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Horizon Constraints and Black Hole Entropy

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Abstract

A question about a black hole in quantum gravity is a conditional question: to obtain an answer, one must restrict initial or boundary data to ensure that a black hole is actually present. For two-dimensional dilaton gravity—and probably for a much wider class of theories—I show that the imposition of a spacelike "stretched horizon" constraint modifies the algebra of symmetries, inducing a central term. Standard conformal field theory techniques then fix the asymptotic density of states, successfully reproducing the Bekenstein-Hawking entropy. The states responsible for black hole entropy can thus be viewed as "would-be gauge" states that become physical because the symmetries are altered.

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Suppose one wishes to ask a question about a quantum black hole. In a semiclassical theory, this is straightforward, at least in principle—one can look at quantum fields and gravitational perturbations around a black hole background. In a full quantum theory of gravity, though, such a procedure is no longer possible—there is no fixed background, and the theory contains both states with black holes and states with none. One must therefore make one's question conditional: "*If* a black hole with property X is present..." Equivalently, one must impose constraints, either on initial data or on boundaries, that restrict the theory to one containing an appropriate black hole [1].

Classical general relativity is characterized by a symmetry algebra, the algebra of diffeomorphisms. But it is well known that the introduction of new constraints can alter such an algebra [2–4]. For the simple model of two-dimensional dilaton gravity, I will show below that the imposition of suitable "stretched horizon" constraints has the effect of adding a central extension to the algebra of diffeomorphisms of the horizon. This is a strong result, because such a centrally extended algebra is powerful enough to almost completely fix the asymptotic behavior of the density of states, that is, the entropy [5,6]. Indeed, I will show that given a reasonable normalization of the "energy," standard conformal field theoretical methods reproduce the correct Bekenstein-Hawking entropy. Moreover, while the restriction to two-dimensional dilaton gravity is a significant one, I will argue that the conclusions are likely to extend to much more general settings.

These results suggest that the entropy of a black hole can be explained by two key features: the imposition of horizon boundary conditions, which can alter the physical content of the theory by promoting "pure gauge" fields to dynamical degrees of freedom, and the existence of a Virasoro algebra, which can control the asymptotic density of states. Both of these features are present for the (2+1)-dimensional black hole [7–9], and a number of authors—see, for example, [10–19]—have suggested that near-horizon symmetries may control generic black hole entropy. In particular, we shall see that the near-horizon conformal symmetry of [20] is closely related to the horizon constraint introduced here.

1. Dilaton Gravity in a Null Frame

We start with canonical two-dimensional dilaton gravity in a null frame, that is, expressed in terms of a null dyad $\{l^a, n^a\}$ with $\ell \cdot n = -1$. The metric is then

$$g_{ab} = -l_a n_b - l_b n_a, \tag{1.1}$$

and "surface gravities" κ and $\bar{\kappa}$ may be defined by

$$\begin{aligned} \nabla_a l_b &= -\kappa n_a l_b - \bar{\kappa} l_a l_b \\ \nabla_a n_b &= \kappa n_a n_b + \bar{\kappa} l_a n_b, \end{aligned} \tag{1.2}$$

where the second of eqns. (1.2) follows from the first. It is easy to check—for instance, by computing $[\nabla_a, \nabla_b]l^b$—that the scalar curvature is

$$R = 2\nabla_a(\kappa n^a - \bar{\kappa}l^a) = -\frac{2}{\sqrt{-g}}\partial_a\left(\kappa\hat{\epsilon}^{ab}n_b + \bar{\kappa}\hat{\epsilon}^{ab}l_b\right) \tag{1.3}$$

where $\hat{\epsilon}^{ab} = -\hat{\epsilon}^{ba}$ is the Levi-Civita density, $\hat{\epsilon}^{uv} = 1$.

