Title
Archimedean Copulas and Temporal Dependence

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Archimedean copulas and temporal dependence

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Abstract: We study the dependence properties of stationary Markov chains generated by Archimedean copulas. Under some simple regularity conditions, we show that regular variation of the Archimedean generator at zero and one implies geometric ergodicity of the associated Markov chain. We verify our assumptions for a range of Archimedean copulas used in applications.

Keywords and phrases: Archimedean copula; geometric ergodicity; Markov chain; mixing; regular variation; tail dependence.

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1 Introduction

During the last five years there has been considerable interest in the statistical and econometric literature with the possibility of modeling the dependence structure of stationary Markov chains using copulas. The allure of this approach is that it facilitates the separate consideration of the dependence structure of the chain, specified using a copula, and the invariant distribution of the chain. This advantage was first emphasized by Darsow, Nguyen and Olsen (1992). Chen and Fan (2006) introduced copula-based Markov models to the econometric literature. Following that contribution, a number of related papers have appeared, including Fentaw and Naik-Nimbalkar (2008), Gagliardini and Gouriéroux (2008), Bouyé and Salmon (2009), Chen, Koenker and Xiao (2009), Chen, Wu and Yi (2009), Ibragimov (2009), Ibragimov and Lentzas (2009), and Beare (2010).

A technical issue that has arisen in this literature is the following: how do the ergodic and mixing properties of a Markov chain relate to the copula describing the dependence between consecutive random variables? Chen and Fan (2006) suggested that Foster-Lyapunov drift conditions of the kind discussed in detail by Meyn and Tweedie (1993) could be used to verify suitable mixing conditions. This approach was used by Gagliardini and Gouriéroux (2008) to obtain conditions under which Markov chains generated by proportional hazard copulas are geometrically ergodic, and by Chen, Wu and Yi (2009) to prove that Markov chains generated by Clayton, Gumbel and t-copulas are geometrically ergodic. Beare (2010) proved that Markov chains generated by copulas with positive symmetric square integrable densities are geometrically ergodic, and commented on the relationship between maximal correlation and ρ-mixing.

In this paper we consider Markov chains generated by copulas that are strictly Archimedean. We identify conditions on the Archimedean generator that ensure the associated Markov chain is geometrically ergodic. These conditions are sufficiently general to encompass eleven of the parametric families of Archimedean copulas listed in Table 4.1 of Nelsen (2006). The key requirement we place upon the Archimedean generator is that it is regularly varying at zero and one. We prove geometric ergodicity by using the theory of regularly varying functions to verify a Foster-Lyapunov drift condition.

In a related contribution that may be of some independent interest, we provide an example of a parametric family of Archimedean copulas that generates a Markov chain that is ergodic but not geometrically ergodic. The key feature of this family is that the Archimedean generator is rapidly varying at zero. Our example is thus suggestive of a link between rapidly varying Archimedean generators and subgeometric rates of ergodicity.

The remainder of the paper is structured as follows. In Section 2 we define the notion of an

Archimedean copula and explain what it means for the generator of an Archimedean copula to vary regularly at zero and one. In Section 3 we present our geometric ergodic theorem for Archimedean copulas, and give a list of eleven parametric copula families to which the theorem applies. In Section 4 we state and discuss our example of an Archimedean copula that generates a subgeometric rate of ergodicity. Section 5 concludes. The proof of our main theorem is contained in the Appendix.

2 Regularly varying Archimedean generators

A copula is a bivariate probability distribution function on the unit square that has uniform marginal distribution functions. An Archimedean copula is a copula that can be defined in terms of a generator function φ in a way to be made precise shortly. Given a continuous, strictly decreasing function $\varphi : [0,1] \to [0,\infty]$ with $\varphi(1) = 0$, let the pseudo-inverse of φ, denoted $\varphi^{[-1]} : [0,\infty] \to [0,1]$, be defined by

$$
\begin{aligned}
\varphi^{[-1]}(u) &= \varphi^{-1}(u) \text{ for } u \in [0, \varphi(0)], \\
&= 0 \text{ for } u \in [\varphi(0), \infty].
\end{aligned}
$$

An Archimedean copula is defined as follows.

Definition 2.1. A copula $C : [0,1]^2 \to [0,1]$ is said to be Archimedean if there exists a continuous, strictly decreasing, convex function $\varphi : [0,1] \to [0,\infty]$ with $\varphi(1) = 0$ such that $C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$ for all $(u,v) \in [0,1]^2$. The function φ is referred to as the Archimedean generator of C. When $\varphi(0) = \infty$, C is said to be strictly Archimedean, and φ is said to be a strict Archimedean generator.

Many examples of and facts about Archimedean copulas may be found in Chapter 4 of Nelsen (2006). Some of those examples may also be found in Section 3 below.

Definition 2.1 states that a copula C is Archimedean if we can find a generator φ such that $C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$ for all $u, v \in [0,1]$. It can be shown (see e.g. Theorem 4.14 in Nelsen, 2006) that if φ is any function satisfying the conditions placed on the Archimedean generator in Definition 2.1, then $(u,v) \mapsto \varphi^{[-1]}(\varphi(u) + \varphi(v))$ is a well-defined copula. This result goes some way toward explaining the apparent popularity of Archimedean copulas in applied work: constructing an Archimedean copula is as simple as choosing a continuous, strictly decreasing, convex function on $[0,1]$ that vanishes at one.

We are concerned in this paper with copulas that are strictly Archimedean. For strict Archimedean generators φ, there is no distinction between the pseudo-inverse $\varphi^{[-1]}$ and ordinary inverse φ^{-1}. The behavior of a strict Archimedean generator φ near the origin turns out to be of critical importance in the study of various limiting phenomena. Juri and Wüthrich (2002) have shown that if φ varies regularly at zero, then the index of regular variation determines the lower tail dependence coefficient of the copula C. More strikingly, they have shown that, under mild regularity conditions, the extreme lower tail dependence copula associated with any Archimedean copula whose generator is regularly varying at zero is a member of the family of Clayton copulas, with the Clayton parameter determined by the index of regular variation of the generator. Complementary results pertaining to upper tail dependence were shown by Juri and Wüthrich (2003) to depend critically on the behavior of φ near one; see also Charpentier and Segers (2007). In this paper we will link regular variation of φ at zero and one to the property of geometric ergodicity in Markov chains whose dependence is characterized by the copula C.

Before we define the notions of regular variation at zero and one, it will be helpful to define the more standard notion of regular variation at infinity. Let f denote a positive measurable real valued function defined on $(1, \infty)$.

Definition 2.2. The function f is said to be regularly varying at infinity with index $\varsigma \in \mathbb{R}$, written $f \in \mathscr{R}_\varsigma(\infty)$, if $f(sx)/f(x) \to s^\varsigma$ as $x \to \infty$, for all $s \in (0, \infty)$. If $f \in \mathscr{R}_0(\infty)$, then f is said to be slowly varying at infinity.

Our choice of $(1, \infty)$ as the domain of f is not, of course, entirely necessary; what matters is that f is defined in a neighborhood of infinity. The property of regular variation is determined solely by the behavior of $f(x)$ as $x \to \infty$. But the domain $(1, \infty)$ is convenient for our purposes.

An extensive treatment of the theory of regular variation has been provided by Bingham, Goldie and Teugels (1987), henceforth referred to as BGT. Here we will require only the most basic elements of this theory. Intuitively, a function is regularly varying at infinity if it behaves like a polynomial in x for large x. More formally, any $f \in \mathscr{R}_\varsigma(\infty)$ satisfies the decomposition $f(x) = x^\varsigma \ell(x)$ for all x, for some $\ell \in \mathscr{R}_0(\infty)$. This decomposition may be proved by noting that $x^{-\varsigma} f(x)$ is a slowly varying function of x at infinity. Functions that are slowly varying at infinity may be viewed as asymptotically akin to a constant. Critically, the logarithm function falls into this category.

