be compact in $\mathrm{Loc}(X)$, because the functor of cohomology $p^h_* := \hom_{\mathrm{Loc}(X)}(k_X, -)$ does not generally preserve colimits (although see Corollary 4.30);

2. the object $x^h_! k$, compact in $\mathrm{Loc}(X)$ by Remark 4.7, has fibers isomorphic to $C_{-\bullet}(\Omega_x X; k)$, which is hardly ever a perfect k-module; thus $x^h_!$ is typically not in the Φ-image of $\mathrm{Loc}^{\mathrm{Perf}}(X)$.

Note that owing to part (1), Φ does not typically admit a right adjoint in $\mathrm{Pr}^{L,\mathrm{st}}$.

These categories $\mathrm{Loc}(X)$ and $\mathrm{Loc}'(X)$ for a space X are the homotopy theory analogs of $\mathrm{QCoh}(X)$ and $\mathrm{IndCoh}(X)$ for a scheme X, respectively, in at least the sense of having similar functoriality. To see the difference, take again the maps $\mathrm{pt} \xrightarrow{x} X \xrightarrow{p} \mathrm{pt}$:

1. as discussed, the theory of Kan extensions gave the adjunctions in $\mathrm{Pr}^{L,\mathrm{st}}$

$$\begin{tikzcd} \mathrm{Loc}(\mathrm{pt}) \arrow[r, "p^*" description] & \mathrm{Loc}(X) \arrow[l, bend right, "p^h_!"'] \arrow[l, bend left, dashed, "p^h_*"] \arrow[r, "x^*" description] & \mathrm{Loc}(\mathrm{pt}) \arrow[l, bend right, "x^h_!"'] \arrow[l, bend left, dashed, "x^h_*"] \end{tikzcd}$$

[1] Alternatively, one can think of Φ as the natural functor from the colimit to the limit

$$\underbrace{\underset{\overrightarrow{X}}{\mathrm{colim}}({}_k\,\mathrm{Mod})}_{\mathrm{Loc}'(X)} \xrightarrow{\mathrm{can}} \underbrace{\underset{\overleftarrow{X}}{\lim}({}_k\,\mathrm{Mod})}_{\mathrm{Loc}(X)}$$

where ${}_k\,\mathrm{Mod}$, thought of as a functor $X \xrightarrow{{}_k\,\mathrm{Mod}} \mathrm{Pr}^{L,\mathrm{st}}$, is simply the constant diagram $\mathrm{pt} \mapsto {}_k\,\mathrm{Mod}$. We learned this from the note [19].

2. On the other hand, the functors of stable small categories

$$\mathrm{Perf}(k) \xrightarrow{\;p^*\;} \mathrm{Loc}^{\mathrm{Perf}}(X) \xrightarrow{\;x^*\;} \mathrm{Perf}(k)$$

yield the adjunctions in $\mathrm{Pr}^{L,\mathrm{st}}$

$$\begin{tikzcd} \mathrm{Loc}(\mathrm{pt}) \arrow[r, shift left=2, "\mathrm{Ind}\, p^*"] & \mathrm{Loc}'(X) \arrow[l, "(\mathrm{Ind}\, p^*)^R"] \arrow[l, shift right=2, dashed, leftarrow, "(\mathrm{Ind}\, p^*)^{RR}"'] \arrow[r, shift left=2, "\mathrm{Ind}\, x^*"] & \mathrm{Loc}(\mathrm{pt}) \arrow[l, "(\mathrm{Ind}\, x^*)^R"] \arrow[l, shift right=2, dashed, leftarrow, "(\mathrm{Ind}\, x^*)^{RR}"'] \end{tikzcd}$$

where the fact that p^* and x^* preserve compact objects translates into the properness of their ind-completions, and thus the existence of colimit-preserving right adjoints $(\mathrm{Ind}\, p^*)^R$ and $(\mathrm{Ind}\, x^*)^R$.

We will not use $\mathrm{Loc}'(X)$ in this document, but one may try to find BGS-type statements (cf. Proposition 2.50) for $\mathrm{Loc}'(X)$ in place of $\mathrm{Loc}(X)$.

4.4 Monadicity for local systems

We come to our first descent result for local systems, which as promised is very general:

Theorem 4.14. *For a path-connected pointed space (X, x) and classical commutative ring k, the functor*

$$\mathrm{Loc}(X) \xrightarrow{\;x^{*,\mathrm{enh}}\;} {}_{C_{-\bullet}(\Omega_x X;k)}\mathrm{Mod},$$

$$\mathscr{L} \longmapsto \mathscr{L}|_x \circlearrowleft C_{-\bullet}(\Omega_x X;k)$$

is an equivalence.

Proof. We first show that the adjunction in Pr^L

$$\begin{tikzcd} \mathrm{Loc}(X) \arrow[r, "x^*"'] & {}_k\mathrm{Mod} \arrow[l, bend right, "x_!^h"'] \end{tikzcd}$$

is monadic, by checking the hypotheses of monadic Barr-Beck-Lurie in Pr^L. This only requires x^* to be conservative, and this holds because X is assumed to be path-connected and because local systems are locally-constant sheaves.

So x^* is monadic, and we now identify the monad $T := x^* x_!^h \curvearrowright {}_k\mathrm{Mod}$. Since $x_!^h, x^*$ preserve colimits, they in particular commute with every functor $(-) \otimes_k V$ for $V \in {}_k\mathrm{Mod}$, and therefore

$$T(V) = T(k \otimes V) = T(k) \otimes V = C_{-\bullet}(\Omega_x X; k) \otimes V,$$

under which we see that ${}_T\mathrm{Mod}\left({}_k\mathrm{Mod}\right) \simeq {}_{C_{-\bullet}(\Omega_x X;k)}\mathrm{Mod}$. The unit for this algebra can be identified with the result of applying the chains functor to the inclusion $\mathrm{pt} \xrightarrow{c_x} \Omega_x X$ of the constant loop at x. ∎

We record the comonadic version as well:

Corollary 4.15. *Under similar hypotheses, the functor* $x^{*\mathrm{enh}}$

$$\begin{tikzcd} \mathrm{Loc}(X) \arrow[r, "x^{*,\mathrm{enh}}"] & {}_{\Omega}\,\mathrm{coMod}, \\ \mathscr{L} \arrow[r, mapsto] & \mathscr{L}|_x \arrow[loop right, "\Omega := x^* \circ x^h_*"] \end{tikzcd}$$

is also an equivalence.

Proof. We check the hypotheses of comonadic Barr-Beck-Lurie, and work in Cat_∞ since the adjoint x^h_* does not generally preserve colimits. But x^* is still conservative and bicontinuous, so $x^{*,\mathrm{enh}}$ is an equivalence. However, while we know that

$$\Omega(k) \simeq C^\bullet(\Omega_x X; k),$$

the functor $x^h_* \simeq \hom_k(C_{-\bullet}(\Omega_x X; k), -)$ always will lack the necessary continuity (since A is never k-compact) for us to be able to say that $\Omega(V) \simeq C^\bullet(\Omega_x X; k) \otimes_k V$ for a general $V \in {}_k\,\mathrm{Mod}$. $\square$

Remark 4.16. *Note that the monadic equivalence* $x^{*,\mathrm{enh}}$ *is* not monoidal. *This is because* $\mathrm{Loc}(X)$ *has the usual fiberwise tensor product of local systems, but* ${}_{C_{-\bullet}(\Omega_x X;k)}\,\mathrm{Mod}$ *is usually not even a monoidal category! Indeed, the* ∞*-group* $\Omega_x X$ *is generally not abelian, and so the algebra* $C_{-\bullet}(\Omega_x X; k)$ *is not commutative, meaning that there is no monoidal structure available. Note that this happens in spite of the original categories* $\mathrm{Loc}(X)$ *and* ${}_k\,\mathrm{Mod}$ *being monoidal, and even the functor* x^* *being a monoidal functor.*

The lesson, as mentioned earlier, is: if the underlying monadic functor R *preserves some structure between the base categories, that structure may either fail to lift to the category of* T*-modules, or fail to be preserved by the monadic equivalence.*

Having proven the monadic descent theorem, we use the result to prove a limit descent theorem. The homotopical cover

$$\mathrm{pt} \xrightarrow{x} X$$

of the path-connected space X produces the following augmented simplicial diagram in spaces:

$$\left[\cdots \underbrace{\mathrm{pt} \times_X \mathrm{pt} \times_X \mathrm{pt}}_{\Omega_x X \times \Omega_x X} \rightrightarrows \underbrace{\mathrm{pt} \times_X \mathrm{pt}}_{\Omega_x X} \overset{p}{\underset{p}{\rightrightarrows}} \mathrm{pt} \right] \xrightarrow{x} X$$

which gives the diagram in Pr^L

$$\begin{tikzcd}
\varprojlim_{\Delta} \mathrm{Loc}(\mathrm{pt}^{\bullet+1}/X) \arrow[dr, dashed, "\mathrm{ev}_0"] \arrow[drr, dashed, bend left, "\mathrm{ev}_1"] \arrow[drrr, dashed, bend left] & & & \\
\mathrm{Loc}(X) \arrow[u, dotted, "\mathrm{can}"] \arrow[r, "x^*"] & \mathrm{Loc}(\mathrm{pt}) \arrow[loop below, "T:=x^*\circ x_!^h\simeq p_!^h\circ p^*"] \arrow[r, shift left, "p^*"] \arrow[r, shift right, "p^*"'] & \mathrm{Loc}(\Omega_x X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}$$

Here is our limit descent result:

Corollary 4.17. *The map* can *is an equivalence.*

Proof. The functor x^* is monadic by the descent result for local systems Theorem 4.14 that we just proved, and the augmented cosimplicial diagram of categories on the second row of the above diagram satisfies the left Beck-Chevalley condition by the base change result for local systems Proposition 4.3. Therefore can is an equivalence by Theorem 3.8 □

4.5 Examples: matching local systems with modules

In this section we get a feel for the equivalence

$$\mathrm{Loc}(X) \simeq {}_{C_{-\bullet}(\Omega_x X;k)}\mathrm{Mod}$$

of Theorem 4.14 for a path-connected X by seeing how it matches certain objects.

A picture

Before doing so, we make a comment to help with the interpretation. The algebra $A := C_{-\bullet}(\Omega_x X; k)$ has the augmentation homomorphism $A \to k$ induced by the map of spaces $p : \Omega_x X \to \mathrm{pt}$, which we think of as building a distinguished "point" called 1 on the noncommutative dg scheme $1 \in \operatorname{Spec} A$; see Figure 4.5.

We may think about this by passing to the classical picture: since it is supported in non-positive degrees, A has a homomorphism

$$A \xrightarrow{\epsilon} \pi_0(A) \cong k[\pi_1(X,x)]$$

to the group algebra of the fundamental group, which is a classical noncommutative algebra. The augmentation $A \to k$ factors through ϵ as the "evaluation at 1" morphism

$$\mathrm{ev}_1 : \sum_{\gamma\in\pi_1(X,x)} c_\gamma \gamma \mapsto \sum_{\gamma\in\pi_1(X,x)} c_\gamma$$

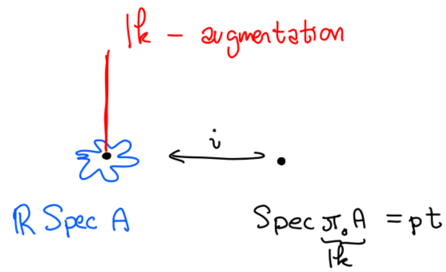


Figure 4.1: Cartoon of $\operatorname{Spec} A$ for a general path-connected and simply-connected X. In this case, the degree zero piece is $\pi_0(A) \simeq k$. In this case, there is only one classical point, which supports the augmentation module. We draw the rest of $\mathbb{R}\operatorname{Spec} A$ with blue negative-degree non-commutative fuzz.

which builds a distinguished "point," which we also denote 1, on the noncommutative classical scheme $1 \in \operatorname{Spec} k[\pi_1(X, x)]$.

This point inherits its meaning from ev_1: a classical local system $\mathscr{L}$ is just the data of a module $\mathscr{L}|_x \in {}_{k[\pi_1(X,x)]}\mathrm{Mod}$, and thus it localizes to a sheaf $\widetilde{\mathscr{L}|_x}$ over $\operatorname{Spec} k[\pi_1(X, x)]$. The fiber $k \otimes_{k[\pi_1(X,x)]} \widetilde{\mathscr{L}|_x}$ of this sheaf over the point 1 is the generalized 1-eigenspace of $\mathscr{L}|_x$. We may now loosely apply this classical reasoning to $\operatorname{Spec} A$ to think of it as *the space of monodromies for local systems on* X, with the augmentation $1 \in \operatorname{Spec} A$ playing the distinguished role of unit monodromy.

Some tautological modules

We now turn to looking for some (left) modules for A. There are several canonical ones. The most basic is A itself. The second most basic is the augmentation module $A \to k$. Its kernel I is yet another module, which gives rise to the modules I^n and A/I^n, as well as the completion

$$A_{\hat{I}} := \varprojlim\Big(\cdots \to A/I^n \to A/I^{n-1} \to \cdots \to A/I = k\Big)$$

Under the equivalence $x^{*\mathrm{enh}}$, we get the identifications

$$\begin{aligned} x_!^h k &\leftrightarrow A, \\ k_X &\leftrightarrow k = A/I, \\ x_*^h k &\leftrightarrow A^* \end{aligned}$$

The identification $k_X \leftrightarrow k = A/I$ fits into the interpretation offered above because k_X is a local system of unit monodromy. The local systems corresponding to A/I^n are some generalizations of k_X, whose monodromy may not be exactly unital but is not far off from it because their classical support is a torsion sheaf based at $1 \in \operatorname{Spec} \pi_0(A)$; we call them *unipotent local systems*. We also define the *universal pro-unipotent local system* to be the canonical limit of the A/I^n:

$$\mathscr{L}_\infty^{\text{pro-uni}} \quad \leftrightarrow \quad A_{\hat{I}} := \varprojlim A/I^n$$

These examples always exist. We now examine what the corresponding local systems look like, in the case of $X = S^1, S^2$.

Local systems on S^1 and modules

As a first example, let us match some other local systems on S^1 with modules for $A \simeq k[t, t^{-1}]$.

The augmentation $k[t, t^{-1}] \to k$ is evaluation at 1, so has kernel

$$I = (t-1) \hookrightarrow k[t, t^{-1}] \xrightarrow{\mathrm{ev}_{t=1}} k$$

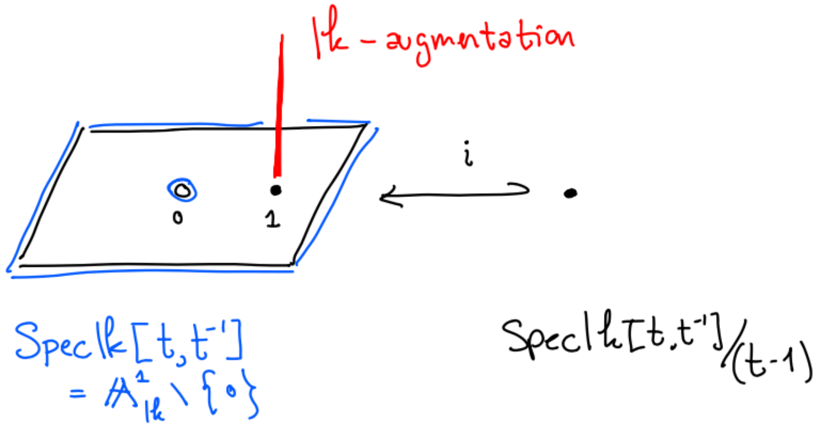


Figure 4.2: Local systems on S^1 correspond to all modules, supported within $\mathbb{A}^1_k \setminus \{0\}$. Ind-unipotent local systems only are supported at $\{1\} \in \mathbb{A}^1_k \setminus \{0\}$.

The indecomposable module $A/I^n \simeq k[t, t^{-1}]/(t-1)^n$ can be identified with the space $k^{\oplus n}$ on which t acts by the single indecomposable $n \times n$ Jordan block

$$J_n \quad := \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which, under the equivalence, corresponds to the rank n indecomposable unipotent local system $\mathscr{L}_n^u$ on S^1 with monodromy given by J_n:

$$A/I^n \quad \leftrightarrow \quad \mathscr{L}_n^u$$

For $n < m$, the non-split short exact sequence

$$A/I^n \hookrightarrow A/I^m \twoheadrightarrow A/I^{m-n}$$

gives the sequence of Jordan blocks

$$J_n \hookrightarrow J_m \twoheadrightarrow J_{m-n}$$

where the first inclusion is as the top-left $n \times n$ corner, and the quotient is onto the bottom-right $(m-n)\times(m-n)$ corner. This gives the non-split extension of indecomposable unipotent local systems

$$\mathscr{L}_n^u \hookrightarrow \mathscr{L}_m^u \twoheadrightarrow \mathscr{L}_{m-n}^u$$

Using the maps

$$J_n \hookrightarrow J_{n+1}, \qquad J_n \twoheadrightarrow J_{n-1}$$

we can build two more "limiting" local systems:

1. as already introduced, the *universal pro-unipotent local system* is defined to be the limit of the projections $J_n \twoheadrightarrow J_{n-1}$:

$$\mathscr{L}^{\text{pro-uni}} := \varprojlim \mathscr{L}_n \quad \leftrightarrow \quad k[t,t^{-1}]_{\widehat{(t-1)}} = k[[t-1]]$$

   It corresponds to the completion at the point $\{1\} \in \mathbb{A}_k^1 \setminus \{0\}$;

2. the *universal ind-unipotent local system* is defined to be the colimit of the inclusions $J_n \hookrightarrow J_{n+1}$:

$$\mathscr{L}^{\text{ind-uni}} := \underset{\longrightarrow}{\text{colim}}\, \mathscr{L}_n \quad \leftrightarrow \quad Dist_1(k[t,t^{-1}])$$

   It corresponds to the locally-nilpotent module of distributions supported at $\{1\} \in \mathbb{A}_k^1 \setminus \{0\}$.

Local systems on S^2 and modules

Here is a calculation that we take for granted:

Proposition 4.18. *For $n \geq 2$ and k a field of characteristic 0, there is a quasi-isomorphism of (noncommutative) $\mathbb{E}_1$ algebras over k:*

$$C_{-\bullet}(\Omega_x S^n; k) \simeq k\langle u\rangle, \qquad |u| = -n+1$$

Taking $n = 2$ gives us the identification

$$C_{-\bullet}(\Omega_x S^2; k) \simeq k\langle u\rangle,$$

where for $|u| = -1$.

Let us examine the local systems that correspond to the modules $A/I^n = k\langle u\rangle/(u^n)$. We already know that

$$A/I \quad \leftrightarrow \quad k_{S^2}$$

We will now see that the next object in line is built by the Hopf fibration:

Lemma 4.19. *Under the equivalence* $\mathrm{Loc}(S^2) \simeq {}_{C_{-\bullet}(\Omega_x S^2;k)}\mathrm{Mod}$*,*

$$A/I^2 \quad \leftrightarrow \quad f_!^h k_{S^3},$$

the homotopy shriek-pushforward of the constant local system k_{S^3} under the Hopf fibration $S^3 \xrightarrow{f} S^2$.

Proof. The Hopf fibration induces a map on basepoints

$$\begin{array}{ccc} S^3 & \xrightarrow{\quad f \quad} & S^2 \\ & & \\ x & \longmapsto & f(x) \end{array}$$

Let us pick the notation

$$k\langle v\rangle \simeq C_{-\bullet}(\Omega_x S^3; k), \quad k\langle u\rangle \simeq C_{-\bullet}(\Omega_{f(x)} S^2; k).$$

We first need to first understand the morphism induced by f on these dgas. Then, writing

$$k_{S^3} \quad \leftrightarrow \quad k = k\langle v\rangle/(v)$$

as the augmentation $k\langle v\rangle$-module, we would need to calculate $f_!^h\big(k\langle v\rangle/(v)\big)$. Let us do both in turn.

The map $f : S^3 \to S^2$ induces a map on loop spaces $f : \Omega_x S^3 \to \Omega_{f(x)} S^2$, which induces a map on chains $f_\bullet : k\langle v\rangle \to k\langle u\rangle$ which is a homomorphism of dgas. By degree reasons it is determined by an element $\alpha \in k$ where $f_\bullet(v) = \alpha u^2$. Since $v \in \pi_3(S^3) \cong \mathbb{Z}$ is the generator and $f_\bullet(v) \in \pi_3(S^2)$ is also a generator, we conclude that α must be a unit, and therefore freely assume $f_\bullet(v) = u^2$.

So the induced map on dgas is $v \mapsto u^2$. To calculate $f_!^h k_{S^3}$, first take the Koszul resolution of the module $k = k\langle v\rangle/(v)$, which is the quasi-free item $(k\langle v\rangle[\varepsilon], d\varepsilon := v)$, where $|\varepsilon| = -3$; thus $\varepsilon^2 = 0$. This gives a model for calculating the derived tensor product classically:

$$\begin{aligned} f_!^h k_{S^3} &\leftrightarrow k\langle v\rangle/v \otimes^L_{C_{-\bullet}(\Omega_x S^3)} C_{-\bullet}(\Omega_{f(x)} S^2) \\ &\simeq k\langle v\rangle[\varepsilon] \otimes_{k\langle v\rangle} k\langle u\rangle \\ &= k\langle u\rangle[\varepsilon], \quad d\varepsilon = f_\bullet(v) = u^2 \\ &\simeq k\langle u\rangle/u^2 \end{aligned}$$

The last line is the statement that the object $k\langle u\rangle[\epsilon]$ is nothing but a quasi-free resolution of the module $k\langle u\rangle/u^2 \in {}_{k\langle u\rangle}\mathrm{Mod}$, which is therefore the module corresponding to the local system $f_!^h k_{S^3}$. $\square$

Let us check this by also calculating the global sections $\pi_!^h f_!^h k_{S^3}$, where $S^2 \xrightarrow{\pi} \mathrm{pt}$, using this module realization. This map induces the dga morphism $\pi_\bullet : k\langle u\rangle \to k = k\langle u\rangle/u$. Using the quasi-free resolution from above gives

$$\begin{aligned}
\pi_!^h f_!^h k_{S^3} &\leftrightarrow k\langle u\rangle/u^2 \otimes^L_{k\langle u\rangle} k\langle u\rangle/u \\
&\simeq k\langle u\rangle[\varepsilon] \otimes_{k\langle u\rangle} k\langle u\rangle/u \\
&= k\langle u\rangle[\varepsilon]/(u), \quad d\varepsilon = 0 \\
&= k[\varepsilon]/(\varepsilon^2), |\varepsilon| = -3 \\
&\simeq C_{-\bullet}(S^3; k) \in {}_k\mathrm{Mod}
\end{aligned}$$

which is what we expected.

Remark 4.20. *Since $A/I^2 \leftrightarrow f_!^h k_{S^3}$, we might be forgiven to think of the local systems that match up with A/I^n as some kinds of "higher" Hopf local systems. Unfortunately, these do not arise as pushforwards under maps from spheres to S^2.*

Digression: the Hopf local system is derived

This section is inessential, and is supposed to be fun.

Given a DG local system $\mathscr{L} \in \mathrm{Loc}(X)$, we can extract its cohomology sheaves $\mathscr{H}^i\mathscr{L} \in \mathrm{Loc}(X)^\heartsuit \simeq \mathrm{Rep}^\heartsuit_{\pi_1(X,x)}$, which are classical local systems on X. We can wonder:

Question 4.21. *How different is $\mathscr{L}$ from the data of its "semi-classical" probes $\{\mathscr{H}^i\mathscr{L}\}_{i\in\mathbb{Z}}$? In other words, is there a quasi-isomorphism*

$$\mathscr{L} \overset{?}{\simeq} \bigoplus_{i\in\mathbb{Z}} \mathscr{H}^i\mathscr{L}[-i]$$

Since we are here, we will now use the example of the Hopf local system $\mathscr{L} := f_!^h k_{S^3}$ to illustrate the point that the two are not quasi-isomorphic.

In the case of smooth fibrations $P \xrightarrow{p} X$, $p_!^h$ can be identified with a more familiar functor:

Lemma 4.22. *If $P \xrightarrow{p} X$ is a submersion in the world of finite-dimensional manifolds, then for any finite-rank unipotent local system $\mathscr{L} \overset{!}{\in} \mathrm{Thick}(k_P)$,*

$$p_!^h\mathscr{L} \simeq p_!(\mathscr{L} \otimes_{k_P} p^! k_X) := p_!(\mathscr{L} \otimes_{k_P} \mathrm{or}_{P/X}[\dim p])$$

Proof. Since both functors are exact, it suffices to show the claim for $\mathscr{L} = k_P$. Both functors satisfy base change, under which the claim reduces to the fiberwise case $X = \mathrm{pt}$, where we know both of the following equivalences independently:

$$p_{x!}^{\,h} k_{P_x} \simeq C_{-\bullet}(P_x; k) \simeq (p_x)_!(p_x)^! k.$$

This proves the claim. □

Let us now apply the above result to $\mathscr{L} = k_{S^3}$ and $p = f$ the Hopf fibration.

First, since f is a submersion with oriented 1-dimensional fibers, we know that $f_!^h k_{S^3} \simeq f_! k_{S^3}[1]$. So let us just look at the more familiar object $f_! k_{S^3}$. If it were true that $f_! k_{S^3} \simeq \bigoplus_{i\in\mathbb{Z}} \mathscr{H}^i f_! k_{S^3}[-i]$, then the sheaves $\mathscr{H}^i f_! k_{S^3}$ would be constant by the fact that S^2 is simply-connected. Since the fiber of f is S^1, both cohomology sheaves would be rank 1, and so would force $f_! k_{S^3} \simeq k_{S^2} \oplus k_{S^2}[-1]$. The global sections of the direct sum are:

$$\Gamma(k_{S^2} \oplus k_{S^2}[-1]) \simeq k \oplus k[-1] \oplus k[-2] \oplus k[-3].$$

On the other hand, we know $\Gamma(k_{S^3}) \simeq C^\bullet(S^3; k) \simeq k \oplus k[-3]$.

Thus $f_! k_{S^3} \not\simeq k_{S^2} \oplus k_{S^2}[-1]$, and so $f_! k_{S^3}$ is fundamentally a derived object.

4.6 Koszul duality for local systems

For a pointed path-connected space (X, x), we considered the fundamental diagram

$$\mathrm{pt} \xrightarrow{x} X \xrightarrow{p} \mathrm{pt}$$

and proceeded to monadically describe $\mathrm{Loc}(X)$ using the map x^*. Can we obtain a (co)monadic description using $p_!^h$ or p_*^h? As mentioned, the answer turns out to be: "almost." There is only a partial answer, and it will take more work than Theorem 4.14. Why the difficulty?

Broadly: since Barr-Beck-Lurie is a precise characterization of descent, any obstacle to monadicity for a functor F must be either the failure of F to be conservative, or the failure of F to preserve enough colimits for monadicity or limits for comonadicity. While $F := x^*$ was conservative for a topological reason and preserved enough (co)limits for a purely abstract reason, the functors $p_!^h$ and p_*^h are neither conservative nor obviously able to preserve the (co)monadically necessary (co)limits. Here is an example illustrating the problem:

Example 4.23. *Consider the case of $X = S^1$ and the right adjoint functor p_*^h. While here the ability of p_*^h to preserve the monadically-necessary colimits is good — it preserves all geometric realizations — the functor fails to be conservative. For example, the local system $\mathscr{L}_{-1}$ of rank 1 and monodromy -1 has no cohomology:*

$$\begin{aligned} p_*^h \mathscr{L}_{-1} &\simeq C^\bullet(S^1; \mathscr{L}_{-1}) \\ &\underset{k}{\simeq} \mathscr{L}_{-1}|_x[0] \oplus \mathscr{L}_{-1}|_x[-1] \\ &\simeq 0 \end{aligned}$$

To study this further, let us reformulate our topological problem in terms of functors between categories of modules.

By monadicity for local systems, a path-connected space X gives $\mathrm{Loc}(X) \simeq {}_A\mathrm{Mod}$ for $A := C_{-\bullet}(\Omega_x X; k)$. This algebra carries an augmentation homomorphism $A \to k$ induced by applying the chains functor to $\Omega_x X \to \mathrm{pt}$, and this allows us to identify the functors $(p_!^h, p^*, p_*^h)$ as follows, in a diagram drawn in $\mathrm{Pr}^{L,\mathrm{st}}$.

$$\begin{tikzcd} \mathrm{Loc}(X) \arrow[r, bend left, "p_!^h = C_{-\bullet}(X;-)"] \arrow[r, bend right, dashed, "p_*^h = C^\bullet(X;-)"'] & \mathrm{Loc}(\mathrm{pt}) \arrow[l, "p^*"'] \end{tikzcd} \quad : \quad \begin{tikzcd} {}_A\mathrm{Mod} \arrow[r, bend left, "k \otimes_A (-)"] \arrow[r, bend right, dashed, "\hom_A(k-)"'] & {}_k\mathrm{Mod} \arrow[l, "\mathrm{fgt}"'] \end{tikzcd}$$

Question 4.24. *In terms of this language, we would like to answer the following:*

1. *When does $p_!^h = k \otimes_A (-)$ preserve totalizations, and on which subcategory is it conservative?*

2. *When does $p_*^h = \hom_A(k, -)$ preserve geometric realizations, and on which subcategory is it conservative?*

We summarize the results that follow:

Theorem 4.25 (Summary of Koszul duality for local systems)**.** *If X is a path-connected space with the homotopy type of a finite CW complex (e.g. a connected compact manifold), then k is an A-perfect module, and thus*

1. *$p_!^h$ preserves totalizations in the stable category of eventually connective modules ${}_A\mathrm{Mod}^{<\infty}$, and is conservative on this subcategory if X is also* simply-connected*;*

2. *p_*^h is a colimit-preserving functor, and is conservative on the stable presentable category ${}^\perp(k^\perp) \subseteq {}_A\mathrm{Mod}$.*

Since both functors thus participate in colimit-preserving adjunctions, the comonadicity result

$$(p_!^h)^{\mathrm{enh}} : \mathrm{Loc}(X)^{<\infty} \xrightarrow{\simeq} co\mathrm{Mod}_{C_{-\bullet}(X;k)}{}^{<\infty}$$

and the monadicity result

$$(p_*^h)^{\mathrm{enh}} : \mathrm{Loc}(X)^{\mathrm{ind\text{-}uni}} \xrightarrow{\simeq} \mathrm{Mod}_{C^\bullet(X;k)}$$

follow.

Proof. The fact that k is A-perfect will follow from Corollary 4.30. Part (1) is the content of Theorem 4.40. Part (2) is the content of Theorem 4.33. The identifications in the monadicity result are the content of Proposition 4.31. $\square$

Example 4.26. *The monadicity result holds for all spheres $S^{n\geq 1}$ and the comonadicity result holds for all spheres $S^{n\geq 2}$.*

Remark 4.27. *Part (1) of the above theorem has a generalization to path-connected and simply-connected X whose homology $H_{-\bullet}(X;k)$ is a perfect k-module in each degree. For example, $X = BG$ for a connected Lie group G. However, we only present the argument for the more minimal statement.*

The rest of this chapter establishes this theorem. Before continuing, we pause to do a useful calculation.

Interlude: identifying the (co)bar constructions in topology

Here we study the objects that we will encounter in our descent questions: the coalgebra $k \otimes_A k$ and the algebra $\mathrm{End}_A(k)$, in the example where $A = C_{-\bullet}(\Omega_x X; k)$ for a path-connected space X.

We calculate these objects in terms of a particular free resolution of k, which we can obtain by a general procedure. Put $G := \Omega_x X$, and consider first the simplicial group

$$E_\bullet G := \qquad \left[\cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} G \times G \times G \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} G \times G \substack{\longrightarrow\\[-1ex]\longrightarrow} G\right]$$

where $E_n G = G^{n+1}$, and where the face and degeneracy maps are all group homomorphisms that are furthermore equivariant with respect to the left action of G on the first factor of each $E_n G$. Its geometric realization, $EG := |E_\bullet G|$, is contractible. Taking chains gives an action $C_{-\bullet}(G; k)$ on the simplicial vector space $C_{-\bullet}(E_\bullet G; k)$, which as a simplicial $C_{-\bullet}(G; k)$-module is

$$C_{-\bullet}(E_\bullet G; k) = C_{-\bullet}(G; k) \otimes_k C_{-\bullet}(B_\bullet G; k) \quad \in {}_{C_{-\bullet}(G;k)}\mathrm{sMod}$$

Since the chains functor $C_{-\bullet}(-; k)$ is colimit-preserving, taking the geometric realization yields an action

$$C_{-\bullet}(G; k) \quad \curvearrowright \quad C_{-\bullet}(EG; k) = C_{-\bullet}(G; k) \otimes_k C_{-\bullet}(BG; k),$$

which, by the contractibility of $|E_\bullet G|$, is therefore a free $C_{-\bullet}(G; k)$-resolution of k.

This free $C_{-\bullet}(G; k)$-resolution of k allows us to calculate

$$\begin{aligned}
\mathrm{Bar}_k(C_{-\bullet}(G; k)) &:= k \otimes_{C_{-\bullet}(G;k)} k \\
&\simeq C_{-\bullet}(BG; k), \\
\mathrm{Cobar}_k(C_{-\bullet}(G; k)) &:= \hom_{C_{-\bullet}(G;k)}(k, k) \simeq C_{-\bullet}(BG; k)^* \\
&=: C^\bullet(BG; k)
\end{aligned}$$

Taking $G := \Omega_x X$ and X path-connected gives

Corollary 4.28. *For a path-connected space X, the bar and cobar constructions are*

$$\begin{aligned} k \otimes_{C_{-\bullet}(\Omega_x X;k)} k &\simeq C_{-\bullet}(X;k) \\ \mathrm{End}_{C_{-\bullet}(\Omega_x X;k)}(k) &\simeq C^{\bullet}(X;k) \end{aligned}$$

Perfectness of k as an A-module and perfectness of the (co)bar as a k-module are related via the following technical lemma:

Lemma 4.29. *Let $A \to k$ be a map of connective $\mathbb{E}_1$ rings, whose underlying map $\pi_0(A) \to \pi_0(k)$ of associative rings is a surjection, with nilpotent kernel $I \subseteq \pi_0(A)$. Then k is an A-perfect module if and only if $k \otimes_A k$ is a k-perfect module.*

Proof. This is an immediate consequence of [15] Proposition 2.7.3.2, part (d). □

We may therefore conclude the following by the identifications in Corollary 4.28:

Corollary 4.30. *Let k be a classical commutative ring, and X a path-connected space. Then k is a perfect $C_{-\bullet}(\Omega_x X;k)$-module if and only if $C_{-\bullet}(X;k)$, or $C^{\bullet}(X;k)$, is a perfect k-module.*

Proof. Apply the above Lemma to $A = C_{-\bullet}(\Omega_x X;k)$, which is possible since the kernel is $I = 0$. □

We will use this in our descent results.

Monadicity for p_*^h

In this section we focus on the subcategory ${}^{\perp}(k^{\perp}) \subseteq {}_A\mathrm{Mod}$, and study the diagram in Pr:

$$\begin{tikzcd}
k^{\perp} \arrow[dr, hook] & {}^{\perp}(k^{\perp}) \arrow[d, hook] \arrow[drr, bend left, "\Phi^{\mathrm{enh}}"] & & \\
& {}_A\mathrm{Mod} \arrow[rr, "\hom_A(k-)^{\mathrm{enh}}"] \arrow[dr, bend right, "\hom_A(k-)"'] & & {}_T\mathrm{Mod} \arrow[ll, dashed, bend right, "{\mathrm{fgt}_k}^{\mathrm{recon}}"'] \arrow[dl, bend left, "\mathrm{fgt}_T"] \\
& & {}_k\mathrm{Mod} \arrow[ul, bend right, "\mathrm{fgt}_k"'] \arrow[ur, bend left, "\mathrm{free}_T"] \arrow[loop below, "T"] &
\end{tikzcd}$$

As mentioned, this diagram is not wholly in Pr^L because $\hom_A(k,-)$ need not preserve colimits. This, together with its failure to be conservative, are the obstructions to it being monadic.

Here is the main result:

Proposition 4.31. *Suppose $A \to k$ is a homomorphism of $\mathbb{E}_1$-algebras over a classical commutative ring k. Then:*

1. *the colimit-preserving functor*

$$\operatorname{Ind}\operatorname{Thick}_A(k) \xrightarrow{\operatorname{Ind}\hom_{\operatorname{Thick}_A(k)}(k,-)} {}_k\operatorname{Mod}$$

is monadic, rendering an equivalence

$$\operatorname{Ind}\operatorname{Thick}_A(k) \simeq \operatorname{Mod}_{\operatorname{End}_A(k)}$$

2. *if k is A-perfect, then the functor $\hom_A(k,-)$ is colimit-preserving, and we may further identify*

$$\operatorname{Ind}\operatorname{Thick}_A(k) \simeq {}^{\perp}(k^{\perp})$$

3. *if X is path-connected and has the homotopy type of a finite CW complex, then we may identify the category of modules over the monad as*

$$\operatorname{Mod}_{\operatorname{End}_A(k)} \simeq \operatorname{Mod}_{C^{\bullet}(X;k)}.$$

Thus, for such spaces,

$$\begin{tikzcd}
\operatorname{Ind}\operatorname{Thick}_{C_{-\bullet}(\Omega_x X;k)}(k_X) \arrow[r, equal] & {}^{\perp}(k_X^{\perp}) \arrow[r, equal] \arrow[d, hook] & \operatorname{Mod}_{C^{\bullet}(X;k)} \\
& \operatorname{Loc}(X) \arrow[r, equal] & {}_{C_{-\bullet}(\Omega_x X;k)}\operatorname{Mod}
\end{tikzcd}$$

Proof. Part (1) was already proven in Theorem 2.45. To see part (2), since $k \in {}^{\perp}(k^{\perp})$ and the latter is closed under finite colimits, certainly there is furthermore always a canonical map

$$\operatorname{Thick}_A(k) \to {}^{\perp}(k^{\perp}).$$

But by the assumption that k is A-perfect, $\operatorname{Thick}_A(k) \subseteq \operatorname{Perf}(A)$, and therefore

$$\operatorname{Ind}\operatorname{Thick}_A(k) \subseteq \operatorname{Ind}\operatorname{Perf}(A) = {}_A\operatorname{Mod}$$

Since ${}^{\perp}(k^{\perp})$ is closed under colimits, this embedding in fact factors through ${}^{\perp}(k^{\perp})$, and therefore indeed

$$\operatorname{Ind}\operatorname{Thick}_A(k) \subseteq {}^{\perp}(k^{\perp}).$$

To show equality, we argue that both categories can be monadically reconstructed from matching monads on the same category, ${}_k\operatorname{Mod}$. To that end, we consider the following diagram, with dashed arrows denoting left adjoints:

$$\begin{tikzcd}[column sep=large, row sep=large]
\operatorname{Ind}\operatorname{Thick}_A(k) \arrow[r, hook, "\iota"] \arrow[dr, bend right, "\Phi"'] & {}^\perp(k^\perp) \arrow[r, "\hom_A(k-)^{\mathrm{enh}}"'] \arrow[d, bend right, "\hom_A(k-)"'] & {}_T\operatorname{Mod} \arrow[l, dashed, bend right, "\mathrm{fgt}_k^{\mathrm{recon}}"'] \arrow[ll, dashed, bend right=40, "(\Phi^L)^{\mathrm{recon}}"'] \arrow[dl, bend left, "\mathrm{fgt}_T"] \\
& {}_k\operatorname{Mod} \arrow[ul, dashed, bend right, "\Phi^L"] \arrow[u, dashed, bend right, "\mathrm{fgt}_k"'] &
\end{tikzcd}$$

The first order of business is to show that these left adjoints actually exist:

1. The adjoint Φ^L exists by the Adjoint Functor Theorem, due to the fact that Φ is a bicontinuous functor between presentable categories. To see that Φ is bicontinuous, we observe that it preserves limits by its definition as the restriction of the limit-preserving functor $\hom_A(k, -)$, and it preserves colimits because the construction of the category $\operatorname{Ind}\operatorname{Thick}_A(k)$ ensures that k is a compact object there. Thus we can sensibly study the monadicity question for $\operatorname{Ind}\operatorname{Thick}_A(k) \xrightarrow{\Phi} {}_k\operatorname{Mod}$.