In addition to the metric, two-dimensional dilaton gravity contains a dilaton field, which I shall call A. For models obtained by dimensionally reducing Einstein gravity, A is just the transverse area. With appropriate field redefinitions [21], the action becomes

$$I = \frac{1}{16\pi G}\int d^2x\sqrt{-g}\,[AR + V(A)] \tag{1.4}$$

$$= \frac{1}{16\pi G}\int d^2x\left[\hat{\epsilon}^{ab}\left(2\kappa n_b\partial_a A - 2\bar{\kappa}l_b\partial_a A\right) + \sqrt{-g}V\right]$$

with a potential $V(A)$ that depends on the specific model. If one now defines components

$$l = \sigma du + \alpha dv, \qquad n = \beta du + \tau dv \tag{1.5}$$

with respect to coordinates (u, v), and chooses units $16\pi G = 1$, the Lagrangian becomes

$$L = \frac{1}{\sigma\tau - \alpha\beta}\left[2\left(\tau\dot{A} - \beta A'\right)(\dot{\alpha} - \sigma')\right.$$

$$\left. + 2\left(\alpha\dot{A} - \sigma A'\right)(\dot{\tau} - \beta')\right] + (\sigma\tau - \alpha\beta)V(A), \tag{1.6}$$

where a dot denotes a derivative with respect to u and a prime a derivative with respect to v. One can immediately read off the canonical momenta,

$$\pi_\alpha = \frac{2}{\sigma\tau - \alpha\beta}\left(\tau\dot{A} - \beta A'\right),$$

$$\pi_\tau = \frac{2}{\sigma\tau - \alpha\beta}\left(\alpha\dot{A} - \sigma A'\right), \tag{1.7}$$

$$\pi_A = \frac{2}{\sigma\tau - \alpha\beta}\left[\tau\left(\dot{\alpha} - \sigma'\right) + \alpha\left(\dot{\tau} - \beta'\right)\right].$$

The variables σ and β appear with no time derivatives, and act as Lagrange multipliers. As in any diffeomorphism-invariant theory, the Hamiltonian is a linear combination of constraints: $H = \sigma C_\perp + \frac{\beta}{\tau}\left(C_\parallel - \alpha C_\perp\right)$, with

$$C_\perp = \pi_\alpha{}' - \frac{1}{2}\pi_\alpha\pi_A - \tau V(A)$$

$$C_\parallel = \pi_A A' - \alpha\pi_\alpha{}' - \tau\pi_\tau{}'. \tag{1.8}$$

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An additional constraint appears because both π_α and π_τ depend only on $\dot{A}$:

$$C_\pi = \tau\pi_\tau - \alpha\pi_\alpha + 2A'. \tag{1.9}$$

$C_\perp$ and $C_\parallel$ are ordinary Hamiltonian and momentum constraints, while C_π is a disguised version of the generator of local Lorentz invariance, appearing because the pair $\{l, n\}$ is invariant under the boost $l \to fl$, $n \to f^{-1}n$. The constraints have the Poisson algebra

$$
\begin{aligned}
&\{C_\perp[\xi], C_\perp[\eta]\} = 0 && \{C_\parallel[\xi], C_\perp[\eta]\} = C_\perp[\xi\eta'] \\
&\{C_\parallel[\xi], C_\parallel[\eta]\} = C_\parallel[\xi\eta' - \eta\xi'] && \{C_\perp[\xi], C_\pi[\eta]\} = C_\perp[\xi\eta] \\
&\{C_\parallel[\xi], C_\pi[\eta]\} = C_\pi[\xi\eta'] && \{C_\pi[\xi], C_\pi[\eta]\} = 0
\end{aligned} \tag{1.10}
$$

where $C[\xi]$ denotes the "smeared" constraint $\int dv\,\xi C$ on a surface of constant u.

2. Horizon Constraints

So far, this has all been standard, albeit in a slightly unusual parametrization. Now, however, let us demand that the initial surface $u = 0$ be a horizon, which we can define in the sense of [22] as a null surface (with null normal l^a) with vanishing expansion ϑ. The usual definition of "expansion" does not apply in two spacetime dimensions, but a straightforward generalization,

$$\vartheta = l^a \nabla_a A / A, \tag{2.1}$$

captures the same information: it describes the fractional change in the transverse area, and is proportional to the conventional expansion in models obtained by dimensional reduction. The imposition of such a horizon constraint is a bit delicate, however, as I shall now describe.