The definition of regular variation at zero is a simple adaptation of the definition of regular variation at infinity. Let φ denote a positive measurable real valued function defined on $(0, 1)$. Alternatively, φ may be a nonnegative measurable extended real valued function defined on $[0, 1]$, provided that it is positive and finite on $(0, 1)$.

Definition 2.3. The function φ is said to be regularly varying at zero with index $\eta \in \mathbb{R}$, written $\varphi \in \mathscr{R}_\eta(0)$, if $\varphi(su)/\varphi(u) \to s^\eta$ as $u \downarrow 0$, for all $s \in (0, \infty)$. If $\varphi \in \mathscr{R}_0(0)$, then φ is said to be slowly varying at zero.

We must also define what it means for φ to be regularly varying at one.

Definition 2.4. The function φ is said to be regularly varying at one with index $\zeta \in \mathbb{R}$, written $\varphi \in \mathscr{R}_\zeta(1)$, if $\varphi(1-su)/\varphi(1-u) \to s^\zeta$ as $u \downarrow 0$, for all $s \in (0, \infty)$. If $\varphi \in \mathscr{R}_0(1)$, then φ is said to be slowly varying at one.

The definitions of regular variation at zero and one derive directly from the definition of regular variation at infinity. Specifically, $\varphi \in \mathscr{R}_\eta(0)$ if and only if the map $x \mapsto \varphi(1/x)$ is in $\mathscr{R}_{-\eta}(\infty)$, while $\varphi \in \mathscr{R}_\zeta(1)$ if and only if the map $x \mapsto \varphi(1-1/x)$ is in $\mathscr{R}_{-\zeta}(\infty)$. We may also decompose functions that are regularly varying at zero or one into the product of polynomials and slowly varying functions, as we did earlier: we have $\varphi \in \mathscr{R}_\eta(0)$ if and only if $\varphi(u) = u^\eta \ell(u)$ for all $u \in (0,1)$ and some $\ell \in \mathscr{R}_0(0)$, and similarly $\varphi \in \mathscr{R}_\zeta(1)$ if and only if $\varphi(u) = (1-u)^\zeta \ell(u)$ for all $u \in (0,1)$ and some $\ell \in \mathscr{R}_0(1)$. In this sense, functions in $\mathscr{R}_\eta(0)$ behave like $u \mapsto u^\eta$ near zero, while functions in $\mathscr{R}_\zeta(1)$ behave like $u \mapsto (1-u)^\zeta$ near one.

If an Archimedean generator φ is regularly varying at zero and/or one, the indices of regular variation must fall within a specified range. When $\varphi \in \mathscr{R}_\eta(0)$, we must have $\eta \leq 0$, since otherwise φ would vanish at zero, violating the assumption of strict monotonicity. And when $\varphi \in \mathscr{R}_\zeta(1)$, we must have $\zeta \geq 1$, since otherwise φ would fail to be convex (if $0 < \zeta < 1$) or fail to vanish at one (if $\zeta < 0$), or fail at least one of these two conditions (if $\zeta = 0$). Theorem 4.4 of Juri and Wüthrich (2003) shows how the indices of regular variation η and ζ determine the upper and lower tail dependence coefficients of the Archimedean copula C generated by φ. When $\varphi \in \mathscr{R}_\eta(0)$, the lower tail dependence coefficient of C is equal to $2^{1/\eta}$ (for $\eta < 0$) or equal to 0 (for $\eta = 0$). And when $\varphi \in \mathscr{R}_\zeta(1)$, the upper tail dependence coefficient of C is equal to $2 - 2^{1/\zeta}$. For a definition and further discussion of tail dependence coefficients, see Section 5.4 in Nelsen (2006).

It turns out that many Archimedean copulas used in practice have generators that are regularly varying at zero and one. Examples are provided in the following section.

3 Geometric ergodic theorem for Archimedean copulas

Let $\{U_t : t \in \mathbb{Z}\}$ be a stationary Markov chain defined on a probability space $(\Omega, \mathscr{F}, P)$. We assume that the invariant distribution of the chain is uniform on $(0,1)$; that is, $U_t \sim U(0,1)$ for each $t \in \mathbb{Z}$. Let $C : [0,1]^2 \to [0,1]$ denote the joint distribution function of (U_0, U_1). Since $\{U_t\}$ is stationary with invariant distribution $U(0,1)$, the joint distribution function C is the unique copula for each consecutive pair (U_t, U_{t+1}), $t \in \mathbb{Z}$. The Markov property ensures that the entire joint distribution of $\{U_t\}$ is uniquely determined by C.

We are concerned in this paper with identifying conditions on C that are sufficient for $\{U_t\}$ to be geometrically ergodic. Let $\mathscr{B}$ denote the σ-field of Borel subsets of $(0,1)$.

Definition 3.1. The stationary Markov chain $\{U_t : t \in \mathbb{Z}\}$ is said to be geometrically ergodic if, for a.e. $u \in (0,1)$, there exists a real number $r > 1$ such that

$$\sum_{k=1}^{\infty} r^k \sup_{B \in \mathscr{B}} |P(U_k \in B | U_0 = u) - P(U_k \in B)| < \infty.$$

Remark 3.1. Definition 3.1 is a minor variation on Definition 15.7 of Meyn and Tweedie (1993). We have dropped those authors' requirement that $\{U_t\}$ be positive Harris recurrent, which is, in any case, implied by Assumption 3.1 below. We have also departed from Meyn and Tweedie's definition of geometric ergodicity by requiring that the summability condition in Definition 3.1 hold only for a.e. $u \in (0,1)$, rather than all $u \in (0,1)$. For our purposes, the distinction is immaterial.

Remark 3.2. Theorem 21.19 of Bradley (2007) implies that $\{U_t\}$ is geometrically ergodic if and only if the β-mixing coefficients for $\{U_t\}$ decay to zero at a rate that is exponential or faster. In an earlier paper (Beare, 2010) we identified conditions on C under which $\{U_t\}$ is β-mixing at such a rate. A key condition was that C is absolutely continuous with square integrable density. Most Archimedean copulas used in applications do not satisfy this condition – indeed, no absolutely continuous copula exhibiting positive tail dependence admits a square integrable density, by Theorem 3.3 in Beare (2010) – so our previous result is of limited applicability in the present context.

Remark 3.3. It is clear from Definition 3.1 that if $\{U_t\}$ is geometrically ergodic then, for any Borel measurable $h : (0,1) \to \mathbb{R}$, $\{h(U_t)\}$ is also geometrically ergodic. This property makes our assumption that $\{U_t\}$ has invariant distribution $U(0,1)$ innocuous. Suppose we have a stationary Markov chain $\{X_t\}$ with continuous invariant distribution function F, and with C the unique copula for (X_0, X_1); that is, C satisfying $P(X_0 \le x_0, X_1 \le x_1) = C(F(x_0), F(x_1))$ for all $x_0, x_1 \in \mathbb{R}$. The entire distribution of this chain is identical to the distribution of the chain $\{Q(U_t)\}$, where $Q : (0,1) \to \mathbb{R}$ is the quantile function corresponding to F. Thus, if C is such that $\{U_t\}$ is geometrically ergodic, then $\{X_t\}$ must also be geometrically ergodic. This conclusion remains true even if F is not continuous, though in this case C may be one of many copulas for (X_0, X_1)

Geometric ergodicity of $\{U_t\}$ will be established under the following assumption on C.

Assumption 3.1. The copula C is strictly Archimedean, with strict Archimedean generator φ satisfying the following conditions.

(i) $\varphi \in \mathscr{R}_\eta(0)$ for some $\eta \in (-\infty, 0]$, and $\varphi \in \mathscr{R}_\zeta(1)$ for some $\zeta \in [1, \infty)$.

(ii) φ is twice continuously differentiable on $(0, 1)$.

(iii) φ'' is monotone in a right-neighborhood of zero and in a left-neighborhood of one.

(iv) φ'' is strictly positive on $(0, 1)$.

(v) If $\eta = 0$, then

- **(a)** $-\varphi' \in \mathscr{R}_{-1}(0)$, and
- **(b)** $u\varphi'(u)$ is bounded away from zero for u in a right-neighborhood of zero.