2. The reconstruction functor $(\Phi^L)^{\mathrm{recon}}$ exists because (Φ^L, Φ) is an adjunction, and because $\operatorname{Ind}\operatorname{Thick}_A(k)$ admits all colimits.

3. The functor fgt_k exists as a functor ${}_k\operatorname{Mod} \xrightarrow{\mathrm{fgt}_k} {}_A\operatorname{Mod}$, but we need to show that it lands inside the category ${}^\perp(k^\perp)$. This follows by adjunction: if $M \in k^\perp$ and $V \in {}_k\operatorname{Mod}$, then
$$\hom_A(\mathrm{fgt}_k V, M) = \hom_k(V, \underbrace{\hom_A(k, M)}_{\simeq 0}) \mathrel{\dot{\simeq}} 0,$$
so $\mathrm{fgt}_k V \in {}^\perp(k^\perp)$. Thus we can sensibly study the monadicity question for ${}^\perp(k^\perp) \xrightarrow{\hom_A(k,-)} {}_k\operatorname{Mod}$ as well.

4. The reconstruction functor $\mathrm{fgt}_k^{\mathrm{recon}}$ exists because $(\mathrm{fgt}_k, \hom_A(k, -))$ is an adjunction, and ${}^\perp(k^\perp)$ admits all colimits.

Now that the functors are all justified, we show that Φ is monadic. As mentioned, it is bicontinuous, and its domain and codomain categories are presentable. Thus it suffices to show that Φ is conservative, but this is true because Φ factors through $\hom_A(k, -)$, which conservative by design.

Next, we show that fgt_k in fact lands inside $\operatorname{Ind}\operatorname{Thick}_A(k)$. For a finitely generated free k-module V, certainly $\mathrm{fgt}_k(V) \in \operatorname{Thick}_A(k) \subseteq {}^\perp(k^\perp)$. Note that ${}_k\operatorname{Mod}$ is the ind-completion of the closure under retracts of finitely generated free modules

$${}_k\operatorname{Mod} \simeq \operatorname{Ind}\left({}_k\operatorname{Mod}_{\mathrm{fg}}^{\mathrm{free}}\right)^{\tau}$$

meaning that ${}_k\,\mathrm{Mod}$ is built out of infinite colimits applied to ${}_k\,\mathrm{Mod}^{\mathrm{free}}$. Since fgt_k preserves colimits, we conclude that it factors through $\mathrm{Ind}\,\mathrm{Thick}_A(k)$. This shows that the functors $\Phi^L = \mathrm{fgt}_k$ are identified, and therefore so are $(\Phi^L)^{\mathrm{recon}} = \mathrm{fgt}_k^{\mathrm{recon}}$.

To conclude that ι is an equivalence, we simply note that $\hom_A(k,-)^{\mathrm{enh}}$ is conservative by design, and $(\Phi^L)^{\mathrm{recon}}$ is an embedding by the monadicity of Φ. Thus $\mathrm{fgt}_k^{\mathrm{recon}}$ is an embedding, and so $\hom_A(k,-)$ is monadic as well. Therefore,

$$\begin{tikzcd}
\mathrm{Ind}\,\mathrm{Thick}_A(k) \arrow[rr, "\iota"] \arrow[dr, equal, "\Phi^{\mathrm{enh}}"'] & & {}^{\perp}(k^{\perp}) \arrow[dl, equal, "\hom_A(k-)^{\mathrm{enh}}"] \\
& {}_T\,\mathrm{Mod} &
\end{tikzcd}$$

and so ι is an equivalence. This finishes the proof of part (2).

To see part (3), by Corollary 4.28 we can identify $T(k) := \mathrm{End}_A(k) \simeq C^\bullet(X;k)$, which by the assumptions on X is a perfect k-module. By Corollary 4.30, this means that k is a perfect A-module, and so we conclude by part (2). $\square$

To see the implication in the topological world, we first make a definition:

Definition 4.32. *The category of ind-unipotent local systems on a path-connected space X is*

$$\mathrm{Loc}(X)^{\text{ind-uni}} := {}^{\perp}({k_X}^{\perp}) \subseteq \mathrm{Loc}(X)$$

Similarly, the category of pro-unipotent local systems is

$$\mathrm{Loc}(X)^{\text{pro-uni}} := ({k_X}^{\perp})^{\perp} \subseteq \mathrm{Loc}(X)$$

Identifying the monad using Corollary 4.28 and applying Proposition 4.31 immediately gives:

Theorem 4.33. *Let X be any path-connected space with the homotopy type of a finite CW complex, and let k be a commutative ring. The functor p_*^h of taking cohomology is monadic on the subcategory of ind-unipotent local systems:*

$$\mathrm{Loc}(X)^{\text{ind-uni}} \overset{p_*^{h,\mathrm{enh}}}{=\!=\!=\!=} \mathrm{Mod}_{C^\bullet(X;k)}$$

This result finally answers the question: what kinds of local systems on X do modules over cochains on X describe? The answer is: exactly those that can be built out of k_X under self-extensions, retracts, and filtered colimits—the so-called ind-unipotent local systems.

Example 4.34. *Taking $X = S^1$ gives $A := C_{-\bullet}(\Omega_x S^1; k) \simeq k[t,t^{-1}]$, with augmentation homomorphism*

$$k[t,t^{-1}] \xrightarrow{\mathrm{ev}_{t=1}} k[t,t^{-1}]/(t-1) \simeq k$$

Inside ${}_{k[t,t^{-1}]}\mathrm{Mod}$, the sector of ind-unipotent local systems is tiny, but admits a descent description

$$\begin{aligned}\mathrm{Loc}(S^1)^{\text{ind-uni}} &\simeq {}_{k[t,t^{-1}]}\mathrm{Mod}^{(t-1)-\text{torsion}}\\ &\simeq k[D], \quad |D| = 1\end{aligned}$$

as either sheaves on $\mathbb{A}^1_k \setminus \{0\}$ whose reduced support is the point $\{1\}$, or as k-modules with a second differential D.

Note that, for example, the universal local system is not describable under this equivalence. The reason why this subcategory of $\mathrm{Loc}(S^1)$ is so small is because S^1 is not simply-connected.

Comonadicity for $p^h_!$

This subsection explores the flipside to the story above: what can be said about comonadicity of the functor $p^h_! = k \otimes_A (-)$?

We originally learned the contents of this subsection from a lovely note by Rok Gregoric; however, we unfortunately now cannot track down. Here we attempt a partial reconstruction of that argument, streamlined slightly for our specific purpose.

Let k be a classical (that is, concentrated in degree 0) commutative ring, and consider a morphism of $\mathbb{E}_1$ k-algebras $A \xrightarrow{f} k$, where A is concentrated in degrees ≤ 0:

$$\begin{tikzcd} \cdots \arrow[r] & A_{-2} \arrow[r] \arrow[d] & A_{-1} \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & k \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

Let us furthermore assume that the induced morphism $\pi_0 A \xrightarrow{\pi_0 f} \pi_0 k = k$ is an isomorphism.

Example 4.35. *The main example we have in mind, as always, is the augmentation map for $A := C_{-\bullet}(\Omega_x X; k) \to k$ induced by the map $\Omega_x X \to \mathrm{pt}$, where X is path-connected and simply-connected. It is the simply-connected assumption that ensures that $\pi_0 f$ is an isomorphism.*

This gives the adjunction between categories of (left) modules

$$\begin{tikzcd} {}_A\mathrm{Mod} \arrow[r, shift left, "k\otimes_A(-)"] & {}_k\mathrm{Mod} \arrow[l, shift left, "\mathrm{fgt}"] \arrow[loop right, "\Omega:=(k\otimes_A k)\otimes_k(-)"] \end{tikzcd}$$

We would like to argue that $k \otimes_A (-)$ is comonadic. Unfortunately, even though we assumed X is simply-connected, this functor is still not generally conservative:

Example 4.36. *For $X = S^2$ and thus $A = k\langle v \rangle$ for $|v| = -1$, the colimit of the infinite chain of Hopf local systems*

$$\operatorname*{colim}_{\longrightarrow}\left[\cdots \longrightarrow A/v^2 \xrightarrow{\cdot v} A/v^2 \xrightarrow{\cdot v} A/v^2 \longrightarrow \cdots\right]$$

is a non-trivial local system on S^2 with zero cohomology.

However, $k \otimes_A (-)$ *is* conservative once we restrict to the subcategory of connective modules. In fact, under another hypothesis, this restriction is comonadic:

Theorem 4.37. *Let $A \to k$ be a map of connective $\mathbb{E}_1$ rings, where k is a classical commutative ring, such that $\pi_0(A) \xrightarrow{\cong} \pi_0(k) = k$ is an isomorphism. Then:*

1. *the adjunction $(k \otimes_A (-), \mathrm{fgt})$ restricts to the subcategories of connective modules, on which $k \otimes_A (-)$ is conservative;*
2. *if furthermore $k \in \mathrm{Perf}(A)$, then $k \otimes_A (-)$ is comonadic between connective categories:*

$$\begin{tikzcd}
{}_A\mathrm{Mod}^{\leq 0} \arrow[rr, shift left, "k\otimes_A(-)^{\mathrm{enh}}"] \arrow[rr, phantom, "\sim"] \arrow[dr, bend right, "k\otimes_A(-)"'] \arrow[dd, hook] & & {}_{k\otimes_A k}\mathrm{coMod}^{\leq 0} \arrow[ll, shift left, "\mathrm{fgt}_k^{\mathrm{recon}}"] \arrow[dl, bend left, "\mathrm{fgt}_\Omega"] \arrow[dd, equal] \\
& {}_k\mathrm{Mod}^{\leq 0} \arrow[dd, hook] & \\
{}_A\mathrm{Mod} \arrow[rr, shift left, "k\otimes_A(-)^{\mathrm{enh}}"] \arrow[dr, bend right, "k\otimes_A(-)"'] & & {}_{k\otimes_A k}\mathrm{coMod} \arrow[ll, shift left, "\mathrm{fgt}_k^{\mathrm{recon}}"] \arrow[dl, bend left, "\mathrm{fgt}_\Omega"] \\
& {}_k\mathrm{Mod} &
\end{tikzcd}$$

Proof. The adjunction certainly restricts. So we turn to verifying the Barr-Beck-Lurie hypotheses:

1. *On ${}_A\mathrm{Mod}^{\leq 0}$, $k \otimes_A (-)$ is conservative.* Suppose that M is a connective A-module so that $k \otimes_A M \simeq 0$; i.e. $\pi_i(k \otimes_A M) \simeq 0$ for each $i \geq 0$. Consider the fiber sequence of A-modules
$$F \to A \xrightarrow{f} k$$
where $\pi_0 F = 0$ by assumption. Apply $(-) \otimes_A M$ and take the long exact sequence of homotopy groups:
$$\begin{tikzcd}
\pi_{-1}(F \otimes_A M) \arrow[r, "g_{-1}"] & 0 \arrow[r] & \underbrace{\pi_{-1}(k \otimes_A M)}_{=0} \\
\pi_0(F \otimes_A M) \arrow[r, "g_0"] & \pi_0(M) \arrow[r] & \underbrace{\pi_0(k \otimes_A M)}_{=0} \arrow[ull, rounded corners] \\
\pi_1(F \otimes_A M) \arrow[r, "g_1"] & \pi_1(M) \arrow[r] & \underbrace{\pi_1(k \otimes_A M)}_{=0} \arrow[ull, rounded corners]
\end{tikzcd}$$

The first column of horizontal morphisms g_i are all isomorphisms, and certainly $\pi_{\leq -1}(F \otimes_A M) = 0$, i.e. $g_{\leq -1} = 0$. The image of g_0 is $I\pi_0(M)$ for $I = \pi_0 A \to \pi_0 k$, which by assumption is 0, and so $\pi_0(M) = 0$. But now we repeat the argument on $M[-1]$, which is connective, to see that $\pi_1(M) = 0$, and thus $\pi_i(M) = 0$ for all i. Thus $M \simeq 0$, confirming conservativity.

Let now $\Delta \xrightarrow{M^\bullet} {}_A\mathrm{Mod}^{\leq 0}$ be a cosimplicial diagram for which $\Delta \xrightarrow{k \otimes_A M^\bullet} {}_k\mathrm{Mod}^{\leq 0}$ splits.

2. *The category ${}_k\mathrm{Mod}^{\leq 0}$ admits the totalization $\varprojlim_\Delta k \otimes_A M^\bullet$.* The limit is the splitting, which by assumption exists in ${}_k\mathrm{Mod}^{\leq 0}$.

3. *The functor $k \otimes_A (-)$ satisfies $k \otimes_A \varprojlim_\Delta M^\bullet \simeq \varprojlim_\Delta k \otimes_A M^\bullet$.* Note the object $\varprojlim_\Delta M^\bullet$ makes sense as an object of ${}_A\mathrm{Mod}$, since it is bicomplete. The result is now an immediate corollary of Lemma 18.2.5.14 in [15], given that $k \otimes_A k$ is perfect, hence almost perfect, over k by assumption. We note that this required each term $M^\bullet$ to be connective.

4. *The totalization $\varprojlim_\Delta M^\bullet$ exists in ${}_A\mathrm{Mod}^{\leq 0}$.* As mentioned, the limit certainly exists in ${}_A\mathrm{Mod}$; it remains to show that it is connective. The only possible obstruction would be a nonzero π_{-1}. Tensor up with the fiber sequence $F \to A \to k$ and use the fact that $k \otimes_A (-)$ preserves the totalization to write down the long exact sequence

$$\begin{tikzcd}
\pi_{-1}(F \otimes_A \varprojlim_\Delta M^\bullet) \arrow[r] & \pi_{-1}(\varprojlim_\Delta M^\bullet) \arrow[r] & \underbrace{\pi_{-1}(\varprojlim_\Delta k \otimes_A M^\bullet)}_{=0} \\
\pi_0(F \otimes_A \varprojlim_\Delta M^\bullet) \arrow[r] & \pi_0(\varprojlim_\Delta M^\bullet) \arrow[r] & \pi_0(\varprojlim_\Delta k \otimes_A M^\bullet) \arrow[uull, bend right]
\end{tikzcd}$$

But since F is supported in strictly negative degrees, $F \otimes_A \varprojlim_\Delta M^\bullet$ is supported in strictly non-positive degrees. Thus $\pi_{-1}(F \otimes_A \varprojlim_\Delta M^\bullet) = 0$, and therefore $\pi_{-1}(\varprojlim_\Delta M^\bullet) = 0$.

$\square$

Remark 4.38. *We note from the proof that connectivity was crucial for both conservativity, and for $k \otimes_A (-)$ to preserve totalizations.*

We now apply the above to produce another comonadicity statement for local systems:

Corollary 4.39. *Let X be a path-connected and simply-connected space with the homotopy type of a finite CW complex, and let k be a classical commutative ring. Then there are equivalences of (unstable) ∞-categories*

$$\begin{tikzcd} & {}_{C_{-\bullet}(\Omega_x X;k)}\mathrm{Mod}^{\leq 0} \arrow[dr, "k\otimes_A(-)^{\mathrm{enh}}"] & \\ \mathrm{Loc}_k(X)^{\leq 0} \arrow[ur, "x^{h,*\mathrm{enh}}"] & & {}_{C_{-\bullet}(X;k)}\mathrm{coMod}^{\leq 0} \end{tikzcd}$$

By definition, $\mathrm{Loc}_k(X)^{\leq 0}$ is the category of all local systems whose fibers are k-modules (equivalently, A-modules) in degree ≤ 0.

Proof. Since X is path-connected, the monadicity theorem above ensures that $x^{h,*\mathrm{enh}}$ restricts to an equivalence on the depicted connective subcategories. The assumptions that X is simply-connected and is a finite CW complex activate Theorem 4.37, which, together with the identification of the comonad $k \otimes_A k \simeq C_{-\bullet}(X;k)$ by Corollary 4.28 gives the other equivalence. $\square$

We can stabilize the above statement by taking the colimit of categories of increasing connectivity degrees, and consequently replace "connective" local systems, modules, and comodules by "eventually connective" ones:

Theorem 4.40. *Let X be a path-connected and simply-connected space with the homotopy type of a finite CW complex, and let k be a classical commutative ring. Then there are equivalences of stable ∞-categories*

$$\begin{tikzcd} & {}_{C_{-\bullet}(\Omega_x X;k)}\mathrm{Mod}^{<\infty} \arrow[dr, "k\otimes_A(-)^{\mathrm{enh}}"] & \\ \mathrm{Loc}_k(X)^{<\infty} \arrow[ur, "x^{h,*\mathrm{enh}}"] & & {}_{C_{-\bullet}(X;k)}\mathrm{coMod}^{<\infty} \end{tikzcd}$$

As mentioned in Remark 4.27, the results that we have just presented are not in their most general form; they are just in the form that we understood how to prove. There is a generalization of the above result to path-connected and simply-connected spaces X for which k is an "almost" perfect A-module (if and only if $k \otimes_A k$ is an "almost" perfect k-module), which is guaranteed by the k-modules $H_i(X;k)$ being finitely generated; such a generalization would allow one to also enjoy the following examples:

Example 4.41. *Let $X = BG$ where G is a connected finite-dimensional Lie group and k a classical ring. This implies that the k-modules $H_i(BG;k)$ are finitely generated, and that BG is path-connected and simply-connected, meaning in particular that k is an almost perfect A-module. Using the identification $\Omega_x BG \simeq G$, one obtains the equivalences of stable bounded above ∞-categories*

$$\begin{tikzcd}
& {}_{C_{-\bullet}(G;k)}\mathrm{Mod}^{<\infty} \arrow[dl, equal, "x^{h,*\mathrm{enh}}"'] \arrow[dr, equal, "k\otimes_A(-)^{\mathrm{enh}}"] & \\
\mathrm{Loc}_k(BG)^{<\infty} & & {}_{C_{-\bullet}(BG;k)}\mathrm{coMod}^{<\infty}
\end{tikzcd}$$

Chapter 5

Examples of Descent II: Covers in Topology

5.1 What is in this chapter?

Chapter 4 discussed descent results for the sheaf theory of local systems of k-modules $\mathrm{Loc}(-)$ on the theory of homotopy types X. This chapter explores descent results for the sheaf theory of all sheaves of k-modules $\mathrm{Sh}(-)$ on the theory of topological spaces and continuous maps.

Given such a map $X \xrightarrow{f} Y$, the goal is to understand when the canonical map can below is an equivalence:

$$\begin{tikzcd}
\varprojlim_{\Delta} \mathrm{Sh}(X^{\bullet+1}/Y) \arrow[drr, dashed, "\mathrm{ev}_0"] \arrow[drrr, dashed, bend left, "\mathrm{ev}_1"] \arrow[drrrr, dashed, bend left] & & & & \\
\mathrm{Sh}(Y) \arrow[u, dotted, "\mathrm{can}"] \arrow[rr, "f^*"] & & \mathrm{Sh}(X) \arrow[r, shift left, "\pi_1^*"] \arrow[r, shift right, "\pi_2^*"'] & \mathrm{Sh}(X \times_Y X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}$$

Assuming that the spaces are locally compact and Hausdorff, this diagram exists in Pr^L, whence all the universal dashed and dotted functors admit (possibly not colimit-preserving) right adjoints. As mentioned previously, we know:

1. by Proposition 3.5, since ev_0 admits a right adjoint ev_0^R, then

$$\varprojlim_{\Delta} \mathrm{Sh}_{\mathscr{S}}(X^{\bullet+1}/Y) \xrightarrow{\mathrm{ev}_0^{\mathrm{enh}}} {}_{\mathrm{ev}_0 \circ \mathrm{ev}_0^R}\mathrm{coMod}\, \mathrm{Sh}_{\mathscr{S}}(X)$$

is automatically an equivalence;

2. by Theorem 3.6, if furthermore a base change result holds for the diagram $X^{\bullet+1}/Y$, then we an rewrite this comonad on $\mathrm{Sh}_{\mathscr{S}}(X)$ as

$$\mathrm{ev}_0 \circ \mathrm{ev}_0^R \quad \simeq \quad \pi_{2*} \circ \pi_1^*;$$

3. by Theorem 3.8, if furthermore (1) f^* is comonadic and (2) the following *additonal* base change result holds

$$\begin{tikzcd} \mathrm{Sh}_{\mathscr{S}}(Y) \arrow[r, "f^*"] \arrow[d, "f^*"] & \mathrm{Sh}_{\mathscr{S}}(X) \arrow[l, dashed, bend left, "f_*"] \arrow[d, "\pi_1^*"] \\ \mathrm{Sh}_{\mathscr{S}}(X) \arrow[r, "\pi_2^*"] & \mathrm{Sh}_{\mathscr{S}}(X \times_Y X) \arrow[l, dashed, bend left, "\pi_{2*}"] \end{tikzcd}$$

then

$$\mathrm{Sh}_{\mathscr{S}}(Y) \xrightarrow{\mathrm{can}} {}_{\pi_{2*}\pi_1^*}\mathrm{coMod}\,\mathrm{Sh}_{\mathscr{S}}(X)$$

is an equivalence. That is, we can identify the desired category $\mathrm{Sh}_{\mathscr{S}}(Y)$ with the category of comodules.

We will organize the results by the kind of map $X \xrightarrow{f} Y$ is:

1. a "nice" cover of good spaces;
2. a general surjective map;
3. a cover by pieces of a stratification.

Before proceeding to these examples, we record some basic properties of f^*:

Observation 5.1. *For a continuous map $X \xrightarrow{f} Y$ of topological spaces, the following is true:*

1. *The functor f^{-1} is classical and is t-exact; in particular, it commutes with taking cohomology sheaves $\mathscr{H}^i$ for all degrees i.*
2. *Conservativity of f^{-1} on the bounded-below derived category of sheaves can be checked on cohomology sheaves; i.e. $f^{-1}F \simeq 0$ iff $f^{-1}\mathscr{H}^i F = 0$ for all i.*
3. *Since a classical sheaf is 0 iff all its stalks are 0, and since the stalk of the preimage is $(f^{-1}F)_y = F_{f(y)}$ for a classical sheaf F, it follows that f^{-1} is conservative iff $f : X \to Y$ is surjective.*
4. *In general, f_* does not preserve all colimits, and therefore does not admit a right adjoint.*

Since we are concerned with the comonadicity of the left adjoint functor f^*, by Barr-Beck-Lurie we are therefore concerned with its ability to preserve certain limits. The following warning tells us that we need to work carefully:

Warning 5.2. *The functor f^* does not generally preserve limits.*

1. *It does not preserve infinite products. For example, consider $\{0\} \xhookrightarrow{i} \mathbb{R}$, and consider the sheaf $\mathscr{C}$ of continuous functions; thus $i^*\mathscr{C} =: \mathscr{C}_0$ is the stalk at 0. There is a natural map*
$$\phi : (\prod_{n\in\mathbb{N}} \mathscr{C})_0 \to \prod_{n\in\mathbb{N}} \mathscr{C}_0,$$
but it is not an isomorphism because it fails to be injective. To see this, take a sequence of functions $f := \{f_n\}_{n\in\mathbb{N}}$ which vanish on $[-1/n, 1/n]$ but are nonzero outside. Then $[f] \neq 0$ since over any open $0 \in U$, there is some $f_n|_U \neq 0$. However, $\phi([f]) = 0$.

2. *It does not preserve cofiltered limits. Consider the same map $\{0\} \xhookrightarrow{i} \mathbb{R}$, and the family of sheaves $k_{[\frac{1}{n},\infty)}$, whose limit can be determined to be $k_{[0,\infty)}$:*
$$k_{[0,\infty)} \to \left[\cdots \to k_{[\frac{1}{n},\infty)} \to \cdots \to k_{[\frac{1}{2},\infty)} \to k_{[1,\infty)} \right]$$
While $i^*k_{[0,\infty)} = k$, the stalk at each term in the diagram $i^*k_{[\frac{1}{n},\infty)} = 0$. This is the failure. However, as we will see in Proposition 5.12, one diagnosis of the inability of i^* to preserve this limit is that the given cofiltered diagram is not j^*-split: this is because a splitting would be a collection of elements of hom$(k_{[\frac{1}{n},\infty)}, k_{(0,\infty)}) = 0$. But zero maps do not assemble into legitimate splitting data.

5.2 For nice covers

The following lemma defines what constitutes "nice":

Lemma 5.3. *If $X \xrightarrow{f} Y$ is a surjective morphism, such that*

1. *base change holds for* Sh$(-)$ *holds over*
$$\begin{tikzcd} X \times_Y X \arrow[r, "f'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

2. *this property is stable under pullback,*

then

$$\mathrm{Sh}(Y) \simeq \varprojlim_{\Delta} \mathrm{Sh}(X^{\bullet+1}/Y) \simeq {}_{f^*\circ f_*}\mathrm{coMod}\,\mathrm{Sh}(X)$$

Proof. The assumptions guarantee that the comonads of the vertical functors below match:

$$\begin{tikzcd}
\mathrm{Sh}(Y) \arrow[rr, "\mathrm{can}"] \arrow[dr, bend right, "f^*"'] & & \varprojlim_{\Delta} \mathrm{Sh}(X^{\bullet+1}/Y) \arrow[dl, bend left, "\mathrm{ev}_0"] \\
& \mathrm{Sh}(X) \arrow[loop below, "f_* f^* = \mathrm{ev}_0\, \mathrm{ev}_0^R"] &
\end{tikzcd}$$

Therefore, by [13] Corollary 4.7.3.16, it suffices to show that can is conservative. Since $f^* = \mathrm{ev}_0 \circ \mathrm{can}$, this is equivalent to showing that f^* is conservative. But it is by the above observation, since f is surjective. □

Corollary 5.4. *If Y is locally contractible and $X \xrightarrow{f} Y$ is*

1. *a bundle map, or*

2. *a proper surjective map*

then

$$\mathrm{Sh}(Y) \simeq \varprojlim_{\Delta} \mathrm{Sh}(X^{\bullet+1}/Y) \simeq {}_{f^*\circ f_*}\mathrm{coMod}\,\mathrm{Sh}(X)$$

Proof. Both satisfy the hypotheses of the above Lemma; for example, if the map is proper, the requisite base change results are filled by proper base change. □

5.3 For a general surjective map

A more general descent result requires a restriction of categories. In contrast to the above deductions, this one is purely a comonadicity statement, with no base change hypotheses.

Proposition 5.5. *If f is surjective, then the adjunction (f^{-1}, f_*) restricted to bounded-below categories*

$$\begin{tikzcd}
\mathrm{Sh}(Y)^+ \arrow[r, shift left, "f^{-1}"] & \mathrm{Sh}(X)^+ \arrow[l, shift left, "f_*"]
\end{tikzcd}$$

is comonadic.

Proof. We learned the following proof strategy from the note [10], where it was used to show that QCoh satisfies fpqc descent; we will reference it again in our future discussion of that as well.

We first check that the adjunction restricts to these subcategories. By definition,

$$\mathrm{Sh}(Y)^+ := \underset{n\to\infty}{\underrightarrow{\mathrm{colim}}}\, \mathrm{Sh}(Y)^{\geq -n}$$

Since f^{-1} is exact and since f_* is defined in terms of a totalization of rightward-growing injective resolutions, both functors restrict to the subcategories $\mathrm{Sh}(-)^{\geq -n}$ for each n, and thus to their colimits $\mathrm{Sh}(-)^+$. So without loss of generality we work with the categories $\mathrm{Sh}(-)^{\geq 0}$, and show that

$$\mathrm{Sh}(Y)^{\geq 0} \xrightarrow{\quad\simeq\quad} {}_{f^{-1}f_*}\,\mathrm{coMod}\,\mathrm{Sh}(X)^{\geq 0}$$

Taking the colimit of this equivalence as $n \to \infty$ would yield the desired statement.

So let $\mathscr{F}^\bullet$ be a cosimplicial diagram in $\mathrm{Sh}(Y)^{\geq 0}$ such that $f^{-1}\mathscr{F}^\bullet$ is split. It suffices to show that the natural map

$$f^{-1}\underset{\Delta}{\underleftarrow{\lim}}\,\mathscr{F}^\bullet \to \underset{\Delta}{\underleftarrow{\lim}}\, f^{-1}\mathscr{F}^\bullet$$

is an equivalence in $\mathrm{Sh}(X)^{\geq 0}$. Thus, we need to show that on the level of cohomology sheaves, the following map is an isomorphism of classical sheaves in $\mathrm{Sh}(X)^{\heartsuit}$:

$$\mathscr{H}^i f^{-1}\underset{\Delta}{\underleftarrow{\lim}}\,\mathscr{F}^\bullet \xrightarrow{\cong} \mathscr{H}^i\underset{\Delta}{\underleftarrow{\lim}}\, f^{-1}\mathscr{F}^\bullet, \quad \text{for all } i \geq 0$$

There is a spectral sequence computing the homotopy groups of totalizations of cosimplicial spectra:

Theorem 5.6 (Bousfield-Kan spectral sequence)**.** *Suppose $\mathscr{E}^\bullet$ is a cosimplicial spectrum, such that the associated cochain complex obtained by applying π_i for each $i \in \mathbb{Z}$ is an acyclic resolution of the kernel $K_i := \mathrm{Ker}(\alpha_i)$ of the first map α_i, i.e. that the following complex of abelian groups is acyclic for each $i \geq 0$:*

$$0 \to K_i \hookrightarrow \pi_i\mathscr{E}_0 \xrightarrow{\alpha_i} \pi_i\mathscr{E}_1 \to \pi_i\mathscr{E}_2 \to \cdots$$

Then the natural map $\pi_i\underset{\Delta}{\underleftarrow{\lim}}\,\mathscr{E}^\bullet \to \pi_i\mathscr{E}_0$ factors through an isomorphism with K_i:

$$\begin{tikzcd}
\pi_i\underset{\Delta}{\underleftarrow{\lim}}\,\mathscr{E}^\bullet \arrow[rr] \arrow[dr, dashed, "\cong"'] & & \pi_i\mathscr{E}^0 \\
& K_i \arrow[ur, hook] &
\end{tikzcd}$$

In other words, the theorem says that the homotopy groups of the limits are calculable as the kernels of maps of homotopy groups.

Using this theorem, we deduce the claim in several steps:

1. The cosimplicial object $\mathscr{F}^\bullet$ yields for each $i \geq 0$ a cochain complex of sheaves

$$0 \to K_i \hookrightarrow \mathscr{H}^i\mathscr{F}^0 \xrightarrow{\alpha_i} \mathscr{H}^i\mathscr{F}^1 \to \mathscr{H}^i\mathscr{F}^2 \to \cdots,$$

   which may not be exact, i.e. a resolution of $K_i := \mathrm{Ker}(\alpha_i)$.

2. On the other hand, $f^{-1}\mathscr{F}^\bullet$ is split, which means that the cochain complex $\mathscr{H}^i f^{-1}\mathscr{F}^\bullet$ is split exact for each $i \geq 0$, and in particular a resolution of its kernel, which we'll call $\widetilde{K}_i$:

$$0 \to \widetilde{K}_i \hookrightarrow \left[\mathscr{H}^i f^{-1}\mathscr{F}^0 \to \mathscr{H}^i f^{-1}\mathscr{F}^1 \to \mathscr{H}^i f^{-1}\mathscr{F}^2 \to \cdots\right]$$

3. Since $\mathscr{H}^i f^{-1} \cong f^{-1}\mathscr{H}^i$, comparing the complexes above gives $\widetilde{K}_i \cong f^{-1}K_i$.

4. Since f^{-1} is conservative, a sequence is exact iff f^{-1} of it is exact. We therefore deduce that the original sequence,

$$0 \to K_i \hookrightarrow \left[\mathscr{H}^i\mathscr{F}^0 \xrightarrow{\alpha_i} \mathscr{H}^i\mathscr{F}^1 \to \mathscr{H}^i\mathscr{F}^2 \to \cdots\right],$$

   was also exact.

5. We now apply the Bousfield-Kan spectral sequence to both $\mathscr{F}^\bullet$ and $f^{-1}\mathscr{F}^\bullet$ to deduce that

$$\mathscr{H}^i \varprojlim_{\Delta} \mathscr{F}^\bullet \xrightarrow{\cong} K_i,$$
$$\mathscr{H}^i \varprojlim_{\Delta} f^{-1}\mathscr{F}^\bullet \xrightarrow{\cong} \widetilde{K}_i$$

   Applying f^{-1} to the first isomorphism and using the fact that $f^{-1}K_i \cong \widetilde{K}_i$, we deduce the desired canonical isomorphism

$$\mathscr{H}^i f^{-1} \varprojlim_{\Delta} \mathscr{F}^\bullet \xrightarrow{\cong} \mathscr{H}^i \varprojlim_{\Delta} f^{-1}\mathscr{F}^\bullet, \quad \text{for all } i \geq 0$$

□

Remark 5.7. *This proof made use of "flatness" of the functor f^{-1} — a compatibility with t-structures — to reduce to a verification in the familiar world of abelian categories. While a similar argument works to show fpqc descent for* QCoh*, it will unfortunately not be a useful argument to mimic for our ultimate purpose of finding descent statements for* Sh_Λ*. See the Remark 6.3 for a discussion of the issues.*

Aside on the Godement envelope

Here is an example that will couch something familiar in the language of the above results. It will also illustrate a useful caution.

Take X a space, and let $f : X^{\text{discrete}} \to X$ be the inclusion of the discretization of X. Here, $\mathrm{Sh}(X^{\text{discrete}}) \simeq \widetilde{\prod_{x\in X}} \mathrm{Sh}(\mathrm{pt})$ where $\widetilde{\prod}$ denotes the categorical product, and the functors can be written as

$$f^{-1}F = \widetilde{\prod_{x\in X}} i_x^* F, \qquad f_*(\widetilde{\prod_{x\in X}} V_x) = \prod_{x\in X} i_{x*}V_x$$

where $\prod$ denotes the product within $\mathrm{Sh}(X)$. Since f is continuous and surjective, Theorem 5.5 guarantees that the functor f^{-1} is comonadic:

$$\mathrm{Sh}(X) \overset{(f^{-1})^{\mathrm{enh}}}{=\!=\!=} {}_{f^{-1}f_*}\mathrm{coMod}\,\mathrm{Sh}(X^{\text{discrete}})$$

This in particular implies something familiar. The monad $T := f_*f^{-1}$ of the adjunction acts on $\mathrm{Sh}(X)$, producing out of any object F its T-cobar diagram

$$F \longrightarrow \left[\textstyle\prod_{x\in X} i_{x*}i_x^*F \rightrightarrows \prod_{x\in X} \left(\prod_{y\in X} i_{y*}i_y^*F \right)_x \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots \right]$$

If $F \in \mathrm{Sh}(X)^\heartsuit$, then turning this cosimplicial diagram into a cochain complex by taking alternating sums of the coface maps gives the familiar Godement resolution of F. This squares with the content of Remark **??**, which records the following consequence of comonadicity: the above tautological coaugmented cosimplicial diagrams are in fact limit diagrams. So we learn that the fact that the Godement resolution is actually a resolution is a shadow of the fact that the functor f^{-1} is comonadic.

Now the caution. The map f generally is not proper, and neither is it a bundle projection. In fact, the worst is true: it fails base change for the fiber square

$$\begin{tikzcd}
X^{\text{discrete}} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\square"] & X^{\text{discrete}} \arrow[d, "f"] \\
X^{\text{discrete}} \arrow[r, "f"] & X
\end{tikzcd}$$

because the inclusion

$$i_x^*F \hookrightarrow \left(\prod_{y\in X} i_y^*F \right)_x$$

is not an isomorphism. However, since

$$(X^{\text{discrete}})^{\times_X^\bullet} = X^{\text{discrete},}$$

the Cech nerve of f collapses to a diagram of identity maps

$$\left[\cdots \;\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}\; X^{\text{discrete}} \;\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}\; X^{\text{discrete}} \rightrightarrows X^{\text{discrete}}\right] \xrightarrow{f} X$$

which in turn ensures that the resulting diagram of categories of sheaves tautologically satisfies the Beck-Chevalley condition, and also that

$$\varprojlim_{\Delta} \mathrm{Sh}(X^{\text{discrete}\bullet}/X) \simeq \mathrm{Sh}(X^{\text{discrete}})$$

But the canonical functor $\mathrm{Sh}(X) \xrightarrow{\Phi} \varprojlim_{\Delta} \mathrm{Sh}(X^{\text{discrete}\bullet}/X)$, which is identified with f, is not an equivalence. The issue was in the failure of base change for the very first square in the coaugmented cosimplicial diagram $\mathrm{Sh}(X) \to \mathrm{Sh}(X^{\text{discrete}\bullet}/X)$, which we drew above. So Theorem 3.8 is actually inapplicable here.

Moral 5.8. *This example illustrates the generality of the above comonadicity result, while simultaneously showing that accompanying base change-dependent results may not hold. In particular, it serves as a warning that, while Beck-Chevalley conditions for a cosimplicial diagram induce a Beck-Chevalley condition on the specific coaugmentation that is the limit of the diagram, they do not induce a Beck-Chevalley condition on general coaugmentations.*

5.4 For a cover by strata

Finally, we look at maps $Y \xrightarrow{f} X$ of the form

$$Y = \bigsqcup_{s\in S} X_s \xrightarrow{k_{s*}} X$$

for $\{X_s \overset{k_s}{\hookrightarrow} X\}_{s\in S}$ a *stratification* of X; see Definition 7.2. The descent question will be whether the functor

$$f^* := \widetilde{\prod_{s\in S} k_s^*}$$

in the adjunction

$$\mathrm{Sh}(X) \underset{\prod_{s\in S} k_{s*}}{\overset{f^*}{\rightleftarrows}} \widetilde{\textstyle\prod_{s\in S}} \mathrm{Sh}(X_s)$$

is comonadic.

Remark 5.9. *Note that we still look at the categories of all sheaves* Sh$(-)$. *The given functors also descent to an adjunction on the* S-constructible categories

$$\mathrm{Sh}_S(X) \underset{\prod_{s\in S} k_{s*}}{\overset{f^*}{\rightleftarrows}} \widetilde{\prod_{s\in S}} \mathrm{Loc}(X_s)$$

and therefore any comonadicity result above will restrict to one here.

We will begin with the comonadicity question for a two-piece stratification $\{Z, U\}$, where Z is thus closed and U is open, using

$$Z \overset{i}{\hookrightarrow} X, \quad U \overset{j}{\hookrightarrow} X$$

as the inclusions $X_s \overset{k_s}{\hookrightarrow} X$.

This example and the method of proof were absolutely crucial to the content of this thesis. In all of the subsequent material, we will take the tools used here, generalize them, and attempt to fit them into a broader framework for application to other comonadicity questions.

Proposition 5.10. *The functor*

$$\mathrm{Sh}(X) \xrightarrow{f^*} \mathrm{Sh}(Z) \boxplus \mathrm{Sh}(U)$$

is comonadic.

Proof. We verify the Barr-Beck-Lurie hypotheses. First note that f^* is conservative because f is surjective on points. The subtle point is to argue that f^* preserves the limits of f^*-split cosimplicial diagrams. Equivalently, we must argue that

1. j^* preserves the limits of cosimplicial diagrams that are both j^*-split and i^*-split, and
2. i^* preserves the limits of cosimplicial diagrams that are both j^*-split and i^*-split.

Part (1) is immediate because $j^* = j^!$ is a right adjoint, and therefore it preserves all limits. Part (2) is the hard part. Indeed, recall Warning 5.2.

So let $\Delta \xrightarrow{F^\bullet} \mathrm{Sh}_S(X)$ be a cosimplicial diagram such that $j^*F^\bullet$ and $i^*F^\bullet$ are split. To conclude that f^* is comonadic, it remains to show that the natural comparison

$$i^* \varprojlim_{\Delta} F^\bullet \xrightarrow{\phi} \varprojlim_{\Delta} i^* F^\bullet$$

is a quasi-isomorphism. Our strategy will be to sandwich i^* between two functors for which the analogous comparison map is a quasi-isomorphism on $F^\bullet$.

We offer two ways to implement this strategy.