First, for a standard canonical approach to work, the initial surface should be spacelike. While a canonical analysis of gravity with a null slicing is possible [23, 24], it introduces considerable complications. We shall avoid these by considering a slightly distorted horizon that is "almost null," requiring that $\alpha = \epsilon_1 \ll 1$ in (1.5).

For such a "stretched horizon," one should not require that the expansion—here, the logarithmic derivative of A at the $u = 0$ surface—vanish, but only that it be sufficiently small. Here a second subtlety arises: the null normal l^a does not have a unique normalization. While the condition of vanishing expansion is independent of the normalization of l^a, the condition of "small" expansion is not; indeed, by simply rescaling l^a by a constant, one can make ϑ arbitrarily large. Fortunately, another quantity, the surface gravity κ, changes in the same way under constant rescalings of l^a. We therefore require that $l^v \nabla_v A / \kappa A = \epsilon_2$, or, from (1.7), $A' - \frac{1}{2}\epsilon_2 A\pi_A = 0$. I show in the appendix that for ϵ_2 small and negative, these constraints yield a spacelike surface that closely traces the true horizon.

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A final ambiguity comes from the existence of the constraint (1.9), which allows us to trade A' in the expansion for the combination $\tau\pi_\tau - \alpha\pi_\alpha$. It is not clear which expression provides the "correct" constraint. We therefore leave the choice open, writing the horizon constraints in the form

$$K_1 = \alpha - \epsilon_1 = 0$$

$$K_2 = A' - \frac{1}{2}\epsilon_2 A_+ \pi_A + \frac{a}{2}C_\pi = 0 \qquad (2.2)$$

where a is an arbitrary constant. I have substituted the horizon value A_+ for A in the second line; this will avoid complicated field redefinitions later without changing the physics, since the difference between A and A_+ is $\mathcal{O}(\epsilon_2)$.

We may now make the fundamental observation that $\{K_i, K_j\} \neq 0$, that is, that the K_i are second class constraints. Indeed, a straightforward computation yields

$$\{K_i(x), K_j(y)\} = \Delta_{ij}(x,y) = \begin{pmatrix} 0 & -\frac{a}{2}\alpha\delta(x-y) \\ \frac{a}{2}\alpha\delta(x-y) & -(1+a)\epsilon_2 A_+\delta'(x-y) \end{pmatrix}. \qquad (2.3)$$

For the Poisson algebra of our remaining observables to be consistent with these constraints, we should replace Poisson brackets by Dirac brackets [2,3],

$$\{P,Q\}^* = \{P,Q\} - \sum_{i,j} \int dx\,dy\{P, K_i(x)\}\Delta_{ij}^{-1}(x,y)\{K_j(y), Q\} \qquad (2.4)$$

so that $\{P, K_i(x)\}^* = 0$. Equivalently, this amounts to replacing every observable P by a combination

$$P^* = P + c_1 K_1 + c_2 K_2, \qquad (2.5)$$

with coefficients c_i chosen so that $\{P^*, K_i(x)\} = 0$ [25]. Since the K_i vanish for physical configurations, P^* is physically equivalent to P; and since P^* has a vanishing bracket with the K_i, we can impose these constraints consistently without changing the meaning of the Poisson bracket.

Now, the horizon constraints (2.2) hold only on the surface $u = 0$, so it is not clear that Dirac brackets are needed for the Hamiltonian constraint $C_\perp$, which evolves quantities off the initial surface. The momentum and Lorentz constraints $C_\parallel$ and C_π, on the other hand, clearly need to be modified appropriately. It is not hard to show that the "starred" constraints are

$$C_\parallel^* = C_\parallel + \frac{4(1+a)}{a^2}\epsilon_2 A_+ \frac{K_1''}{\epsilon_1} - \frac{2}{a}K_2'$$

$$C_\pi^* = C_\pi + \frac{2}{a}\left(1 + \frac{2}{a}\right)\epsilon_2 A_+ \frac{K_1'}{\epsilon_1} - \frac{2}{a}K_2 \qquad (2.6)$$