(vi) If $\zeta = 1$, then φ' and φ'' are bounded away from zero in a left-neighborhood of one.

We shall shortly provide a number of examples of Archimedean copulas satisfying Assumption 3.1. First, we make the following remarks.

Remark 3.4. Theorem 1 of Genest and MacKay (1986b) states that an Archimedean copula C with twice continuously differentiable generator φ is absolutely continuous if and only if $\lim_{u\downarrow 0} \varphi(u)/\varphi'(u) = 0$. Under Assumptions 3.1(i),(ii), since $\varphi \in \mathscr{R}_\eta(0)$ and φ' is monotone, the Monotone Density Theorem (see Theorem 1.7.2 in BGT) implies that $-\varphi' \in \mathscr{R}_{\eta-1}(0)$, provided that $\eta < 0$. When $\eta = 0$, the same is true by Assumption 3.1(v)(i). With $\varphi \in \mathscr{R}_\eta(0)$ and $-\varphi' \in \mathscr{R}_{\eta-1}(0)$, it is easy to show that $-\varphi(\cdot)/\varphi'(\cdot) \in \mathscr{R}_1(0)$, from which it follows that $\lim_{u\downarrow 0} \varphi(u)/\varphi'(u) = 0$. Thus, under Assumption 3.1, C is absolutely continuous, and we may obtain its density c on $(0, 1)^2$ by differentiation:

$$c(u, v) = -\frac{\varphi''(C(u, v))\varphi'(u)\varphi'(v)}{\varphi'(C(u, v))^3} \text{ for } (u, v) \in (0, 1)^2. \tag{3.1}$$

As noted by Genest and MacKay (1986b), $c(u, v) > 0$ for all u, v such that $\varphi(u)+\varphi(v) < \varphi(0)$. C is strictly Archimedean under Assumption 3.1, so $\varphi(0) = \infty$, and $c > 0$ on $(0, 1)^2$.

Remark 3.5. As noted in Remark 3.4, the Monotone Density Theorem ensures that $-\varphi' \in \mathscr{R}_{\eta-1}(0)$ when $\eta < 0$. The point of Assumption 3.1(v)(a) is to ensure that this is also true when $\eta = 0$. In fact, we are unaware of any example of a strict Archimedean generator in $\mathscr{R}_0(0)$ that is twice continuously differentiable and violates Assumption 3.1(v)(a), and must confess there is some possibility that Assumption 3.1(v)(a) is redundant. Charpentier and Segers (2007) provide an example of a continuously differentiable strict Archimedean generator φ such that $\varphi \in \mathscr{R}_0(0)$ and $-\varphi' \notin \mathscr{R}_{-1}(0)$; however, this generator is not twice differentiable and so would not satisfy Assumption 3.1(ii) above.

Remark 3.6. Assumptions 3.1(v)(b) and 3.1(vi) are not always satisfied. We shall provide an example of a copula that violates both of them, while satisfying the remaining parts of Assumption 3.1. Consider the strict Archimedean generator

$$\varphi(u) = \log(1 - \log u).$$

The Archimedean copula corresponding to this generator is a member of the so-called Gumbel-Barnett family of copulas, which forms the ninth entry in Table 4.1 of Nelsen (2006); we have set the parameter value θ equal to one. It is easily verified that φ satisfies Assumptions 3.1(i) through 3.1(v)(a) with $\eta = 0$ and $\zeta = 1$. Differentiating φ, we obtain

$$\varphi'(u) = \frac{-1}{u(1 - \log u)},$$

implying that $\lim_{u \downarrow 0} u\varphi'(u) = 0$. Thus, Assumption 3.1(v)(b) is violated. Differentiating φ again, we find that

$$\varphi''(u) = \frac{-\log u}{u^2(1 - \log u)^2}.$$

We can see that $\lim_{u \uparrow 1} \varphi''(u) = 0$, and so Assumption 3.1(vi) is also violated.

Remark 3.6 notwithstanding, there are many well-known families of Archimedean copulas that satisfy Assumption 3.1. We shall provide eleven examples of such families, each of which may be found in Table 4.1 of Nelsen (2006). Sometimes we must restrict the parameter space given by Nelsen to ensure that Assumption 3.1 is satisfied. To conserve space, we do not give details of precisely how Assumption 3.1 is verified in each example. Typically, verification can be achieved by differentiating φ twice and perhaps applying l'Hôpital's rule or a Taylor expansion where appropriate.

Example 3.1. Consider the family of Archimedean generators

$$\varphi(u) = \frac{1}{\theta}\left(u^{-\theta} - 1\right), \, \theta \in [-1, \infty) \setminus \{0\}.$$

The corresponding family of copulas is known as the Clayton family, and forms the first entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = -\theta$ and $\zeta = 1$, provided that $\theta \in (0, \infty)$. When $\theta \in [-1, 0)$, $\varphi(0) = -1/\theta$, and so φ is not strict and Assumption 3.1 does not hold.

Example 3.2. Consider the family of Archimedean generators

$$\varphi(u) = \log \frac{1-\theta(1-u)}{u}, \, \theta \in [-1, 1).$$

The corresponding family of copulas is known as the Ali-Mikhail-Haq family, and forms the third entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = 1$, provided that $\theta \in (-1, 1)$. When $\theta = -1$, $\lim_{u \uparrow 1} \varphi''(u) = 0$, and so Assumption 3.1(vi) is violated.

Example 3.3. Consider the family of Archimedean generators

$$\varphi(u) = (-\log u)^{\theta}, \, \theta \in [1, \infty).$$

The corresponding family of copulas is known as the Gumbel family, or Gumbel-Hougaard family, and forms the fourth entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = \theta$.

Example 3.4. Consider the family of Archimedean generators

$$\varphi(u) = -\log u \frac{\mathrm{e}^{-\theta u} - 1}{\mathrm{e}^{-\theta} - 1}, \, \theta \in (-\infty, \infty) \setminus \{0\}.$$

The corresponding family of copulas is known as the Frank family, and forms the fifth entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = 1$.

Example 3.5. Consider the family of Archimedean generators

$$\varphi(u) = -\log\left(1 - (1-u)^{\theta}\right), \, \theta \in [1, \infty).$$

The corresponding family of copulas is known as the Joe family, and forms the sixth entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = \theta$

Example 3.6. Consider the family of Archimedean generators

$$\varphi(u) = \log\left(2u^{-\theta} - 1\right), \, \theta \in (0, 1].$$

The corresponding family of copulas forms the tenth entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = 1$, provided that $\theta \in (0, 1)$. When $\theta = 1$, $\lim_{u \uparrow 1} \varphi''(u) = 0$, and so Assumption 3.1(vi) is violated.

Example 3.7. Consider the family of Archimedean generators

$$\varphi(u) = \left(u^{-1} - 1\right)^{\theta}, \theta \in [1, \infty).$$

The corresponding family of copulas forms the twelfth entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = -\theta$ and $\zeta = \theta$.

Example 3.8. Consider the family of Archimedean generators

$$\varphi(u) = (1 - \log u)^{\theta} - 1, \theta \in (0, \infty).$$

The corresponding family of copulas forms the thirteenth entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = 1$, provided that $\theta \in [1, \infty)$. When $\theta \in (0, 1)$, $\lim_{u \downarrow 0} u\varphi'(u) = 0$, and so Assumption 3.1(v)(b) is violated.

Example 3.9. Consider the family of Archimedean generators

$$\varphi(u) = \left(u^{-1/\theta} - 1\right)^{\theta}, \theta \in [1, \infty).$$

The corresponding family of copulas forms the fourteenth entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = -1$ and $\zeta = \theta$.

Example 3.10. Consider the family of Archimedean generators

$$\varphi(u) = \left(\theta u^{-1} + 1\right)(1 - u), \theta \in [0, \infty).$$

The corresponding family of copulas forms the sixteenth entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = -1$ and $\zeta = 1$, provided that $\theta \in (0, \infty)$. When $\theta = 0$, $\varphi(0) = 1$, and so φ is not strict and Assumption 3.1 does not hold.