Method 1: We thank Germán Stefanich for explaining the following strategy to us, and thereby setting in motion the tinkering that led to the contents of the last chapter. Consider the fiber sequence

$$i_! i^! \to \mathrm{Id} \to j_* j^*$$

of functors on $\mathrm{Sh}(X)$. Applying i^* yields the fiber sequence

$$i^! \to i^* \to i^* j_* j^*$$

on $\mathrm{Sh}(Z)$. This is the sandwiching we want. To see what it does, apply it to the diagram $F^\bullet$, and build the following canonical comparison between fiber sequences in $\mathrm{Sh}(Z)$, with our target map ϕ in the center:

$$\begin{tikzcd} i^! \varprojlim_\Delta F^\bullet \arrow[r] \arrow[d, equal, "(*)"] & i^* \varprojlim_\Delta F^\bullet \arrow[r] \arrow[d, "\phi"] & i^* j_* j^* \varprojlim_\Delta F^\bullet \arrow[d, equal, "(**)"] \\ \varprojlim_\Delta i^! F^\bullet \arrow[r] & \varprojlim_\Delta i^* F^\bullet \arrow[r] & \varprojlim_\Delta i^* j_* j^* F^\bullet \end{tikzcd}$$

The canonical map $(*)$ is an equivalence because $i^!$ preserves all limits as a right adjoint. The canonical map $(**)$ is the chain of equivalences

$$\varprojlim_\Delta i^* j_* j^* F^\bullet \simeq i^* j_* \varprojlim_\Delta j^* F^\bullet \simeq i^* j_* j^* \varprojlim_\Delta F^\bullet,$$

where the first equivalence comes from the assumption that $j^* F^\bullet$ is split, and the second by the already-mentioned fact that j^* preserves all limits.

Thus, ϕ is also an equivalence.

Method 2: Consider the other fiber sequence

$$j_! j^! \to \mathrm{Id} \to i_* i^*$$

of functors on $\mathrm{Sh}(X)$. This also is an adequate sandwiching map for $F^\bullet$, which builds the comparison between fiber sequences in $\mathrm{Sh}(X)$, with $i_*\phi$ on the right side:

$$\begin{tikzcd} j_! j^! \varprojlim_\Delta F^\bullet \arrow[r] \arrow[d, equal, "(*)"] & \varprojlim_\Delta F^\bullet \arrow[r] \arrow[d, equal] & i_* i^* \varprojlim_\Delta F^\bullet \arrow[d, "i_*\phi"] \\ \varprojlim_\Delta j_! j^! F^\bullet \arrow[r] & \varprojlim_\Delta F^\bullet \arrow[r] & \varprojlim_\Delta i_* i^* F^\bullet \end{tikzcd}$$

This time, one of the legs (the center) is an equivalence for free. The map $(*)$ is the chain of equivalences

$$j_! j^! \varprojlim_\Delta F^\bullet \simeq j_! \varprojlim_\Delta j^! F^\bullet \simeq \varprojlim_\Delta j_! j^! F^\bullet$$

where the first holds because $j^* = j^!$ preserves limits, and the second holds because $j^! F^\bullet = j^* F^\bullet$ is assumed split.

Thus, the map

$$i_*(i^* \varprojlim_\Delta F^\bullet) \xrightarrow{i_*\phi} i_*(\varprojlim_\Delta i^* F^\bullet) \simeq \varprojlim_\Delta i_* i^* F^\bullet$$

is an equivalence; we have identified its target using the assumption that $i^* F^\bullet$ is split.

Finally, since i_* is conservative, the fact that $i_*\phi$ is an isomorphism means that ϕ must have been an isomorphism. $\square$

Remark 5.11. *We will generalize Method 2 and apply it to prove all ensuing comonadicity statements.*

A slight modification gives the more general result:

Proposition 5.12. *This implies comonadicity for sheaves* $\mathrm{Sh}(X)$ *for a cover by any* finite *stratification* S *of* X*.*

Proof. We induct on the length of the stratification.

Base case: A 1-step stratification, for which the result is trivial.

Inductive step: Take a finite-step filtration of X by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset X_{\leq 1} \subset \cdots \subset X_{\leq k} \subset \cdots \subset X_{\leq n-1} \subset X_{\leq n} = X,$$

which yields a cover by the locally-closed pieces $X_k := X_{\leq k} \setminus X_{\leq k-1}$:

$$p : \bigsqcup_k X_k \to X$$

Time to verify the Barr-Beck conditions. First, since p is surjective, p^* is conservative. So it remains to check that p^* preserves the limits of p^*-split cosimplicial diagrams. So suppose $F^\bullet$ is a cosimplicial diagram such that $p^* F^\bullet$ is split. Equivalently, we assume:

$$\begin{cases} j_n'^* p^* F^\bullet \simeq j_n^* F^\bullet & \text{is split (1),} \\ i_n'^* p^* F^\bullet \simeq p_{\leq n-1}^* i_n^* F^\bullet & \text{is split (2)} \end{cases}$$

where the notation comes from the following diagram, where the bottom row is the open-closed decomposition produced by the top stratum, and the rest:

$$\begin{tikzcd} X_n \arrow[r, hook, "j_n'"] \arrow[d, "="'] \arrow[dr, phantom, "\square"] & \bigsqcup_k X_k \arrow[d, "p"] \arrow[dr, phantom, "\square"] & \bigsqcup_{k \leq n-1} X_k \arrow[l, hook', "i_n'"'] \arrow[d, "p_{\leq n-1}"] \\ X_n \arrow[r, hook, "j_n"] & X & X_{\leq n-1} \arrow[l, hook', "i_n"'] \end{tikzcd}$$

Thus we need to show that the following are equivalences:

$$j_n^* \varprojlim_{\Delta} F^\bullet \xrightarrow{\simeq} \varprojlim_{\Delta} j_n^* F^\bullet$$
$$i_n'^* p^* \varprojlim_{\Delta} F^\bullet \xrightarrow{\simeq} \varprojlim_{\Delta} i_n'^* p^* F^\bullet$$

The first is true because j_n^* preserves all limits. For the second, note first that $(*)$ below follows as in the previous argument from the assumption (1) that $j_n^* F^\bullet$ is split:

$$\begin{tikzcd}
i_n'^* p^* \varprojlim_{\Delta} F^\bullet \arrow[r] \arrow[d, equal] & \varprojlim_{\Delta} i_n'^* p^* F^\bullet \arrow[d, equal] \\
p^*_{\leq n-1} i_n^* \varprojlim_{\Delta} F^\bullet \arrow[r] \arrow[d, equal, "(*)"] & \varprojlim_{\Delta} p^*_{\leq n-1}(i_n^* F^\bullet) \\
p^*_{\leq n-1} \varprojlim_{\Delta} (i_n^* F^\bullet) \arrow[ur, bend right, "\phi"'] &
\end{tikzcd}$$

Finally, the sheaf $i_n^* F^\bullet \in Sh(X_{\leq n-1})$ is amenable to the inductive hypothesis, hence the proof would conclude once it is known to be $p^*_{\leq n-1}$-split. But that is guaranteed by assumption (2) on $F^\bullet$. □

Finally, we generalize the statement slightly:

Corollary 5.13. Sh$(-)$ *satisfies descent for (locally finite) stratifications.*

Proof. Let $p : X \to Y$ be a locally finite stratification, and $f : U \to Y$ be an open cover such that the base change p' is a disjoint union of finite stratifications:

$$\begin{tikzcd}
\coprod_{\alpha \in A} U_\alpha \times_Y X \arrow[r, equal] \arrow[d, "\coprod_{\alpha \in A} p'_\alpha"] & U \times_Y X \arrow[r, "f'"] \arrow[d, "p'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\
\coprod_{\alpha \in A} U_\alpha \arrow[r, equal] & U \arrow[r, "f"] & Y
\end{tikzcd}$$

We know that the open covers f, f' are coonadic, as is each p'_α from earlier, and thus p' since the p'_α are disjoint. The only thing to note is that the adjunction (f^*, f_*) extends to a triple (left adjoints on top)

$$\begin{tikzcd}
\mathrm{Sh}(Y) \arrow[r, "f^* = \widetilde{\prod}_\alpha j_\alpha^*"] & \widetilde{\prod}_{\alpha \in A} \mathrm{Sh}(U_\alpha) \arrow[l, bend right, "f_! := \oplus_\alpha j_{\alpha !}"'] \arrow[l, bend left, "f_* = \prod_\alpha j_{\alpha *}"] \arrow[r, equal] & \mathrm{Sh}(U)
\end{tikzcd}$$

where $\widetilde{\prod}_\alpha$ denotes the product in $\mathrm{Pr}^{L,\mathrm{st}}$ and $\oplus_\alpha$ and $\prod_\alpha$ denote the coproduct and product within $\mathrm{Sh}(Y)$. Thus, f^* is bicontinuous, and we now argue essentially by Lemma 3.12, but reproduce all the details here: taking a p^*-split cosimplicial diagram $\Delta \xrightarrow{\mathscr{F}^\bullet} \mathrm{Sh}(Y)$ gives, after applying f'^*,

$$\begin{tikzcd}
f'^* p^* \varprojlim_{\Delta} \mathscr{F}^\bullet \arrow[r, "f'^* \mathrm{can}"] \arrow[d, equal, "(1)"] & f'^* \varprojlim_{\Delta} p^* \mathscr{F}^\bullet \arrow[d, equal] \\
p'^* f^* \varprojlim_{\Delta} \mathscr{F}^\bullet \arrow[d, equal, "(2)"] & \varprojlim_{\Delta} f'^* p^* \mathscr{F}^\bullet \arrow[dddl, equal, bend left, "\therefore"'] \\
p'^* \varprojlim_{\Delta} f^* \mathscr{F}^\bullet \arrow[d, equal, "(3)"] & \\
\varprojlim_{\Delta} p'^* f^* \mathscr{F}^\bullet \arrow[d, equal, "(4)"] & \\
\varprojlim_{\Delta} f'^* p^* \mathscr{F}^\bullet &
\end{tikzcd}$$

For the right-hand column, the justification is that $p^*\mathscr{F}^\bullet$ is an absolute limit diagram. For the left-hand column, the justifications top-to-bottom, are as follows:

1. Commutativity of the square;
2. f^* preserves all limits;
3. p'^* is comonadic and $f^*\mathscr{F}^\bullet$ is p'^*-split, so this limit is preserved;
4. Commutativity of the square.

Since furthermore f'^* is conservative, this implies the desired outcome:

$$p^* \varprojlim_{\Delta} \mathscr{F}^\bullet \xrightarrow{\mathrm{can}\, \simeq} \varprojlim_{\Delta} p^* \mathscr{F}^\bullet$$

□

Corollary 5.14. $\mathrm{Sh}(-)$ *also satisfies descent for stratified bundles* $p : X \to Y$.

Proof. There is a stratification $f : S \to Y$ that base changes p to a split map. Since the base change f' is also a stratification of X, we conclude descent by Lemma 3.13. □

Chapter 6

Examples of Descent III: Zariski Comonadicity, Two Ways

6.1 What is in this chapter?

Zariski descent is the model descent statement and is the inspiration for this entire work. We will take it to mean the following two things:

Theorem 6.1. *Let X be a scheme and $j : U := \sqcup_{i\in I} U_i \to X$ a finite qcqs open cover, giving the diagram*

$$
\begin{tikzcd}
\varprojlim_{\Delta} \mathrm{QCoh}(U^{\bullet+1}/X) \arrow[dr, dashed, "\mathrm{ev}_0"] \arrow[drr, dashed, bend left, "\mathrm{ev}_1"] \arrow[drrr, dashed, bend left] & & & \\
\mathrm{QCoh}(X) \arrow[u, dotted, "\mathrm{can}"] \arrow[r, "j^*"] & \mathrm{QCoh}(U) \arrow[loop below, "\Omega:=j^*j_*"] \arrow[r, shift left, "\pi_1^*"] \arrow[r, shift right, "\pi_2^*"'] & \mathrm{QCoh}(U \times_X U) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}
$$

where for readability we have omitted the colimit-preserving right adjoints. Then

1. *(Zariski comonadicity) j^* is comonadic; and*
2. *(Zariski limit descent)* can *is an equivalence.*

Part (2) follows from part (1) by faithfully flat base change, via Theorem 3.8. Thus, the focus of this chapter will be on exploring part (1) in detail. More specifically, we present two ways of arguing for Zariski comonadicity:

1. Way 1 is through *t-structures*. Its advantage is that it is fairly tidy, and works more generally to show descent for fpqc covers. The downside, however, is that in making reference to t-structures, as a method of proof it is not immediately adapted to producing descent statements in the context of sheaves with prescribed singular support.

2. Way 2 is through *recollement and the study of localizations*. This has the limitation of not working for general fpqc covers. But what we lose in beautiful generality that is useful for algebraic geometry we make up for in an instant ability to transfer the argument to the mirror world of sheaves with prescribed singular support. That is because, unlike the pullback p^* for a general fpqc cover $Y \xrightarrow{p} X$, both the pullback

$$\mathrm{QCoh}(X) \xrightarrow{j^*} \prod_{i \in I} \mathrm{QCoh}(U_i)$$

for an open cover $\sqcup_{i \in I} U_i \xrightarrow{j} X$ and the total wrapping functor

$$\mathrm{Sh}_\Lambda(M) \xrightarrow{L} \prod_{i \in I} \mathrm{Sh}_{\Lambda_i}(M)$$

for a closed cover $\sqcup_{i \in I} \Lambda_i \subseteq \Lambda$ are products of localizations.

We now present the two ways of arguing for Zariski comonadicity in turn.

6.2 Way 1: using t-structures

As mentioned, this method of argument is more general and works for fpqc covers. We include it here for visibility and completeness, and with the hope that someone may find it adaptable to other contexts. We then comment on why it is not immediately adaptable to the context of sheaves with prescribed singular support.

Let $p : Y \to X$ be an fpqc cover of the scheme of interest X, which produces the adjunction

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r, shift left, "p^*"] & \mathrm{QCoh}(Y) \arrow[l, shift left, "p_*"] \end{tikzcd}$$

in $\mathrm{Pr}^{L,\mathrm{st}}$. We jump straight into it:

Theorem 6.2 (fpqc descent). *The functor p^* is comonadic.*

Proof. Writing

$$\mathrm{QCoh}(-) = \mathrm{QCoh}(-)^{>(-\infty)} \simeq \varinjlim_{n \to \infty} \mathrm{QCoh}(-)^{\geq -n}$$

and noting that the adjunction (p^*, p_*) restricts to each of these categories, we proceed to work only with $\mathrm{QCoh}(-)^{\geq 0}$. This boundedness will be useful, since the thrust of the strategy is to move calculations into the world of abelian categories by extracting cohomology sheaves.

We first establish that p^* is conservative. So suppose $p^*F \simeq 0$ for some $\mathscr{F} \in \mathrm{QCoh}(X)^{\geq 0}$. Since p is flat, it commutes with the operation of taking cohomology sheaves $\mathscr{H}^i p^* \mathscr{F} \cong p^* \mathscr{H}^i \mathscr{F}$, and so $p^* \mathscr{H}^i \mathscr{F} \cong 0$. Faithfulness of p^* implies that all $\mathscr{H}^i \mathscr{F} \cong 0$, which by boundedness implies that $\mathscr{F} \simeq 0$. Thus p^* is conservative.

Now suppose that $\Delta \xrightarrow{\mathscr{F}^\bullet} \mathrm{QCoh}(X)^{\geq 0}$ is a cosimplicial diagram such that $p^*\mathscr{F}^\bullet$ is split. We wish to show that the canonical map $p^* \varprojlim_{\Delta} \mathscr{F}^\bullet \xrightarrow{\phi} \varprojlim_{\Delta} p^* \mathscr{F}^\bullet$ is a quasi-isomorphism. Taking cohomology sheaves, this is equivalent to showing that the following maps in $\mathrm{QCoh}(Y)^\heartsuit$ are isomorphisms of classical sheaves:

$$\mathscr{H}^i p^* \varprojlim_{\Delta} \mathscr{F}^\bullet \xrightarrow{\mathscr{H}^i \phi} \mathscr{H}^i \varprojlim_{\Delta} p^* \mathscr{F}^\bullet, \quad \text{for all } i \geq 0$$

The rest of the proof proceeds verbatim as in Theorem 5.5, again taken from the note [10]. ☐

As an immediate corollary, since $j = \sqcup_{i \in I} j_i$ for a (not necessarily finite) open cover by j_i is fpqc, we get Theorem 6.1.

Having seen the proof, we make the following remark without introducing the notation, as it may only be relevant for Chapter 9. It is meant to illustrate the limitations of this method of argument in a desired context, as mentioned earlier:

Remark 6.3. *Two issues arise in trying to employ the above argument for the categories* $\mathrm{Sh}_\Lambda(M)$ *(for notation, see the Chapter 7). The first is an annoyance but not prohibitive, while the second is a real problem:*

1. *Taking cohomology sheaves does not respect singular support conditions. For example, consider* $M = \mathbb{R}^2$ *with the closed singular support condition* $\Lambda = SS(\mathbb{C}_{\mathbb{R}^2}) \cup SS(\mathbb{C}_{x \geq 0}) \cup SS(\mathbb{C}_{y \geq 0})$*, and the cofiber sequence*

$$\mathbb{C}_{\mathbb{R}^2} \to \mathbb{C}_{x \geq 0} \oplus \mathbb{C}_{y \geq 0} \to \mathscr{F} \xrightarrow{+1}$$

   in $\mathrm{Sh}_\Lambda(\mathbb{R}^2)$*. The cohomology sheaves are*

$$\mathscr{H}^{-1}\mathscr{F} = \mathbb{C}_{x,y<0}, \quad \mathscr{H}^0 \mathscr{F} = \mathbb{C}_{x,y \geq 0},$$

   and in fact $\mathscr{F}$ *is the nontrivial extension of one by the other*

$$\mathbb{C}_{x,y<0}[1] \to \mathscr{F} \to \mathbb{C}_{x,y \geq 0}.$$

   But these sheaves do not belong to $\mathrm{Sh}_\Lambda(\mathbb{R}^2)$ *because, in the cotangent fiber* $T_0^*\mathbb{R}^2$*, their singular supports contain more covectors than* Λ *allows.*

2. *The functors of "wrapping" (i.e. the left adjoints to the embeddings of categories that are induced by inclusions of singular support conditions) do not commute with the functor of taking cohomology sheaves. For example, consider* $M = \mathbb{R}$ *with the skyscraper sheaf*

$\mathbb{C}_0$, whose wrapping into the zero section is $L\mathbb{C}_0 = \mathbb{C}_{\mathbb{R}}[1]$ is a shifted *constant sheaf. Thus*

$$\mathscr{H}^0(L\mathbb{C}_0) = \mathscr{H}^0\mathbb{C}_{\mathbb{R}}[1] = 0, \quad \text{while} \quad L\mathscr{H}^0(\mathbb{C}_0) = L\mathbb{C}_0 = \mathbb{C}_{\mathbb{R}}[1].$$

In some sense, the issue here is that a pullback f^* is a "local" operation on sheaves, and thus commutes with the local operation $\mathscr{H}^\bullet$, while wrappings L are very non-local, or "global," which is "why" they do not commute with $\mathscr{H}^\bullet$.

The second difficulty shows that wrapping functors are not t-exact (or, suggestively, t-"flat") with respect to the standard t-structure on $\mathrm{Sh}_\Lambda(M)$. This means that we cannot use them to reduce the problem to a calculation in abelian categories. However, looking for t-structures on $\mathrm{Sh}_\Lambda(M)$ adapted to this proof is one possible area of future research that might lead to finding more general conditions for comonadicity for a cover of Λ.

6.3 Prelude to Way 2: localizations in algebraic geometry

To prepare for Way 2 of proving Zariski comonadicity, we begin with a discussion of how localizations arise in algebraic geometry. The goal is to understand the statement of Corollary 6.7.

Formal completions along a closed subscheme

Let X be a scheme and Z a closed subscheme, with $U := X \setminus Z$ the complement. We can enrich the picture by considering a completion of X along Z. If we do this naively by looking at the colimit

$$X^\wedge_{Z,\,\text{naive}} := \varinjlim \left[\operatorname{Spec} \mathscr{O}_X/I \to \operatorname{Spec} \mathscr{O}_X/I^2 \to \operatorname{Spec} \mathscr{O}_X/I^3 \to \cdots \right]$$

in the category Sch of *schemes*, then the category $\mathrm{QCoh}(X^\wedge_{Z,\,\text{naive}})$ will end up not participating in a useful localization with $\mathrm{QCoh}(X)$. Instead, we look at the *formal* completion $X^\wedge_Z$ of X along Z: that is, we simply think of the whole diagram of growing nilpotent thickenings of $Z := \operatorname{Spec} \mathscr{O}_X/I$ in X

$$X^\wedge_Z := \left[\operatorname{Spec} \mathscr{O}_X/I \to \operatorname{Spec} \mathscr{O}_X/I^2 \to \operatorname{Spec} \mathscr{O}_X/I^3 \to \cdots \right] := \text{“}\varinjlim\text{''} \operatorname{Spec} \mathscr{O}_X/I^n$$

as formally a single *object* in the category

$$\operatorname{Ind}\mathrm{Sch} \subseteq \operatorname{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set}) =: \mathrm{PSh}\,\mathrm{Sch}$$

of ind-schemes rather than schemes. These two categories are related by the Yoneda embedding Y and its left adjoint localization L,

$$\begin{tikzcd} \mathrm{Sch} \arrow[r, hook, "Y"] & \mathrm{Ind}\,\mathrm{Sch} \arrow[l, bend right, "L"'] \end{tikzcd}$$

which simply evaluates the formal colimit as the colimit inside Sch:

$$L(\text{``}\varinjlim\text{''} X_n) := \varinjlim^{\mathrm{Sch}}(X_n).$$

Thus, $L(X_Z^\wedge) \simeq X_{Z,\mathrm{naive}}^\wedge$. On the other hand, $Y(X_{Z,\mathrm{naive}}^\wedge) \not\simeq X_Z^\wedge$, owing to the fact that Y does not preserve colimits.

We therefore get the following diagram in $\mathrm{Ind}\,\mathrm{Sch}$:

$$\begin{tikzcd} X_Z^\wedge \arrow[r, "\hat{i}"] & Y(X) & Y(U) \arrow[l, hook', "Yj"'] \\ Y(Z) \arrow[u] \arrow[ur, hook, "Yi"] & & \end{tikzcd}$$

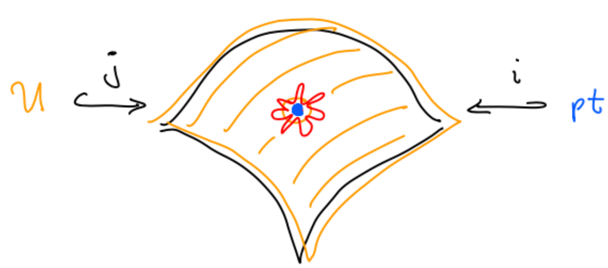


Figure 6.1: An open-closed decomposition of the scheme X. Formal neighborhood $X_Z^\wedge$ of Z is cartoonized in red.

where we will drop the Yoneda functor Y in the future.

The purpose of having looked at the formal completion instead of the naive one is that the category of sheaves on the formal completion,

$$\mathrm{QCoh}(X_Z^\wedge) := \varprojlim_{n\to\infty} \mathrm{QCoh}(\mathrm{Spec}\,\mathscr{O}_X/I^n),$$

can be fortuitously identified with the full subcategory $\mathrm{QCoh}\,X^{I-\mathrm{complete}}$ of I-complete modules in $\mathrm{QCoh}\,X$. Believing this, this set-up now yields the following diagram of categories in $\mathrm{Pr}^{L,\mathrm{st}}$, where $\hat{i}_*$ denotes the embedding above:

$$\begin{tikzcd}
\mathrm{QCoh}(X^\wedge_Z) \arrow[r, hook, "\hat{i}_*"] & \mathrm{QCoh}(X) \arrow[r, bend left, "j^*"] & \mathrm{QCoh}(U) \arrow[l, hook', "j_*"'] \\
\mathrm{QCoh}(Z) \arrow[u] \arrow[ur, "i_*"] & &
\end{tikzcd}$$

To flesh out this picture, we could like to identify the kernel[1] of j^*, and to write a formula for how to project onto it from $\mathrm{QCoh}(X)$. This will lead us to a pair of localizations, between $\mathrm{QCoh}(X)$ and either of $\mathrm{QCoh}(U), \mathrm{QCoh}(X^\wedge_Z)$. These localizations will prove to be very useful in crafting descent statements.

A recollement fiber sequence for QCoh

The adjunction (j^*, j_*) is already a localization; in this section, we show that $\mathrm{QCoh}(X)$ also localizes onto $\mathrm{QCoh}(X^\wedge_Z)$. To simplify our analysis, we work affine-locally, and therefore restrict to the case of an affine $X = \operatorname{Spec} R$. The closed subscheme Z is cut out by an ideal $I \subseteq R$ be an ideal, and with structure sheaf $\mathscr{O}_Z := A := R/I$. Let

$$E := \mathrm{End}_R(A)$$

be the *derived* endomorphism algebra of A, and define adjoint functors (T, D, C)

$$\begin{aligned} T &= (-) \otimes_E A \\ D &= \hom_R(A, -) \\ C &= \hom_E(A^\#, -) \end{aligned}$$

We use this set-up to define the following subcategories of $\mathrm{QCoh}(X)$:

$$\begin{aligned} A^\perp &:= \mathrm{Ker}(D) \\ \mathrm{QCoh}(X)^{I-\text{torsion}} &:= {}^\perp(A^\perp) \\ \mathrm{QCoh}(X)^{I-\text{complete}} &:= (A^\perp)^\perp \end{aligned}$$

These are called the *A-trivial modules*, the *I-torsion modules*, and the *I-complete modules*, respectively. Dwyer-Greenlees situate them in the following diagram:

$$\begin{tikzcd}
& \mathrm{QCoh}(X) \arrow[d, "D"] & \\
\mathrm{QCoh}(X)^{I-\text{torsion}} \arrow[r, bend right, "D"'] \arrow[loop below, "\Gamma_I(-):=TD"] & \mathrm{Mod}_E(\mathrm{QCoh}(X)) \arrow[l, bend right, "T"'] \arrow[u, bend left=40, "T"] \arrow[u, bend right=40, "C"'] \arrow[r, bend right, "C"'] & \mathrm{QCoh}(X)^{I-\text{complete}} \arrow[l, bend right, "D"'] \arrow[loop below, "(-)^\wedge:=CD"]
\end{tikzcd}$$

[1]Note that functor $\mathrm{QCoh}(Z) \xrightarrow{i_*} \mathrm{QCoh}(X)$ factors through $\mathrm{Ker}(j^*)$, but the functor $\mathrm{QCoh}(X^\wedge_Z) \xrightarrow{\hat{i}_*}$ does not.

We use the notation

1. $\Gamma_I M := TD(M)$, read as "(derived) I-torsion" and interpreted as I-power torsion, or I^∞-torsion[2]. This is because its classical piece is

$$\begin{aligned} H^0(TD(M)) &= \{m : I^k m = 0 \text{ for some } k > 0\} \\ &= \bigcup_k M^{I^k} =: M^{I^\infty} \subseteq M \end{aligned}$$

2. $N^\wedge := CD(N)$, read as "(derived) I-completion."

The following theorem relates these two full subcategories:

Theorem 6.4 ([3], Theorem 2.1)**.** *In the given set-up, assume that $A \in \mathrm{Perf}(R)$. Then the following hold:*

1. *all the functors in the bottom row of the diagram are equivalences;*

2. *the functors Γ_I and $(-)^\wedge$ share the following compatibility:*

$$\Gamma_I(M^\wedge) = \Gamma_I M, \qquad (\Gamma_I M)^\wedge = M^\wedge$$

Proof. (Partial) We offer a monadic proof of part of (1), just to illustrate the perspective. For the closed embedding $\mathrm{Spec}\, A \overset{i}{\hookrightarrow} \mathrm{Spec}\, R$, consider the following functors:

$$\begin{tikzcd} {}_R\mathrm{Mod} \arrow[r, shift right, "i^! = \hom_R(A -)"'] & {}_A\mathrm{Mod} \arrow[l, shift right, "i_* = \mathrm{fgt}"'] \arrow[r, shift right, "i_* = \mathrm{fgt}"'] & {}_R\mathrm{Mod} \arrow[l, shift right, "i^* = A \otimes_R (-)"'] \end{tikzcd}$$

All of these functors preserve colimits: the only one to check is $i^!$, but this is true because A is compact and ${}_R\mathrm{Mod}$ is compactly generated. The bottom functors preserve limits. Composing the two functors in each direction gives an adjunction in $\mathrm{Pr}^{L,\mathrm{st}}_\omega$:

$$\begin{tikzcd} {}_R\mathrm{Mod} \arrow[r, shift right, "R := i_* i^!"'] & {}_R\mathrm{Mod} \arrow[l, shift right, "L := i_* i^*"'] \end{tikzcd}$$

[2] To pick up I^∞ torsion, it was crucial here that E was the full derived algebra of endomorphisms. Taking just $E^0 := \mathrm{End}^0_R(A)$ produces the "non-local" version as the classical piece:

$$H^0 \Gamma'_I(M) := \{m : Im = 0\} =: M^I.$$

We study the question of monadicity for $R := i_* i^!$, and build the following diagram:

$$\begin{tikzcd}
\underbrace{A^\perp}_{\mathrm{Ker}(R)} \arrow[dr, hook] & {}^\perp(A^\perp) \arrow[d, hook] \arrow[drr, bend left, "R^{\mathrm{enh}}|_{{}^\perp(A^\perp)}"] & & \\
& \mathrm{QCoh}(X) \arrow[rr, "R^{\mathrm{enh}}:=(i_*i^!)^{\mathrm{enh}}"] \arrow[dr, "R:=i_*i^!"] & & {}_T\,\mathrm{Mod}(\mathrm{QCoh}(X)) \arrow[dl, "\mathrm{fgt}"] \\
& & \mathrm{QCoh}(X) \arrow[ul, bend left, "L:=i_*i^*"] \arrow[ur, bend right, "\mathrm{free}"'] \arrow[loop below, "T"] &
\end{tikzcd}$$

By the perfectness of A, the endofunctor T has the formula

$$\begin{aligned}
TM = i_* i^! i_* i^* M &= \hom_R(A, A \otimes_R M) \\
&= \hom_R(A, A) \otimes_R M \\
&= E \otimes_R M
\end{aligned}$$

The reconstruction functor L^{recon} takes a T-module M with structure map $E \otimes_R M \to M$, and produces

$$\varinjlim \mathrm{colim}\left(\cdots \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}\; A \otimes_R E \otimes_R E \otimes_R M \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; A \otimes_R E \otimes_R M \;\rightrightarrows\; A \otimes_R M \right)$$

which means that

$$L^{\mathrm{recon}} = A \otimes_E (-),$$

as claimed earlier.

The functor $R := i_* i^!$ preserves limits and colimits, but is not conservative: its kernel is $A^\perp$. The claim now is that restricting to the subcategory ${}^\perp(A^\perp)$ restores conservativity. We apply Barr-Beck-Lurie:

1. To show R is conservative on this subcategory, let M be an object in it such that $RM \simeq 0$, i.e. $M \in A^\perp \cap {}^\perp(A^\perp)$. Examining the identity morphism

   $$\underbrace{M}_{\in {}^\perp(A^\perp)} \xrightarrow{\mathrm{Id}_M} \underbrace{M}_{\in A^\perp}$$

   shows that $\mathrm{Id}_M \simeq 0$, meaning that $M \simeq 0$.

2. To show that the category ${}^\perp(A^\perp)$ admits the colimits of R-split simplicial diagrams, we simply note that the condition of being in the left orthogonal ${}^\perp(A^\perp)$ is closed under all colimits available in the ambient category.

3. To show that L lands inside ${}^\perp(A^\perp)$, let $M \in \mathrm{QCoh}(X)$, and consider $N \in A^\perp$. Then

$$\hom_R(LM, N) := \hom_R(A \otimes_R M, N) \simeq \hom_R(M, \hom_R(A, N)) \simeq 0$$

by assumption. Thus $LM \in {}^\perp(A^\perp)$.

This concludes the proof that $R^{\mathrm{enh}}|_{{}^\perp(A^\perp)}$ is an equivalence. □

Remark 6.5. *A consequence of the proof is a formula for the projection* Γ_I *onto the subcategory* ${}^\perp(A^\perp)$*: it is the reconstruction functor* $L^{\mathrm{recon}} = A \otimes_E (-)$*. Similarly, a formula for the projection* $(-)^\wedge$ *onto the subcategory* $(A^\perp)^\perp$ *is* $\hom_E(A, -)$.

Corollary 6.6. *Since* $\mathrm{Mod}_E(\mathrm{QCoh}(X))$ *is presentable for* $A \in \mathrm{Perf}(X)$*, so are* $\mathrm{QCoh}(X)^{I-\mathrm{torsion}}$ *and* $\mathrm{QCoh}(X)^{I-\mathrm{complete}}$.

In summary, $\mathrm{QCoh}(X)$ has two isomorphic subcategories, embedded in a variety of interesting ways:

1. the coreflective (in Pr^L) subcategory of I-torsion modules, which participates in the following diagram in Pr^L (the top embedding is the standard embedding):

$$\begin{tikzcd} \mathrm{QCoh}(X)^{I-\mathrm{torsion}} \arrow[r, hook, bend left] \arrow[r, dashed, bend right, "(-)^\wedge"'] & \mathrm{QCoh}(X) \arrow[l, "\Gamma_I"] \end{tikzcd}$$

where the dashed functor is not necessarily colimit-preserving;

2. the reflective (in Pr) subcategory of I-complete modules, which participates in the following diagram in Pr^L (the bottom embedding is the standard embedding):

$$\begin{tikzcd} \mathrm{QCoh}(X)^{I-\mathrm{complete}} \arrow[r, bend left, "\Gamma_I"] \arrow[r, hook, dashed, bend right] & \mathrm{QCoh}(X) \arrow[l, "(-)^\wedge"] \end{tikzcd}$$

These are the structures that we will exploit in our proof of Zariski comonadicity. We encase them in the following statement:

Corollary 6.7. *The main consequence is that for* $\mathrm{QCoh}(X)$*, there is the following diagram in* Pr^L

$$\begin{tikzcd} \mathrm{QCoh}(X)^{I-\mathrm{torsion}} \arrow[r, hook] & \mathrm{QCoh}(X) \arrow[r, "j^*"] \arrow[l, bend left, "\Gamma_I"] & \underbrace{\mathrm{QCoh}(U)}_{\mathrm{QCoh}(X)/\langle A \rangle} \arrow[l, hook, bend left, "j_*"] \end{tikzcd}$$

that is the source of fiber sequences

$$\Gamma_I M \to M \to j_* j^* M \xrightarrow{+1}$$

for any $M \in \mathrm{QCoh}(X)$. *To mimic notation from topology, we will use* $\hat{i}_!$ *to denote the usual embedding of* I*-torsion modules into* $\mathrm{QCoh}(X)$, *and* $\hat{i}^!$ *to denote the functor* Γ_I *of taking the* I*-torsion; in this notation, the above fiber sequence takes on the form*

$$\hat{i}_! \hat{i}^! M \to M \to j_* j^* M \xrightarrow{+1}$$

Since $\mathrm{QCoh}(X)^{I-\mathrm{torsion}}$ *is presentable, it is bicomplete. Here is how we can calculate colimits and limits in it:*

1. *To calculate a colimit in it, simply calculate the colimit inside* $\mathrm{QCoh}(X)$*; this is valid because the embedding is colimit-preserving.*

2. *To calculate a limit in it, first calculate the limit inside* $\mathrm{QCoh}(X)$*, and then apply the coreflector* Γ_I *to the result to build the correct limit.*

Example 6.8. *Consider* $X = \operatorname{Spec} k[x]$ *and* $A = k[x]/x$. *Certainly* A *is perfect, and so in this case,*

$$\mathrm{QCoh}(X)^{I-\mathrm{torsion}} = \langle k[x]/x \rangle_{colimits}$$

is just the subcategory of $\mathrm{QCoh}(X)$ *generated under colimits by the object* $k[x]/x \simeq k$. *Let us understand some objects in this category.*

Since Γ_I *is essentially surjective, every object in* $\mathrm{QCoh}(X)^{I-\mathrm{torsion}}$ *can be built as* $\Gamma_I M$ *for some object* $M \in \mathrm{QCoh}(X)$. *So certainly, the objects* $\Gamma_I(k[x]/x^n) = k[x]/x^n$ *are in this category. To see another object, consider* $M = k[x]$. *Looking in the abelian category* $\mathrm{QCoh}(X)^\heartsuit$ *gives the short exact sequence*

$$0 \to k[x] \hookrightarrow k[x, x^{-1}] \to k[x, x^{-1}]/k[x] \to 0$$

which we identify with the (rotated) distinguished triangle

$$k[x] \to j_* j^* k[x] \to \Gamma_I(k[x])[1] \xrightarrow{+1}$$

in $\mathrm{QCoh}(X)$. *This cone* $k[x, x^{-1}]/k[x]$ *goes by many names:*

1. *it is the locally nilpotent module, where the action of* x *can be written in the natural* k*-basis as*

$$\begin{tikzcd} \cdots \arrow[r, bend left, "\cdot x"] & k \cdot x^{-3} \arrow[r, bend left, "\cdot x"] & k \cdot x^{-2} \arrow[r, bend left, "\cdot x"] & k \cdot x^{-1} \arrow[r, bend left, "0"] & 0; \end{tikzcd}$$

2. *it is the colimit of inclusions*

$$\varinjlim \left[k[x]/x^n \xhookrightarrow{x^k \mapsto x^{k+1}} k[x]/x^{n+1} \right];$$

3. it is the module of distributions supported at $0 \in \mathbb{A}^1_k$, which we might denote $Dist_0(\mathbb{A}^1_k)$.

This module, under the change of coordinates $x \mapsto t-1$, already appeared in Chapter 4.5 as the universal ind-unipotent local system $\mathscr{L}^{\text{ind-uni}}$ on S^1.

Remark 6.9. *To calculate $\Gamma_I M$, it can help to view j_*j^*M as the endpoint of the infinite process of repeatedly tensoring F with the sheaf $\mathscr{O}(Z)$:*

$$j_*j^*F \simeq \underrightarrow{\operatorname{colim}}\Big[F \otimes \mathscr{O}(Z) \to F \otimes \mathscr{O}(2Z) \to F \otimes \mathscr{O}(3Z) \to \dots\Big]$$

Each term $F \otimes \mathscr{O}(n\mathbb{Z})$ permits poles of order up to n along Z, and thus the colimit permits poles of arbitrary finite order along Z.

Functors that do not preserve limits

To close this section, we record some examples from algebraic geometry of pullback functors that do not preserve limits. This is similar in purpose to Warning 5.2 from topology, because our comonadicity arguments in algebraic geometry will have to reckon with the same reality.

Warning 6.10 (Failure to preserve injective limits)**.** *This is a crucial example. Let*

$$j : U \hookrightarrow X$$

be an affine open embedding into a scheme. The functor j^* does not preserve all limits. For example, let $X = \operatorname{Spec} k[x]$, $U = k[x, x^{-1}]$, and consider the diagram of modules $F : \mathbb{N}^{\text{op}} \to \operatorname{QCoh}(X)$ given by

$$F^\bullet = \Big[\cdots \to k[x]/x^n \to k[x]/x^{n-1} \to \cdots \to k[x]/x\Big]$$

Then certainly $j^*F^\bullet \simeq 0^\bullet$, so $\underleftarrow{\lim}_{\mathbb{N}^{\text{op}}} j^*F^\bullet \simeq 0$. But

$$j^*\underleftarrow{\lim}_{\mathbb{N}^{\text{op}}} F^\bullet \simeq j^*k[[x]] \simeq k[[x]][x^{-1}] \not\simeq 0$$

is Laurent series. We note that the fact that the morphisms in $F^\bullet$ were surjections meant that the diagram $F^\bullet$ was already fibrant, and therefore was suitable for calculating the homotopy limit as $\underleftarrow{\lim} F^\bullet \simeq k[[x]]$.

Thus, j^* does not preserve all cosimplicial limits. This fact will be the biggest obstacle to reckon with in the upcoming recollement-based proof of Zariski comonadicity.

Warning 6.11 (Failure to preserve products)**.** *Consider*

$$p : \operatorname{Spec} k[x] \to \operatorname{Spec} k$$

the projection. It does not preserve arbitrary infinite products: the object

$$p^*(\prod_{\mathbb{Z}} k) := (\prod_{\mathbb{Z}} k) \otimes_k k[x]$$

consists of (finite) polynomials with coefficients in $V := \prod_{\mathbb{Z}} k$*, while*

$$\prod_{\mathbb{Z}} (k \otimes_k k[x]) = \prod_{\mathbb{Z}} k[x]$$

contains more general items, such as the infinite series $\sum_i e_i x^i$*; here* e_i *is the* i*th standard basis vector of* V*.*

The above warnings illustrate two different ways in which an fpqc cover $p : X \to Y$ can have a pullback p^* which does not preserve limits.