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and that the Dirac brackets become

$$\{C_\|[\xi], C_\|[\eta]\}^* = C_\|[\xi\eta' - \eta\xi'] - \frac{2(1+a)}{a^2}\epsilon_2 A_+ \int dv(\xi'\eta'' - \eta'\xi'')$$

$$\{C_\|[\xi], C_\pi[\eta]\}^* = C_\pi[\xi\eta'] + \frac{2}{a}\left(\frac{2}{a}+1\right)\epsilon_2 A_+ \int dv\,\xi'\eta'$$

$$\{C_\pi[\xi], C_\pi[\eta]\}^* = -\frac{2}{a^2}\epsilon_2 A_+ \int dv(\xi\eta' - \eta\xi'). \tag{2.7}$$

With the choice $a = -2$, the anomalous term in the transformation of C_π vanishes, and this algebra has a simple conformal field theory interpretation [26]: the $C_\|$ generate a Virasoro algebra with central charge

$$\frac{c}{48\pi} = -\frac{1}{2}\epsilon_2 A_+, \tag{2.8}$$

while C_π is an ordinary primary field of weight one. (Recall that $\epsilon_2 < 0$, so the central charge is positive.)

Thus far we have freely used integration by parts. To compute the entropy associated with the horizon, we will also need the classical value of $C_\|^*$. At first sight this must vanish, since the constraint is zero for any classical configuration. As usual in general relativity, though, $C_\|^*$ has a nontrivial boundary term, which gives a nonvanishing classical contribution.

In particular, suppose that our "stretched horizon" has an end point at the actual horizon, at $v = v_+$. To obtain the boundary term at this point, one must decide what is held fixed at the boundary. Here we can take a hint from [28], where it is shown that the boundary conditions that give the correct horizon contribution to the Euclidean path integral are those that hold fixed the variable conjugate to A. Moreover, since we are limiting ourselves to configurations for which the constraints K_1 and K_2 vanish, we can take $\delta K_1 = \delta K_2 = 0$ at v_+. With these boundary conditions, it is evident that the variation of $C_\|^*[\xi]$ contains a boundary term $\xi\pi_A\delta A|_{v_+}$, which must be canceled off to obtain well-defined Poisson brackets [29]. Hence

$$C_{\|\,bdry}^* = -\,\xi\pi_A A|_{v=v_+}. \tag{2.9}$$

3.   Computing the Entropy

We are now in a position to compute the entropy associated with the stretched horizon. The key ingredient is the Cardy formula [5,6], which states that for a conformal field theory with central charge c, the number of states with eigenvalue Δ of the Virasoro operator L_0 has the asymptotic form

$$\rho(\Delta) = \exp\left\{2\pi\sqrt{\frac{c_{eff}\Delta}{6}}\right\} \tag{3.1}$$

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with $c_{eff} = c - 24\Delta_0$, where Δ_0 is the smallest eigenvalue of L_0. For us, c is given by (2.8), while Δ is the boundary contribution (2.9) to $C_\parallel^*[\xi_0]$, where ξ_0 is the generator of "constant" translations in v. These "classical" values may be subject to quantum corrections, but for macroscopic black holes, these will be small. Indeed, making the usual quantum substitution $\{\,,\,\} \to \frac{1}{i\hbar}[\,,\,]$ and restoring the factors of $16\pi G$, the first equation in (2.7) becomes

$$\left[\frac{1}{16\pi\hbar G}C_\parallel[\xi], \frac{1}{16\pi\hbar G}C_\parallel[\eta]\right] = \frac{i}{16\pi\hbar G}C_\parallel[\xi\eta' - \eta\xi'] + \frac{i\epsilon_2 A_+}{32\pi\hbar G}\int dv(\xi'\eta'' - \eta'\xi''), \quad (3.2)$$

from which we can read off the values

$$c = -\frac{3\epsilon_2 A_+}{2\hbar G}, \qquad \Delta = -\frac{1}{16\pi\hbar G}\,\xi_0 \pi_A A|_{v=v_+}. \quad (3.3)$$

The classical central charge will thus dominate as long as $|\epsilon_2|$ is large compared to the tiny quantity A_{Planck}/A_+. For Planck-sized black holes, on the other hand, we might expect quantum corrections to become important, leading to a breakdown in this analysis.