Example 3.11. Consider the family of Archimedean generators

$$\varphi(u) = -\log \frac{(1+u)^{-\theta} - 1}{2^{-\theta} - 1}, \theta \in (-\infty, \infty) \setminus \{0\}.$$

The corresponding family of copulas forms the seventeenth entry in Table 4.1 of Nelsen (2006). This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = 1$.

We have seen that many families of Archimedean copulas satisfy Assumption 3.1 over much or all of their parameter space. The following theorem, which is the main result of the paper, states that Archimedean copulas satisfying Assumption 3.1 generate geometrically ergodic Markov chains. The proof is deferred to the Appendix.

Theorem 3.1. *Suppose $\{U_t : t \in \mathbb{Z}\}$ is a stationary Markov chain whose invariant distribution is uniform on $(0, 1)$. Let C denote the joint distribution function of (U_0, U_1). If C satisfies Assumption 3.1, then $\{U_t : t \in \mathbb{Z}\}$ is geometrically ergodic.*

Remark 3.7. Theorem 3.1 shows that the eleven families of Archimedean copulas listed in Examples 3.1-3.11 generate geometrically ergodic Markov chains over the stated parameter ranges. For three of those families, this result was known already. Theorem 2.1 of Chen, Wu and Yi (2009) established geometric ergodicity for the Clayton and Gumbel families, and Theorem 3.1 of Beare (2010) established geometric ergodicity for the Frank family. To the best of our knowledge, geometric ergodicity for the remaining eight families has not been previously established.

Remark 3.8. Theorem 3.1 is an application of the Geometric Ergodic Theorem, discussed in detail in the text of Meyn and Tweedie (1993). The proof involves verifying that the one-step dependence characterized by C satisfies a Foster-Lyapunov drift condition. Chen, Wu and Yi (2009) used precisely this approach to prove geometric ergodicity for the Clayton and Gumbel families. Our proof is based loosely on theirs, though the conditions we impose on C are much weaker.

4 An example of subgeometric ergodicity

In the previous section we saw that many families of Archimedean copulas can be used to generate Markov chains that are geometrically ergodic. In this section we identify a family of Archimedean copulas for which the associated rate of ergodicity is subgeometric.

Example 4.1. Consider the family of Archimedean generators

$$\varphi(u) = \exp(u^{-\theta}) - \mathrm{e},\ \theta \in (0, \infty).$$

The corresponding family of copulas forms the twentieth entry in Table 4.1 of Nelsen (2006). Clearly φ is not regularly varying at zero, and so Theorem 3.1 cannot be applied. In fact, $\log \varphi \in \mathscr{R}_{-\theta}(0)$, and φ is said to be rapidly varying at zero; see Section 2.4 in BGT for a formal definition of rapid variation, and further discussion. We will show that, when $\theta > 1$, a Markov chain generated by φ is not geometrically ergodic.

Let $\{U_t : t \in \mathbb{Z}\}$ be a stationary Markov chain with the joint distribution of U_0 and U_1 given by C, the Archimedean copula generated by φ. Geometric ergodicity of $\{U_t\}$ is equivalent to exponential decay of the β-mixing coefficients associated with $\{U_t\}$; see e.g. Theorem 21.19 in Bradley (2007). The β-mixing coefficients for $\{U_t\}$ are bounded from below by the corresponding α-mixing coefficients. To disprove geometric ergodicity, it therefore suffices to demonstrate that the α-mixing coefficients for $\{U_t\}$ do not decay to zero at an exponential rate. In fact, we will show that $\liminf_{k\to\infty} k\alpha_k \geq 1$, demonstrating that the decay rate of α_k is no faster than k^{-1}.

The kth α-mixing coefficient α_k for $\{U_t\}$ is defined as the supremum of $|P(A \cap B) - P(A)P(B)|$ over all $A \in \sigma(U_t : t \leq 0)$ and $B \in \sigma(U_t : t \geq k)$. Therefore, for $k \in \mathbb{N}$, we have

$$\alpha_k \geq |P(U_0 \leq k^{-1}, U_k \leq k^{-1}) - P(U_0 \leq k^{-1})P(U_k \leq k^{-1})| = |C_k(k^{-1}, k^{-1}) - k^{-2}|, \quad (4.1)$$

where C_k denotes the joint distribution function of U_0 and U_k. By elementary arguments,

$$\begin{aligned} C_k(k^{-1}, k^{-1}) &\geq P(U_0 \leq k^{-1}, U_{k-1} \leq k^{-1}, U_k \leq k^{-1}) \\ &\geq P(U_0 \leq k^{-1}, U_{k-1} \leq k^{-1}) + P(U_{k-1} \leq k^{-1}, U_k \leq k^{-1}) - P(U_{k-1} \leq k^{-1}) \\ &= C_{k-1}(k^{-1}, k^{-1}) + C(k^{-1}, k^{-1}) - k^{-1}. \end{aligned}$$

On recursion, we obtain

$$C_k(k^{-1}, k^{-1}) \geq kC(k^{-1}, k^{-1}) - 1 + k^{-1}. \quad (4.2)$$

Convexity of φ implies that

$$\varphi\left(k^{-1} + \frac{\varphi(k^{-1})}{\varphi'(k^{-1})}\right) \geq \varphi(k^{-1}) + \varphi'(k^{-1}) \cdot \frac{\varphi(k^{-1})}{\varphi'(k^{-1})} = 2\varphi(k^{-1}), \quad (4.3)$$

provided of course that $k^{-1} + \varphi(k^{-1})/\varphi'(k^{-1}) > 0$. Since $\log\varphi \in \mathscr{R}_{-\theta}(0)$ and $\log\varphi$ is convex, the Monotone Density Theorem implies that $-\varphi'(\cdot)/\varphi(\cdot) \in \mathscr{R}_{-\theta-1}(0)$. It follows that $k^\gamma\varphi(k^{-1})/\varphi'(k^{-1}) \to 0$ as $k \to \infty$ for any $\gamma < \theta+1$, and so we have $k^{-1}+\varphi(k^{-1})/\varphi'(k^{-1}) > 0$ for all k sufficiently large. From (4.3) we obtain

$$C(k^{-1}, k^{-1}) = \varphi^{-1}\left(2\varphi(k^{-1})\right) \geq k^{-1} + \frac{\varphi(k^{-1})}{\varphi'(k^{-1})} \quad (4.4)$$

for all k sufficiently large. Combining (4.4) with (4.2) yields

$$C_k(k^{-1}, k^{-1}) \geq k^{-1} + \frac{k\varphi(k^{-1})}{\varphi'(k^{-1})}$$

for all k sufficiently large. Recalling that $k^\gamma\varphi(k^{-1})/\varphi'(k^{-1}) = o(1)$ for any $\gamma < \theta + 1$, and our assumption that $\theta > 1$, we deduce that $C_k(k^{-1}, k^{-1}) \geq k^{-1} + o(k^{-1})$. In view of (4.1), this proves that $\liminf_{k\to\infty} k\alpha_k \geq 1$.

We conclude this section with some remarks on Example 4.1.

Remark 4.1. Inspection of our demonstration that $\{U_t\}$ is not geometrically ergodic reveals that only three features of the generator φ were essential to our argument. They are: (1) φ is differentiable; (2) $\log\varphi \in \mathscr{R}_\eta(0)$ for some $\eta < -1$; (3) $\log\varphi$ is convex. In fact, property (3) was used only to justify the application of the Monotone Density Theorem, and thus need hold only locally to zero. Our argument thus demonstrates that any reasonably behaved Archimedean generator that diverges sufficiently rapidly at zero will generate a Markov chain that fails to be geometrically ergodic.