6.4 Way 2: using localizations

Armed with the pair of localizations determined by an open subscheme $U \subset X$, are now ready to tackle Zariski comonadicity. We start very explicitly with the simplest version, and then tackle the general finite cover case.

Zariski comonadicity for a two-piece cover

Proposition 6.12. *Suppose*

$$X = U_1 \cup U_2$$

is an open covering of a variety X*. Then the functor* $L := j_1^* \boxplus j_2^*$ *is comonadic.*

Proof. The proof here is very much in the spirit of Proposition 5.12.

We verify the hypotheses of Barr-Beck-Lurie for $L = j_1^* \boxplus j_2^* = (j_1 \sqcup j_2)^*$. First, since $j_1 \sqcup j_2 : U_1 \sqcup U_2 \to X$ is surjective, that means that L is conservative. So it remains to show, if $F^\bullet : \Delta \to \operatorname{QCoh} X$ is a diagram such that $j_1^* F^\bullet, j_2^* F^\bullet$ are split, that the canonical map $\phi_i : j_i^* \varprojlim_{\Delta} F^\bullet \xrightarrow{\simeq} \varprojlim_{\Delta} j_i^* F^\bullet$ is an equivalence for $i = 1, 2$. Recalling Warning 6.10, we will need to proceed carefully. Since j_{i*} is conservative, this is equivalent to showing that

$$j_{i*}\phi_i : j_{i*} j_i^* \varprojlim_{\Delta} F^\bullet \xrightarrow{\simeq} j_{i*} \varprojlim_{\Delta} j_i^* F^\bullet, \qquad \text{for } i = 1, 2.$$

As announced, the strategy will be to prove this using the fiber sequences from the previous section.

Put $Z_1 := U_1 \setminus U_2, Z_2 = U_2 \setminus U_1$. Recalling the notation in Corollary 6.7, this gives two "distribution" fiber sequences[3]:

$$\underbrace{\hat{i}_{2!}\hat{i}_2^! F^\bullet}_{(B)} \to F^\bullet \to \underbrace{j_{1*}j_1^* F^\bullet}_{(A)}$$

$$\hat{i}_{1!}\hat{i}_1^! F^\bullet \to F^\bullet \to j_{2*}j_2^* F^\bullet$$

The idea is simple: to show that $\varprojlim_{\Delta}$ commutes with the functors in (A), use the split hypothesis and functor compatiblities to argue that $\varprojlim_{\Delta}$ commutes with the functors in (B) (see Figure 6.2).

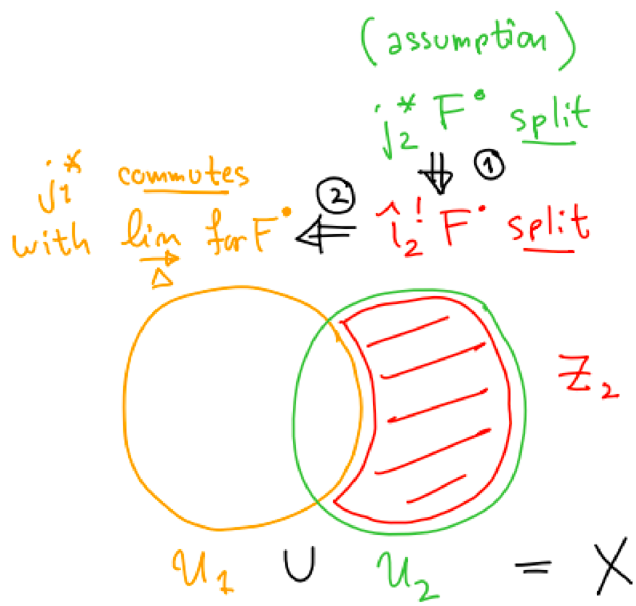


Figure 6.2: The logical flow of the argument for comonadicity. The implication (1) is due to the compatibility $\hat{i}_2^! F^\bullet = \hat{i}_2^! j_{2*}j_2^* F^\bullet$ together with the fact that the image under any functor of a split diagram is still split, and the implication (2) is due to the analysis below of the "distribution" fiber sequence

$$\hat{i}_{2!}\hat{i}_2^! F^\bullet \to F^\bullet \to j_{1*}j_1^* F^\bullet$$

Here are the details. Since $j_1^* F^\bullet$ is split, then so is $\hat{i}_1^! j_1^* F^\bullet$, which—importantly—is the same as $\hat{i}_1^! F^\bullet$. Similarly, $\hat{i}_2^! F^\bullet$ is split. Therefore, taking limits of the first triangle gives:

[3] To recap: we use the notation $\hat{i}_Z^!$ in place of $\Gamma_{I_Z}(-)$ to denote "taking the derived I_Z-torsion," and $\hat{i}_{Z!}$ in place of ι to denote the canonical embedding.

$$\begin{tikzcd}
\hat{i}_{2!}\hat{i}_2^! \varprojlim_\Delta F^\bullet \arrow[r] \arrow[d, equal, "(*)"] & \varprojlim_\Delta F^\bullet \arrow[r] \arrow[dd, equal] & j_{1*}j_1^* \varprojlim_\Delta F^\bullet \arrow[dd, "j_{1*}\phi_1"] \\
\hat{i}_{2!} \varprojlim_\Delta \hat{i}_2^! F^\bullet \arrow[d, equal, "(**)"] & & \\
\varprojlim_\Delta \hat{i}_{2!}\hat{i}_2^! F^\bullet \arrow[r] & \varprojlim_\Delta F^\bullet \arrow[r] & \varprojlim_\Delta j_{1*}j_1^* F^\bullet
\end{tikzcd}$$

where $(*)$ is from the fact that $\hat{i}_2^!$ is a right adjoint, and $(**)$ is from the fact that $\hat{i}_2^! F^\bullet$ is split. Thus, $j_{1*}\phi_1$ is an equivalence. Since this holds for the other index as well, this concludes the proof. □

Remark 6.13. *It is crucial to our strategy that the above proof was completely mechanical once we knew the following four facts for $k = 1, 2$:*

1. j_k^* *are jointly conservative;*
2. j_{k*} *are conservative;*
3. $\hat{i}_k^! j_k^* \simeq \hat{i}_k^!$*;*
4. $\hat{i}_k^!$ *are right adjoints.*

Zariski comonadicity for a finite cover

We now set up some notation for the general proof. For opens $V \subset U \subset X$, let $j_{V \subset U} : V \hookrightarrow U$ denote the inclusion. For a subscheme Z which is closed in an open U and which further is closed in an open $V \subset U$, write $i_{Z \subset U} = j_{V \subset U} \circ i_{Z \subset V}$. These have the following important compatibilities:

I. $j_{V \subset U}^* j_{W \subset V}^* \simeq j_{W \subset U}^*$

II. $\hat{i}_{Z \subset U}^! \simeq \hat{i}_{Z \subset V}^! j_{V \subset U}^*$

By repeatedly using the active ingredient, contained in the above two-piece cover proof and enshrined abstractly in Lemma 8.6, we have a proof of Zariski comonadicity for finite covers:

of Theorem 6.1, part (1). Let $X = U_1 \cup \cdots \cup U_n$ be an open cover. For $K \subseteq [n]$, put $U_K := \cup_{k \in K} U_k$. For $K \subseteq L$, put $j_{K \subset L} : U_K \hookrightarrow U_L$. Again, we argue for descent by using recollement technology, whose consequenes we have repackaged into Lemma 8.6.

Assume that $j^*_{k\in[n]}F^\bullet$ is split for all k; we wish to show that $j^*_{k\in[n]}\varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} j^*_{k\in[n]}F^\bullet$. WLOG, up to re-indexing, we wish show that $j^*_{[1]\subset[n]}\varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} j^*_{[1]\subset[n]}F^\bullet$. The strategy is to first use Compatibility I to factor

$$j^*_{[1]\subset[n]} \simeq j^*_{[1]\subset[2]}j^*_{[2]\subset[3]}\cdots j^*_{[n-1]\subset[n]}$$

and show that the successive pullbacks preserve the limit. Here is how that goes:

1. First, look at $j := j_{[n-1]\subset[n]}$; the complement Z_n (inside $U_{[n]} = X$) includes as a closed subscheme into U_n, and thus by the Compatibility II, the restriction $\hat{i}^!_{Z_n\subset[n]}F^\bullet \simeq \hat{i}^!_{Z_n\subset U_n}j^*_{n\in[n]}F^\bullet$ to this complement is split. By Lemma 8.6, we conclude that
$$j^*_{[n-1]\subset[n]}\varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} j^*_{[n-1]\subset[n]}F^\bullet$$

2. Next, look at $j := j_{[n-2]\subset[n-1]}$; the complement Z_{n-1} (inside $U_{[n-1]}$) includes as a closed subscheme into U_{n-1}, and thus by Compatibility I and II, the restriction
$$\hat{i}^!_{Z_{n-1}\subset[n-1]}j^*_{[n-1]\subset[n]}F^\bullet \simeq \hat{i}^!_{Z_{n-1}\subset U_{n-1}}j^*_{(n-1)\in[n-1]}j^*_{[n-1]\subset[n]}F^\bullet \simeq \hat{i}^!_{Z_{n-1}\subset U_{n-1}}j^*_{(n-1)\in[n]}F^\bullet$$
to this complement is split. By Lemma 8.6, we conclude that
$$j^*_{[n-2]\subset[n-1]}\varprojlim_{\Delta} j^*_{[n-1]\subset[n]}F^\bullet \simeq \varprojlim_{\Delta} j^*_{[n-2]\subset[n-1]}j^*_{[n-1][n]}F^\bullet$$
We can chain this together with the result in (1) to obtain, using Compatibility I,
$$j^*_{[n-2]\subset[n]}\varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} j^*_{[n-2]\subset[n]}F^\bullet$$

3. Repeat the reasoning in the above step finitely many times to yield $j^*_{[1]\subset[n]}\varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} j^*_{[1]\subset[n]}F^\bullet$, which is what we wanted to show.

$\square$

This argument will be recast in more general language in Proposition 8.19.

6.5 Appendix: non-comonadic covers

Zariski comonadicity states that that an open cover can be used to comonadically describe QCoh(X). One might ask:

Question 6.14. *Can other kinds of covers of X in algebraic geometry be comonadic?*

Here is an example of a cover that falls short of comonadicity:

Example 6.15 (Open-closed decomposition)**.** *Let $\mathscr{C} = \mathrm{QCoh}(\mathbb{A}^1) = {}_{k[x]}\mathrm{Mod}$, and take the open-closed decomposition*

$$\{0\} \overset{i}{\hookrightarrow} \mathbb{A}^1, \quad U := \mathbb{A}^1 \setminus \{0\} \overset{j}{\hookrightarrow} \mathbb{A}^1$$

Thus, $\{0\}, \mathbb{A}^1 \setminus \{0\}$ is a cover of $\mathbb{A}^1$, at least in the sense that the functor

$$\mathrm{QCoh}(\mathbb{A}^1) \xrightarrow{L := i^* \boxplus j^*} \mathrm{QCoh}(\{0\}) \boxplus \mathrm{QCoh}(U)$$

is conservative.

However, it is not comonadic because it fails to preserve enough limits.

Let us back up a bit. To see that it is conservative, suppose that $F \in \mathrm{QCoh}(\mathbb{A}^1)$ is such that $i^*F \simeq 0$. Equivalently, $F \xrightarrow{\cdot x} F$ is a quasi-isomorphism, which means that $F \simeq j_*j^*F$. So if $j^*F \simeq 0$ as well, then $F \simeq 0$.

However, L is not comonadic. As proposed by the two comonadic versions of Barr-Beck from Chapter 2.5, we can examine this failure from two angles:

1. *One perspective on the failure is that L^{enh} is not an embedding, i.e. that there exists an object F that is not the limit of its T-cobar. Let us find such an object. To arrive there, note that since $i^*j_* \simeq 0$ and $j^*i_* \simeq 0$, the subcategories $i_*\,\mathrm{QCoh}(\{0\})$ and $j_*\,\mathrm{QCoh}(\mathbb{A}^1)$ are orthogonal and the monad T is diagonal:*

$$T = \begin{bmatrix} i_*i^* & 0 \\ 0 & j_*j^* \end{bmatrix}$$

Consider now $F = \mathscr{O}$ the structure sheaf, and build its T-cobar $T^{\bullet+1}\mathscr{O}$:

$$\mathscr{O} \longrightarrow i_*i^*\mathscr{O} \oplus j_*j^*\mathscr{O} \rightrightarrows i_*i^*i_*i^*\mathscr{O} \oplus j_*j^*j_*j^*\mathscr{O} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

By orthogonality, the limit of the cobar decomposes as a direct sum:

$$\begin{aligned} \varprojlim_{\Delta} T^{\bullet+1}\mathscr{O} &\simeq \varprojlim_{\Delta} (i_*i^*)^{\bullet+1}\mathscr{O} \oplus \varprojlim_{\Delta} (j_*j^*)^{\bullet+1}\mathscr{O} \\ &\simeq \mathscr{O}_0^{\wedge} \oplus j_*j^*\mathscr{O} \end{aligned}$$

But certainly, $\mathscr{O}$ does not e.g. have $j_*j^*\mathscr{O}$ as a direct summand, so $\mathscr{O} \neq \varprojlim_{\Delta} T^{\bullet+1}\mathscr{O}$. Thus L is not comonadic. As we will see in Theorem **??***, this particular failure is the same as the failure of R^{recon} to be an embedding.*

2. *Another perspective on the failure is that R^{recon} is not an embedding. This failure would be implied by the existence of a cosimplicial diagram that is L-split, but whose limit*

is not preserved by L. We can mine our previous example to get one here; take the following cosimplicial diagram, which we have decorated with its limit:

$$\mathscr{O}_0^\wedge \longrightarrow \left[i_* i^* \mathscr{O} \rightrightarrows i_* i^* i_* i^* \mathscr{O} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \cdots \right]$$

We first check that it is L-split. As the cobar for the adjunction (i^*, i_*), it is i^*-split. The cosimplicial piece is also trivially j^*-split, because its j^*-image is the zero cosimplicial diagram. Though L-split, its limit is not preserved, because $j^*\mathscr{O}_0^\wedge \not\simeq 0$. Indeed, this is the same example as Warning 6.10, but in the guise of a cosimplicial diagram; taking its tower of truncated totalizations gives the injective limit diagram in the warning.

We now look at both failures simultaneously, by first identifying

$$_\Omega\, \mathrm{coMod}\left(\mathrm{QCoh}(\{0\}) \boxplus \mathrm{QCoh}(U)\right) \quad \text{with} \quad \mathrm{QCoh}(\mathbb{A}^1)^{I-\mathrm{complete}} \boxplus \mathrm{QCoh}(U),$$

and then building the comonadicity diagram in Pr^L

$$\begin{tikzcd}
\mathrm{QCoh}(\mathbb{A}^1) \arrow[rr, shift left, "L^{\mathrm{enh}} = (-)^\wedge \boxplus j^*"] \arrow[dr, shift left, "i^* \boxplus j^*"] & & \mathrm{QCoh}(\mathbb{A}^1)^{I-\mathrm{complete}} \boxplus \mathrm{QCoh}(U) \arrow[ll, shift left, "R^{\mathrm{recon}} = \oplus"] \arrow[dl, "\mathrm{fgt}"] \\
& \mathrm{QCoh}(\{0\}) \boxplus \mathrm{QCoh}(U) \arrow[ul, shift left, "i_* \oplus j_*"] &
\end{tikzcd}$$

This illustrates both failures: L^{enh} is conservative but not an embedding, and the same is true of R^{recon}.

Remark 6.16. *While j^* is a localization, i^* is not: its right adjoint i_* is not an embedding because*

$$\hom(i_* k, i_* k) = k \oplus T_0 \mathbb{A}^1[-1].$$

However, the enhanced version $(i^*)^{\mathrm{enh}} \simeq (-)^\wedge$ is *a localization.*

Though the diagram above exhibits the failure of $i^* \boxplus j^*$ to be comonadic, it also offers a correction: instead of considering the functor i^* to $\mathrm{QCoh}(\{0\})$, consider its completed version $(-)^\wedge$ to $\mathrm{QCoh}(\mathbb{A}^1)^{I-\mathrm{complete}}$. That is, make the replacement

$$i^* \boxplus j^* \quad \rightsquigarrow \quad (-)^\wedge \boxplus j^* =: L^{\mathrm{enh}}$$

One may then prove the following generalization, at least for the case where $Z \overset{i}{\hookrightarrow} X$ is a Cartier divisor on a classical scheme X:

Theorem 6.17 (Beauville-Laszlo)**.** *The functor L^{enh}, i.e. the cover*

$$X_Z^\wedge \sqcup U \quad \xrightarrow{\hat{i} \sqcup j} \quad X$$

is comonadic.

Chapter 7

Singular Support

7.1 What is in this chapter?

This chapter introduces the notion of constructible sheaf, and explores the concept of its singular support. The ultimate purpose is to build the conceptual base for stating the comonadic problem of interest.

The central notion is that of a microstalk of sheaves. The remainder of this section will attempt to motivate this construction, which is analogous to that of the directional derivative of functions.

Let us now build this analogy. First, given a manifold M and a C^1 function $f : M \to \mathbb{R}$, at any point $x \in M$ the function has an associated value, $f(x) \in \mathbb{R}$. If M is given a metric, then f has a gradient vector field ∇f, and so for any point $(x, \xi) \in T^*M$, f also has a directional derivative at x in the codirection ξ:

$$\xi(\nabla_x f) \in \mathbb{R}.$$

We now *categorify* this construction from analysis to the setting of classical sheaves on M $\mathrm{Sh}(M)^\heartsuit$. The function f now becomes a sheaf $F \in \mathrm{Sh}(M)^\heartsuit$. The value of f at $x \in M$ becomes the stalk of F at x, F_x. To calculate the "directional derivative" of F at (x, ξ), let us pick a small open ball $B \subseteq M$ around x, and consider an auxiliary C^1 function $\phi : B \to \mathbb{R}$ such that

$$\phi(x) = 0 \quad \text{and} \quad d_x\phi = \xi.$$

Then it makes sense to consider the kernel and cokernel of the restriction map $\mathrm{res}^B_{B_{\phi<0}}$ from sections over B to sections over the half-ball

$$B_{\phi<0} := \{m \in B \,:\, \phi(m) < 0\},$$

which fill out the exact sequence

$$\mathrm{Ker}(\mathrm{res}^B_{B_{\phi<0}}) \hookrightarrow \left[F(B) \xrightarrow{\mathrm{res}^B_{B_{\phi<0}}} F(B_{\phi<0})\right] \twoheadrightarrow \mathrm{Coker}(\mathrm{res}^B_{B_{\phi<0}})$$

The size of the cokernel measures the extent to which sections over $B_{f<0}$ fail to propagate to the rest of B. The size of the kernel measures the extent to which sections that do propagate fail to propagate uniquely.

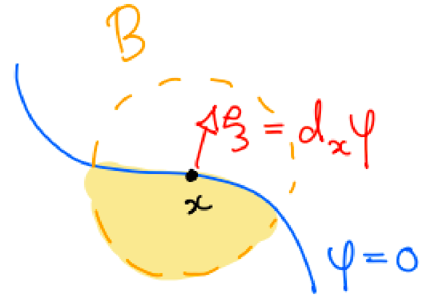


Figure 7.1: Set-up for taking the directional derivative of a sheaf. The goal is to see which sections propagate from the orange-colored region $B_{\phi<0}$ across the blue frontier $\phi = 0$ to the rest of the ball B, in the codirection ξ that is "normal" to the frontier at x.

We amalgamate the kernel and the cokernel into the homotopically-savvy construction of the *cone*, or *cofiber*, of $\mathrm{res}^B_{B_{f<0}}$, which now makes good sense for any sheaf F in the ∞-category $\mathrm{Sh}(M)$. In fact, we can remove the dependence of B by taking the colimit as these opens B shrink to x to get something depending only on the shred of M around x:

Definition 7.1. *Let $F \in \mathrm{Sh}(M)$, $(x, \xi) \in T^*M$, and $\phi \in C^1(M)_x$ a germ of a C^1 function at x satisfying $\phi(x) = 0$ and $d_x\phi = \xi$. The* microstalk of F at x in the codirection ξ, with respect to ϕ, *is the (well-defined) k-module*

$$\begin{aligned} \mu^\phi_{(x,\xi)}(F) &:= \varinjlim_{B \ni x} \mathrm{Cone}(\mathrm{res}^B_{B_{\phi<0}}) \\ &\simeq \mathrm{Cone}(F|_x \xrightarrow{(\mathrm{res}^B_{B_{\phi<0}})|_x} j_*j^*F|_x) \end{aligned}$$

where in the second line, a particular $B \overset{j}{\hookrightarrow} M$ around x and a particular $\phi : B \to \mathbb{R}$ was chosen, before taking the stalk at x; this is because taking stalks is an operation that preserves finite limits like cones.

So it is a "directional derivative" of sorts: it is a k-module that tracks the extent to which the map $\mathrm{res}^B_{B_{\phi<0}}$, in the limit of shrinking B, fails to be a quasi-isomorphism. That is, it measures the extent to which sections over $B_{\phi<0}$ fail to propagate uniquely past the frontier $\{\phi = 0\}$ to the rest of B, in the limit of shrinking B.

However, as declared in the notation, this construction depends not just on the point (x, ξ), but on a germ ϕ. There is a wiser definition that will be presented later that is, up to a shift that is interesting yet irrelevant for our purposes, truly independent of the auxiliary function ϕ. And it will be particularly calculable for constructible sheaves, to which we now turn in detail.

7.2 Constructible sheaves

A previous chapter discussed local systems on a topological space X, which we now take to be a C^1 manifold M. A constructible sheaf on M is the next simplest thing to a local system: loosely speaking, it is a patchwork of local systems.

To be precise, we first describe how M is to be patched together. This is via a *stratification.*

Definition 7.2. *A* stratification S *of* M *is a collection* $\{M_s\}_{s\in S}$ *subsets subsets of* M*, called* strata*, such that on the level of underlying sets,*

$$\bigsqcup_{s\in S} M_s = M.$$

We demand a few other properties of S*:*

1. *the strata* M_s *are locally closed subspaces of* M*, and are smooth manifolds in their own right;*
2. *the stratification is locally finite;*
3. *the* frontier condition *is satisfied: every* $\overline{M_s} \setminus M_s$ *is a disjoint (in the space* M*) union of strata in* S*. This imbues* S *with the structure of a poset, where* $s \leq t$ *if and only if* $M_s \subseteq \overline{M_t}$

We can now define the key objects:

Definition 7.3. *Given a stratification* S*, a sheaf* F *is said to be* S-constructible *if* $F|_{M_s}$ *is a local system for all* $s \in S$[1]*. Denote by*

$$\mathrm{Sh}_S(M) \subseteq \mathrm{Sh}(M)$$

the full subcategory of S*-constructible sheaves. We say that a sheaf* F *is* constructible *if it is* S*-constructible for some stratification* S*.*

There is one more condition on our stratifications that we will require, and it is a regularity condition that has useful microlocal implications (see Proposition 7.25):

[1]In the literature, one may see this as the definition of a *weakly* S-constructible sheaf, with the term "S-constructible sheaf" reserved for those weakly S-constructible sheaves that have perfect stalks. We do not make this assumption.

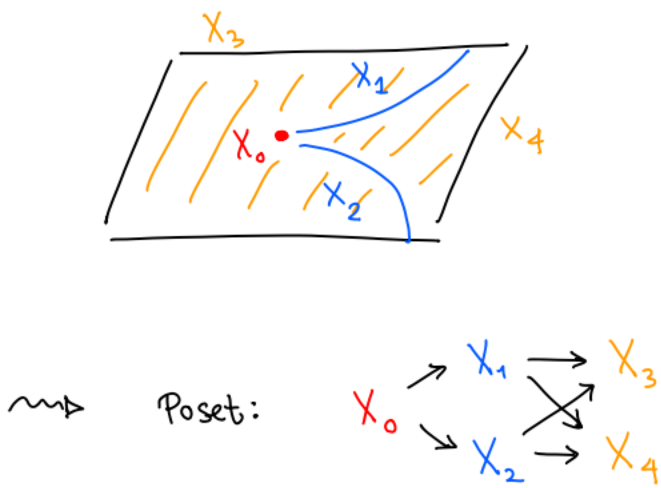


Figure 7.2: An example of a stratification of $\mathbb{R}^2$ meeting all of the above criteria, together with the associated poset.

Definition 7.4. *A stratification S of M is a* Whitney stratification *if, for any pair of strata M_s and M_t satisfying $\overline{M_s} \supseteq M_t$ and any pair of sequences $x_n \in M_s, y_n \in M_t$ that satisfy the conditions that*

1. *both sequences converge $x_n, y_n \to x$ to the same point,*

2. *the tangent planes $T_{x_n}M_s$ on the bigger stratum converge to a subspace $T \subseteq T_xM$,*

3. *the secant lines (in any Riemannian metric on M) $\overline{x_n y_n}$ converge to a line $\ell \subset T_xM$,*

the containment $\ell \subseteq T$ holds.

In other words, limiting secant lines, when they exist, between two neighboring strata must be contained in the limiting tangent planes, when they exist, for the bigger stratum. For example, all stratifications of two-dimensional spaces M are vacuously Whitney stratifications, but starting in three dimensions there are stratifications that are not Whitney (see Figure 7.3).

Let $T^*_S M$ denote the union

$$T^*_S M := \bigcup_{s \in S} T^*_{M_s} M$$

of conormals to the strata. It satisfies the following properties:

1. since $T^*_{M_s}M$ is a smooth conic Lagrangian for each M_s, $T^*_S M$ is a singular conic Lagrangian in T^*M;

2. since S satisfies the Whitney condition, $T^*_S M$ is a closed subset.

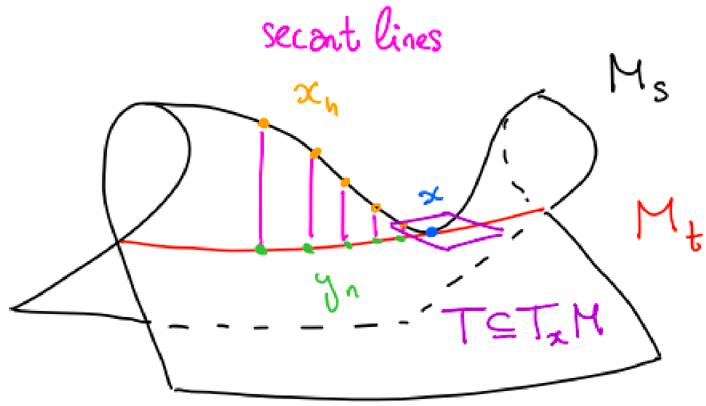


Figure 7.3: The Whitney cusp gives an example of a stratification of $\mathbb{R}^3$ by a line (red), a surface (black), and the complement, which is not a Whitney stratification: the depicted sequences of x_n, y_n have vertical secant lines (fuchsia), but the tangent planes $T_{x_n}M_s$ converge to a horizontal tangent plane (purple). This beautiful image appears in the note [23].

7.3 Singular support

Let F be a sheaf on M, and $\phi : M \to \mathbb{R}$ a C^1 function. Choose a $t \in \mathbb{R}$, and define the following closed submanifolds (with boundary) of M:

1. the level set $M_t := \{m : \phi(m) = t\} \xhookrightarrow{i_{M_t}} M$; and

2. the suplevel set $M_{\geq t} := \{m : \phi(m) \geq t\} \xhookrightarrow{i_{M_{\leq t}}} M$.

Now, pick a point $x \in M_t$.

Definition 7.5. *We say that x is a* cohomologically F-critical point *of ϕ if*

$$(i^!_{M_{\geq t}} F)_x \not\simeq 0.$$

In other words, by first situating F into the open-closed decomposition triangle

$$i_{M_{\geq t}!} i^!_{M_{\geq t}} F \to F \to j_{M_{<t}*} j^*_{M_{<t}} F \xrightarrow{+1}$$

and then taking the stalk at x

$$\underbrace{(i_{M_{\geq t}!} i^!_{M_{\geq t}} F)_x}_{(i^!_{M_{\geq t}} F)_x} \to F_x \to (j_{M_{<t}*} j^*_{M_{<t}} F)_x \xrightarrow{+1}$$

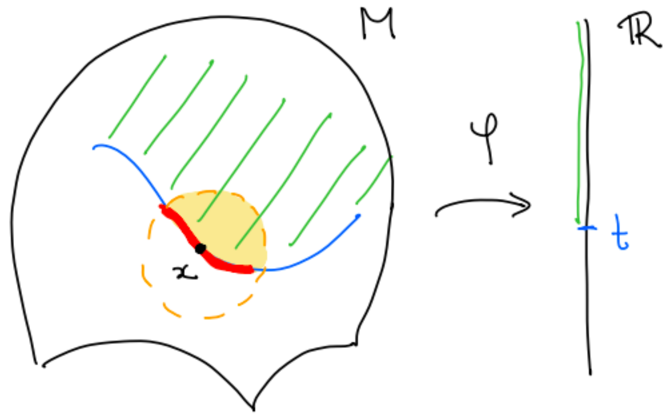


Figure 7.4: Calculation of $(i^!_{M\geq t}F)_x$ as the derived sections of F in a small half-open half-ball $B_\epsilon(x) \cap M_{\geq t}$ around x (shaded orange), relative to its frontier $B_\epsilon(x) \cap M_t$ (red).

we see that the k-module $(i^!_{M_{\geq t}}F)_x$ measures the extent to which germs of sections around x fail to propagate across the hypersurface M_t, in the direction of increasing ϕ.

We can use this to define a subset of the cotangent bundle:

Definition 7.6. *The* singular support*, or* microsupport*, of F is the subset of T^*M that is defined to be the closure of the locus of the differentials of C^1 functions at their cohomologically F-critical points:*

$$SS(F) := \overline{\bigcup_{\phi \in C^1(M)} \{(\phi(t), d_{\phi(t)}\phi) \,:\, (i^!_{M_{\geq t}}F)_{\phi(t)} \not\simeq 0\}}$$

In words, it is the closure of the locus of points and codirections (x, ξ) for which there is a frontier (say, cut out by a function ϕ with $\phi(x) = 0, d_x\phi = \xi$) past which local sections of F do not propagate uniquely. It satisfies the following basic properties:

Theorem 7.7. *Some fundamental properties of singular support:*

1. *$SS(F)$ is a closed subset which is* conic*, i.e., stable under the scaling action $\mathbb{R}^{>0} \curvearrowright T^*M$;*

2. *$SS(F) \cap 0_M = \iota_M \operatorname{supp}(M)$, where $0_M \subseteq T^*M$ denotes the zero section, the image of the embedding $M \xhookrightarrow{\iota_M} T^*M$. Thus $\pi SS(F) = \operatorname{supp}(M)$, where $\pi : T^*M \to M$ is the projection;*

3. *([9], Theorem 6.5.4) $SS(F)$ is always a singular coisotropic subset, i.e. a union of smooth coisotropic submanifolds of T^*M. In particular, its smooth locus $SS(F)^{\text{smooth}}$ is a smooth coisotropic submanifold, and hence is of dimension $\geq \dim_{\mathbb{R}} M$.*

We also record some basic estimates on the size of the singular support for certain limits and colimits of sheaves:

Theorem 7.8. *The following singular support estimates hold:*

1. *([9], Proposition 5.1.3) If $F \to G \to H \xrightarrow{+1}$ is a triangle in $\mathrm{Sh}(M)$, then*

$$SS(F) \triangle SS(H) \subseteq SS(G) \subseteq SS(F) \cup SS(G)$$

2. *([9], Exercise V.7) Let I be a set and $\{F_i\}_{i \in I}$ be a family in $\mathrm{Sh}(M)$ indexed by I. Then*

$$SS(\bigoplus_{i \in I} F_i) \subseteq \overline{\bigcup_{i \in I} SS(F_i)}$$

and

$$SS(\prod_{i \in I} F_i) \subseteq \overline{\bigcup_{i \in I} SS(F_i)}$$

Example 7.9. *Here are singular supports to closed and open submanfiolds of M:*

1. *Let $Z \subseteq M$ be a C^1 closed submanifold. Then the singular support of k_Z is*

$$SS(k_Z) = T^*_Z M,$$

the conormal bundle of Z. For example, for a point $m \in M$, the singular support of the skyscraper k_m is

$$SS(k_m) = T^*_m M,$$

the contangent fiber.

2. *Let $U \subseteq M$ be an open subset with C^1 boundary. Then*

$$SS(k_U) = 0_U \cup T^{*,\geq 0}_{\partial U} M,$$

the union of the zero section and the outward conormal at ∂U.

We can use singular support conditions to define subcategories of $\mathrm{Sh}(M)$:

Definition 7.10. *Given any conic subset $C \subseteq T^*M$, define the full subcategory of* sheaves with singular support in C *to be*

$$\mathrm{Sh}_C(M) := \{F \,:\, SS(F) \subseteq C\}$$

Both of the kinds of sheaves we have encountered previously, namely local systems and constructible sheaves, are in fact cut out by microlocal conditions:

Theorem 7.11. *Let $F \in \mathrm{Sh}(M)$.*

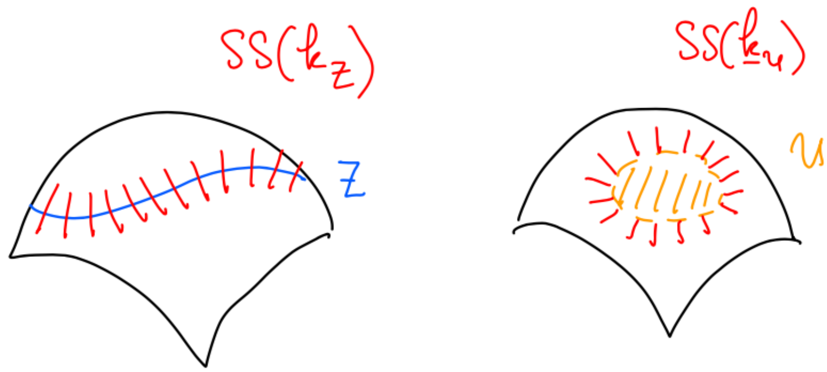


Figure 7.5: Singular supports (nonzero covectors in red) for k_Z and k_U for Z a closed C^1 submanifold and U an open subset with C^1 boundary, respectively. Images are depicted on the ambient manifold M.

1. *F is a local system iff $SS(F) = 0_M$. In other words,* $\mathrm{Loc}(X) = \mathrm{Sh}_{0_M}(M)$*;*
2. *F is Whitney constructible if and only if $SS(F)$ is a singular Lagrangian subset.*

In other words, a sheaf with a sufficiently regular conic Lagrangian singular support condition is constructible. These will be the singular support conditions we are interested in:

Definition 7.12. *A* Lagrangian skeleton $\Lambda \subseteq T^*M$ *is*

1. *a closed conic half-dimensional subanalytic subset stratified by isotropic submanifolds*
2. *which contains the zero section 0_M.*

We may also sometimes call such a Λ a conic Lagrangian*.*

As a consequence of Theorem 7.8, $\mathrm{Sh}_\Lambda(M)$ is closed under finite limits, and infinite products and coproducts. Since we require Λ to contain the zero section, Theorem 7.11 also implies that

$$\mathrm{Sh}_{0_M}(M) = \mathrm{Loc}(M) \subseteq \mathrm{Sh}_\Lambda(M).$$

Furthermore, $\mathrm{Sh}_\Lambda(M)$ is compactly generated; see the Appendix for this section and Corollary 7.31 for an argument.

7.4 Microstalks

In general, the k-modules $(i^!_{M_{\geq t}} F)_x$ depend on the choice of function ϕ, and not just on the point (x, ξ). However, if (x, ξ) is a smooth point of Λ, then if we agree to only consider those local functions ϕ for which $\{(x, \xi)\}$ represents a transverse intersection between $\Gamma_{d\phi}$ and Λ^{smooth}, then we create an essentially well-defined object. In more detail:

Definition 7.13. *Let $\Lambda \subseteq T^*M$ be a Lagrangian skeleton. Let $f : U \to \mathbb{R}$ be a C^1 function defined on some open subset $U \subseteq M$.*

1. *A point $x \in U$ is called a* Λ-critical point *of ϕ if the graph of df intersects Λ above x:*

$$\Gamma_{d\phi} \cap \Lambda \cap T^*_x M = \{(x, d_x\phi)\}$$

2. *A Λ-critical point x of ϕ is called* Morse *if $(x, d_x\phi) \in \Lambda^{\text{smooth}}$, and $\Gamma_{d\phi} \cap \Lambda$ is transverse at $(x, d_x\phi)$.*

Proposition 7.14 ([9], Proposition 7.5.3)**.** *Let $\Lambda \subseteq T^*M$ be a Lagrangian skeleton, and suppose f is a local function on M that has a Morse Λ-critical point at $(x, d_x\phi)$. Then for any $F \in \mathrm{Sh}_\Lambda(M)$, the object $(i^!_{M_{\geq t}} F)_x$ is independent of such ϕ, up to a shift.*

We use these ϕ to define the microstalk functors:

Definition 7.15. *Let $\Lambda \subseteq T^*M$ be a Lagrangian skeleton, and $(x, \xi) \in \Lambda^{\text{smooth}}$ a smooth point. For $F \in \mathrm{Sh}_\Lambda(M)$, the* microstalk functor

$$\begin{array}{rcl} \mu_{(x,\xi)} : \mathrm{Sh}_\Lambda(M) & \to & {}_k\mathrm{Mod} \\ F & \mapsto & (i^!_{M_{\geq t}} F)_x \end{array}$$

is independent, up to a shift, of the choice of ϕ for which $\Gamma_{d\phi}$ has a transverse intersection with Λ^{smooth} at (x, ξ).

The shift will not matter to us, so we will freely choose any suitable local function ϕ to calculate this functor at smooth points $(x, \xi) \in \Lambda^{\text{smooth}}$, and smugly call it *the* microstalk at (x, ξ).

Example 7.16. *Let $M = \mathbb{R}$ and consider $\Lambda = 0_\mathbb{R} \cup T^*_0\mathbb{R}$. Then*

$$\Lambda^{\text{smooth}} = \Lambda \setminus \{(0, 0)\},$$

which has four pieces. There are four co-cores, depicted in the following figure:

Example 7.17. *Let $M = \mathbb{R}$ and consider $\Lambda = 0_\mathbb{R} \cup T^{*,\geq 0}_0\mathbb{R} \cup T^{*,\geq 0}_1\mathbb{R}$. Then*

$$\Lambda^{\text{smooth}} = \Lambda \setminus \{(0, 0), (1, 0)\}$$

and the singular supports of the co-cores to the two non-zero codirections are depicted in purple; see Figure 7.7.

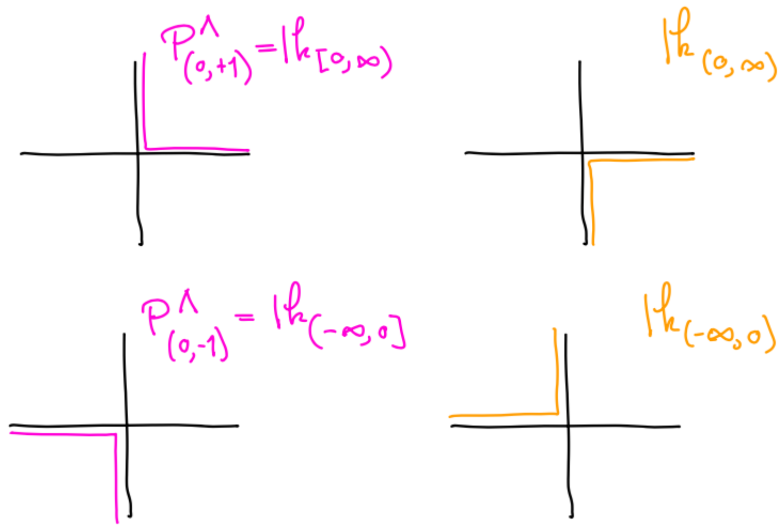


Figure 7.6: The four co-cores for $\Lambda \subseteq T^*\mathbb{R}$ the cross. The co-cores for the two zero codirections are the corepresentatives $k_{(-\infty,0)}, k_{(0,\infty)}$ of the stalks at the two points $-1, +1 \in \mathbb{R}$, respectively.