It remains to determine the parameter ξ_0 in (3.2). To define energy in an asymptotically flat spacetime, one may fix the normalization of ξ_0 at infinity. Working only on the stretched horizon, though, we do not have that luxury; as in the "isolated horizons" program [22], we must cope with an uncertainty in the normalization of the horizon diffeomorphisms. As noted in [20], though, there is one natural choice associated with a stretched horizon:

$$z = e^{2\pi i A/A_+} \quad (3.4)$$

gives us a good intrinsic coordinate associated with such a surface. We can therefore parametrize diffeomorphisms by vector fields

$$\xi_n = \frac{A_+}{2\pi A'}z^n, \quad (3.5)$$

where the prefactor has been chosen to ensure that $[\xi_m, \xi_n] = i(n - m)\xi_{m+n}$. Of course, these fields are only defined on a stretched horizon—A' is zero at the actual horizon—but our canonical formalism breaks down on the actual horizon as well. (Note that $z = 1$ at the horizon, so the diffeomorphisms (3.5) reduce to a single constant, albeit infinite, shift.)

Inserting (3.5) into (3.2), we see that

$$\Delta = -\frac{1}{16\pi\hbar G}\,\xi_0 \pi_A A|_{v_+} = -\frac{A_+^2}{32\pi^2\hbar G}\frac{\pi_A}{A'}\bigg|_{v_+} = -\frac{A_+}{16\pi^2\epsilon_2\hbar G} \quad (3.6)$$

where I have used the constraint $K_2 = 0$ in the last equality. Combining (3.1), (3.3), and (3.6), and assuming that $\Delta_0 = 0$, we obtain a microcanonical entropy

$$S = \ln\rho(\Delta) = 2\pi\sqrt{\frac{\epsilon_2 A_+}{4\hbar G} \cdot \frac{A_+}{16\pi^2\epsilon_2\hbar G}} = \frac{A_+}{4\hbar G} \quad (3.7)$$

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which agrees precisely with the expected Bekenstein-Hawking entropy.

We thus see that the entropy of a two-dimensional black hole can, indeed, be determined almost uniquely from the imposition of appropriate "horizon constraints." The Cardy formula does not tell us *what* states are being counted; for that, one needs a detailed microscopic theory. It does, however, fix the actual microscopic density of states: we have not merely reproduced black hole thermodynamics, but have made a genuine statement about the underlying statistical mechanics.

Furthermore, this derivation gives us *some* information about the microscopic states. In standard approaches to quantum gravity, physical states are annihilated by all of the constraints, including the diffeomorphism constraint $C_\parallel^*$. But the appearance of a central extension in the constraint algebra makes this impossible: the condition $C_\parallel^*|phys\rangle = 0$ is inconsistent with the brackets (3.2). One must instead impose a less restrictive condition, for example

$$\langle phys|C_\parallel^*|phys\rangle = 0, \tag{3.8}$$

which can be satisfied by states that would otherwise be discarded as "pure gauge." We thus confirm the picture of [1], that black hole entropy comes from "would-be pure gauge degrees of freedom" that become physical because boundary conditions relax the constraints.

These results agree with those of Ref. [20], although our central charge (2.8) and classical "energy" (3.6) differ by (opposite) factors of two from that paper. The difference can be traced to the fact that Virasoro generator (2.6) of this paper *nearly* generates the near-horizon conformal symmetry of [20], but with a factor of $1/2$. This, in turn, may reflect a difference in the choice of what is held fixed at the stretched horizon—in [20], l^a is fixed—but a more detailed understanding would be desirable. In particular, it has recently been shown that a near-horizon conformal symmetry exists for any Killing horizon [30]; one might hope that this symmetry could be related to the existence of horizon constraints of the type explored here.