Remark 4.2. Perhaps the best known nontrivial example of a Markov chain that is not geometrically ergodic is the stationary linear process $X_t = \sum_{j=0}^{\infty} \varepsilon_{t-j}$, $t \in \mathbb{Z}$, formed from independent innovations ε_t, $t \in \mathbb{Z}$, that are each equal to 0 with probability 1/2 and 1/2 with probability 1/2. It is known (see e.g. Andrews, 1984) that this process is not α-mixing, and in fact satisfies $\alpha_k = 1/4$ for all k. As noted in Remark 4.2 in Beare (2010), the unique copula for (X_0, X_1) is absolutely singular with respect to Lebesgue measure on the unit square. In contrast, the Archimedean copula in Example 4.1 is absolutely continuous with respect to Lebesgue measure on the unit square, and admits a density that is positive almost everywhere. Using, for instance, Theorems 21.3 and 21.5 in Bradley (2007), one may show that this property implies that $\{U_t\}$ is ergodic and β-mixing. The rates of ergodicity and β-mixing are, however, subexponential.

Remark 4.3. It is not clear from our discussion whether geometric ergodicity obtains in Example 4.1 when $\theta \in (0, 1)$. What we do know in this case is that $\{U_t\}$ fails to be ρ-mixing, and has $\rho_k = 1$ for all k. See Beare (2010) for further discussion of ρ-mixing in copula-based Markov models. The failure of ρ-mixing is a consequence of the fact that the copula in Example 4.1 exhibits perfect lower tail dependence, which is itself a consequence of the rapid variation of φ at zero; see Theorem 3.9 of Juri and Wüthrich (2002).

Remark 4.4. Given that the rate of α-mixing in Example 4.1 has been shown to be no faster than k^{-1}, it is tempting to describe $\{U_t\}$ as exhibiting long memory of some form. Ibragimov and Lentzas (2009) considered the possibility that copulas may be used to generate "long memory-like" behavior in Markov chains. Nevertheless, the traditional definition of long memory concerns the summability of autocovariances, and it is not clear to us that the nonsummability of α-mixing coefficients implies that the autocovariances of $\{U_t\}$, or indeed of $\{U_t^p\}$ for some power p, are themselves nonsummable. We therefore refrain from suggesting a connection between long memory and rapid variation of φ at zero.

5 Conclusion

In this paper we have identified conditions under which a Markov chain whose dependence is characterized by an Archimedean copula will be geometrically ergodic. These conditions are sufficiently general to encompass eleven families of Archimedean copulas described in the monograph of Nelsen (2006), over a range of possible parameter values. Nevertheless, they are far from necessary, and substantial scope exists for generalizing our main result. In particular, it would be useful to weaken our conditions to allow for Archimedean copulas that are not strict, as several of the families listed by Nelson (2006) are of this kind. We leave this task to future research.

A Appendix: Proof of Theorem 3.1

In our proof of Theorem 3.1 we shall employ five supplementary lemmas. Proving these lemmas requires multiple applications of the Monotone Density Theorem and Potter's Theorem. For a statement of these results, refer to Theorem 1.7.2 and Theorem 1.5.6 in BGT.

Lemma A.1. *Under Assumption 3.1, for $p \in [0, 1 - 1/\eta)$, with $1/\eta := -\infty$ when $\eta = 0$, we have*

$$\lim_{u \downarrow 0} \int_0^1 \frac{\varphi''(su)}{\varphi''(u)} \left(\frac{\varphi'(su)}{\varphi'(u)} \right)^{-2} \left(\frac{\varphi(su)}{\varphi(u)} - 1 \right)^p ds = \int_0^1 s^{-\eta}(s^\eta - 1)^p ds.$$

Proof. The integrand on the left-hand side of the equation to be proved is written as the product of three terms. Since $\varphi \in \mathscr{R}_\eta(0)$, the third term satisfies

$$\lim_{u \downarrow 0} \left(\frac{\varphi(su)}{\varphi(u)} - 1 \right)^p = (s^\eta - 1)^p$$

pointwise in s. We know from the Monotone Density Theorem (when $\eta < 0$) or by Assumption 3.1(v)(a) (when $\eta = 0$) that $-\varphi' \in \mathscr{R}_{\eta-1}(0)$, so the second term satisfies

$$\lim_{u \downarrow 0} \left(\frac{\varphi'(su)}{\varphi'(u)} \right)^{-2} = s^{2-2\eta}$$

pointwise in s. Since $-\varphi' \in \mathscr{R}_{\eta-1}(0)$ and $\eta - 1 < 0$, the Monotone Density Theorem also implies that $\varphi''(u) \in \mathscr{R}_{\eta-2}(0)$, and so the first term satisfies

$$\lim_{u \downarrow 0} \frac{\varphi''(su)}{\varphi''(u)} = s^{\eta-2}$$

pointwise in s. Consequently, our integrand converges pointwise to $s^{-\eta}(s^\eta - 1)^p$ as $u \downarrow 0$.

Using Potter's Theorem, we can show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that our integrand is bounded by $2s^{(p-1)\eta-\varepsilon}$ for all $u \in (0, \delta)$. Since $(p-1)\eta > -1$, we may choose ε small enough to make this bound integrable on $(0, 1)$. The Dominated Convergence Theorem now delivers our desired result. $\square$

Lemma A.2. *Under Assumption 3.1, for $p \in (0, 1)$ we have*

$$\lim_{u \downarrow 0} \varphi(u)^{-p} \int_0^1 \frac{\varphi''(su)}{\varphi''(u)} \left(\frac{\varphi'(su)}{\varphi'(u)} \right)^{-2} (\varphi(su) - \varphi(u))^{-p} \, ds = 0.$$

Proof. Convexity of φ implies the inequality $\varphi(su) - \varphi(u) \geq -(1-s)u\varphi'(u)$, valid for $s \in (0, 1)$. We know from the Monotone Density Theorem (when $\eta < 0$) or by Assumption 3.1(v)(b) (when $\eta = 0$) that $u\varphi'(u)$ is bounded away from zero in a neighborhood of zero. Since $\lim_{u\downarrow 0} \varphi(u) = \infty$, it remains only to show that

$$\limsup_{u \downarrow 0} \int_0^1 \frac{\varphi''(su)}{\varphi''(u)} \left(\frac{\varphi'(su)}{\varphi'(u)} \right)^{-2} (1-s)^{-p} \, \mathrm{d}s < \infty.$$

Using the Monotone Density Theorem, we can show that the above integrand converges pointwise to $s^{-\eta}(1-s)^{-p}$ as $u \downarrow 0$. And using Potter's Theorem, we can show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that our integrand is bounded by $2s^{-\eta-\varepsilon}(1-s)^{-p}$ for all $u \in (0, \delta)$. Since $p < 1$, we may choose $\varepsilon < 1 - \eta$ to make this bound integrable in s. The Dominated Convergence Theorem thus yields

$$\lim_{u \downarrow 0} \int_0^1 \frac{\varphi''(su)}{\varphi''(u)} \left(\frac{\varphi'(su)}{\varphi'(u)} \right)^{-2} (1-s)^{-p} \, \mathrm{d}s = \int_0^1 s^{-\eta}(1-s)^{-p} \mathrm{d}s.$$

Since $p < 1$, the limiting integral is finite, and we are done. $\square$

Lemma A.3. *Under Assumption 3.1, if $\zeta > 1$ then for $p < 1 - 1/\zeta$ we have*

$$\lim_{u \downarrow 0} \int_1^{\frac{1}{u}} \frac{\varphi''(1-su)}{\varphi''(1-u)} \left(\frac{\varphi'(1-su)}{\varphi'(1-u)} \right)^{-2} \left(\frac{\varphi(1-su)}{\varphi(1-u)} - 1 \right)^p ds = \int_1^\infty s^{-\zeta}(s^\zeta - 1)^p \, ds.$$

Proof. Since $\varphi \in \mathscr{R}_\zeta(1)$ and $\zeta > 1$, we know from the Monotone Density Theorem that $-\varphi' \in \mathscr{R}_{\zeta-1}(1)$ and $\varphi'' \in \mathscr{R}_{\zeta-2}(1)$. Consequently, as $u \downarrow 0$, the integrand on the left-hand side of the equation to be proved converges to $s^{-\zeta}(s^\zeta - 1)^p$ pointwise on $(1, \infty)$. Using Potter's Theorem, we can show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that our integrand is bounded by $2s^{(p-1)\zeta+\varepsilon}$ for all $u \in (0, \delta)$. Since $(p-1)\zeta < -1$, we may choose ε small enough to make this bound integrable on $(1, \infty)$. The Dominated Convergence Theorem now delivers our desired result. $\square$