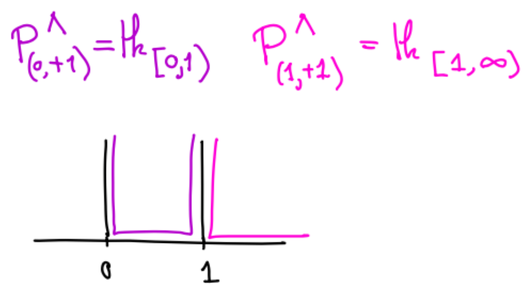


Figure 7.7: The two co-cores to the nonzero codirections in Λ of Example 7.17.

7.5 Wrapping, and the comonadicity problem

In this section, we define the fundamental functor of "wrapping," and pose the comonadicity problem.

Given $\Lambda \subseteq T^*X$ a Lagrangian skeleton, recall that we have defined the full subcategory of $\mathrm{Sh}(X)$

$$\mathrm{Sh}_\Lambda(X) := \{F \,:\, SS(F) \subseteq \Lambda\}$$

If $\Lambda' \subseteq \Lambda$ is an inclusion of Lagrangian skeleta, then the embedding

$$\iota : \mathrm{Sh}_{\Lambda'}(X) \hookrightarrow \mathrm{Sh}_{\Lambda}(X)$$

is also bicontinuous. Since $\mathrm{Sh}_{\Lambda'}(X), \mathrm{Sh}_{\Lambda}(X) \in \mathrm{Pr}^L$ are compactly generated by Corollary 7.31, the Adjoint Functor Theorem guarantees that ι has both adjoints

$$\begin{tikzcd} \mathrm{Sh}_{\Lambda'} \arrow[r, hook, "\iota"] & \mathrm{Sh}_{\Lambda} \arrow[l, bend right, "\iota^L"'] \arrow[l, bend left, dashed, "\iota^R"] \end{tikzcd} \qquad \text{in } \mathrm{Pr}^{L,\mathrm{st}}$$

where we emphasize that ι^R may not be a left adjoint, and thus draw it dashed.

Definition 7.18. *We call the left adjoint ι^L* wrapping, *or* stop-removal.

We now have the language to formulate our fundamental descent question:

Question 7.19. *Let $\{\Lambda_i\}_{i\in I}$ be a (finite) closed cover of Λ by closed conic Lagrangians (which we allow to not contain the entire zero section). Amalgamate all the wrapping functors ι_i^L into the left adjoint functor[2] L*

$$\mathrm{Sh}_{\Lambda} \xrightarrow{L := \prod_{i\in I} \iota_i^L} \prod_{i\in I} \mathrm{Sh}_{\Lambda_i}$$

Which assumptions on $\{\Lambda_i\}_{i\in I}$ would ensure that L is comonadic?

We pursue this question in the final chapter. For the remainder of this chapter, let us simply try to get a sense of what the wrapping functor ι^L does by looking at some examples.

Example 7.20. *Let $M = S^1$ and $\Lambda = 0_{S^1} \cup T_0^* S^1$ for the basepoint $\mathrm{pt} \xrightarrow{0} S^1$, and $\Lambda' = 0_{S^1}$ is just the zero section. Then the wrapping $\iota^L(k_0)$ of the skyscraper k_0 is:*

$$\iota^L(k_0) \simeq \exp_! k_{\mathbb{R}}[1] \simeq \mathscr{L}^{\mathrm{univ}}[1] := 0_!^h k[1]$$

i.e. is (up to a shift) the universal local system on S^1, which we encountered in an earlier chapter.

Remark 7.21. *Wrapping, thought of as an endofunctor on $\mathrm{Sh}_{\Lambda}(M)$ via the embedding*

$$\mathrm{Sh}_{\Lambda}(M) \xrightarrow{\iota^L} \mathrm{Sh}_{\Lambda'}(M) \xhookrightarrow{\iota} \mathrm{Sh}_{\Lambda}(M),$$

is a "non-local," or "global," operation in the sense that to know $\iota^L(F)|_U$, i.e. the outcome of $\iota^L(F)$ over an open subset $U \subseteq M$, it is generally not enough to simply know $F|_U$. For

[2]This is a functor in Pr^L by Corollary 7.31 and the fact that Pr^L is closed under limits.

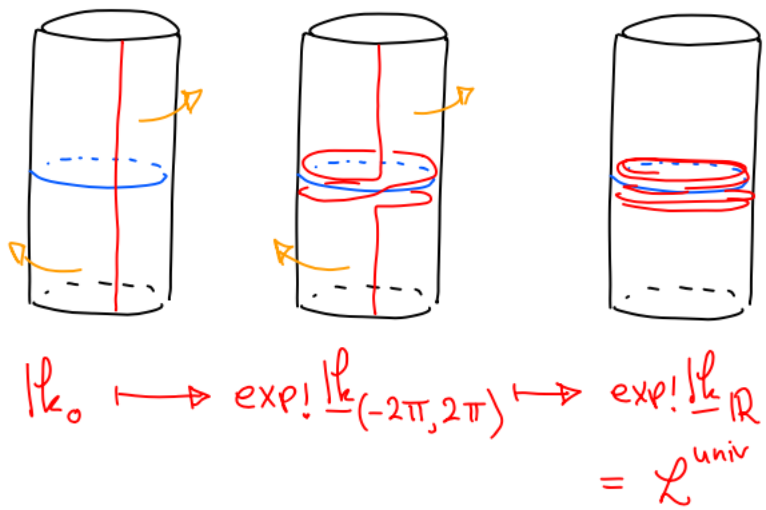


Figure 7.8: Wrapping of the skyscraper k_0 into 0_{S^1}, with singular supports of the sheaves drawn in red. It is not, so to speak, a "finite" procedure: one can think of it as the colimit of the "partial wrappings"

$$\left[k_0 \to \exp_! k_{(-1,1)}[1] \to \exp_! k_{(-2,2)}[1] \to \cdots\right] \to \exp_! k_{\mathbb{R}}[1]$$

example, if Λ is the cross $T^*_S\mathbb{R}$ for the stratification $S = \{0, \mathbb{R}^{>0}, \mathbb{R}^{<0}\}$ and Λ' is the zero section $0_{\mathbb{R}}$, then wrapping the skyscraper at $0 \in \mathbb{R}$ results in

$$\iota^L(k_0) = k_{\mathbb{R}}[1]$$

which, over $U := \mathbb{R}^{>0}$, certainly could not have been constructed from knowledge of the sheaf $k_0|_U \simeq 0$.

7.6 Aside: a first comonadicity result

This section is non-essential, and in retrospect quite silly-looking, but represents one of our first lines of thinking about the kinds of two-piece covers $\Lambda = \Lambda_1 \cup \Lambda_2$ of Lagrangian skeleta lead to comonadic descriptions of $\mathrm{Sh}_\Lambda(X)$ using adjunctions with $\mathrm{Sh}_{\Lambda_i}(X)$. It is an outgrowth of the following simple question.

Inside $T^*\mathbb{R}$, consider the cover $\Lambda = Z \cup C$ given by the zero section and the cotangent fiber at $0 \in \mathbb{R}$:

$$Z := T^*_{\mathbb{R}}\mathbb{R}, \quad C := T^*_0\mathbb{R}$$

Is this cover comonadic?

It turns out to be so. Put

$$\begin{tikzcd} \{0\} \arrow[r, hook, "i"] & \mathbb{R} \arrow[d, "p"] \\ & \mathrm{pt} \end{tikzcd}$$

By looking at the generators of the constructible category $\mathrm{Sh}_{Z\cup C}(\mathbb{R})$ and examining the effects of various functors on the generators, the localization functors L_Z, L_C can be identified with

$$L_Z \simeq p^! p_! \simeq p^*[1]p_!, \quad L_C \simeq i_* i^*$$

as part of the following diagram:

$$\begin{tikzcd} W_0 & \mathrm{Sh}_C(\mathbb{R}) \arrow[r, hook] & \mathrm{Sh}_{Z\cup C}(\mathbb{R}) \arrow[l, bend right, "L_C"'] \arrow[r, bend left, "L_Z"] \arrow[dl, bend right, "i^*"'] \arrow[dr, bend left, "p_!"] & \mathrm{Sh}_Z(\mathbb{R}) \arrow[l, hook'] & V[1] \\ W \arrow[u, mapsto] & \mathrm{Sh}(\mathrm{pt}) \arrow[u, "\simeq"] \arrow[ur, hook, "i_*"'] & & \mathrm{Sh}(\mathrm{pt}) \arrow[u, "\simeq"'] \arrow[ul, hook', "p^!"] & V \arrow[u, mapsto] \end{tikzcd}$$

A specific identification is the data of a trivialization of the $\mathbb{Z}/2$-torsor $\mathrm{or}_{\mathbb{R}}$, because it affords the counit isomorphism of the adjunction $p_! p^! \cong \mathrm{Id}$.

Here is the simple result:

Lemma 7.22. *$L_Z \boxplus L_C$ is comonadic.*

Proof. The identifications $L_Z \simeq p^! p_!$ and $L_C \simeq i_* i^*$, together with the identifications $p_! \simeq i^!$ and $p^! \simeq p^*[1]$ in this case, show that L_Z, L_C preserve all limits and colimits. For (co)monadicity, it therefore remains to check conservativity. Supposing that $i^*F \simeq 0$, then $F \simeq j_! j^! F$ where $j : \mathbb{R} \setminus \{0\} \hookrightarrow \mathbb{R}$ is the embedding of the open complement, in which case

$$p_! F \simeq \big(F_{-1} \oplus F_1\big)[1]$$

is the direct sum of the fibers at ± 1. Using the fact that any F is the data of the quiver representation

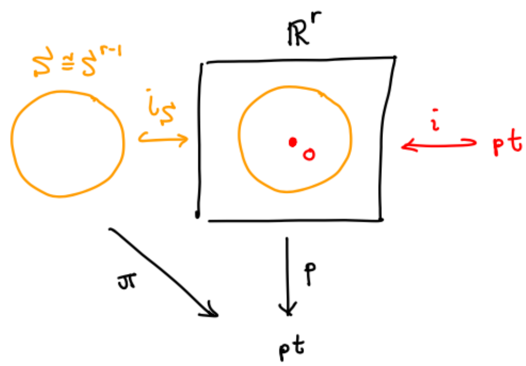
$$F_{-1} \xleftarrow{r_-} F_0 \xrightarrow{r_+} F_1,$$

the further assumption $p_! F \simeq 0$ shows that $F \simeq 0$. □

The next result is both a higher-dimensional and a family version of this example. To set it up, consider the decomposition of $\mathbb{R}^r$

$$\begin{tikzcd} \{0\} \arrow[r, hook] & \mathbb{R}^r \arrow[d, "p"] & \mathbb{R}^r \setminus \{0\} \arrow[l, hook', "j"] & S^{r-1} \arrow[l, hook'] \arrow[dll, bend left, "\pi"] \\ & \mathrm{pt} \end{tikzcd}$$

as in the picture:



Put $\Lambda = Z \cup C = 0_{\mathbb{R}^r} \cup T_0^*\mathbb{R}^r$, and define $\mathrm{Sh}_\Lambda(\mathbb{R}^r)^{\text{ind-uni}}$ to be the stable presentable category appearing as the fiber product

$$\begin{tikzcd}
\mathrm{Sh}_\Lambda(\mathbb{R}^r)^{\text{ind-uni}} \arrow[rr, hook] \arrow[d, "j^*"] \arrow[drr, phantom, "\square"] & & \mathrm{Sh}_\Lambda(\mathbb{R}^r) \arrow[d, "j^*"] \\
{}^\perp(k_{S^{r-1}}^\perp) = \mathrm{Loc}(S^{r-1})^{\text{ind-uni}} \arrow[rr, hook] \arrow[dr, "p_*^h"'] & & \mathrm{Loc}(S^{r-1}) \arrow[dl, "p_*^h"] \\
& \mathrm{Loc}(\mathrm{pt}) &
\end{tikzcd}$$

where we recall that ${}^\perp(k_{S^{r-1}}^\perp) \subseteq \mathrm{Loc}(S^{r-1})$ is the subcategory consisting of those local systems on S^{r-1} that have non-trivial cohomology, i.e. the subcategory on which p_* is conservative. So $\mathrm{Sh}_\Lambda(\mathbb{R}^r)^{\text{ind-uni}}$ is the category consisting of all Λ-constructible sheaves that either are the skyscrapers at 0 or restrict to cohomologically-nontrivial local systems on the unit sphere $S^{r-1} \subseteq \mathbb{R}^r$.

The result below is about a family version of the construction $\mathrm{Sh}_\Lambda(\mathbb{R}^r)^{\text{ind-uni}}$:

Proposition 7.23. *Let $Y \subset X$ be a regularly embedded codimension r closed submanifold such that $X \simeq N_{Y/X}$. Put*

$$\Lambda = 0_X \cup T_Y^*X := Z \cup C \subseteq T^*X,$$

and let $\mathrm{Sh}_\Lambda(X)^{\text{ind-uni}}$ denote the stable presentable full subcategory of $\mathrm{Sh}_\Lambda(X)$ consisting of those objects that restrict to objects of $\mathrm{Sh}_{\Lambda \cap T^*(N_{Y/X}|y)}(\mathbb{R}^r)^{\text{ind-uni}}$ on each fiber of the normal bundle $N_{Y/X} \to Y$; that is, sheaves that restrict to ind-unipotent local systems on the links of Y inside X (see Figure 7.9). Then the functor

$$\mathrm{Sh}_\Lambda(X)^{\text{ind-uni}} \xrightarrow{L_Z \boxplus L_C} \mathrm{Sh}_Z(X) \boxplus \mathrm{Sh}_C(X)$$

is comonadic.

Proof. The proof mirrors the argument in Lemma 7.22. It suffices to show the fiberwise statement, that the functor

$$\mathrm{Sh}_\Lambda(\mathbb{R}^r)^{\text{ind-uni}} \xrightarrow{L_Z \boxplus L_C} \mathrm{Loc}(X) \boxplus \mathrm{Sh}_C(X)$$

is comonadic. First, by the identifications of L_Z, L_C above, they are biadjointable. Therefore, $L := L_Z \boxplus L_C$ preserves all limits and colimits, can be identified with $L \simeq p_! \boxplus i^*$, and has a right adjoint $R := p^! \oplus i_*$ which does indeed land in the requisite category.

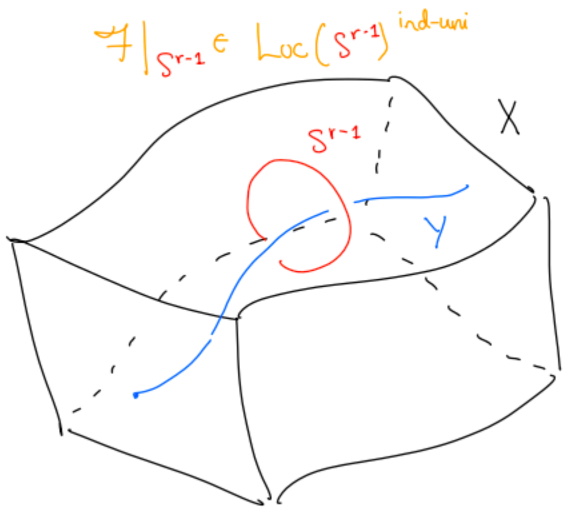


Figure 7.9: The condition defining an object of $\mathrm{Sh}_\Lambda(X)^{\text{ind-uni}}$ for $\Lambda = 0_X \cup T^*_Y X$.

It remains to verify that the domain of L was properly rigged to make L conservative. Any object $\mathscr{F} \in \mathrm{Sh}_\Lambda(\mathbb{R}^r)^{\text{ind-uni}}$ fits into a fiber sequence

$$j_! j^! \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \xrightarrow{+1},$$

whose image under $p_!$ is the fiber sequence

$$\underbrace{p_! j_! j^! \mathscr{F}}_{\pi_* i_S^*[-1]\mathscr{F}} \longrightarrow p_! \mathscr{F} \longrightarrow \underbrace{p_! i_* i^* \mathscr{F}}_{i^* \mathscr{F}}$$

Assuming that $L_Z \mathscr{F} = L_C \mathscr{F} = 0$, the identifications mean that the last two terms of the above sequence vanish, meaning that the first one does as well. But this exactly means that $i_S^* \mathscr{F}$ is a π_*-trivial local system, which by definition are excluded from the category $\mathrm{Sh}_\Lambda(\mathbb{R}^r)^{\text{ind-uni}}$. Thus $\mathscr{F}$ must be stitched together from a skyscraper at 0 and a zero local system in the complement of 0, meaning that $\mathscr{F} \simeq 0$. $\square$

Example 7.24. *Taking $r = 2$, we see that $\mathrm{Sh}_\Lambda(\mathbb{R}^2)$ cannot be comonadically described using the functors $p_!$ and i^* because e.g. the object $j_! k^{\mu=-1}_{\mathbb{R}^2\setminus\{0\}}$, which is the shriek extension of the rank 1 local system on $S^1 \simeq \mathbb{R}^2 \setminus \{0\}$ with mondromy -1, vanishes under the total wrapping functor $p_! \boxplus i^*$. In general, the two functors $p_!, i^*$ can recover*

1. *the global sections of a constructible sheaf $\mathscr{F}$, and*
2. *the global sections of the local system $j^*\mathscr{F}$, which are only the monodromy invariants of any fiber $(j^*\mathscr{F})|_{\neq 0}$.*

For a general local system, it is impossible to reconstruct the whole fiber and the monodromy from just the monodromy invariants. But, especially after accounting for the comonad structure, it is hopefully more believable that this is possible for the less variable ind-unipotent local systems.

7.7 Appendix: compact generation of $\mathrm{Sh}_\Lambda(M)$

In this section we record important formal properties of the categories $\mathrm{Sh}_\Lambda(M)$, culminating with the result that it is compactly generated for Lagrangian skeleta Λ. For an even better exposition of the basics, see Chapter 3 of the PhD thesis of Christopher Kuo [11].

We first record some fundamental results about the interaction between Lagrangian skeleta and C^1 Whitney stratifications:

Proposition 7.25. *For a C^1 Whitney stratification S of a C^1 manifold M, the following categories coincide:*

$$\mathrm{Sh}_S(M) = \mathrm{Sh}_{T^*_S M}(M)$$

Lemma 7.26. *Let Λ be a Lagrangian skeleton. Then there exists a C^1 Whitney stratification S such that $\Lambda \subseteq T^*_S M$. In fact, the S can be chosen to be a Whitney* triangulation.

In other words, S-constructible sheaves for a Whitney stratification are cut out by a microlocal condition, and any sheaf F for which $SS(F)$ lies within a Lagrangian skeleton is in fact S-constructible for some Whitney stratification.

Further regularizing to Whitney triangulations affords these categories a particularly nice combinatorial presentation:

Proposition 7.27. *Let S be a C^1 Whitney triangulation. Then there is an equivalence*

$$\mathrm{Sh}_{T^*_S M}(M) \xrightarrow{\simeq} {}_{S^{\mathrm{op}}}\mathrm{Mod}$$

$$k_{M_s} \longmapsto k_s$$

where we think of S as a poset as we did in the definition of stratification, and where k_s is the indicator defined by

$$k_s(t) := \begin{cases} k & t \leq s, \\ 0 & \text{otherwise} \end{cases}$$

Since ${}_S\operatorname{Mod} \simeq \operatorname{Ind}\operatorname{Perf}(S)$ is compactly generated, this immediately implies

Corollary 7.28. *The presentable category $\operatorname{Sh}_{T^*_S M}(M)$ is in fact compactly generated. Its compact objects $\operatorname{Sh}_{T^*_S M}(M)^c$ are those sheaves that have compact support and perfect stalks.*

Compact generation implies the following important result:

Lemma 7.29. *For a C^1 Whitney triangulation S of M, the following hold:*

1. *for any covector $\xi \in \operatorname{Sh}_S(M)$, the microlocal stalk functor*

$$\operatorname{Sh}_S(M) \xrightarrow{\mu_\xi} {}_k\operatorname{Mod}$$

admits a left adjoint μ_ξ^L and a right adjoint μ_ξ^R;

2. *the object*

$$P_\xi^S := \mu_\xi^L(k)$$

is compact, and corepresents μ_ξ;

3. *the objects P_ξ^S for ξ in the smooth locus*

$$T^*_S M^{\text{smooth}} \subseteq T^* M$$

generate $\operatorname{Sh}_S(M)$.

Proof. For (1), both the source and target of μ_ξ are compactly generated, so by the Adjoint Functor Theorem 1.7 it would suffice to show that μ_ξ preserves all limits and colimits. This holds because μ_ξ can be calculated as the fiber of two stalk functors, each of which is bicontinuous because S is a triangulation. This produces the adjunction

$$\begin{tikzcd} \operatorname{Sh}_{T^*_S M}(M) \arrow[r, "\mu_\xi"] & {}_k\operatorname{Mod} \arrow[l, bend right, "\mu_\xi^L"'] \arrow[l, bend left, dashed, "\mu_\xi^R"] \end{tikzcd}$$

in Pr^L. Part (2) follows from

$$\hom_{\operatorname{Sh}_{T^*_S M}(M)}(\mu_\xi^L(k), F) \simeq \hom_{{}_k\operatorname{Mod}}(k, \mu_\xi F) = \mu_\xi F,$$

which shows that $\mu_\xi^L(k)$ is the corepresentative. It is compact because μ_ξ preserves colimits. Part (3) holds because, by definition,

$$\left\{P_\xi^S \,:\, \xi \in T_S^* M^{\text{smooth}}\right\}^\perp = \mathrm{Sh}_I(M)$$

for $I \subseteq T^*M$ the isotropic subset, of dimension less than $\dim_\mathbb{R} M$, of vectors in the non-smooth locus of T_S^*M. But by Theorem 7.7 there are no sheaves with properly isotropic singular support, so $\mathrm{Sh}_I(M) \simeq 0$. □

Having established compact generation for the categories $\mathrm{Sh}_S(M)$ for S a C^1 Whitney triangulation, we proceed to establish it for certain categories $\mathrm{Sh}_\Lambda(M)$ with finer singular support conditions Λ by embedding them into categories of the form $\mathrm{Sh}_S(M)$ and using the following lemma:

Lemma 7.30. *Let $\mathscr{C}'$ be a reflective subcategory of $\mathscr{C}$ in Pr^L:*

$$\begin{tikzcd} \mathscr{C}' \arrow[r, hook, "\iota"] & \mathscr{C} \arrow[l, bend right, "\iota^L"'] \arrow[l, bend left, dashed, "\iota^R"] \end{tikzcd} \qquad \text{in } \mathrm{Pr}^L$$

If $\mathscr{C}$ is compactly generated, then so is $\mathscr{C}'$.

Proof. Consider any object $F \in \mathscr{C}'$. Its image ιF can be written as a filtered colimit of compact objects in $\mathscr{C}$

$$\iota F \simeq \varinjlim F_i \quad \text{for } F_i \in \mathscr{C}^c.$$

Since ι^L is a colimit-preserving localization, applying it to the above isomorphism gives the following formula for F:

$$\begin{aligned} F \simeq \iota^L \iota F &\simeq \iota^L \varinjlim F_i \\ &\simeq \varinjlim \iota^L F_i. \end{aligned}$$

The objects $\iota^L F_i$ are compact in $\mathscr{C}'$ because ι^L has a colimit-preserving right adjoint ι. □

We apply this strategy to arrive at the concluding result of this section:

Corollary 7.31. *Let Λ be a Lagrangian skeleton. Then the category $\mathrm{Sh}_\Lambda(M)$ is compactly generated.*

Proof. We offer a proof that at least looks morally correct. Another proof can be found in [17], Section A.1.4.

By Lemma 7.26, there exists a C^1 Whitney triangulation S for which $\Lambda \subseteq T_S^*M$, which gives the embedding

$$\mathrm{Sh}_\Lambda(M) \overset{\iota}{\hookrightarrow} \mathrm{Sh}_{T_S^*M}(M) \simeq {}_S\mathrm{Mod}$$

into a category that is compactly generated by Proposition 7.27. Since Λ is closed and conic, ι furthermore preserves all finite limits and colimits, as well as all products and coproducts, by Theorem 7.8. If we could guarantee the existence of a left adjoint ι^L, and thus of a reflective localization in Pr^L

$$\begin{tikzcd} \mathrm{Sh}_\Lambda(M) \arrow[r, hook, "\iota"] & \mathrm{Sh}_{T^*_S M}(M) \simeq {}_S\,\mathrm{Mod} \arrow[l, bend right, "\iota^L"'] \arrow[l, bend left, dashed, "\iota^R"] \end{tikzcd}$$

then we would conclude by Lemma 7.30 that $\mathrm{Sh}_\Lambda(M)$ is also compactly generated. By the Adjoint Functor Theorem 1.7, it suffices to show that $\mathrm{Sh}_\Lambda(M)$ is presentable and that the colimit-preserving functor ι is accessible.

Since $T^*_S M$ is a Lagrangian skeleton, by Proposition 7.14 the microstalk functors μ_ξ, for $\xi \in (T^*_S M)^{\mathrm{smooth}}$, are well-defined on the category $\mathrm{Sh}_{T^*_S M}$. Since each of them is calculated as a finite limit of stalk functors, which are bicontinuous on the constructible category $\mathrm{Sh}_S(M)$, the microstalk functors

$$\mathrm{Sh}_{T^*_S M}(M) \xrightarrow{\mu_\xi} {}_k\,\mathrm{Mod}$$

are therefore themselves bicontinous functors of presentable categories. Since Pr^L is closed under products, the funcor

$$\mathrm{Sh}_{T^*_S M}(M) \xrightarrow{\Phi := \prod_{\xi \in (T^*_S M)^{\mathrm{smooth}} \setminus \Lambda} \mu_\xi} \prod_{\xi \in (T^*_S M)^{\mathrm{smooth}} \setminus \Lambda} {}_k\,\mathrm{Mod}$$

is also a left adjoint functor, and therefore $\mathrm{Sh}_\Lambda(M)$, cut out by the conditions of having zero microstalk in codirections $\xi \in (T^*_S M)^{\mathrm{smooth}} \setminus \Lambda$,[3] appears as the fiber product of the following cospan in Pr^L:

$$\begin{tikzcd} \mathrm{Sh}_\Lambda(M) \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Sh}_{T^*_S M}(M) \arrow[d, "\Phi"] \\ 0 \arrow[r] & \prod_{\xi \in (T^*_S M)^{\mathrm{smooth}} \setminus \Lambda} {}_k\,\mathrm{Mod} \end{tikzcd}$$

The category $\mathrm{Sh}_\Lambda(M)$ is thus presentable by [14], Proposition 5.5.3.12, and ι is a bicontinuous functor. $\square$

This leads us to one final important result:

[3] This is the part of the proof where we are uncertain!

Corollary 7.32. *For $\Lambda' \subseteq \Lambda$ an inclusion of Lagrangian skeleta, the kernel $\mathscr{D}$ of the localization* $\mathrm{Sh}_\Lambda(M) \xrightarrow{L} \mathrm{Sh}_{\Lambda'}(M)$,

$$\begin{tikzcd} \mathscr{D} \arrow[r, hook, "\tilde{\iota}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Sh}_\Lambda(M) \arrow[d, "L"] \\ \mathrm{pt} \arrow[r, "0"] & \mathrm{Sh}_{\Lambda'}(M) \end{tikzcd}$$

is compactly generated, and its embedding $\tilde{\iota}$ admits a right adjoint.

Proof. Being a limit in $\mathrm{Pr}^{L,\mathrm{st}}$, certainly it is stable and presentable. By Corollary 7.31 and Lemma 7.29, $\mathscr{D}$ is generated by the (small) set of co-cores P_ξ^Λ for $\xi \in \Lambda^{\mathrm{smooth}} \setminus \Lambda'$. These objects inherit the property of being compact in $\mathscr{D}$ from $\mathrm{Sh}_\Lambda(M)$. $\square$

Chapter 8

Preparation: Results on Recollement Squares

8.1 What is in this chapter?

This interlude develops basic tools that are useful for proving comonadicity results, by exploiting some kind of adjointability. It is also a historical record: it represents our first attempts at finding ways of arguing for comonadicity. The material here is likely very well known to experts, but since we did not find a reference, we attempt a detailed presentation.

The first punchline is the key Lemma 8.6, which was already used to prove Zariski comonadicity.

The second punchline is a characterization of pullback squares for localizations in terms of orthogonality, which can be easier to check:

Proposition 8.1. *Let $\{\mathscr{C} \xrightarrow{L_i} \mathscr{C}/\mathscr{D}_i\}_{i=1,2}$ be two reflective localizations in $\mathrm{Pr}^{L,\mathrm{st}}$, using which we can draw the commuting square*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "L_2"] \arrow[d, "L_1"] & \mathscr{C}/\mathscr{D}_2 \arrow[d] \\ \mathscr{C}/\mathscr{D}_1 \arrow[r] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle \end{tikzcd}$$

The following are equivalent:

1. *this is a pullback square;*
2. *$\mathscr{D}_1 \perp \mathscr{D}_2$ are orthogonal;*
3. *the functors L_1, L_2 are jointly conservative, and the square is right-biadjointable (see Definition 8.8).*

The third and final punchline is the use of (right-bi)adjointability in proving comonadicity:

Proposition 8.2. *In the same set-up as above, if the square is a pullback square, then*

$$\mathscr{C} \xrightarrow{L_1 \boxplus L_2} \mathscr{C}/\mathscr{D}_1 \boxplus \mathscr{C}/\mathscr{D}_2$$

is comonadic.

This result is superseded by Corollary 9.2, but as mentioned, pullback squares were the first tool that we learned for proving comonadicity results, and we think they illustrate a useful method of argument.

We now begin at the beginning, with the notion of recollement. We should admit outright that our use of the word "recollement" will be a bit of a misnomer, as the traditional recollement set-up requires more properties than we ask for; in fact, what we call "recollement" is what is often called a *semiorthogonal decomposition*. Nonetheless, *we locally appropriate the name because we like it.*

8.2 Basics of recollement

Everything will occur in $\mathrm{Pr}^{L,\mathrm{st}}$. Consider a reflective localization ι^L:

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\iota^L"] & \mathscr{C}' \arrow[l, hook, bend left, "\iota"] \end{tikzcd}$$

We may form the pullback in Pr^L

$$\begin{tikzcd} \mathscr{D} := \mathrm{Ker}(\iota^L) \arrow[r, hook, "\tilde{\iota}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{C} \arrow[d, "\iota^L"] \\ 0 \arrow[r] & \mathscr{C}' \end{tikzcd}$$

which means the following are true:

1. the category $\mathscr{D}$ is presentable, and is the full subcategory on objects $c \in \mathscr{C}$ such that $\iota^L c \simeq 0$, hence the notation of $\mathrm{Ker}(\iota^L)$;

2. the functor $\tilde{\iota}$ admits a right adjoint $\tilde{\iota}^R$ which a priori may not be colimit-preserving, giving the diagram

$$\begin{tikzcd} \mathscr{D} \arrow[r, hook, "\tilde{\iota}"] & \mathscr{C} \arrow[l, dashed, bend left, "\tilde{\iota}^R"] \arrow[r, "\iota^L"] & \mathscr{C}' \arrow[l, hook, bend left, "\iota"] \end{tikzcd}$$

3. the functor ι can be realized as the embedding of the right-orthogonal subcategory $\mathscr{D}^\perp \simeq \mathscr{C}'$; we may therefore use the quotient notation for

$$\mathscr{C}' \simeq \mathscr{C}/\mathscr{D}$$

4. Any object $F \in \mathscr{C}$ fits into a distinguished triangle that we call the *recollement triangle*:

$$\tilde{\iota} \circ \tilde{\iota}^R F \to F \to \iota \circ \iota^L F \xrightarrow{+1}$$

5. The endofunctors of $\mathscr{C}$ called $Q := \iota \circ \iota^L$ and $P := \tilde{\iota} \circ \tilde{\iota}^R$ are projections

$$Q^2 \simeq Q, \qquad P^2 \simeq P,$$

and furthermore $PQ \simeq QP \simeq 0$.

Example 8.3. *The first main example of a reflective localization is in algebraic geometry, arising from the inclusion of an open subscheme $U \overset{j}{\hookrightarrow} X$ into a qcqs scheme X:*

$$\begin{tikzcd} \mathrm{QCoh}(X)^{I-\mathrm{torsion}} \arrow[r, hook, "\hat{i}_!"] & \mathrm{QCoh}(X) \arrow[r, "j^*"] \arrow[l, bend left, "\hat{i}^!"] & \mathrm{QCoh}(U) \arrow[l, hook, bend left, "j_*"] \end{tikzcd}$$

This is in the notation of Corollary 6.7. As a consequence of Thomason's compact generation theorem [21], as generalized to qcqs schemes by Bondal-Van den Bergh [1] and to the derived setting by Toën [22], the kernel subcategory is compactly generated, and Γ_I exists in $\mathrm{Pr}^{L,\mathrm{st}}$.

Example 8.4. *The second main example is in symplectic topology, arising from an inclusion of closed conic Lagrangians $\Lambda' \hookrightarrow \Lambda \subseteq T^*M$:*

$$\begin{tikzcd} \langle P_\xi^\Lambda \,:\, \xi \in \Lambda^{\mathrm{smooth}} \setminus \Lambda' \rangle_{\mathrm{colimits}} \arrow[r, hook] & \mathrm{Sh}_\Lambda(M) \arrow[r, "\iota^L"] \arrow[l, bend left, "\tilde{\iota}^R"] & \mathrm{Sh}_{\Lambda'}(M) \arrow[l, hook, bend left] \end{tikzcd}$$

Here, the kernel $\mathscr{D}$ is the full subcategory generated under ambient colimits by the co-cores

$$P_\xi^\Lambda := \mu_\xi^L(k) \qquad \text{for } \xi \in \Lambda^{\mathrm{smooth}} \setminus \Lambda'$$

where μ_ξ^L is the left adjoint to the microstalk functor $\mathrm{Sh}_\Lambda(M) \xrightarrow{\mu_\xi} {}_k\mathrm{Mod}$. *By Corollary 7.31, as long as Λ, Λ' are subanalytic conic closed Lagrangians, $\mathscr{D}$ is compactly generated by the P_ξ^Λ and $\tilde{\iota}^R$ exists in* $\mathrm{Pr}^{L,\mathrm{st}}$.

Remark 8.5. *The upshot of the examples above is that we will always assume that the entire diagram*

$$\begin{tikzcd}
\mathscr{D} \arrow[r, hook, "\tilde{\iota}"] \arrow[d] & \mathscr{C} \arrow[l, bend left, "\tilde{\iota}^R"] \arrow[d, "\iota^L"] \\
0 \arrow[r] & \mathscr{C}' \arrow[u, bend left, "\iota"]
\end{tikzcd}$$

is in $\mathrm{Pr}^{L,\mathrm{st}}$*, i.e. that* ιi^R *is colimit-preserving. For some arguments, we might assume that all categories are compactly generated.*

To prove comonadicity results using Barr-Beck-Lurie, as we have already seen, we will be interested in showing that a reflective localization ι^L commutes past certain totalizations. The following lemma, which was already used in the guise of Method 2 of Proposition 5.12 and for the proof of Theorem 6.1 part (1), and to be used repeatedly later, is the most important part of this section, and tells us one situation in which this is possible:

Lemma 8.6. *In a recollement, if* $F^\bullet$ *is a cosimplicial diagram in* $\mathscr{C}$ *for which* $\tilde{\iota} \circ \tilde{\iota}^R F^\bullet$ *(equivalently,* $\tilde{\iota}^R F^\bullet$*) is an* absolute limit diagram*, then* $\iota^L \varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} \iota^L F^\bullet$.

Proof. First, the equivalence of the statements: certainly if $\tilde{\iota}^R F^\bullet$ is an absolute limit diagram, then so is $\tilde{\iota} \circ \tilde{\iota}^R F^\bullet$. The converse holds because in this case, $\tilde{\iota}^R \circ \tilde{\iota} \circ \tilde{\iota}^R \simeq \tilde{\iota}^R$. Now let $\phi : \iota^L \varprojlim_{\Delta} F^\bullet \to \varprojlim_{\Delta} \iota^L F^\bullet$ denote the canonical map. By comparing the distinguished triangles before and after taking the limit, we get:

$$\begin{tikzcd}
\tilde{\iota}\,\tilde{\iota}^R \varprojlim_{\Delta} F^\bullet \arrow[r] \arrow[d, equal, "(*)"] & \varprojlim_{\Delta} F^\bullet \arrow[r] \arrow[dd, equal] & \iota^L \iota \varprojlim_{\Delta} F^\bullet \arrow[r, "+1"] \arrow[dd, "\iota^L \phi"] & \tilde{\iota}\,\tilde{\iota}^R \varprojlim_{\Delta} F^\bullet[1] \\
\tilde{\iota} \varprojlim_{\Delta} \tilde{\iota}^R F^\bullet \arrow[d, equal, "(**)"] & & & \\
\varprojlim_{\Delta} \tilde{\iota}\,\tilde{\iota}^R F^\bullet \arrow[r] & \varprojlim_{\Delta} F^\bullet \arrow[r] & \varprojlim_{\Delta} \iota^L \iota\, F^\bullet \arrow[r, "+1"] & \varprojlim_{\Delta} \tilde{\iota}\,\tilde{\iota}^R F^\bullet[1]
\end{tikzcd}$$

where $(*)$ is from the fact that $\tilde{\iota}^R$ is a right adjoint, and $(**)$ is from the hypothesis that $\tilde{\iota}^R F^\bullet$ is an absolute limit diagram. Thus $\iota^L \phi$ is an equivalence, and since ι^L is conservative (it is an embedding), we deduce that ϕ is an equivalence. □

As a reminder, our favorite kinds of absolute limit diagrams are *split* cosimplicial diagrams. However, the more general language here will be useful in the future.

8.3 Recollement squares

Suppose now we have a commuting square of reflective localizations

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "b"] \arrow[d, "a"] & \mathscr{C}/\mathscr{D}_2 \arrow[d, "d"] \\
\mathscr{C}/\mathscr{D}_1 \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle
\end{tikzcd}$$

which we call a *recollement square*; the subcategory on the bottom-right is the full subcategory on all objects generated under all colimits in $\mathscr{C}$.

Remark 8.7. *This is not necessarily a pullback square. For example, take $M = \mathbb{R}$, $\Lambda_1 = 0_{\mathbb{R}}$ and $\Lambda_2 = T_0^*\mathbb{R}$. Then the square is*

$$\begin{tikzcd}
\mathrm{Sh}_{\Lambda_1 \cup \Lambda_2} \mathbb{R} \simeq \mathrm{Rep}_k \left[\bullet \leftarrow \bullet \rightarrow \bullet \right] \arrow[r] \arrow[d] & \mathrm{Sh}_{\Lambda_1} \mathbb{R} \simeq {}_k \mathrm{Mod} \arrow[d] \\
\mathrm{Sh}_{\Lambda_2} \mathbb{R} \simeq {}_k \mathrm{Mod} \arrow[r] & \mathrm{Sh}_{\Lambda_1 \cap \Lambda_2} \mathbb{R} \simeq 0
\end{tikzcd}$$

However, the category $\mathrm{Sh}_{\Lambda_1 \cup \Lambda_2}$ in this case still admits a comonadic description.

For visualization, we complete the diagram to include the four recollements, together with primed functors between them, defined by $a' := \tilde{c}^R a \tilde{b}$, $a'' := \tilde{b}^R a^R \tilde{c}$, and b', b'' similarly, giving the following picture (right adjoints and double-primed functors are omitted for clarity):

$$\begin{tikzcd}
 & \mathscr{D}_1 \arrow[r, "b'"] \arrow[d, hook, "\tilde{a}"'] & \mathscr{D}_1' \arrow[d, hook, "\tilde{d}"'] \\
\mathscr{D}_2 \arrow[r, hook, "\tilde{b}"] \arrow[d, "a'"'] & \mathscr{C} \arrow[r, "b"] \arrow[d, "a"'] & \mathscr{C}/\mathscr{D}_2 \arrow[d, "d"] \\
\mathscr{D}_2' \arrow[r, hook, "\tilde{c}"] & \mathscr{C}/\mathscr{D}_1 \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle
\end{tikzcd}$$

It follows from commutativity data for the localizations $ca \simeq db$ that there is also commutativity data for the embedding $a^R c^R \simeq b^R d^R$. Note the following negative statements:

1. a', a'' are not necessarily adjoints;
2. a' is not necessarily conservative; and
3. a'' is not necessarily an embedding.

We will be interested in situations when some or all of these actually hold.