A key question is how sensitive this derivation is to the details of the stretched horizon constraints. Some flexibility certainly exists. One can, for instance, choose $a \neq -2$ in (2.2), and remove the resulting anomaly in the $\{C_\parallel, C_\pi\}^*$ bracket by defining an "improved" generator $\widehat{C}_\parallel \sim C_\parallel + bC_\pi'$. The resulting central charge then agrees with (2.8) for any value of a. Still, a more systematic understanding would be helpful. A possible avenue would be to repeat this analysis in a genuine light-cone quantization at the true horizon, thus removing any ambiguity in the definition of the stretched horizon. The appearance of new second class constraints makes such a program difficult [23, 24], but it might be manageable in our relatively simple two-dimensional setting.

While the final expression (3.7) for the entropy is independent of the "stretching parameters" ϵ_1 and ϵ_2, one might worry that the central charge and the eigenvalue Δ depend on ϵ_2, and that the central charge becomes very small as we approach the true horizon. This behavior may indicate that we are missing some important physics;

one might hope that an analysis in light-cone quantization, for example, will give a cutoff-independent value for the central charge. It is worth noting, though, that in a one-dimensional conformal theory with no obvious preferred periodicity, c and Δ do not have clear independent physical meanings: as Bañados has observed [27], the transformation

$$L_n \to \frac{1}{k} L_{kn} \tag{3.9}$$

gives a new Virasoro algebra with $c \to kc$ and $\Delta \to \Delta/k$, while preserving the density of states (3.1). A similar ambiguity occurs in other approaches to near-horizon state-counting, such as those of [10,11,13,16,20], in which c and Δ each depend on an arbitrary parameter but have an unambiguous product.

It would also be of interest to extend this picture to the Euclidean analysis of Ref. [28], to explore its relationship to the path integral approach to black hole entropy. The constraint K_2 takes a particularly natural form in that setting. The analog of the expansion is $\vartheta = n^a \nabla_a A/A$, while $\kappa = n^a \nabla_a N$, where n^a is the unit radial normal and N is the lapse. Our stretched horizon constraint then describes circles in the r–t plane of constant proper distance ρ from the horizon, with $\epsilon_2 = (\partial_r A_+/A_+)\rho$.

Finally, let me briefly address the limitations coming from the restriction to a two-dimensional model. While it would clearly be desirable to extend this analysis to higher dimensions, there are good reasons to expect that dilaton gravity captures the essential elements. Indeed, general relativity in any dimension can be dimensionally reduced to a two-dimensional model near the horizon of a black hole, and there are strong indications that the resulting Liouville-like theory captures the salient features [11,16,31]. The Euclidean path integral approach [28] similarly indicates that the dynamics of the conjugate variables A and κ in the r–t plane determines the entropy. It is thus plausible that the two-dimensional constraint analysis developed here will extend to black holes in any dimension.

Acknowledgments

Appendix.   The Stretched Horizon

In this appendix I describe the stretched horizon determined by the constraints (2.2) in more detail. A general two-dimensional nonextremal black hole with a horizon at $r = r_+$ can be described in Kruskal-like coordinates by a metric

$$ds^2 = -2H(r)dUdV \qquad \text{with} \quad UV = 2\kappa(r - r_+)J(r), \tag{A.1}$$

where H and J are finite and regular at the horizon. To switch to coordinates (u,v) for which $u = 0$ can be spacelike, let

$$U = u + \epsilon_1 f(v), \quad V = v. \tag{A.2}$$

From (1.1) and (1.5), it is easy to see that $\beta = 0$, $\sigma\tau = H$, and $\alpha\tau = \epsilon_1 H f'$, or with $K_1 = 0$,

$$\alpha = \epsilon_1, \quad \beta = 0, \quad \sigma = 1/f', \quad \tau = Hf'. \tag{A.3}$$

Note that the induced metric at $u = 0$ is $ds^2 = -2\epsilon_1 H f' dv^2$, which will be spacelike as long as $\epsilon_1 f' > 0$, and nearly null as long as $\epsilon_1 \ll 1$.