Lemma A.4. *Fix $u_0 \in (0,1)$. Under Assumption 3.1, for $u \in [u_0, 1)$ and $p \in [0, 1-1/\eta)$ we have*

$$\int_0^1 \left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{w}\right)\right) - \varphi(u)\right)^p dw \le -\varphi'(u_0) \int_0^{u_0} \frac{\varphi(v)^p \varphi''(v)}{\varphi'(v)^2} dv < \infty.$$

Proof. Since φ is decreasing and strictly convex, $\varphi(\varphi'^{-1}(\varphi'(\cdot)/w))$ is decreasing for each $w \in (0,1)$. Combined with the nonnegativity of φ, we find that

$$\int_0^1 \left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{w}\right)\right) - \varphi(u)\right)^p \mathrm{d}w \le \int_0^1 \varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u_0)}{w}\right)\right)^p \mathrm{d}w.$$

The first inequality to be proved follows easily using the change of variables $w = \varphi'(u_0)/\varphi'(v)$. It remains to show that $\varphi(\cdot)^p \varphi'(\cdot)^{-2} \varphi''(\cdot)$ is integrable on $(0, u_0)$. Twice continuous differentiability of φ on $(0,1)$ ensures integrability provided that our integrand does not diverge too rapidly at the origin. In fact, Assumption 3.1(v)(a) and the Monotone Density Theorem imply that $\varphi \in \mathscr{R}_\eta(0)$, $-\varphi' \in \mathscr{R}_{\eta-1}(0)$ and $\varphi'' \in \mathscr{R}_{\eta-2}(0)$, and so we have $\varphi(\cdot)^p \varphi'(\cdot)^{-2} \varphi''(\cdot) \in \mathscr{R}_{\eta(p-1)}(0)$. Since $\eta(p-1) > -1$, integrability holds. □

Lemma A.5. *Fix $u_0, u_1 \in (0,1)$ with $u_0 < u_1$. Under Assumption 3.1, for $u \in [u_0, u_1]$ and $p \in [0,1)$ we have*

$$\begin{aligned} &\int_0^1 \left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{w}\right)\right) - \varphi(u)\right)^{-p} dw \\ \le\ & (1-\sqrt{p})^{-\sqrt{p}} u_1^{\sqrt{p}-p} (-\varphi'(u_0))^{1-p} \left(\int_0^{u_1} \left(\frac{\varphi''(v)}{\varphi'(v)^2}\right)^{\frac{1}{1-\sqrt{p}}} dv\right)^{1-\sqrt{p}} < \infty. \end{aligned}$$

Proof. Combining the change of variables $w = \varphi'(u)/\varphi'(su)$ with the inequality $\varphi(su) - \varphi(u) \ge -(u - su)\varphi'(u)$, valid for $s \in (0,1)$ due to the convexity of φ, we obtain

$$\int_0^1 \left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{w}\right)\right) - \varphi(u)\right)^{-p} \mathrm{d}w \le u^{1-p}(-\varphi'(u))^{1-p} \int_0^1 (1-s)^{-p} \frac{\varphi''(su)}{\varphi'(su)^2} \mathrm{d}s.$$

Applying Hölder's inequality and the change of variables $v = su$,

$$\begin{aligned} \int_0^1 (1-s)^{-p} \frac{\varphi''(su)}{\varphi'(su)^2} \mathrm{d}s \ &\le\ \left(\int_0^1 (1-s)^{-\sqrt{p}} \mathrm{d}s\right)^{\sqrt{p}} \left(\int_0^1 \left(\frac{\varphi''(su)}{\varphi'(su)^2}\right)^{\frac{1}{1-\sqrt{p}}} \mathrm{d}s\right)^{1-\sqrt{p}} \\ &=\ (1-\sqrt{p})^{-\sqrt{p}}\, u^{\sqrt{p}-1} \left(\int_0^u \left(\frac{\varphi''(v)}{\varphi'(v)^2}\right)^{\frac{1}{1-\sqrt{p}}} \mathrm{d}v\right)^{1-\sqrt{p}}. \end{aligned}$$

The first inequality to be proved now follows from the inequalities $u_0 \leq u \leq u_1$ and $-\varphi'(u) \leq -\varphi'(u_0)$. It remains to show that $(\varphi'(\cdot)^{-2}\varphi''(\cdot))^{1/(1-\sqrt{p})}$ is integrable on $(0, u_1)$. Twice continuous differentiability of φ on $(0,1)$ ensures integrability provided that our integrand does not diverge too rapidly at the origin. In fact, Assumption 3.1(v)(a) and the Monotone Density Theorem imply that $-\varphi' \in \mathscr{R}_{\eta-1}(0)$ and $\varphi'' \in \mathscr{R}_{\eta-2}(0)$, and so we have $(\varphi'(\cdot)^{-2}\varphi''(\cdot))^{1/(1-\sqrt{p})} \in \mathscr{R}_{-\eta/(1-\sqrt{p})}(0)$. Since $-\eta/(1-\sqrt{p}) > -1$, integrability holds. $\square$

The proof of Theorem 3.1 involves an application of the Geometric Ergodic Theorem. This result is presented in many ways and discussed in great detail in the book of Meyn and Tweedie (1993), which we shall henceforth refer to as MT. A version of the Geometric Ergodic Theorem is given below as Theorem A.1. First, we require an additional definition.

Definition A.1. A set $S \in \mathscr{B}$ is said to be small if there exists a nontrivial measure ν on $\mathscr{B}$ such that $P(U_1 \in B | U_0 = u) \geq \nu(B)$ for a.e. $u \in S$ and all $B \in \mathscr{B}$.

The above definition of a small set differs somewhat from the definition given by Meyn and Tweedie (1993). Aside from the a.e. qualifier, our definition is more narrow than theirs. But it is sufficient for our purposes.

The statement of Theorem A.1 employs the notions of irreducibility and aperiodicity. For definitions, we refer the reader to MT. Here, we note only that our Markov chain $\{U_t : t \in \mathbb{Z}\}$ is irreducible and aperiodic whenever C admits a density c that is positive on $(0,1)^2$.

Theorem A.1. *Suppose $\{U_t : t \in \mathbb{Z}\}$ is irreducible and aperiodic, and there exists a function $V : (0,1) \to [1, \infty)$, a small set $S \in \mathscr{B}$, and constants $a < 1$, $b < \infty$ such that*

$$E(V(U_1)|U_0 = u) \leq aV(u) + b1_S(u) \tag{A.1}$$

for a.e. $u \in (0,1)$. Then $\{U_t : t \in \mathbb{Z}\}$ is geometrically ergodic.

Proof of Theorem A.1. By Proposition 5.5.3 in MT, every small set is petite (defined on p. 124 in MT), and so the assumptions of Theorem A.1 are stronger than those of Theorem 16.0.1 of MT (aside from the a.e. qualifier in Definition A.1, which we may safely ignore). Thus, the equivalence of (ii) and (iv) in Theorem 16.0.1 of MT implies that

$$|P(U_k \in B | U_0 = u) - P(U_k \in B)| \leq V(u) A \mathrm{e}^{-\gamma k}$$

for a.e. $u \in (0,1)$, all $B \in \mathscr{B}$, all $k \in \mathbb{N}$, and some $A < \infty$ and $\gamma > 0$. It follows immediately that $\{U_t\}$ is geometrically ergodic. $\square$

We are now in a position to provide a proof of Theorem 3.1

Proof of Theorem 3.1. Our proof consists of verifying the conditions of Theorem A.1. As noted above, irreducibility and aperiodicity of $\{U_t\}$ hold if C admits a density c that is positive on $(0,1)^2$. Recalling Remark 3.4, this is indeed the case under Assumption 3.1. It remains for us to verify the drift condition (A.1) for suitably chosen V, S, a and b. Our choice of these objects will depend critically on whether $\zeta = 1$ or $\zeta > 1$. We therefore separate the remainder of the proof into two parts.