Definition 8.8. *We say the recollement square is* right-biadjointable *if both $ab^R \simeq c^R d$ and $ba^R \simeq d^R c$; i.e. the following dashed square commutes*

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "b"] \arrow[d, "a"] & \mathscr{C}/\mathscr{D}_2 \arrow[l, dashed, bend left, "b^R"] \arrow[d, "d"] \\
\mathscr{C}/\mathscr{D}_1 \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle \arrow[l, dashed, bend left, "c^R"]
\end{tikzcd}$$

as does

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "b"] \arrow[d, "a"'] & \mathscr{C}/\mathscr{D}_2 \arrow[d, "d"'] \\
\mathscr{C}/\mathscr{D}_1 \arrow[r, "c"] \arrow[u, dashed, bend right, "a^R"'] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle \arrow[u, dashed, bend right, "d^R"']
\end{tikzcd}$$

Here are some ways we can characterize this:

Lemma 8.9. *Consider a recollement square.*

1. *It is right-biadjointable iff the top right and the bottom left squares are both right adjointable.*
2. *It is right-biadjointable iff both $\tilde{c}^R ab^R \tilde{d} \simeq 0$ and $\tilde{d}^R ba^R \tilde{c} \simeq 0$.*
3. *It is right-badjointable iff both $ab^R \tilde{d} \simeq 0$ and $ba^R \tilde{c} \simeq 0$.*
4. *It is right-biadjointable iff you mix and match the above conditions.*

Proof. Part (1). We measure the distance from half of right-biadjointability $ab^R \simeq c^R d$ by the cone of the natural map $\theta : ab^R \to c^R d$, gotten by subjecting ab^R to c-recollement, and using commutativity:

$$\underbrace{\tilde{c}\tilde{c}^R ab^R F}_{\text{measurement}} \to ab^R F \xrightarrow{\theta} \underbrace{c^R cab^R}_{c^R dbb^R = c^R d} F \to$$

Thus, since $\tilde{c}$ is an embedding, θ is an equivalence iff $\tilde{c}^R ab^R \simeq 0$.

Similarly, we can measure the distance from right-adjointability of the bottom left square by taking the cone of $\phi : a'\tilde{b}^R = \tilde{c}^R a\tilde{b}\tilde{b}^R \to \tilde{c}^R a$, which is sourced from b-recollement:

$$\tilde{c}^R a\tilde{b}\tilde{b}^R F \xrightarrow{\phi} \tilde{c}^R aF \to \underbrace{\tilde{c}^R ab^R bF}_{\text{measurement}} \to$$

Thus, since b is essentially surjective, ϕ is an equivalence iff $\tilde{c}^R ab^R \simeq 0$, which we saw holds iff θ is an equivalence. A symmetric argument now shows the characterization of the other half of right-biadjointability.

Part (2). Above we learned that right-biadjointability is equivalent to both $\tilde{c}^R ab^R \simeq 0$ and $\tilde{d}^R ba^R \simeq 0$. Certainly this implies that both $\tilde{c}^R ab^R \tilde{d} \simeq 0$ and $\tilde{d}^R ba^R \tilde{c} \simeq 0$. So now suppose that $\tilde{c}^R ab^R \tilde{d} \simeq 0$; we wish to show that $\tilde{c}^R ab^R F \simeq 0$ for any $F \in \mathscr{C}/\mathscr{D}_2$. Apply $\tilde{c}^R ab^R$ to its d-recollement, and use commutativity:

$$\underbrace{\tilde{c}^R ab^R \tilde{d}\tilde{d}^R F}_{\text{assume } \simeq 0} \to \tilde{c}^R ab^R F \to \underbrace{\tilde{c}^R ab^R d^R}_{\tilde{c}^R aa^R c^R = \tilde{c}^R c^R = 0} dF \to$$

Thus $\tilde{c}^R ab^R \simeq 0$. The other half is symmetric.

Part (3) is similar to part (2).

Part (4) is true because no single equivalent characterization depends on any other; it just uses recollement and commutativity. □

We introduce one more term, which we can cutely phrase by completing the big diagram with a 0 at the missing corner:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathscr{D}_1 \arrow[r, "b'"] \arrow[d, hook, "\tilde{a}"'] & \mathscr{D}_1' \arrow[d, hook, "\tilde{d}"'] \\
\mathscr{D}_2 \arrow[r, hook, "\tilde{b}"] \arrow[d, "a'"'] & \mathscr{C} \arrow[r, "b"] \arrow[d, "a"'] & \mathscr{C}/\mathscr{D}_2 \arrow[d, "d"] \\
\mathscr{D}_2' \arrow[r, hook, "\tilde{c}"] & \mathscr{C}/\mathscr{D}_1 \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle
\end{tikzcd}$$

The reason for doing this is that right-biadjointability of the top-left square is then equivalent to the condition that both $\tilde{b}^R \tilde{a} \simeq 0$ and $\tilde{a}^R \tilde{b} \simeq 0$. So:

Definition 8.10. *We say the recollement square is* strongly right-biadjointable *if both the bottom-right and the top-left squares are right-biadjointable.*

Example 8.11. *If $X = U_1 \cup U_2$ is a union of open subschemes, then the resulting recollement square is strongly right-biadjointable.*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & QC_{U_2 \setminus U_1}(X) \arrow[r, "b'"] \arrow[d, hook, "\tilde{a}"'] & QC_{U_2 \setminus U_1}(U_2) \arrow[d, hook, "\tilde{d}"'] \\
QC_{U_1 \setminus U_2}(X) \arrow[r, hook, "\tilde{b}"] \arrow[d, "a'"'] & QC(X) \arrow[r, "b"] \arrow[d, "a"'] & QC(U_2) \arrow[d, "d"] \\
QC_{U_1 \setminus U_2}(U_1) \arrow[r, hook, "\tilde{c}"] & QC(U_1) \arrow[r, "c"] & QC(U_1 \cap U_2)
\end{tikzcd}$$

Indeed, the nice properties of this diagram, and their relation to excision and descent for Zariski open covers, was the inspiration for this entire approach to descent on the "A-side."

Lemma 8.12. *(Semi-orthogonality)*

1. $\tilde{b}^R\tilde{a} \simeq 0$ *iff* $\hom_{\mathscr{C}}(\mathscr{D}_2, \mathscr{D}_1) \simeq 0$ *i.e. are semi-orthogonal.*
2. $\tilde{b}^R\tilde{a} \simeq 0$ *implies that* $a'' \simeq (a')^R$ *and* $b'' \simeq (b')^R$*; i.e. it gives adjunctions on kernels.*
3. $\tilde{b}^R\tilde{a} \simeq 0$ *iff* $\tilde{a} \simeq b^R b\tilde{a}$.

Thus, if the top-left square is right-biadjointable, then

1. $\mathscr{D}_1 \perp \mathscr{D}_2$*;*
2. (a', a'') *and* (b', b'') *are adjunctions;*
3. $\tilde{a}$ *lands inside* $Im(b^R)$*, and* $\tilde{b}$ *lands inside* $Im(a^R)$*.*

Proof. Part (1) is adjunction. For Part (2), consider the case of a; there are canonical maps

$$\mathrm{Id} \xrightarrow{\eta} a''a', \quad a'a'' \xrightarrow{\epsilon} \tilde{b}^R a^R a\tilde{b} \xleftarrow{\phi_a} \tilde{b}^R\tilde{b} = \mathrm{Id}$$

coming from (co)units. The morphism ϕ_a is a quasi-isomorphism iff $\tilde{b}^R\tilde{a}\tilde{a}^R\tilde{b} \simeq 0$. The given hypothesis guarantees this, and by symmetry also guarantees that the analogous map ϕ_b for the base of b', b'' also is a quasi-isomorphism. Thus, both (a', a'') and (b', b'') are adjunctions. Part (3) follows immediately from b-recollement applied to $\tilde{a}F$:

$$\tilde{b}\tilde{b}^R\tilde{a}F \to \tilde{a}F \to b^R b\tilde{a}F \to$$

□

Lemma 8.13. *Some basic facts about recollement squares:*

1. $a' := \tilde{c}^R a\tilde{b}$ *is the formula for the functor induced by* a*;*
2. $\tilde{d}^R b a^R \tilde{c} \simeq 0$ *iff the top square is right-adjointable (i.e.* $b'\tilde{a}^R \simeq \tilde{d}^R b$*), iff* $b a^R \tilde{c} \simeq 0$*;*
3. *if* $\tilde{d}^R b a^R \tilde{c} \simeq 0$ *and* $\tilde{a}^R\tilde{b} \simeq 0$*, then* a' *and* a'' *are inverse equivalences;*
4. *if* $\tilde{d}^R b a^R \tilde{c} \simeq 0$ *and* $\tilde{a}^R\tilde{b} \simeq 0$ and *if* $\tilde{c}^R a b^R \tilde{d} \simeq 0$ *and* $\tilde{b}^R\tilde{a} \simeq 0$*, then* $(a', a''), (b', b'')$ *are pairs of inverse equivalences;*
5. *if* $(a', a''), (b', b'')$ *are pairs of inverse equivalences, then* $\tilde{b}^R\tilde{a} \simeq 0$ *and* $\tilde{a}^R\tilde{b} \simeq 0$ *iff* $\tilde{d}^R b a^R \tilde{c} \simeq 0$ *and* $\tilde{c}^R a b^R \tilde{d} \simeq 0$.

Proof. We prove each part. (Some of these proofs are redundant.)

1. For any $F \in \mathscr{D}_2$, $a\tilde{b}F = \tilde{c}G$ for some $G \in \mathscr{D}'_2$, because $c(a\tilde{b}F) = d(b\tilde{b}F) \simeq 0$. By the lower recollement triangle, we have

$$\tilde{c}\tilde{c}^R a\tilde{b}F \to a\tilde{b}F \to \underbrace{c^R ca\tilde{b}F}_{\simeq 0} \to,$$

which means, since $\tilde{c}$ is an embedding, that G must be $\simeq \tilde{c}^R a\tilde{b}F$. This also means that we can write $a' = \tilde{c}^R c\tilde{c}^R a\tilde{b}$. Note however that, without further hypotheses, the natural functor $\tilde{b}^R a^R \tilde{c}$ going the other direction is not a right adjoint.

2. The counit for $(\tilde{a}, \tilde{a}^R)$ gives the comparison map $b'\tilde{a}^R := \tilde{d}^R b\tilde{a}\tilde{a}^R \to \tilde{d}^R b$, whose cone is $\tilde{d}^R ba^R a$. Since a is essentially surjective, this shows that the square is right-adjointable iff $\tilde{d}^R ba^R \simeq 0$. We're not done, since we wish to show that $\tilde{d}^R ba^R \tilde{c} \simeq 0$ is sufficient to imply this. Well, split F up by c, and apply $\tilde{d}^R ba^R$:

$$\underbrace{\tilde{d}^R ba^R \tilde{c}\tilde{c}^R F}_{=0} \to \tilde{d}^R ba^R F \to \underbrace{\tilde{d}^R ba^R c^R cF}_{\simeq \tilde{d}^R bb^R d^R cF} \to,$$

where the second brace is due to the "backwards commutativity" of the right adjoints in the square. But now, $\tilde{d}^R bb^R d^R cF \simeq \tilde{d}^R d^R cF \simeq 0$, showing that indeed $\tilde{d}^R ba^R \simeq 0$.

Lastly, suppose $\tilde{d}^R ba^R \tilde{c} \simeq 0$; then by recollement, $ba^R \tilde{c}F \simeq d^R dba^R \tilde{c}F$. By commutativity, the latter is $d^R caa^R \tilde{c}F \simeq d^R c\tilde{c}F \simeq 0$, as desired.

3. Let $G \in \mathscr{D}'_2$, so $\tilde{c}G \in \mathscr{C}/\mathscr{D}_1$. Thus $a : a^R \tilde{c}G \mapsto \tilde{c}G$. We can break up this input object inside $\mathscr{C}$ under the horizontal recollement:

$$\tilde{b}\tilde{b}^R a^R \tilde{c}G \to a^R \tilde{c}G \to b^R \underbrace{ba^R \tilde{c}G}_{\tilde{d}\tilde{d}^R ba^R \tilde{c}G} \to,$$

where the underbrace uses the fact that $caa^R \tilde{c}G \simeq c\tilde{c}G \simeq 0$, whence $d(ba^R \tilde{c}G) \simeq 0$. The left is $\tilde{b}$ applied to an object from $\mathscr{D}_2$; therefore, if $\tilde{d}^R ba^R \tilde{c} \simeq 0$, then it tells us that $a^R \tilde{c}G$ never has a $b^R b$-component in its recollement — i.e. $a^R \tilde{c}$ is a section of $\tilde{c}^R a$; furthermore, $\tilde{b}$ is essentially surjective onto the image of this section, since $a^R \tilde{c}G \simeq \tilde{b}(\tilde{b}^R a^R \tilde{c}G) =: \tilde{b}a''G$.

What this implies is that a', a'' are inverses: first,

$$a'a''G := \tilde{c}^R a \underbrace{\tilde{b}\tilde{b}^R a^R \tilde{c}G}_{a^R \tilde{c}G} \simeq \tilde{c}^R aa^R \tilde{c}G \simeq \tilde{c}^R \tilde{c}G \simeq G;$$

and second,

$$a''a'F := \tilde{b}^R a^R \underbrace{\tilde{c}\tilde{c}^R a\tilde{b}F}_{a\tilde{b}F} \simeq \tilde{b}^R a^R a\tilde{b}F,$$

which we can stick into the vertical recollement

$$\tilde{b}^R \tilde{a} \quad \underbrace{\tilde{a}^R \tilde{b}F}_{=0 \text{ by assumption}} \to \underbrace{\tilde{b}^R \tilde{b}F}_{F} \to \underbrace{\tilde{b}^R a^R a\tilde{b}F}_{a''a'F} \to$$

and therefore $a''a'F \simeq F$.

4. This is just the previous proposition twice.

5. Suppose first that $\tilde{a}^R\tilde{b}F \simeq 0$; this means that $\tilde{b}F \simeq a^R a\tilde{b}F$. We therefore have

$$\tilde{d}^R ba^R\tilde{c}a'F \simeq \tilde{d}^R ba^R\tilde{c}\tilde{c}^R a\tilde{b}F \simeq \tilde{d}^R ba^R a\tilde{b}F \simeq \tilde{d}^R b\tilde{b}F \simeq 0$$

   Next, suppose that $\tilde{d}^R ba^R\tilde{c}G \simeq 0$. Then

$$b'\tilde{a}^R\tilde{b}a''G := b'\tilde{a}^R\tilde{b}\tilde{b}^R a^R\tilde{c}G \simeq \tilde{d}^R b\,\underbrace{\tilde{b}\tilde{b}^R a^R\tilde{c}G}_{=a^R\tilde{c}G} \simeq \tilde{d}^R ba^R\tilde{c}G \simeq 0;$$

   here, the first equality is a definition, the second is due to right-adjointability (which holds by part (2)), and the third is because a'' is an equivalence.

   Now that we have this, the rest of the argument is symmetric.

□

Pullback squares and recollement

We now move towards descent-type statements that we can make in our set-up by comparing strong right-biadjointability to being a pullback square.

Lemma 8.14. *If a recollement square is right-biadjointable and both a', b' are conservative, then in fact a', b' are equivalences (equivalently, it is strongly right-biadjointable).*

Proof. By the above lemma, it suffices to show that $\tilde{b}^R\tilde{a} \simeq 0$ and $\tilde{a}^R\tilde{b} \simeq 0$. We just show the first. Since a' is conservative, it suffices to show that $a'\tilde{b}^R\tilde{a} \simeq 0$. But by Lemma 8.13.(2) above, $a'\tilde{b}^R \simeq \tilde{c}^R a$, which kills the image of $\tilde{a}$. □

Lemma 8.15. *A recollement square is strongly right-biadjointable iff it is a pullback square.*

Proof. Suppose first that $\mathscr{C} \simeq \mathscr{C}/\mathscr{D}_1 \times^h_{\mathscr{C}/\langle\mathscr{D}_1,\mathscr{D}_2\rangle} \mathscr{C}/\mathscr{D}_2$ is the pullback, via the identification $\Phi(F) := (aF, bF, id : caF \simeq dbF)$. The diagram looks like

$$\begin{tikzcd}
 & \mathrm{Ker}(\pi_1) \arrow[r, "\pi_2'"] \arrow[d, hook, "\tilde{\pi}_1"] & \mathscr{D}_1' \arrow[d, hook, "\tilde{d}"] \\
\mathrm{Ker}(\pi_2) \arrow[r, hook, "\tilde{\pi}_2"] \arrow[d, "\pi_1'"] & \mathscr{C}/\mathscr{D}_1 \times^h_{\mathscr{C}/\langle\mathscr{D}_1,\mathscr{D}_2\rangle} \mathscr{C}/\mathscr{D}_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathscr{C}/\mathscr{D}_2 \arrow[l, bend left, "\pi_2^R"] \arrow[d, "d"] \\
\mathscr{D}_2' \arrow[r, hook, "\tilde{c}"] & \mathscr{C}/\mathscr{D}_1 \arrow[r, "c"] \arrow[u, bend right, "\pi_1^R"'] & \mathscr{C}/\langle\mathscr{D}_1,\mathscr{D}_2\rangle
\end{tikzcd}$$

where the formulae are

$$\pi_1^R(F_1) := (F_1, d^R cF_1, \eta_d : cF_1 \simeq dd^R cF_1), \qquad \pi_2^R(F_2) := (c^R dF_2, F_2, \eta_c : cc^R dF_2 \simeq dF_2)$$

This shows that right-biadjointability

$$\pi_1\pi_2^R \simeq c^R d, \qquad \pi_2\pi_1^R \simeq d^R c$$

holds. Finally, $\mathrm{Ker}(\pi_2)$ consists of those objects of the form $(F_1, 0, 0 : 0 \simeq 0)$, on which π_1' is certainly conservative. So by the lemma above, the square is strongly right-biadjointable.

For the converse, suppose we have a strongly right-biadjointable recollement square with $\mathscr{C}$ at the corner. Let $\Phi : \mathscr{C} \to \mathscr{C}/\mathscr{D}_1 \times^h_{\mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2\rangle} \mathscr{C}/\mathscr{D}_2$ denote the canonical map

$$F \mapsto (aF, bF, caF \xrightarrow{\phi = id} dbF),$$

and let $\Psi : \mathscr{C}/\mathscr{D}_1 \times^h_{\mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2\rangle} \mathscr{C}/\mathscr{D}_2 \to \mathscr{C}$ be the map

$$(G, H, cG \xrightarrow{\phi} dH) \mapsto \mathrm{Cone}(a^R G \oplus b^R H \xrightarrow{\theta} a^R c^R cG)$$

These are adjoints, so it suffices to show that one of them is an equivalence. First, we check that $\Psi \circ \Phi(F) \simeq F$. We do this by using the octahedral axiom:

$$\begin{tikzcd}
a^R aF \arrow[r, "id\oplus 0"] \arrow[dr, bend right, "\eta_b|_{a^R aF}"'] & a^R aF \oplus b^R bF \arrow[r] \arrow[d, "\theta"] & b^R bF \arrow[d, dashed] \\
& \underbrace{a^R c^R caF}_{(*)\, b^R b a^R aF} \arrow[r] \arrow[dr, bend right] & \underbrace{\tilde{b}\tilde{b}^R a^R aF}_{(**)\, \tilde{b}\tilde{b}^R F}[1] \arrow[d, dashed] \\
& & \Psi \circ \Phi(F)
\end{tikzcd}$$

Here, $(*)$ is due to right-biadjointability, and $(**)$ is due to the "strong" part of strong right-biadjointability. Thus $\Psi \circ \Phi(F)$ fits into the same recollement triangle as F, whence canonically $\Psi \circ \Phi(F) \simeq F$.

Thus, Ψ is essentially surjective and full. To conclude showing that Ψ is an equivalence, we should argue that it is faithful. In our set-up of $\mathrm{Pr}^{L,\mathrm{st}}$, this will be implied by conservativity. But conservativity holds by right-biadjointability. □

Before proceeding, we use the above lemma to collect a proof of Proposition 8.1 from the beginning:

Proof. (of Proposition 8.1) Take localizations $\mathscr{C} \xrightarrow{a,b} \mathscr{D}_1, \mathscr{D}_2$ of a stable presentable category $\mathscr{C}$, and build the following commuting diagram, which is based around a recollement square:

$$\begin{tikzcd}
 & \mathscr{D}_1 \arrow[r, "b'"] \arrow[d, hook, "\tilde{a}"] & \langle \mathscr{D}_1, \mathscr{D}_2 \rangle / \mathscr{D}_2 \arrow[d, hook, "\tilde{d}"] \\
\mathscr{D}_2 \arrow[r, hook, "\tilde{b}"] \arrow[d, "a'"] & \mathscr{C} \arrow[r, "b"] \arrow[d, "a"] & \mathscr{C}/\mathscr{D}_2 \arrow[d, "d"] \\
\langle \mathscr{D}_1, \mathscr{D}_2 \rangle / \mathscr{D}_1 \arrow[r, hook, "\tilde{c}"] & C/\mathscr{D}_1 \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle
\end{tikzcd}$$

By the above Lemma 8.15, it suffices to show that $\mathscr{D}_1 \perp \mathscr{D}_2$ iff the square is strongly right-biadjointable.

If the square is strongly right-biadjointable, then certainly $\mathscr{D}_1 \perp \mathscr{D}_2$.

To argue for the converse, let us first see how far from right-biadjointable the bottom-right square is. By Lemma 8.9, this is equivalent to $ba^R\tilde{c} \simeq 0$ and $ab^R\tilde{d} \simeq 0$; let us just look at the first. We note that a^R includes $\mathscr{C}/\mathscr{D}_1$ as $\mathscr{D}_1^\perp \subseteq \mathscr{C}$, thus

$$\mathrm{Im}(a^R\tilde{c}) = \{X : \hom_{\mathscr{C}}(\mathscr{D}_1, X) = 0,\ X \in \langle \mathscr{D}_1, \mathscr{D}_2 \rangle\}$$

Note that thickness of $\langle \mathscr{D}_1, \mathscr{D}_2 \rangle$ guarantees that, since it is closed under $\tilde{b}\tilde{b}^R$ and $\tilde{a}\tilde{a}^R$, it is also closed under a^Ra, b^Rb. Furthermore, b-recollement on $a^R\tilde{c}Y$ gives

$$\underbrace{b^Rba^R\tilde{c}Y}_{\in \mathscr{D}_2^\perp} \to \underbrace{a^R\tilde{c}Y}_{\in \mathscr{D}_1^\perp} \to \underbrace{\tilde{b}\tilde{b}^Ra^R\tilde{c}Y}_{\in \mathscr{D}_2}$$

Furthermore, note that

$$\hom_{\mathscr{C}}(\tilde{a}Z, \tilde{b}\tilde{b}^Ra^R\tilde{c}Y) \simeq \hom_{\mathscr{D}_1}(Z, \underbrace{\tilde{a}^R\tilde{b}}_{!}\,\tilde{b}^Ra^R\tilde{c}Y)$$

Thus, supposing that $\mathscr{D}_1 \perp \mathscr{D}_2$, the fact that $\tilde{a}^R\tilde{b} \simeq 0$ implies that $\tilde{b}\tilde{b}^Ra^R\tilde{c}Y \in \mathscr{D}_1^\perp$, whence

$$\mathrm{Im}(b^Rba^R\tilde{c}) \subset \{X : \hom_{\mathscr{C}}(\mathscr{D}_1, X) = 0,\ \hom_{\mathscr{C}}(\mathscr{D}_2, X) = 0, X \in \langle \mathscr{D}_1, \mathscr{D}_2 \rangle\},$$

which by orthogonality is the 0 category. So $ba^R\tilde{c} \simeq 0$. This implies the equivalence of hypotheses (1) and (2).

Hypotheses (2) and (3) are equivalent by Lemma 8.14. □

Descent for a square

We are ready to return to what we were initially after, and make our first link with comonadicity. It is the "recollement implies Zariski descent for a two-piece cover" argument, in the abstract:

Lemma 8.16 (Restatement of Proposition 8.2)**.** *Consider a commuting square of reflective localizations. Suppose that it is strongly right-biadjointable. Then* $\mathscr{C} \xrightarrow{a \boxplus b} \mathscr{C}/\mathscr{D}_1 \boxplus \mathscr{C}/\mathscr{D}_2$ *is comonadic.*

Proof. Here is the diagram again:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathscr{D}_1 \arrow[r, "b'"] \arrow[d, hook, "\tilde{a}"'] & \mathscr{D}_1' \arrow[d, hook, "\tilde{d}"'] \\
\mathscr{D}_2 \arrow[r, hook, "\tilde{b}"] \arrow[d, "a'"'] & \mathscr{C} \arrow[r, "b"] \arrow[d, "a"'] & \mathscr{C}/\mathscr{D}_2 \arrow[d, "d"] \\
\mathscr{D}_2' \arrow[r, hook, "\tilde{c}"] & \mathscr{C}/\mathscr{D}_1 \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle
\end{tikzcd}$$

By Barr-Beck-Lurie, we need to show that $L := a \boxplus b$ is conservative, and preserves certain limits.

First we show conservativity, so let $F \in \mathscr{C}$ be such that $aF \simeq 0$ and $bF \simeq 0$. This assumption shows that

$$\tilde{b}\tilde{b}^R F \simeq F \simeq \tilde{a}\tilde{a}^R F,$$

and, by substituting one into the other, that

$$F \simeq \tilde{b}(\underbrace{\tilde{b}^R \tilde{a}}_{\simeq 0})\tilde{a}^R F \simeq 0$$

by orthogonality. So, L is conservative.

Now, suppose that $F^\bullet : \Delta \to \mathscr{C}$ is such that $aF^\bullet, bF^\bullet$ are split. We wish to show that $L \varprojlim_\Delta F^\bullet \simeq \varprojlim_\Delta LF^\bullet$, or in other words that both $a \varprojlim_\Delta F^\bullet \simeq \varprojlim_\Delta aF^\bullet$ and $b \varprojlim_\Delta F^\bullet \simeq \varprojlim_\Delta bF^\bullet$.

Let us show that $a \varprojlim_\Delta F^\bullet \simeq \varprojlim_\Delta aF^\bullet$, assuming $bF^\bullet$ is split; the argument for b will be symmetric. By Lemma 8.6, it suffices to show that $\tilde{a}^R F^\bullet$ is split.

$$\begin{tikzcd}
\Delta \arrow[ddr, "F^\bullet"] \arrow[ddrrr, dotted, bend left, "\text{split}"] & & & & \\
& & \mathscr{D}_1 \arrow[rr, equal, "b'"] \arrow[dl, hook', "\tilde{a}"'] & & \mathscr{D}_1' \arrow[dl, hook', "\tilde{d}"'] \\
& \mathscr{C} \arrow[ur, dashed, bend right, "\tilde{a}^R"'] \arrow[rr, "b"] \arrow[d, "a"] & & \mathscr{C}/\mathscr{D}_2 \arrow[ur, dashed, bend right, "\tilde{d}^R"'] \arrow[d, "d"] & \\
& \mathscr{C}/\mathscr{D}_1 \arrow[rr, "c"] & & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle &
\end{tikzcd}$$

Since b' is an equivalence, thus in particular an embedding, this is the same as showing that $b'\tilde{a}^R F^\bullet$ is split. Right-adjointability of the square at $\mathscr{C}/\mathscr{D}_1$ means that the upper square with dashed arrows commutes:

$$b'\tilde{a}^R F^\bullet \simeq \tilde{d}^R \underbrace{bF^\bullet}_{\text{assumed split}}.$$

So, as the image of a split diagram, $b'\tilde{a}^R F^\bullet$ is split. □

Remark 8.17. *The converse is not true. Consider the example of $M = \mathbb{R}$, $\Lambda_1 = SS(k_{(-\infty,0)})$, $\Lambda_2 = SS(k_{[0,\infty)})$. Then*

$$\mathrm{Sh}_{\Lambda_1\cup\Lambda_2}(M) \xrightarrow{L:=j_-^! \boxplus i_+^*} \mathrm{Sh}_{\Lambda_1}(M) \boxplus \mathrm{Sh}_{\Lambda_2}(M)$$

is comonadic: L is conservative, and the functors have left adjoints

$$(j_-^!)^L = j_{-!}, \quad (i_+^*)^L = \text{include as zero section}$$

and thus preserve all limits. But it does not yield a pullback square.

Another example is $M = \mathbb{R}, \Lambda_1 = SS(k_{\mathbb{R}}), \Lambda_2 = SS(k_0)$.

Interpretation for sheaves with prescribed singular support

We have reformulated the pullback property of a recollement square in terms of a concrete orthogonality condition. Though strictly stronger than comonadicity, this condition is hopefully easier to check.

Let us interpret this condition in the context of $\mathrm{Sh}_\Lambda(M)$, and a two-piece closed conic Lagrangian cover $\Lambda = \Lambda_1 \cup \Lambda_2$. All the categories below are compactly generated by Corollary 7.31, with compact generators their co-cores, and assemble into a recollement square that we decorate with the kernels:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathscr{D}_1 = \langle P^{\Lambda}_{\eta\in\Lambda^{\mathrm{smooth}}\setminus\Lambda_1}\rangle \arrow[r, "b'"] \arrow[d, hook, "\tilde{a}"] & \mathscr{D}_1' = \langle P^{\Lambda_2}_{\eta\in\Lambda_2^{\mathrm{smooth}}\setminus\Lambda_1}\rangle \arrow[d, hook, "\tilde{d}"] \\
\mathscr{D}_2 = \langle P^{\Lambda}_{\xi\in\Lambda^{\mathrm{smooth}}\setminus\Lambda_2}\rangle \arrow[r, hook, "\tilde{b}"] \arrow[d, "a'"] & \mathrm{Sh}_\Lambda(M) \arrow[r, "b"] \arrow[d, "a"] & \mathrm{Sh}_{\Lambda_2}(M) \arrow[d, "d"] \\
\mathscr{D}_2' = \langle P^{\Lambda_1}_{\xi\in\Lambda_1^{\mathrm{smooth}}\setminus\Lambda_2}\rangle \arrow[r, hook, "\tilde{c}"] & \mathrm{Sh}_{\Lambda_1}(M) \arrow[r, "c"] & \mathrm{Sh}_{\Lambda_1\cap\Lambda_2}(M)
\end{tikzcd}$$

Thus, (semi)orthogonality of $\mathscr{D}_1$ and $\mathscr{D}_2$ is governed by (semi)orthogonality of co-cores. The following simple result demonstrates a further reinterpretation, in terms of singular supports:

Lemma 8.18. *The condition $\mathscr{D}_1 \perp \mathscr{D}_2$ is equivalent to the following:*

1. *categorically,*

$$\hom_\Lambda(P^{\Lambda}_{\xi\in\Lambda^{\mathrm{smooth}}\setminus\Lambda_2}, P^{\Lambda}_{\eta\in\Lambda^{\mathrm{smooth}}\setminus\Lambda_1}) \simeq 0, \quad \hom_\Lambda(P^{\Lambda}_{\eta\in\Lambda^{\mathrm{smooth}}\setminus\Lambda_1}, P^{\Lambda}_{\xi\in\Lambda^{\mathrm{smooth}}\setminus\Lambda_2}) \simeq 0$$

2. *in terms of singular support,*

$$SS(P^{\Lambda}_{\xi\in\Lambda^{\mathrm{smooth}}\setminus\Lambda_2}) \subseteq \Lambda_1, \qquad SS(P^{\Lambda}_{\eta\in\Lambda^{\mathrm{smooth}}\setminus\Lambda_1}) \subseteq \Lambda_2$$

Proof. Since the co-core corepresents the microstalk functor, property (1) being true is equivalent to

$$(\Lambda^{\text{smooth}} \setminus \Lambda_2) \cap SS(P^{\Lambda}_{\eta \in \Lambda^{\text{smooth}} \setminus \Lambda_1}) = \emptyset$$

(together with the symmetric statement for ξ's), which is equivalent to

$$\underbrace{SS(P^{\Lambda}_{\eta \in \Lambda^{\text{smooth}} \setminus \Lambda_1})}_{\text{closed skeleton}} \subseteq \underbrace{\Lambda_2}_{\text{closed skeleton}} \cup (\underbrace{\Lambda \setminus \Lambda^{\text{smooth}}}_{\text{proper isotropic}})$$

Since $\Lambda \setminus \Lambda^{\text{smooth}}$ is a proper isotropic subvariety and both $SS(P^{\Lambda}_{\eta \in \Lambda^{\text{smooth}} \setminus \Lambda_1})$ and Λ_2 are *closed* Lagrangian skeleta, we conclude that in fact

$$SS(P^{\Lambda}_{\eta \in \Lambda^{\text{smooth}} \setminus \Lambda_1}) \subseteq \Lambda_2,$$

which is property (2). The opposite implication is immediate. $\square$

In other words, orthogonality is equivalent to the condition that each piece Λ_i contains the entire singular support of the co-core to each codirection that is unique to it.

8.4 General shelling argument

Proposition 8.2 has a generalization to multiple pieces, which already appeared in the proof of Theorem 6.1. We simply rework that proof to look more general:

Proposition 8.19 (General shelling argument)**.** *Let $\{\mathscr{D}_i\}_{i \in I}$ be a finite collection of coreflective subcategories of $\mathscr{C}$ in $\mathrm{Pr}^{L,\mathrm{st}}$ such that for every $j \in I$ and $K \subset I$, the square*

$$\begin{tikzcd}
\mathscr{C}/(\mathscr{D}_j \cap \bigcap_{k \in K} \mathscr{D}_k) \arrow[r, "L_{j \subset jK}"] \arrow[d, "L_{K \subset jK}"] & \mathscr{C}/\mathscr{D}_j \arrow[d] \\
\mathscr{C}/(\bigcap_{k \in K} \mathscr{D}_k) \arrow[r] & \mathscr{C}/\langle \mathscr{D}_j, \bigcap_{k \in K} \mathscr{D}_k \rangle
\end{tikzcd}$$

is a pullback square. Assume furthermore that $\bigcap_{i \in I} \mathscr{D}_i = 0$. Then

$$\mathscr{C} = \mathscr{C}/(\bigcap_{i \in I} \mathscr{D}_i) \xrightarrow{L := \prod_{i \in I} L_{i \subset I}} \prod_{i \in I} \mathscr{C}/\mathscr{D}_i$$

is comonadic.

Proof. Since we can identify $\mathrm{Ker}(L) = \bigcap_{i \in I} \mathscr{D}_i$, by assumption the functor L is therefore conservative. So it remains to show that each $L_{i \subset I}$ preserves the limit of an L-split diagram $\Delta \xrightarrow{F^\bullet} \mathscr{C}$.

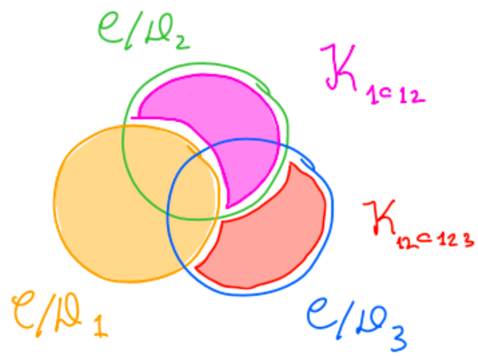


Figure 8.1: A cartoon of the strategy for the shelling argument. To show that the localization $L_{1\subset 123}$ onto $\mathscr{C}/\mathscr{D}_1$ commutes with the limit: (1) first show that $L_{12\subset 123}$ commutes with the limit by using the fact that projection onto the red category within $\mathscr{C}/(\mathscr{D}_1 \cap \mathscr{D}_2 \cap \mathscr{D}_3)$ is split; (2) then show that $L_{1\subset 12}$ commutes with this restricted limit by using the fact that the projection onto the fuchsia category within $\mathscr{C}/(\mathscr{D}_1 \cap \mathscr{D}_2)$ is split.

Consider for the sake of illustration the case of the three-piece cover; the general argument easily follows this blueprint. So let $F^\bullet : \Delta \to \mathscr{C} = \mathscr{C}/(\mathscr{D}_1 \cap \mathscr{D}_2 \cap \mathscr{D}_3)$ be a diagram such that $L_{i\subset 123}F^\bullet$ is split, for all $i \in \{1,2,3\}$. We will show that

$$L_{1\subset 123}\varprojlim_{\Delta} F^\bullet \overset{?}{\simeq} \varprojlim_{\Delta} L_{1\subset 123}F^\bullet$$

(see Figure 8.1 for an outline of the strategy, and Figure 8.2 to follow the notation).

For this, note that $L_{1\subset 123} \simeq L_{1\subset 12}L_{12\subset 123}$. We intend to use this factorization to show that $L_{1\subset 123}$ commutes past the limit first with the factor $L_{12\subset 123}$, and then with the factor $L_{1\subset 12}$:

1. To show that $L_{12\subset 123}\varprojlim_{\Delta}(F^\bullet) \simeq \varprojlim_{\Delta} L_{12\subset 123}(F^\bullet)$, by Key Lemma 8.6, it suffices to show that
$$\tilde{L}_{12\subset 123}\tilde{L}^R_{12\subset 123}\varprojlim_{\Delta}(F^\bullet) \simeq \varprojlim_{\Delta} \tilde{L}_{12\subset 123}\tilde{L}^R_{12\subset 123}(F^\bullet),$$
for which it suffices to show that $\tilde{L}_{12\subset 123}\tilde{L}^R_{12\subset 123}(F^\bullet)$ is split. By the compatibility of the top square (recall our earlier notation in this chapter of the $'$ and $''$ functors being inverses), we have
$$\begin{aligned} \tilde{L}_{12\subset 123}\tilde{L}^R_{12\subset 123}F^\bullet &\simeq \tilde{L}_{12\subset 123}L''_{3\subset 123}L'_{3\subset 123}\tilde{L}^R_{12\subset 123}(F^\bullet) \\ &\simeq \tilde{L}_{12\subset 123}L''_{3\subset 123}\phi_1^R\underbrace{L_{3\subset 123}(F^\bullet)}_{\text{split}} \end{aligned}$$

so $\tilde{L}_{12\subset 123}\tilde{L}^R_{12\subset 123}(F^\bullet)$ is indeed split. So we now know that

$$L_{1\subset 12}L_{12\subset 123}\varprojlim_{\Delta}(F^\bullet) \simeq L_{1\subset 12}\varprojlim_{\Delta} L_{12\subset 123}(F^\bullet)$$

$$\begin{tikzcd}
\Delta \arrow[ddr, "F^\bullet"] \arrow[ddrr, dotted, bend left, "\text{split}"] \arrow[ddddrr, dotted, bend left, "\text{split}"] & & & & \\
& & K_{12\subset 123} \arrow[dl, hook, "\tilde{L}_{12\subset 123}"'] \arrow[rr, equal, "L'_{3\subset 123}"] & & K'_{12\subset 123} \arrow[dl, hook, "\phi_1"] \\
& \mathscr{C}/(\mathscr{D}_1\cap\mathscr{D}_2\cap\mathscr{D}_3) \arrow[rr, "L_{3\subset 123}"] \arrow[dd, "L_{12\subset 123}"] & & \mathscr{C}/\mathscr{D}_3 & \\
& & K_{1\subset 12} \arrow[dl, hook, "\tilde{L}_{1\subset 12}"'] \arrow[rr, equal, "L'_{2\subset 12}"] & & K'_{1\subset 12} \arrow[dl, hook, "\phi_2"] \\
& \mathscr{C}/(\mathscr{D}_1\cap\mathscr{D}_2) \arrow[rr, "L_{2\subset 12}"] \arrow[d, "L_{1\subset 12}"] & & \mathscr{C}/\mathscr{D}_2 & \\
& \mathscr{C}/\mathscr{D}_1 & & &
\end{tikzcd}$$

Figure 8.2: The shelling argument, for a three-piece cover. The dotted arrows are split diagrams by assumption. The horizontal squares are pullback squares, which carry the necessary adjointability for Key Lemma 8.6 to progressively allow deeper partial localizations to commute past the limit.