We next compute the momentum π_A. Equation (1.7) yields

$$\pi_A = \frac{2f''}{f'} + 2\epsilon_1 f' \frac{\dot{H}}{H}. \tag{A.4}$$

Moreover, since H is a function of r, and therefore of UV, $\dot{H}$ and H' are not independent:

$$\dot{H} = \frac{v}{\epsilon_1 (vf)'} H' \tag{A.5}$$

at $u = 0$. Inserting (A.4) and (A.5) into the constraint $K_2 = 0$, we obtain

$$\frac{A'}{\epsilon_2 A_+} - \frac{f''}{f'} - \frac{vf'}{(vf)'} \frac{H'}{H} = 0. \tag{A.6}$$

Now assume that our $u = 0$ surface is initially near the horizon, and write

$$2\kappa(r - r_+) = -\epsilon_2 x. \tag{A.7}$$

Then from (A.1) and (A.2),

$$UV = \epsilon_1 vf = 2\kappa J(r)(r - r_+) = -\epsilon_2 J(r)x \tag{A.8}$$

at $u = 0$, and

$$r' = -\frac{\epsilon_2}{2\kappa} x', \quad A' = -\epsilon_2 \frac{\partial_r A}{2\kappa} x', \quad H' = -\epsilon_2 \frac{\partial_r H}{2\kappa} x'. \tag{A.9}$$

The constraint (A.6) then becomes

$$\frac{\partial_r A}{2\kappa A_+} x' + \frac{f''}{f'} + \epsilon_2 \frac{vf'}{(vf)'} \frac{\partial_r H}{2\kappa H} x' = 0. \tag{A.10}$$

To lowest order, we see that the constraint is independent of H—this is a sort of Rindler approximation—and (A.10) is easily integrated:

$$f'(v) = c \exp\left\{ -\frac{\partial_r A}{2\kappa A_+} x \right\} = c \exp\left\{ \frac{\epsilon_1}{\epsilon_2} \frac{\partial_r A}{2\kappa A_+} \frac{1}{J_+} vf \right\}. \tag{A.11}$$
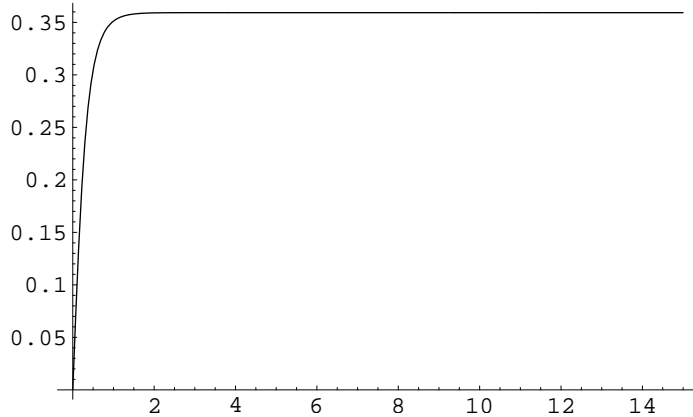
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Figure 1: $y' = e^{-axy}$ with $y(0) = 0$ and $a = 10$

If $\epsilon_2 < 0$, equation (A.11) is of the form $y' = e^{-axy}$. Numerical integration shows that f initially rises linearly, but very rapidly levels off to a nearly constant value f_0 of order $a^{-1/2}$ (see figure 1). If f is initially near zero, its asymptotic value is

$$f_0 \sim \left(\frac{\epsilon_1}{|\epsilon_2|} \frac{\partial_r A}{2\kappa A_+} \frac{1}{J_+} \right)^{-1/2}. \tag{A.12}$$

By (A.2), $u = 0$ corresponds to $U = \epsilon_1 f(V)$; our stretched horizon thus departs from the true horizon $U = 0$, but rapidly approaches a lightlike surface $U = \epsilon_1 f_0 \propto \sqrt{\epsilon_1 |\epsilon_2|} \ll 1$.

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