Case 1: $\zeta = 1$. Fix a number $p \in (0, -1/\eta)$; here and in what follows, $1/\eta$ should be interpreted as $-\infty$ when $\eta = 0$. For our drift function V we choose $V(\cdot) = \varphi(\cdot)^p + 1$. As a first step towards verifying (A.1) for this choice of V we shall investigate the behavior of $E\left(V(U_1)|U_0 = u\right)$ as $u \downarrow 0$. Following the construction on p. 157 of Genest and MacKay (1986a), we may express the relationship between U_0 and U_1 in the nonlinear autoregressive form

$$U_1 = \varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(U_0)}{W}\right)\right) - \varphi(U_0)\right), \tag{A.2}$$

where W is a $U(0,1)$ random variable distributed independently of U_0. Using (A.2), for $u \in (0,1)$ we may write

$$E\left(\varphi(U_1)^p|U_0 = u\right) = \int_0^1 \left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{w}\right)\right) - \varphi(u)\right)^p \mathrm{d}w. \tag{A.3}$$

Applying the change of variables $w = \varphi'(u)/\varphi'(su)$ to the integral in (A.3) and rearranging terms, we obtain

$$\frac{E\left(\varphi(U_1)^p|U_0 = u\right)}{\varphi(u)^p} = -\frac{u\varphi''(u)}{\varphi'(u)}\int_0^1 \frac{\varphi''(su)}{\varphi''(u)}\left(\frac{\varphi'(su)}{\varphi'(u)}\right)^{-2}\left(\frac{\varphi(su)}{\varphi(u)} - 1\right)^p \mathrm{d}s. \tag{A.4}$$

Since $-\varphi' \in \mathscr{R}_{\eta-1}(0)$, the Monotone Density Theorem implies that $\lim_{u\downarrow 0} u\varphi''(u)/\varphi'(u) = \eta - 1$. Combining this result with Lemma A.1, we obtain

$$\lim_{u\downarrow 0}\frac{E\left(\varphi(U_1)^p|U_0 = u\right)}{\varphi(u)^p} = (1-\eta)\int_0^1 s^{-\eta}(s^\eta - 1)^p \mathrm{d}s = \int_0^1 \left(r^{\frac{\eta}{1-\eta}} - 1\right)^p \mathrm{d}r =: \xi_0, \tag{A.5}$$

where we have used the change of variables $s = r^{1/(1-\eta)}$. Clearly $\xi_0 \geq 0$, with $\xi_0 = 0$ when $\eta = 0$. When $\eta < 0$, since $p \in (0, -1/\eta)$, Hölder's inequality implies that

$$\xi_0 < \left(\int_0^1 \left(r^{\frac{\eta}{1-\eta}} - 1\right)^{-1/\eta} \mathrm{d}r\right)^{-p\eta} = \left(\frac{\eta - 1}{\eta}\int_0^1 q^{-1/\eta}\mathrm{d}q\right)^{-p\eta} = 1,$$

where we have used the change of variables $r = (1-q)^{(\eta-1)/\eta}$. Hence $\xi_0 \in [0,1)$.

We have shown that $E\left(\varphi(U_1)^p|U_0 = u\right)/\varphi(u)^p \to \xi_0 \in [0,1)$ as $u \downarrow 0$. Since $\varphi(u)^p \to \infty$

as $u \downarrow 0$, it follows easily that $E\left(V(U_1)|U_0 = u\right)/V(u) \to \xi_0$ as $u \downarrow 0$. Consequently, for any arbitrary constant $a \in (\xi_0, 1)$, there must exist $u_0 \in (0,1)$ such that

$$E\left(V(U_1)|U_0 = u\right) \leq aV(u) \text{ for all } u \in (0, u_0). \tag{A.6}$$

Lemma A.4 and (A.3) ensure the existence of $b < \infty$ such that

$$E\left(V(U_1)|U_0 = u\right) \leq b \text{ for all } u \in [u_0, 1). \tag{A.7}$$

Combining (A.6) and (A.7), we obtain

$$E\left(V(U_1)|U_0 = u\right) \leq aV(u) + b1_{[u_0,1)}(u) \text{ for all } u \in (0, 1).$$

To verify the drift condition (A.1), it remains only to show that $[u_0, 1)$ is a small set. Consider the expression for the copula density c given in (3.1). Since $C > 0$ on $[u_0, 1)^2$, the denominator on the right-hand side of (3.1) is bounded away from $-\infty$ on $[u_0, 1)^2$. Further, since $\zeta = 1$, Assumptions 3.1(vi), 3.1(v)(a) and 3.1(vi) jointly imply that φ' and φ'' are bounded away from zero, implying that the numerator on the right-hand side of (3.1) is also bounded away from zero. Hence, c is bounded away from zero on $[u_0, 1)^2$. Let $\kappa = \inf_{(u,v)\in[u_0,1)^2} c(u, v) > 0$, and for $B \in \mathscr{B}$ let $\nu B = \kappa \int_B 1_{[u_0,1)}(v)\mathrm{d}v$. Clearly ν is a nontrivial measure on $\mathscr{B}$. For any $u \in [u_0, 1)$ and any $B \in \mathscr{B}$ we have

$$P(U_1 \in B|U_0 = u) = \int_B c(u, v)\mathrm{d}v \geq \int_B c(u, v)1_{[u_0,1)}(v)\mathrm{d}v \geq \kappa \int_B 1_{[u_0,1)}(v)\mathrm{d}v = \nu B,$$

implying that $[u_0, 1)$ is small. Our desired result now follows from Theorem A.1 for the case where $\zeta = 1$.

Case 2: $\zeta > 1$. This time we fix $p \in (0, \min\{-1/\eta, 1/\zeta, 1 - 1/\zeta\})$, and for our drift function V we choose $V(u) = \varphi(u)^p + \varphi(u)^{-p}$. We will investigate the behavior of $E(V(U_1)|U_0 = u)$ as $u \downarrow 0$ and as $u \uparrow 1$, beginning with the former scenario. The proof that $\lim_{u\downarrow 0} E\left(\varphi(U_1)^p|U_0 = u\right)/\varphi(u)^p = \xi_0 \in [0, 1)$ given for Case 1 continues to apply here. Trivially modifying (A.4), we have

$$\frac{E\left(\varphi(U_1)^{-p}|U_0 = u\right)}{\varphi(u)^p} = -\frac{u\varphi''(u)}{\varphi'(u)}\varphi(u)^{-p}\int_0^1 \frac{\varphi''(su)}{\varphi''(u)}\left(\frac{\varphi'(su)}{\varphi'(u)}\right)^{-2}\left(\varphi\left(su\right) - \varphi\left(u\right)\right)^{-p}\mathrm{d}s.$$

As noted in the proof for Case 1, $\lim_{u\downarrow 0} u\varphi''(u)/\varphi'(u) = \eta - 1$, and so Lemma A.2 implies that $\lim_{u\downarrow 0} E\left(\varphi(U_1)^{-p}|U_0=u\right)/\varphi(u)^p = 0$. We have now established that

$$\begin{aligned}
\lim_{u\downarrow 0}\frac{E\left(V(U_1)|U_0=u\right)}{V(u)} &= \lim_{u\downarrow 0}\frac{\varphi(u)^p}{\varphi(u)^p+\varphi(u)^{-p}}\left(\frac{E\left(\varphi(U_1)^p|U_0=u\right)}{\varphi(u)^p}+\frac{E\left(\varphi(U_1)^{-p}|U_0=u\right)}{\varphi(u)^p}\right)\\
&= 1\cdot(\xi_0+0)=\xi_0\in[0,1).
\end{aligned}$$