2. It remains to show that $L_{1\subset 12}\varprojlim_{\Delta}(L_{12\subset 123}F^\bullet) \simeq \varprojlim_{\Delta} L_{1\subset 12}(L_{12\subset 123}F^\bullet)$. Again by Key Lemma 8.6, it suffices to show that

$$\tilde{L}_{1\subset 12}\tilde{L}^R_{1\subset 12}\varprojlim_{\Delta}(L_{12\subset 123}F^\bullet) \simeq \varprojlim_{\Delta}\tilde{L}_{1\subset 12}\tilde{L}^R_{1\subset 12}(L_{12\subset 123}F^\bullet),$$

for which it suffices to show that $\tilde{L}_{1\subset 12}\tilde{L}^R_{1\subset 12}(L_{12\subset 123}F^\bullet)$ is split. By the compatibility of the middle square, we have

$$\begin{aligned}
\tilde{L}_{1\subset 12}\tilde{L}^R_{1\subset 12}(L_{12\subset 123}F^\bullet) &\simeq \tilde{L}_{1\subset 12}L''_{2\subset 12}L'_{2\subset 12}\tilde{L}^R_{1\subset 12}(L_{12\subset 123}F^\bullet) \\
&\simeq \tilde{L}_{1\subset 12}L''_{2\subset 12}\phi_2^R\underbrace{L_{2\subset 12}(L_{12\subset 123}F^\bullet)}_{L_{2\subset 123}F^\bullet,\ \text{split}}
\end{aligned}$$

so $\tilde{L}_{1\subset 12}\tilde{L}^R_{1\subset 12}(L_{12\subset 123}F^\bullet)$ is indeed split.

□

Chapter 9

Math That Might Be New

9.1 What is in this chapter?

Consider a collection of reflective localizations in $\mathrm{Pr}^{L,\mathrm{st}}$

$$\left\{ \mathscr{C} \underset{R_i}{\overset{L_i}{\rightleftarrows}} \mathscr{C}/\mathscr{D}_i \right\}_{i\in I}$$

We have in mind the following example: for a closed cover $\{\Lambda_i\}_{i\in I}$ of a singular support Lagrangian $\Lambda \subseteq T^*X$, the collection

$$\left\{ \mathrm{Sh}_{\Lambda}(X) \underset{R_i}{\overset{L_i}{\rightleftarrows}} \mathrm{Sh}_{\Lambda_i}(X) \right\}_{i\in I}$$

In this final chapter we record all descent-related results that we have been able to find for collections of localizations. In particular, we

1. present several criteria for
   - rendering $\mathscr{C} \xrightarrow{L:=\prod_{i\in I} L_i} \prod_{i\in I} \mathscr{C}/\mathscr{D}_i$ comonadic, and
   - assuring that the coaugmented cosimplicial diagram

     $$\mathscr{C} \xrightarrow{L} \left[\prod_{|J|=1} \mathscr{C}/\langle \mathscr{D}_j \rangle_{j\in J} \substack{\longrightarrow \\ \dashleftarrow \\ \longrightarrow} \prod_{|J|=2} \mathscr{C}/\langle \mathscr{D}_j \rangle_{j\in J} \substack{\longrightarrow \\ \dashleftarrow \\ \longrightarrow \\ \dashleftarrow \\ \longrightarrow} \cdots \right]$$

     satisfies the Beck-Chevalley conditions; and
2. interpret the criteria in the language of the running example of covers of singular support Lagrangians, and
3. present some applications in the microlocal setting.

9.2 Comonadicity criterion I: semi-orthogonality

This section describes a comonadicity criterion that can be viewed as the reason Zariski open cover yields comonadicity. It directly generalizes Proposition 8.1. The active ingredient is semi-orthogonality of kernels, illustrated in the following lemma that serves as the base case for the general argument.

Lemma 9.1. *Suppose* $\hom_{\mathscr{C}}(\mathscr{D}_2, \mathscr{D}_1) = 0$ *for the square*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "b"] \arrow[d, "a"] & \mathscr{C}/\mathscr{D}_2 \arrow[d, "d"] \\ \mathscr{C}/\mathscr{D}_1 \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle \end{tikzcd}$$

and let $\Delta \xrightarrow{F^\bullet} \mathscr{C}$ *be a cosimplicial diagram. Then* $\varprojlim_\Delta F^\bullet \to F^\bullet$ *is an absolute limit diagram iff* $\varprojlim_\Delta aF^\bullet \to aF^\bullet$ *and* $\varprojlim_\Delta bF^\bullet \to bF^\bullet$ *are absolute limit diagrams.*

Proof. The "only if" part holds because the property of being an absolute limit diagram is preserved by any (exact) functor. Conversely, suppose $\varprojlim_\Delta aF^\bullet \to aF^\bullet, \varprojlim_\Delta bF^\bullet \to bF^\bullet$ are both absolute limit diagrams. The orthogonality hypothesis says that $\tilde{b}^R\tilde{a} \simeq 0$, so first place $F^\bullet$ into b-recollement (horizontal), and then place its left-hand term into a-recollement (vertical):

$$\begin{tikzcd} \underbrace{\tilde{b}\tilde{b}^R\tilde{a}\tilde{a}^R F^\bullet}_{\simeq 0} \arrow[d] & & & \\ \tilde{b}\tilde{b}^R F^\bullet \arrow[r] \arrow[d, equal, "\therefore"] & F^\bullet \arrow[r] & \underbrace{b^R bF^\bullet}_{absolute} \arrow[r, "+1"] & \tilde{b}\tilde{b}^R F^\bullet[1] \\ \underbrace{\tilde{b}\tilde{b}^R a^R aF^\bullet}_{absolute} \arrow[d, "+1"] & & & \\ 0[1] & & & \end{tikzcd}$$

This situates $F^\bullet$ as the cone of two cosimplicial diagrams whose augmentations by their limits are absolute limit diagrams; therefore the limits arrange into a fiber square,

$$\begin{tikzcd}
\varprojlim_{\Delta} \tilde{b}\tilde{b}^R a^R a F^\bullet \arrow[r] \arrow[d] & \varprojlim_{\Delta} F^\bullet \arrow[r] \arrow[d] & \varprojlim_{\Delta} b^R b F^\bullet \arrow[r, "+1"] \arrow[d] & \varprojlim_{\Delta} \tilde{b}\tilde{b}^R a^R a F^\bullet[1] \arrow[d] \\
\tilde{b}\tilde{b}^R a^R a F^\bullet \arrow[r] & F^\bullet \arrow[r] & b^R b F^\bullet \arrow[r, "+1"] & \tilde{b}\tilde{b}^R a^R a F^\bullet[1]
\end{tikzcd}$$

and since exact functors commute with cones, this realizes $\varprojlim_{\Delta} F^\bullet \to F^\bullet$ as an absolute limit diagram. □

We summarize the above result by saying that semiorthogonal pairs of localizations reflect absolute limit diagrams. This has the following consequence:

Corollary 9.2. *Suppose* $\hom_{\mathscr{C}}(\mathscr{D}_2, \mathscr{D}_1) = 0$ *Then* $L := a \boxplus b$ *is comonadic.*

Proof. The same proof as in Lemma **??** shows that L is conservative. By Lemma 9.1, if $aF^\bullet, bF^\bullet$ are split, then $\varprojlim_{\Delta} F^\bullet \to F^\bullet$ is an absolute limit diagram, in which case $a\varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} aF^\bullet$ and $b\varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} bF^\bullet$. □

Example 9.3. *Consider the skeleton* $\Lambda \subset T^*\mathbb{R}$ *from Example 7.17, and take as a cover*

$$\Lambda_1 = 0_{\mathbb{R}} \cup T_0^{*,\geq 0}\mathbb{R}, \qquad \Lambda_2 = 0_{\mathbb{R}} \cup T_1^{*,\geq 0}\mathbb{R}$$

We illustrate the effect of wrapping on the co-cores in the various subcategories in Figure 9.1. The functor $L_1 \boxplus L_2$ *is conservative, and as depicted, the kernels are semiorthogonal. Thus, this cover is comonadic.*[1]

We now generalize Corollary 9.2 to multiple pieces. The idea of the argument is that semiorthogonality assures that the property of being an absolute limit "propagates upstream" of localizations.

Theorem 9.4. *Suppose that there is a "shelling" of* $\mathscr{C}$ *by thick subcategories* $\mathscr{D}_1, \dots, \mathscr{D}_n$ *in the sense that each span in the following diagram is semi-orthogonal:*

[1] In fact, comonadicity for this example is even simpler: the functors L_i are both bicontinuous.

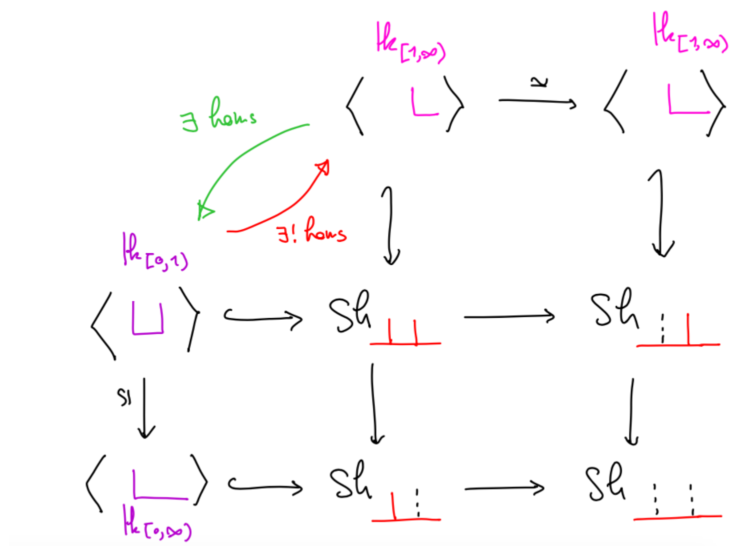


Figure 9.1: The square diagram for Example 7.17. Note that there is semi-orthogonality of the co-cores

$$\begin{tikzcd}
\mathscr{C} \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_n \\
\mathscr{C}/(\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_{n-1}) \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_{n-1} \\
\vdots \arrow[r] \arrow[d] & \vdots \\
\mathscr{C}/(\mathscr{D}_1 \cap \mathscr{D}_2 \cap \mathscr{D}_3) \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_3 \\
\mathscr{C}/(\mathscr{D}_1 \cap \mathscr{D}_2) \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_2 \\
\mathscr{C}/\mathscr{D}_1 &
\end{tikzcd}$$

Then $L : \mathscr{C} \to \prod_{i=1}^{n} \mathscr{C}/\mathscr{D}_i$ *is comonadic.*

Proof. First we show that L is conservative. It suffices to show that all the spans are jointly conservative. But this is true because all the spans are semiorthogonal (in fact, all the lower spans are jointly conservative by definition).

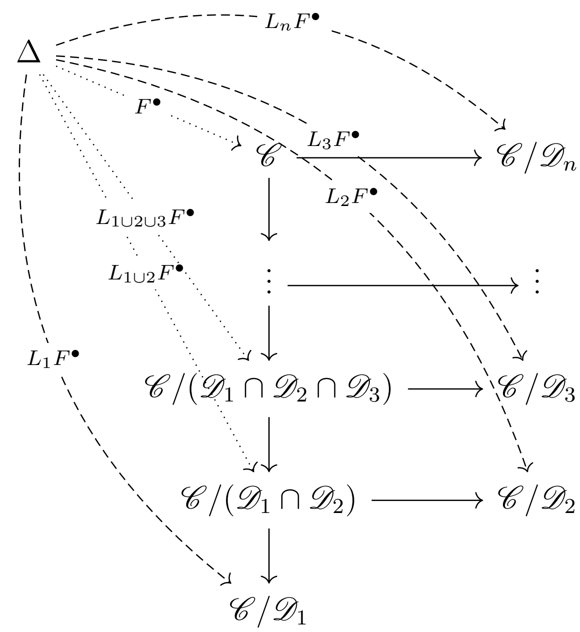


Figure 9.2: The dashed arrows $L_i F^\bullet$ are split diagrams by assumption, and the dotted arrows are absolute limit diagrams as a consequence.

Next, suppose that $\Delta \xrightarrow{F^\bullet} \mathscr{C}$ is a diagram such that $L_i F^\bullet \in \mathscr{C}/\mathscr{D}_i$ is split for all i; we will show that $F^\bullet$ is itself an absolute limit diagram, from which would immediately follow that $L_i \varprojlim_{\Delta} F^\bullet \simeq \varprojlim_{\Delta} L_i F^\bullet$ for all i. Begin at the bottom of the diagram: since $L_1 F^\bullet$ and $L_2 F^\bullet$ are split, by Lemma 9.1 it means that $L_{1\cup 2} F^\bullet$ is an absolute limit diagram. On the next level of the diagram, since $L_{1\cup 2} F^\bullet$ and $L_3 F^\bullet$ are absolute limit diagrams, it means that $L_{1\cup 2\cup 3} F^\bullet$ is one too. Following this all the way up shows that $F^\bullet$ itself must be an absolute limit diagram. We conclude by Barr-Beck-Lurie that L is comonadic. $\square$

Familiar examples

We use the above theorem to deduce comonadicity results in some familiar examples. The first is the comonadic form of Zariski descent for a finite open cover:

Corollary 9.5 (Zariski comonadicity). *For an open cover $\{U_i\}_{i=1}^n$ of a qcqs scheme X, the functor*

$$\mathrm{QCoh}(X) \xrightarrow{L=\prod_{i=1}^n j_i^*} \prod_{i=1}^n \mathrm{QCoh}(U_i)$$

is comonadic.

Proof. In the notation of the theorem above,

$$\mathscr{C}/(\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_k) = \mathrm{QCoh}(U_1 \cup \cdots \cup U_k),$$

and so the spans are

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[dr, "j^*_{1\ldots k}"] \arrow[drr, bend left, "j^*_k"] \arrow[ddr, bend right, "j^*_{1\ldots k-1}"'] & & \\
& \mathrm{QCoh}(U_1 \cup \cdots \cup U_k) \arrow[r, "\bar{j}^*_k"] \arrow[d, "\bar{j}^*_{1\ldots k-1}"] & \mathrm{QCoh}(U_k) \\
& \mathrm{QCoh}(U_1 \cup \cdots \cup U_{k-1}) &
\end{tikzcd}$$

The subcategories $\mathscr{D}_k$ and $\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_{k-1}$ within $\mathrm{QCoh}(U_1 \cup \cdots U_k)$ can be identified with the full subcategories on those quasicoherent sheaves that are supported on the two disjoint pieces (closed in $U_1 \cup \cdots \cup U_k$) of the symmetric difference

$$U_k \bigtriangleup U_1 \cup \cdots \cup U_{k-1}$$

Any pair of objects, one from each category, share no morphisms between them, and thus these subcategories are orthogonal. In particular, they are semi-orthogonal. □

For the next example, consider a strongly semi-orthogonal decomposition of a category $\mathscr{C}$:

$$\mathscr{C} = \langle \mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_n \rangle$$

Recall that this means that the derived hom space $\hom_{\mathscr{C}}(\mathscr{D}_i, \mathscr{D}_j)$ is 0 if $i > j$:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, equal] & \langle \mathscr{D}_1 & \mathscr{D}_2 \arrow[l, bend right, "0"'] & \cdots \arrow[l, bend right, "0"'] & \mathscr{D}_n \rangle \arrow[l, bend right, "0"']
\end{tikzcd}$$

In particular, $\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_{k-1} = 0$, and so certainly

$$\hom_{\mathscr{C}}(\mathscr{D}_k, \mathscr{D}_1 \cap \cdots \cap \mathscr{D}_{k-1}) = 0$$

Thus the semiorthogonal shelling that a strongly semiorthogonal decomposition determines is quite degenerate:

$$\begin{tikzcd}
\mathscr{C} \arrow[r] \arrow[d, equal] & \mathscr{C}/\mathscr{D}_n \\
\mathscr{C} \arrow[r] \arrow[d, equal] & \mathscr{C}/\mathscr{D}_{n-1} \\
\vdots \arrow[r] \arrow[d, equal] & \vdots \\
\mathscr{C} \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_2 \\
\mathscr{C}/\mathscr{D}_1 &
\end{tikzcd}$$

This immediately demonstrates the following comonadicity result:

Corollary 9.6. *For a strongly semiorthogonal decomposition $\mathscr{C} = \langle \mathscr{D}_1, \dots, \mathscr{D}_n \rangle$, there is an equivalence*

$$\mathscr{C} \xlongequal{L^{\mathrm{enh}}} {}_{\Omega}\operatorname{coMod} \textstyle\prod_{i=1}^{n} \mathscr{C}/\mathscr{D}_i$$

where the comonad Ω is strictly lower triangular:

$$\Omega = \begin{bmatrix} \mathrm{Id} & 0 & \cdots & 0 \\ L_2 R_1 & \mathrm{Id} & \ddots & 0 \\ \vdots & & \ddots & 0 \\ L_n R_1 & \cdots & \cdots & \mathrm{Id} \end{bmatrix}$$

Example 9.7. *Here is how this looks like in the simplest example where $\mathscr{C} = \operatorname{QCoh} \mathbb{P}^1$ and $\mathscr{D}_1 = \langle \mathscr{O}(-1) \rangle, \mathscr{D}_2 = \langle \mathscr{O} \rangle$. These fit into the square of reflective localizations*

$$\begin{tikzcd}
\operatorname{QCoh} \mathbb{P}^1 \arrow[r, "L_2"] \arrow[d, "L_1"] & \operatorname{QCoh} \mathbb{P}^1/\langle \mathscr{O} \rangle \arrow[d] \\
\operatorname{QCoh} \mathbb{P}^1/\langle \mathscr{O}(-1) \rangle \arrow[r] & \operatorname{QCoh} \mathbb{P}^1/\langle \mathscr{O}(-1), \mathscr{O} \rangle \simeq 0
\end{tikzcd}$$

where the bottom-right corner is trivial by Beilinson's theorem. The square can be rewritten as

$$\begin{tikzcd}
\operatorname{QCoh} \mathbb{P}^1 \arrow[r, "p_*(- \otimes \mathscr{O}(-1))"] \arrow[d, "p_*"] & {}_{\mathbb{C}}\operatorname{Mod} \arrow[d] \\
{}_{\mathbb{C}}\operatorname{Mod} \arrow[r] & 0
\end{tikzcd}$$

from which it follows that it is not a pullback square. Nonetheless, since $\hom_{\mathscr{C}}(\mathscr{D}_2, \mathscr{D}_1) = 0$, *the functor* $L_1 \boxplus L_2$ *is still comonadic by the above corollary. To see what comonadicity means here, we identify the adjoints as*

$$\begin{aligned} (L_1 = p_*)^R &= p^! := p^*(-) \otimes_{\mathscr{O}} \mathscr{O}(-2)[1], \\ (L_2 = p_*(- \otimes \mathscr{O}(-1)))^R &= p^! \otimes \mathscr{O}(1) = p^*(-) \otimes_{\mathscr{O}} \mathscr{O}(-1)[1] \end{aligned}$$

which gives the comonad

$$\Omega = \begin{bmatrix} \mathrm{Id} & 0 \\ \mathbb{C}^2 \otimes (-) & \mathrm{Id} \end{bmatrix} \quad \curvearrowright \quad {}_{\mathbb{C}}\mathrm{Mod} \boxplus_{\mathbb{C}} \mathrm{Mod}$$

A counital coassociative comodule for Ω *is thus a pair of vector spaces* (V, W) *together with two maps* $V \to W$*, which recovers the Beilinson quiver description.*

9.3 Beck-Chevalley criterion: separability

In the previous section we established a criterion for comonadic descent for a collection of localizations. In this section, we establish a criterion in the form of Theorem 9.15 for limit descent for a collection of localizations.

We begin by interpreting the Beck-Chevalley condition in our context:

Lemma 9.8. *Let* $C^\bullet : \Delta_+ \to \mathrm{Pr}^{L,\mathrm{st}}$ *denote the coaugmented cosimplicial diagram*

$$\mathscr{C} \xrightarrow{L} \Big[\textstyle\prod_{|J|=1} \mathscr{C}/\langle \mathscr{D}_j \rangle_{j \in J} \rightrightarrows \prod_{|J|=2} \mathscr{C}/\langle \mathscr{D}_j \rangle_{j \in J} \;\substack{\to\\\to\\\to}\; \cdots \Big]$$

In this case, the Beck-Chevalley condition translates to the condition that for all $J \subset I$ *and* $i, j \in I$*, the following diagram is (right-)adjointable:*

$$\begin{tikzcd} \mathscr{C}/\langle \mathscr{D}_J \rangle \arrow[r, "b"] \arrow[d, "a"] & \mathscr{C}/\langle \mathscr{D}_{J \cup i} \rangle \arrow[d, "d"] \arrow[l, dashed, bend left, "b^R"] \\ \mathscr{C}/\langle \mathscr{D}_{J \cup j} \rangle \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_{J \cup i \cup j} \rangle \arrow[l, dashed, bend left, "c^R"] \end{tikzcd}$$

Proof. For reference, in our running example this diagram reads

$$\begin{tikzcd} \mathrm{Sh}_{\Lambda_{\cap J}} \arrow[r, "b"] \arrow[d, "a"] & \mathrm{Sh}_{\Lambda_{\cap Ji}} \arrow[d, "d"] \arrow[l, dashed, bend left, "b^R"] \\ \mathrm{Sh}_{\Lambda_{\cap Jj}} \arrow[r, "c"] & \mathrm{Sh}_{\Lambda_{\cap Jij}} \arrow[l, dashed, bend left, "c^R"] \end{tikzcd}$$

Stacking diagrams shows that verifying the Beck-Chevalley condition for all α reduces to verifying it for $\alpha = d^i$ coface and $\alpha = s^j$ codegeneracy maps.

For coface maps we have $[0] \star d^i = d^{i+1}$. Unwinding the definition of $d^0, d^i, R_m = (d^0)^R$ in our cosimplicial diagram gives the above condition. On the other hand, for codegeneracy maps we have $[0] \star s^j = s^{j+1}$, and unwinding the definitions here gives a condition that is tautologically satisfied by our system of localizations. $\square$

The goal of this section is to show that adjointability of all the $J = \emptyset$ squares

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "b"] \arrow[d, "a"] & \mathscr{C}/\langle \mathscr{D}_i \rangle \arrow[l, dashed, bend left, "b^R"] \arrow[d, "d"] \\
\mathscr{C}/\langle \mathscr{D}_j \rangle \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_i, \mathscr{D}_j \rangle \arrow[l, dashed, bend left, "c^R"]
\end{tikzcd}$$

"propagates down" to assure adjointability of all deeper localization squares.

To do this, we first introduce the related notion of "separability" for the collection $\{L_i\}_{i \in I}$ and show that separability "propagates down." We then show that adjointability coincides with separability.

To that end, consider the projection endofunctors of $\mathscr{C}$

$$Q_i := R_i L_i$$

coming from a collection of localizations

$$\begin{tikzcd}
\mathscr{C} \arrow[loop left, "Q_i"] \arrow[r, bend left, "L_i"] & \mathscr{C}/\mathscr{D}_i \arrow[l, hook', "R_i"]
\end{tikzcd}$$

The following insightful definition was already made and studied in [18] in an investigation of descent for sheaves on noncommutative schemes. It has proved to us to be a particularly useful lens for thinking about the active ingredients in comonadicity for localizations:

Definition 9.9. *We call the collection of localizations* $\{L_i\}_{i \in I}$ separable *if*

$$Q_i Q_j \simeq Q_j Q_i$$

Note that pairwise adjointability certainly implies separability. Separability is related to commutativity of the other available collection of projections—the ones onto $\mathscr{D}_i := \mathrm{Ker}(L_i)$:

$$\begin{tikzcd}
 & \mathscr{D}_1 \arrow[r, "L_2'"] \arrow[d, hook, "\tilde{L}_1"] & \langle \mathscr{D}_1, \mathscr{D}_2 \rangle / \mathscr{D}_2 \arrow[d, hook] \\
\mathscr{D}_2 \arrow[r, hook, "\tilde{L}_2"] \arrow[d, "L_1'"] & \mathscr{C} \arrow[r, "L_2"] \arrow[d, "L_1"] & \mathscr{C}/\mathscr{D}_2 \arrow[d] \\
\langle \mathscr{D}_1, \mathscr{D}_2 \rangle / \mathscr{D}_1 \arrow[r, hook] & C/\mathscr{D}_1 \arrow[r] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle
\end{tikzcd}$$

Lemma 9.10. *In the notation of the above diagran, define*

$$\tilde{Q}_1 := \tilde{L}_1\tilde{L}_1^R, \qquad \tilde{Q}_2 := \tilde{L}_2\tilde{L}_2^R$$

Then

$$Q_1Q_2 \simeq Q_2Q_1 \quad \Leftrightarrow \quad \tilde{Q}_1\tilde{Q}_2 \simeq \tilde{Q}_2\tilde{Q}_1$$

Proof. A simple calculation. $\square$

We now collect a range of elementary results about the interplay between squares being pullbacks, adjointable, separable, conservative, and semiorthogonal.

Lemma 9.11. *The following are equivalent for the square*

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "b"] \arrow[d, "a"] & \mathscr{C}/\mathscr{D}_2 \arrow[d, "d"] \\
\mathscr{C}/\mathscr{D}_1 \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle
\end{tikzcd}$$

1. *it is a pullback square;*
2. *it is conservative (i.e. a, b jointly conservative) and adjointable;*
3. *it is conservative and separable;*
4. *it is semiorthogonal and separable;*
5. *it is conservative and adjointable;*
6. *it is semiorthogonal and adjointable.*

Proof. (1) $\Leftrightarrow$ (2) was proven above. (2) $\Rightarrow$ (3) because adjointable implies separable.

We turn to proving (3) $\Rightarrow$ (1). So suppose the square is conservative and separable. By symmetry, it suffices to show that $\tilde{a}^R\tilde{b} \simeq 0$. The conservativity assumption implies that a', b' are conservative, and thus it suffices to show that $a'\tilde{a}^R\tilde{b} \simeq 0$. Finally, since $\tilde{b}^R$ is essentially surjective, it suffices to show that $a'\tilde{a}^R\tilde{b}\tilde{b}^R \simeq 0$. But

$$a'\tilde{a}^R\tilde{b}\tilde{b}^R := \tilde{d}^R b(\tilde{a}\tilde{a}^R)(\tilde{b}\tilde{b}^R) \simeq \tilde{d}^R \underbrace{b(\tilde{b}}_{=0}\,\tilde{b}^R)(\tilde{a}\tilde{a}^R) \simeq 0$$

Thus, $\mathscr{D}_1$ and $\mathscr{D}_2$ are orthogonal.

Finally, we prove (1) $\Leftrightarrow$ (4). Certainly (1) $\Rightarrow$ (4). Suppose that the square is separable, and $\tilde{b}^R\tilde{a} \simeq 0$; we wish to show that $\tilde{a}^R\tilde{b} \simeq 0$. Consider the two recollement squares

$$\begin{tikzcd}
\tilde{a}\tilde{a}^R\tilde{b}\tilde{b}^R F \arrow[r] \arrow[d, equal, "\therefore"] & \tilde{b}\tilde{b}^R F \arrow[r] \arrow[d, equal] & a^R a\tilde{b}\tilde{b}^R F \arrow[r, "+1"] \arrow[d, equal, "\text{separable}"] & (1) \\
\tilde{b}\,\underbrace{\tilde{b}^R\tilde{a}}_{=0}\,\tilde{a}^R F \arrow[r] & \tilde{b}\tilde{b}^R F \arrow[r] & \tilde{b}\tilde{b}^R a^R a F \arrow[r, "+1"] & (1)
\end{tikzcd}$$

which are matched by the fact that the righthand objects are isomorphic by the separability assumption; thus the objects in the lefthand column are isomorphic. Thus the top-left object is 0, which implies that $\tilde{a}^R\tilde{b} \simeq 0$, as desired.

Parts (5) and (6) can each be easily seen to be equivalent to any of the others. $\square$

Lemma 9.12. *Consider the commutative diagram of reflective localizations*

$$\begin{tikzcd}
\mathscr{C} \arrow[dr, "\phi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathscr{C}' \arrow[r] \arrow[d] & \mathscr{C}_2 \arrow[d] \\
& \mathscr{C}_1 \arrow[r] & \mathscr{C}_{12}
\end{tikzcd}$$

Then the outer square is P iff the inner square is P, where $P \in \{$separable, adjointable$\}$.

Proof. Adjointability is easy. For separability, note that ϕ, like every localization, is essentially surjective. $\square$

Corollary 9.13. *The square*

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "b"] \arrow[d, "a"] & \mathscr{C}/\mathscr{D}_2 \arrow[d, "d"] \\
\mathscr{C}/\mathscr{D}_1 \arrow[r, "c"] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle
\end{tikzcd}$$

is separable iff it is adjointable.

Proof. Adjointability implies separability. To see the converse, form the conservative inner square

$$\begin{tikzcd}
\mathscr{C} \arrow[dr, "\phi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathscr{C}/(\mathscr{D}_1 \cap \mathscr{D}_2) \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_2 \arrow[d] \\
& \mathscr{C}/\mathscr{D}_1 \arrow[r] & \mathscr{C}/\langle \mathscr{D}_1, \mathscr{D}_2 \rangle
\end{tikzcd}$$

Separability of the outer square implies separability of the inner square, which by the above Lemma 9.12, thus adjointable. Adjointability of the inner square now implies adjointability of the outer square. $\square$

Corollary 9.14. *Consider a collection of thick categories $\mathscr{D}_{i\in I}$ inside $\mathscr{C}$. If all squares*

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_j \arrow[d] \\ \mathscr{C}/\mathscr{D}_i \arrow[r] & \mathscr{C}/\langle \mathscr{D}_i, \mathscr{D}_j \rangle \end{tikzcd}$$

are separable, then every "deeper intersection" is (right-)adjointable

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathscr{C}/\langle \mathscr{D}_{k\in K} \rangle \arrow[d] \\ \mathscr{C}/\langle \mathscr{D}_{j\in J} \rangle \arrow[r] & \mathscr{C}/\langle \mathscr{D}_{i\in J\cup K} \rangle \end{tikzcd}$$

for $J, K \subseteq I$.

Proof. We can translate from adjointability to separability, which is far easier to check. To commute $\prod_{j\in J} Q_j$ past $\prod_{k\in K} Q_k$, simply move each Q_j past $\prod_{k\in K} Q_k$ using the pairwise commutativity assumption. $\square$

This helps establish the following main result of this section:

Theorem 9.15. *Consider a collection of thick subcategories $\mathscr{D}_{i\in I}$ of $\mathscr{C}$. If*

1. *$\mathscr{C} \xrightarrow{L} \prod_{i\in I} \mathscr{C}/\mathscr{D}_i$ is conservative; and*
2. *all squares*

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_j \arrow[d] \\ \mathscr{C}/\mathscr{D}_i \arrow[r] & \mathscr{C}/\langle \mathscr{D}_i, \mathscr{D}_j \rangle \end{tikzcd}$$

   are separable (equivalently, adjointable),

then the coaugmented cosimplicial diagram

$$\mathscr{C} \xrightarrow{L} \left[\prod_{|J|=1} \mathscr{C}/\langle \mathscr{D}_j \rangle_{j\in J} \rightleftarrows \prod_{|J|=2} \mathscr{C}/\langle \mathscr{D}_j \rangle_{j\in J} \cdots \right]$$

is a limit diagram. In particular, L is comonadic.

Proof. The above Corollary can be rephrased to say that assumption (2) is equivalent the coagumented cosimplicial diagram above satisfying the Beck-Chevalley condition. So by Theorem 3.8, it only remains to show that L is comonadic.

Since L is conservative, the following diagram is a "shelling" of $\mathscr{C}$ by jointly conservative spans:

$$\begin{tikzcd}
\mathscr{C} \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_n \\
\mathscr{C}/(\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_{n-1}) \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_{n-1} \\
\vdots \arrow[r] \arrow[d] & \vdots \\
\mathscr{C}/(\mathscr{D}_1 \cap \mathscr{D}_2 \cap \mathscr{D}_3) \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_3 \\
\mathscr{C}/(\mathscr{D}_1 \cap \mathscr{D}_2) \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{D}_2 \\
\mathscr{C}/\mathscr{D}_1 &
\end{tikzcd}$$

By Theorem 9.4 it would suffice to show that each of these spans (now also decorated with C as a tip)

$$\begin{tikzcd}
\mathscr{C} \arrow[dr, "L_{1\dots k}"] \arrow[drr, bend left, "L_k"] \arrow[ddr, bend right, "L_{1\dots k-1}"'] & & \\
& \mathscr{C}/(\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_k) \arrow[r, "\overline{L}_k"] \arrow[d, "\overline{L}_{1\dots k-1}"] & \mathscr{C}/\mathscr{D}_k \\
& \mathscr{C}/(\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_{k-1}) &
\end{tikzcd}$$

satisfies the property

$$(R_jL_j)(R_{1\dots k-1}L_{1\dots k-1}) \stackrel{?}{=} (R_{1\dots k-1}L_{1\dots k-1})(R_jL_j) \quad \text{for all } j \in \{1, \dots, n\}$$

i.e. that

$$Q_jQ_{1\dots k-1} \stackrel{?}{=} Q_{1\dots k-1}Q_j \quad \text{for all } j \in \{1, \dots, n\}.$$

Assumption (2) forms the base case, where $k = 2$. We now take as our inductive hypothesis the assumption that

$$Q_jQ_{1\dots k-2} = Q_{1\dots k-2}Q_j \quad \text{for all } j \in \{1, \dots, n\}$$

Observe that $L_{1\ldots k-2} \times L_{k-1}$ factors through $L_{1\ldots k-1}$:

$$\begin{tikzcd}
\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_{k-1} \arrow[rr] \arrow[d, hook] & & 0 \arrow[d] \\
\mathscr{C} \arrow[rr, "L_{1\ldots k-2} \times L_{k-1}"] \arrow[d, "L_{1\ldots k-1}"] & & \mathscr{C}/(\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_{k-2}) \times \mathscr{C}/\mathscr{D}_{k-1} \\
\mathscr{C}/(\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_{k-1}) \arrow[urr, bend right, "\overline{L}_{1\ldots k-2} \times \overline{L}_{k-1}"'] & &
\end{tikzcd}$$

Consider now the triangle

$$\begin{tikzcd}
F \arrow[r, "\phi"] \arrow[d, equal] & \underbrace{(\overline{L}_{1\ldots k-2} \times \overline{L}_{k-1})^R \circ (\overline{L}_{1\ldots k-2} \times \overline{L}_{k-1})F}_{\overline{R}_{1\ldots k-2}\overline{L}_{1\ldots k-2}F \oplus \overline{R}_{k-1}\overline{L}_{k-1}F} \arrow[r] \arrow[d, equal] & \underbrace{\mathrm{Cone}(\phi)}_{\overline{R}_{1\ldots k-2}\overline{L}_{1\ldots k-2}\overline{R}_{k-1}\overline{L}_{k-1}F} \arrow[r, "+1"] \arrow[d, equal] & F[1] \arrow[d, equal] \\
F \arrow[r, "\phi"] & \overline{Q}_{1\ldots k-2}F \oplus \overline{Q}_{k-1}F \arrow[r] & \overline{Q}_{1\ldots k-2}\overline{Q}_{k-1}F \arrow[r, "+1"] & F[1]
\end{tikzcd}$$

whose cone has the above formula as a consequence of the inductive hypothesis that the $k-1$ level span completed to a pullback square. Now insert this triangle into the purported equality:

$$\begin{tikzcd}
(R_jL_j)(R_{1\ldots k-1}L_{1\ldots k-1})F \arrow[r, equal, "?"] \arrow[d, "\phi_1"] & (R_{1\ldots k-1}L_{1\ldots k-1})(R_kL_k)F \arrow[d, "\phi_2"] \\
(R_jL_j)(\underbrace{R_{1\ldots k-1}(\overline{L}_{1\ldots k-2} \times \overline{L}_{k-1})^R}_{(L_{1\ldots k-2}\times L_{k-1})^R}\underbrace{(\overline{L}_{1\ldots k-2} \times \overline{L}_{k-1})L_{1\ldots k-1}}_{L_{1\ldots k-2}\times L_{k-1}})F \arrow[r, equal, "(*)"] \arrow[d] & (\underbrace{R_{1\ldots k-1}(\overline{L}_{1\ldots k-2} \times \overline{L}_{k-1})^R}_{(L_{1\ldots k-2}\times L_{k-1})^R}\underbrace{(\overline{L}_{1\ldots k-2} \times \overline{L}_{k-1})L_{1\ldots k-1}}_{L_{1\ldots k-2}\times L_{k-1}})(R_jL_j)F \\
(R_jL_j)(\underbrace{R_{1\ldots k-1}\big(\overline{R}_{1\ldots k-2}}_{R_{1\ldots k-2}}\underbrace{\overline{L}_{1\ldots k-2}\overline{R}_{k-1}}_{L_{1\ldots k-2}R_{k-1}}\underbrace{\overline{L}_{k-1}\big)L_{1\ldots k-1}}_{L_{k-1}})F \arrow[r, equal] & (\underbrace{R_{1\ldots k-1}\big(\overline{R}_{1\ldots k-2}}_{R_{1\ldots k-2}}\underbrace{\overline{L}_{1\ldots k-2}\overline{R}_{k-1}}_{L_{1\ldots k-2}R_{k-1}}\underbrace{\overline{L}_{k-1}\big)L_{1\ldots k-1}}_{L_{k-1}})(R_jL_j)F
\end{tikzcd}$$

which can be rewritten as

$$\begin{tikzcd}
Q_jQ_{1\ldots k-1}F \arrow[r, equal, "?"] \arrow[d, "\phi_1"] & Q_{1\ldots k-1}Q_jF \arrow[d, "\phi_2"] \\
Q_j(Q_{1\ldots k-2}F \oplus Q_{k-1}F) \arrow[r, equal] \arrow[d] & (Q_{1\ldots k-2}F \oplus Q_{k-1})Q_jF \arrow[d] \\
Q_jQ_{1\ldots k-2}Q_{k-1}F \arrow[r, equal] & Q_{1\ldots k-2}Q_{k-1}Q_jF
\end{tikzcd}$$

The bottom two rows are isomorphisms by the inductive hypothesis, and therefore so is the top row. $\square$

9.4 Application: comonadicity and limit descent for skeleta locally of FLTZ type

In this section, we will describe a broad class of examples of pairs $(\Lambda, \{\Lambda_i\}_{i\in I})$ of skeleta and closed covers, which we will call "locally of FLTZ type," to which our descent criteria will apply. To set the stage and to explain the name, we first recall the construction of the skeleton from toric mirror symmetry [4].

Moment map and fan structure

Let Σ be a toric fan in $N_{\mathbb{R}} \cong \mathbb{R}^n$, where N is a rank n lattice, $N_{\mathbb{R}} = N \otimes \mathbb{R}$. Let $M = N^\vee$, $M_{\mathbb{R}} = M \otimes \mathbb{R}$ and $M_{\mathbb{T}} = M \otimes \mathbb{T}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong S^1$. The FLTZ skeleton for Σ is

$$\Lambda_\Sigma := \sqcup_{\sigma\in\Sigma}\Lambda_\sigma, \quad \text{where } \Lambda_\sigma := (M_\sigma + \sigma^\perp)/M \times \operatorname{Int}(\sigma) \subset M_{\mathbb{T}} \times N_{\mathbb{R}} = T^*M_{\mathbb{T}}$$

where $M_\sigma \subset M_{\mathbb{R}}$ is some refinement of lattice M.

Under toric mirror symmetry [4] [5] [12] [24], Λ_σ corresponds to torus orbit O_σ, and the closed subskeleton $\Lambda_{\leq\sigma} := \cup_{\tau\leq\sigma}\Lambda_\tau$ corresponds to the toric Zariski-open neighborhood U_σ of O_σ given by $U_\sigma = \cup_{\tau\leq\sigma} O_\tau$. Then

$$\mathrm{Sh}_{\Lambda_{\leq\sigma}}(M_{\mathbb{T}}) \cong \mathrm{QCoh}(U_\sigma).$$

We now generalize this class of skeleta Λ_Σ and their covers, in two steps. We first consider the skeleton supported on a hyperplane arrangement on $M_{\mathbb{R}}$ with codirections specified by Σ, and then we turn that into a local condition to define a large class of examples:

Definition 9.16. *Let Σ be a rational polyhedral fan in $N_{\mathbb{R}}$. The* local FLTZ skeleton *for Σ is*

$$\Lambda_\Sigma^{loc} := \bigcup_{\sigma\in\Sigma} \sigma^\perp \times \sigma \subset M_{\mathbb{R}} \times N_{\mathbb{R}} \cong T^*M_{\mathbb{R}}.$$

We say a skeleton $\Lambda \subset T^*M_{\mathbb{R}}$ is locally of FLTZ type Σ *if for any $p \in M_{\mathbb{R}}$, there is an open ball U_p around p such that $\Lambda|_{U_p} = \Lambda^{loc}_{\Sigma_p}$ for some subfan $\Sigma_p \subset \Sigma$.*

The following result follows easily from the definition:

Lemma 9.17. *For a subfan $\Sigma' \subseteq \Sigma$, put $\Lambda_{\Sigma'} := \mu^{-1}(\Sigma') \cap \Lambda_\Sigma$. This is a skeleton locally of FLTZ type Σ'.*

We now identify a cover of skeleta locally of FLTZ type that are adapted to the projection $\mu : T^*M_{\mathbb{R}} \to N_{\mathbb{R}}$, which is the moment map for translation by $M_{\mathbb{R}}$ on $T^*M_{\mathbb{R}}$. If Λ is locally of FLTZ type Σ, it has the following set-theoretic decomposition:

$$\Lambda = \bigsqcup_{\sigma\in\Sigma} \Lambda_\sigma, \quad \Lambda_\sigma := \mu^{-1}(\operatorname{Int}(\sigma)) \cap \Lambda. \tag{9.1}$$

The following key technical result holds for these covers. We prove it in the appendix to this chapter.

Theorem 9.18. *Take*

1. Σ *a rational polyhedral fan in* $N_{\mathbb{R}}$,
2. $\Lambda \subset T^*M_{\mathbb{R}}$ *a skeleton of locally of FLTZ type* Σ, *and*
3. $\sigma \in \Sigma$ *any cone.*

Put

$$\mathrm{star}_{\Sigma}(\sigma) := \cup_{\tau \geq \sigma} \tau.$$

Then for any $\xi \in \Lambda_{\sigma}^{\mathrm{smooth}}$, *the support of the co-core* P_{ξ}^{Λ} *is bounded above in the following sense:*

$$\mu(SS(P_{\xi}^{\Lambda})) \subseteq \mathrm{star}_{\Sigma}(\sigma)$$

Therefore, the μ-image of the singular support of P_{ξ}^{Λ} always is at least as big as the cone σ for which $\xi \in \Lambda_{\sigma}^{\mathrm{smooth}}$, and is guaranteed to be no bigger if σ is maximal.

Definition 9.19. *We say* σ *and* τ *are* adjacent *if there exists a cone* κ *such that* $\tau \leq \kappa \geq \sigma$. *In other words, if* σ *and* τ *are the faces of a common bigger cone.*

An immediate corollary is the following orthogonality result:

Corollary 9.20. *If* σ *and* τ *are non-adjacent cones in* Σ, *then for any* $\xi \in \Lambda_{\sigma}^{\mathrm{smooth}}$ *and* $\eta \in \Lambda_{\tau}^{\mathrm{smooth}}$, *we have*

$$\mathrm{hom}_{\mathrm{Sh}_{\Lambda}(M)}(P_{\xi}^{\Lambda}, P_{\eta}^{\Lambda}) = \mathrm{hom}_{\mathrm{Sh}_{\Lambda}(M)}(P_{\eta}^{\Lambda}, P_{\xi}^{\Lambda}) = 0.$$

Proof. By definition,

$$\mathrm{hom}_{\mathrm{Sh}_{\Lambda}(M)}(P_{\xi}^{\Lambda}, P_{\eta}^{\Lambda}) = \mu_{\xi} P_{\eta}^{\Lambda} = \begin{cases} \not\simeq 0 & \text{if } \xi \in SS(P_{\eta}^{\Lambda}) \\ 0 & \text{otherwise} \end{cases}$$

but the non-adjacency assumption ensures that $\mu(\xi)$ and $\mu(SS(P_{\eta}^{\Lambda}))$ are disjoint. □

For example, this happens when σ, τ are maximal cones; see Figure 9.3.

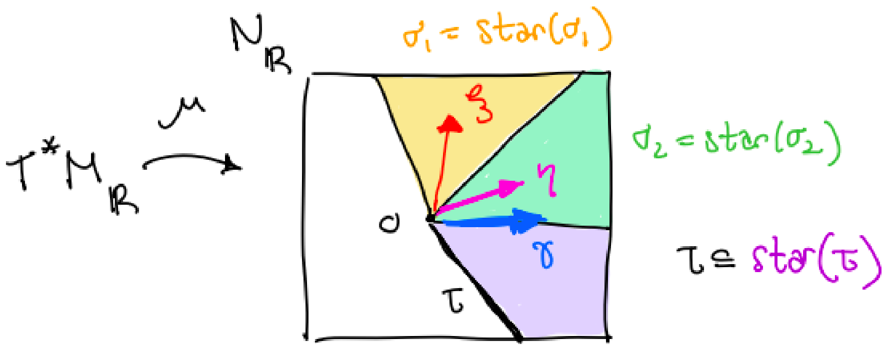


Figure 9.3: A fan Σ with eight cones. The images of $\xi, \eta, \gamma \in \Lambda$ are depicted as lying inside cones σ_1, σ_2, and a 1-d cone, respectively. If $\Lambda \subset T^*M_\mathbb{R}$ is a skeleton locally of FLTZ type Σ, then $\mathrm{Sh}_\Lambda(M)$ would have $P_\xi \perp P_\eta$ and $P_\xi \perp P_\gamma$, but not necessarily $P_\eta \perp P_\gamma$.

Corollary 9.21. *Suppose $\Sigma_1, \Sigma_2 \subseteq \Sigma$ are two subfans, and Λ_Σ is a skeleton locally of FTLZ type Σ. Then in the notation of Lemma 9.17, the following is a pullback square:*

$$\begin{tikzcd}
\mathrm{Sh}_{\Lambda_{\Sigma_1 \cup \Sigma_2}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Sh}_{\Lambda_{\Sigma_2}} \arrow[d] \\
\mathrm{Sh}_{\Lambda_{\Sigma_1}} \arrow[r] & \mathrm{Sh}_{\Lambda_{\Sigma_1 \cap \Sigma_2}}
\end{tikzcd}$$

Proof. Any pair of cones σ_1, σ_2 in $\Lambda_{\Sigma_1} \Delta \Lambda_{\Sigma_2}$ is non-adjacent inside $\Lambda_{\Sigma_1 \cup \Sigma_2}$, and therefore by Corollary 9.20 there is orthogonality of co-cores

$$P_\xi^{\Lambda_{\Sigma_1 \cup \Sigma_2}} \perp P_\eta^{\Lambda_{\Sigma_1 \cup \Sigma_2}}$$

for any $\xi \in \Lambda_{\sigma_1}^{\text{smooth}}$ and $\eta \in \Lambda_{\sigma_2}^{\text{smooth}}$. $\square$

As a consequence, we learn the following main result.

Theorem 9.22. *For a skeleton Λ_Σ locally of FLTZ type Σ, the cover by $\Lambda_\sigma := \mu^{-1}(\sigma) \cap \Lambda_\Sigma$ for $\sigma \in \Sigma_{max}$ is comonadic:*

$$\mathrm{Sh}_{\Lambda_\Sigma} \xrightarrow[\quad]{L^{\text{enh}}} {}_\Omega\,\mathrm{coMod}\left(\prod_{\sigma \in \Sigma_{max}} \mathrm{Sh}_{\Lambda_\sigma} \right)$$

In fact, the following is a limit diagram:

$$\mathrm{Sh}_{\Lambda_\Sigma} \xrightarrow{L} \left[\prod_{\sigma \in \Sigma_{max}} \mathrm{Sh}_{\Lambda_\sigma} \rightrightarrows \prod_{\sigma_1, \sigma_2 \in \Sigma_{max}} \mathrm{Sh}_{\Lambda_{\sigma_1 \cap \sigma_2}} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots \right]$$

Proof. Apply Theorem 9.15 to Corollary 9.21. □

This Theorem 9.22 was a motivating question for this thesis, in the sense that we did not know of a *direct* proof that the FLTZ skeleta with their covers from toric mirror symmetry were comonadic. Certainly, though, there was a beautiful proof through the FLTZ mirror equivalence: the equivalence linked such FLTZ covers to Zariski open covers on the mirror side, and so comonadicity for FLTZ skeleta and their covers was a consequence of the well-known comonadicity for Zariski open toric covers.

Remark 9.23. *The honest FLTZ skeleton Λ_Σ associated to a rational polyhedral fan Σ, together with its cover $\{\Lambda_\sigma\}_{\sigma\in\Sigma}$ as detailed [4], is in particular a skeleton locally of FLTZ type Σ with its canonical cover. Thus, Theorem 9.22 proves comonadicity and limit descent for the FLTZ set-up $(\Lambda_\Sigma, \{\Lambda_\sigma\}_{\sigma\in\Sigma})$ in a way that is independent of, though heavily inspired by, Zariski comonadicity as transported via the FLTZ mirror equivalence.*

9.5 Comonadicity criterion II: limit cobars

In this section, we return to something we saw in Chapter 2: the cobar characterization of comonadicity. Though simple and likely well known, we have not been able to locate a mention or proof of it in the literature. For the convenience of the reader, we give a brief reminder on the set-up, and then restate the result.

As a recollection, the result is an outgrowth of a basic observation: comonadicity of $L : \mathscr{C} \to \mathscr{D}$ implies that T-cobars on all objects $c \in \mathscr{C}$

$$c \longrightarrow Tc \rightrightarrows TTc \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

are limit diagrams. We ask: is it possible to use this to formulate a sort of converse?

First, we consecrate these important objects into named subcategories:

Definition 9.24. *Let $L : \mathscr{C} \to \mathscr{D} : R$ be an adjunction in Cat_∞, giving a monad $T \curvearrowright \mathscr{C}$.*

1. *Define*
$$\mathscr{C}^T_{\text{limit cobars}} \subseteq \mathscr{C}$$
to be the full subcategory on all objects $c \in \mathscr{C}$ whose T-cobars are limit diagrams.

2. *Define*
$$\mathscr{C}^T_{\text{absolute}} \subseteq \mathscr{C}^T_{\text{limit cobars}} \subseteq \mathscr{C}$$
to be the full subcategory on all objects $c \in \mathscr{C}$ whose T-cobars are absolute *limit diagrams.*

And now, the result:

Theorem 9.25 (Restatement of Theorem 2.26)**.** *Let $L : \mathscr{C} \leftrightarrow \mathscr{D} : R$ be an adjunction in* Cat_∞*, and assume that the right adjoint R^{recon} to L^{enh} exists (for example, if all categories were presentable). Then*

1. *L^{enh} is an embedding (with coreflector R^{recon}) if and only if*

$$\mathscr{C}^T_{\text{limit cobars}} = \mathscr{C}$$

2. *L is comonadic if and only if (a) $\mathscr{C}^T_{\text{limit cobars}} = \mathscr{C}$ and (b) R^{recon} is conservative.*

We summarize the basic result in the following diagram:

$$\begin{tikzcd}
R\mathscr{D} \arrow[rr, hook, shift left, "L^{\mathrm{enh}}|_{R\mathscr{D}}"] \arrow[d, hook] & & {}_{\Omega}\,\mathrm{coMod}^{\mathrm{cofree}}\,\mathscr{D} \arrow[ll, hook', shift left, "R^{\mathrm{recon}}|_{{}_{\Omega}\,\mathrm{coMod}^{\mathrm{cofree}}\,\mathscr{D}}"] \arrow[dd] \\
\mathscr{C}^T_{\text{limit cobars}} \arrow[d, hook] \arrow[rrd, hook, shift left, "L^{\mathrm{enh}}|_{\mathscr{C}^T_{\text{limit cobars}}}"] & & \\
\mathscr{C} \arrow[rr, "L^{\mathrm{enh}}"] \arrow[dr, bend right, "L"'] & & {}_{\Omega}\,\mathrm{coMod}\,\mathscr{D} \arrow[llu, shift left, "R^{\mathrm{recon}}"] \arrow[dl, bend left, "\mathrm{fgt}"] \\
 & \mathscr{D} &
\end{tikzcd}$$

We will now use this characterization of comonadicity to present two criteria. To state them, we first introduce a final definition:

Definition 9.26. *Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is* closed under tensors *if for any $S \in \mathrm{Set}$ and any $c \in \mathscr{C}$, the object*

$$c \otimes S := c^{\oplus S}$$

also exists in $\mathscr{C}$. In other words, arbitrary direct sums can be taken of any single object.

The subcategories of objects with limit and absolute limit T-cobars are closed under several operations:

Lemma 9.27. *Let $L : \mathscr{C} \to \mathscr{D} : R$ be an adjunction in $\mathrm{Pr}^{L,\mathrm{st}}$ of stable, presentable categories. The subcategories $\mathscr{C}^T_{\text{limit cobars}}$ and $\mathscr{C}^T_{\text{absolute}}$ enjoy the following properties:*

1. *$\mathscr{C}^T_{\text{limit cobars}}$ is closed under finite limits and retracts.*

2. *$\mathscr{C}^T_{\text{absolute}}$ is closed under finite limits, retracts, and tensors.*

Proof. For part (1), the closure under finite limits follows from the fact that T preserves finite limits, by virtue of being a functor in $\mathrm{Pr}^{L,\mathrm{st}}$. Closure under retracts follows from no hypotheses on the kind of category, for if $c \in \mathscr{C}^T_{\text{limit cobars}}$ and

$$c' \xhookrightarrow{s} c \xrightarrow{r} c'$$

a retract, the retraction r builds maps to c', rendering it the limit of its cobar.

For part (2), similar arguments, but now with arbitrary exact functors $E : \mathscr{C} \to \mathscr{E}$ thrown in, show closure of $\mathscr{C}^T_{\text{absolute}}$ under finite limits and retracts. Finally, for $S \in \text{Set}$, since

$$\mathscr{C} \xrightarrow{(-)^{\oplus S}} \mathscr{C}$$

is an exact functor, it transports absolute limit diagrams to absolute limit diagrams, and thus an absolute limit cobar

$$c \longrightarrow Tc \rightrightarrows TTc \rightrightarrows\!\!\!\!\!\rightarrow \cdots$$

is sent to an absolute limit diagram

$$c^{\oplus S} \longrightarrow (Tc)^{\oplus S} \rightrightarrows (TTc)^{\oplus S} \rightrightarrows\!\!\!\!\!\rightarrow \cdots$$

which, by the fact that T preserves colimits, is the cobar of $c^{\oplus \dim_k V}$:

$$c^{\oplus S} \longrightarrow T(c^{\oplus S}) \rightrightarrows TT(c^{\oplus S}) \rightrightarrows\!\!\!\!\!\rightarrow \cdots$$

Thus $c^{\oplus S} \in \mathscr{C}^T_{\text{absolute}}$. □

Remark 9.28. *In the language of Chapter 2, the above Lemma in particular says that both* $\mathscr{C}^T_{\text{limit cobars}}$ *and* $\mathscr{C}^T_{\text{absolute}}$ *are* thick *subcategories of* $\mathscr{C}$*.*

The purpose of the above result is to conclude the following:

Corollary 9.29. *The closure of $R\mathscr{D}$ under finite limits, tensors, and retracts is contained in* $\mathscr{C}^T_{\text{absolute limit}}$*:*

$$\langle R\mathscr{D}\rangle_{\triangle,\otimes,\tau} \subseteq \mathscr{C}^T_{\text{absolute limit}} \subseteq \mathscr{C}^T_{\text{limit cobars}} \subseteq \mathscr{C}$$

Proof. Since $R\mathscr{D} \subseteq \mathscr{C}^T_{\text{absolute}}$, the claim follows from the closure of the codomain under $\triangle, \otimes, \tau$ by Lemma 9.27. □

We gather everything into an easy corollary:

Corollary 9.30. *Let $L : \mathscr{C} \to \mathscr{D} : R$ be an adjunction in* $\text{Pr}^{L,\text{st}}$*, and suppose that*

1. $\langle R\mathscr{D}\rangle_{\triangle,\otimes,\tau} = \mathscr{C}$*, or* $\mathscr{C}^T_{\text{absolute}} = \mathscr{C}$*, and*

2. *R is conservative.*

Then L is comonadic.

Proof. Assumption (1) implies by Corollary 9.29 that $\mathscr{C}^T_{\text{limit cobars}} = \mathscr{C}$, and therefore by Theorem 2.26 ensures that L^{enh} is an embedding.

Thus it suffices to show that R^{recon} is conservative. By assumption (1), L is conservative, and therefore by assumption (2) $\Omega := LR$ is conservative, as is the functor cofree. Therefore, R^{recon} is conservative. Thus L^{enh} is an equivalence. □

We now come to our first criterion.

Criterion II(a)

Suppose

$$\begin{tikzcd} \mathscr{C} \arrow[r, "L_1"] \arrow[d, "L_2"] & \mathscr{C}/\mathscr{D}_1 \\ \mathscr{C}/\mathscr{D}_2 & \end{tikzcd}$$

are two localizations in $\mathrm{Pr}^{L,\mathrm{st}}$. One general idea for exploiting Corollary 9.30 is to observe the following:

Observation 9.31. *If it can be guaranteed that $\mathscr{D}_1 \subseteq \mathscr{C}^T_{\text{limit cobars}}$, then in fact $\mathscr{C}^T_{\text{limit cobars}} = \mathscr{C}$.*

This works because, for any $c \in \mathscr{C}$, the L_1-recollement

$$\underbrace{\widetilde{L}_1 \widetilde{R}_1 c}_{\in \mathscr{D}_1} \to c \to \underbrace{R_1 L_1 c}_{\in R_1 \mathscr{C}/\mathscr{D}_1} \xrightarrow{+1}$$

shows that c is therefore a finite colimit of objects in $\mathscr{C}^T_{\text{limit cobars}}$, meaning that $c \in \mathscr{C}^{\text{limit cobars}}$. Therefore $\mathscr{C} = \mathscr{C}^T_{\text{limit cobars}}$ by Corollary 9.29.

Here is one way of arranging this:

Proposition 9.32. *Suppose $\mathscr{D}_1 := \operatorname{Ker} L_1 = \langle d \rangle_\oplus$, i.e. $\mathscr{D}_1$ is generated by an exceptional object d. If $d \in \mathscr{C}^T_{\text{absolute}}$, then*

$$\mathscr{C} \xrightarrow{L := L_1 \boxplus L_2} \mathscr{C}/\mathscr{D}_1 \boxplus \mathscr{C}/\mathscr{D}_2$$

is comonadic.

Proof. Let R_1, R_2 denote the adjoints, and $R := R_1 \oplus R_2$ the right adjoint to L. Since R_i are embeddings, R is conservative. Thus by Corollary 9.30, it remains to show that $\mathscr{C}^T_{\text{absolute}} = \mathscr{C}$.

Certainly $R_i \mathscr{C}/\mathscr{D}_i \subseteq \mathscr{C}^T_{\text{absolute}}$, since the cobars on such objects are in fact split. Furthermore, since for any $S \in \mathrm{Set}$,

$$\mathscr{C} \xrightarrow{(-)^{\oplus S}} \mathscr{C}$$

is an exact functor, it follows by the hypothesis and Lemma 9.27 that $\mathscr{D}_1 \subseteq \mathscr{C}^T_{\text{absolute}}$ as well. Therefore, by the immediately preceding discussion, $\mathscr{C}^T_{\text{absolute}} = \mathscr{C}$. □

Remark 9.33. *We can view Corollary 9.2 as another way of instantiating the above strategy. Indeed, semiorthogonality guarantees that*

$$\mathscr{D}_1 \subseteq \langle R_1 \mathscr{C}/\mathscr{D}_1 \oplus R_2 \mathscr{C}/\mathscr{D}_2 \rangle_{\triangle,\tau},$$

which certainly lies inside $\mathscr{C}^T_{\text{absolute}}$.

Remark 9.34. *Note that it was important for the above proof idea that* d *belonged to* $\mathscr{C}^T_{\text{absolute}}$ *rather than to just* $\mathscr{C}^T_{\text{limit cobars}}$. *The reason is that we wish to also force the entire subcategory* $\langle d\rangle_{\text{colimit}}$ *to also be contained* $\mathscr{C}^T_{\text{limit cobars}}$, *which would involve trying to commute infinite colimits (of* d*) past infinite limits (cobars). This typically requires some strong hypotheses, such as the assumption* $d \in \mathscr{C}^T_{\text{absolute}}$ *that we made.*

Application

The above result grew out of a strangely fanatic attempt to see whether the following simple-looking collection of examples was comonadic:

Proposition 9.35. *For* $n \geq 1$, *fix a point* $0 \in S^n$ *and let* $\Lambda = \Lambda_1 \cup \Lambda_2$ *be the following cover of a Lagrangian skeleton inside* T^*S^n*:*

$$\Lambda_1 := T^*_0 S^n, \quad \Lambda_2 := T^*_{S^n} S^n$$

Then the functor

$$\mathrm{Sh}_\Lambda(S^n) \xrightarrow{L := L_1 \boxplus L_2} \mathrm{Sh}_{\Lambda_1}(S^n) \boxplus \mathrm{Sh}_{\Lambda_2}(S^n)$$

is comonadic.

Proof. We can identify the localizations and their right adjoints with

$$\begin{tikzcd} \mathrm{Sh}_\Lambda(S^n) \arrow[r, shift left, "i^*"] \arrow[d, shift right, "W"'] & \mathrm{Loc}(\mathrm{pt}) \arrow[l, hook, shift left, "i_*"] \\ \mathrm{Loc}(S^n) \arrow[u, hook, shift right, "\iota"'] & \end{tikzcd}$$

(see Figure 9.4 below for a more detailed diagram in the case $n = 1$). Let $U := S^n \setminus \mathrm{pt} \overset{j}{\hookrightarrow} S^n$ denote the open complement of $\mathrm{pt} \overset{i}{\hookrightarrow} S^n$. Then

$$\mathscr{D}_1 := \mathrm{Ker}(i^*) = \langle j_! k_U \rangle_\oplus$$

is generated by an exceptional object. The open-closed distinguished triangle

$$j_! k_U \to \underbrace{k_{S^n}}_{\in \mathscr{C}^T_{\text{absolute}}} \to \underbrace{i_* i^* k_{S^n}}_{\in \mathscr{C}^T_{\text{absolute}}} \xrightarrow{+1}$$

presents it as a finite limit of objects in $\mathscr{C}^T_{\text{absolute}}$, and therefore $j_! k_U \in \mathscr{C}^T_{\text{absolute}}$. $\square$

Example 9.36. *The* $n = 1$ *picture translates into algebraic geometry under FLTZ mirror symmetry, alias the coherent-constructible correspondence, as follows: consider the diagram*

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "p"] & \mathrm{pt} \\ U := \mathbb{P}^1 \setminus (\{0\} \cup \{\infty\}) \arrow[u, hook, "j"] & \end{tikzcd}$$

Then the localization functors matching the $n = 1$ diagram above are, at least up to an autofunctor,

$$\begin{tikzcd} \mathrm{QCoh}(\mathbb{P}^1) \arrow[r, shift left, "p_*"] \arrow[d, shift right, "j^*"'] & \mathrm{QCoh}(\mathrm{pt}) \arrow[l, shift left, "p^!"] \\ \mathrm{QCoh}(U) \arrow[u, shift right, "j_*"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sh}_\Lambda(S^1) \arrow[r, shift left, "i^*"] \arrow[d, shift right, "W"'] & \mathrm{Loc}(\mathrm{pt}) \arrow[l, shift left, "i_*"] \\ \mathrm{Loc}(S^1) \arrow[u, shift right, "\iota"'] & \end{tikzcd}$$

and the upshot is that $\mathrm{QCoh}(\mathbb{P}^1) \xrightarrow{p_* \boxplus j^*} \mathrm{QCoh}(\mathrm{pt}) \boxplus \mathrm{QCoh}(U)$ *is comonadic.*

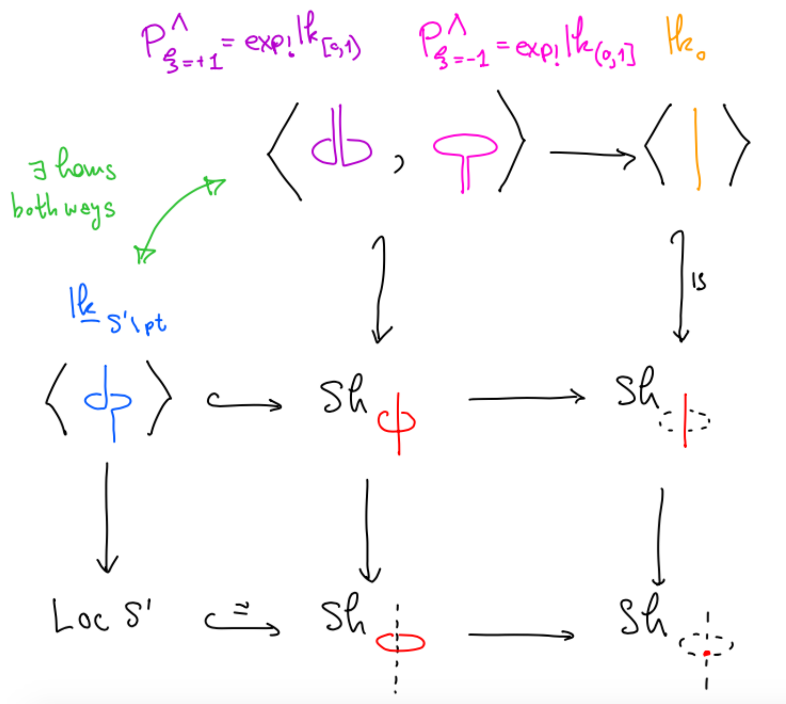


Figure 9.4: The diagram for $n = 1$, with $\Lambda_1 = 0_{S^1}$ and $\Lambda_2 = T_0^* S^1$. Note that $\mathrm{Sh}_{\Lambda_1 \cap \Lambda_2 = \mathrm{pt}}(S^1) \simeq 0$ since the intersection is properly isotropic. This example is inaccessible to the semiorthogonality Criterion I, but is covered by the more general Criterion II.

Remark 9.37. *These examples illustrate the fact that comonadicity does not require the pieces of a Lagrangian skeleton $\Lambda = \Lambda_1 \cup \Lambda_2$ to have the same homotopy type.*

Remark 9.38. *While these examples are comonadic, they fail the Beck-Chevalley condition, and thus do not give rise to limit descent. That is, the category $\mathrm{Sh}_{0_{S^n} \cup T_0^* S^n}(S^n)$ is not the pullback of the categories $\mathrm{Sh}_{0_{S^n}}(S^n)$ and $\mathrm{Sh}_{T_0^* S^n}(S^n)$ over their intersection category $\{0\}$.*

Criterion II(b)

We thank Germán Stefanich for the following observation:

Proposition 9.39. *Let $L : \mathscr{C} \to \mathscr{D} : R$ be an adjunction in $\mathrm{Pr}^{L,\mathrm{st}}$, and suppose that $\mathscr{C}$ is a smooth k-linear category over k. If*

1. *the compact objects $\mathscr{C}^0 \subseteq \mathscr{C}^T_{\mathrm{absolute}}$, and*
2. *R is conservative,*

then L is comonadic.

Proof. The fact that $\mathscr{C}$ is smooth implies that $\mathscr{C}$ has a compact generator c such that

$$\mathscr{C} = \langle c \rangle_{\triangle,\otimes,\tau}$$

The result now follows by Corollary 9.30. □

We would like to find a use for this criterion!

9.6 Questions for further research

We end the main part of this thesis by posing a couple questions:

Question 9.40. *If $\{\mathscr{C} \xrightarrow{L_i} \mathscr{C}/\mathscr{D}_i\}_{i\in I}$ is a collection of localizations in $\mathrm{Pr}^{L,\mathrm{st}}$ (perhaps even take compactly generated categories) which are jointly conservative, is it true that $L := \boxplus_{i\in I} L_i$ is comonadic, with no other hypotheses?*

If the above is not true, then here is a possible strategy for finding descent results:

Question 9.41. *Is it possible to find a t-structure on $\mathrm{Sh}_\Lambda(M)$ adapted to wrappings $\{L_i\}$ for which the Way 1 method of proof of Zariski descent could work to give comonadicity results?*

Best of luck to anyone out there who is interested!

9.7 Appendix: toric stuff

The purpose of this section is to prove Theorem 9.18. The argument is mainly a combination of Theorem 5.2 in [4] which provides a set of generators for the category Sh_Λ, and a formal result on building a co-representing object for a functor over a poset.

Shard sheaves

Let Σ be a simplicial rational fan in $N_{\mathbb{R}}$, where each k-dimensional cone σ is generated by k rays. Let $\Sigma^{(k)}$ be the set of all k-dimensional cone, in particular $\Sigma^{(1)}$ is the set of rays.

Next, given a collection of 'shift parameter', we construct a locally FLTZ-type skeleton. For each ray $\rho \in N_{\mathbb{R}}$ in Σ, we have linear hyperplane $\rho^{\perp} \subset M_{\mathbb{R}}$. For each $b \in M/\rho^{\perp}$, we have an affine hyperplane $b + \rho^{\perp} \subset M_{\mathbb{R}}$. The shift parameter is a collection of subsets S_ρ index by rays ρ, where $S_\rho M/\rho^{\perp}$ is a finite subset. The shift parameters for ρ determines the shift parameter for the cones.

For ρ a ray in Σ, the orthogonal complement

$$\rho^{\perp} = \{x \in M_{\mathbb{R}} \mid \langle v, x \rangle = 0, \quad \forall v \in \rho\}$$

is a hyperplane passing through origin. We further equip this hyperplane with a co-direction ρ. We also want to consider various affine translate of this hyperplane, e.g. $x + \rho^{\perp}$ for $x \in M_{\mathbb{R}}$. Such affine hyperplanes are indexed by $[x] \in M_{\mathbb{R}}/\rho^{\perp} = (\mathbb{R}\rho)^{\vee}$.

Let $\sigma \in \Sigma^{(k)}$ be a simplicial k-dimensional cone, and let $\sigma^{(1)}$ denote the set of rays it contains. Then

$$S_\sigma := \prod_{\rho \in \sigma^{(1)}} S_\rho \subset M_{\mathbb{R}}/\sigma^{\perp}.$$

If $\sigma = 0$, the 0-dimensional cone, we have $S_0 = M_{\mathbb{R}}/M_{\mathbb{R}} = \{0\}$.

Let $S_{max} = \{(s, \sigma) \mid \sigma \in \Sigma, s \in S_\sigma\}$ be the full collection of shift parameters. It is a poset under the relation

$$(s_1, \sigma_1) \le (s_2, \sigma_2) \Leftrightarrow s_1 + \sigma_1^{\vee} \supset s_2 + \sigma_2^{\vee}.$$

Let $S \subset S_{max}$ be any $\le$-*saturated* subset: that is, if $p \in S$ and $q \in S_{max}$ with $q \le p$, then $q \in S$.

Given any $\le$-saturated sub-poset S, we can define a skeleton

$$\Lambda_{\Sigma,S} = \bigsqcup_{(s,\sigma) \in S} (s + \sigma^{\perp}) \times \mathrm{Int}(\sigma) M_{\mathbb{R}} \times N_{\mathbb{R}} = T^* M_{\mathbb{R}}.$$

Following [4], we introduce the notion of "shard sheaves:"[2]

Definition 9.42. *For any $\sigma \in \Sigma$ and $s \in S_\sigma$, let $s + \sigma^{\perp}$ be a "shard," and let*

$$\Delta_{s,\sigma} := \mathbb{C}_{s+\sigma^{\vee}},$$

be the associated shard sheaf*, where*

$$\sigma^{\vee} := \{x \in M_{\mathbb{R}} \mid \langle x, y \rangle \ge 0, \quad \forall y \in \sigma\}$$

is the closed cone dual to σ.

[2] Our shard sheaf differs from [4] by Verdier duality.

The collection of shards forms a partially ordered set by inclusion. Namely, the following result holds, whose proof we leave to the reader:

Lemma 9.43 ([4]).

$$\hom(\Delta_{s_1,\sigma_1}, \Delta_{s_2,\sigma_2}) \cong \begin{cases} \mathbb{C} & (s_1,\sigma_1) \leq (s_2,\sigma_2) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 9.44. *The following singular support estimates hold:*

$$\mu(SS(\mathbb{C}_{s+\sigma^\vee})) = \sigma.$$

In particular, if $\hom(\Delta_{s_1,\sigma_1}, \Delta_{s_2,\sigma_2}) \neq 0$*, we have*

$$\mu(SS(\Delta_{s_1,\sigma_1})) \subset \mu(SS(\Delta_{s_2,\sigma_2})).$$

Proposition 9.45. *Any constructible sheaf in* $F \in \mathrm{Sh}_{\Lambda_{\Sigma,S}}$ *with finite-dimensional stalks can be represented by a chain complex of shard sheaves.*

Proof. This is essentially due to a devissage argument in [4], Theorem 5.2. For any $F \in \mathrm{Sh}_\Lambda(M)$, its Verdier dual $\mathbb{D}F \in \mathrm{Sh}_{-\Lambda}(M)$ is expressible as a finite complex of co-standard shard sheaves $\omega_{s+\mathrm{Int}(\sigma^\vee)}$. So apply $\mathbb{D}$ again to express F as a finite complex of standard shard sheaves $\mathbb{C}_{s+\sigma^\vee}$. □

Building a resolution of a co-core

We would now like to use the singular support estimates on shard sheaves from Lemma 9.44, together with the guarantee of a resolution by shard sheaves offered by Proposition 9.45, to find a resolution of a co-core P_Φ by shard sheaves, and use that to bound its singular support.

For the time being, we hop into a more general set-up. Let $(Q, \leq)$ be a finite poset, where we assume that if $v \leq w$ and $w \leq v$, then $v = w$. If $v \leq w$ and $v \neq w$, we write $v < w$. We abuse notation and use Q to denote the $\mathbb{C}$-linear category that it determines, with the set of objects Q, and with $\hom(v,w) = \mathbb{C}$ if and only if $v \leq w$.

Let $\mathrm{Mod}_Q = \mathrm{Fun}(Q^{\mathrm{op},\mathrm{Vect}\,\mathbb{C})}$ be the module category over Q. Then we have the Yoneda embedding

$$h : Q \to \mathrm{Mod}_Q, \quad v \mapsto \hom(-, v).$$

Let $bi\mathrm{Mod}_Q = \mathrm{Fun}(Q^{\mathrm{op}\times Q,\mathrm{Vect}\,\mathbb{C})}$ be the category of bimodules over Q. A bimodule B takes two input q_1, q_2, and output a vector space $B(q_1,q_2)$, covariant in q_2 and contravariant in q_1. The diagonal bimodule Δ_Q is defined as $\Delta_Q(q_1,q_2) = \hom(q_1,q_2)$. Given a Q-bimodule B, we can obtain a tautological functor

$$B : Q \to \mathrm{Mod}_Q, \quad v \mapsto B(-, v).$$

Given a pair of objects, $v, w \in Q$, we may define the following Yoneda bimodule:

$$v \otimes w^\vee : (q_1, q_2) \mapsto \hom(q_1, v) \otimes \hom(w, q_2).$$

Lemma 9.46. *The diagonal bimodule Δ_Q can be resolved as a finite chain complex of Yoneda bimodules.*

$$\left(\cdots \to \bigoplus_{q_1<q_1<q_2} q_0 \otimes q_2^\vee \to \bigoplus_{q_0<q_1} q_0 \otimes q_1^\vee \to \bigoplus_{q_0} q_0 \otimes q_0^\vee \right) \cong \Delta_Q \tag{9.2}$$

where

$$q_0 \otimes q_0^\vee \to \Delta_Q$$

is given by the composition $\hom(v, q_0) \otimes \hom(q_0, w) \to \hom(v, w)$, *and the maps within the chain complex is given by*

$$[q_0 < q_1 < \cdots < q_k] \to \sum_{i=0^k} (-1)^k [q_0 < \cdots < \hat{q}_i < \cdots < q_k]$$

where $[q_0 < q_1 < \cdots < q_k]$ *represent the term* $q_0 \otimes q_k^\vee$ *in the direct sum. and* $\hat{q}_i$ *means omit that term.*

Proof. It is clear that the above rule gives a chain complex. We now need to show that this chain complex is indeed acylic. We view this chain complex of bimodules as a chain complex of functors from Q to Mod_Q, and we verify that this is 'pointwise' acylic by applying it to each element $v \in Q$. To simplify notation, we will identity $q \in Q$ with its Yoneda image $\hom(-, q)$ in Mod_Q. We have

$$\begin{aligned} \Delta_Q(v) &= v \\ \bigoplus_{q_0<q_1<\cdots<q_k} q_0 \otimes q_k^\vee(v) &= \bigoplus_{q_0<q_1<\cdots<q_k\leq v} q_0 =: \oplus[q_0 < \cdots < q_k \leq v] \end{aligned}$$

where we introduce the notation $[q_0 < \cdots < q_k \leq v]$ for the summand q_0 indexed by $q_0 < q_1 < \cdots < q_k \leq v$.

We claim that there is a degree -1 map h,

$$h : [q_0 < \cdots < q_k \leq v] \to \begin{cases} (-1)^{k+1}[q_0 < \cdots < q_k < v \leq v] & \text{if } q_k < v \\ 0 & \text{if } q_k = v \end{cases}$$

such that $dh + hd = id$. We leave the verification for the interested reader. This finishes the proof of the lemma. □

Corollary 9.47. *Let Q be a finite poset, and let $\Phi : Q \to \mathrm{Vect}\,\mathbb{C}^{f.d.}$ be a functor to finite-dimensional vector spaces. Then, we have a resolution of Φ:*

$$\left(\cdots \to \bigoplus_{q_1<q_1<q_2} \Phi(q_0) \otimes q_2^\vee \to \bigoplus_{q_0<q_1} \Phi(q_0) \otimes q_1^\vee \to \bigoplus_{q_0} \Phi(q_0) \otimes q_0^\vee \right) \cong \Phi.$$

If we define P_Φ as the finite complex

$$P_\Phi := \left(\bigoplus_{q_0} \Phi(q_0)^\vee \otimes q_0 \to \bigoplus_{q_0 < q_1} \Phi(q_0)^\vee \otimes q_1 \to \cdots \right) \tag{9.3}$$

then we have

$$\mathrm{Hom}(P_\Phi, -) \cong \Phi.$$

Proof. The resolution of Φ is obtained by composing Φ with the identity functor $Q \to Q$, and resolving the identity functor using (9.2).

Since $\Phi(q_0)$ is assumed finite-dimensional for any $q_0 \in Q$, we get

$$\Phi(q_0) \otimes q_k^\vee = \mathrm{hom}(q_k, -) = \mathrm{hom}(\Phi(q_0)^\vee \otimes q_k, -).$$

Hence, we have

$$\begin{aligned} \Phi &\cong \cdots \to \bigoplus_{q_0 < q_1} \Phi(q_0) \otimes q_1^\vee \to \bigoplus_{q_0} \Phi(q_0) \otimes q_0^\vee \\ &\cong \cdots \to \mathrm{hom}(\bigoplus_{q_0 < q_1} \Phi(q_0)^\vee \otimes q_1, -) \to \mathrm{hom}(\bigoplus_{q_0} \Phi(q_0)^\vee \otimes q_0, -) \\ &\cong \mathrm{hom}(\cdots \leftarrow \bigoplus_{q_0 < q_1} \Phi(q_0)^\vee \otimes q_1 \leftarrow \bigoplus_{q_0} \Phi(q_0)^\vee \otimes q_0, -) \\ &\cong \mathrm{hom}(P_\Phi, -). \end{aligned}$$

This proves the last part of the claim. □

Now we are ready to prove Theorem 9.18.

Proof. (of Theorem 9.18) Let $(x, \xi) \in \Lambda^{\mathrm{smooth}}_{\Sigma,S}$, and $\xi \in \mathrm{Int}(\sigma)$. Let

$$\Phi = \Phi_{(x,\xi)} : \mathrm{Sh}_\Lambda(M) \to {}_k\,\mathrm{Mod}$$

be the microstalk functor, which is ind-extended from its restriction

$$\Phi : Q \to \mathrm{Perf}\, k$$

to the poset Q of shard sheaves. Then, we have resolution of P_Φ as in Eq (9.3). We have two properties:

1. if $p_0 \le p_1$ in Q, then $\mu(SS(p_0)) \subseteq \mu(SS(p_1))$;
2. for any $p \in Q$, if $\Phi(p) \neq 0$, then $\mu(SS(p)) \supset \sigma$.

Hence, for any k-chain $p_0 < \cdots < p_k$ in Q, if $\Phi(p_0) \neq 0$, then

$$\sigma \subset \mu(SS(p_0)) \subseteq \mu(SS(p_k)).$$

Since the μ-image of the singular support $\mu(SS(p))$ for any shard sheaf p is a cone in Σ, it means that $\mu(SS(P_\Phi)) \subset \mathrm{star}(\sigma)$. This concludes the proof. □

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