Next consider the behavior of $E\left(V(U_1)|U_0=u\right)$ as $u\uparrow 1$. Applying the change of variables $w=\varphi'(1-u)/\varphi'(1-su)$ to the integral in (A.3) with $1-u$ in place of u, and rearranging terms, we obtain

$$\begin{aligned}
&\frac{E\left(\varphi(U_1)^p|U_0=1-u\right)}{\varphi(1-u)^p}\\
&= -\frac{u\varphi''(1-u)}{\varphi'(1-u)}\int_1^{\frac{1}{u}}\frac{\varphi''(1-su)}{\varphi''(1-u)}\left(\frac{\varphi'(1-su)}{\varphi'(1-u)}\right)^{-2}\left(\frac{\varphi\left(1-su\right)}{\varphi\left(1-u\right)}-1\right)^p \mathrm{d}s. \qquad \text{(A.8)}
\end{aligned}$$

The Monotone Density Theorem implies that $\lim_{u\downarrow 0} u\varphi''(1-u)/\varphi'(1-u)=1-\zeta$, while Lemma A.3 implies that the integral in (A.8) converges to $\int_1^\infty s^{-\zeta}(s^\zeta-1)^p\mathrm{d}s$ as $u\downarrow 0$. Since $p<1-1/\zeta$, this integral is finite. We have thus shown that

$$\lim_{u\uparrow 1}\frac{E\left(\varphi(U_1)^p|U_0=u\right)}{\varphi(u)^p}=(\zeta-1)\int_1^\infty s^{-\zeta}(s^\zeta-1)^p\mathrm{d}s. \qquad \text{(A.9)}$$

In fact, by an identical argument, (A.9) remains true with $-p$ in place of p. Consequently,

$$\begin{aligned}
\lim_{u\uparrow 1}\frac{E\left(V(U_1)|U_0=u\right)}{V(u)} &= \lim_{u\uparrow 1}\left(\frac{\varphi(u)^p}{\varphi(u)^p+\varphi(u)^{-p}}\cdot\frac{E\left(\varphi(U_1)^p|U_0=u\right)}{\varphi(u)^p}\right)\\
&\quad+\lim_{u\uparrow 1}\left(\frac{\varphi(u)^{-p}}{\varphi(u)^p+\varphi(u)^{-p}}\cdot\frac{E\left(\varphi(U_1)^{-p}|U_0=u\right)}{\varphi(u)^{-p}}\right)\\
&= (\zeta-1)\int_1^\infty s^{-\zeta}(s^\zeta-1)^{-p}\mathrm{d}s =: \xi_1. \qquad \text{(A.10)}
\end{aligned}$$

Applying the change of variables $s=r^{1/(1-\zeta)}$ to the integral defining ξ_1 in (A.10), we obtain

$$\xi_1=\int_0^1\left(r^{\frac{\zeta}{1-\zeta}}-1\right)^{-p}\mathrm{d}r,$$

which is well defined since $\zeta>1$. Clearly $\xi_1>0$. And since $p\in(0,1/\zeta)$, Hölder's inequality implies that

$$\xi_1<\left(\int_0^1\left(r^{\frac{\zeta}{1-\zeta}}-1\right)^{-1/\zeta}\mathrm{d}r\right)^{p\zeta}=\left(\frac{\zeta-1}{\zeta}\int_0^1 q^{-1/\zeta}\mathrm{d}q\right)^{p\zeta}=1,$$

where we have used the change of variables $r = (1-q)^{(\zeta-1)/\zeta}$. Hence $\xi_1 \in (0,1)$.

We have now shown that $E\left(V(U_1)|U_0 = u\right)/V(u) \to \xi_0 \in [0,1)$ as $u \downarrow 0$, and that $E\left(V(U_1)|U_0 = u\right)/V(u) \to \xi_1 \in (0,1)$ as $u \uparrow 1$. Consequently, there exists $a \in (\max\{\xi_0, \xi_1\}, 1)$ and $u_0, u_1 \in (0,1)$ with $u_0 < u_1$ such that

$$E\left(V(U_1)|U_0 = u\right) \leq aV(u) \text{ for all } u \in (0, u_0) \cup (u_1, 1). \tag{A.11}$$

Lemma A.4, Lemma A.5 and (A.3) ensure the existence of $b < \infty$ such that

$$E\left(V(U_1)|U_0 = u\right) \leq b \text{ for all } u \in [u_0, u_1]. \tag{A.12}$$

Combining (A.11) and (A.12), we obtain

$$E\left(V(U_1)|U_0 = u\right) \leq aV(u) + b1_{[u_0,u_1]}(u) \text{ for all } u \in (0,1).$$

To verify the drift condition (A.1), it remains only to show that $[u_0, u_1]$ is a small set. Recalling the proof that $[u_0, 1)$ was small in Case 1, it should be clear that we need only show that c is bounded away from zero on $[u_0, u_1]^2$. But this is obvious from (3.1) in view of the fact that $-\varphi'$ and φ'' are continuous and strictly positive on $[u_0, u_1]$ (recall Assumptions 3.1(ii) and 3.1(iv)), while $-\varphi' \circ C$ is continuous and therefore bounded on $[u_0, u_1]^2$. We may therefore apply Theorem A.1 to obtain our desired result for the case where $\zeta > 1$ also. $\square$

References

Andrews, D. W. K. (1984). Non strong mixing autoregressive processes. *Journal of Applied Probability* **21** 930–934.

Beare, B. K. (2010). Copulas and temporal dependence. *Econometrica* **78** 395–410.

Bingham, N. H., Goldie, C. M. and Teugels, J. L. (1987). *Regular Variation.* Cambridge University Press, Cambridge.

Bouyé, E. and Salmon, M. (2009). Dynamic copula quantile regressions and tail area dynamic dependence in Forex markets. *European Journal of Finance* **15** 721–750.

Bradley, R. C. (2007). *Introduction to Strong Mixing Conditions,* v. 1-3. Kendrick Press, Heber City.

Charpentier, A. and Segers, J. (2007). Lower tail dependence for Archimedean copulas: characterizations and pitfalls. *Insurance: Mathematics and Economics* **40** 525–532.

Chen, X. and Fan, Y. (2006). Estimation of copula-based semiparametric time series models. *Journal of Econometrics* **130** 307–335.

Chen, X., Koenker, R. and Xiao, Z. (2009). Copula-based nonlinear quantile autoregression. *Econometrics Journal* **12** S50–S67.

Chen, X., Wu, W. B. and Yi, Y. (2009). Efficient estimation of copula-based semiparametric Markov models. *Annals of Statistics* **37** 4214–4253.

Darsow, W. F., Nguyen, B. and Olsen, E. T. (1992). Copulas and Markov processes. *Illinois Journal of Mathematics* **36** 600–642.

Fentaw, A. and Naik-Nimbalkar, U. V. (2008). Dynamic copula-based Markov time series. *Comm. Statist. Theory Methods* **37** 2447–2460.

Gagliardini, P. and Gouriéroux, C. (2008). Duration time-series models with proportional hazard. *Journal of Time Series Analysis* **29** 74–124.

Genest, C. and MacKay, J. (1986a). Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données. *Canadian Journal of Statistics* **14** 145–159.

Genest, C. and MacKay, J. (1986b). The joy of copulas: bivariate distributions with uniform marginals. *American Statistician* **40** 280–283.

Ibragimov, R. (2009). Copula-based characterizations for higher-order Markov processes. *Econometric Theory* **25** 819–846.

Ibragimov, R. and Lentzas, G. (2009). Copulas and long memory. Harvard Institute of Economic Research Discussion Paper No. 2160.

Juri, A. and Wüthrich, M. V. (2002). Copula convergence theorems for tail events. *Insurance: Mathematics and Economics* **30** 411–427.

Juri, A. and Wüthrich, M. V. (2003). Tail dependence from a distributional point of view. *Extremes* **6** 213–246.

Meyn, S. P. and Tweedie, R. L. (1993). *Markov Chains and Stochastic Stability.* Springer-Verlag, London.

Nelsen, R. B. (2006). *An Introduction to Copulas.* 2nd edition. Springer-Verlag, New York.